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MATCHING, HETEROGENEITY AND THE EVOLUTION OF INCOME DISTRIBUTION

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Abstract

This paper presents a model in which firms and workers have to engage in costly search to find a production partner. In this setting, the skill, job and wage distributions and their evolutions are endogenized. The presence of search costs implies that there are two redistributive forces in the labor market. First, because skilled workers sometimes produce with jobs intended for unskilled workers and vice versa, the gap between skilled and unskilled workers gets compressed. Second, because skilled workers always have a better outside option, unskilled wages are pushed down. We show that these forces can lead to a non-ergodic equilibrium process whereby at high levels of initial inequality, wage inequality can keep rising whereas it would be decreasing starting from lower levels of inequality. The model predicts that increasing wage inequality is more likely to arise in economies with less frictional labor markets, less redistributive taxation and less public schooling. These predictions are in line with the diverse cross-country patterns that we observe.

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I. Introduction

Income inequality in the U.S. has risen considerably over the past two and a half decades. The main component of this change is identified as increased earnings and wage inequality (see for instance Levy, 1989, Levy and Murnane, 1992, Juhn, Murphy and Pierce, 1993). Similar trends are observed in other OECD countries but appear much less pronounced than in the U.S. For instance Katz, et al (1992) report that the logarithm of the wage differential for the workers on the 90th and 10th percentiles of the wage distribution in the U.S. has increased from 1.23 to 1.38 between 1979 and 1987. The same statistic has changed from 0.88 to 1.10 in the U.K., from 1.19 to 1.22 in France and from .95 to 1.04 in Japan. The premium that college graduates earn over non-college graduates (between-group wage inequality) has also soared in the U.S. and to some extent in the U.K., while in Japan it has only changed by a small amount, and in France, it has not increased at all. The same pattern emerges from the study of Card et al (1995) who find the highest increase in wage inequality in the U.S., a moderate increase in Canada and not much action in France. Davis' (1993) who analyzes wage changes in West Germany, Netherlands, Austria and Sweden as well as the above countries, also confirms the conclusion that wage inequality has risen mostly in the Anglo-Saxon economies.

These trends towards higher inequality in a number of countries pose many academic and policy questions. Although empirical labor research has accomplished reasonable success in accounting for these changes in terms of relative supply and demand of skills (e.g. Katz and Murphy, 1992, Katz et al, 1992), we still lack an understanding of why relative demands and supplies have moved differently in different countries. For instance, Katz et al report that from 1979 to 89, the growth rate of the relative supply of college educated workers has been 0.023 in the U.S. as compared to 0.037 in the U.K., 0.050 in France and 0.029 in Japan. But in a general equilibrium model, these changes are *endogenous*, and they can be a consequence as much as a cause of the wage inequality. Similarly, the relative demand for skilled labor is endogenously determined and it is influenced by the composition of the labor force and the institutional set-up of the labor market[1]. Further, it is widely believed that the U.S. has a less

[1] Card (1990) finds that after the Mariel Boat lift in Miami despite the unprecedented increase in unskilled labor force, there was no change in unskilled relative wage nor unemployment, thus

frictional labor market than other OECD economies [this is also suggested to be true for the U.K. during the last decade or so]. It is then natural to ask whether this institutional difference is one of the main factors underlying the increase in wage inequality in the U.S. Again, we need a well-formulated general equilibrium model to answer this question. The objective of this paper is to provide such a theoretical framework to evaluate these facts while also matching the salient patterns that arise from a cross-country comparison.

Our model builds on the work of Diamond (1982), Jovanovic (1979), Mortensen (1982) and Pissarides (1985) where workers and firms have to search for the right partner in the labor market. We extend this framework in three respects: first, both workers and firms are potentially heterogenous, thus workers of different skill levels and jobs of different qualities may be simultaneously searching. Second, the composition of jobs is endogenized; firms, knowing the distribution of skills in the economy, and the wages they will pay in equilibrium, decide what kind of jobs to open. Third, as it is the case in the data (for instance Cameron and Heckman, 1992), the distribution of income is allowed to influence the distribution of skills; in other words, rich parents will be able to buy more education (skills) for their offsprings[2].

In our model frictional labor markets lead to a feature we call *mismatch.* A firm will stop looking before it finds the worker that the Walrasian auctioneer would have allocated to it; similarly, a worker will not look until he finds the job where his marginal product is highest. Therefore, compared to the frictionless market, our economy will have a degree of mismatch between the skills of workers and the requirements of jobs. The main results of this paper will be derived from the fact that the level of income inequality, through its impact on the distribution of skills, will influence the degree of mismatch in the labor market, and mismatch will introduce forces towards *redistribution* across agents of different income levels.

Our main conclusions are: *first,* more heterogeneity in the form of larger gaps between the income and thus skill level of workers will lead to more mismatch which means that more workers will be either under- or over-educated for their jobs. Additionally, since mismatch

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relative demand appears to have adjusted to accommodate the shift of relative supply.

[2] Two other recent papers are related to our work. Davis (1995) analyzes the composition of jobs and the impact of bargaining on wages, and Saint-Paul (1993) discusses the impact of a skill biased technological shock on unskilled wages and unemployment. Neither of these models endogenize the distribution of skills and income and the underlying mechanisms are different.

reduces profitability of jobs, aggregate investment falls and the economy grows less. And yet, *second*, in itself, mismatch is a stabilizing force on income distribution; mismatch implies that highly skilled workers sometimes end-up working in unskilled jobs and receive lower wages while less skilled workers get matched to higher paying jobs; therefore, the gap between the two groups becomes narrower over time. *Thirdly*, however, there is another aspect which we call the *outside option effect*: wages in a non-competitive economy are in general influenced by the outside options of workers (and firms) and these outside options counteract the equalizing effect of mismatch; because the outside option of the skilled workers is always better than that of the unskilled, unskilled wages decrease relative to the earnings of the skilled workers. It is shown that this outside option effect increases with the degree of inequality, and while an economy with limited inequality will exhibit decreasing inequality over time and have a steady growth, the same economy, when it starts with higher initial inequality, will have increasing wage and income inequality and lower growth. Put differently, the impact of labor market frictions on wage determination makes the dynamics of inequality *non-ergodic,* and leads to multiple limiting distributions of income.

The presence of two counteracting redistributive forces implies that our model can account for both increasing and decreasing wage inequality depending on the institutional features and the starting level of inequality of the economy. [This contrasts with the simplest form of relative supply-demand story which needs repeated positive shocks to the relative demand for skilled workers]. The model also predicts that economies with less frictional labor markets, less redistributive taxation and less public schooling should experience more wage inequality than the rest (or should have increasing wage inequality while the rest may have no change). These findings accord with the cross-country patterns that we observe [i.e. the U.S., the U.K., Canada versus France, Japan, Germany and Sweden[3]]. The model also suggests a negative correlation between the growth rate and the level of inequality, as found by the cross-country studies of Persson and Tabellini (1993), Perotti (1994,95).

Finally, labor economists have observed that in most periods within group and between

[3] An alternative approach is to explain the cross-country patterns by different technology shocks for different countries. This approach is however not very satisfactory since OECD countries should be affected by the same shocks. Indeed Card et al (1995) find that the measures of skilled biased technological change to be identical for the U.S., Canada and France.

group inequality move in the same direction; that is, when wage differences increase between workers of different skill levels, so do wage differences among equally skilled workers (see for instance Goldin and Margo, 1992, for the 50s and 60s and Juhn, Murphy and Pierce, 1993). Yet, despite the importance of the changes in this aspect of inequality, in Levy and Murnane's words (1992, p.1372) *"the most important unresolved puzzle concerns the reasons for the almost 20 years trend towards increased within group inequality"*. Our theory can be useful here. The pattern we predict is that when inequality of skills increases, firms start creating a more diverse job distribution, and workers of the same skill level sometimes end-up working with jobs of different qualities; this therefore explains the rise in within group inequality. However, we will also suggest a mechanism that can account for within and between group wage inequalities moving in opposite directions as in the U.S. economy of the 1970s.

Before proceeding with the analysis it is useful to ask whether the microfoundations that we build on are plausible. Considerable evidence suggests that job search is an important and costly activity. Ruhm (1991), Topel and Ward (1992) show how workers are only slowly allocated to the jobs most suited to them. Barron et al (1989) find that employers spend considerable resources in recruiting activities, and this is confirmed by the case studies in Levy and Murnane (1995). The findings of high correlation between profitability of firms and wages both in union and non-union sectors [e.g. Abowd and Lemieux, 1993, Dickens and Katz, 1987, Estevao and Tevlin, 1995, Katz and Summers, 1989] support the view that there is rent-sharing in the labor market, thus bargaining powers and outside options should indeed be expected to play an important role in wage determination. Finally, there is a large empirical literature on over- and under-education (e.g. Rumberger, 1981) which suggests that the mismatch even with respect to easily observed characteristics such as years of schooling is very common. The most interesting piece for us is a careful paper using the PSID by Sicherman (1991). He finds that workers who have more education than the reported amount that is required for the job (over-educated workers) earn more than other workers doing the same job but less than a typical worker with the same education level and characteristics. Conversely, workers who appear under-educated earn less than others doing the same job but more than a typical worker with the same characteristics. Although the evidence could also be interpreted as unobserved heterogeneity, i.e. over-educated workers having less "unobserved capital", this is not consistent with the rest of Sicherman's results. In particular, he also finds that over-educated workers have a significantly higher probability of moving to a better job, which means that on average these workers are truly over-qualified for the job they are performing and more importantly, this reallocation is quite slow implying that "mismatch" is not a very transitory phenomenon. To give a concrete example, mismatch would arise if a firm which has installed a new equipment accepts workers who require training while some other workers who can use this equipment with less training are also available somewhere in the labor market.

The plan of the paper is as follows. The next section presents a simple dynamic model and characterizes the equilibrium of this economy in the presence of frictionless markets as a benchmark case. Section III analyzes the dynamics of inequality with random matching in the context of a two-class economy. This section also conducts the comparative static exercises regarding labor market efficiency, redistributive taxation and public education. Section IV analyzes the dynamics of inequality starting from general income distributions. Section V turns to different matching technologies. It demonstrates the robustness of our results and also links within group inequality to changes in between group inequality. Section VI concludes. Appendix A contains the proofs. Appendix B extends our results to economies with more general transmission technologies. Appendix C which is available upon request includes the game form used for wage determination and some further robustness results.

II. The Basic Environment and the Walrasian Equilibrium

II.1. Preferences, Timing of Events, Technology

Each agent lives for three periods. The first is youth in which he (he or she) acquires human capital. In the second, middle age, he works and decides how much to consume. In this period, each agent also conceives a unique offspring, and decides how much to spend on his offspring's education and also how much to save for the final, retirement, period of his life out of his wage income. We refer to an agent as of generation t, if he is middle aged at time t. In each generation, there is a continuum of agents with size normalized to 1. The utility function of agent j of generation t is;

(1) $$(1-\delta-\gamma)\log c_{j,t}+\delta\log e_{j,t+1}+\gamma\log c^{O}_{j,t+1}$$

where c denotes consumption, the superscript O is used for old age, and e denotes the education expenditure. The utility function exhibits impure altruism as the agent does not obtain utility from the welfare of his child but from the education he gives him.

All that the agent decides to save this period is invested at the gross market rate of return R_{t+1}. Thus the budget constraint of the agent is

(2) $$c_{j,t}+\frac{c^{O}_{j,t+1}}{R_{t+1}}+e_{j,t+1}=w_{j,t},$$

where $w_{j,t}$ is the wage income of the individual j, which is his only income when he is middle aged (only the retired are rentiers).

With this specification, we obtain the following simple decision rules for agent j;

(3) $$c_{j,t}=(1-\delta-\gamma)w_{j,t},\quad c^{O}_{j,t+1}=R_{t+1}\gamma w_{j,t},\quad e_{j,t+1}=\delta w_{j,t},\quad s_{j,t}=\frac{1}{R_{t+1}}(c^{O}_{j,t+1})=\gamma w_{j,t},$$

where $s_{j,t}$ denotes the savings. The human capital of the next generation is determined by the education level (expenditure) given to them by their parents. More precisely,

(4) $$h_{j,t+1}=\begin{cases} e_{j,t+1} & if \quad e_{j,t+1}\geq h_{\min} \\ h_{\min} & if \quad e_{j,t+1}<h_{\min}, \end{cases}$$

In other words, there is a certain level of human capital, h_{min}, below which agents do not fall -- e.g. what they learn from compulsory schooling or from their friends, etc. This is a very simple specification for the intergenerational transmission of human capital; in particular, the human capital of the offspring is linear in the wage of the parent, which will in turn be linear in the human capital of the parent in the competitive equilibrium. Appendix B shows the robustness of our results to a general transmission rule of the form $h_{t+1}=\psi(w_t,h_t)$.

We assume that there is a continuum of firms equal to 1, and that each firm employs one worker, thus we can think of these firms as jobs. The measure of firms is taken to be equal to that of workers so as to avoid issues of unemployment[4]. A firm (job) i, if matched

[4] Although unemployment is closely related to the issues of wage inequality [because unemployed workers are poor and also because unskilled workers are more likely to be unemployed], the issues of unemployment are distinct from the basic redistributive forces that frictions in the labor market introduce. Therefore, in order to focus on our primary interest, we have chosen an economy without unemployment. The issues of unemployment are analyzed in similar but simpler setting in Acemoglu (1995b).

with worker j, produces

(5) $$y_{i,j,t}=Ak_{i,t}^{1-\alpha}h_{j,t}^{\alpha},$$

where $k_{i,t}$ is the capital associated to this job, and $h_{j,t}$ is the human capital that worker j brings to the relation. The production function has constant returns to scale and complementarity between physical and human capital. As a result, there are diminishing returns to human capital. In this frictionless (competitive) benchmark case, it is important that the production function has non-increasing returns to scale, because otherwise a competitive equilibrium would not exist, and also, since there is nothing that pins down jobs in this economy, it should not be possible to produce more output by combining two jobs.

The capital that the firms will use at time t comes from the savings of generation t-1 workers. All workers invest their savings in a mutual fund which is forced to make zero profits by the threat of potential entry (a perfectly contestable market). The savings are then allocated to the firms. The intermediation by the mutual fund removes all uncertainty from the rate of return on savings. We also assume that the economy is small and open, thus it faces a constant world interest rate R at which the mutual fund can borrow or invest, therefore, the cost of capital to a firm and the rate of return on savings are fixed at R[5].

We assume that firms hire their capital stock at the beginning of period t and this decision is completely *irreversible*. Thus at that point, while they know the distribution of human capital in the labor market, the do not necessarily know which worker they will employ. In practice k_t does not only correspond to capital stock but also to the quality of the equipment, the type of job, the location, etc, and so it is natural that it has a high degree of irreversibility. The result of the firms' choices of capital stock will in general give a distribution of jobs and we denote this distribution by $P_t(k)$.

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[5] An alternative is to have a linear household technology with a rate of return R. Then a simple condition, namely $[(1-\alpha)A/R]^{1/\alpha}\alpha\delta\leq\gamma$, ensures that the economy always invests some funds in this linear technology and thus fixes the cost of capital at R. In the absence of a linear technology and other investment opportunities, all our result would hold except those regarding the growth rates. This is of course natural; growth in this economy is driven by the total amount of investment and in the absence of an alternative investment opportunity, all that is saved will be invested and moreover, with logarithmic preferences savings are independent of the rate of return - i.e. (4) -, hence the economy would achieve the same growth rate irrespective of the rate of return on capital.

II.2. The Walrasian Equilibrium

In this section we will model the labor market as frictionless. Thus, all firms compete a la Bertrand for all the workers in the labor market. This implies that workers will be paid their marginal product, and the allocation of the heterogeneous workers to the heterogenous firms is the same as the one that a Walrasian auctioneer (or a Social Planner) would choose.

Given the distribution of human capitals, $F_t(h)$, firms' investment choices determine $P_t(k)$. Then given the complementarity of human and physical capital, the highest human capital worker is allocated to the highest physical capital firm (e.g. Sattinger, 1993 - see section V.1. for a formal definition). If according to this allocation rule, a worker with human capital h is allocated to a job with physical capital k at time t, then we use the notation, $(h,k)\in\mathcal{P}_t$. Then since workers are paid their marginal product, the wage of a worker of generation t that has a human capital h is given by

$$w_t(h)=\alpha A k^{1-\alpha} h^{\alpha-1} \quad where \ (h,k)\in\mathcal{P}_t. \tag{6}$$

We also introduce the variable θ such that $h_{j,t}=\tilde{h}_t\theta_{j,t}$ where $\tilde{h}_t$ is the median level of human capital at time t, and thus the distribution of θ measures inequality of skills relative to the median. This distribution can be simply obtained from the distribution of h and we denote it by $G(\theta)$. Let us also denote the starting distribution of human capital by $F_0(.)$ and the corresponding distribution of inequality by $G_0(.)$. Further, we assume as in the rest of the paper that even the poorest agent j* starts with $h_{j^*,0}>>h_{min}$. Then we can state[6];

Proposition 1: *Let $G_0(\theta)$ be the distribution of initial inequality of human capital as defined above. Then, with frictionless labor markets;*

(i) The physical to human capital ratio for all jobs is given by $\frac{k}{h}=\left(\frac{(1-\alpha)A}{R}\right)^{\frac{1}{\alpha}}$.

(ii) $\forall$ G_0, $G_t=G_0$ for all t, thus inequality in this economy self-replicates.

(iii) The growth rate of the economy is always $g^C=\delta\alpha A\{(1-\alpha)AR^{-1}\}^{\frac{1-\alpha}{\alpha}}-1$.

The proof of this proposition, like all the others in this paper, is in Appendix A. Here

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[6] If the poorest agent has less than h_{min}, then in the first period there will be a contraction in inequality and inequality would self-replicate forever from then on. The reason we start away from h_{min} is to avoid having the max{.,.} notation.

it suffices to give the main intuition. The economy is linear and this is the feature that leads to steady growth as in the model of Rebelo (1991). More importantly for the focus of this paper, given constant returns to scale, the rate of return on human capital is constant; that is, a worker with twice as much human capital as another works with twice as much physical capital and thus earns twice as much. With logarithmic preferences (which would also be true with CRRA preferences), the accumulation rules are also linear. Thus the offspring of worker j who has twice as much capital as j' will also have twice as much capital as the offspring of j'. This result emphasizes that we obviously have a special model here, but it is only chosen to make our results in the rest of the paper as transparent as possible, and this can be seen in Appendix B where the robustness of our results to non-linearities is demonstrated.

Note finally that in this economy, there are many missing markets: young generations cannot pay their parents to get more human capital in their youth (a form of intergenerational credit constraints). But despite these missing markets, inequality is not harmful to aggregate performance. The economy has the same growth rate irrespective of the level of inequality.

III. Inequality With Random Matching: A Two Class Economy

III.1. Preliminaries

We now analyze the economy outlined in the last section but with a frictional labor market. The key assumption is that it is costly for agents to engage in search activities. In the beginning of the period, all firms and workers are matched 1-to-1, thus each job will have a worker, and each worker will have a job. At this point, either party can terminate the relation and look for a new partner. Costly matching refers to the fact that engaging in further search is costly to both firms and workers. We model this by assuming that workers' and firms' working lives are divided into two segments of length $1-\eta$ and η. If the worker (or the firm) decides to look for another match, they do not produce in the first segment and only meet another partner from the pool of unmatched agents for the second segment of their lives; after this they cannot break the match again[7]. Thus η captures the degree of frictions in the labor

[7] Appendix C shows that all our results are unchanged if parties can change as many partners as they like - the reason we limit ourselves to one round of change is to simplify the expressions and the discussion in the text. The advantage of having infinite sampling as in Appendix C is that the economy would converge to the perfectly competitive case analyzed in the last section as $\eta \to 1$ (e.g.

market. When $\eta=0$, mobility is very costly; after separation no output is produced.

In this section we start with the *extreme assumption* that matching is random. This implies that any two workers, even if they have unequal human capital levels, face exactly the same probability of meeting a given firm in the economy and conversely, the same also applies to firms. This assumption will be relaxed in section V.

There are a number of other issues to deal with. First, since there are costs of changing partners, each match has some surplus to be shared, thus a bargaining rule needs to be assumed. We will follow the usual practice of using Nash bargaining [a game theoretic justification is given in appendix C]. The second issue is that when η is near 1, some firms and workers will want to separate from their first match and look for another partner. We want to avoid this issue as it will lead to unemployment. Therefore, throughout the paper we will deal with the case where $\eta<\eta^*$ such that all firms and workers produce with the first partner they meet [this is trivially true when $\eta=0$, Appendix C derives the exact expression for η^*]. As a result of this assumption in the class of economies that we analyze, outside options will influence wages through the threat of separations, but in equilibrium, there will be no actual separations. This introduces a minor problem; since there are no separations in equilibrium, a worker (or a firm) who becomes separated will not be able to find a new partner. To avoid this problem we assume that each worker and firm face a probability ν of not getting matched in the first segment of his or its life, and this event is independent of the human or physical capital level of the agent. We will analyze this economy as $\nu\rightarrow 0$, thus the set of agents who are unmatched will be of measure zero. Also since the event of being unmatched is independent of characteristics, the distribution of these unmatched agents is exactly the same as their initial distributions, thus F_t for the workers and P_t for the firms.

With this formulation, the wage level of a worker with human capital $h_{j,t}$ who is matched with a firm of physical capital $k_{i,t}$, when the distribution of human and physical capital are respectively $F_t(h)$ and $P_t(k)$, is given by [see Appendix C for details];

$$(7)\qquad w[h_{j,t},k_{i,t},F_t(h),P_t(k)]=\beta A h_{j,t}^{\alpha}k_{i,t}^{1-\alpha}-\beta(1-\beta)\eta A k_t^{1-\alpha}\int h^{\alpha}dF_t(h)+(1-\beta)\beta\eta\int A k^{1-\alpha}h_{j,t}^{\alpha}dP_t(k).$$

Gale, 1987) and this shows that the only difference between the economy of section II and the one we are analyzing now arises from the labor market frictions.

The first term is the share of total output that the worker gets. The second is the threat point of the firm and is thus deducted from the wages, and the third is the worker's threat point and adds to his share of the surplus [since we have transferable utility]. Recall that we are looking at an equilibrium in which there are no separations. Thus, if a firm deviates and separates, it will meet one of the unmatched workers, and these workers have a distribution of human capital given by $F_t(h)$. Since there is only one more round of matching, it has to produce with the worker it meets and as there is no possibility of separation, the worker obtains a proportion β of the surplus and the firm gets the rest. This term is subtracted from the total surplus that the parties share and since the worker obtains a proportion β of the total surplus, it is also multiplied by β. The third term is explained similarly as the deviation of the worker to make one more round of search among the pool of unmatched firms, $P_t(k)$ and this is subtracted from the total surplus and then also added to the total wage of the worker as his threat point. The sum of these three terms give (7). The firm's profit function is

$$\pi[k_t,F_t(h),P_t(k)]=\int\left\{Ak_t^{1-\alpha}h^{\alpha}-w[h,k_t,F_t(h),P_t(k)]\right\}dF_t(h)-Rk_t, \tag{8}$$

which the firm will maximize by choosing k_t. It is easily seen that (8) is strictly concave in k_t. Thus, given the assumption that $\eta<\eta^*$ which ensures that there will be no separations, all firms choose the same level of physical capital investment equal to;

$$k_t=\left[\frac{(1-\beta)(1+\beta\eta)(1-\alpha)\int h^{\alpha}dF_t(h)}{R}\right]^{\frac{1}{\alpha}}. \tag{9}$$

We can note a number of important features in equation (9). First because the firm is unable to capture the full marginal product of its investment, it will always underinvest [see Acemoglu, 1995a] and for this reason, the growth rate of this economy will always different than the growth rate of the competitive economy, g^C. More interestingly, however, firm level investments also depend on the distribution of human capital, $F_t(.)$. We can see immediately from (9) that a mean-preserving spread of F_t will reduce investment. The intuition is that, a mean-preserving spread which maintains the total amount of human capital the same but makes it more unequally distributed, increases *mismatch* and thus reduces profits. Mismatch in this economy is reflected by the distribution of human to physical capital ratios and recall that in the Walrasian case these ratios are always constant. In contrast, in this economy, since all firms choose to have the same level of physical capital, F_t also gives the distribution of the human to physical capital ratios and an increase in inequality in the distribution of human capital increases mismatch. A slightly different way of seeing the intuition is to note that because there are decreasing returns to human capital, a firm loses more from a low human capital workers than it gains from a high human capital worker.

III.2. Analysis of Inequality and Growth Dynamics with Two Classes

To simplify the analysis we start with the case where the economy starts with two groups of workers; rich and poor. We denote the human capital of the rich by h_{2t} and that of the poor by h_{1t}; we then study the dynamics of $\phi_t = h_{1t}/h_{2t}$, the ratio of human capital and income between the two groups. We let the proportion of poor agents be λ. This proportion will never change, but poor agents may become gradually less poor relative to the rich.

From equation (7), wages of each group, $j=1,2$, can be written as

$$w_{jt} = \beta A k_t^{1-\alpha} h_{jt}^{\alpha} - \beta(1-\beta)\lambda\eta A k_t^{1-\alpha} h_{1t}^{\alpha} - \beta(1-\beta)(1-\lambda)\eta A k_t^{1-\alpha} h_{2t}^{\alpha} + \beta(1-\beta)\eta A k_t^{1-\alpha} h_{jt}^{\alpha}. \tag{10}$$

Now to characterize the evolution of this system, as long as h_{1t} is away from h_{min}, we can write the law of motion of $\phi_t = h_{1t}/h_{2t}$ as follows;

$$\phi_{t+1} = \frac{h_{1,t+1}}{h_{2,t+1}} = \frac{\delta w_{1t}}{\delta w_{2t}} = \frac{[1+(1-\beta)(1-\lambda)\eta]\phi_t^{\alpha} - (1-\beta)(1-\lambda)\eta}{1+(1-\beta)\lambda\eta - (1-\beta)\lambda\eta\phi_t^{\alpha}}. \tag{11}$$

Therefore, we have a simple non-linear first-order difference equation describing the evolution of inequality. In other words, equation (11) gives the law of motion of the human capital ratio of the poor to that of the rich and then from (9) and (10) we can see what wages are in any given period. This equation, (11), obviously has a stationary point at full equality, $\phi=1$. The questions to ask are (i) whether there are others; (ii) whether this stationary point is (locally and globally) stable. Using subscript 0 to denote initial values, we have;

Proposition 2: (i) If $\eta=0$, $\forall\phi_0$, ϕ_t *monotonically converges to* $\phi_\infty=1$ *and the growth rate* g_t *monotonically converges (from below) to* $g_\infty^*=\beta\delta A\{(1-\alpha)(1-\beta)AR^{-1}\}^{\frac{1-\alpha}{\alpha}}-1$.
(ii) If $\eta>0$ *and* $\alpha[1+\eta(1\text{-}\beta)]<1$, *then* $\exists\ \phi(\lambda)\in(0,1)$: $\forall\ \phi_0\in(\phi(\lambda),1]$, ϕ_t *monotonically converges to* $\phi_\infty=1$ *and the growth rate* g_t *converges (from below) to* $g_\infty^*=\beta\delta A\{(1-\alpha)(1-\beta)(1+\beta\eta)AR^{-1}\}^{\frac{1-\alpha}{\alpha}}-1$. $\forall\ \phi_0<\phi(\lambda)$, ϕ_t *monotonically converges to* $\phi_\infty=\phi_{min}>0$, *and* g_t *converges (from above) to* $g_\infty=0$.
(iii) If $\eta>0$ *and* $\alpha[1+\eta(1\text{-}\beta)]>1$, *then* $\exists\ \phi^*\in(0,1]$ *such that* $\forall\ \phi_0<\phi^*$, $\phi_t\to\phi_\infty=\phi_{min}$ *and* $g_\infty=0$. *If* $\phi^*<1$, *then* $\forall\ \phi_0\in(\phi^*,1)$, *we have* $0<\phi_\infty<1$ *and* $0<g_\infty<g_\infty^*$.

The technical intuition of the result can be obtained from Figure 1; the broken line represents $\eta=0$, and all possible inequality levels (all ϕ's) are in the basin of attraction of full equality. However, as soon as $\eta>0$, the curve starts below the 45^0 line and it approaches $\phi_t=1$ from either below or above the 45^0 line. If it approaches from below, full equality will be unstable. To be able to approach it from above, it needs to cut the 45^0 line at some point A, and now only points to the right of A are in the basin of attraction of full equality. To the left of A, wage and income inequality increase over time. Whether this line approaches $\phi_t=1$ from above or below 45^0 can be determined by looking at

$$\frac{\partial\phi_{t+1}(\phi_t=1)}{\partial\phi_t}=\alpha+\alpha(1-\beta)\eta. \tag{12}$$

For (12) to be less than 1 - i.e. local stability of $\phi=1$ -, the condition in case (ii) of Proposition 2 needs to hold [note that when the poor hit h_{min}, the curve no longer applies and the system converges to ϕ_{min} – see proof of Proposition 2 in Appendix A].

Proposition 2 is the main result of the paper. (i) is the case where the frictions are so high that there are no *outside option* effect on wages. The wage rate is simply βy for each worker where y is the output produced. Since all firms have the same level of capital, this implies that high human capital workers (h_2) produce more than low-human capital workers (h_1) and get higher wages. However, given decreasing returns to human capital (constant returns to scale), a worker who has twice as much human capital, does not produce twice as much output. Since the accumulation rules (h_{t+1} as a function w_t) are linear, this means that his offspring is not twice as rich, and therefore the gap is getting narrower. Overtime, this process takes us to full equality. Expressed differently, when $\eta=0$, the only redistributive force that emerges from our frictional labor market is *mismatch*. Compared to the frictionless economy where the human to physical capital ratio was constant in all firms, in this economy, high human capital agents are working with lower physical to human capital ratio than low human capital agents. This is because firms not knowing who they will match with choose a certain level of capital stock that they cannot change.

Next consider the case in which $\eta > 0$. There is now an additional redistributive force due to costly matching: *the outside option effect* which increases the relative wages of the skilled workers. Intuitively, a firm, when bargaining with a skilled worker, has a worse outside option if it leaves the worker the next worker will be worse on average. In contrast, it has a strong bargaining position against the unskilled worker because the next worker will be better on average. This outside option effect, which redistributes from the poor to the rich, is stronger the larger is the gap between the skilled and the unskilled. This gives the intuition for why the mismatch effect dominates at low levels of inequality while the outside option effect dominates at high levels.

A possible conjecture after realizing the presence of two counteracting effects could have been that the mismatch effect would always dominate the outside option effect. But this conjecture is wrong. Why? The intuition lies in realizing that the distribution of jobs is *endogenous* in this economy, and outside options are determined by the *composition* of jobs. In the frictionless economy, there are firms who prefer to employ the unskilled workers (low physical capital firms targeting to employ low skill workers - see also Section V.1). In contrast, in the economy of this section all firms *dislike* employing unskilled workers (but given that $\eta < \eta^*$, they are happy <u>not</u> to segregate into two groups, one targeting the unskilled) and this depresses the relative wages of the unskilled below their level in the frictionless economy.

Note also that as we converge to full equality, the growth rate of the economy is increasing until it finally reaches g_∞^*. Also as the economy converges to maximal inequality, the growth rate is decreasing. The intuition of this result can be obtained from expression (9). In this economy, accumulation is the engine of growth, and the more is the inequality of skills, the less is investment because firms anticipate the mismatch that will reduce their profitability. This is the feature that leads to a negative link between inequality and growth in our model [see the cross-country evidence on this respect discussed in the introduction].

V.3. Remarks and Discussion

First, increasing the number of workers per firm would not change our results as long as there are decreasing returns to human capital, i.e. the return to the human capital of one worker is decreasing once all the other workers and the irreversible attributes of the job are in place. The interesting question is whether a large firm could avoid the mismatch problems: for instance, by opening a series of jobs simultaneously, a firm may mitigate the matching imperfections to some extent. For instance, a firm that will employ two workers can open one skilled and one unskilled job. But, this will clearly not solve all problems. When two skilled or two unskilled workers arrive, there will again be mismatch. It can also be thought that in the limit, if a firm could employ a continuum of workers, there would be no uncertainty regarding the qualifications of new employees. However, in practice there are limits to how large firms can become, especially when they have diverse jobs and workforces. And even large firms do not have a large number of job openings at the same time; for instance, a large firm in need of an engineer in a given period would face the same problems.

Secondly, decreasing returns to human capital plays an important role in this result. In this respect the model has a close link to the work of Benabou (1993,1995), Durlauf (1995), Fernandez and Rogerson (1994) where a concavity in one the key functions creates costs of inequality and to Tamura (1991) where it leads to convergence to full equality [see Benabou 1995 for general conditions for convergence in economies with local externalities]. The difference is that in our case, all we need is the usual assumption of decreasing returns to a factor, human capital, and in fact, without this assumption, the Walrasian analogue of this economy analyzed in section II would not be well-defined. Another related difference is that in contrast to the technological externalities in these models, in our economy all the external effects are pecuniary and are derived from the frictional nature of the labor market. A second literature that our results relate to is the one of credit market imperfections and inequality [see among others Galor and Zeira, 1992, Banerjee and Newman, 1993, Piketty, 1995]. In this literature, it is emphasized that the dynamics of income distribution can be non-ergodic, however, these papers have no frictions in the labor market, and there is no direct relation to wage inequality. However, as in our paper, non-ergodicity is due to the fact that agents of different wealth levels receive different returns on their assets.

Finally, note that in this economy, human capital investments are completely backward looking; parents, when deciding how much education to give to their offspring, do not consider the rate of return on human capital. For our purposes here, it suffices to say that our qualitative results would hold as long as there is some backward looking element in these investments; i.e. as long as the income level of the parents influences the human capital of their offspring [see Acemoglu, 1995a, for forward-looking human capital investments].

III.4. The Importance of Institutions and Implications for Cross-country Trends

Labor Market Efficiency: Labor market institutions differ greatly across countries. Large differences between US and European labor markets are often emphasized and the less "efficient" labor markets of European countries are suggested as the reason for the limited increase in inequality (e.g. Katz et al, 1994 and Bertola and Ichino, 1995). In our model, the parameter η can be treated as a measure of the degree of inefficiency of the labor market. The greater is η, the less are mobility costs and the closer are wages to marginal product. Thus a labor market that has easier turnover and is less unionized is naturally thought to correspond to one with lower η. Looking at equations (11) and (12), we can see that the higher is η (as long as it is less than η^* - thus we are not letting the economy become frictionless or near-frictionless), the more likely is inequality to increase. In particular, the system is more likely to have a stable region for lower η. A reduction in η shifts the solid curve in Figure 1 up and enlarges the basin of attraction of full equality (the region of decreasing wage inequality). Thus if we consider two identical economies *a* and *b* except that $\eta_b < \eta_a < \eta^*$ [i.e. *a* has lower search costs], then it is immediate from Figure 1 that *a* will always have more inequality in the sense of a lower ϕ_t than economy *b*. Moreover, there exists a set of initial inequality levels $F_0=F_0^a=F_0^b$ such that starting from these inequality levels, *b* will converge to full equality and steady growth, while economy *a* converges to maximum inequality and no growth.

In other words, since the main role of higher η in this economy is to redistribute income from the poor to the rich (or slow down the reverse redistribution), a higher level of η is harmful to equalization of incomes, thus it creates more inequality and more mismatch.

Implication III.1: *Inequality is more likely to increase with more efficient labor markets.*

Implication III.2: *More efficient labor markets can lead to poor long-run performance.*

These implications are in line with the view that the increasing wage and income inequality in the US are due to more efficient labor market institutions (easy turnover, low firing costs and weak unions), however, they question the conventional wisdom that this is necessarily a desirable outcome.

Redistributive Taxation: We now turn to a simple characterization of the impact of different levels of redistributive taxation. Let us assume that there is a flat tax rate equal to τ on labor income that is then redistributed in a lamp-sum fashion[8]. This means that there will be redistribution from the high human capital (rich) agents to poor agents. Since there are no wealth effects, the tax redistribution is directly reflected in the transmission of human capital. More specifically,

$$\phi_{t+1}=\frac{h_{1,t+1}}{h_{2,t+1}}=\frac{\delta w_{1t}^{p}}{\delta w_{2t}^{p}}=\frac{(1-\tau)w_{1t}+\lambda\tau w_{1t}+(1-\lambda)\tau w_{2t}}{(1-\tau)w_{2t}+\lambda\tau w_{1t}+(1-\lambda)\tau w_{2t}}=\frac{w_{1t}+\tau(1-\lambda)(w_{2t}-w_{1t})}{w_{2t}-\tau\lambda(w_{2t}-w_{1t})} \tag{13}$$

where w^p denotes the post tax income and as before, w denotes the actual wages and the subscripts are again used to distinguish the two groups. Now substituting from (10), we obtain

$$\phi_{t+1}=\frac{[1+(1-\beta)(1-\lambda)\eta]\phi_t^{\alpha}-(1-\beta)(1-\lambda)\eta+(1-\lambda)\tau[1+(1-\beta)\eta](1-\phi_t^{\alpha})}{+(1-\beta)\lambda\eta-(1-\beta)\lambda\eta\phi_t^{\alpha}-\lambda\tau[1+(1-\beta)\eta](1-\phi_t^{\alpha})} \tag{14}$$

This difference equation, like (11), has a steady state at $\phi=1$. To investigate the stability properties, we again look at;

$$\frac{\partial\phi_{t+1}(\phi_t=1)}{\partial\phi_t}=\alpha[1+(1-\beta)\eta-\tau(1+(1-\beta)\eta)] \tag{15}$$

Thus, the economy with redistributive taxation will tend to be more stable since the slope of the curve $\phi_{t+1}(\phi_t)$ near the steady state $\phi=1$ is always decreasing in τ. Also,

$$\phi_{t+1}(\phi_t=0)=\frac{-(1-\beta)(1-\lambda)\eta+\tau(1-\lambda)[1+(1-\beta)\eta]}{1+(1-\beta)\lambda\eta-\tau\lambda[1+(1-\beta)\eta]} \tag{16}$$

is increasing in τ. Thus an increase in τ shifts the solid curve in Figure 1 upward (and also

[8] There is also the issue of capital income taxation. We can assume that the capital income is taxed in the same proportion and redistributed among the retired generation or is not taxed at all. This does not change our results. The only short-cut we are taking here is that we are ignoring the possible effects of labor income taxation on the Nash Bargaining between firms and workers.

flattens it). Moreover, $\phi_{t+1}(\phi_t=0)$ is no longer always negative, and for high enough values of τ, the system may become globally stable, hence converge to full equality from all starting distributions. Thus;

Implication III.3: *Redistributive taxation reduces the forces that lead to inequality in the pre-tax income and wage distribution, thus stabilizes the system. It also tends to increase growth in the long-run.*

Therefore, this comparative static suggests that societies with less redistributive tax systems, again the U.S. and the U.K., should have had a more pronounced increase in their *pre-tax* wage and income inequality, and perhaps more adverse effects on their overall performance. Also, interestingly, despite the obvious and often emphasized disincentive effects of redistributive activities on growth, Perotti (1995) finds that measures of such activities are positively correlated with subsequent growth.

Public Education: A similar analysis can be conducted for the impact of public education. The effects are quite intuitive. More public education, similar to redistributive taxation, reduces the link between a parent's wage and the offspring's human capital, as a result more reliance on public education reduces the forces that lead to increasing wage inequality and thus increases growth. This is again in line with the pattern of cross-country trends that we observe whereby the U.S., the economy with the least amount of public education (especially at the college level), has experienced the most dramatic increase in wage inequality.

IV. Inequality Dynamics With General Income Distributions

In this section, we will demonstrate that the results of the previous section are robust to richer starting distributions of inequality than two classes. Our analysis will also give us some new predictions. We still assume that wages are given by (7) [that is, η is smaller than an appropriately defined η^{**} such that again there are no separations along the equilibrium path]. We start with an arbitrary distribution $F_t(h)$ and a corresponding distribution of relative wealth $G_t(\theta)$. The question is when $G_{t+1}(\theta)$ will be more disperse than $G_t(\theta)$.

As long as the poorest agent is away from h_{min}, the dynamics of inequality can now be determined from the following equation;

(17)
$$\theta_{j,t+1}=\frac{h_{j,t+1}}{\bar{h}_{t+1}}=\frac{\beta[1+(1-\beta)\eta]h_{j,t}^{\alpha}-\beta(1-\beta)\eta\int h^{\alpha}dF_t(h)}{\beta[1+(1-\beta)\eta]\bar{h}_t^{\alpha}-\beta(1-\beta)\eta\int h^{\alpha}dF_t(h)}=\frac{\beta[1+(1-\beta)\eta]\theta_{j,t}^{\alpha}-\beta(1-\beta)\eta\int\theta^{\alpha}dG_t(\theta)}{\beta[1+(1-\beta)\eta]-\beta(1-\beta)\eta\int\theta^{\alpha}dG_t(\theta)}$$

where recall that $\bar{h}_t$ is the median of the human capital distribution at time t. In contrast, when the poorest agent hits h_{min}, then for that agent we have $\theta_{j,t}=\theta_{min}=\frac{h_{min}}{\bar{h}}$. Therefore, the dynamics of inequality are determined by a dynamic functional equation. Although it is not possible to solve these kinds of equations, it is still possible to ascertain a number of useful features about the dynamics of inequality. The main results are summarized in Proposition 3.

Proposition 3: A) Suppose $\eta=0$, then $\forall G_0$, G_t converges monotonically to full equality and the growth rate of the economy converges monotonically to $g_\infty{}^*$.

B) Suppose $\eta>0$. Then;

(i) Full equality is always a stationary distribution.

(ii) There always exist a stationary distribution with two or three groups. One group is with positive measure at h_{min}; one group is with positive measure at some h above the median; and one more group may also have positive measure at the median.

(iii) No stationary distributions other than the ones in (i) and (ii) exist.

(iv) If $\alpha[1+(1-\beta)\eta]<1$, then full equality is locally stable and otherwise, it is not.

(v) Maximal inequality with two groups is always locally stable.

The first part of the result is immediate. With $\eta=0$, only the mismatch effect is present as before, this leads to global stability of full equality. With $\eta>0$, full equality is always a stationary distribution as inspection of (17) shows immediately. Also, the exact same condition that we had in the two-class economy ensures the local stability of this stationary distribution. To see what other stationary distributions are possible, we make use of Figure 2. This Figure plots (17) with $\theta_{j,t+1}$ on the vertical and $\theta_{j,t}$ on the horizontal axis for a given distribution of inequality at time t [for a given value of the integral $\int\theta^{\alpha}dG_t(\theta)$]. The curve is upward sloping and strictly concave. It always intersects the 45^0 line at $\theta=1$ and never for $\theta<1$ and it may intersect 45^0 one more time at some $\theta > 1$. A stationary distribution must have the property that $\theta_{j,t+1} = \theta_{j,t}$, for all j, thus we cannot have a stationary distribution that has positive weight at a level below the median unless for this group, this equation does not apply, i.e. a group at h_{min}. Also, by the same argument, the stationary distribution can have positive weight on one group above the median.

IV.2. Which Distributions are Likely to Lead to Increasing Wage Inequality

What types of distributions of income and skill create forces that lead to inequality? In terms of equation (17) this corresponds to investigating the role that the integral term plays in determining convergence. To answer this question we investigate how the conditions for a "poor" agent getting richer are affected by the integral term.

Lemma 1: Suppose infimum $\{\theta_t\} < \alpha^{\frac{1}{1-\alpha}}$, *then* $\exists \bar{\theta}_t$, *such that* $\theta_t < \bar{\theta}_t$ *get poorer and* $\theta \in (\bar{\theta}_t, 1)$ *get richer.* $\bar{\theta}_t$ *is increasing in* $\int \theta^\alpha dG_t(\theta)$.

The economic intuition of this result is straightforward. The integral term is related to the outside option of firms [observe that when $\eta = 0$, these terms disappear]. If the outside option of firms is sufficiently high - a large value of the integral -, then there exists a section at the bottom of the distribution who have to accept a very low wage to get employed and therefore, this group will get poorer while the rest of the population converges to a higher level of skills and income.

When is the integral term (the outside option) high? Intuitively, the outside option of the firms will be high if they can get a highly skilled worker with a high likelihood. Therefore, when the distribution of skills is unequal, the outside option of firms is not high. Since it is the differential outside options of firms that introduces the forces towards increasing wage inequality, convergence to full equality is most difficult when the distribution is skewed to the right [that is, when there are a large number of rich agents with a smaller group of agents who are sufficiently poor relative to them]. In this case, the integral term would be large [thus not much uncertainty about the workers that the firm can get at the second round of matching] but there will still exist some workers who are poor enough that they have to accept lower wages and thus get poorer over time.

Implication IV.1: Increasing wage inequality is likely to be a more serious problem when the distribution of income and skills are skewed to the right.

If we think of real world economies, this is a blessing and a curse. Firstly, real world income distributions are skewed to the left and thus according to our result, increasing wage inequality will be a less serious problem. However, in the real world income and skill inequality are not as closely related as in our model. The recent increase in college attendance may suggest that the skill distribution is increasingly skewed to the right and this would increase the risk of increasing wage inequality.

IV.3. Inequality Cycles

Proposition 3 only dealt with stationary distributions. However, a dynamic system may also settle into a cycle. Whether our economy can generate endogenous cycles is of some interest as it illustrates the interaction of the counteracting forces in the model. The discussion around Lemma 1 suggests that inequality cycles may be possible in this model. Intuitively, poor agents are more likely to become poor when income inequality is limited, but when they get poorer, income inequality increases, and because the integral term in (17) falls, they may again get richer. However, we know that with two-groups, Proposition 2 gave the complete characterization and there were no cycles. In fact, we will find that two groups is a special case and with richer starting distributions of income, cycles are a generic possibility.

We prove the possibility of two-cycles by constructing an example. Consider the following economy consisting of three groups: $\theta^H > \theta^M = 1 > \theta^L$ with proportions λ^H, λ^M and λ^L such that $\lambda^L\theta^L+\lambda^M+\lambda^H\theta^H=1$ and $\lambda^L+\lambda^M+\lambda^H=1$. Figure 3 draws the possibility of a two-cycle. In odd numbered periods, inequality increases and the share of high group goes up from θ_2^H to θ_1^H. Similarly, the share of the Low group goes from θ_2^L to θ_1^L. In contrast, in even numbered periods, the share of the Low group goes up to $\theta_2^L > \theta_1^L$ and that of the High falls to $\theta_2^H < \theta_1^H$ - and of course that of the median/mean group is always at 1. Since inequality is higher in odd numbered periods, the curve in Figure 3 that describes transitions is the broken one. Whereas in even numbered periods, inequality decreases, thus the solid curve in Figure 3 applies, and this curve is a tilted image of the broken one around $\theta = 1$. The intuition of this figure follows from Lemma 1. If the integral increases, the poor get poorer and the rich get richer and hence the tilt around $\theta = 1$ to the solid curve.

It can also be verified that when we have more than three groups, three-period cycles are possible and thus from Sarkovski's Theorem [see Grandmont, 1985, Li and York, 1975], cycles of any periodicity and chaotic behavior can arise in this economy.

Proposition 4: (i) In a three group economy, there always exists a vector $(\lambda^L, \lambda^H, \theta_1^L, \theta_1^H, \theta_2^L, \theta_2^H)$ *thus a corresponding set of starting values, which lead a two-cycle.*

(ii) In a four group economy, there exists a set of starting distributions that lead to cycles of any periodicity and chaotic behavior.

V. More General Matching Technologies, Mismatch and Implications for Within versus Between Group Inequality

The premise that an increase in heterogeneity will lead to more mismatch has played an important role in our analysis so far and was derived for the special case of random matching. However, it is quite clear that for many situations random matching is not an appropriate assumption. For instance, skilled workers do not look for unskilled jobs nor do firms post vacancies that are open to all skill categories of workers. Of equal importance is that so far a change in the skill composition affected the investment of firms but not the diversity of jobs that were offered, and this is again unrealistic. In this section we want to analyze these issues. We will find that more general matching technologies do not change the key results of the previous sections. Also, an interesting new prediction will follow; as higher inequality leads to more types of jobs becoming available, the between and within group measures of inequality will be linked in a way that is observed in the data.

We start by defining the polar extreme to the random matching technology; efficient matching. If the matching technology of the economy is efficient, then the highest skilled worker is allocated to the firm with the highest amount of physical capital, then the second most skilled is allocated to the second highest and so on. If there is a separation, then the same rule applies within the set of separated workers. This is clearly the same allocation that the Walrasian auctioneer chose in the equilibrium of section II. We can give the definition of efficient matching a little more formally,

Definition: *Let us define for every worker j,* $\Omega_W(j)=\int_{s\in S_W:h_s>h_j} ds$ *and similarly for each firm i,* $\Omega_F(i)=\int_{s\in S_F:k_s>k_i} ds$ *where* S_F *and* S_W *are the sets of firms and workers who are looking for a match. Then* $\Omega_F(i)$ *and* $\Omega_W(j)$ *are respectively the ranks of firm i and worker j in the set of firms, and in the set of workers looking for a match. Let us also again use the notation* $(i,j)\in\mathcal{P}$ *if i and j will be matched together. The matching technology is efficient* <u>*iff*</u> $(i,j)\in\mathcal{P} \Leftrightarrow$

either (i) $\Omega_W(j)=\Omega_F(i)$

or (ii) if $\Omega_W(j)<\Omega_F(i) \Rightarrow \forall\, i^*:\ \Omega_F(i^*)<\Omega_F(i)$ *and* $(i^*,j^*)\in\mathcal{P}$*, then* $\Omega_W(j^*)\leq\Omega_W(j)$.

or (iii) if $\Omega_W(j)<\Omega_F(i) \Rightarrow \forall\, j^*:\ \Omega_W(j^*)<\Omega_W(j)$. *and* $(i^*,j^*)\in\mathcal{P}$*, then* $\Omega_F(i^*)\leq\Omega_F(i)$.

The matching technology we will use is a hybrid between the efficient and random technologies. In particular, we assume that at the beginning of the period, a random proportion (1-q) of firms and of workers are chosen to match randomly among themselves, and the remaining q of the firms and workers are matched efficiently. As a motivation consider the case where a worker looks at all the firms before deciding which one to apply to. However, there is only a probability q that he will correctly assess the firm's type and otherwise, with probability (1-q), he will be in effect applying to a random firm.

However, the degree of efficiency of the matching technology is a separate issue from the costs of search, and the economy we analyze still has costly search. In fact, to simplify the analysis, we take the costs of mobility to be sufficiently high so that there is no second stage matching, that is $\eta=0$. This removes the outside option effect from the model and simplifies the exposition. It is clear to see that the presence of the outside option effects ($\eta>0$) would again add an additional force towards increasing inequality, but would not change the rest of our results. For the analysis it is also important that, since there is a continuum of workers, the subsamples selected for random and efficient matching are identical to the overall distributions. Further, we use the notation $(k',h') \in\mathcal{P}_t$, to denote h' and k' matching together at time t, if they were both chosen to match *efficiently*.

Lemma 2: *For all* $q>0$, $\forall\ (k',h') \in\mathcal{P}_t$, $F_t(h')=P_t(k')$.

This lemma states that if a firm and a worker will match together when selected for efficient matching, then they must have *exactly* the same rank in their respective distributions (i.e. in terms of the definition in the above paragraph, we will always be in case (i)). It is not surprising that given efficient matching and complementarities between human and physical capital, a highly skilled worker and low quality worker will not match together, but the result of the lemma is stronger than this: it rules out the possibility of *clustering* by firms; for instance, the possibility of two firms choosing the same level of physical capital and one matching with a high skill worker, and the other with a less skilled one is ruled out. Why? To see this suppose that two firms of the same capital level k expect to match with two workers of different levels of human capital in the case when they are both selected for efficient matching. But the profits of a firm are increasing in the human capital of the worker it employs, and the firm that matches with the less skilled worker could increase its investment by ϵ and since $q>0$, this would give the firm a strictly higher probability of matching with the better worker. For all $q>0$, we can find ϵ small enough that this is profitable, hence the result of no clustering. Moreover, we know that the subsamples of firms and workers selected for efficient matching are identical to the initial distributions F_t and P_t, therefore, these distributions must have the same form as each other [that is F_t must be a one-to-one transformation of P_t], and thus we know a lot about the equilibrium distributions.

Now the profits of a firm with physical capital k are given by

$$\text{(18)} \qquad \pi_t(k)=(1-q)(1-\beta)Ak^{1-\alpha}\int v^{\alpha}dF_t(v)+q(1-\beta)Ak^{1-\alpha}h^{\alpha}-Rk \quad s.t.\ (k,h)\in\mathcal{P}_t$$

It is clear that all firms have to make the same level of profits. Thus, $\pi_t(k)=\pi_t$ for all levels of investment k that are chosen in equilibrium.

Next we can also write the first-order condition for a typical firm:

$$\text{(19)} \quad (1-\alpha)(1-q)(1-\beta)Ak^{-\alpha}\int v^{\alpha}dF_t(v)+(1-\alpha)q(1-\beta)Ak^{-\alpha}h^{\alpha}-R+\alpha q(1-\beta)Ak^{1-\alpha}h^{\alpha-1}\frac{dh}{dk}\Big|_{\mathcal{P}_t}=0.$$

Lemma 3: For $q=1$, the physical to human capital ratio, $\mu(k)$, for workers matching efficiently is constant. For all $q<1$, this ratio, $\mu(k)$, is decreasing in k.

This is a very strong result. It demonstrates that for $q<1$, as well as the workers matching randomly, those allocated to efficient matching will create forces towards equalization of incomes. Let us first understand the result of the constant human to physical capital ratio with full efficient matching ($q=1$). In this case, we are very close to the Walrasian allocation with the only difference that because of the ex post bargaining, physical capital does not receive its full marginal product and thus firms underinvest. In fact, by inspecting the first order condition (19) with $q=1$, we can see that all firms have to work at a constant human to physical capital ratio and make zero profits. Now consider $q<1$, and suppose $\mu(k)$ is constant; but each firm can also be selected for random matching and, a firm with a high level of physical capital will lose more from being randomly allocated – strict concavity of the profit function in the random matching case as in equation (8). Thus to be compensated, such a firm, when matching efficiently, should work with a higher human to physical capital ratio.

Next, given mismatch established in Lemma 3, it is straightforward that:

Lemma 4: *For all $q<1$, a mean-preserving spread of $F_t(.)$ reduces investment and output.*

Proposition 5: *Suppose $\eta=0$. Consider an initial human capital distribution F_0 and a corresponding distribution of relative wealth G_0.*

(i) Suppose $q=1$, then $\forall G_0$, $G_t=G_0$ for all t, thus inequality self-replicates and the growth rate $g_t=g_\infty^*$ for all t.

(ii) $\forall$ $0<q<1$ and $\forall$ G_0, $G_t \rightarrow G_\infty$ where G_∞ exhibits full equality. The growth rate of the economy, g_t, is always less than g_∞^* and monotonically converges to it.

In this economy with hybrid matching technology, all the key results of the section III hold as long as $q<1$ [the limiting case of fully efficient technology, $q=1$, is obviously unrealistic as it relies on an invisible hand arranging the right matches]. Also, the degree of efficiency of the matching technology is another measure of labor market efficiency and we can see that a higher q (more efficient matching technology) makes inequality more long-lasting. Therefore this result complements the one obtained in section III.3 that more efficient labor markets are more likely to lead to higher wage inequality.

V.2. Between and Within Group Inequality

Finally, from the analysis of this section we can draw some implications about the interaction of between group and within group inequality. In this section both of these types of wage inequality are present. Let us define the expected wage of a worker with human capital h_j at time t by $W_t(h_j)$. This wage depends on the equilibrium choices of firms. In particular let P_t be the equilibrium distribution of firms and k_j be such that $(h^j,k^j) \in \mathcal{P}_t$. Then

$$W_t(h_j)=(1-q)\beta A(h_j)^\alpha \int k^{1-\alpha} dP_t(k)+q\beta A(h_j)^\alpha (k_j)^{1-\alpha}. \tag{20}$$

And clearly for $h_1 < h_2$, $W_t(h_2)$ - $W_t(h_1) > 0$. If these two skill groups are observable, this difference is what we will measure as the "skill" premium or as the between group wage inequality.

There is also within group wage inequality in this economy. To see this, note that the wage distribution for workers who have human capital h_1 has a mass of q at $\beta A(h_1)^\alpha (k_1)^{1-\alpha}$ and the rest of its mass is distributed as $\beta A(h_1)^\alpha k^{1-\alpha}$ where k has distribution $P_t(k)$. Now, suppose we take a mean-preserving spread of F_t. This will mean that there is more inequality and in fact, the gap between the less and the more skilled is larger, hence more *between group* inequality. But when F_t undergoes a mean-preserving spread, then from Lemma 2 P_t, also becomes more spread (though its mean falls too), and the *within* group wage inequality increases. Intuitively, higher between group inequality leads to a more diverse distribution of jobs. But, sometimes (with probability 1-q) two identical workers end-up randomly allocated to these jobs, and hence, more diversity in these jobs implies a larger wage gap between these two identical workers.

Finally, although it is often the case, between and within group inequality do not always move together. For instance, in the 1970s the U.S. experienced an increase in within group inequality but no increase in between group wage inequality. Such a pattern can be explained if q is changing [e.g. as a consequence of the changing organization of firms and labor markets]. From (20) we can see that an increase in q will increase the wages of the h_2 [the high skilled] group more than that for h_1 [the low skilled] group, leading to an increase in between group inequality but it will also increase the proportion that are matching efficiently, thus will generally lead to a compression of within group inequality.

VI. Concluding Comments

This paper has presented a general equilibrium model with endogenous job and skill distribution. The model provides a simple framework with which to analyze the interaction between labor markets and the evolution of income and wage inequality. Our investigation has revealed that labor market frictions introduce a number of redistributive forces. The first one, which we dubbed mismatch, redistributes from rich to poor workers while the second, the outside option effect, redistributes from the poor to the rich. Moreover, we found that the second effect was increasing with the degree of inequality and this led to the conclusion that the dynamics of income and wage inequality are non-ergodic: inequality is decreasing at low levels but can increase starting from high initial levels of inequality.

This framework has also enabled us to carry out simple comparative statics. We found that wage inequality is more likely to increase in economies with more efficient labor markets, less redistributive taxation, and less public schooling. Interestingly, these patterns are in line with the diverse experiences of OECD countries over the past twenty-five years. Finally, we also found that our model implies that within and between group inequality should move together which again appears to be the case in the data.

This paper is a first attempt at a framework for the theoretical analysis of wage inequality and labor market organization. As such, it leaves many issues unresolved. First, we have restricted our attention to parameter values for which there are no separations or disagreements along the equilibrium path. Thus our model is not well-suited to analyze issues of unemployment and inequality which are interesting and important. Introducing separations makes the model technically much more complicated [see for instance the full dynamic models of search with ex ante heterogeneity of Sattinger (1995), Shimer and Smith (1995), Burdett and Coles (1995)], but this is an important extension to consider [Acemoglu, 1995b]. Second, it is natural to question whether segmentation in the labor market is likely to arise as a way of limiting mismatch and which of our conclusions will be affected. Future research in this area may also open the way for a theory of endogenous segmentation in the labor market. Finally, our model poses a number of new empirical questions; can we find a good measure of mismatch? Is mismatch higher in labor markets with more inequality? Is performance in labor markets with higher inequality worse?

Appendix A: Proofs of Lemmas and Propositions

Proof of Proposition 1: (i) Take the equilibrium with the physical to human capital ratio as in Proposition 1 and suppose all workers are paid their marginal product as in (6). Then a firm with capital level k makes profits equal to

(A1)
$$\begin{aligned}\pi(k) &= Ak^{1-\alpha}h^{\alpha} - Rk - w(h)h \\ &= \frac{1}{1-\alpha}R - Rk - \frac{\alpha}{1-\alpha}Rk\ . \\ &= 0\end{aligned}$$

irrespective of the value of k. Thus at the allocation where all firms are allocated according to the Walrasian rule (see the definition in section V.1) and k/h is constant, all firms make zero profit. Given the value of h, each firm would also exactly choose this level of capital ratio since

(A2)
$$\pi'(k) = 0 \ \ at \ \ k = \left(\frac{(1-\alpha)A}{R}\right)^{\frac{1}{\alpha}} h.$$

Thus no firm wants to deviate and change its capital stock, and the allocation is an equilibrium. At no other capital stock, a firm would be maximizing profit, thus there is no equilibrium in which firms are at a different capital ratio.

(ii) Given a constant physical to human capital ratio, we can write the wage of worker with human capital $h_{j,t}$ as

(A3)
$$w(h_{j,t}) = \alpha A \left(\frac{(1-\alpha)A}{R}\right)^{\frac{1-\alpha}{\alpha}} h_{j,t}.$$

Thus from (3) in the text;

(A4)
$$h_{j,t+1} = \delta\alpha A \left(\frac{(1-\alpha)A}{R}\right)^{\frac{1-\alpha}{\alpha}} h_{j,t}.$$

for all j. Next, $\theta_{j,t} = \frac{h_{j,t}}{\bar{h}_t}$ and since all transitions are linear, $\bar{h}_{t+1} = \delta\alpha A \left(\frac{(1-\alpha)A}{R}\right)^{\frac{1-\alpha}{\alpha}} \bar{h}_t$ and hence, $\theta_{j,t+1} = \theta_{j,t}$.

(iii) From the above equations, the human capital of all agents grows at the rate $g^{C} = \delta\alpha\left(\frac{(1-\alpha)A}{R}\right)^{\frac{1-\alpha}{\alpha}} - 1$ and given that the capital ratio is constant all the time, this is the rate at which firm level capital stocks grow too, hence also the output growth rate. □

Proof of Proposition 2: (i) Consider Figure 1 in the text; stationary points (distributions) correspond to the intersection of $\phi_{t+1}(\phi_t)$ with the 45^0 line and $\phi_{t+1}(\phi_t = 1) = 1$, thus full equality is always a stationary point.

Now consider

$$\phi_{t+1}(\phi_t=0) = \frac{-(1-\beta)(1-\lambda)\eta}{1+\eta(1-\beta)\lambda}. \tag{A6}$$

This is negative for all $\eta > 0$ and equal to zero for $\eta = 0$.

$$\frac{\partial\phi_{t+1}}{\partial\phi_t} = \frac{[\alpha\phi^{\alpha-1}+(1-\beta)(1-\lambda)\eta\alpha\phi_t^{\alpha-1}]\times[1-(1-\beta)\lambda\eta\phi_t^{\alpha}+(1-\beta)\lambda\eta]}{[1-(1-\beta)\lambda\eta\phi_t^{\alpha}+(1-\beta)\lambda\eta]^2} + \frac{[(1-\beta)\lambda\eta\alpha\phi_t^{\alpha-1}]\times[\phi_t^{\alpha}+(1-\beta)(1-\lambda)\eta\phi_t^{\alpha}-(1-\beta)(1-\lambda)\eta]}{[1-(1-\beta)\lambda\eta\phi_t^{\alpha}+(1-\beta)\lambda\eta]^2}. \tag{A7}$$

This derivative in (A7) always exists and is positive, and when $\eta = 0$, it starts from the origin and never falls below the 45^0 line. Thus the system is globally stable. The growth rate at the limit is obtained straightforwardly from (9) and (4) for $\eta = 0$.

(ii) Next for all $\eta > 0$, the curve in Figure 1 starts negative at $\phi = 0$ and thus if it is going to approach $\phi = 1$ from above, it must cut the 45^0 line at least once before $\phi = 1$. Evaluating (A7) at $\phi_t = 1$, we get (12). This implies that for $\alpha[1+(1-\beta)\eta] < 1$, the function in Figure 1 cuts the 45^0 line at $\phi_t = 1$ from above (with a slope of less than 1). Differentiation shows that either $\partial\phi_{t+1}{}^2/\partial\phi_t{}^2 < 0$ or $\partial\phi_{t+1}{}^2/\partial\phi_t{}^2 > 0$ but then $\partial^3\phi_{t+1}/\partial\phi_t{}^3$ is unambiguously positive [full details available upon request]. Therefore, the function in Figure 1 cannot turn from convex to concave and hence, the shape of the function can only be as in Figure 1.

As a result, the curve must cut the 45^0 line once at some point A. Let us call the horizontal coordinate of A, $\phi(\lambda)$. Then, $\forall\ \phi_t \in (\phi(\lambda),1)$, $d[\phi_{t+1},1] < d[\phi_t,1]$ where d[.,.] is the Euclidian distance between two points. Thus, all points to the right of A are in the basin of attraction of full equality; or in other words, $\phi_0 \in (\phi(\lambda),1)$, $\phi_t \rightarrow \phi_\infty = 1$. And again the growth rate at the limit is given by (9) and (4); and before this limit is reached, inspection of (9) shows that investment is less, thus growth is lower.

Similarly, $\forall\ \phi_t \in (\phi_{min}, \phi(\lambda))$ where ϕ_{min} will be defined below, $d[\phi_{t+1},1] > d[\phi_t,1]$ and thus for all ϕ_0 to the left of A, we diverge from full equality. The dynamics for both groups are defined by continuous functions, and $\{\phi_t\}$ forms a monotonic sequence and is defined between 0 and 1, thus over a closed and bounded set. This implies that $\{\phi_t\}$ must have a convergent subsequence, therefore, we must converge to a certain point. Since $h_{1,t}$ is bounded below by h_{min}, this sequence $\{\phi_t\}$ can only tend to zero, if $h_{2,t}$ goes to infinity. Thus we need to check whether growth in $h_{2,t}$ can be sustained with a proportion (1-λ) of the agents accumulating and the rest at the lower bound h_{min}. In this case the law of motion of h_{2t} would be given by;

(A8) $$h_{2,t+1}=\delta w_{2t}=\beta h_t^{\alpha} k_t^{1-\alpha}-\beta(1-\beta)\eta k_t^{1-\alpha}[\lambda h_{min}^{\alpha}+(1-\lambda)h_{2t}^{\alpha}]+(1-\beta)\beta\eta A k_t^{1-\alpha}h_{2t}^{\alpha}$$

We will show that $h_{2,t}$ growing without a bound (at a constant rate) is not possible by contradiction: $h_{2,t}$ can grow only if k_t is linear in $h_{2,t}$. So let us suppose that k_t is linear in $h_{2,t}$. This would imply from (A8) that $h_{2,t+1}=Bh_{2,t}+Ch_{2,t}^{\alpha}$ where B and C are suitably defined constants. But this implies that constant growth is not possible for $h_{2,t}$ and there exists a unique level of h_2 such that $h_{2,t+1}=h_{2,t}=h_2$. Similarly, if k_t were an everywhere concave function of $h_{2,t}$, the same conclusion would apply a fortiori. From (9), k_t is immediately seen to be a concave function of $h_{2,t}$ and therefore, there is a unique level of h_2 such that $h_{2t}=h_{2,t+1}=h_2$ and thus the human capital of the rich class stops growing when the poor hit their lower bound, h_{min}. Then, $\phi_{min}=h_{min}/h_2$ is a stationary point. By definition there can be no other stationary point when $\phi_t < \phi(\lambda)$. Thus, the economy that starts with a level of inequality more than that represented by $\phi(\lambda)$ exhibits no long-run growth. This proves part (ii)).

(iii) Finally, we turn to the alternative configuration with $\alpha[1+(1-\beta)\eta]>1$. In this case, full equality is not a stable ergodic set. If the function never intersects the 45^0 line before $\phi=1$, the unique ergodic set is maximum inequality which corresponds to $\phi^*=1$ and the economy has a unique stable limiting distribution that is the one described above in (ii). Alternatively, if the curve intersects the 45^0 line, it must do so twice, thus $\phi^*<1$ and then there are two locally stable ergodic sets; (a) maximum inequality without any growth and (b) another locally stable ergodic set with a certain degree of inequality but also positive growth but at a rate less than g_∞. This proves case (iii). □

Proof of Proposition 3: A) $\eta=0$, then

(A9) $$\theta_{j,t+1}=\beta\theta_{j,t}^{\alpha} \quad for\ h_{j,t} \gg h_{min}.$$

Thus $\theta_{jt}>1$ implies that $\theta_{j,t+1}<\theta_{j,t}$ and for $\theta_{jt}<1$, $\theta_{j,t+1}>\theta_{j,t}$ hence convergence to full equality.

B) (i) $\theta_{j,t}=1$, $\forall$j implies that $\theta_{j,t+1}=1$ $\forall$j. Thus full equality is a stationary point.

(ii) $\theta_{j,t}=1$, implies that $\theta_{j,t+1}=1$ irrespective of the distribution. Thus, the group at the median stays there. Take a proportion of the population λ_L to be at h_{min} and suppose a proportion λ_H is above the median at h_2. This will be a stationary distribution iff

(A10)
$$h_2=\beta[1+(1-\beta)\eta]h_2^{\alpha}-\beta(1-\beta)\eta[\lambda_H h_2^{\alpha}+(1-\lambda_H-\lambda_L)+\lambda_L h_{\min}^{\alpha}]$$
$$h_{\min}\geq\beta[1+(1-\beta)\eta]h_{\min}^{\alpha}-\beta(1-\beta)\eta[\lambda_H h_2^{\alpha}+(1-\lambda_H-\lambda_L)+\lambda_L h_{\min}^{\alpha}]$$

Thus any vector (λ_L, λ_H, h_2) that satisfies (A10) will constitute a stationary distribution and given these two equations such vectors always exist.

(iii) This follows from the inspection of equation (17) and Figure 2. (17) can have at most one intersection with the 45^0 above $\theta=1$, hence at most one group above the median. Also it cannot have any intersections below $\theta=1$, thus no group below the median other than one at h_{min}. Finally, we can have a third group at the median.

(iv) Take $\theta_t<1$. For convergence we need $\theta_{t+1}>\theta_t$. This implies

(A11)
$$\frac{\beta[1+(1-\beta)\eta]\theta_t^{\alpha}-\beta(1-\beta)\eta\int\theta^{\alpha}dG_t(\theta)}{\beta[1+(1-\beta)\eta]-\beta(1-\beta)\eta\int\theta^{\alpha}dG_t(\theta)}>\theta_t.$$

or, equivalently,

(A12)
$$\Theta^{-}=\beta[1+(1-\beta)\eta](\theta_t^{\alpha}-\theta_t)-\beta(1-\beta)\eta(1-\theta_t)\int\theta^{\alpha}dG_t(\theta)>0.$$

Similarly, for $\theta_t>1$, convergence requires $\theta_{t+1}<\theta_t$, thus

(A13)
$$\Theta^{+}=\beta[1+(1-\beta)\eta](\theta-\theta^{\alpha})+\beta(1-\beta)\eta(\theta-1)\int\theta^{\alpha}dG_t(\theta)>0.$$

For local stability, the relevant conditions can be written as

(A14)
$$\Theta^{-}(\theta_t^{i}=1\ \ \forall i\neq j,\theta_t^{j}\to 1^{-})>0$$
$$\Theta^{+}(\theta_t^{i}=1\ \ \forall i\neq j,\theta_t^{j}\to 1^{+})<0$$

These in turn are equivalent to $\frac{\partial\Theta^{-}(\theta_t^i=1)}{\partial\theta_t}<0$ *and* $\frac{\partial\Theta^{+}(\theta_t^i=1)}{\partial\theta_t}>0$. Thus, local stability requires

(A15) $$\frac{\partial\Theta^{-}(\theta_t^i=1)}{\partial\theta_t}=\beta(\alpha-1)+\beta(1-\beta)\eta(\alpha-1)+\beta(1-\beta)\eta<0.$$

and

(A16) $$\frac{\partial\Theta^{+}(\theta_t^i=1)}{\partial\theta_t}=\beta(1-\alpha)+\beta(1-\beta)\eta(1-\alpha)+\beta(1-\beta)\eta>0.$$

(A15) and (A16) are equivalent to each other and in turn to our condition for the stability of full equality in the text $\alpha+\alpha(1-\beta)\eta<1$.

(v) The diagrammatic representation of the solution to (A10) corresponds to Figure 2b (a intersection above the median is necessary), hence the local stability. □

Proof of Lemma 1: From (A12), we obtain

(A17) $$\frac{\partial\Theta^{-}}{\partial\theta_t}=\beta[1+(1-\beta)\eta](\alpha\theta_t^{\alpha-1}-1)+\beta(1-\beta)\eta\int\theta^{\alpha}dG_t(\theta)$$

This expression is unambiguously negative if $\theta_t<\alpha^{\frac{\alpha}{1-\alpha}}$. That is if relative inequality is sufficiently large, it is most difficult to close the gap between the poorest agents and the mean level of income (and skills). This expression is also increasing in the integral term, thus its zero $\bar{\theta}_t$ is decreasing in the amount of inequality. □

Proof of Proposition 4: (i) From the text, we can see the condition for a cycle to be

(A18) $$\theta_1^H=\frac{\beta[1+(1-\beta)\eta](\theta_2^H)^{\alpha}-\beta(1-\beta)\eta[\lambda^L(\theta_2^L)^{\alpha}+(1-\lambda^L-\lambda^H)+\lambda^H(\theta_2^H)^{\alpha}]}{\beta[1+(1-\beta)\eta]-\beta(1-\beta)\eta[\lambda^L(\theta_2^L)^{\alpha}+(1-\lambda^L-\lambda^H)+\lambda^H(\theta_2^H)^{\alpha}]}$$

(A19) $$\theta_1^L=\frac{\beta[1+(1-\beta)\eta](\theta_2^L)^{\alpha}-\beta(1-\beta)\eta[\lambda^L(\theta_2^L)^{\alpha}+(1-\lambda^L-\lambda^H)+\lambda^H(\theta_2^H)^{\alpha}]}{\beta[1+(1-\beta)\eta]-\beta(1-\beta)\eta[\lambda^L(\theta_2^L)^{\alpha}+(1-\lambda^L-\lambda^H)+\lambda^H(\theta_2^H)^{\alpha}]}.$$

(A20) $$\theta_2^H=\frac{\beta[1+(1-\beta)\eta](\theta_1^H)^{\alpha}-\beta(1-\beta)\eta[\lambda^L(\theta_1^L)^{\alpha}+(1-\lambda^L-\lambda^H)+\lambda^H(\theta_1^H)^{\alpha}]}{\beta[1+(1-\beta)\eta]-\beta(1-\beta)\eta[\lambda^L(\theta_1^L)^{\alpha}+(1-\lambda^L-\lambda^H)+\lambda^H(\theta_1^H)^{\alpha}]}$$

(A21)
$$\theta_2^L=\frac{\beta[1+(1-\beta)\eta](\theta_1^L)^\alpha-\beta(1-\beta)\eta[\lambda^L(\theta_1^L)^\alpha+(1-\lambda^L-\lambda^H)+\lambda^H(\theta_1^H)^\alpha]}{\beta[1+(1-\beta)\eta]-\beta(1-\beta)\eta[\lambda^L(\theta_1^L)^\alpha+(1-\lambda^L-\lambda^H)+\lambda^H(\theta_1^H)^\alpha]}.$$

(A22)
$$\lambda^L\theta_1^L+\lambda^H\theta_1^H=\lambda^L+\lambda^H.$$

(A23)
$$\lambda^L\theta_2^L+\lambda^H\theta_2^H=\lambda^L+\lambda^H.$$

Thus the problem of finding a cycle is to find a vector (λ^L, λ^H, θ_1^L, θ_1^H, θ_2^L, θ_2^H) to satisfy these six equations, (A18)-(A23). For all values of α, β, η these six equations are continuous and map from the bounded, closed, convex set $[0,1]^6$ into itself. Thus by Brouwer's fixed-point theorem such a vector will always exist. Thus a two-cycle always exists.

(ii) This part of the proposition follows part (i) with nine equations in nine unknowns. The details are omitted. □

Proof of Lemma 2: First suppose F(.) has no atoms. We will first show that in this case P(k) cannot have any atoms either. Then, by definition of the efficient matching technology, the highest human capital will match with the highest physical capital and so on and thus, by definition, $F(h)=P(k)\ \forall(h,k)\in\mathcal{P}$.

We will now show that P(k) has no atoms by contradiction. Suppose P(k) has an atom at k', then $\exists$ a set of firms $\sigma(k')$ such that all $i\in\sigma(k')$ have the same capital level. Since F(.) has no atoms, with positive probability q, firms in $\sigma(k')$ will match with workers of human capital levels h_1 and h_2 where $h_2>h_1$. But, this cannot be an equilibrium because one of the firms in $\sigma(k)$ could increase its capital by ϵ and make sure that it meets h_2 whenever it is selected for efficient matching. For all $q>0$, we can find a small enough ϵ such that this strategy is profitable.

Now suppose that F(.) has an atom at h' and denote the measure of this by f(h'). To show that $F(h)=P(k)\ \forall(h,k)\in\mathcal{P}$, it is sufficient to prove that (i) if $(h',k')\in\mathcal{P}$ and $(h',k'')\in\mathcal{P}$, then $k'=k''$; <u>and</u> (ii) if $(h',k')\in\mathcal{P}$ and $(h'',k')\in\mathcal{P}$, then $h'=h''$. (PS: in other words, P(.) must have an atom of exactly the same size at k' which is the level of physical capital that matches with h' when matching efficiently).

(i) Suppose ∃ k'', such that (h',k'')∈𝒫. But given the human capital at h', the profits π(k) is concave, thus either k' or k'' is not profit maximizing. Hence a contradiction.

(ii) Suppose ∃ i ∈ σ(k') such that (h,k')∈𝒫 for h different from h'. If h>h' then some of i ∈ σ(k') would increase its investment by ϵ and match with h with probability q. If h<h', then i would increase its investment by ϵ and match with h'. Hence a contradiction. □

Proof of Lemma 3: (i) Take two workers with different human capital levels $h_1 < h_2$ and denote the capital stocks that these two workers will work respectively with by $k_1 < k_2$ and the corresponding physical to human capital ratios by μ_1 and μ_2. From the fact that both firms will make the same profit level, we can write

(A24) $$k_1[(1-\beta)A\mu_1^{-\alpha}-R]=k_2[(1-\beta)A\mu_2^{-\alpha}-R].$$

Since $k_2 > k_1$, as long as both firms are making positive profits, $\mu_1 < \mu_2$, thus the physical to human capital ratio is decreasing. And if the profits are equal to zero, $\mu_1 = \mu_2$, i.e. all firms have constant human to physical capital ratios (this can also be checked directly from (19)).

Now re-write (19) by noting that dk/dh$\leq\mu$(k), i.e. a unit increase in capital will increase human capital by more than the current human to physical capital ratio [this is by definition of the human to physical capital ratio being increasing from (A24)]. Hence

(A25) $$(1-\alpha)(1-\beta)A\mu(k)^{-\alpha}+\alpha(1-\beta)A\mu(k)^{1-\alpha}\mu(k)^{-1}\leq R$$

But this implies that the firm is making negative profits. Thus all firms must be making zero-profits and μ(k) is constant. Since μ(k) is constant and independent of the distribution of human capital, the rate of return on physical capital and total output are independent of heterogeneity.

(ii) Take a range of values of k and evaluate (19) in the text and let us call this T(k,h). Take k_1 and $k_2>k_1$ and $h_1<h_2$ such that $(h_1,k_1)\in\mathcal{P}$ and $(h_2,k_2)\in\mathcal{P}$. Then, by definition $T(h_1,k_1)=T(h_2,k_2)=0$. T(.,.) has three terms (not counting R which is constant): the first term is decreasing in k and does not depend on h, this therefore implies that the sum of the other terms have to be higher for k_2 than for k_1. This implies that either μ(k), the physical to human capital ratio, has to be decreasing in k or dh/dk have to be increasing in k. But for k_1 and k_2 sufficiently close to each other μ(k) is decreasing if and only if dh/dk is increasing (since by

Lemma 2 dh/dk is continuous) and therefore $\mu(k)$ has to be decreasing. □

Proof of Lemma 4: This follows from equation (19). □

Proposition 5: (i) q=1: since human capital ratios are constant from Lemma 2, wages are linear in human capital, thus inequality self-replicates as in Proposition 1.

(ii) From Lemma 3, wages are a concave function of human capital, thus the argument of Proposition 3 for the case of $\eta=0$ applies. □

Appendix B: Dynamics With Non-linear Transitions

So far we have analyzed a simple economy in order to clearly illustrate the innovation of this paper. In particular, the decision rules and the accumulation equations were linear and this led to the result that inequality self-replicates in the competitive economy. However, it is well-known that the competitive economy also includes certain redistributive forces that often lead towards equality (e.g. Tamura, 1991). Thus it may be wondered how the new forces we proposed here would interact with these. In particular, it is important to determine whether our main conclusions are robust. In this appendix we take general non-linear transitions that transform the human capital and the wage of a worker into a human capital level for the offspring. Using this framework, we demonstrate the robustness of our results. That is; (i) mismatch introduces forces towards equality, thus the economy with mismatch converges faster to full equality; (ii) outside options introduce forces towards inequality and may lead to a multiplicity of limiting distributions.

In the text we had $e_{t+1}=\delta w_t$, thus the education expenditure was linear in wage income. And also $h_{t+1}=\min\{h_{\min},e_{t+1}\}$, thus the human capital of the offspring was linear in education expenditure and in particular, it did not depend on the human capital of the parent directly. We know both of these assumptions to be untrue (see Borjas, 1992, Cameron and Heckman, 1992 for empirical evidence). Instead here, we assume

(B1) $$h_{j,t+1}=\psi(w_{j,t},h_{j,t}),$$

where the range of $\psi(.,.)$ is $[h_{\min},+\infty)$.

Further we assume that ψ is a continuous, strictly concave and strictly increasing in both arguments. Also for notational convenience define $a \equiv A\left(\frac{(1-\alpha)A}{R}\right)^{\frac{1-\alpha}{\alpha}}$. Then;

Proposition B.1: *Consider the frictionless economy of section IV with a starting distribution of inequality G_0. Then, for all $\psi(.,.)$ such that $\psi(\alpha a h_{min}, h_{min}) > h_{min}$, G_t converges to a unique stationary limiting distribution G_∞ that exhibits full equality.*

Proof: The labor market and wage determination in the competitive economy are unchanged, thus from our analysis of section 2, $w_t = a h_t$. Concavity of ψ ensures that there is a point such that $h^* = \psi(\alpha a h^*, h^*)$ and since $\psi(\alpha a h_{min}, h_{min}) > h_{min}$, all dynasties will reach this level as $t \to \infty$. □

Proposition B.2: *Consider the economy of section IV, with frictions and $\eta = 0$. Then, for all $\psi(.,.)$ such that $\psi(\alpha a h_{min}, h_{min}) > h_{min}$, and starting distribution of inequality G_0, G_t converges to G_∞ that exhibits full equality. Convergence to full equality is always faster than the frictionless economy.*

Proof: The labor market is unchanged, in particular $w_t = \beta y_t$. Thus

(B2)
$$h_{j,t+1} = \psi\left(\beta(1-\beta)^{\frac{1-\alpha}{\alpha}} a\left[\int h^\alpha dF_t(h)\right]^{\frac{1-\alpha}{\alpha}} h_{j,t}^\alpha \ , \ h_{j,t}\right),$$

and convergence to full equality is immediate.

To see that it is faster than in the competitive economy, take two agents $h_{1t} < h_{2t}$ and look at the distance between their offsprings in the competitive economy and the mismatch economy given respectively by $\left(\frac{h_{2t+1}}{h_{1t+1}}\right)^c$ *and* $\left(\frac{h_{2t+1}}{h_{1t+1}}\right)^m$. Thus we are comparing

(B3)
$$\frac{\psi(\alpha a h_{2t}, h_{2t})}{\psi(\alpha a h_{1t}, h_{1t})} \quad \text{and} \quad \frac{\psi\left(\beta(1-\beta)^{\frac{1-\alpha}{\alpha}} a\left[\int h^\alpha dF_t(h)\right]^{\frac{1-\alpha}{\alpha}} h_{2t}^\alpha \ , \ h_{2t}\right)}{\psi\left(\beta(1-\beta)^{\frac{1-\alpha}{\alpha}} a\left[\int h^\alpha dF_t(h)\right]^{\frac{1-\alpha}{\alpha}} h_{1t}^\alpha \ , \ h_{1t}\right)},$$

since ψ is always increasing in its first argument, the second term in (B3) is smaller irrespective of the values of h_{1t} and h_{2t}, and thus the faster convergence. □

This proposition establishes the generality of our result that mismatch introduces *additional* forces towards equalization of income, or alternatively that it redistributes from the relatively rich to the poor.

We now turn to the economy with outside options. Investment in this economy is still given by (9) since the production side is exactly the same as in the main text. Thus;

(B4)
$$w(h_t)=c\left[\int h^{\alpha}dF_t(h)\right]^{\frac{1-\alpha}{\alpha}}h_t^{\alpha}-d\left[\int h^{\alpha}dF_t(h)\right]^{\frac{1}{\alpha}},$$

where $c=\beta+\beta(1-\beta)\eta$ *and* $d=\beta(1-\beta)\eta$.

Proposition B.3: Consider the economy of section IV, with frictions and $\eta>0$. *Then, there exists an open set of continuous functions* $\psi(.,.)$ *with* $\psi(\alpha ah_{min},h_{min})>h_{min}$, *and starting distributions of inequality* G_0, *such that* G_t *converges to a stationary distribution* G_{∞} *that exhibits limiting inequality.*

Proof: It suffices to show that a limiting stationary distribution with inequality is possible. Consider the following where a proportion λ of agents are at h_{min} and $(1-\lambda)$ are at h^*. For this we require,

(B5)
$$\begin{aligned} h_{\min} &\geq \psi\left(c\left[\lambda h_{\min}^{\alpha}+(1-\lambda)h^{*\alpha}\right]^{\frac{1-\alpha}{\alpha}}h_{\min}^{\alpha}-d\left[\lambda h_{\min}^{\alpha}+(1-\lambda)h^{*\alpha}\right]^{\frac{1}{\alpha}},h_{\min}\right) \\ h^{*} &= \psi\left(c\left[\lambda h_{\min}^{\alpha}+(1-\lambda)h^{*\alpha}\right]^{\frac{1-\alpha}{\alpha}}h_{\min}^{\alpha}-d\left[\lambda h_{\min}^{\alpha}+(1-\lambda)h^{*\alpha}\right]^{\frac{1}{\alpha}},h_{\min}\right) \end{aligned}.$$

Choose $\psi(\alpha ah_{min},h_{min})=h_{min}+\epsilon$, then for h^* large enough, the first equation in (B5) can be satisfied. We can choose the rest of $\psi(.,.)$ in any form (in an open set) to satisfy the second equation. □

Therefore, this proposition establishes that outside options in general introduce forces that redistribute from the poor to the rich, hence even when the competitive economy exhibits decreasing wage and income inequality everywhere, the frictional economy with outside options can have increasing wage and income inequality. It thus follows that our conclusions are independent of the linear transitions utilized in the paper.

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FIGURE 1

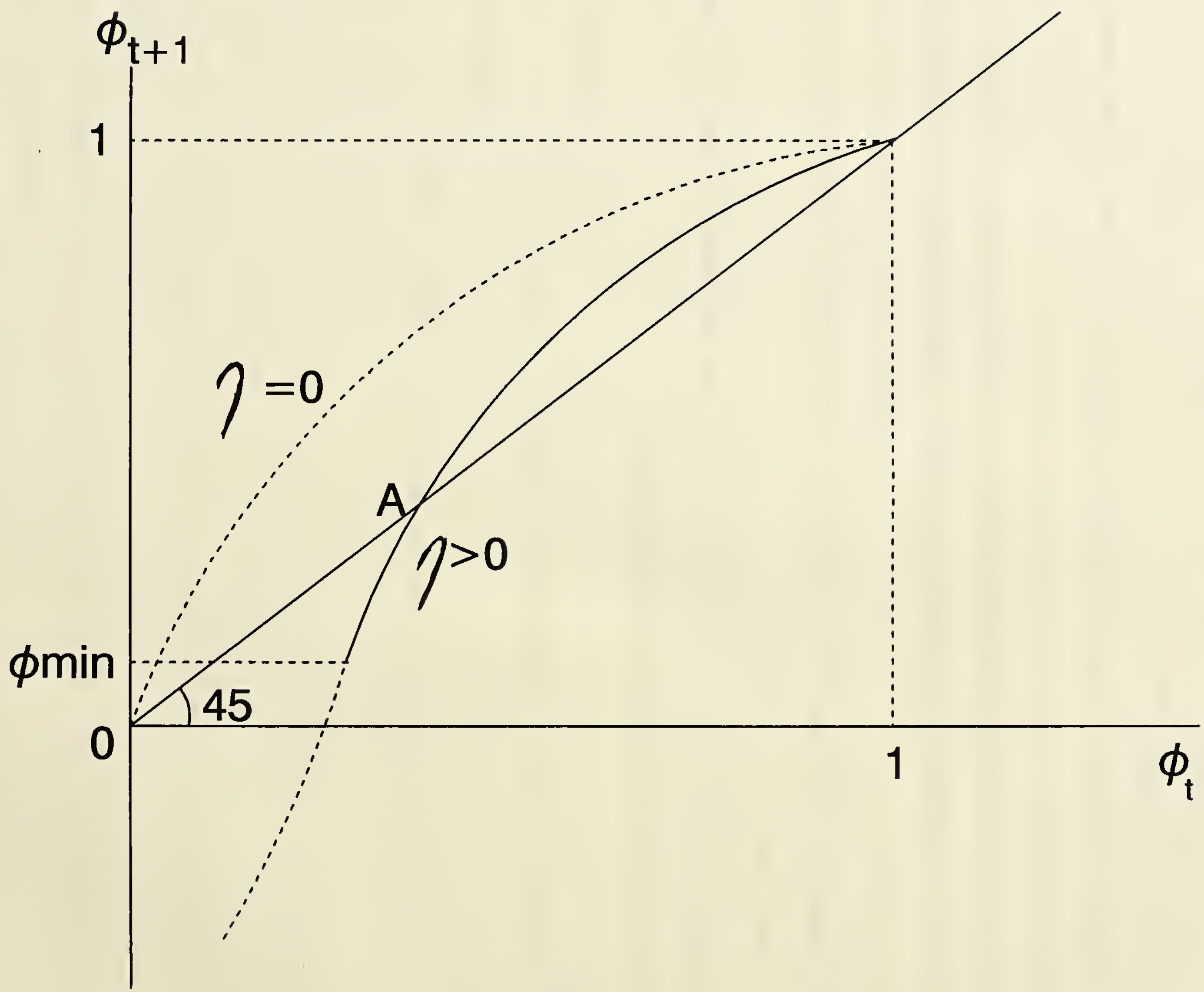
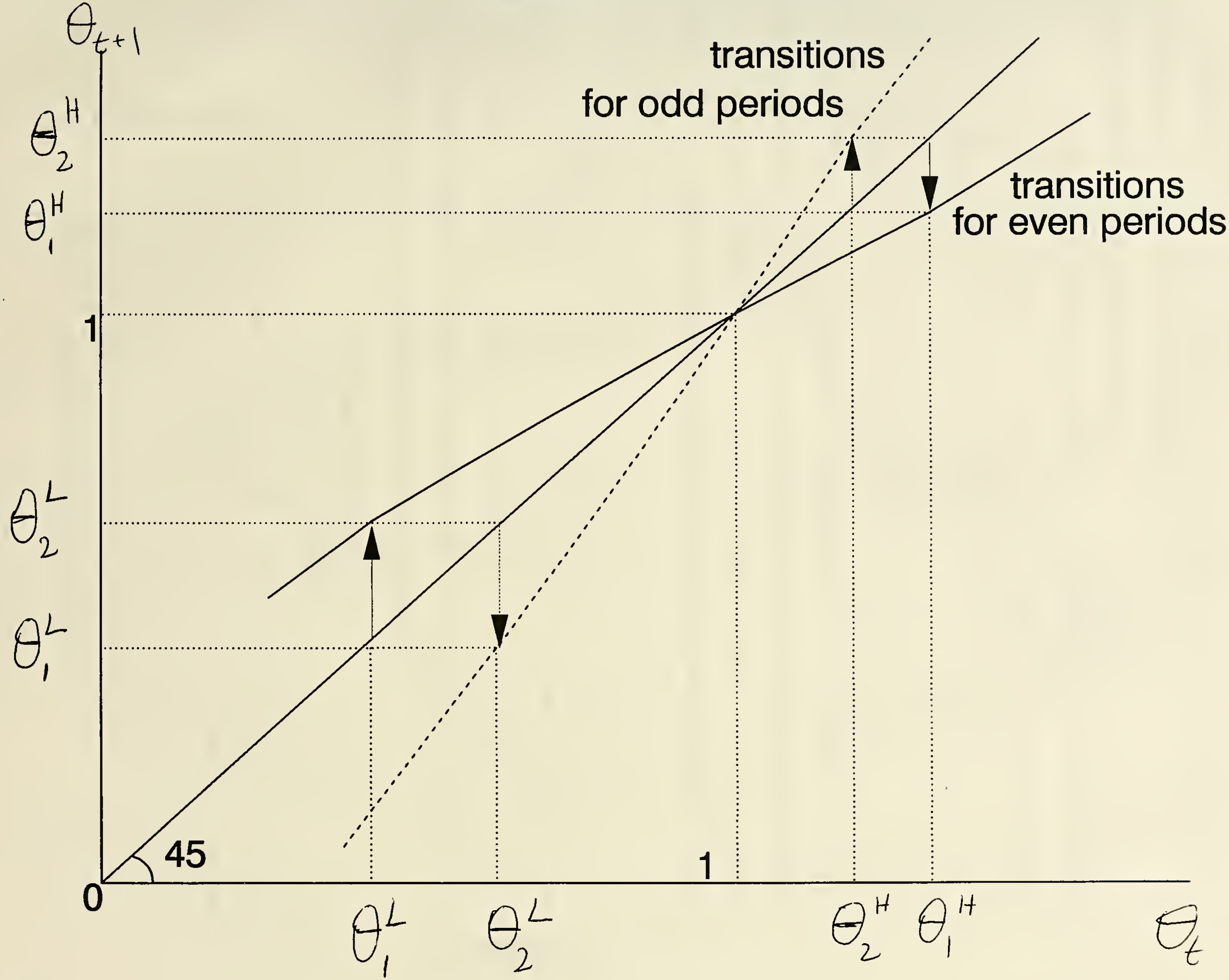


FIGURE 3



θ_{t+1}
θ_2^H
θ_1^H
1
θ_2^L
θ_1^L
0
45
transitions
for odd periods
transitions
for even periods
1
θ_1^L
θ_2^L
θ_2^H
θ_1^H
θ_t

FIGURE 2

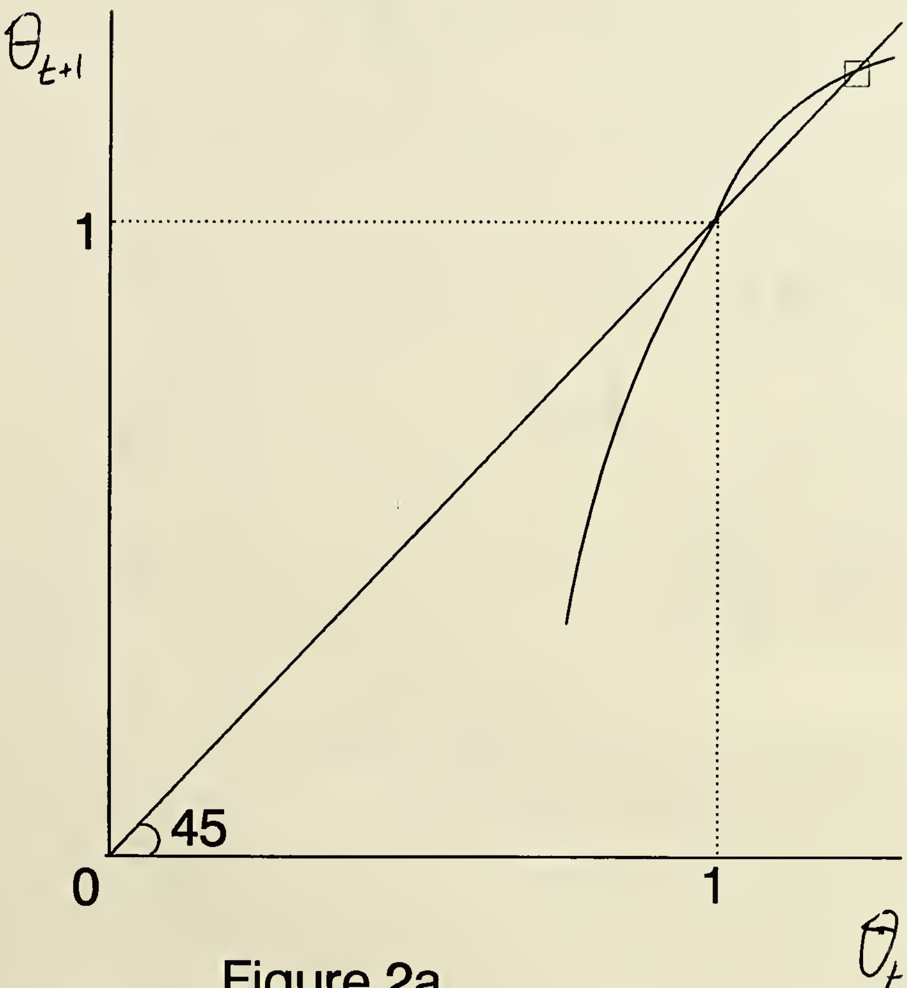


Figure 2a

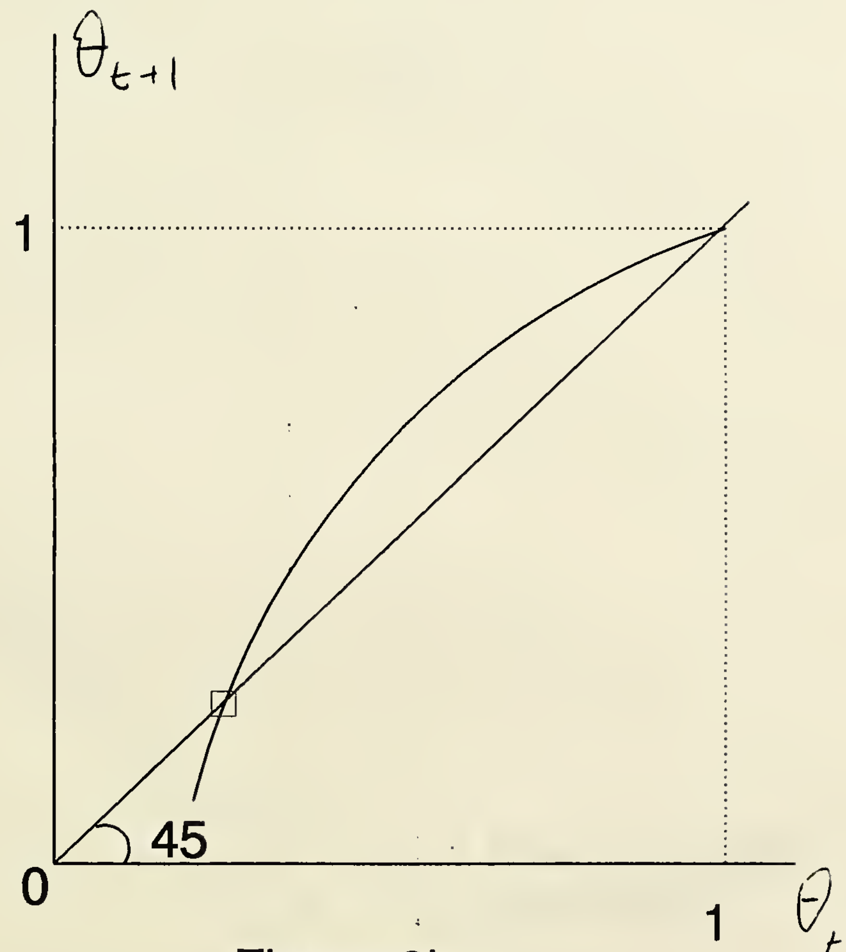


Figure 2b

Appendix C: Wage Determination, Investment Decision By Firms and Stability Conditions for Infinite Sampling

C.1) Equilibrium Wage Determination

Consider the game illustrated in Figure C1. Nature moves first and decides with respective probabilities β and 1-β whether the worker or the firm will make the first offer. This introduces the feature that the bargaining powers are potentially unequal. After the offer of the worker (firm), the firm (worker) can say Yes in which case there is agreement at the offered wage. Or the answer may be Out in which case both parties look for a new partner. Finally, the answer can be No in which Nature again randomly picks another party to make offers. However, as in Binmore, Rubinstein and Wolinsky (1986), after the No decision, there is a probability s that the relationship will come to an end before the next round of offers and if this is the case, again both parties have to look for a new partner. We will analyze this game for any s and then let s tend to zero.

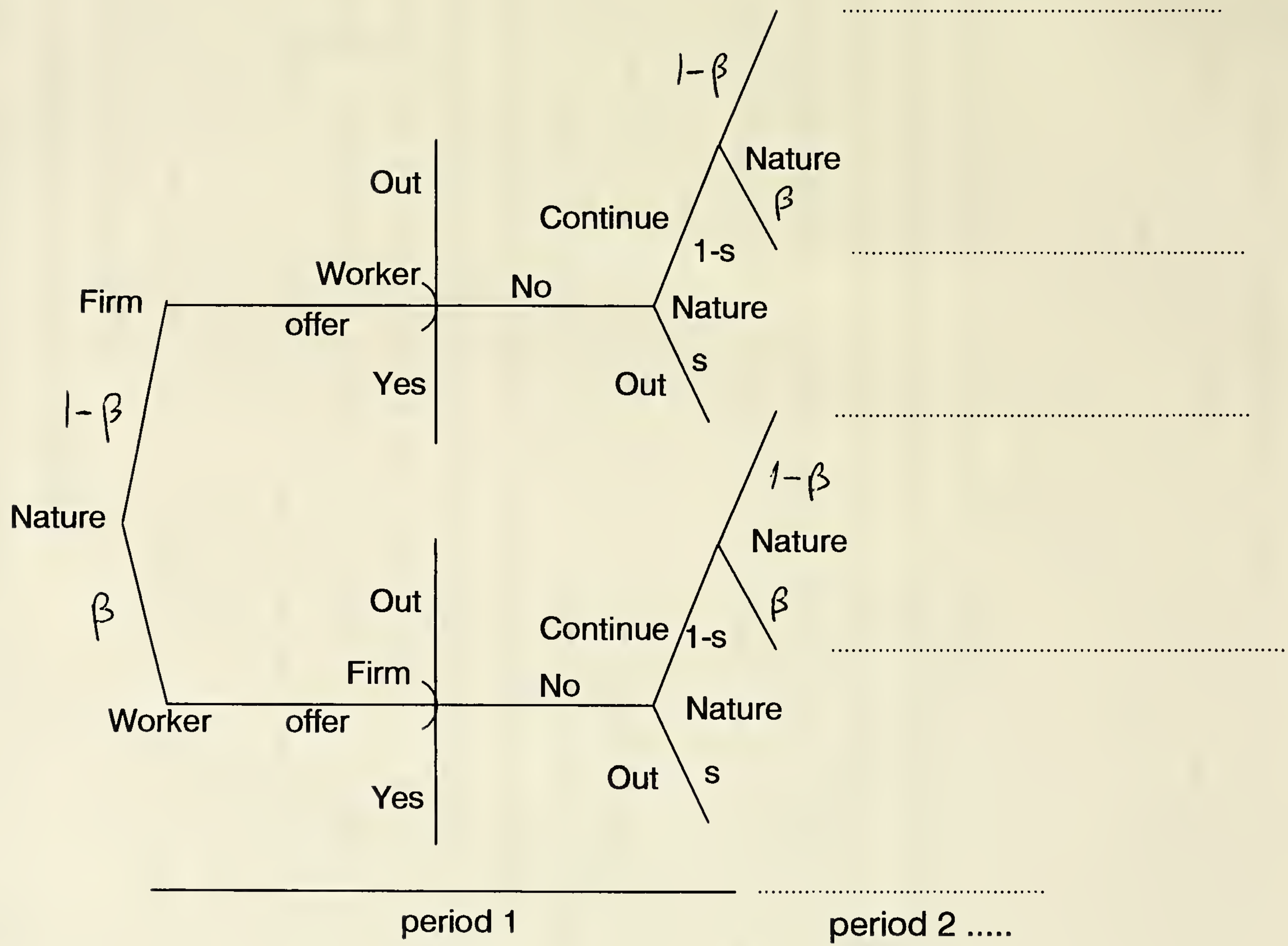
Let the value of the game to the firm be V_F and that to the worker V_W. These determine in the distribution of total output y_t. And also denote the value to the parties if the relation ends by W_W and W_F (obviously these values are the same irrespective of whether it is a voluntary end to the relation or a separation due to Nature's choice). And given our assumptions in the text, these "disagreement" values are as follows:

$$W_W=\beta\eta A h_t^{\alpha}\int k^{1-\alpha}dP_t(k)$$
$$W_F=(1-\beta)\eta A k_t^{1-\alpha}\int h^{\alpha}dF_t(h)$$

Also to start with, suppose that the relation is jointly beneficial, so that $y > W_W + W_F$. Let us also suppose that we are on the higher branch so that the firm is making the offer and has offered w to the worker. If the worker says Yes he gets w. If he says No, he obtains, $(1-s)V_W+sW_W$ and if he says No he would get W_W. Thus, the firm would offer

(C1) $$w_F=\max\{W_W\ ;(1-s)V_W+sW_W\}$$

FIGURE C1


Firm
Worker
offer
Out
Yes
No
Continue
1-s
Nature
s
Out
1-β
β
Nature
1-β
β
Worker
offer
Firm
Out
Yes
No
Continue
1-s
Nature
s
Out
1-β
Nature
β
period 1
period 2 .....

Similarly, on the lower branch the worker would offer,

(C2) $$w_W = y - \min\{W_F\ ;(1-s)V_F + sW_F\}.$$

And obviously by construction both of these would be accepted. Since Nature always chooses between the firms and the worker making offers (the two branches) with probabilities (1-β) and β, then

(C3) $$V_W = \beta w_W + (1-\beta) w_F.$$

Now suppose that $V_W \geq W_W$ and $V_F \geq W_F$, then, by rearranging, s cancels and we obtain

(C4) $$V_W = \beta y + (1-\beta) W_W - \beta W_F,$$

This is obviously greater than W_W (if it were less $y > W_W + W_F$ would be violated) and similarly $V_F \geq W_F$.

Next suppose that $W_W > V_W$. Then we can write

(C5) $$W_W = (1-\beta) W_W + \beta[y - (1-s)V_F - sW_F].$$

which on rearranging terms yields $W_W + W_F = y$, thus a contradiction and therefore it must be the case that $W_W \leq V_W$. By a similar argument, $W_F \leq V_F$ can be established.

Next, let $s \to 0$, from (C1) we get $w_F = w_W = V_W$ and $y\text{-}w_F = y\text{-}w_W = V_F$. Thus both parties make exactly the same offer which is the one given in the text (equation (7)); thus irrespective of who makes the offer, we have the same equilibrium: the worker obtains a proportion β of the "net surplus" plus his "reservation utility" W_F; this is of course exactly the Nash Bargaining Solution, given in (7) in the text.

C.2) Investment By Firms:

Now we will establish that there exists a cut-off level η^* such that for $\eta < \eta^*$, in the model of section III, all firms will choose the same investment level and accept all workers. Suppose a firm adopts the strategy used in the text, which is to accept all workers, then this firm will choose its level of capital, k_n, by solving

$$\text{(C6)} \quad \begin{aligned} &\max_{k_n}\{\lambda[(1-\beta)Ah_1^{\alpha}k_n^{1-\alpha}+(1-\beta)\beta\eta\lambda Ah_1^{\alpha}k_n^{1-\alpha} \\ &+(1-\beta)\beta\eta(1-\lambda)Ah_2^{\alpha}k_n^{1-\alpha}-(1-\beta)\beta\eta Ah_1^{\alpha}k_0^{1-\alpha}] \\ &+(1-\lambda)[(1-\beta)Ah_2^{\alpha}k_n^{1-\alpha}+(1-\beta)\beta\eta\lambda Ah_1^{\alpha}k_n^{1-\alpha} \\ &+(1-\beta)\beta\eta(1-\lambda)Ah_2^{\alpha}k_n^{1-\alpha}-(1-\beta)\beta\eta Ah_2^{\alpha}\bar{k}^{1-\alpha}]-Rk_n\}. \end{aligned}$$

where $\bar{k}$ is the capital stock that all other firms have chosen. This maximization will lead to a mapping level of k_n which does not depend on $\bar{k}$ since there is no term in which these two interact. Then the equilibrium is given by $k_n=\bar{k}$ Therefore,

$$\text{(C7)} \quad E\pi(\bar{k})=\lambda(1-\beta)Ah_1^{\alpha}\bar{k}^{1-\alpha}+(1-\lambda)(1-\beta)Ah_2^{\alpha}\bar{k}^{1-\alpha}-R\bar{k}\}.$$

Suppose now a firm deviates from the strategy of accepting all workers and decides to turn down low skill workers and wait for one more round of sampling. Obviously, such a deviant firm would also choose a different level of capital which we denote by k_d. Then the profits of this deviant are given by

$$\text{(C8)} \quad \begin{aligned} &E\pi^d(k_d)=\max_{k_d}\{\lambda[(1-\beta)\eta\lambda Ah_1^{\alpha}k_d^{1-\alpha}+(1-\beta)\eta(1-\lambda)Ah_2^{\alpha}k_d^{1-\alpha}] \\ &+(1-\lambda)[(1-\beta)Ah_2^{\alpha}k_d^{1-\alpha}+(1-\beta)\beta\eta\lambda Ah_1^{\alpha}k_d^{1-\alpha} \\ &+(1-\beta)\beta\eta(1-\lambda)Ah_2^{\alpha}k_d^{1-\alpha}-(1-\beta)\beta\eta Ah_2^{\alpha}\bar{k}^{1-\alpha}]-Rk_d\}. \end{aligned}$$

(C8) has exactly the same last two lines as (C6) and thus the only difference between the two expressions is that the first four terms in (C6) have been replaced by the first two terms in (C8). Obviously, the deviant can only make more profits if $k_d>\bar{k}$ since it is producing with high skilled workers more often. Now look at the first order conditions for k and k_d from (C6) and (C8).

$$\text{(C6')} \quad \begin{aligned} &(1-\alpha)\{\lambda[(1-\beta)Ah_1^{\alpha}k_n^{-\alpha}+(1-\beta)\beta\eta\lambda Ah_1^{\alpha}k_n^{-\alpha} \\ &+(1-\beta)\beta\eta(1-\lambda)Ah_2^{\alpha}k_n^{-\alpha}] \\ &+(1-\lambda)[(1-\beta)Ah_2^{\alpha}k_n^{-\alpha}+(1-\beta)\beta\eta\lambda Ah_1^{\alpha}k_n^{-\alpha} \\ &+(1-\beta)\beta\eta(1-\lambda)Ah_2^{\alpha}k_n^{-\alpha}]\}-R=0. \end{aligned}$$

And for the deviant:

$$(C8') \quad \begin{aligned}(1-\alpha)\{\lambda[(1-\beta)\eta\lambda Ah_1^{\alpha}k_d^{-\alpha}+(1-\beta)\eta(1-\lambda)Ah_2^{\alpha}k_d^{-\alpha}] \\ +(1-\lambda)[(1-\beta)Ah_2^{\alpha}k_d^{-\alpha}+(1-\beta)\beta\eta\lambda Ah_1^{\alpha}k_d^{-\alpha} \\ +(1-\beta)\beta\eta(1-\lambda)Ah_2^{\alpha}k_d^{-\alpha}]\}-R=0.\end{aligned}$$

Comparing (C6') and (C8'), we can see that now only the first three terms are different and for $k_d>\bar{k}$, we need these three terms in (C8') to be larger. Thus, a sufficient condition for a deviation not to be profitable is

$$(C9) \quad \begin{aligned}(1-\beta)Ah_1^{\alpha}+(1-\beta)\beta\eta\lambda Ah_1^{\alpha}+(1-\beta)\beta\eta(1-\lambda)h_2^{\alpha}> \\ \lambda(1-\beta)\eta Ah_1^{\alpha}+(1-\lambda)(1-\beta)\eta Ah_2^{\alpha}.\end{aligned}$$

Or, $\eta\leq\frac{1}{(1-\beta)\lambda+(1-\beta)(1-\lambda)\phi^{-\alpha}}$ and recalling that there will be a maximum value, $\phi_{\min}$, that ϕ can take in any limiting distribution, $\eta\leq\eta^{*}=\frac{1}{(1-\beta)\lambda+(1-\beta)(1-\lambda)\phi_{\min}^{-\alpha}}$ is sufficient for the equilibrium in which all firms to choose the same level of capital stock. Also note that when $h_{\min}\rightarrow 0$, $\phi_{\min}\rightarrow\infty$ and thus $\eta^{*}\rightarrow 0$. Finally, since all firms choose the same level of investment, they will be all happy to accept all workers in the first round.

Remark: It should also be noted that because the dynamics of the system are backward looking, even if $\eta<\eta^{*}$ does not hold (for instance because $h_{\min}=0$), in the region where the two groups are close to each other, the results in the text exactly apply. Only after inequality reaches a certain level that firms start choosing different capital levels.

C.3) Conditions For Local Stability of Full Equality With Infinite Sampling

In the text we analyzed the simplified case where agents had only one more chance to meet a new partner after they separated from their first match. This was done for simplicity of exposition and the more realistic case of course corresponds to the one where a firm can decide to wait an arbitrarily large number of periods for a high skill worker if it chooses to. Again under a suitably chosen condition on η, this will not happen along the equilibrium path but this possibility will influence the wages. Especially, as this channel from the overall distribution of skills to wages is the crucial one that leads to non-convergence in section III, it is important to check that it is not driven by the two-period assumption. In this part of the appendix, we investigate this question.

As before, we investigate wage determination under the hypothesis that all firms choose the same level of capital, k_t, and this will lead to a similar condition to the one in C.2 above that η should be less than a cut-off level η^*.

With the infinite sampling assumption, the wage of a worker with human capital h_t is given as

(C10) $$w(h_t)=\beta A h_t^{\alpha} k_t^{1-\alpha}-\beta\eta\{\int[Ah^{\alpha}k_t^{1-\alpha}-w(h)]dF_t(h)\}+(1-\beta)\eta w(h_t).$$

The explanation for this equation is straightforward. The worker gets a proportion β of the total surplus minus β times the threat point of the firm and plus (1-β) times his own threat point. The threat point of the firm is the average of the output it would produce with a randomly selected worker minus the wage it would pay to this worker, all multiplied by η since the firm loses potential output in the process of waiting for the next partner. Note that this averaging uses F_t as the relevant distribution, because in equilibrium all firms and workers agree and as in the text, there is a set of agents of measure υ who are unmatched and $\upsilon \to 0$; if a firm deviates and breaks the match, it would meet with one of those and the distribution of this measure zero set is assumed to be exactly the same as the unconditional distribution, F_t. Finally, the threat point of the worker, the last term, is straightforward; since all firms are the same and the problem faced by the worker is an infinite horizon one with constant discount factor, η, in the Markov Perfect equilibrium, the worker will agree to the same wage now as he would in the second period [see Osborne and Rubinstein, 1990, or Fudenberg and Tirole, 1991].

In the case of two types, we can write the wages as

(C11) $$w_{jt}=Ah_{jt}^{\alpha}k_t^{1-\alpha}-\beta\eta[\lambda(Ah_{1t}^{\alpha}k_t^{1-\alpha}-w_{1t})+(1-\lambda)(Ah_{2t}^{\alpha}k_t^{1-\alpha}-w_{2t})+(1-\beta)\eta w_{jt}.$$

for j=1 and 2.

Combining this equation for j=1 and j=2, we can write

(C12) $$[1-(1-\beta)\eta][w_{2t}-w_{1t}]=\beta A h_{2t}^{\alpha}k_t^{1-\alpha}-\beta A h_{1t}^{\alpha}k_t^{1-\alpha}.$$

Then, the wages of the two types of workers will be given as

(C13) $$w_{1t}=\frac{1}{1-\eta}\left\{\beta A h_{1t}^{\alpha}k_t^{1-\alpha}-\beta\eta\lambda A h_{1t}^{\alpha}k_t^{1-\alpha}-\beta^2\eta\frac{(1-\lambda)A h_{1t}^{\alpha}k_t^{1-\alpha}}{1-(1-\beta)\eta}-\beta\eta(1-\lambda)\left[\frac{1-(1-\beta)\eta-\beta}{1-(1-\beta)\eta}A h_{2t}^{\alpha}k_t^{1-\alpha}\right]\right\},$$

and

(C14) $$w_{1t}=\frac{1}{1-\eta}\left\{\beta A h_{2t}^{\alpha}k_t^{1-\alpha}-\beta\eta(1-\lambda)A h_{2t}^{\alpha}k_t^{1-\alpha}-\beta^2\eta\frac{\lambda A h_{2t}^{\alpha}k_t^{1-\alpha}}{1-(1-\beta)\eta}-\beta\eta\lambda\left[\frac{1-(1-\beta)\eta-\beta}{1-(1-\beta)\eta}A h_{1t}^{\alpha}k_t^{1-\alpha}\right]\right\}.$$

Proceeding in a similar fashion to the analysis in the text, we can write

(C15) $$\begin{aligned}\phi_{t+1}=\frac{w_{1t}}{w_{2t}}&=\frac{Ch_{1t}^{\alpha}-C\eta\lambda h_{1t}^{\alpha}-\beta\eta(1-\lambda)h_{1t}^{\alpha}-\eta(1-\lambda)(C-\beta)h_{2t}^{\alpha}}{Ch_{2t}^{\alpha}-(C-\beta)\eta\lambda h_{1t}^{\alpha}-\beta\eta\lambda h_{2t}^{\alpha}-\eta(1-\lambda)Ch_{2t}^{\alpha}}\\&=\frac{C\phi_t^{\alpha}-C\eta\lambda\phi_t^{\alpha}-\beta\eta(1-\lambda)\phi_t^{\alpha}-\eta(1-\lambda)(C-\beta)}{C-(C-\beta)\eta\lambda\phi_t^{\alpha}-\beta\eta\lambda-\eta(1-\lambda)C}\end{aligned}.$$

where $C=1-(1-\beta)\eta$. Again, it can be checked readily that $\phi_{t+1}(\phi_t=1)=1$ and that[1]

(C16) $$\phi_{t+1}(\phi_t=0)=\frac{-\eta(1-\lambda(C-\beta)}{C-\beta\eta\lambda-\eta C(1-\lambda)}<0.$$

Thus, we find that as in the text, for $\eta>0$, the difference equation that related ϕ_{t+1} to ϕ_t starts below the 45^0 and thus as in the proof of Proposition 2, we can see that there must be a set of inequality level from which full equality, $\phi=1$, will not be reached.

Next, to investigate the local stability of full equality, we look at

[1] To see that the term in (C16) is negative note that the numerator is always negative and the denominator is positive. The positivity of the denominator, D, follows from the fact that when $\eta=1$, D=0 and $\partial D/\partial\eta<0$ and as η is always less than 1, D>0.

(C17) $$\frac{\partial\phi_{t+1}}{\partial\phi_t}=\frac{\alpha C\phi_t^{\alpha-1}-\alpha C\eta\lambda\phi_t^{\alpha-1}-\alpha\beta\eta(1-\lambda)\phi_t^{\alpha-1}}{C-(C-\beta)\eta\lambda\phi_t^{\alpha}-\beta\eta\lambda-\eta(1-\lambda)C}+\frac{\alpha\eta\lambda(C-\beta)\phi_t^{\alpha-1}\phi_{t+1}}{C-(C-\beta)\eta\lambda\phi_t^{\alpha}-\beta\eta\lambda-\eta(1-\lambda)C},$$

thus

(C18) $$\frac{\partial\phi_{t+1}(\phi_t=1)}{\partial\phi_t}=\frac{\alpha C-\alpha C\eta\lambda-\alpha\beta\eta(1-\lambda)}{C-(C-\beta)\eta\lambda-\beta\eta\lambda-\eta(1-\lambda)C}+\frac{\alpha\eta\lambda(C-\beta)}{C-(C-\beta)\eta\lambda-\beta\eta\lambda-\eta(1-\lambda)C}=\frac{\alpha(1-(1-\beta)\eta-\eta\beta)}{C(1-\eta)}=\frac{\alpha}{1-(1-\beta)\eta}.$$

Therefore, the stability condition becomes $\alpha<1-(1-\beta)\eta$ rather than $\alpha<\frac{1}{1+(1-\beta)\eta}$. These conditions are very similar to each other and have the same comparative static properties, in particular, a high α or a high η make it harder for the system to be stable. Therefore, with infinite sampling all the conclusions of Proposition 2 hold. □