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Regularity in Overlapping Generations Exchange Econcmies

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Abstract

In this paper we develop a regularity theory for stationary overlapping generations economies. We show that generically there are an odd number of steady states in which a non-zero amount of nominal debt (fiat money) is passed from generation to generation and an odd number in which there is no nominal debt. We are also interested in non-steady state perfect foresight paths. As a first step in this direction we analyze the behavior of paths near a steady state. We show that generically they are given by a second order difference equation that satisfies strong regularity properties. Economic theory alone imposes little restriction on these paths: With n goods, for example, the only restriction on the set of paths converging to the steady state is that they form a manifold of dimension no less than one, no more than 2n.

Regularity in Overlapping Generations Exchange Economies

by

Timothy J. Kehoe and David K. Levine*

1. INTRODUCTION

The theory of regularity developed by Debreu (1970) for static exchange economies has played an important role in recent studies of the comparative statics properties of general equilibrium models. In this paper we develop a regularity theory for stationary overlapping generation exchange economies.

We begin by studying steady states. We show that generically there are an odd number of steady states in which a non-zero amount of nominal debt (fiat money) is passed from generation to generation and an odd number in which there is no nominal debt. Generically, these latter steady states have price levels that explode either to zero or to infinity. We are also interested in non-steady state perfect foresight paths. As a first step in this direction we analyze the behavior of paths near a steady state. We show that generically they are given by a second order difference equation that satisfies strong regularity properties. Economic theory alone imposes little restriction on these paths: With n goods, for example, the only restriction on the set of paths converging to the steady state is that they form a manifold of dimension no less than one, no more than 2n.

The regularity theory we develop here can be applied to analyze the response of an overlapping generations economy to unanticipated shocks.

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Kehoe and Levine (1982a) consider the impact of shocks under alternative assumptions about the types of contractual arrangements existing before the shock and the process by which perfect foresight forecasts are formed.

2. THE MODEL

We analyze a stationary overlapping generations model that generalizes that introduced by Samuelson (1958). In each period there are n goods. Each generation t is identical and consumes in periods t and t + 1. The consumption and savings decisions of the (possibly many different types of) consumers in generation t are aggregated into excess demand functions $y(p_t, p_{t+1})$ in period t and $z(p_t, p_{t+1})$ in period t + 1. The vector $p_t = (p_t^1, \dots, p_t^n)$ denotes the prices prevailing in period t. Excess demand is assumed to satisfy the following assumptions:

(A.1) (Differentiability) $y, z\colon R_{++}^{2n} \to R^n$ are smooth (that is, C^1) functions.

(A.2) (Walras's law) $p_t' y(p_t, p_{t+1}) + p_{t+1}' z(p_t, p_{t+1}) = 0$.

(A.3) (Homogeneity) y, z are homogeneous of degree zero.

(A.4) (Boundary) $\|(y(q_k), z(q_k))\| \to \infty$ as $q_k \to q$, $q \in \partial R_+^{2n} \setminus \{0\}$. (y, z) is bounded from below, however, for all $q \in R_{++}^{2n}$.

A.1 has been shown by Debreu (1972) and Mas-Colell (1974) to entail little loss of generality. A.2 implies that each consumer faces an ordinary budget constraint in the two periods of his life. As we later show, this is equivalent to assuming a fixed (possibly zero or negative) stock of fiat money. A.3 is standard. As we shall see, A.4 is used only to guarantee the existence of interior steady states. Although the theory can be extended to analyze free goods, we do not attempt to do so here. Muller (1983) has, in fact, extended the type of results presented in this paper to economies with general activity analysis production technologies that include free disposal.

Note that we consider only pure exchange economies and two period lived consumers. We do, however, allow many goods and types of consumers, and the multi-period consumption case can easily be reduced to the case we consider: If consumers live m periods, we simply redefine generations so that consumers born in periods 1,2,...,m-1 are generation 1, consumers born in periods -m+2,-m+3,...,0 and m,m-1,...,2m-2 are generations 0 and 2 respectively, and so forth. In this reformulation each generation overlaps only with the next generation. (See Balasko, Cass and Shell (1980).)

The space of feasible economies $\mathcal{E}$ are the pairs (y, z) which satisfy A.1 - A.5. This is a topological space in the weak C^1 topology described, for example, by Hirsch (1976). Roughly, two economies (y^1, z^1) and (y^2, z^2) are close if the functions and their first derivatives are close.

A.1 - A.4 are naturally satisfied by any demand function derived by aggregating the individual demand functions of utility maximizing consumers. Furthermore, Debreu (1974) has demonstrated that, for any (y, z) that satisfy A.1 - A.3, there exists a generation of 2n utility maximizing consumers whose aggregate excess demands (y^*, z^*) agree with (y, z) on any compact subset of R^{2n}_{++}. Since homogeneity allows us to restrict our attention to prices

$q \varepsilon R^{2n}_{++}$ that satisfy such a price normalization as $\Sigma^{2n}_{i=1} q^i = 1$, this means that problems can occur only as some relative prices approach zero. As we point out in the next section, however, this minor technical problem plays no role in our study of steady states or of equilibrium price paths near steady states. Consequently, we are justified in viewing A.1 - A.4 as completely characterizing demand functions derived from utility maximization by heterogenous consumers.

3. STEADY STATES

A steady state of an economy $(y, z) \varepsilon \mathcal{E}$ is a relative price vector $p \varepsilon R^n_{++}$ and price level growth factor $\beta > 0$ such that

$$(3.1) \qquad z(p, \beta p) + y(\beta p, \beta^2 p) = z(p, \beta p) + y(p, \beta p) = 0.$$

In other words, if relative prices in each period are given by p and the price level grows at β, the market is always in equilibrium. Since claims to good i now cost p^i and claims to good i next period cost βp^i, $1/\beta - 1$ is the steady state rate of interest.

Notice that any steady state price vector $(p, \beta p)$ is a special case of a price vector $q \varepsilon R^{2n}_{++}$ that satisfies $z(q) + y(q) = 0$. We are now in a position to argue that, for our purposes, A.1 - A.4 completely characterize excess demand functions derived from utility maximization, and that we need not worry about problems near the boundary of R^{2n}_{+}: If (y, z) satisfies A.1 - A.3, then there exists a (y*, z*), derived from utility maximization by 2n consumers, that agrees with (y, z) on $S_\varepsilon = \{q \varepsilon R^{2n} | q'e = 1, q^i > \varepsilon\}$ for any $\varepsilon > 0$. Here $e = (1, \ldots, 1)$. If (y, z) satisfies A.4, then as $q_k \to q$, $q \in \partial R_+^{2n} \setminus \{0\}$, $e'(z(q) + y(q)) \to \infty$. Consequently, S_ε can be chosen large enough so that $e'(z(q) + y(q)) > 0$ for all $q \in S_0 \setminus S_\varepsilon$. This obviously implies that we can choose S_ε large enough so that every steady state of (y, z) lies in its interior. Mas-Colell (1977) has further demonstrated that, for any $\varepsilon > 0$ and any (y, z) that satisfies A.1 - A.4 and the condition that $e'(z(q) + y(q)) > 0$ for all $q \in S_0 \setminus S_\varepsilon$, there exists (y^*, z^*), derived from utility maximization by 2n consumers, that agrees with (y, z) on S_ε and also satisfies $e'(z^*(q) + y^*(q)) > 0$ for all $q \in S_0 \setminus S_\varepsilon$. Consequently, the only steady states of (y^*, z^*) are those of (y, z). Furthermore, S_ε can be chosen large enough so that (y, z) and (y^*, z^*) agree on any open neigborhood of these steady states in S_0.

The nominal steady state savings for the entire economy is $\mu = -p'y(p, \beta p)$. There are two kinds of steady states: <u>real</u> steady states in which $\mu = 0$ and monetary, or <u>nominal</u>, steady states in which $\mu \neq 0$. Gale (1973) refers to real steady states as <u>balanced</u>. By Walras's law, $p'(y + \beta z) = 0$, which implies $\beta p'z = -p'y = \mu$. By the equilibrium condition, $p'(z + y) = 0$, which implies $p'z = \mu$. Consequently, $(\beta - 1)\mu = 0$, and in a monetary steady state the interest rate must be zero. We shall see that a real steady state has $\beta = 1$ purely by coincidence. We therefore refer to a steady state with $\beta = 1$ as a nominal steady state. Gale refers to these as <u>golden rule</u> steady states since they maximize a weighted sum of utilities subject to the steady state consumption constraint.

We now examine the number of steady states. We first separate the nominal and real cases. If both $\beta = 1$ and $\mu = -p'y = 0$ at a steady state, then

$$z(p, p) + y(p, p) = 0 \tag{3.2}$$
$$-p'y(p, p) = 0.$$

By virtue of Walras's law, the first n equations may be viewed as a system of n - 1 equations while, by homogeneity, p constitutes n - 1 independent variables. 3.2 may therefore be regarded as n equations in n - 1 unknowns. Let us assume that

(R.1) System 3.2 has no solution.

The importance of this regularity assumption is that it is satisfied by almost all $(y, z) \in \mathcal{E}$. Here "almost all" means an open dense subset of $\mathcal{E}$. We call a property generic if it is satisfied by an open dense subset of a topological space. Note that we can easily show that genericity in $\mathcal{E}$ is equivalent to genericity in the space of excess demand functions derived from utility maximization (see the discussion of the boundary condition above). This has implications for economies parameterized by utility functions and endowments (see Mas-Colell (1974)). The principal tool that we use to prove genericity is the following result from differential topology (see Guillemin and Pollack (1974, pp. 67-69)).

TRANSVERSALITY THEOREM: Let M, V, N be smooth manifolds where dim M = m and dim N = n. Let $y \in N$. Suppose that $f: M \times V \to N$ is a C^r map, where $r > \max[0, m - n]$, such that for every (x, v) that satisfies f(x, v) = y, rank Df(x, v) = n; then the set of $v \in V$ for which f(x, v) = y implies rank $D_1 f(x, v) = n$ has full Lebesgue measure. In other words, if y is a regular value of f, then, for all $v \in V$ in a set of full Lebesgue measure, it is a regular value of f_v.

Since a set of full Lebesgue measure is dense, we can use this theorem to prove the density of sets that satisfy some property. Openness usually follows trivially from definitions. Notice that, since $Df = [D_1 f \;\; D_2 f]$, it suffices to demonstrate that rank $D_2 f(x, v) = n$ to prove that, for almost all $v \in V$, $Df_v(x)$ has rank n whenever $f_v(x) = y$.

PROPOSITION 3.1: The set of economies that satisfy R.1 is open and dense in $\mathcal{E}$.

PROOF: Openness is obvious. To prove density, we let $v_1 \in R^n$, $v_2 \in R$ and construct the perturbation

$$y_v^i = y^i + \frac{\Sigma_{j=1}^n p_1^j v_1^j}{\Sigma_{j=1}^n p_1^j} - v_1^i + \frac{p_2^i}{p_1^i} v_2 \tag{3.3}$$

$$z_v^i = z^i - v_2.$$

A check shows that, for v small enough, $(y_v, z_v) \in \mathcal{E}$; in other words, A.1 - A.5 are satisfied. To show the set of economies that satisfy R.1 is dense, it suffices by the transversality theorem to show that the derivative of the system in 3.2 with respect to v has rank n at any solution: The only way 0 can be a regular value of $f_v(p) = z_v(p, p) + y_v(p, p), -p'y_v(p, p)$ is for there to be no p for which $f_v(p) = 0$. Writing out this derivative, we have

$$\begin{bmatrix} ep' - I & 0 \\ 0 & -1 \end{bmatrix}$$

for any $p \in S_\varepsilon$. This matrix has rank n as required.

Q.E.D.

Nominal steady states are characterized by $z(p, p) + y(p, p) = 0$. Since $z(p, p) + y(p, p)$ has the formal properties of the excess demand function of a static exchange economy with n goods, the theory of nominal steady states carries over directly from the static theory. For the sake of completeness we prove the following proposition:

PROPOSITION 3.2: Every economy $(y, z) \in \mathcal{E}$ has a steady state in which $\beta = 1$.

PROOF: Let S_ε now denote the set $\{p \in R^n | p'e = 1, p^i > \varepsilon\}$. Choose ε small enough so that $e'(z(p, p) + y(p, p)) > \alpha$ for all $p \in S_0 \setminus S_\varepsilon$ and some $\alpha > 0$. S_ε is obviously compact, convex, and, choosing $\varepsilon < 1/n$, non-empty. For any $p \in S_\varepsilon$, define $f(p)$ as the vector in S_ε that is closest to $p + z(p, p) + y(p, p)$ in terms of euclidean distance. $f: S_\varepsilon \to S_\varepsilon$ is obviously continuous and, hence, by Brouwer's fixed point theorem, has a fixed point.

At any $p \in S_\varepsilon$ $f(p)$ is the vector that solves the problem of minimizing $1/2||f - p - z(p, p) - y(p, p)||^2$ subject to the constraints $f > \varepsilon e$ and $f'e = 1$. By the Kuhn-Tucker theorem such a point satisfies

$$(3.5) \qquad f - p - z(p, p) - y(p, p) - \lambda_1 + \lambda_2 e = 0$$

and $(f - \varepsilon e)'\lambda_1 = 0$ for some $\lambda_1 \in R^n_+$, $\lambda_2 \in R$. At a fixed point $f = p$. Post-multiplying 3.5 by $(p - \varepsilon e)'$ yields $(1 - n\varepsilon)\lambda_2 = -e'(z(p, p) + y(p, p))$; post-multiplying by p' yields $\lambda_2 = p'\lambda_1$. If $p \in \partial S_\varepsilon$, then $0 > \alpha > (1 - n\varepsilon)\lambda_2 = (1 - n\varepsilon)p'\lambda_1 > 0$, which is impossible. Consequently, since any fixed point p lies in the interior of S_ε, $\lambda_1 = 0$, which implies $\lambda_2 = 0$, and 3.5 is the steady state condition.

Q.E.D.

Following Debreu (1970), we impose the regularity assumption

(R.2) $D_1 z(p, p) + D_2 z(p, p) + D_1 y(p, p) + D_2 y(p, p)$ has rank $n - 1$ at nominal steady states.

Since the map from $\mathcal{E}$ to static exchange economies with n goods is a continuous open map, R.2 is generic in $\mathcal{E}$.

We can use the fixed point index theorem developed by Dierker (1972) to prove that R.2 implies that there are an odd number of nominal steady states. Let $J = D_1 z + D_2 z + D_1 y + D_2 y$, evaluated at a nominal steady state p. If we define index (p) = $\text{sgn}(\det[-\bar{J}])$, where $\bar{J}$ is the $(n - 1) \times (n - 1)$ matrix formed by deleting the first row and column from J, then index theory implies that $\Sigma\,\text{index}(p) = +1$, where the sum is over all nominal steady states. For example, if (y, z) exhibits gross substitutability, which implies that $\det[-\bar{J}] > 0$, then there is a unique nominal steady state.

Real steady states are characterized by the equations

$$z(p, \beta p) + y(p, \beta p) = 0$$

(3.5)

$$-p'y(p, \beta p) = 0.$$

Walras's law implies that $p'z(p, \beta p) = 0$ at the steady state and, consequently, that (p, β) solves 3.5 if and only if it solves

$$(I - ep')(z(p, \beta p) + y(p, \beta p)) = 0$$

(3.6)

$$-p'y(p, \beta p) = 0.$$

PROPOSITION 3.3: Every economy has a steady state in which $\mu = -p'y(p, \beta p) = 0$.

PROOF: The proof of this proposition is similar to that of Proposition 3.2: We find a non-empty, compact, convex set whose interior contains all steady states that satisfy 3.6. We then define a continuous mapping of this set into itself whose fixed points are steady states.

We begin by putting bounds on p: A.4 implies that there exists some $\varepsilon > 0$ such that $e'(z(p, \beta p) + y(p, \beta p)) > \alpha > 0$ for all $p \in S_0 \setminus S_\varepsilon$ and all $\beta > 0$. To see why, suppose instead that there exists a sequence $(p_k, \beta_k) \in S_0 \times R_{++}$ such that $e'(z(p_k, \beta_k p_k) + y(p_k, \beta_k p_k)) < \alpha$ and $p_k \to p \in \partial S_0$. Now there is either a subsequence of (p_k, β_k) for which β_k converges or one for which $1/\beta_k$ converges. In the first case, the associated subsequence $(p_k, \beta_k p_k)$ provides an example of a price sequence that converges to a point on the boundary of R_+^{2n} and violates A.4. In the second case, $((1/\beta_k)p_k, p_k)$ provides such an example. Consequently, we can find an $\varepsilon > 0$

such that all steady states (p, β) have $p \in S_\varepsilon$. It is now easy to put bounds on β: A.4 implies that for any $p \in S_\varepsilon$ $p'z(p, \beta_k p) \to \infty$ as $\beta_k \to 0$ and, similarly, $p'y((1/\beta_k)p, p) \to \infty$ as $\beta_k \to \infty$. Since S_ε is compact, we can find some $\beta > 0$ such that $-p'y(p, \beta p) > 0$ for all $\beta > \bar{\beta}$ and all $p \in S_\varepsilon$ and some $0 < \underline{\beta} < \bar{\beta}$ such that $-p'y(p, \beta p) < 0$ for all $\beta < \underline{\beta}$ and all $p \in S_\varepsilon$.

Consider now the set $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$. It is non-empty, compact, and convex. Furthermore steady states that satisfy 3.6, if any exist, lie in its interior. For any $(p, \beta) \in S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$ we define $f(p, \beta)$ as the vector in $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$ that is closest to $[p + (I - ep')(z(p, \beta p) + y(p, \beta p))]$, $\beta - p'y(p, \beta p))$ in terms of euclidean distance. Again using the Kuhn-Tucker theorem to chartacterize $f(p, \beta)$, we establish that any fixed point $(p, \beta) = f(p, \beta)$ must satisfy

$$
\text{(3.7)} \qquad \begin{aligned} &-(I - ep')(z(p, \beta p) + y(p, \beta p)) - \lambda_1 + \lambda_2 e = 0 \\ &p'y(p, \beta p) - \lambda_3 + \lambda_4 = 0 \end{aligned}
$$

$(p - \varepsilon e)'\lambda_1 = 0$, $(\beta - \underline{\beta})\lambda_3 = 0$, and $(\bar{\beta} - \beta)\lambda_4 = 0$ for some $\lambda_1 \in R^n_+$, $\lambda_2 \in R$, and $\lambda_3, \lambda_4 \in R_+$. The choice of $\underline{\beta}$ and $\bar{\beta}$ implies that $\lambda_3 = \lambda_4 = 0$. An argument identical to that in Proposition 3.2 implies $\lambda_1 = 0$ and $\lambda_2 = 0$. Consequently, a fixed point of f, which necessarily exists, is a steady state in which $\mu = -p'y(p, \beta p) = 0$.

Q.E.D.

The relevant regularity condition is

$$\text{(R.3)} \quad \begin{bmatrix} (I - ep')(D_1 z + \beta D_2 z + D_1 y + \beta D_2 y) & (I - ep')(D_2 z + D_2 y)p \\ -y' - p'(D_1 y + \beta D_2 y) & -p'D_2 yp \end{bmatrix} \text{ has rank n.}$$

Since $S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$ is compact, a standard argument implies that economies that satisfy R.3 at every real steady state have only a finite number of real steady states. Define index(p, β) to be +1 or -1 according to whether the sign of the determinant of the negative of the above matrix with its first row and column deleted is positive or negative. Another standard argument then implies that Σindex(p, β) = +1 when summed over all equilibria. This implies there is an odd number of real steady states, and indeed a unique real steady state if index(p, β) = +1 at every possible steady state.

PROPOSITION 3.4: Given R.1, R.3 is also generic.

PROOF: The openness of R.3 is immediate from the stability of transversal intersections and the continuity of the derivatives of (y, z). To prove density, we use the same perturbation as that used in the proof of Proposition 3.1. Differentiating the system in 3.6 with respect to v, we obtain

$$\begin{bmatrix} ep' - I & (I - ep')(\beta - 1)e \\ 0 & -\beta \end{bmatrix}$$

at a steady state (p, β). Since this matrix has rank n, the proposition now follows from the transversality theorem.

Q.E.D.

Let $\mathcal{E}^R$ be the subset of $\mathcal{E}$ that satisfies R.1 - R.3. We can summarize the discussion with the following result.

PROPOSITION 3.5: $\mathcal{E}^R$ is open dense in $\mathcal{E}$. Every economy in $\mathcal{E}^R$ has an odd number of real steady states and an odd number of nominal steady states. No real steady state has $\beta = 1$. Furthermore, the number of steady states of each type are constant on connected components of $\mathcal{E}^R$ and vary continuously with the economy.

Suppose we want to show that for a generic economy certain properties are satisfied at all steady states. Mathematically, it is more convenient to prove that for a generic economy these properties are satisfied at a particular steady state. A useful fact about regular economies is that the latter property implies the former. To formalize this let $\mathcal{J}^R \subset \mathcal{E}^R \times S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$ be the set of (y, z, p, β) for which (p, β) is a steady state of (y, z). Let $\mathcal{J}^G$ be open dense in $\mathcal{J}^R$. Define $\mathcal{E}^G$ to be the subset of $\mathcal{E}^R$ such that, if $(y, z) \in \mathcal{E}^G$ and $(y, z, p, \beta) \in \mathcal{J}^R$, then $(y, z, p, \beta) \in \mathcal{E}^G$. It follows directly from Proposition 3.5 and the fact that finite intersections of open dense sets are open dense that $\mathcal{E}^G$ is open dense in $\mathcal{E}$. Consequently, in the sequel, we prove all theorems about genericity in $\mathcal{J}^R$, with the understanding that this carries over into $\mathcal{E}$.

4. RESTRICTIONS ON DEMAND DERIVATIVES

We are interested in discovering the properties of the demand derivatives D_1y, D_2y, D_1z, and D_2z evaluated at steady states (p, β). The most convenient way to do this is to introduce the jet mapping $d: \mathcal{J} \to \mathcal{D}$ where is a subset of the space of six-tuples $(D_1y, D_2y, D_1z, D_2z, p, \beta)$ and the mapping d applied to (y, z, p, β) yields the excess demand derivatives evaluated at (p, β).

What restrictions should we place on the elements of $\mathcal{D}$? Differentiating Walras's law, we see that

$$(4.1)\qquad \begin{aligned} y' + p'D_1y + \beta p'D_1z &= 0 \\ z' + p'D_2y + \beta p'D_2z &= 0. \end{aligned}$$

But the steady state condition says that $z' + y' = 0$. Consequently, we can rewrite Walras's law as

$$(4.2)\qquad p'(D_1y + D_2y + \beta D_1z + \beta D_2z) = 0.$$

Differentiating the homogeneity assumption, we can rewrite it as

$$(4.3)\qquad \begin{aligned} (D_1y + \beta D_2y)p &= 0 \\ (D_1z + \beta D_2z)p &= 0. \end{aligned}$$

Now let us restrict attention to economies with steady states in $S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$. We define $\mathcal{D}$ to be the six-tuples that satisfy 4.2 and 4.3 and for which $(p, \beta) \in S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$. The following theorem implies that the space $\mathcal{D}$ captures all the important restrictions on demand derivatives.

PROPOSITION 4.1: The jet mapping d is a continuous open mapping of $\mathcal{F}^R$ onto an open dense subset of $\mathcal{D}$.

PROOF: Continuity of d is obvious. To prove the remainder of the proposition we need to know how to convert elements of $\mathcal{D}$ into elements of $\mathcal{F}$. Suppose $d \in \mathcal{D}$. Let us renormalize prices $q \in R^{2n}_{++}$ by setting $q^1 = 1$. Let $\bar{X}_d$ be the matrix of demand derivatives with first row and column deleted. Using 4.1, we see that we should define $y' = -p'(D_1 y + \beta D_1 z)$ and $z' = -p'(D_2 y + \beta D_2 z)$. Let $\bar{q}$ be the vector $(p, \beta p)$ with the first component deleted, and let $\bar{x}_d(\bar{q})$ be the vector (y, z) with the first component deleted. Let $\bar{q}_t$ be an arbitrary $2n - 1$ vector. We define the linear affine function $\bar{x}_d$: $R^{2n-1} \to R^{2n-1}$ by the rule $\bar{x}_d(\bar{q}_t) = \bar{x}_d(\bar{q}) + \bar{X}_d(\bar{q}_t - \bar{q})$. Suppose that $x \in \mathcal{E}$ and that $\bar{x}$ is the last $n - 1$ components of x viewed as a function of R^{2n-1}_{++} by setting $q^1 = 1$. We define $\bar{x}_\lambda$ to be the weighted average

$$(4.4) \qquad \bar{x}_\lambda(\bar{q}_t) = \lambda(\bar{q}_t)\bar{x}_d(\bar{q}_t) + (1 - \lambda(\bar{q}_t))\bar{x}(\bar{q}_t).$$

Let $B \subset R^{2n}_{++}$ be the open ball of radius $\varepsilon > 0$ around q. We can construct λ: $R^{2n-1} \to R$ so that it is C^1 and satisfies $0 < \lambda(\bar{q}_t) < 1$, $\lambda(\bar{q}) = 1$, and $\lambda(\bar{q}_t) = 0$ for $\bar{q}_t \notin B$. Furthermore, we can choose λ so that $D\lambda(\bar{q}) = 0$ and $||D\lambda(\bar{q}_t)|| < 3/\varepsilon$ (see Hirsch (1976, pp. 41-42)). Consequently, $\bar{x}_\lambda$ coincides with $\bar{x}$ outside of B, but $\bar{x}_\lambda(\bar{q}) = \bar{x}_d(\bar{q})$ and $D\bar{x}_\lambda(\bar{q}) = \bar{X}_d$. There is a unique extension of $\bar{x}_\lambda$ to $x_\lambda : R^{2n}_{++} \to R^{2n}$ that satisfies Walras's law and homogeneity. Furthermore, for ε small enough, the boundary assumption is satisfied. Consequently, we may assume $x_\lambda \in \mathcal{E}$. Finally, a direct computation shows that $d(x_\lambda, p, \beta) = d$.

Let us first use this construction to show that d is open. Let $d = d(x, p, \beta)$, let $d^k \to d$, and let $\varepsilon^k = \max\{||q^k - q||, ||\bar{x}^k_d(\bar{q}^k) - \bar{x}(\bar{q})||, ||\bar{x}^k_d - D\bar{x}(\bar{q})||\}$. Then $\varepsilon^k \to 0$. Furthermore, a computation using the mean value theorem shows $x^k_\lambda \to x$. Since $\mathcal{E}^R$ is open in $\mathcal{E}$, x^k_λ is eventually in $\mathcal{E}^R$. This implies that d is open.

Next we show $d(\mathcal{F}^R)$ is dense in $\mathcal{E}$. Indeed, suppose $d \notin d(\mathcal{F}^R)$. Since $x_\lambda \in \mathcal{E}$, there is $x^k \to x_\lambda$ with $x^k \in \mathcal{E}^R$. By construction, however, the steady state (p, β) is itself a regular steady state of x_λ in the ball B of fixed radius ε. Thus, the x^k must have a steady state $(p^k, \beta^k) \to (p, \beta)$. Therefore, $(x^k, p^k, \beta^k) \in \mathcal{F}^R$ and $d(x^k, p^k, \beta^k) \to d = d(x_\lambda, p, \beta)$.

Q.E.D.

This result says that any generic set in $\mathcal{D}$ corresponds to a generic property in $\mathcal{E}$. Furthermore, any open set in $\mathcal{D}$ corresponds to a non-void open set in $\mathcal{E}$. It enables us to restrict our study entirely to the space $\mathcal{D}$.

It is of interest to see what R.1 to R.3 mean in $\mathcal{D}$. 4.1 implies that $p'y = 0$ if and only if $p'(D_1y + \beta D_1z)p = 0$. R.1 is therefore equivalent to the assumption that $p'(D_1y + D_1z)p = 0$ implies $\beta \neq 1$. Let us define $J = D_1z + \beta D_2z + D_1y + \beta D_2y$. Homogeneity implies that $Jp = 0$. At steady states where $\beta = 1$ R.2 is equivalent to the assumption that J has rank $n - 1$. At steady states where $\beta \neq 1$ Walras's law implies the matrix in R.3 equals

$$\begin{bmatrix} (I - ep')J & (I - ep')(D_2z + D_2y)p \\ \beta p'(D_1y - D_2z) & -p'D_2yp \end{bmatrix}.$$

A second application of Walras's law shows that this has the same rank as

$$\begin{bmatrix} J & (D_2 z + D_2 y)p \\ \beta p'(D_1 y - D_2 z) & -p' D_2 y p \end{bmatrix}.$$

It also implies that if $Jx = 0$ then $p'(D_1 y - D_2 z)x = 0$. Consequently, R.3 implies that J has rank $n - 1$. Observe that, if there is a vector x such that $x'J = 0$ and $x'(D_2 z + D_2 y)p \neq 0$ and J has rank $n - 1$, then R.3 is satisfied. It is straightforward to show that the former condition is generic given the latter.

5. PATHS NEAR STEADY STATES

A (perfect foresight) equilibrium price path is a finite or infinite sequence of prices $\{\ldots, p_{t-1}, p_t, p_{t+1}, \ldots\}$ such that $p_t \in R^n_{++}$ and

$$(5.1) \qquad z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0.$$

Our goal is to find generic conditions under which paths near steady states are well behaved, which means that they should follow a nice second order difference equation.

Fix a steady state (p, β). The equilibrium condition 5.1 can be linearized as

$$(5.2) \qquad D_1 z(p_{t-1} - \beta^{t-1} p) + (D_2 z + \beta^{-1} D_1 y)(p_t - \beta^t p) + \beta^{-1} D_2 y(p_{t+1} - \beta^{t+1} p) = 0.$$

Here all derivatives are evaluated at $(p, \beta p)$ and we use the fact that excess demand derivatives are homogeneous of degree minus one. Suppose that the following condition holds.

(R.4) D_2y is non-singular.

Then the linearized system can be solved to find

$$(5.3) \qquad (q_{t+1} - \beta^{t+1}q) = G(q_t - \beta^t q)$$

where $G = \begin{bmatrix} 0 & I \\ G_1 & G_2 \end{bmatrix}$, $G_1 = -\beta D_2y^{-1}D_1z$, $G_2 = -D_2y^{-1}(\beta D_2z + D_1y)$,

$q = (p, \beta p)$, and $q_t = (p_{t-1}, p_t)$. A direct implication of the implicit function theorem is

PROPOSITION 5.1: If R.4 holds, then there is an open cone $U \subset R^{2n}_{++}$ around q and a unique function g: $U \to R^{2n}_{++}$, which is smooth, homogeneous of degree one, and such that

(a) If $\{p_t\}$ is an equilibrium price path and $q_t, q_{t+1} \in U$, then $q_{t+1} = g(q_t)$.

(b) If $\{p_t\}$ has $q_t \in U$ at all times and $q_{t+1} = g(q_t)$, then it is an equilibrium price path. Furthermore, $Dg(q) = G$.

Our goal is to establish that there are generic restrictions on the demand derivatives D_1y, D_2y, D_1z, D_2z such that R.4 holds and such that G is a nice matrix, and to prove that under these conditions g is a nice dynamical system.

6. RESTRICTIONS ON THE LINEARIZED SYSTEM

We are interested in discovering the properties of the linearized system as represented by the matrix G. It is convenient to work in the subset $\mathcal{D}^R$ of $\mathcal{D}$ for which R.1 - R.4 and the following restriction hold:

(R.5) $\quad K = D_1 y + D_2 y + \beta D_1 z + \beta D_2 z$ has rank n - 1.

Note that Walras's law implies that $p'K = 0$, so K cannot have full rank.

PROPOSITION 6.1: $\mathcal{D}^R$ is open dense in $\mathcal{D}$.

PROOF: Openness is obvious. To demonstrate the density of R.4, let us define $D_1 y_v = D_1 y + v\beta I$, $D_2 y_v = D_2 y - vI$, $D_1 z_v = D_1 z - v\beta I$, and $D_2 z_v = D_2 z + vI$. Leave p and β fixed. It is easy to verify that $(D_1 y_v, D_2 y_v, D_1 z_v, D_2 z_v, p, \beta)$ is an element of $\mathcal{D}$ if $(D_1 y, D_2 y, D_1 z, D_2 z, p, \beta)$ is. Let λ have the smallest absolute value of any non-zero, real eigenvalue of $D_2 y$. Obviously, $D_2 y_v$ is non-singular for any v such that $0 < |v| < \lambda$.

To demonstrate the density of R.5, let us define $D_1 y_v = D_1 y - v(I - ep')$ where $p'e = 1$. Observe that $(D_1 y_v, D_2 y, D_1 z, D_2 z, p, \beta)$ still satisfies the relevant versions of Walras's law, 4.2, and the homogeneity assumption, 4.3. Now $K_v = K + v(I - ep')$. Let $p_0' K_v = 0$. We know that $p'K = 0$. If p_0 is necessarily proportional to p, then K_v has rank n - 1. But $(p_0' - (p_0' e)p')(K - vI) = 0$. Since $K - vI$ is non-singular for $v \neq 0$ small enough, $p_0 = (p_0' e)p$.

Q.E.D.

Our next step is to consider the mapping $h: \mathcal{D}^R \to \mathcal{H}$ where the elements of $\mathcal{H}$ are six-tuples $(D_1y, D_2y, G_1, G_2, p, \beta)$ that satisfy the appropriate conditions. The map h is the identity on the first two and last two components. G_1 and G_2 are defined as $G_1 = -\beta D_2y^{-1}D_1z$ and $G_2 = -D_2y^{-1}(\beta D_2z + D_1y)$. Since D_2y is non-singular on $\mathcal{D}^R$, h is obviously continuous. Equally important, it has a continuous inverse on $h(\mathcal{D}^R)$ given by the identity on the first two and last two components and by

$$(6.1) \qquad [D_1z \quad D_2z] = -(1/\beta)[D_1y \quad D_2y]G$$

where $G = \begin{bmatrix} 0 & I \\ G_1 & G_2 \end{bmatrix}$ as in 5.3.

Thus, h is a homeomorphism onto $\mathcal{H}^R = h(\mathcal{D}^R)$. It remains to identify $\mathcal{H}^R$. Walras's law A.2 holds if and only if

$$(6.2) \qquad p'D_2y[I - G_1 - G_2] = 0.$$

Note that this implies $p'[D_2yG_1 \quad D_2y]G = p'[D_2yG_1 \quad D_2y]$, and, therefore, that G has an eigenvalue equal to one. The homogeneity condition A.3 holds if and only if

$$(6.3) \qquad \begin{aligned} &(D_1y + D_2y)p = 0 \\ &Gq = \beta q \end{aligned}$$

where $q = (p, \beta p)$. Consequently, G has an eigenvalue equal to β. R.4 is unchanged while R.5 becomes

$$(6.4) \qquad I - G_1 - G_2 \text{ has rank } n - 1.$$

6.2 - 6.4 and R.4 completely characterize $\mathcal{H}^R$.

Finally, we focus in on G itself, considering $\gamma: \mathcal{H}^R \to \mathcal{D}$ where the elements of $\mathcal{D}$ are three-tuples of the form (G, p, β) and γ is the projection map. γ is obviously continuous; we want to show that it is an open map onto $\gamma(\mathcal{H}^R)$.

We examine 6.3 first. Since D_1y does not appear except in this condition, $(D_1y + \beta D_2y)p = 0$ serves only to determine D_1y once D_2y is given. Obviously, D_1y may be locally chosen as a continuous function of β, D_2y, and p. The second condition is $Gq = \beta q$. Now consider 6.4. Notice that this condition implies that G has a unit root.

We claim that this is all: 6.3 and 6.4 uniquely characterize $\mathcal{D}$, and γ is open. To prove this let x be in the left null space of $I - G_1 - G_2$. We think of x as lying in the manifold formed by identifying radially opposite points on the unit sphere. Since $I - G_1 - G_2$ has rank $n - 1$, x is a continuous function of G. We need to be able to locally map vectors x and p continuously into non-singular matrices D_2y such that $p'D_2y = x$. This, however, is obviously possible. We summarize our arguments with the following proposition:

PROPOSITION 6.2: Let $\mathcal{D}$ be the space of (G, p, β) such that G has one unit root (counting geometric multiplicity), $Gq = \beta q$, and $I - G_1 - G_2$ has rank $n - 1$. Then the mapping of $\mathfrak{D}^R$ taking excess demand derivatives to coefficient matrices of the linearized system is continuous open and onto $\mathcal{D}$.

In particular, G is a coefficient matrix of a linearized system of a steady state q if and only if G has one unit root and $Gq = \beta q$.

7. RESTRICTIONS ON EIGENVALUES

We now examine the implication of the restrictions on G for its eigenvalues. It is convenient to work in the subspace $\mathcal{Y}^R$ of $\mathcal{Y}$ for which $\beta^2 I - G_1 - \beta G_2 = \beta D_2 y^{-1}(D_1 z + \beta D_2 z + D_1 y + \beta D_2 y)$ has rank n - 1. Since this condition is already generic in $\mathcal{D}$, it is generic in $\mathcal{Y}$. Let $\mathcal{S}$ be the manifold of eigenvalues of 2n x 2n matrices: This is the subset of 2n-tuples of complex numbers in which complex numbers occur only in conjugate pairs and in which vectors which differ only by the order of components are identified. The eigenvalue evaluation map σ maps 2n x 2n matrices to $\mathcal{S}$ and is known to be continuous. We now consider the set $\mathcal{J} \subset \mathcal{S} \times [\underline{\beta}, \overline{\beta}]$ whose elements (s, β) have a component equal to one and an additional component equal to β if $\beta \neq 1$. We extend σ to $\tau: \mathcal{Y}^R \to \mathcal{J}$. We claim that the only restrictions on the eigenvalues of G are that one equal unity and one equal β. If $\beta = 1$, then there is only one restriction. To justify this claim we use the following result.

PROPOSITION 7.1: τ is a continuous open mapping of $\mathcal{Y}^R$ onto an open dense subset of $\mathcal{J}$.

PROOF: τ is obviously continuous. To show τ is open, let $(s, \beta) = \tau(G, p, \beta)$, and suppose $(s^k, \beta^k) \to (s, \beta)$. We construct $G^k \to G$ with $\tau(G^k, p^k, \beta^k) = (s^k, \beta^k)$. Set $G^k = H^k C^k (H^k)^{-1}$. Given C^k can we choose H^k so that G^k has the partitioned structure corresponding to a second order difference equation? Obviously G^k is the unique solution of $G^k H^k = H^k C^k$. Writing this out in partitioned form, we see

$$
(7.1) \quad
\begin{aligned}
&\begin{bmatrix} 0 & I \\ G_1^k & G_2^k \end{bmatrix}
\begin{bmatrix} H_{11}^k & H_{12}^k \\ H_{21}^k & H_{22}^k \end{bmatrix}
=
\begin{bmatrix} H_{21}^k & H_{22}^k \\ * & * \end{bmatrix}
= \\
&\begin{bmatrix} H_{11}^k & H_{12}^k \\ H_{21}^k & H_{22}^k \end{bmatrix}
\begin{bmatrix} C_{11}^k & C_{12}^k \\ C_{21}^k & C_{22}^k \end{bmatrix}
=
\begin{bmatrix} H_{11}^k C_{11}^k + H_{21}^k C_{12}^k & H_{11}^k C_{12}^k + H_{12}^k C_{22}^k \\ * & * \end{bmatrix},
\end{aligned}
$$

from which it follows that G^k has the correct structure if and only if

$$
(7.2) \quad
\begin{aligned}
H_{21}^k &= H_{21}(C^k, H_{11}^k, H_{12}^k) = H_{11}^k C_{11}^k + H_{12}^k C_{21}^k \\
H_{22}^k &= H_{22}(C^k, H_{11}^k, H_{12}^k) = H_{11}^k C_{12}^k + H_{12}^k C_{22}^k
\end{aligned}
$$

Now let H be a basis for R^{2n} such that $C = H^{-1}GH$ is in real canonical form. Obviously, $\sigma(C) = \sigma(G) = s$. Hirsch and Smale (1973, pp. 153-157) show how to construct a sequence of real matrices $C^k \to C$ with $\sigma(C^k) = s^k$. Set $H_{11}^k = H_{11}$, $H_{12}^k = H_{12}$ and H_{21}^k, H_{22}^k as defined above. By continuity $H^k \to H$ and is eventually non-singular, so G^k is well-defined and, by construction, has the proper structure. Furthermore, since components of s^k are one and β, G^k has them as eigenvalues. Observe that, since G^k has a unit root, $I - G_1^k - G_2^k$ is singular, but, since $G^k \to G$, it has rank $n - 1$. Next, the structure of G^k implies that there is an eigenvector corresponding to β^k that has the form $q^k = (p^k, \beta^k p^k)$. We think of this as lying on the unit sphere with radial identification and thus being unique. Further, since $G^k \to G$, p^k is the unique component in the right null space of $\beta^2 I - G_1 - \beta G_2$ and, therefore, converges to p. Consequently, $(G^k, p^k, \beta^k) \to (G, p, \beta)$, and the map is open.

Finally, we want $\tau(\mathcal{D}^R)$ to be open dense in $\mathcal{J}$. Only density remains to be shown; we do this by constructing an open dense subset of $\mathcal{J}$, denoted $\mathcal{J}^R$, such that $\mathcal{J}^R \subset \tau(\mathcal{D}^R)$. Let $(s, \beta) \in \mathcal{J}$. Arranging diagonal blocks, we can construct a block diagonal matrix

$$(7.3) \qquad C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

in real canonical form where $\sigma(C) = s$ and where the first diagonal entry of C is β. We define $\mathcal{J}^R$ to be the subset of $\mathcal{J}$ for which the above construction can yield a matrix C such that $C_1 - C_2$ is non-singular and for which there is only one unit eigenvalue and one eigenvalue β. Clearly, $\mathcal{J}^R$ is an open dense subset of $\mathcal{J}$. Choose $p \in S_\varepsilon$, let H_{11} be a non-singular matrix with first column equal to p, and let $H_{12} = H_{11}$. Using 7.2, we set $H_{21} = H_{11}C_1$ and $H_{22} = H_{12}C_2$. Since $C_1 - C_2$ is non-singular, so is

$$(7.4) \qquad H = \begin{bmatrix} H_{11} & H_{12} \\ H_{11}C_1 & H_{12}C_2 \end{bmatrix}.$$

Assuming that C has only one unit eigenvalue implies that $I - G_1 - G_2$ has rank $n - 1$. Consequently, $(HCH^{-1}, p, \beta) \in \mathcal{D}$. Similarly, assuming that C has only one eigenvalue β implies that $\beta^2 I - G_1 - \beta G_2$ has rank $n - 1$, and in fact $(HCH^{-1}, p, \beta) \in \mathcal{D}^R$.

Q.E.D.

8. NOMINAL DYNAMICS

Until now we have largely combined the study of real and nominal steady states. The dynamics near each type of steady state are, however, rather different. We begin by studying the nominal case. Here we know only that G has one unit root.

It is useful to define the money supply $m(q_t) = p_t' z(p_{t-1}, p_t)$. This is homogeneous of degree one. Walras's law implies that this equals $-p_{t-1}' y(p_{t-1}, p_t)$, and the equilibrium condition implies $p_t' z(p_{t-1}, p_t) = -p_t' y(p_t, p_{t+1})$. Consequently, $m(q_t) = m(g(q_t))$; the money supply is constant along equilibrium price paths. At a nominal steady state $\mu = m(q) \neq 0$. The homogeneity condition implies that, if $m(q_t) = \mu$, $Dm(q_t)q_t = \mu \neq 0$ and, therefore $m(q_t) = \mu$ defines a 2n - 1 submanifold $Q_\mu \subset R^{2n}_{++}$ that is transversal to the steady state ray and invariant under g. We denote the restriction of g to Q_μ by g_μ.

All interest focuses on g_μ. If $\operatorname{sgn} \mu_1 = \operatorname{sgn} \mu_2$ then g_{μ_1} and g_{μ_2} exhibit the same dynamics except that the price level is increased by a factor of μ_1/μ_2. Examining the linearization, we see that Dg_μ is G restricted to $Dm(q)q_t = 0$. Since Q_μ is invariant and tranversal to the steady state ray, it follows that the generalized eigenspace of G that excludes the eigenvector q spans the space $Dm(q)q_t = 0$ and that G restricted to this space has the eigenvalues of G excluding the one unit root known a priori to exist. Furthermore, the results of the previous section imply that the remaining eigenvalues are unrestricted. Let n^s be the number of these eigenvalues inside the unit circle. Using standard results, such as those in Irwin (1980), we can easily prove the following proposition.

PROPOSITION 8.1: There is an open dense set of economies that satisfy the following conditions at all nominal steady states:

(a) g_μ is a local diffeomorphism; that is, G is non-singular.

(b) g_μ has no roots on the unit circle; that is, g_μ is hyperbolic.

(c) g_μ has an n^s dimensional stable manifold W_s of $q_0 \varepsilon Q_\mu$ for which $g_\mu^t(q_0) \to q$.

(d) g_μ has a $2n - n^s - 1$ dimensional unstable manifold W_u of $q_0 \varepsilon Q_\mu$ for which $g_\mu^{-t}(q_0) \to q$;

(e) (Hartmann's theorem) There is a smooth coordinate change c(q) such that $c o g_\mu o c^{-1} = G$ on W_s, and for a residual set of economies this holds on all of Q_μ (and thus R^{2n}_{++}).

One warning should be given about the genericity of these results: They hold for almost all economies when the only restrictions that we place on excess demands are A.1 - A.5. Suppose, however, that we restrict our attention to economies with a single, two period lived consumer in each generation who has an intertemporally separable utility function. Then both D_2y and D_1z have at most rank one, and R.4 is violated. Since the set of economies that satisfy these restrictions is closed and nowhere dense, none of our previous analysis applies. Kehoe and Levine (1982b) analyze this case and show that it is essentially the same as that of an economy with one good in every period.

9. REAL DYNAMICS

We now study the neighborhood of a steady state $q = (p, \beta p)$ with $m(q) = 0$

and $\beta \neq 1$. In this case prices are not stationary at a steady state, but grow or decline exponentially. Let $b\colon R^{2n}_{++} \to R$ be a function that is homogenous of degree one. We can normalize prices to focus on the convergence of relative prices. Define g^b on $Q^b = \{q_t \in Q \mid b(q_t) = 1\}$ by $g^b(q_t) = g(q_t)/b(g(q_t))$. If b is monotonically increasing, then it can be naturally thought of as a price index. As it is, it provides a one dimensional restriction on relative prices. Homogeneity implies that $b(g^b(q_t)) = 1$. We say that an equilibrium price path converges to q if $q_t/b(q_t) \to q$. This is true of a path beginning at q_0 if and only if the path under g^b starting at $q_0/b(q_0)$ converges to q.

What is the linear approximation to g^b? It is $(1/\beta)(I - q'B)G$, where $B = Db(q)$, restricted to $Bq_t = 0$. Choosing b so that $Bq_t = 0$ defines the generalized eigenspace of G in which the eigenvector q is excluded, we see that the eigenvalues of g^b are those of $(1/\beta)G$, excluding the unit eigenvalue that arises from the eigenvalue β corresponding to q. One of these values is equal to $1/\beta$; the remaining $2n - 2$ are unrestricted. Let $\bar{n}^s$ be the number of these remaining eigenvalues inside the unit circle. Then g^b generically is hyperbolic with an $\bar{n}^s$ dimensional stable manifold and a $2n - \bar{n}^s - 1$ dimensional unstable manifold if $\beta < 1$. Similarly g^b has a $\bar{n}^s + 1$ dimensional stable manifold and a $2n - \bar{n}^s - 2$ dimensional unstable manifold if $\beta > 1$. Furthermore, g^b is linearizable by a smooth coordinate change on the stable manifold.

It is useful also to distinguish between initial conditions with $m(q_0) = 0$ (real initial conditions) and those with $m(q_0) \neq 0$ (nominal initial conditions). Observe that $Dm(q) = (-p'\beta D_1 z, p'D_2 y)$, which, by R.4, generically does not vanish. Thus, generically $Dm(q_t) = 0$ defines a $2n - 1$ cone $Q_0 \subset R^{2n}_{++}$ invariant under g. This is transversal to Q^b and, consequently, intersects it in a $2n - 2$ manifold Q^b_0 invariant under g^b. Furthermore, a simple computation shows that Q_0 is tangent to the eigenvectors of g except the one having the unit root; thus Q^b_0 is tangent to the eigenvectors of g^b except the eigenvector with root $1/\beta$. Since Q^b_0 is invariant and, for $q_t \in Q_0$, $m(q_t) = 0$, nominal initial q_0 (those with $m(q_0) \neq 0$) can approach q only if $\beta > 1$; otherwise, if $\beta < 1$, nominal paths cannot approach the real steady state. On the other hand, in Q^b_0 the linearized system has all the eigenvalues of $(1/\beta)G$ except 1 and $1/\beta$. The real system on the invariant manifold Q^b_0 is, therefore, generically hyperbolic and has an $\bar{n}^s$ dimensional stable manifold and a $2n - \bar{n}^s - 2$ dimensional unstable manifold. Furthermore, it is linearizable on the stable manifold.

10. PARETO EFFICIENCY AND FIAT MONEY

Consider an infinite price sequence $\{p_1, p_2, p_3, \ldots\}$ that satisfies the conditions $(p_t, p_{t+1}) \in R^{2n}_{++}$ and

(10.1) $$z_0(p_1) + y(p_1, p_2) = 0$$

(10.2) $$z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0.$$

In other words, $\{p_1, p_2, p_3, \ldots\}$ is an equilibrium price path for the economy specified by the demand functions y and z and a demand function z_0 for the old generation alive in the first period. For such an economy, where each generation consists of a representative consumer, Balasko and Shell (1980) have established that a necessary and sufficient condition for pareto efficiency is that the infinite sum $\Sigma 1/||p_t||$ diverges. They require that a certain uniform curvature condition on indifference surfaces be satisfied. This condition, while restrictive in non-stationary models, is naturally satisfied in a stationary model such as ours. This result can easily be extended to economies with many consumers in each generation. Consequently, steady states with $\beta < 1$, with a non-negative interest rate, are pareto efficient. So are paths that converge to them. An economy always has a pareto efficient steady state since it always has a steady state where $\beta = 1$. Is there anything more we can say? Can we, for example, guarantee the existence of a pareto efficient steady state where $\mu > 0$?

To answer these questions, let us rephrase the conditions that characterize a steady state. Consider pairs (p, β) that satisfy the price normalization $(p'e) = 1$. Let $f\colon S_\varepsilon \times [\underline{\beta}, \overline{\beta}] \to R^{n-1}$ be given by the first $n-1$ coordinate functions of $(I - ep')(z(p, \beta p) + y(p, \beta p))$. In other words,

$$(10.3) \qquad f(p, \beta) = L(I - ep')(z(p, \beta p) + y(p, \beta p))$$

where L is the projection operator that can be represented in standard coordinates by the $(n - 1) \times n$ matrix.

$$(10.4) \qquad L = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} .$$

We work with the function $(I - ep')(z + y)$ because, unlike $(z + y)$

itself, its first (n - 1) coordinates are equal to zero only if its last coordinate is equal to zero. This is because $p'(I - ep')(z(p, \beta p) + y(p, \beta p)) = 0$. Also, like $(z + y)$ itself, $(I - ep')(z + y)$ has the property that we can select $\varepsilon > 0$ small enough so that $e'(I - ep')(z(p, \beta p) + y(p, \beta p)) > \alpha > 0$ for all $p \in S_0 S_\varepsilon$ and any $\underline{\beta} < \beta < \bar{\beta}$. To see why, suppose instead that $e'(I - ep_k')(z(p_k, \beta_k p_k) + y(p_k, \beta_k p_k)) < 0$ for a sequence $(p_k, \beta_k) \to (p, \beta)$, $p \in S_0$. Since $e'(z(p_k, \beta_k p_k) + y(p_k, \beta_k p_k)) \to \infty$ and $z + y$ is bounded from below, this implies $p_k'(z(p_k, \beta_k p) + y(p_k, \beta_k p_k)) \to \infty$. This can only happen if $\beta \neq 1$. Walras's law can be used to rewrite this expression as either $(1 - \beta_k)p_k'z(p_k, \beta_k p_k)$ or as $(1/\beta_k)(\beta_k - 1)p_k'y(p_k, \beta_k p_k)$. If $\beta > 1$, then $(1 - \beta_k)p_k'z(p_k, \beta_k p_k)$ is bounded from below. Similarly, if $\beta < 1$, then $(1/\beta_k)(\beta_k - 1)p_k'y(p_k, \beta_k p_k)$ is bounded from above. In either case $p_k'(z(p_k, \beta_k p_k) + y(p_k, \beta_k p_k))$ is bounded from above, which is a contradiction.

In what follows, it is important that f be C^2. To ensure this, we assume that y and z are not only C^1 but also C^2. We need to assume that f is C^2 so that we can use the transversality theorem to prove that 0 is generically a regular value of f. Indeed, for $v \in R^{n+1}$, we define

$$(10.5) \qquad f_v(p, \beta) = L(I - ep')(z_v(p, \beta p) + y_v(p, \beta p))$$

where y_v and z_v are defined as in the proof of Proposition 3.1. Differentiating f_v with respect to v, we obtain the n x (n + 1) matrix

$$\left[L\left(\frac{1}{e'p}ep' - I\right) \qquad L(I - ep')(\beta - 1)e\right].$$

Notice that $x'(\frac{1}{e'p}ep' - I) = 0$ implies that x is a scalar multiple of p. Since $u'L = [u_1 \; u_2 \; . \; . \; . \; u_{n-1} \; 0]$ for any $u \in R^{n-1}$, however, this implies that, for all $p \in S_\varepsilon$, $u'L(\frac{1}{e'p}ep' - I) = 0$ only if $u = 0$. Consequently, this matrix has rank $n - 1$, and 0 is a regular value of f_v for all v in a subset of R^{n+1} of full Lesbesgue measure. It is now, as before, a straightforward matter to demonstrate that 0 is a regular value of f for all (y, z) in an open dense subset of $\mathcal{E}$.

What does the pre-image of 0 under f look like? Obviously, $f^{-1}(0)$ is compact since $S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$ is compact and f is continuous. Since $f(p, \beta)$ cannot equal zero for any p on the boundary of S_ε, the only points in $f^{-1}(0)$ on the boundary of $S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$ are those where β equals $\underline{\beta}$ or $\overline{\beta}$. We have argued that 0 is generically a regular value of f on the interior of $S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$. Our argument also implies that 0 is generically a regular value of f restricted to $S_\varepsilon \times \{\beta\}$ for almost all fixed β; in particular, 0 is generically a regular value of f on the boundary of $S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$. Unfortunately, $S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$ is not a smooth manifold with boundary because it has corners. Since $f^{-1}(0)$ stays away from these corners, however, it is a smooth one dimensional manifold with boundary whose boundary is contained in the boundary of $S_\varepsilon \times [\underline{\beta}, \overline{\beta}]$. Furthermore, using index theory we can show that $f(p, \beta) = 0$ has an odd number of solutions when $\beta = \underline{\beta}$ and an odd number of solutions when $\beta = \overline{\beta}$.

Define $m(p, \beta) = -p'y(p, \beta)$ for all $(p, \beta) \in f^{-1}(0)$. There are two distinct ways for $(p, \beta) \in f^{-1}(0)$ to be an equilibrium: $m(p, \beta) = 0$ or

$\beta = 1$. In either case, Walras's law implies that $(z(p, \beta) + y(p, \beta))$ is equal to 0.

Consider now the graph of m, $\{(p, \beta, m) \varepsilon S_\varepsilon \times [\underline{\beta}, \overline{\beta}] \times R \mid f(p, \beta) = 0, m = m(p, \beta)\}$: It is obviously a smooth one dimensional manifold with boundary diffeomorphic to $f^{-1}(0)$. Steady states of (y, z) are points where the graph of m intersects either the n - 1 dimensional submanifold of $S_\varepsilon \times [\underline{\beta}, \overline{\beta}] \times R$ where $m = 0$ or the n - 1 dimensional submanifold where $\beta = 1$. We can picture these intersections graphically if we project $S_\varepsilon \times [\underline{\beta}, \overline{\beta}] \times R$ onto $[\underline{\beta}, \overline{\beta}] \times R$. Under this projection the graph of m need not be an embedded submanifold, of course, because it may contain points of self-intersection. It is, however, an immersed submanifold. The self-intersections are generically transversal, but this is not important for our arguments.

Figure 10.1

R.1 says that the graph of m does not pass through (1, 0); R.2 says that it intersects the line $\beta = 1$ transversally; and R.2 says that it intersects the line $m = 0$ transversally. Considering diagrams like that in Figure 10.1, we can see why every economy does, in fact, have at least one steady state where $\beta < 1$ and $\mu > 0$: There are an odd number of points in $f^{-1}(0)$ where $\beta = \underline{\beta}$. Because of the boundary condition at all of these $m(p, \beta) > 0$. An even number, possibly zero, of these points are the endpoints of paths that return to the boundary $\beta = \underline{\beta}$. An odd number, at least one, must be endpoints

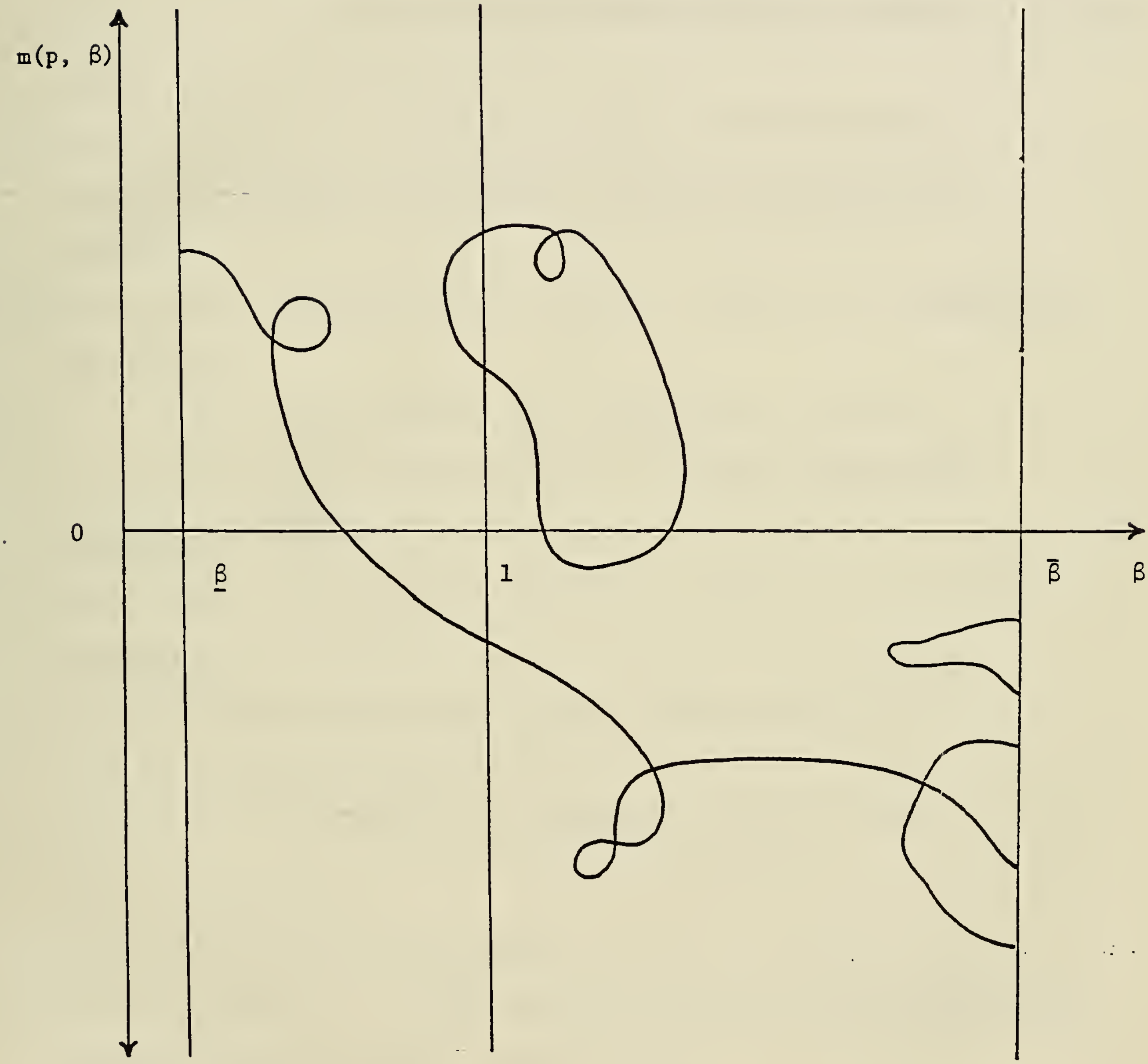

Figure 10.1

of paths that lead to the boundary $\beta = \bar{\beta}$, where $m(p, \beta) < 0$. Such a path must either cross the line $m = 0$ where $\beta < 1$ or cross the line $\beta = 1$ where $m > 0$. This same sort of argument can be used to demonstrate that every economy has at least one steady state where $\beta > 1$ and $\mu < 0$.

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