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Necessary and Sufficient Conditions for
the Perfect Finite Horizon Folk Theorem

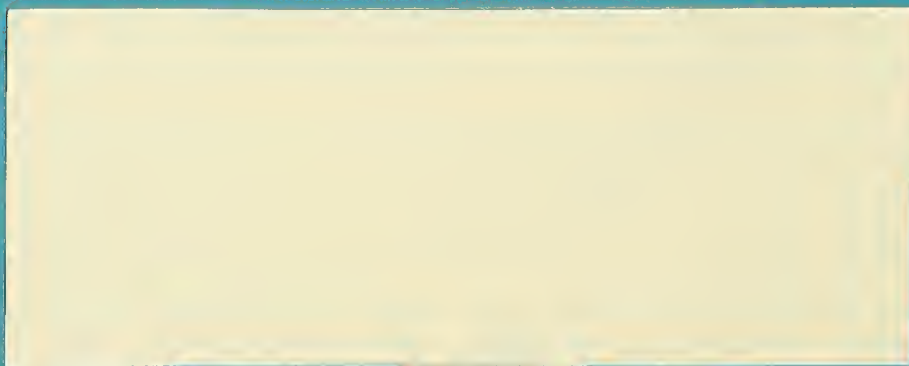
Lones Smith

No. 93-6

May 1993

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Necessary and Sufficient Conditions for the Perfect Finite Horizon Folk Theorem

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May 7, 1993

Abstract

Benoit and Krishna (1985) proved a finite-horizon n -player perfect folk theorem that assumed that every player has distinct Nash payoffs in the stage game and (essentially) that it satisfy the conditions of the infinite-horizon folk theorem. Abreu, Dutta and Smith (1993) have recently provided the necessary and sufficient condition (NEU) for the infinite-horizon folk theorem. We do prove that NEU is necessary for the finite-horizon folk theorem, but more importantly, this note substitutes the distinct Nash payoff condition requirement with a weaker necessary and sufficient condition, that players have *recursively distinct Nash payoffs*. As a consequence, we show how the n -player finite-horizon folk theorem might even obtain if only one player has distinct Nash payoffs in the stage game. Conversely, when the stage game satisfies NEU but not our condition, the folk theorem's failure is rather dramatic: In particular, at least one player's limit equilibrium payoff set is a singleton.

*The motivation for this note, namely seeking the *necessary* conditions for a folk theorem, stemmed from collaboration with Dilip Abreu and Prajit Dutta, and in particular from their earlier work Abreu and Dutta (1991).

1. INTRODUCTION

For a given n -player normal form game G , let $G(\delta, T)$ be the T -fold repeated game where the objective function is the average discounted sum of stage payoffs. For this finitely-repeated game, the perfect folk theorem is said to hold if the set of subgame perfect equilibrium payoffs includes any feasible and strictly individually rational payoff vector of G for large enough $T < \infty$ and $\delta < 1$. It has been long known that in the finitely-repeated Prisoners' Dilemma, the only subgame perfect equilibrium is 'defect' every period for any $T < \infty$. This is the ineluctable consequence of a simple backwards induction argument. Perhaps motivated by this fact, Benoit and Krishna (1985) [hereafter, BK] were led to correctly conclude that a perfect folk theorem for $G(\delta, T)$ obtains when (i) G satisfies the sufficient conditions for the infinite-horizon folk theorem — namely, the full-dimensionality condition of Fudenberg and Maskin (1986), and (ii) each player has distinct Nash payoffs in G .

Recently, Abreu et al. (1993) discovered an (essentially) necessary and sufficient condition for (i), which they termed NEU, for *non-equivalent utilities*, that neatly supplants the full-dimensionality condition. While we do prove that NEU is necessary for the finite-horizon folk theorem, this note primarily provides a *necessary and sufficient* condition — *recursively distinct Nash payoffs* — for the finite-horizon folk theorem that replaces (ii), thus extending BK to a complete characterization.

The contribution of this note is purely conceptual: The characterization is intended to finish the work of BK, thus completing the perfect information folk theorem program. For just as NEU only differed from full-dimensionality by a nongeneric class of games, so too, within the class of stage games with *recursively distinct Nash payoffs*, the measure with distinct Nash payoffs for all players is zero. In addition to the stated goal of this note, our condition helps explain how finite-horizon subgame perfect equilibria behave as the horizon grows.

The logic of the BK folk theorem, as best exemplified in Krishna (1989) or Smith (1990/92) is rather simple: Late in the repeated game, because all players have distinct Nash payoffs, the behaviour of any one of them can be leveraged by threatening to finish off with a fixed number (say S) of plays of that player's worst Nash payoff, rather than cycle through all the players' best Nash payoffs. Away from the end of the game, the infinite-horizon punishments work perfectly well. And because S is fixed independent of the horizon length, the effect on the average

payoff can be made arbitrarily small. This essentially is their proof.

Rather than formally define our condition, we first motivate it with the following three-player example game G .

2,2,3	2,2,2	2,2,2	2,1,-1	-1,-1,-1	1,-1,-1
2,2,2	2,2,2	2,2,2	-1,-2,-1	-2,-1,-1	1,-2,-1
2,2,3	2,2,2	-1,-1,0	-1,1,-1	-2,1,-1	0,0,-1

In this game, 1 chooses rows (actions U, M, D), 2 chooses columns (actions ℓ, m, r), and 3 chooses matrices (actions L, R). Player 3 strictly prefers L to R , while action D (resp. r) is weakly dominated for player 1 (resp. player 2). Thus it is easy to see that the only Nash (pure or mixed) payoffs are all convex combinations of $(2, 2, 2)$ and $(2, 2, 3)$. Furthermore, the mutual minimax payoff is 0 for all three players.¹

Because players 1 and 2 have unique Nash payoffs, G does not satisfy the BK condition (ii). Nonetheless, a folk theorem does obtain! For G satisfies NEU, and thus the infinite-horizon folk theorem applies to G . Next, player 3 enjoys the distinct (extremal) Nash payoffs of 2 and 3, so that his behaviour is leveraged near the end of the game: We need only threaten to switch from (U, ℓ, L) to (M, ℓ, L) for the S -period phase. By choosing S large enough, 3 is willing to play R for as many periods, say S' , as we wish just prior to this Nash phase. When player 3 plays R irrespective of what players 1 and 2 do, a new game $G(R)$ is *induced* for players 1 and 2. It has the unique Nash equilibrium payoff vector $(2, 1)$. So player 2's Nash payoff from $G(R)$ is 1, which differs from his unique Nash payoff in G (i.e. when player 3 plays L). We have now leveraged the behaviour of player 2 near the end of the game! Iterate this process. By choosing S' (and hence S) large enough, we can induce player 2 to play ℓ, m , or r for as many periods, say S'' , as we wish just prior to this penultimate 'Nash' phase. In particular, players 2 and 3 are willing to play (r, R) irrespective of what player 1 does, yielding a new (one-player) game $G(r, R)$ for player 1. It has the unique optimal payoff of 0, which differs from 1's unique optimal payoff of 2 in $G(\ell, L)$. We have now also leveraged the behaviour of player 1 near the end of the game! That G satisfies a folk theorem now follows by the same proof as before.

¹But observe that no one player can simultaneously minimax the other two, a fact of some importance later on in our proof of necessity.

We summarize the above procedure by saying that G has *recursively distinct Nash payoffs*. If G satisfies NEU, it is a *sufficient* condition for the BK result: So long as such a recursive procedure eventually leverages the behaviour of all players, then a perfect folk theorem obtains. Moreover, if G satisfies NEU, our new condition is *necessary* too for the perfect finite-horizon folk theorem. Namely, if for any such chain of recursive reductions, we cannot leverage the behaviour of all players, then a folk theorem does not obtain.

The intuition behind the necessity is also rather simple. For any given horizon length T , a player's set of subgame perfect equilibrium payoffs is either point-valued, or it is not. It is evident that if the behaviour of a player can be leveraged, then his payoffs are multivalued for large enough T , while if his behaviour cannot be leveraged, then he entertains a unique payoff for all T . Clearly in the latter case, we cannot possibly hope for anything more. In the above game G with no discounting, for instance, player 3 has a multivalued SPE payoff set for all T , while players 2 and 1 only receive distinct SPE payoffs for $T \geq 4$ and $T \geq 8$, respectively.

The special payoffs in G might suggest the nongeneric way in which distinct Nash payoff requirement can fail. Indeed, if one player has distinct Nash payoffs, then the game has at least two Nash equilibria. Thus, for generic games satisfying our condition, all players enjoy distinct Nash payoffs.

In section 2, we focus on the stage game, and precisely define our iterative procedure. The folk theorem is established in section 3.1. We prove the necessity of our condition and of NEU in section 3.2.

2. THE STAGE GAME

2.1 Basic Definitions

Consider a finite normal form n -player game $G = \langle A_i, \pi_i; i = 1, \dots, n \rangle$, where A_i is player i 's finite set of actions, and $A = \prod_1^n A_i$. Player i 's utility function is $\pi_i : A \rightarrow \mathbb{R}$. Let M_i be the set of player i 's mixed strategies and let $M = \prod_1^n M_i$. Simply write $\pi_i(\mu)$ for i 's expected payoff under the mixed strategy $\mu = (\mu_1, \dots, \mu_n) \in M$. Let m_{-i}^i be an $(n-1)$ -profile of mixed strategies which *minimax* player i , and m_i^i a best response for i when being *minimax*. Normalize $\pi_i(m^i) = 0$ for all i . Define $F = \text{co}\{\pi(\mu) : \mu \in M\}$, so that the set of feasible and *strictly individually rational* payoffs is $F^* = \{w \in F : w_i > 0, \text{ for all } i\} \neq \emptyset$, to sidestep trivialities.

Denote the set of players as $\mathcal{I} = \{1, 2, \dots, n\}$. For a given strict subset of players $\mathcal{I}' = \{i_1, i_2, \dots, i_m\} \subsetneq \mathcal{I}$ and corresponding choice of (possibly mixed) actions

$$a_{\mathcal{I}'} \equiv (a_{i_1}, a_{i_2}, \dots, a_{i_m}) \in M_{i_1} \times M_{i_2} \times \dots \times M_{i_m} \equiv M_{\mathcal{I}'},$$

let $G(a_{\mathcal{I}'})$ be the induced $(n - m)$ -player game for players $\mathcal{I} \setminus \mathcal{I}'$ obtained from G when the actions of players \mathcal{I}' are fixed to $a_{\mathcal{I}'}$.

2.2 Two Properties of Stage Games

1. The game G has *non-equivalent utilities* (or satisfies NEU) if no two players' von Neumann Morgenstern utility functions are equivalent, i.e. for all i and j , $\pi_i(\cdot)$ is not a positive affine transformation of $\pi_j(\cdot)$.

2. The game G has *recursively distinct Nash payoffs* if for some $h \geq 1$, there exists an increasing sequence of h nonempty subsets of players from \mathcal{I} , namely

$$\{\emptyset = \mathcal{I}_0 \subsetneq \mathcal{I}_1 \subsetneq \mathcal{I}_2 \subsetneq \dots \subsetneq \mathcal{I}_h = \mathcal{I}\}, \quad (1)$$

and actions $a_{\mathcal{I}_g} \in M_{\mathcal{I}_g}$ and $b_{\mathcal{I}_g} \in M_{\mathcal{I}_g}$ for each $g = 1, 2, \dots, h$, with some pair of Nash payoff vectors $y(a_{\mathcal{I}_g})$ of $G(a_{\mathcal{I}_g})$ and $y(b_{\mathcal{I}_g})$ of $G(b_{\mathcal{I}_g})$ different exactly for players in $\mathcal{I}_g \setminus \mathcal{I}_{g-1}$, i.e.

$$y(a_{\mathcal{I}_g})_i \neq y(b_{\mathcal{I}_g})_i \quad (2)$$

for all $i \in \mathcal{I}_g \setminus \mathcal{I}_{g-1}$. Even if $\mathcal{I}_r \subsetneq \mathcal{I}$, call such a sequence $\{\mathcal{I}_g\}$ a *Nash decomposition* of G .

Observe that if all players have distinct Nash payoffs as assumed in Benoit and Krishna (1985), then our condition holds. The converse is not true, as illustrated earlier.

It later proves fruitful to consider the case where G does *not* have recursively distinct Nash payoffs. In that case, for every Nash decomposition and corresponding choice of Nash equilibrium payoffs $y(a_{\mathcal{I}_g})$ and $y(b_{\mathcal{I}_g})$, $\mathcal{I}_h \subsetneq \mathcal{I}$ always obtains. Nonetheless, it is conceivable that the set of *all* such Nash decompositions could include every player. In fact this cannot occur.

Lemma 1 *There is a well-defined maximal set of players $\mathcal{J} \subseteq \mathcal{I}$ who have recursively distinct Nash payoffs.*

The proof is in the appendix.

3. FINITELY-REPEATED GAMES

3.1 Sufficiency of NEU and Recursively Distinct Nash Payoffs

We will analyze finitely-repeated games with *perfect monitoring*, allowing each player to condition his current actions on the past actions of all players. Also, for simplicity, *public randomization* is allowed: In every period players publicly observe the realization of an exogenous continuous random variable and can condition on its outcome.

Denote by $\alpha_i = (\alpha_{i1}, \dots, \alpha_{iT})$ a behavior strategy for player i and by $\pi_{it}(\alpha)$ his expected payoff in period t given the strategy profile α . Each player i 's objective function in $G(\delta, T)$ is the expected discounted sum of his payoffs: $\frac{1-\delta}{1-\delta^T} \sum_0^{T-1} \delta^t \pi_{it}(\alpha)$. The set of subgame perfect equilibrium payoffs is $V(\delta, T)$.

Beyond the replacement of the distinct Nash payoff requirement, Theorem 1 differs from Theorem 3.7 of Benoit and Krishna (1985) on a few fronts. First, it admits payoff discounting. As such, it is a *uniform* folk theorem, meaning that the discount factor and horizon length can vary independently over the relevant range. Second, NEU replaces full-dimensionality, as already noted. Third, unlike the (more intuitive) use of long deterministic cycles in Benoit and Krishna (1985), we simply rely upon correlated outcomes.²

Theorem 1 (The Folk Theorem) *Suppose that the stage game G satisfies NEU and has recursively distinct Nash payoffs. Then for the finitely-repeated game $G(\delta, T)$, $\forall u \in F^*$ and $\forall \varepsilon > 0$, $\exists T_0 < \infty$ and $\delta_0 < 1$ so that $T \geq T_0$ and $\delta \in [\delta_0, 1] \implies \exists v \in V(\delta, T)$ with $\|v - u\| < \varepsilon$.*

Proof: We suppose first that mixed strategies are observable, and discuss how to modify the argument later.

Fix a Nash decomposition (1) of players for which inequality (2) obtains, and define

$$c_g = \min_{i \in \mathcal{I}_g \setminus \mathcal{I}_{g-1}} \|y(a_{\mathcal{I}_g};)_i - y(b_{\mathcal{I}_g};)_i\| > 0$$

²It is also possible to take advantage of the correlating device to produce an exact, rather than an approximate, folk theorem. We omit such a laborious exercise.

for $g = 1, 2, \dots, h$. Let the $\rho > 0$ be the largest payoff range for any player in G . By means of public randomizations, let y^g (resp. z^g) be a payoff vector according equal weight to the *preferred* (resp. *less preferred*) Nash payoff vectors amongst $y(a_{\mathcal{I}_g})$ and $y(b_{\mathcal{I}_g})$ for players in $\mathcal{I}_g \setminus \mathcal{I}_{g-1}$. Then, without payoff discounting, any player in $\mathcal{I}_g \setminus \mathcal{I}_{g-1}$ is strictly willing to suffer through k periods of any action profile if the alternative is that at least $f_g(k) \equiv \lfloor kn\rho/c_j \rfloor + 1$ periods of y^g are switched to z^g , i.e.

$$-k\rho + f_g(k)y_i^g > f_g(k)z_i^g. \quad (3)$$

For any $m > 0$, recursively define $s_h(m) = f_h(m)$ and

$$s_g(m) = f_g(m + s_{g+1}(m) + \dots + s_h(m))$$

for $g = h - 1, h - 2, \dots, 1$. Define $t_0(m) = 0$, and $t_g(m) = s_1(m) + \dots + s_g(m)$ for $g = 1, 2, \dots, h$. The T -period equilibrium outcome sequence is

$$u, \dots, u; y^h, \dots, y^h; \dots; y^1, \dots, y^1$$

where each y^g phase lasts $s_g(q+r)$ periods, and $u \in F^*$ is played $T - t_h(q+r)$ times.

Abreu et al. (1993) have shown that NEU implies the existence of feasible payoff vectors x^1, x^2, \dots, x^n such that for all $i \neq j$, we have $x^i \gg 0$ (strict individual rationality), $x_i^i < x_i^j$ (payoff asymmetry), and $x_i^i < u_i$ (target payoff domination). These inequalities are later essential.

We now explicitly describe the players' strategies which support this equilibrium.³ For ease of exposition, *late* deviations are those occurring during the final $q+r+t_h(q+r)$ periods of the repeated game; all others are called *early* deviations.

1. Play u until period $T - t_h(q+r) - 1$. [If any player i deviates early, start 3; if some player in $\mathcal{I}_{g'}$ deviates late, start 5.]
2. For $g = h, h-1, \dots, 1$: Play y^g in periods $T - t_g(q+r), \dots, T - t_{g-1}(q+r) - 1$. [If some player in $\mathcal{I}_{g'}$ deviates late, where $g' < g$, start 5.]
3. Play m^i for q periods. [If player $j \neq i$ deviates, start 4.] Then set $j \leftarrow i$.

³Below, i and j denote arbitrary players. Also, for clarity, we use the simple notation $j \leftarrow i$ to mean "assign j the value i ." Also, steps always follow sequentially, unless otherwise indicated. Bracketed remarks refer to off-path play, i.e. following deviations.

4. Play x^j for r periods. [If any player i deviates early, restart 3; if some player in $\mathcal{I}_{g'}$ deviates late, start 5.] Then return to step 2.
5. Play $z^{g'}$ until period $T - t_{g'-1}(q+r) - 1$. [If some player in $\mathcal{I}_{g''}$ deviates, where $g'' < g'$, set $g' \leftarrow g''$ and restart 5.] Then go to step 2.

The verification that these strategies constitute a subgame perfect equilibrium for suitable choices of q, r, δ_0 and T_0 is found in the appendix. \diamond

Remark: If the minimax strategy m^i entails some punishers playing completely mixed strategies, then unobservable deviations by all players j within the support of m_j^i must somehow be deterred. Fudenberg and Maskin (1986) and Abreu et al. (1993) show how,⁴ in an infinite-horizon context, to make players indifferent across their pure strategies: After playing m^i , play proceeds to step 4, but with x^i probabilistically replaced by one of $n - 1$ possible substitute reward phases, with probabilities chosen as to make each minimaxing player $j \neq i$ indifferent over all actions in the support of m_j^i . This technique applies here equally well.

In our finitely repeated context, we must also deter late unobserved deviations from any mixed strategies required in step 2. For this we provide another way to maintain the incentives of the minimaxing players that takes advantage of the recursively distinct Nash payoffs and not of NEU. First replace y^g with the new payoff vector $\tilde{y}^g = y^g/2 + z^g/2$ that is strictly worse for every player $j \in \mathcal{I}_g \setminus \mathcal{I}_{g-1}$ than \tilde{y}^g , and yet still strictly preferred to z^g .⁵ Second, to reward player $j \in \mathcal{I}_g \setminus \mathcal{I}_{g-1}$ for playing a less preferred action in the support of $a_{\mathcal{I}_{g'}} \in M_{\mathcal{I}_{g'}}$, where $g' > g$, we may probabilistically replace \tilde{y}^g in step 2 with player i 's best Nash outcome among $y(a_{\mathcal{I}_g})$ and $y(b_{\mathcal{I}_g})$. Since $g' > g$, any such substitution cannot possibly affect the incentives of the original deviant in $\mathcal{I}_{g'} \setminus \mathcal{I}_{g'-1}$. Finally, it is possible to define in a fashion analogous to Abreu et al. (1993) a proper probabilistic transition function that will maintain all incentives. We omit the tedious algebraic verification.

3.2 Necessity of NEU and Recursively Distinct Nash Payoffs

Abreu et al. (1993) have already established the (near) necessity of NEU for the infinite-horizon folk theorem. That proof easily extends to our finite-horizon case.

⁴The first authors discovered the idea, while the latter modified their argument to handle stage games that do not satisfy full-dimensionality, but only the weaker NEU assumption.

⁵The definition of $f_g(k)$ will have to be modified to $f_g(k) \equiv \lfloor 2kn\rho/c_g \rfloor + 1$.

For consider the infinitely-repeated game with stage game G as a sequence of games $G(\delta, T)$. Clearly, if u is a subgame perfect discounted average payoff for $G(\delta, T)$, then by concatenation of the equilibria, it is also one for the infinitely repeated game with stage game G . This establishes

Theorem 2 (Necessity of NEU) *Suppose that no two players can be simultaneously held to their lowest feasible payoff in G . Then NEU is necessary for the conclusion of the folk theorem.*

If a game G , however, does not have recursively distinct Nash payoffs, the consequences are rather stark. The following result has the flavour of the “zero-one” laws of probability theory. Namely, in a T -period repeated game $G(\delta, T)$, either the set of subgame perfect equilibrium payoffs of all players tends to the strictly individually rational payoff set F^* of G , or some players receive a *unique* equilibrium payoff. There is no middle ground.

Theorem 3 (Necessity of Recursively Distinct Nash Payoffs) *In any subgame perfect equilibrium of $G(\delta, T)$, for any T , players in $\mathcal{I} \setminus \mathcal{J}$ receive a payoff equal to their unique Nash equilibrium payoff of G .*

Proof: Assume that every player in $\mathcal{I} \setminus \mathcal{J}$ has a unique subgame perfect equilibrium payoff of $G(\delta, T)$ equal to his Nash payoff in G , for $T = 1, 2, \dots, T_0$. (Since $\mathcal{I} \setminus \mathcal{J} \neq \emptyset$, we know this to be true for $T_0 = 1$.) Then in the first period of $G(\delta, T_0 + 1)$, players in $\mathcal{I} \setminus \mathcal{J}$ must play a Nash equilibrium of $G(a_{\mathcal{J}})$, for any $a_{\mathcal{J}} \in M_{\mathcal{J}}$, because they have a unique (subgame perfect) continuation payoff by assumption. By induction, the result obtains for all T . ◇

As a simple corollary, we can offer some insight into the behaviour of the payoff set $V(\delta, T)$ as T grows. If many possible Nash decompositions exist, we have little to say, but otherwise, we can sometimes make rather sharp predictions.

Corollary *If there exist a unique Nash decomposition of G , and it satisfies $\mathcal{I}_g \setminus \mathcal{I}_{g-1} = i_g$ for all $g = 1, 2, \dots, h$, then there exist times $T_1 < T_2 < \dots < T_h$ such that $\text{proj}_{i_g} \langle V(\delta, T) \rangle$ is single valued for all periods $T < T_g$, and otherwise multi-valued.*

4. CONCLUDING REMARKS

This note has illustrated an easily-verifiable, and yet necessary and sufficient replacement of the distinct Nash payoff requirement of Benoit and Krishna (1985). Although its theoretical advantage is nongeneric, it hopefully underscores that folk theorems only imply that players' behaviour is *iteratively* leveraged near the end of the repeated game. This conceptual contribution might have generic application in other circumstances.

The analysis has also offered insight into the behaviour of games only partially fulfilling our condition: When not all players can be leveraged, their payoff sets are singletons. This contrasts with recent work by Wen (1993), who has pointed out that the consequences of the partial failure of NEU might not be so dramatic: Players can be held down not to their minimax payoff level, but their (newly defined) *effective minimax* payoff level.⁶

5. APPENDIX

5.1 Proof of Lemma 1

If for some Nash decomposition we have $\mathcal{I}_h = \mathcal{I}$, then unambiguously $\mathcal{J} = \mathcal{I}$, and we are done.

Otherwise, let $\mathcal{E}_i(G)$ be the set of all Nash payoffs for player i in the game G . Consider the alternative *unique* nested sequence of h' nonempty sets $\{\mathcal{I}_g\}$ satisfying

$$\{\emptyset = \mathcal{J}_0 \subsetneq \mathcal{J}_1 \subsetneq \mathcal{J}_2 \subsetneq \cdots \subsetneq \mathcal{J}_{h'} \equiv \mathcal{J} \subsetneq \mathcal{I}\},$$

given by the following (well-defined) recursion: Given \mathcal{J}_g , the set $\mathcal{J}_{g+1} \setminus \mathcal{J}_g$ is all players $i \in \mathcal{I} \setminus \mathcal{J}_g$ for whom $\{\mathcal{E}_i(G(a_{\mathcal{I}_g})), a_{\mathcal{I}_g} \in M_{\mathcal{I}_g}\}$ does not consist of a singleton. If no such players $\mathcal{J}_{g+1} \setminus \mathcal{J}_g$ exist, set $h' = g$.

Note that it may not be possible to find only two Nash equilibria providing distinct payoffs simultaneously for all players $i \in \mathcal{J}_{g+1} \setminus \mathcal{J}_g$. That is, $\{\mathcal{J}_g\}$ need not coincide with *any* Nash decomposition $\{\mathcal{I}_g\}$. Still, we can show that $\mathcal{J}_{h'} = \mathcal{I}_h$ for all Nash decompositions $\{\mathcal{I}_g\}$. For it is clear that $\mathcal{I}_h \subseteq \mathcal{J}_{h'}$. Moreover, for any

⁶This equals $\min_{a_{-I(i)}} \max_{j \in I(i)} \max_{a_j} \pi_i(a_j, a_{-j})$, where $I(i)$ are all players with equivalent utilities to i .

such nested sequence (1), all players in any $\mathcal{J}_{g'}$ are eventually included in *some* \mathcal{I}_g . For if $\{\mathcal{E}_i(G(a_{\mathcal{I}'})), a_{\mathcal{I}'} \in M_{\mathcal{I}'}\}$ does not consist of a singleton, then neither does $\{\mathcal{E}_i(G(a_{\mathcal{I}''})), a_{\mathcal{I}''} \in M_{\mathcal{I}''}\}$ if $\mathcal{I}' \subsetneq \mathcal{I}''$.

5.2 Verification of Strategies for Theorem 1

We proceed recursively as we ensure subgame perfection. First note that *we only check that all deterrents are strict by some positive margin, say 1, without discounting*. Given continuity of discounted sums in δ , they will thus remain strict for any level of discounting $\delta \in [\delta_0, 1]$, for some $\delta_0 < 1$. It will follow that for this profile, after no history consisting of isolated deviations does a one-shot deviation yield a higher payoff. By the “unimprovability” criterion, the profile will be a subgame perfect equilibrium.

We now select q and r . In light of $x_i^i > 0$, let q satisfy⁷

$$w_i^i + qx_i^i > \beta_i^i + 1 \quad (4)$$

for all j , where β^i is the best payoff vector for player i in G . Since $x_i^i < u_i$ too, (4) simultaneously renders steps 3 and 4 a strict deterrent to deviations from steps 1 and 2, for any r . Next, step 4 deters deviations by the punishers from step 3 if r is large enough that

$$qw_j^j + rx_j^i > \beta_j^j + rx_j^j + (q-1)u_j + 1$$

for all $j \neq i$. This is possible because $x_j^i > x_j^j$.

Next, we verify that step 5 deters all late deviations. Indeed, this is an immediate consequence of inequality (3), which upon substitution yields

$$[q + r + t_h(q + r) - t_g(q + r)](-\rho) + s_g(q + r)y_i^g > s_g(q + r)z_i^g + 1$$

for all $i \in \mathcal{I}_g$, and $g = 1, 2, \dots, h$.

Given q and r as defined above, the above program is feasible for any repeated context if $T_0 \geq p + q + t_h(p + q)$. For $T \geq T_0$, each deterrent remains strict for all

⁷The inequality (4) — in particular, why the left-hand side is *not* $(q+1)x_i^i$ — reflects the fact that obeying the random correlating device generating x^i might sometimes require i to play his worst possible outcome.

$\delta \in [\delta_0, 1]$, for some $\delta_0 < 1$. Finally, for $T_0 < \infty$ and $\delta_0 < 1$ large enough, if the resulting average payoff vector is $v \in F^*$, we have $\|u - v\| < \varepsilon$, since p and q are fixed.

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