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THE PURE THEORY OF THE NATIONAL-OUTPUT DEFLATOR*

by

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No. 59 August, 1970

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THE PURE THEORY OF THE NATIONAL-OUTPUT DEFLATOR

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1. Introduction

While the economic theory of the cost-of-living index is firmly grounded in the theory of the consumer,[1] apparently no similar foundation for the other principal type of price index number construction -- the deflation of money output by a price index to obtain a measure of real output -- exists. Yet the construction of a national output deflator is an enterprise of equal importance to the construction of a cost-of-living index, and it therefore seems desirable to consider that construction in the light of the theory of production in a fashion parallel to the consideration of the cost-of-living index in the light of the theory of consumption. Among other things, such a consideration should illuminate the relationships between the deflators actually constructed with market data and the production-theoretic index of indices that one might wish to construct given more complete information about the technical capabilities of the economy. It should also assist in the analysis of the effects which technical change and the introduction of new goods ought to have on such actually constructed deflators.

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[1] Ignoring aggregation and other difficulties. See, for example, Samuelson [5]; v. Hofsten [2]; Fisher and Shell [1].

This paper provides such a theory, which turns out to be isomorphic in most important respects to the pure theory of the cost-of-living index. That isomorphism allows us to draw on the latter theory, particularly in the consideration of technical change where the interpretation of our earlier work on taste changes in the theory of the cost-of-living index [1] turns out to have a natural and immediate interpretation.

There are two points which it will be wise to get out of the way before beginning our analysis, however. As must already be clear, we believe that it is important to ground price index construction in economic theory, in particular, to ground the construction of output deflators in the theory of production. While we are not the first to take this view,[1] we are going to emphasize it and to argue that existing practices should be considered only as approximations to a theoretically sound index. Yet, unlike the case of the cost-of-living index, where such a view is both natural and long accepted, this view of output deflation may not seem instantly compelling, especially to those directly concerned with index number construction. This is not unreasonable. Any scalar measure of real output and associated measure of price change must contain some arbitrary elements. If one wishes to take the position that what he "really" means by real GNP is current output valued in base-year prices, that position is entirely tenable and it is fruitless to argue that what is 'really" meant is something else. While, of course, we believe that the indices here discussed are in some sense more natural and appealing than are real GNP (in the usual sense) and the corresponding GNP deflator, and that the usually computed indices should be con-

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[1] See, for example, Richter [3].

sidered as approximations to ours in the same way as the consumer price index is considered an approximation to the true cost-of-living index, there is no point in inviting semantic controversy, and the definitions of the GNP deflator and real GNP are well established. For this reason in the following analysis, we shall refer to indices of national output and to a national output deflator.

Secondly, we shall be considering the production-theoretic analysis of national output deflation. A different analysis can (and has) been given in terms of the arithmetic properties that one might want such a deflator to have (transitivity, for example). While our indices do have some of those properties (when properly considered), our analysis is properly an exercise in production theory rather than in the theory of "ideal index number" construction.

2. Real Output Indices and Production Possibility Maps

Suppose that there are factors of production, $v_1, \ldots, v_m$ and outputs $x_1, \ldots, x_r$. Let v and x be the column vectors of the v_i and x_i, respectively. Let the technology at time 0 (the base year) be given by:

$$(2.1) \qquad F(x, v) = 0 \quad .$$

For the present, assume that at time 0, the factor endowments v are inelastically supplied and fixed in amount,[1] say at v^0. Then the production possibility frontier (PPF) at time 0 is given by:

$$(2.2) \qquad F(x, v^0) = 0 \quad .$$

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[1] This is not an innocuous assumption and we shall have to deal with it later.

Consider now all the PPF which could be generated by factor endowments in the proportions given by v^0 but in different absolute amounts. That is, consider for any positive scalar, μ, the PPF given by:

$$F(x, \mu v^0) = 0 \quad . \tag{2.3}$$

We shall call the set of such frontiers the production possibility map (PPM) at time 0. It shows what efficient combinations of outputs would be with the technology of time 0 and "doses" of the factor endowments of that time applied in different amounts. Naturally, if all production functions exhibit constant returns, F will be homogeneous of degree zero in x and μ, and the PPM will be homothetic. Such homotheticity can also arise in other ways. For the present, we need not assume homotheticity.

It will be convenient and reasonable, but not strictly necessary, to assume that (2.3) can be solved for μ as a function of x and v^0. Since v^0 is constant, we shall suppress it and write the PPF for any value of μ as the set of x such that:

$$\phi(x) = \mu \quad . \tag{2.4}$$

The function ϕ can be regarded as a factor requirements function which for any bundle of outputs, x, gives the minimum dosage of factors (in the proportions stated by v^0) which is required to produce it.

Now suppose that, with $\mu = 1$, output is originally at some bundle, x^A (See Figure 2.1). Prices are p^A (a column vector) and money output is

$$y^A \equiv p^{A\prime} x^A \quad . \tag{2.5}$$

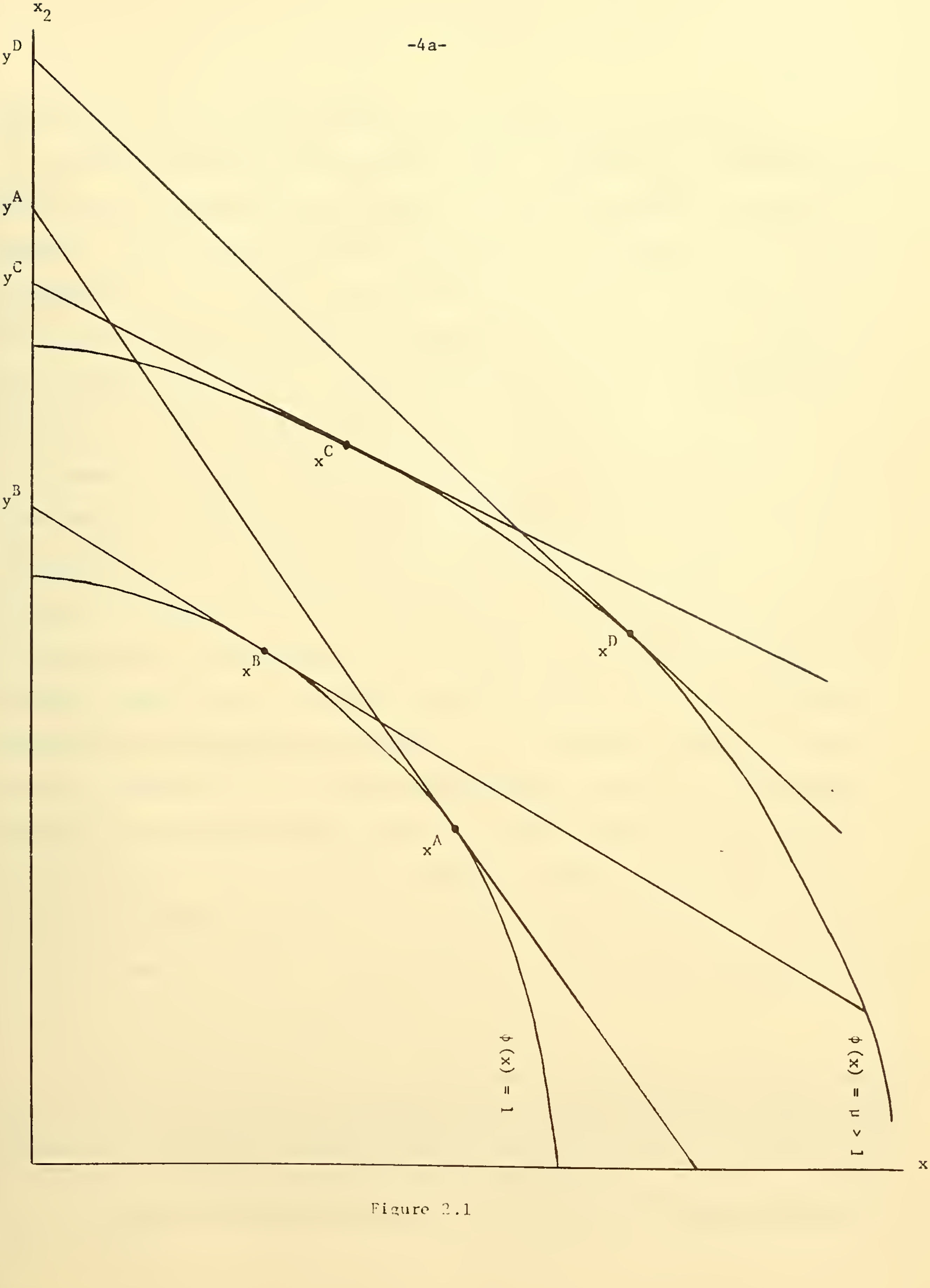

Figure 2.1

Naturally, the line (or hyperplane) (2.5) is tangent to the PPF at x^A.[1]

Suppose that there is no change in either technology or the proportions in which factor endowments occur, so that the PPM and the function $\phi(x)$ are unchanged. Suppose further that output then moves to a different point, x^B, on the <u>same</u> PPF ($\mu = 1$) as x^A. This occurs because prices have shifted from p^A to p^B. It is thus natural (but not inevitable) to regard the resulting shift in money output, from y^A to $y^B \equiv p^{B\prime}x^B$ as entirely a "monetary" phenomenon and to say that real output has not changed at all. We shall adopt this position.[2]

Reasonable as this position is, however, it carries implications which at first may seem disturbing. If x^A and x^B are judged to have the same real national output because they are on the same PPF, it follows immediately that comparison of real national output at those (or any) points cannot depend solely on observed prices and outputs. It also depends on the PPM with which the comparison is made. Thus, for example, there is no reason why with a different technology or a different set of relative factor endowments, the PPF through x^A could not pass above x^B. In this case, even though prices p^B might call forth outputs x^B, we would certainly say that real national output had decreased in the move from x^A to x^B. Similarly, a case can readily be generated in which real national output increases in the same move. There is nothing particularly disturbing about this once we realize that real national output comparisons must

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[1] This is natural only if some efficiency assumptions are made about production. If there is not perfect competition or if, for any reason, resources are underutilized, production will generally not be on the PPF. Since output indices and deflators have to be constructed even in imperfect worlds, we discuss these problems below but assume them absent for the time being.

[2] Of course, we are not the first to do so. See, for example, Richter [3].

necessarily be arbitrary in some degree and that the natural-appearing choice which we have adopted makes such comparisons from the point of view of a particular PPM. In effect, in each comparison we are asking whether the production system that produced x^A would have to be expanded, contracted, or left unchanged to produce x^B.

Such dependence on the PPM can readily be used to generate seeming anomalies.[1] Thus, for example, suppose that the initial situation (in which x^A was produced) is that pictured in Figure 2.1. Then a movement to x^B will be regarded as no change in real national output. Suppose, however, that after the move takes place, the PPM shifts and the new PPF through x^B passes above x^A. Then we shall apparently be saying that real national output at x^A has two different values. Obviously, this comes from trying to chain together two different things: a comparison made with the PPM which obtained when x^A was produced and one made with the PPM which obtained when x^B was. Had the first map remained unchanged, the comparison would clearly have been transitive.

Moreover, as we shall see in more detail a bit later on, if, with the second map, production had shifted back to x^A, then there is a real sense in which the entire move from x^A to x^B and back to x^A has decreased real national output. The prices at which x^A will be produced with the new PPM will no longer be p^A, but a different set of prices; and had those prices obtained originally, the value of production would have been greater than it now is at x^A. But of this , more in a moment.

We return now to the case of an unchanging PPM and inquire about compari-

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[1] It is a basic source of the lack of independence of path observed by Richter [3].

sons of points not on the same PPF. Thus, suppose that with no change in technology or factor endowment proportions, output is observed to move from x^A to x^C, not on the same PPF as x^A (see Figure 2.1). Suppose, as before, that prices move from p^A to p^B. Money output becomes $y^C \equiv p^{B\prime}x^C$. Clearly, in this situation, it is natural to regard relative money output as composed of:

$$(2.6) \qquad (y^C/y^A) = (y^B/y^A)\ (y^C/y^B) \quad ,$$

and to regard the first factor on the right as the appropriate money-output deflator and the second as a measure of increase in real output.

Now, we have not drawn x^C on the same ray through the origin as x^B in Figure 2.1, because we have not assumed the PPM to be homothetic. This immediately raises the possibility that our decomposition (2.6) of the move from x^A to x^C may not be unique. Thus, consider the PPF through x^C ($\phi(x) = \mu > 1$). This is tangent at x^C to the price line corresponding to prices p^B. We can clearly start with this PPF as a base and look for that point on it which would have been produced with the original prices, p^A. Call that point x^D and the associated value $y^D \equiv p^{A\prime}x^D$. Then the relative money output from y^A to y^C can be decomposed as:

$$(2.7) \qquad (y^C/y^A) = (y^D/y^A)\ (y^C/y^D)$$

and the first factor on the right measures the increase in real output while the second is a money-output deflator.

If the PPM is homothetic, then the two decompositions, (2.6) and (2.7), generate the same real output index and the same deflator. If the map is not homothetic, then two different indices are generated even from a single PPM. The situation is precisely the same as in the theory of the cost-of-living index where a nonhomothetic indifference map generates two equally sound indices, one starting from today's and one from yesterday's indifference curve.[1] The case in the present analysis differs only in that there may be more reason to be interested in homotheticity of the PPM than of the indifference map.

An illuminating way to look at the matter in the nonhomothetic case is to see that the two sets of indices answer different questions. The decomposition in (2.6) corresponds to asking what the economy which possessed just the right resources to produce x^A when prices were p^A would have produced if prices had been p^B. The decomposition (2.7), on the other hand, corresponds to asking what the economy which in fact produced x^C when prices were p^B would have produced had prices been p^A. In the homothetic case, the indices derived from the answers to these questions happen to coincide, but they are, of course, different questions nonetheless.

Considering the matter in this way, it becomes clear how to deal with the problem when the PPM changes. (This is of crucial importance not only because technology changes over time but also because the very process of investment alters the vector v^0 and thus the PPM). Nothing in the above discussion really depended on an unchanged PPM. There is no reason why the PPF through x^C had to come from the same PPM as that through x^A. If it does not, then the two decompositions (2.6) and (2.7) correspond to two sets of output indices and deflators coming from two different PPM.

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[1] See [1], especially pp. 98-103.

It is clear, of course, that neither real output index depends solely on the PPM. That would be impossible, in general, given the scalar nature of an index. Rather each index even of <u>real</u> output depends also on prices. In particular, the real output index starting with x^A depends on p^B and the real output index starting with x^C depends on p^A. The reason for this has already been indicated, but a slightly different discussion may make it appear more reasonable.

What we count as part of real output does not depend solely on production possibilities. The fact that many widgets could be produced does not affect real output unless widgets happen to be something that people are interested in buying. Similarly, items which consumers value highly should and do count more heavily in a real output index than items which they value relatively little. Prices, of course, give the rates at which purchasers are just willing to trade one good for another and it is clear that with a given set of prices, a particular iso-value line gives the locus of points of equal real output by giving the locus of points which purchasers value equally. It is true that the trade-offs given by the price ratios in fact hold only locally, but, just as in constructing a cost-of-living index we ignore production (except as revealed through prices) and take a given price line as representing the opportunity locus, so in constructing a real output index we ignore tastes (except as revealed through prices) and take a given price line as representing a locus of constant value.

Looked at in this way, the index beginning with the PPF through x^A is relevant in answering the following question: What is the capacity of the economy which produced value y^A when prices were p^A to produce the things now valued at prices p^B. A similar question corresponds to the other index.[1]

The fact that these two indices are designed to answer different questions and that they depend on prices and the PPM can be well pointed up by explicit consideration of an apparently anomalous case. In Figure 2.2, the initial PPM is drawn. With prices p^A, production is at x^A and money output at y^A. Now suppose that prices change to p^B and that we observe the <u>same</u> production point as before, x^A, but a new money output, y^C. Using the initial PPF, we conclude that the change in money output is not merely a price phenomenon. With that frontier, had prices been p^B, production would have been at x^B and money output at y^B. The fact that money output is only at y^C means that real output has decreased. This may seem strange, but it is entirely reasonable. The capacity of the economy to produce value at prices p^B has decreased. In terms of the valuation now placed by purchasers on goods, real output has indeed gone down; the new PPF lies inside the old PPF in the relevant range.

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[1] It should be noted that, apart from the two sets of indices under discussion, two additional ones can be similarly constructed. This corresponds to the fact that even with an unchanged PPM there would be two indices. Thus, one might construct an index in the following way. Consider the PPM obtaining when x^C was produced. To compare y^A and y^C using that PPM, we might begin by finding the PPF tangent to the price line through x^A which has slope corresponding to p^A. In general, the tangency will <u>not</u> occur at x^A. Starting with this PPF, consider what money output would have been had prices been p^B. Call that money output $y^{B}{'}$. Then the movement from y^A to $y^{B}{'}$ could be considered purely a price movement and that from $y^{B}{'}$ to y^C a real output change. Similarly, an index could be constructed using the PPM which obtained when x^A was produced and finding the PPF tangent to the price line through x^C corresponding to prices p^B.

These two indices do not seem to us to be of much direct interest (although the reader is free to differ). Unlike the two indices with which we have been concerned, these new ones employ a PPF which never existed. On the other hand, if the PPM are homothetic, then the new indices will be the same as the old ones and we shall want to make use of this property.

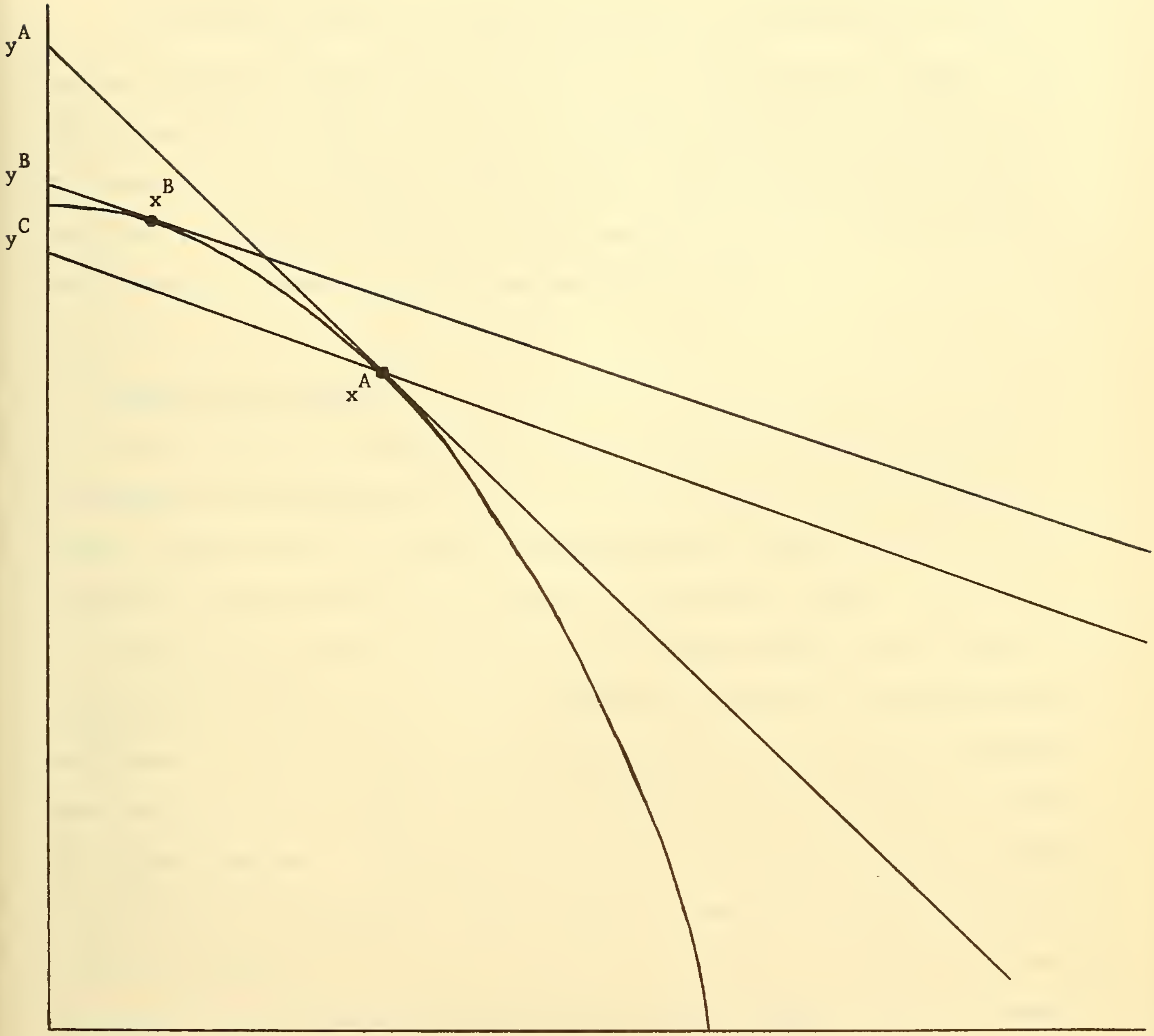

Figure 2.2

On the other hand, if we begin with prices p^B and money output y^C and use the new PPF (not drawn) we will find that real output has increased. But this is real output valued at prices p^A, and this is not the same thing at all. The capacity of the economy to produce value, given the old valuations has indeed gone up and the new PPF lies outside the old PPF in the range relevant to that valuation. There is nothing contradictory in this.

3. Which Index is Relevant?

The example just given suggests that while our two indices have equal intellectual footing, they may not be of equal relevance for all purposes. Clearly, such equality must appear when geographical comparisons are in view; it need not appear when we are interested in changes over time.

The reason for this is the fact that one index compares real outputs using today's valuations while the other makes the comparisons using yesterday's valuations. Yet if we consider planning ahead, these are not equally relevant. Thus, suppose that we consider alternative policies which will result in different outputs tomorrow. If we are interested in which policy will give the higher real output it is clear that we want to make the comparison using the valuations which will then obtain.[1] The fact that Policy I leads to an economy whose ability to produce items now valued is less than that of the economy which will result from Policy II is of no moment if those valuations will no longer apply. Crudely

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[1] Naturally, alternative policies will generally lead to alternative prices. This is largely beside the point, however. The focus of real output indices is to take tastes as given and reflected through fixed demand prices just as the focus of cost-of-living indices is to take production possibilities as given and reflected through fixed supply prices.

put, the two policies ought to be compared as to ability to produce what people want, not as to ability to produce what they used to want. It is possible but not particularly appealing to say that real output has gone down because suits of armor are no longer produced. Thus the index based on yesterday's capacity to produce items valued today seems the more relevant one.[1]

Naturally, however, the other index is not without interest and we shall want to see in what way the two indices differ. This will be of practical importance, for it bears on the properties of Paasche and Laspeyres indices.

4. Paasche and Laspeyres Indices

It is obviously of interest to see how our indices are related to the usual Paasche and Laspeyres indices. This is quite easy to do. In Figure 4.1, with prices p^A and the PPF as drawn, production was initially at x^A and money output at y^A. Prices now shift to p^B. With the same PPF, production would be at x^B and money output at y^B, so our deflator based on yesterday's PPM and today's valuations will be given by (y^B/y^A).

Now consider the Laspeyres price index given by $(y^{B'}/y^A)$ where $y^{B'} \equiv p^{B'}x^A$. Since x^B maximizes value at prices p^B subject to the constraint of being on the PPF, and since x^A is on that PPF, it is clear that $y^{B'} \leq y^B$, with the strict inequality holding if the production set is strictly convex.

Thus, <u>a Laspeyres price index understates the deflator based on yesterday's PPM and today's valuations</u>. Since the product of a Laspeyres <u>price</u> and a Paasche <u>quantity</u> index gives the change in money output, it follows that <u>a Paasche quan-</u>

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[1] The situation is the same (but the result as to which map should be used, different) in the case of a cost-of-living index when tastes change. See Fisher and Shell [1]. In both cases, it is the relevance of current tastes that matters.

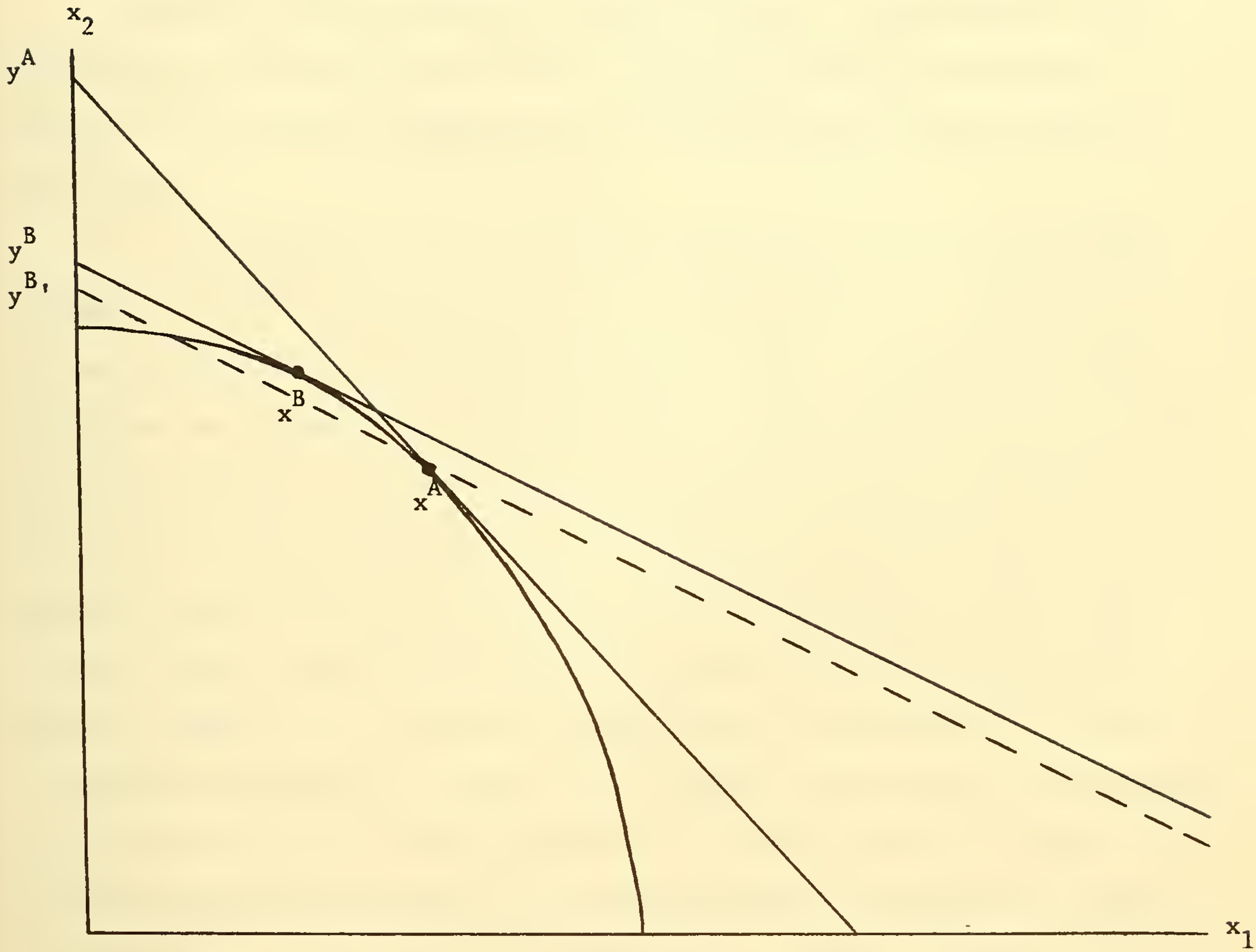

Figure 4.1

tity index overstates the change is real output measured on the same basis.

Similarly, by an essentially identical argument, a Paasche price index overstates the deflator based on today's PPM and yesterday's valuations; a Laspeyres quantity index understates the correspondingly computed change in real output.

It is interesting to note that these relationships are just the reverse of those which occur in the case of a cost-of-living index.[1] In that case, as is well known, a Paasche price index understates and a Laspeyres price index overstates the relevant cost-of-living index. In that case, also, there is not one but two indices which are involved with the Paasche index bounding below the cost-of-living index based on today's budget constraint and today's tastes and the Laspeyres index bounding above the cost-of-living index based on yesterday's budget constraint and yesterday's tastes. (If the indifference map does not change, then "indifference curve" should be substituted for "tastes" in the above sentence.) We argued in [1] that if tastes change, the interesting cost-of-living index is the one bounded by the Paasche price index rather than the one bounded by the Laspeyres. In the present case, the reverse is true if the PPM changes. It is the index bounded by the Laspeyres price index which is the relevant deflator.[2]

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[1] This comes about, of course, because in one case value is maximized subject to a production constraint concave to the origin while in the other, cost is minimized subject to an indifference curve constraint which is convex to the origin. The price line is a separating hyperplane for the two relevant convex sets.

[2] The possibility that there is a contradiction and that we have shown that a Paasche index lies above a Laspeyres index which also lies above it is not a real one. Aside from the fact that cost-of-living and real output indices are not defined over the same commodities (which is not a valid answer to such an objection), prices can only change if either indifference maps or the PPM change. But if prices do not change, all these indices coincide and if either the PPM or indifference map changes, one of the inequalities ceases to be established.

We may remark that the actual practice is to compute the Consumer Price Index as a Laspeyres index and the Implicit GNP Deflator as a Paasche index. If it is accepted that the former index is an approximation to the cost-of-living index based on consumer theory and the latter an approximation to the real output deflator based on production theory as here discussed, then in both cases the published indices have an inflationary bias, despite (indeed because of) their different methods of computation. If, in addition, we accept the arguments in [1] as to which of the two theoretically based cost-of-living indices is the relevant one when tastes change and the argument of the present paper as to which of the two real output deflators is the appropriate one when the PPM changes, then in both cases the published index imparts an inflationary bias to a theoretically based index which is not the relevant one. Their relation to the relevant indices when tastes or production possibilities change is in general unknown.[1] It would seem to be better to compute the Consumer Price Index as a Paasche and the GNP Deflator as a Laspeyres index.

Returning to real output deflation, as we have just seen, a Laspeyres price index bounds the more relevant theoretical index from below while a Paasche price index bounds the less relevant one from above. Clearly, it would be desirable to have an upper bound on the more relevant index as well. To accomplish this (in effect, to see how a Paasche index might be altered to make it bound that index) is possible if one knows something about the way in which the PPM changes. A principal aim of the present paper is to analyze some leading cases of such changes; this is done in later sections.

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[1] It is therefore not strictly correct to say that the published indices have an inflationary bias as compared with the relevant theoretically based index.

5. Market Imperfections and Underutilized Resources

Before proceeding, it may be useful to clear up one point by indicating that our analysis is in fact not wholly dependent on a restrictive-appearing assumption. This is the assumption of perfect competition -- more generally, the assumption that the economy operates on the PPF.

This is not hard to handle, at least in part. Recall that our PPM is generated by taking different values of the "dose" level μ in the equation

$$(5.1) \qquad \phi(x) = \mu$$

where the function $\phi(x)$ is defined relative to a vector of fixed available resources, v^0. The PPF is obtained by setting $\mu = 1$. If the economy operates on the PPF, then, as seen above, we can form our indices by using that frontier -- formally, by using the production possibility curve which is just tangent to the value line $y = p'x$ where p and y are actual prices and money output, respectively.

If, on the other hand, the economy operates inside the frontier, then one possibility is to continue to form our indices in essentially the same manner. In this case, however, the production possibility curve will not be the production possibility frontier ($\mu = 1$), but the curve for some lower value of μ. (That value can, incidentally, be taken as an index of the underutilization or misallocation of resources.) It is the case, of course, that particularly when markets are imperfect, the relevant point of tangency of that curve and the actual value line will not generally occur at the point of actual production, but this is not important for most of our analysis. On the other hand, it is important for the relations between our indices and Paasche and Laspeyres indices, as can be seen from an examination of the argument in the preceding section. If we proceed as just described, those relations cease to be known when production is not on the frontier.

Are there then alternative ways to proceed? The alternative just discussed essentially proceeded by making the inefficient economy equivalent to an efficient one with the same PPM but a "dose" level μ just enough to enable it to produce the same value. We might also consider a comparison with an efficient economy with the same map and a level of μ just sufficient to enable it to reach the same production point. That is, we might use the production possibility curve which passes through the point x^A of actual production. This situation is pictured in Figure 5.1. Actual production is x^A and money output is y^A. An efficient economy with the pictured PPF passing through x^A and facing the same prices would produce money output $y^{A\prime}$. Provided we recognize that such an efficient economy would in fact have a higher real output by the factor $(y^{A\prime}/y^A)$, we can use the frontier as the base for much of our analysis. On the other hand, the fact that x^A (the point of actual production) is not a point of tangency between the production possibility curve and the value line means again in this case, as in the former one, that the relations between our indices and Paasche and Laspeyres indices cease to be known. Further, such lack of tangency presents difficulties for our analysis of the effects of changes in the PPM, given below.[1] In view of this, and in view of the somewhat artificial (but still possible) necessity of remembering that two points (generating money outputs y^A and $y^{A\prime}$) do

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[1] In that analysis, we begin with a point of tangency to a given value line. We could certainly take that value line to correspond to $y^{A\prime}$ instead of to y^A; the value of $y^{A\prime}$, however, itself depends on the PPM, and the change in that value resulting from a change in the PPM would have to be ignored were we to follow this procedure. If the PPM is homothetic, however, the deflator using the PPF tangent to the value line corresponding to $y^{A\prime}$ will be identical with that using the PPF tangent to the value line corresponding to y^A, the preferred alternative in the text.

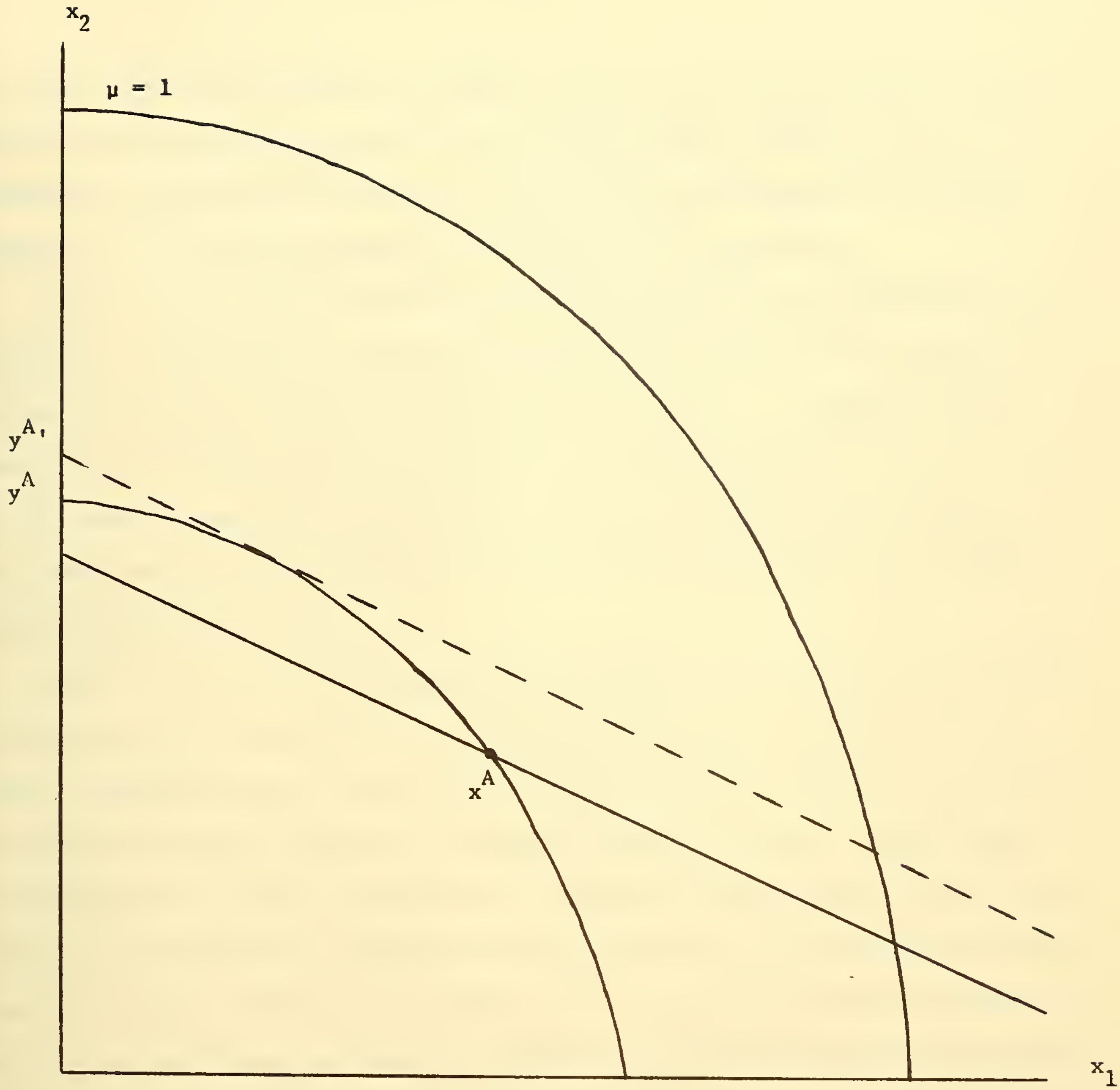

Figure 5.1

not have the same real output even though they are on the same production possibility curve because some of that curve is unobtainable, it seems to us that the alternative given above (of taking as a base the production possibility curve tangent to the actual value line) is preferable to the present one.

For the case in which inefficiency reflects general under-employment rather than market imperfections, a third alternative is available which is clearly superior to either of the others in most respects. This is to construct the PPM itself with v^0 defined not as the vector of resources available, but as the vector of resources actually utilized. This obviously results in a PPF tangent to the actual value line at the point of actual production and our entire analysis goes through without change.

Unfortuantely, this alternative is not available in the case of inefficiency caused by market imperfections. In that case, the first alternative listed is still a reasonable one and permits both the definitions of our indices and almost the entire analysis of this paper to proceed without difficulty. Nevertheless, in the presence of market imperfections, no procedure seems available which will allow us to preserve the relationships between Paasche and Laspeyres indices and those here defined which otherwise obtain. If we agree that our indices reasonably capture the production-theoretic content of real national output and national output deflator indices, then this fact should be regarded not as a defect in our indices but as a result about Paasche and Laspeyres indices. In the presence of market imperfections, such indices still approximate ours (both output indices measure a shift in the PPF in some manner) but whether from above or below is not known without further assumptions, as it is in the perfectly competitive case. Further analysis of this problem seems desirable.

For the remainder of this paper, we assume that production is efficient.

6. <u>The Indices: Formal Description</u>

We now give a formal description of our real output index and deflator. This need only be done for the deflator, the real output index being obtained by division into the relative change in money output.

We are given initial prices, $\hat{p}$, money value of output, $\hat{y}$, and a second set of prices p. Given also a particular PPM given by $\{(x, \mu) : \phi(x) = \mu > 0\}$ we begin by finding that set of outputs, $\hat{x}$, which solves the following problem:

$$\text{Minimize } \mu \text{ subject to } \hat{p}'\hat{x} = \hat{y} \quad . \tag{6.1}$$

Call the resulting value of μ, $\hat{\mu}$. Now find that vector of outputs, x, which solves the problem:

$$\text{Maximize } p'x \text{ subject to } \phi(x) = \hat{\mu} \quad . \tag{6.2}$$

Let y be the resulting value of $p'x$; then the deflator is $(y/\hat{y})$.

This apparently cumbersome description is needed for the following reason. When the PPM used is that which actually obtained when $\hat{y}$ was produced at prices $\hat{p}$, the procedure described clearly amounts to finding the actual PPF (and actual outputs $\hat{x}$) and forming the deflator as previously described. When, on the other hand, the PPM used is that which obtained when prices were p, then, <u>provided that the PPM is homothetic</u>, the described procedure generates the deflator based on that map and on the actual PPF holding at that time. In the latter case, $\hat{x}$ is not a vector of actually produced outputs but the vector of outputs which would have been produced at prices $\hat{p}$ had the production possibility curve on the new map been just sufficient to generate money output $\hat{y}$ at those prices.

The latter situation is pictured in Figure 6.1. The actual PPF is the outer

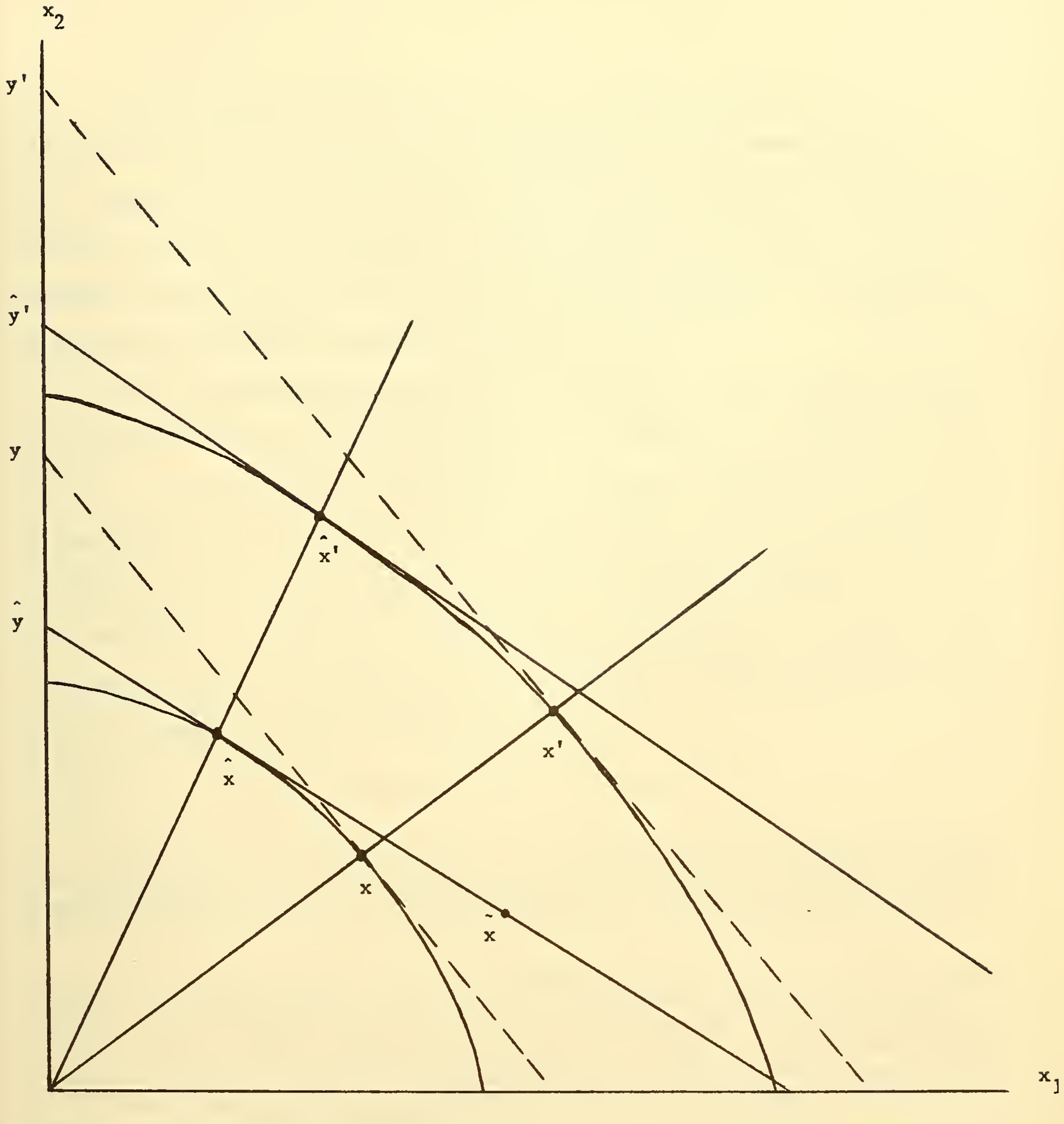

Figure 6.1

one. Actual production at prices p is at x' and money output at y'. Had prices been $\hat{p}$, actual production would have been at $\hat{x}'$ and money output at $\hat{y}'$. Clearly, the deflator based on this map is $(y'/\hat{y}')$ and this is the same as $(y/\hat{y})$ if the map is homothetic. Actual production in the initial period, however, need not have been at $\hat{x}$, since the PPM was different. In general, it was at some other point on the same value line, say $\tilde{x}$.

If we assume homotheticity, then, the described formal procedure enables us to make comparisons of deflators based on different maps by using the calculus. If we do not assume homotheticity, then what is compared is the appropriate deflator from one map with the deflator from the other map, discussed briefly in Section 2, which uses as a base the production possibility curve just sufficient to produce the observed value rather than the actual PPF. The comparison made in the homothetic case is clearly more interesting and we shall henceforth generally assume homotheticity.[1, 2]

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[1] Note that only homotheticity of the second map need be assumed. This apparent asymmetry is only apparent. If the first map is homothetic, we could begin by redefining $\hat{y}$ to be the value which the second economy would actually have produced at prices $\hat{p}$. Everything would then go through as before.

[2] As already suggested, the formal description just given is entirely isomorphic to the description of the cost-of-living index. This can be seen by changing minima to maxima and the reverse and considering x as a vector of outputs, $\phi(x)$ as a utility function, and y as money income. (In the case of the cost-of-living index, homotheticity is not a particularly appealing assumption; on the other hand, in that case, separate interest does attach to a comparison based on a given budget constraint.)

7. Hicks-Neutral Technological Change

Assume that the output of the ith good (i = 1, 2,..., r) depends upon the amounts of the m factors devoted to the production of that good. If there are no external economies (or diseconomies) in production, then base-period production possibilities can be described by the system

$$(7.1)\qquad \begin{aligned} & x_i = g^i\,(v_{i1},\dots, v_{im}) \\ & \sum_{i=1}^{r} v_{ij} \le v_j, \quad v_{ij} \ge 0, \text{ for } i = 1, 2,\dots, r;\ j = 1, 2,\dots, m. \end{aligned}$$

v_{ij} is the amount of the jth factor allocated to the production of the ith good, and $g^i(\cdot)$ is the base-period production function for the ith good.

In this section, we focus on an economy which experiences Hicks-neutral technological change (at possibly varying rates) in each of the sectors. For ease of exposition, we begin with the case in which factors are inelastically supplied in equal amounts during the base and current periods, and in which only the production of the first good experiences (Hicks-neutral) technological change. In this special case, the system (7.1) can be rewritten as

$$(7.1)'\qquad \begin{aligned} & x_1 = ag^1\,(v_{11},\dots, v_{1m}) \\ & x_1 = g^i\,(v_{i1},\dots, v_{im}) \\ & \sum_{i=1}^{r} v_{ij} \le v_j,\ v_{ij} \ge 0, \text{ for } 1, 2,\dots, r, \text{ and } j = 1, 2,\dots, m. \end{aligned}$$

a is then the Hicks-neutral technological efficiency parameter applied to the production of the first good, and units have been chosen so that a = 1 in the base period. If, for example, $a > 1$ in the current period, then we say that the production of the first good has experienced Hicks-neutral technological progress. Using the notation developed above, it is easy to see that the PPM based on current technology is given by

$$\text{PPM} \equiv \{(x_1, x_2, \ldots, x_r; \mu) : \phi(x_1/a, x_2, \ldots, x_r) = \mu > 0\}$$

with the production possibility map depending on the parameter a.

We now turn to the analysis of this economy in terms of the formalism developed in Section 6. The r-dimensional column vector of base-period prices $\hat{p}$, base-period "money" income $\hat{y}$, the r-dimensional column vector of current prices p, and the PPM defined by current technology are all given data. The first step is to solve for the r-dimensional column vector of outputs $\hat{x}$ that minimizes factor dosage $\phi(x_1/a, x_2, \ldots, x_r)$ subject to the requirement that "money" output in base-period prices, $\hat{p}'\hat{x}$, be equal to the value produced in the base-period, $\hat{y}$. $\hat{x}$ is found by solving the first-order system

$$(7.2) \qquad \begin{pmatrix} \hat{p}'\hat{x} \\ \hat{\phi}_1/a \\ \hat{\phi}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{\phi}_r \end{pmatrix} - \begin{pmatrix} \hat{y} \\ --- \\ \\ \hat{\lambda}\hat{p} \\ \\ \\ \\ \end{pmatrix} = 0 \quad ,$$

where $\hat{\phi}_i (i = 1, 2, \ldots, r)$ denotes the derivative of $\phi(\cdot)$ with respect to its ith argument evaluated at $\hat{x}$. Under current technological conditions but with prices at $\hat{p}$ and income at $\hat{y}$, $\hat{\phi}_i$ $(i = 2,\ldots, r)$ is the required increase in factor dosage μ necessitated by a first-order increase in the production of the ith good; $\hat{\phi}_1/a$ is the required increase in factor dosage necessitated by a first-order increase in the output of the first good. $\hat{\lambda}$ is a nonnegative scalar Lagrange multiplier. $(1/\hat{\lambda})$ can be interpreted as the current marginal cost of factor dosage (in "money" units) when prices are $\hat{p}$ and value of production is $\hat{y}$.[1]

Defining

$$\hat{\mu} = \phi(\hat{x}_1/a, \hat{x}_2,\ldots, \hat{x}_r) \quad , \tag{7.3}$$

the next step is to find the r-dimensional column vector of outputs $x = (x_1, x_2,\ldots, x_r)'$ which maximizes the "money" value of output subject to the dosage requirement $\phi(x_1/a, x_2,\ldots, x_r) = \hat{\mu}$. Thus x satisfies the first-order conditions

$$\begin{pmatrix} \phi \\ \phi_1/a \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \phi_r \end{pmatrix} - \begin{pmatrix} \hat{\mu} \\ \text{---} \\ \lambda p \end{pmatrix} = 0 \quad , \tag{7.4}$$

---

[1] We assume unless otherwise stated that solutions to first-order conditions are interior. Corner solutions are discussed in Section 12.

where $\phi_i(i = 1, 2,\ldots, r)$ denotes the derivative of $\phi(\cdot)$ with respect to its ith argument evaluated at x, and λ is a nonnegative scalar Lagrange multiplier. As before, $(\phi_1/a, \phi_2,\ldots, \phi_r)'$ can be interpreted as the vector of marginal dosage requirements, while $(1/\lambda)$ is interpreted as the marginal "money" cost of a dose when prices are p and factor availability (measured in doses) is $\hat{\mu}$.

We now observe that the formal analysis of the pure theory of the true national-output deflator under Hicks-neutral technological change is isomorphic to that of the pure theory of the true cost-of-living index under good-augmenting taste change. See Fisher-Shell [1], especially Section 3. Maximal utility $\hat{u}$ in the former theory, (evaluated at base-period prices and income, but current tastes) is replaced in the present context by the minimal factor dosage $\hat{\mu}$. Current utility at current prices $u(\cdot)$ is replaced by dosage requirement $\phi(\cdot)$. The taste change parameter, b, is replaced with the inverse of the technological parameter, $(1/a)$. We have chosen notation that should allow the reader easily to establish the remainder of the isomorphism.

It should be noted that a similar isomorphism can be established regardless of the choice of preference map on the one hand, or the choice of PPM on the other hand. Nor is the isomorphism special to the good-augmenting taste change and Hicks-neutral technological change cases. The isomorphism between the pure theory of the true cost-of-living index and the pure theory of the true national-output deflator is completely general, as is the isomorphism between the theory of the utility-maximizing consumer and the theory of the cost-minimizing firm.

Defining $y = p'x$, the true national-output deflator is $(y/\hat{y})$. We are interested in how the true national-output deflator is affected by technological change. Therefore, we must calculate the total derivative of y with respect to the technological parameter a. Base-period "money" output $\hat{y}$, base-period prices $\hat{p}$, and current prices p are the given data.

<u>Theorem 7.1</u>: $$(\partial y/\partial a) = \frac{p_1 x_1}{a}\left(1 - \frac{\hat{x}_1 \hat{\phi}_1}{x_1 \phi_1}\right) .$$

<u>Proof:</u> This is an immediate consequence of the isomorphism just discussed and Theorem 3.1 of Fisher and Shell [1], <u>mutatis</u> <u>mutandis</u>. It can also easily be proved as follows:

y is the solution to a constrained maximum problem in which the Lagrangian is

$$L_2 = p'x - \frac{1}{\lambda}[\phi(x_1/a, x_2, \ldots, x_n) - \hat{\mu}] \tag{7.5}$$

Hence, by the Envelope Theorem,[1]

$$\frac{\partial y}{\partial a} = \frac{\partial L_2}{\partial a} = \frac{1}{\lambda}\left(\frac{\phi_1 x_1}{a^2} + \frac{\partial \hat{\mu}}{\partial a}\right) . \tag{7.6}$$

However, $\hat{\mu}$ is the solution to a constrained minimum problem with Lagrangian,

$$L_1 = \phi(\hat{x}_1/a, \hat{x}_2, \ldots, \hat{x}_n) - \hat{\lambda}(\hat{p}'\hat{x} - \hat{y}) \quad . \tag{7.7}$$

Hence, by the same Envelope Theorem

$$\frac{\partial \hat{\mu}}{\partial a} = \frac{\partial L_1}{\partial a} = \frac{-\hat{\phi}_1 \hat{x}_1}{a^2} \quad . \tag{7.8}$$

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[1] See Samuelson [5], pp. 34-35.

Substituting, we obtain

$$(7.9) \qquad \frac{\partial y}{\partial a} = \frac{1}{\lambda}\left(\frac{\phi_1 x_1}{a^2} - \frac{\hat{\phi}_1 \hat{x}_1}{a^2}\right) .$$

Use of the first-order conditions (7.2) and (7.4) now immediately yields the theorem.

The alternative form of the result just obtained in the proof [equation (7.9)] has a straightforward interpretation. We note that $(\hat{x}_1\hat{\phi}_1/a^2)$ is the ceteris paribus decrease in factor dosage when prices are $\hat{p}$ and income is $\hat{y}$ due to a first-order increase in the efficiency parameter a. $(x_1\phi_1/a^2)$ is the ceteris paribus decrease in factor dosage when prices are p and factor dosage is $\hat{\mu}$. Therefore, the terms in the parenthesis on the RHS of (7.9) measure the total change in current factor dosage required to meet the constraint that $\phi(x_1/a, x_2, \ldots, x_r) = \phi(\hat{x}_1/a, \hat{x}_2, \ldots, \hat{x}_r)$. Since $(1/\lambda)$ is the current marginal cost (in units, say, of dollars per dose), the entire right-hand side of (7.9) gives the additional dollar value which would be produced today if the economy producing $\hat{y}$ yesterday had in both periods a small change in a.

Corollary 7.1: If $p = \hat{p}$, then $(\partial y/\partial a) = 0$ for all a.

Proof: The corollary is an immediate consequence of Theorem 7.1. By (7.2) and (7.4), if $p = \hat{p}$, then $y = \hat{y}$, so that $x_1 = \hat{x}_1$ and $\phi_1 = \hat{\phi}_1$.

Corollary 7.1 is also an obvious consequence of our definition of the true national-output deflator. If prices are unchanged, then in order for the budget lines to be tangent to the same PPF, income must also be unchanged -- no matter which PPM is employed as a frame of reference.

We know that $(\partial y/\partial a)$ is zero when current output prices are the same as base-period output prices. Therefore, in studying the effect of technological change upon the national-output deflator, it will be helpful to study the qualitative and quantitative behavior of $(\partial y/\partial a)$ as current prices p are displaced from their base-period values $\hat{p}$. First we derive results concerning the sign of $(\partial y/\partial a)$ for values of p different from $\hat{p}$. To do this, it is convenient to define $z(p) = x_1 \phi_1$, and to study the effects of price changes upon z(p).

Several of the results that follow depend upon elasticities of output supply. It will be helpful to agree on some definitions. First define θ_{i1},

$$(7.10) \qquad \theta_{i1} = \frac{p_1}{x_i}\left(\frac{\partial x_i}{\partial p_1}\right)_{y\ \text{const.}}, \quad i = 1, 2, \ldots, r.$$

This is the supply elasticity of the ith good with respect to the price of the first good when "money" value (equal to "money" income) is held constant while total factor dosage is allowed to vary. Next, define η_{i1},

$$(7.11) \qquad \eta_{i1} = \frac{p_1}{x_1}\left(\frac{\partial x_i}{\partial p_1}\right)_{\phi=\hat{\mu}\ \text{const.}}, \quad i = 1, 2, \ldots, r.$$

This is the supply elasticity of the ith good with respect to the price of the first good when factor dosage (and thus over-all factor supply) is held constant while "money" value (equal to "money" income) is allowed to vary.

By analogy with consumer theory, θ_{i1} can be thought of as a gross supply elasticity since value y is held constant, while η_{i1} can be thought of as a net supply elasticity since the calculation is restricted to a given production curve and "dosage" effects are suppressed. Both θ_{i1} and η_{i1} are in principle obtainable from econometric supply studies. Which of the two supply elasticities is easier to work with in practice? No clear-cut answer can be given to this question. If an econometric study is based on data for which factor supplies change relatively little from observation to observation, then the net supply elasticity η_{i1} is probably more natural to estimate. On the other hand, if the data are such that from observation to observation, the "money" value of output is relatively unchanged, then the gross supply elasticity θ_{i1} may be more natural to estimate. Since one can be obtained from the other by a known transformation, neither can be said to be truly preferable; nevertheless, η_{i1} seems to us to measure something more interesting than does θ_{i1}. It measures the supply response of the economy with a fixed production capacity to changes in price.

A third elasticity is often estimated in applied studies, the output supply elasticity where factor supplies are perfectly elastic at fixed prices. However, such partial equilibrium estimates ignore changes in factor prices which must be taken into account in a general equilibrium framework such as the present one. The following theoretical results are thus in terms of θ_{i1} and η_{i1}. Note that, by the Slutsky equation,

(7.12) $$\theta_{i1} = \eta_{i1} - \frac{p_1 x_1}{x_i}\left(\frac{\partial x_i}{\partial y}\right)_{p\ \mathrm{const.}}$$

Lemma 7.1: If $z(p) = x_1\phi_1$, then

$$\frac{\partial z}{\partial p_i} = \frac{x_i\phi_1}{p_1}\{\theta_{i1} + 1\} \quad \text{and}$$

$$\frac{\partial z}{\partial p_i} = \frac{x_1\phi_1}{p_1}\,\theta_{i1} \qquad i = 2,\ldots, r.$$

<u>Proof</u>: This follows directly from Fisher-Shell [1], Lemma 3.6, <u>mutatis mutandis</u>.

<u>Lemma 7.2</u>: If $z = x_1\phi_1$, then

$$\frac{\partial z}{\partial p_1} = \frac{x_1\phi_1}{p_1}\left\{\eta_{11} - p_1 \left.\frac{\partial x_1}{\partial y}\right|_{p\ \mathrm{const.}}\right\} + 1$$

$$\frac{\partial z}{\partial p_i} \quad \frac{x_1\phi_1}{\;}\left\{\eta_{i1} - \frac{p_1 x_1}{x_i} \left.\frac{\partial x_i}{\partial y}\right|_{p\ \mathrm{const.}}\right\}, \text{ for } i = 2,\ldots, r.$$

<u>Proof</u>: The lemma follows after substituting (7.12) into the results of lemma 7.1. Next, define η_{1i}, the net supply elasticity of the first good in terms of the price of the ith good by

$$\eta_{1i} = \frac{p_i}{x_1}\left(\frac{\partial x_1}{\partial p_i}\right)_{\phi = \hat{\mu}\ \mathrm{const.}} , \quad i = 1, 2,\ldots, r.$$

Notice, of course, that while substitution effects are symmetric, it is not necessary that the elasticities η_{i1} and η_{1i} be equal, although they certainly must possess the same sign.

<u>Lemma 7.3</u>: If $z = x_1\phi_1$, then

$$\frac{\partial z}{\partial p_i} = \frac{x_1\phi_1}{p_i}\left\{\eta_{1i} - p_i \left.\frac{\partial x_i}{\partial y}\right|_{p\ \mathrm{const.}}\right\} \quad \text{for } i = 2,\ldots, r.$$

Proof: The lemma follows from Lemma 7.2 using the definition of η_{1i} and the symmetry of substitution effects.

We must now agree on some terminology. Define the ith good's share of national income, α_i, by

$$\alpha_i = p_i x_i / y \qquad \text{for } i = 1, 2, \ldots, r. \tag{7.13}$$

Note that the PPM is homothetic if and only if $(\partial x_i/\partial y)_{p\text{ const.}} = (x_i/y)$ for $i = 1, 2, \ldots, r$, and for all positive p and y.

Lemma 7.4: If the PPM is homothetic, then

$$\frac{\partial z}{\partial p_1} = \frac{x_1 \phi_1}{p_1}\left\{\eta_{11} + 1 - \alpha_1\right\} \quad \text{and}$$

$$\frac{\partial z}{\partial p_i} = \frac{x_i \phi_1}{p_1}\left\{\eta_{i1} - \alpha_1\right\} = \frac{x_1 \phi_1}{p_i}\left\{\eta_{1i} - \alpha_i\right\}, \text{ for } i = 2, \ldots, r.$$

Proof: $(\partial x_i/\partial y)_{p\text{ const.}} = (x_i/y)$ as a consequence of the homotheticity of the PPM. Substituting in Lemma 7.2 and Lemma 7.3 and using definition (7.13), completes the proof.

Since substitution effects are symmetric, we know that η_{i1} and η_{1i} share the same sign. From Lemma 7.4, we can further deduce that sign $(\eta_{i1} - \alpha_1)$ = sign $(\eta_{1i} - \alpha_i)$.[1]

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[1] This fact follows, as it must, from the symmetry of the substitution terms. Thus:

$$(\eta_{i1} - \alpha_1)\frac{\alpha_i}{\alpha_1} = \left\{\left(\frac{\partial x_i}{\partial p_1}\right)_{\hat{\phi}=\mu\text{ const.}} \frac{p_1}{x_i} - \frac{p_1 x_1}{y}\right\}\frac{p_i x_i}{p_1 x_1}$$

$$= \left\{\left(\frac{\partial x_1}{\partial p_i}\right)_{\hat{\phi}=\mu\text{ const.}} \frac{p_i}{x_1} - \frac{p_i x_i}{y}\right\} = \eta_{1i} - \alpha_i \quad .$$

In the remainder of this section, we study the implications for the national output deflator of Hicks-neutral technological change under various production conditions. First we study the qualitative effect of Hicks-neutral technological change on the national output deflator for the case in which there is information about the gross supply elasticities, θ_{i1}, $i = 1, 2, \ldots, r$. It is now not assumed that the PPM is homothetic. The reinstatement of that assumption below enables us greatly to strengthen the results.

Theorem 7.2: (A) Suppose $p_i = \hat{p}_i$ for $i = 2, \ldots, r$. If $\theta_{11} > -1$, then $(\partial y/\partial a)$ has the same sign as $(p_1 - \hat{p}_1)$. If $\theta_{11} < -1$, then $(\partial y/\partial a)$ has the same sign as $(\hat{p}_1 - p_1)$. If $\theta_{11} = -1$, then $(\partial y/\partial a) = 0$.

(B) Suppose $p_i = \hat{p}_i$ for $i = 1, \ldots, r$, $i \neq j$, $j \neq 1$. If the jth good is a gross complement to the production of the first good $(\theta_{j1} > 0)$, then $(\partial y/\partial a)$ has the same sign as $(p_j - \hat{p}_j)$. If the jth good is a gross substitute to the production of the first good then $(\theta_{j1} < 0)$, then $(\partial y/\partial a)$ has the same sign as $(\hat{p}_j - p_j)$. If $\theta_{j1} = 0$, then $(\partial y/\partial a) = 0$.

(C) If $p_i = k\hat{p}_i$, $i = 1, 2, \ldots, r$, where k is a positive scalar constant, then $(\partial y/\partial a) = 0$.

Proof: (A) and (B) follow from Theorem 7.1, Corollary 7.1, and Lemma 7.1.

(C) Total differentiation of z with respect to k yields

$$k\frac{\partial z}{\partial k} = \phi_1 x_1 + \phi_1 \sum_{i=1}^{r} p_i \left(\frac{\partial x_i}{\partial p_1}\right)_{y\ \mathrm{const.}} \tag{7.14}$$

by Lemma 7.1 since $k(\partial p_i/\partial k) = p_i$ by hypothesis. Since $y = p'x$,

$\sum_{i=1}^{r} p_i \left(\frac{\partial x_i}{\partial p_1}\right)_{y\ \mathrm{const.}} = -x_1$. Theorem 7.2 (C) follows from Theorem 7.1 and

Corollary 7.1.

Notice that Theorem 7.2 (A) is a <u>global</u> result (that is, a result that holds for all values of p_1) when the sign of $(\theta_{11} + 1)$ is independent of the value of p_1. Theorem 7.2 (B) is a global result when the sign of θ_{j1} is independent of the value of p_j. Theorem 7.2 (C) is an extension of Corollary 7.1. No matter what the level of the technological parameter, a, if current prices are all k times base-period prices, current value, y, must be equal to k times base-period value, $\hat{y}$, in order that tangency with the same PPF be achieved.

The practical importance of Theorem 7.2 (A) and (B) is limited by our information on the constant-value supply elasticities θ_{i1}, $i = 1, 2, \ldots, r$. This is probably a severe limitation, since it is difficult to imagine a situation in which it would be natural to estimate directly the supply elasticities θ_{i1}. However, if the θ_{i1} are known, important practical implications follow. Suppose, for example, that all prices save the jth are the same in the two periods. If technology had not changed ($a = 1$), the only change in the national-output deflator would be due to the change in the value of the price of the jth good from $\hat{p}_j$ to p_j. If $(\partial y/\partial a)$ has the same sign as $(p_j - \hat{p}_j)$, the effect of technological progress ($a > 1$) is to magnify the effect of the change in p_j. In other words, the jth good receives increased weight in the index when $(\partial y/\partial a)$ and $(p_j - \hat{p}_j)$ agree in sign. That is, if $(\partial y/\partial a)$ and $(p_j - \hat{p}_j)$ agree in sign the jth good ought to receive more weight in a national-output index because of Hicks neutral technological progress in the production of the first good. Similarly, if $(\partial y/\partial a)$ and $(p_j - \hat{p}_j)$ disagree in sign, then technological change reduces the effect of the change in p_j and the jth good ought to receive reduced weight.

For our purposes, a change in more than one price can be thought of as a series of individual price changes. Therefore, these conclusions are not restricted to cases in which only one price changes between the two periods.

Notice that a (weak) sufficient condition for $(\partial y/\partial a)$ and $(p_1 - \hat{p}_1)$ to share the same sign is that θ_{11} be nonnegative. Thus, if the supply curve ("money" value of output held constant) is not downward sloping, then technological progress reduces the weight of a change in p_1, and the first good ought to receive less weight in a national-output index. It will turn out below that this is guaranteed in the homothetic case.

If the PPM $\equiv \{(x_1, x_2, \ldots, x_r; \mu): \phi(x_1/a, x_2, \ldots, x_r) = \mu > 0\}$ is homothetic, then we know that the national output deflator $(y/\hat{y})$ is such that

$$(7.15) \qquad \left(\frac{p'\hat{x}}{\hat{p}'\hat{x}}\right) \leqq \left(\frac{y}{\hat{y}}\right) \leqq \left(\frac{p'x}{\hat{p}'x}\right) \quad ,$$

because the price indices on the left and the right do not account for substitution effects.[1] The price index on the right in (7.15) is the Paasche Index (weighted by the vector of current outputs). If technology had not changed ($a = 1$), the price index on the left would be a Laspeyres Index (weighted by the vector of base-period outputs). In the case of unchanging technology $\hat{x} = \tilde{x}$, where $\tilde{x}$ is an r-dimensional vector with the element $\tilde{x}_i$ denoting the quantity of the ith good ($i = 1, 2, \ldots, r$) actually produced during the base period. Since the vector $\hat{x}$ is not observed while the vector $\tilde{x}$ is observed, it is important to

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[1] See the discussion in Section 4. The inequalities are strict in the usual case where equal factor-dosage loci in $(x_1, x_2, \ldots, x_r)$-space are <u>strictly</u> convex.

know the relationship of the Laspeyres Index $(p'\tilde{x}/\hat{p}'\tilde{x})$ to the unobserved price index $(p'\hat{x}/\hat{p}'\hat{x})$. This is the purpose of the following theorem, which of course also provides guidance as to the even more important question of the relationship of the Paasche Index to the deflator based on yesterday's PPM.[1]

<u>Theorem 7.3</u>:
$$\frac{\partial x_1}{\partial a} = \frac{\hat{x}_1}{a}(1 + \theta_{11}), \text{ and}$$

$$\frac{\partial \hat{x}_i}{\partial a} = \frac{\hat{x}_i}{a}\theta_{i1}, \text{ for } i = 1, 2, \ldots, r.$$

<u>Proof</u>: Theorem 7.3 can be easily proved after manipulation of equation (7.2). It is more interesting, however, to study the problem when the output of the first good is deflated by the technological parameter a. Let $\hat{x}_1^* = \hat{x}_1/a$ be the amount of the first good produced (in deflated units) when prices are $\hat{p}$. $\hat{p}_1^* = \hat{p}_1 a$ is the price per deflated unit of the first good. Equilibrium output of the various goods measured in deflated units depends only on prices per deflated unit and the "money" value of output $\hat{y}$. Therefore we conclude that

$$(7.16) \qquad \left(\frac{\partial \hat{x}_1^*}{\partial a}\right)\left(\frac{\partial a}{\partial \hat{p}_1^*}\right)_{\hat{p}_1 \text{ const.}} = \left(\frac{\partial \hat{x}_1^*}{\partial \hat{p}_1}\right)\left(\frac{\partial \hat{p}_1}{\partial \hat{p}_1^*}\right)_{a \text{ const.}}$$

and

$$(7.17) \qquad \left(\frac{\partial \hat{x}_i}{\partial a}\right)\left(\frac{\partial a}{\partial p_1^*}\right)_{\hat{p}_1 \text{ const.}} = \left(\frac{\partial \hat{x}_i}{\partial \hat{p}_1}\right)\left(\frac{\partial \hat{p}_1}{\partial \hat{p}_1^*}\right)_{a \text{ const.}}, \quad i = 2, \ldots, r.$$

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[1] See Section 4. We have chosen to work directly with the Laspeyres Index and the deflator based on today's PPM so as not to burden the notation.

Using the definitions of $\hat{x}_1^*$, $\hat{p}_1^*$, and θ_{i1} in equations (7.16) and (7.17) completes the proof of the theorem. In Theorem 7.3, the gross supply elasticities ("money" value of output constant) θ_{i1} $(i = 1, 2, \ldots, r)$ are evaluated at $\hat{p}$, $\hat{x}$, and $\hat{y}$.

If the PPM is homothetic, we can derive from Theorem 3.3 sufficient conditions for the observed Laspeyres Price Index to bound the national-output deflator based on today's PPM from below. For example, if $\theta_{11} > -1$ and $\theta_{i1} > 0$ $(i = 2, \ldots, r)$, then by Theorem 3.3, $\hat{x}_i > \tilde{x}_i$ $(i = 1, 2, \ldots, r)$ if there has been technological progress in the production of the first good $(a > 1)$. Then, if the price of the first good has risen, $p_1 > \hat{p}_1$, we have that $(p'\hat{x}/\hat{p}'\hat{x}) > (p'\hat{x}/\hat{p}'\hat{x})$. In this special case, therefore, the observed Laspeyres Price Index lies below the unobserved price index $(p'\hat{x}/\hat{p}'\hat{x})$. Further, if the PPM is homothetic, we have shown that in this special case the national-output deflator must lie between the observed Laspeyres Price Index and the observed Paasche Index,

$$\left(\frac{p'\tilde{x}}{\hat{p}'\tilde{x}}\right) \leq \left(\frac{y}{\hat{y}}\right) \leq \left(\frac{p'x}{\hat{p}'x}\right) .$$

This result can also be deduced from Theorem 7.2, because $(\partial y/\partial a) > 0$ in this special case.

Theorem 7.3 reinforces Theorem 7.2. For example, Theorem 7.3 states that with $\theta_{11} > -1$, had the current technology $(a > 1)$ been in practice during the base period, output of the first good would have been higher. For the case $\theta_{i1} > 0$ $(i = 2, \ldots, r)$, had the current technology been in practice, output of the ith good $(i = 2, \ldots, r)$ would have been greater. Therefore, in constructing a Laspeyres Price Index, the first good should receive increased (decreased) weight if $\theta_{i1} > -1$ ($\theta_{i1} < -1$). The ith good ($i = 2,\dots, r$) should receive more (less) weight if $\theta_{i1} > 0$ ($\theta_{i1} < 0$). According to Theorem 7.2, similar changes must be made in the national-output deflator. The Paasche Price Index retains its property as an upper bound on the deflator based on today's PPM (given homotheticity) without corrections.

Obviously, reversal of dates and of the movement in a yields parallel results as to the crucial relation between the Paasche Price Index and the deflator based on yesterday's PPM.

The case in which the PPM is homothetic is worthy of detailed analysis. This, of course, is the case in which in the absence of technological change the Laspeyres and Paasche Price Indices bound the national-output deflator. More importantly, many leading descriptions of technology imply homotheticity of the PPM. For example, if each of the r production functions, $g^i(\cdot)$ ($i = 1, 2,\dots, r$) in equation (7.1) exhibits constant returns then the associated PPM is homothetic, and there are other cases as well. The purpose of the following theorem is to exploit the special information available when the PPM is homothetic. The results depend on the net supply elasticities η_{i1}, η_{1i}, and on the shares $\alpha_i = p_i x_i / y$.

<u>Theorem 7.4</u>: Assume that the PPM is homothetic.

(A) Suppose $p_i = \hat{p}_i$ for $i = 2,\dots, r$. Then $(\partial y/\partial a)$ and $(p_1 - \hat{p}_1)$ share the same sign.

(B) Suppose $p_i = \hat{p}_i$ for $i = 1, 2,\dots, r$ and $i \neq j$, $j \neq 1$. If $\eta_{j1} > \alpha_1$, then $(\partial y/\partial a)$ and $(p_j - \hat{p}_j)$ share the same sign. If $\alpha_1 > \eta_{j1}$, then $(\partial y/\partial a)$ and $(\hat{p}_j - p_j)$ share the same sign. If $\eta_{j1} = \alpha_1$, then $(\partial y/\partial a) = 0$.

(C) Suppose $p_i = \hat{p}_i$ for $i = 1, 2, \ldots, r$ and $i \neq j$, $j \neq 1$. If $\eta_{1j} > \alpha_j$, then $(\partial y/\partial a)$ and $(p_j - \hat{p}_j)$ share the same sign. If $\alpha_j > \eta_{1j}$, then $(\partial y/\partial a)$ and $(\hat{p}_j - p_j)$ share the same sign. If $\eta_{1j} = \alpha_j$, then $(\partial y/\partial a) = 0$.

Proof: (A) follows from Lemma 7.4. Note that the net supply elasticity η_{11} is always positive[1] while the share of output $0 < \alpha_1 < 1$. (B) and (C) also follow from Lemma 7.4 and definitions (7.11) and (7.13). Note here that $0 < \alpha_j < 1$.

For the interesting case in which the PPM is known to be homothetic, important practical implications follow. The good whose production has experienced Hicks neutral technological progress (the first good) should receive more weight in a national-output deflator based on today's PPM than in one based on yesterday's PPM. For $j \neq 1$, if the net supply elasticity η_{j1} is greater (less) than the first good's share of "money" output, α_1, then the jth good should receive more (less) weight. An equivalent statement is that if the net supply elasticity η_{1j} is greater (less) than the jth good's share of "money" output, α_j, then the jth good should receive more (less) weight.[2]

Theorems 7.2 - 7.4 yield sharp qualitative results. It may be of some interest to know how the effects described in these theorems vary with the size of the price change $(p_j - \hat{p}_j)$. It is therefore helpful to study the second-order partial derivative $(\partial^2 y/\partial a \partial p_j)$.

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[1] It is an own substitution elasticity differing only in sign from the similar term in consumer theory.

[2] These statements are equivalent, by the remarks following Lemma 7.4 above.

Lemma 7.5:
$$\frac{\partial^2 y}{\partial a \partial p_1} = \frac{1}{p_1}(\eta_{11} + 1)(\partial y/\partial a) + \hat{x}_1\hat{\phi}_1\left(\frac{p_1}{a x_1 \phi_1^2}\right)(\partial z/\partial p_1)$$

and

$$\frac{\partial^2 y}{\partial a \partial p_j} = \frac{1}{p_j}\eta_{j1}(\partial y/\partial a) + \hat{x}_1\hat{\phi}_1\left(\frac{p_1}{a x_1 \phi_1^2}\right)(\partial z/\partial p_j) \qquad (i = 2,\ldots, r.$$

Proof: The lemma follows from Theorem 7.1, remembering that, by definition, $z(p) = x_1\phi_1$.

Lemma 7.6: (A) Suppose $p_i = \hat{p}_i$ for $i = 2,\ldots, r$. For p_1 sufficiently close to $\hat{p}_1$,

$$\text{sign}\,(\partial^2 y/\partial a\partial p_1) = \text{sign}\,(\partial z/\partial p_1) \quad .$$

(B) Suppose $p_i = \hat{p}_i$ for $i \neq j$, $j = 2,\ldots, r$. For p_j sufficiently close to $\hat{p}_j$,

$$\text{sign}\,(\partial^2 y/\partial a\partial p_j) = \text{sign}\,(\partial z/\partial p_j) \qquad (i = 2,\ldots, r).$$

Proof: This follows from Corollary 7.1 and Lemma 7.5.

Theorem 7.5: (A) Suppose $p_i = \hat{p}_i$ for $i = 2,\ldots, r$. For p_1 sufficiently close to $\hat{p}_1$,

$$(\partial^2 y/\partial a\partial p_1) \gtrless 0 \text{ as } \theta_{11} \gtrless -1 \quad .$$

Moreover, this holds for all $p_1 > \hat{p}_1$.

If the PPM is homothetic, then for all p_1,

$$(\partial^2 y/\partial a\partial p_1) > 0 \quad .$$

(B) Suppose $p_i = \hat{p}_i$ for $i \neq j$, $j = 2,\ldots, r$. For p_j sufficiently close to $\hat{p}_j$,

$$(\partial^2 y/\partial a \partial p_j) \gtreqless 0 \text{ as } \theta_{j1} \gtreqless 0 \quad (j = 2,\ldots, r).$$

If the PPM is homothetic, then for p_j sufficiently close to $\hat{p}_j$,

$$(\partial^2 y/\partial a \partial p_j) \gtreqless 0 \text{ as } \eta_{j1} > \alpha_1 \quad (j = 2,\ldots, r).$$

If the PPM is homothetic and p_j is in a sufficiently small neighborhood of $\hat{p}_j$ then it is also true that

$$(\partial^2 y/\partial a \partial p_j) \gtreqless 0 \text{ as } \eta_{1j} > \alpha_j \quad (j = 2,\ldots, r).$$

Furthermore, if $\eta_{j1} \geq 0$, then all the above statements hold for all $p_j > \hat{p}_j$. If $\eta_{j1} \leq 0$, then all the above statements hold for all $p_j < \hat{p}_j$.

Proof: The theorem follows from Theorems 7.2 and 7.4, Lemmas 7.1, and 7.4 - 7.6, and the fact that $\eta_{11} > 0$.

For at least these particular cases studied in Theorem 7.5, second-order effects reinforce first-order effects. The effects of technological change on proper weights in the national-output deflator are larger for larger price changes.

8. Changing Factor Supplies and Factor-Augmenting Technological Change: The Two-Sector Model

We now turn to the investigation of two formally equivalent cases: (1) that in which the relative supplies of factors change from period to period, and (2) that in which technological change is purely factor-augmenting and does not depend upon the sector in which the factor is employed. For ease of exposition and the sake of concreteness, much of the analysis is in terms of the well-known two-sector model.[1]

In the next section, we generalize to the case of r outputs and m factors. We then study the case in which the rate of factor-augmenting technological change varies from sector to sector and return again in Section 10 to the two-sector model to obtain more detailed results.

We now outline the two-sector model.[2] In any period, the quantity of (say) consumption goods, Y_C, produced depends on the quantities of (say) capital and labor devoted to that sector, K_C and L_C, respectively.

(8.1) $$Y_C = F_C(K_C, L_C).$$

Similarly, the quantity of (say) investment goods produced, Y_I, is given by

(8.2) $$Y_I = F_I(K_I, L_I),$$

where K_I and L_I are respectively the quantities of capital and labor devoted to the I-sector. The consumption value of output, Y, is given by

(8.3) $$Y = Y_C + p\, Y_I \quad ,$$

where p is the consumption goods price of investment. If factors are fully

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[1] In order to avoid confusion, we will adopt the widely accepted notation and terminology of Uzawa's article [7]. Although our references usually will be to Uzawa [7], many of the comparative statics theorems appeared first in Rybczynski [4]. We are indebted to Winston W. Chang for helpful suggestions on applications of two-sector theory.

[2] In adopting the notation of [7], we introduce some contradiction with our earlier notation, but no confusion should arise.

employed,

$$K_I + K_C = K \quad ,$$
(8.4)
$$L_I + L_C = L \quad ,$$

where K and L are respectively the amounts of capital and labor (completely inelastically) supplied in the period.

If factors are mobile and efficiently allocated, and if production is not completely specialized, then

$$\frac{\partial F_C}{\partial L_C} = p\frac{\partial F_I}{\partial L_I} = w \quad ,$$
(8.5)
$$\frac{\partial F_C}{\partial K_C} = p\frac{\partial F_I}{\partial K_I} = r \quad ,$$

where w and r are respectively the competitive wage rate for labor and rental rate on capital.

We now assume that there is constant returns-to-scale in production, so that the analysis can proceed in terms of intensive units. We define

$$k = K/L, \qquad y = Y/L$$

and

$$k_i = K_i/L_i, \qquad y_i = Y_i/L_i, \qquad \ell_i = L_i/L, \qquad i = I, C$$

and

$$\omega = w/r.$$

The equations (8.1) and (8.2) can be rewritten as

(8.6) $$y_i = \ell_i f_i(k_i) \qquad i = I, C$$

where

$$f_i(k_i) = F_i(k_i, 1).$$

It is assumed that $f_i(\cdot)$ is twice-continuously-differentiable and that

(8.7) $\qquad f_i(k_i) > 0, \qquad f_i'(k_i) > 0, \qquad f_i''(k_i) < 0, \qquad \text{for } 0 < k_i < \infty,$

where primes denote differentiation.

Conditions (8.1) through (8.6) reduce to:

(8.8) $$\omega = \frac{f_i(k_i)}{f_i'(k_i)} - k_i \quad , \qquad i = I, C$$

(8.9) $$p = \frac{f_C'(k_C)}{f_I'(k_I)} \quad ,$$

(8.10) $$y = y_C + py_I \quad ,$$

(8.11) $$y_I = f_I(k_I)\,\frac{k_C - k}{k_C - k_I} \qquad\qquad y_C = f_C(k_C)\,\frac{k - k_I}{k_C - k_I} \quad .$$

Differentiation in (8.8) shows that k_i is uniquely determined by and is increasing in the wage-rental ratio, ω,

(8.12) $$\frac{dk_i}{d\omega} = \frac{-\,[f_i'(k_i)]^2}{f_i(k_i)\ f_i''(k_i)} > 0 \quad ,$$

from (8.7). Capital intensities, $k_i(\omega)$, are fundamental properties of the production functions given in (8.1) and (8.2), and they will play an important role in the following analysis. Before proceeding, we will present a basic result in two-sector comparative statics.[1]

Lemma 8.1: $(dp/d\omega) \gtrless 0$ as $k_C(\omega) \gtrless k_I(\omega)$.

Proof: Logarithmic differentiation of (8.9) and substitution in (8.8) and (8.12) yield

$$\frac{1}{p}\frac{dp}{d\omega} = \frac{1}{k_I+\omega} - \frac{1}{k_C+\omega} \quad ,$$

which shares the sign of $(k_C - k_I)$.

A remark about the special case where $k_C(\omega) = k_I(\omega)$ is in order. In this case, the PPF is a straight line segment; thus, there is one and only one price ratio, p, consistent with nonspecialized production. Therefore, analysis of nontrivial equal capital intensity cases will necessarily involve the study of specialized production.[2] If, on the other hand, $k_C(\omega) \neq k_I(\omega)$, then along the PPF, y_I is a strictly concave function of y_C, so that there are a whole range of positive and finite price ratios that are consistent with nonspecialization of production.[3] For the remainder of this section, we assume $k_C(\omega) \neq k_I(\omega)$.

We now turn to the analysis of the two-sector production model in terms of

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[1] See, e.g., Rybczynski [4] and Uzawa [7].

[2] We treat "corner" cases in Section 12.

[3] The assertions in this paragraph are formally proved in Rybczynski [4] and Uzawa [7].

the formalism developed in Section 6. The aggregate capital-labor ratio k determines the production curve in (y_C, y_I)-space. From Lemma (8.1), we know that the price ratio, p, uniquely determines the wage-rental ratio, ω. From (8.12), ω uniquely determines the capital intensities, $k_C(\omega)$ and $k_I(\omega)$. Since we have assumed that the technology exhibits constant returns-to-scale, specification of k then determines the full PPM in (Y_C, Y_I)-space. The PPF is then determined by scale, i.e., by L or by K (= kL).

We analyze the effect on the true national-output deflator of a change in the overall capital-labor ratio, k. Such a change may be thought of as due to investment (an increase in K) or as due to a change in the labor force (L), or as stemming from a factor-augmenting technological change in both sectors with the amount of augmentation the same in both sectors. In the last case, factors must be considered as measured in efficiency units.

Given the base-period price ratio, $\hat{p}$, and the PPM based upon the current capital-labor ratio, k, the problem is to minimize labor employment, L, subject to producing a consumption value of output $\hat{Y}$. The relevant Lagrangean is

$$\Lambda_1 = L - \hat{\lambda}\{L[y_C(k, \hat{p}) + \hat{p}\, y_I(k, \hat{p})] - Y\} \quad . \tag{8.13}$$

$\hat{\lambda}$ is a nonnegative Lagrange multiplier. $y_i (i = I, C)$ is uniquely determined by k and $\hat{p}$ from Lemma 8.1 and equations (8.11). If $\hat{L}$ is the solution to this constrained minimization problem, then the first-order condition yields

$$\hat{L} = \hat{\lambda}\,\hat{Y} \quad , \tag{8.14}$$

so that $\hat{\lambda}$ can be interpreted as the marginal labor cost of expanding the consumption value of output when the capital-labor ratio is fixed at k, the price ratio is $\hat{p}$, and the consumption value of output is $\hat{Y}$.

By the Envelope Theorem,

$$\partial\hat{L}/\partial k = \partial\Lambda_1/\partial k = -\hat{\lambda}\hat{L}\hat{r} \quad , \tag{8.15}$$

where $\hat{r} = f_C'[k_C(\omega)] = \hat{p}\ f_I'[k_I(\omega)]$ is the rental rate on capital when the price ratio is $\hat{p}$. $\hat{r}$ is the first-order increase in the consumption value of output per head due to an increase in k, ceteris paribus, when the price ratio is p. Thus, $\hat{L}\hat{r}$, is the first-order increase in Y due to an increase in k, ceteris paribus. But, since $\hat{\lambda}$ is the marginal labor cost of output, (8.15) says that the first-order reduction in $\hat{L}$ due to an increase in k, mutatis mutandis, is equal to the first-order reduction in $\hat{L}$ due to an increase in k, ceteris paribus.

Given the current price ratio, p, the problem is to maximize the consumption value of output, when labor is constrained to be equal to $\hat{L}$ (or capital input constrained to be equal to $k\hat{L}$). The Lagrangean expression is

$$\Lambda_2 = L\ [y_C(p,\ k) + py_I(p,\ k)] - \frac{1}{\lambda}\ (L-\hat{L}) \quad , \tag{8.16}$$

where $\lambda = L/Y = \hat{L}/Y$ is a nonnegative Lagrange multiplier, which is interpreted as the marginal labor cost of output when the price ratio is p and labor input is $\hat{L}$.

By the Envelope Theorem, $\partial Y/\partial k = \partial\Lambda_2/\partial k$, so that

$$\partial Y/\partial k = Lr + (1/\lambda)(\partial\hat{L}/\partial k) \quad , \tag{8.17}$$

$$\partial Y/\partial k = Lr\ (1 - \frac{\hat{r}y}{r\hat{y}}) \quad , \tag{8.18}$$

where $\hat{y} = y_C(\hat{p}, k) + \hat{p}y_I(\hat{p}, k)$.

An alternate form of this result is

$$\frac{\partial Y}{\partial k} = \hat{L}\hat{r}\left(\frac{r}{\hat{r}} - \frac{y}{\hat{y}}\right) \quad , \tag{8.18'}$$

using the fact that in (8.16), L is constrained to equal $\hat{L}$. We shall comment on this form in the next section. In the present, two-sector context, it is more convenient to observe that, by Walras' Law, $y = rk + w$ and $\hat{y} = \hat{r}k + \hat{w}$, so (8.18) reduces to

$$\partial Y/\partial k = Lr\left(1 - \frac{\omega+k}{\hat{\omega}+k}\right) \quad . \tag{8.19}$$

<u>Theorem 8.1</u>: If $k_C > k_I$, the sign $(\partial Y/\partial k)$ = sign $(\hat{p} - p)$. If, on the other hand, $k_I > k_C$, then sign $(\partial Y/\partial k)$ = sign $(p - \hat{p})$.

<u>Proof</u>: The theorem follows from combining equation (8.19) with Lemma 8.1.

<u>Corollary 8.1</u>: If $p = \hat{p}$, then $(\partial Y/\partial k) = 0$.

<u>Proof</u>: The corollary follows immediately from equation (8.19) and the fact that the price ratio uniquely determines the wage-rentals ratio.

Theorem 8.1 and Corollary 8.1 are <u>global</u> results (<u>i.e</u>., they hold for all values of p and $\hat{p}$) as long as capital intensities do not cross.[1]

Theorem 8.1 and Corollary 8.1 have important practical implications. Let

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[1] Capital intensities are said to cross if there exist $\omega^{\dagger}$ and $\omega^{\dagger\dagger}$ such that $k_C(\omega^{\dagger}) > k_I(\omega^{\dagger})$ while $k_I(\omega^{\dagger\dagger}) < k_C(\omega^{\dagger\dagger})$.

the "money" price of the investment good be p_I, the "money" price of the consumption good be p_C, $p_I/p_C = p$. If the effective capital-labor ratio (in efficiency units) is unchanged from base period to current period, then the only change in true national output would be due to either a change in p_I or in p_C. The change in the true-national-output deflator due to a change in p_i ($i = I, C$) would be in the same direction as the change in p_i. If k (in efficiency units) is increasing through time, then the good whose production is more (less) capital intensive should receive increased (decreased) weight in the construction of the true-national-output deflator.[1]

It should be stressed that the analysis of this section applies to any two-sector model in which relative factor supplies change from period to period and in which technological change is purely factor-augmenting at the same rate in each sector. k must, then, be interpreted as the ratio (in efficiency units) of the quantity of the first factor to the quantity of the second factor. y_C and y_I are then respectively the ratio of the quantity produced of the first good to the aggregate supply of the second factor and the ratio of the quantity produced of the second good to the aggregate supply of the second factor.

9. Changing Factor Supplies and Factor-Augmenting Technological Change: The General Case

In this section, we generalize the results of the preceding section to the many-sector model and consider, as far as possible, the case in which the pro-

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[1] Because sign $(\partial Y/\partial k)$ = sign $[(k_C - k_I)(\frac{\hat{p}_I}{\hat{p}_C} - \frac{p_I}{p_C})]$.

duction function for a single good exhibits factor-augmenting technological change at a rate not identical with that exhibited by the production functions for other goods. The price we pay for such generality, however, is fairly substantial. In the case in which a factor increases in supply (or has the same rate of augmentation in all sectors), we are able to generalize the preceding results through equation (8.18) or (8.18'), and to give an economic interpretation to that result. As one might expect, however, the strong result of Theorem 8.1 seems to have no simple extension to the multi-sector case, although factor intensities clearly play an important role. In the case of factor-augmenting change in just one sector, the latter problem becomes even harder, although rather natural generalizations of all the other results to this case are readily available.

We begin, then, with the case in which there is no technological change, but the supply of some factor, say the mth, increases. (As already noted, this is equivalent to assuming that every sector experiences an mth factor-augmenting technological change at the same rate.) Unfortunately, whereas in the case of a Hicks-neutral technological change, an easy parametrization of the resulting shift in the PPM was available, that is not the case here and we must work with the underlying production functions.

Those production functions are denoted, as before, by:

$$x_i = g^i(v_{i1}, \ldots, v_{im}) \qquad (i = 1, \ldots, r) \tag{9.1}$$

where x_i is the amount of the ith good produced and v_{ij} is the amount of the jth factor used in its production.

The constraints on the system are given by:

$$(9.2) \qquad \sum_{i=1}^{r} v_{ij} = v_j; \qquad v_{ij} \geq 0 \qquad (i = 1,\ldots, r;\ j = 1,\ldots, m)$$

where the v_j denote the total amount of the jth factor used. The PPM is generated by considering those outputs (x_i) which can be produced when the v_j are given by

$$(9.3) \qquad v_j = \mu\, v_j^0 \qquad (j = 1,\ldots, m)$$

where v_j^0 is the amount of the jth factor available and μ (the factor "dosage") is allowed to vary over positive scalars.

We are going to investigate the effect on the national output deflator of a change in v_m^0. The deflator itself is formally constructed from the production system and constraints (9.1) - (9.3) instead of directly from the PPM as in Section 6 as follows.

First, given base period prices, $\hat{p}$, and base-period value of production, $\hat{y}$, we find the minimum μ for which $\hat{y}$ could be produced. Then, given that μ, which we shall call $\hat{\mu}$, we maximize value of production at current period prices, p. The resulting value, y, divided by the base period value, $\hat{y}$, is the index.

Accordingly, we must first set up the Lagrangean corresponding to the minimization of μ, given the base period value. This is most easily accomplished by minimizing the ratio of v_1 to v_1^0 (the choice of which factor to use is arbitrary), while constraining all ratios of v_j to v_j^0 to be the same. Obviously, the common value of such ratios is μ. The appropriate Lagrangean is therefore:

$$(9.4)\qquad \Lambda_1 = \frac{\sum_{i=1}^{r} v_{i1}}{v_1^0} + \sum_{j=2}^{m} \eta_j \left[\frac{\sum_{i=1}^{r} v_{ij}}{v_j^0} - \frac{\sum_{i=1}^{r} v_{i1}}{v_1^0} \right] - \lambda(\hat{p}'\hat{x} - \hat{y}) \quad ,$$

where λ and the η_j are Lagrangean multipliers.

Calling the resulting minimized value of μ, $\hat{\mu}$, the Lagrangean for the maximization of y is considerably simpler. It is:

$$(9.5)\qquad \Lambda_2 = p'x - \sum_{j=1}^{m} \pi_j \left(\sum_{i=1}^{r} v_{ij} - \hat{\mu}\, v_j^0 \right) \quad ,$$

the π_j being Lagrangean multipliers.

By the Envelope Theorem, applied to (9.5),

$$(9.6)\qquad \frac{\partial y}{\partial v_m^0} = \frac{\partial \Lambda_2}{\partial v_m^0} = \pi_m \hat{\mu} + (\partial\hat{\mu}/\partial v_m^0) \sum_{j=1}^{m} \pi_j v_j^0 \quad .$$

To evaluate $(\partial\hat{\mu}/\partial v_m^0)$, we apply the Envelope Theorem again, this time to (9.4), obtaining:

$$(9.7)\qquad \frac{\partial \mu}{\partial v_m^0} = \frac{\partial \Lambda_1}{\partial v_m^0} = \frac{-\eta_m \sum_{i=1}^{r} v_{im}}{(v_m^0)^2} = \frac{-\eta_m \hat{\mu}}{v_m^0} \quad ,$$

where the last equality follows from the definition of $\hat{\mu}$. Substituting (9.7) into (9.6), we obtain

<u>Lemma 9.1</u>: $$\frac{\partial y}{\partial v_m^0} = \pi_m \hat{\mu} - (\eta_m \hat{\mu}/v_m^0) \sum_{j=1}^{r} \pi_j v_j^0 \quad .$$

We now proceed to simplify and interpret this result. In so doing, it will be convenient to assume constant returns so that total factor payments equal total value of production. We shall assume constant returns for the remainder of this and the next section. (If constant returns are not assumed, our results below still go through, but with factor wages interpreted as shadow wages and total values (y and $\hat{y}$) interpreted as total factor payments at shadow wages.)

<u>Lemma 9.2</u>: $$y = \hat{\mu} \sum_{j=1}^{r} \pi_j v_j^0 \quad .$$

<u>Proof</u>: Differentiating (9.5) with respect to any v_{ij} and setting the result equal to zero yields as part of the first-order conditions for a maximum:

$$p_i(\partial g^i/\partial v_{ij}) = \pi_j \qquad (i = 1,\ldots, r;\ j = 1,\ldots, m) \tag{9.8}$$

whence it is clear that π_j is the wage of the jth factor in the second period. Since the total amount of the jth factor employed in that period is $(\hat{\mu}\, v_j^0)$, the lemma follows immediately from the constant-returns assumption.

Denote the wage of the jth factor in the second period by w_j, and its wage in the base period by $\hat{w}_j$. Then, as just remarked, $w_j = \pi_j$. We now seek an expression for $\hat{w}_j$.

<u>Lemma 9.3</u>: Denoting the wage of the jth factor in the base period by $\hat{w}_j$,

$$\hat{w}_1 = \frac{1 - \sum_{j=2}^{m} \eta_j}{\lambda\, v_1^0}, \quad \hat{w}_j = \frac{\eta_j}{\lambda\, v_j^0} \qquad (j = 2,\ldots, m).$$

Proof: Differentiating (9.4) with respect to any v_{ij} and setting the result equal to zero yields as part of the first-order conditions for a maximum:

$$(9.9) \qquad p_i(\partial g^i/\partial v_{i1}) = \frac{1 - \sum_{j=2}^{m} \eta_j}{\lambda\, v_1^0} \qquad (i = 1,\ldots, r)$$

and

$$(9.10) \qquad p_i(\partial g^i/\partial v_{ij}) = \frac{\eta_j}{\lambda\, v_j^0} \qquad (i = 1,\ldots, r;\ j = 2,\ldots, m)$$

which is equivalent to the lemma.

Lemma 9.4: $\lambda = \hat{\mu}/\hat{y}$.

Proof: By constant returns, $\hat{y}$ equals total factor payments in the base period. The total amount of the jth factor employed in that period is $\hat{\mu}v_j^0$, so that Lemma 9.3 yields

$$(9.11) \qquad \hat{y} = \sum_{j=1}^{m} \hat{w}_j(\hat{\mu}\, v_j^0) = (\hat{\mu}/\lambda)(1 - \sum_{j=2}^{m} \eta_j + \sum_{j=2}^{m} \eta_j) = \hat{\mu}/\lambda \quad ,$$

proving the lemma.

It is now easy to prove

Theorem 9.1: $$\frac{\partial y}{\partial v_m^0} = \hat{\mu}\, \hat{w}_m \left(\frac{w_m}{\hat{w}_m} - \frac{y}{\hat{y}}\right) \quad .$$

Proof: Combining Lemmas 9.1 and 9.2 and using the fact that $\pi_m = w_m$ yields:

(9.12)
$$\frac{\partial y}{\partial v_m^0} = w_m \hat{\mu} - \frac{\eta_m}{v_m^0} y \quad .$$

Application of Lemma 9.3 and then Lemma 9.4 shows:

(9.13)
$$\eta_m / v_m^0 = \lambda \hat{w}_m = \hat{\mu} \hat{w}_m / \hat{y} \quad .$$

Substituting (9.13) into (9.12) and rearranging yields the statement of the theorem.

Before interpreting Theorem 9.1, we note that it is the generalization of equation (8.18') to the present case. The two expressions differ only in notation and in the way in which the problem is stated. Thus, in deriving (8.18'), we were concerned with the effects of an increase in the capital-labor ratio, which may be interpreted as an increase in capital, given labor. The return to capital appears in (8.18') in precisely the same way as the returns to the changing factor (the mth) appears in the theorem. The remaining difference is the appearance in (8.18') of $\hat{L}$ in place of $\hat{\mu}$, which reflects the fact that in the two-sector model, we were able to avoid the complicated constraints involved in (9.4). In that model, we could in effect choose convenient units by taking the reference amount of labor available (the equivalent of v_1^0) to be unity, thus making $L = \hat{\mu}$.

Corollary 8.1 also carries over immediately to the present case:

<u>Corollary 9.1</u>: If $p = h\,\hat{p}$ for some scalar $h > 0$, then $(\partial y / \partial v_m^0) = 0$.

<u>Proof</u>: It is obvious that if $p = h\hat{p}$, then $w_m = wh\hat{w}_m$ and $y = h\hat{y}$.

The corollary is obvious in any case, since, if all prices are multiplied by h, the national output deflator will be equal to h regardless of what PPM is used.

We can now proceed to interpret Theorem 9.1. To do so, it will be convenient to define $\beta_m = (w_m v_m/y)$ and $\hat{\beta}_m = (\hat{w}_m v_m/\hat{y})$ as the shares of the mth factor in the current and base periods, respectively.[1] Then Theorem 9.1 can be restated as

<u>Theorem 9.1'</u>: $$\frac{\partial y}{\partial v_m^0} = (y/v_m^0)(\beta_m - \hat{\beta}_m) \quad ,$$

from which it is clear that what matters is whether the share of the mth factor goes up or down as a result of the price change. That is: <u>If, with prices</u> p, <u>the share of the</u> mth <u>factor is greater (less) than with prices</u> $\hat{p}$, <u>the effect of an increase in the supply of that factor will be to increase (decrease) the national output deflator when the new rather than the old PPM is used.</u>

The case of a proportional price change, covered in Corollary 9.1, is, of course, a case in which the price change leaves all factor shares unaltered.

What does this mean in terms of the relative importance of the various goods in the deflator before and after the price change? We can best investigate this by examining cases in which p differs from $\hat{p}$ in only one component.

<u>Theorem 9.2</u>: If a rise in the price of the <u>kth</u> good (k = 1,..., r) would increase (decrease) the share of the <u>mth</u> factor, then an increase in the supply of the <u>mth</u> factor leads to an increase (decrease) in the relative importance of the <u>kth</u> price in the national output deflator.

<u>Proof</u>: First, suppose that all prices except the <u>kth</u> remain the same but that

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[1] Note that the amount of the mth factor is $v_m = \hat{\mu} v_m^0$ in both periods.

the kth price rises, i.e., $p_i = \hat{p}_i$, $i = 1,\ldots, r$, $k \neq i$, and $p_k > \hat{p}_k$. Then, with either PPM, $y > \hat{y}$ and the deflator will exceed unity. If the rise in the ith price leads to an increase in the share of the mth factor, then, by Theorem 9.1', the effect of a rise in the supply of the mth factor will be to increase the deflator. On the other hand, suppose that with all other prices constant, $p_k < \hat{p}_k$. Then with either PPM, $y < \hat{y}$, and the deflator will be less than unity. However, the decrease in p_k will, by assumption, decrease the share of the mth factor, so that Theorem 9.1' shows that an increase in the supply of that factor will decrease the deflator in this situation. Thus, if increases in the price of the ith good increase the share of the mth factor, the effect of an increased supply of that factor will be to magnify the effect of the kth price on the deflator. Similarly, it is clear that if an increase in the ith price decreases the share of the mth factor, then an increase in the supply of that factor will diminish the effect of changes in the kth price on the deflator.

It may be remarked that this result is global rather than local, in the sense that it holds over any region of price and factor supply changes in which the indicated relationships remain valid.

Further insight into Theorem 9.2 can be gained by considering the way in which a Laspeyres price index would have to be changed to reflect the conditions prevailing with the new PPM (or, equivalently, the way in which a Paasche price index would have to be changed to reflect the conditions prevailing with the old PPM). We shall prove a theorem reinforcing Theorem 9.2 in precisely the same way that Theorem 7.3 reinforces Theorem 7.2, and shall show that had the mth factor supply been greater in the initial period, the production of the ith good would have been greater (less) relative to total income, y, if and only if a rise in the price of that good would have increased (decreased) the share of the mth factor.

To do this directly would require examination of the way in which $\hat{\beta}_m$ varies with $\hat{p}_i$ and $(\hat{x}_i/\hat{y})$ varies with v_m^0, when the maximizing problem is that whose Lagrangean is given by (9.4). This is moderately inconvenient, however, and the presence of constant returns makes it just as acceptable to work with (9.5) and to examine the variation of p_m with p_i and of (x_i/y) with v_m^0. We first prove a lemma reflecting the duality between factor supplies and outputs on the one hand and wages and prices on the other.

<u>Lemma 9.5</u>: $$\frac{\partial x_k}{\partial(\hat{\mu v}_m^0)} = \frac{\partial w_m}{\partial p_k} \qquad (k = 1, \ldots, r)$$

<u>Proof</u>: By the Envelope Theorem applied to (9.5),

$$\partial y/\partial p_k = \partial \Lambda_2/\partial p_k = x_k \tag{9.14}$$

and

$$\partial y/\partial(\hat{\mu v}_m^0) = \partial \Lambda_2/\partial(\hat{\mu v}_m^0) = \pi_m = w_m \quad . \tag{9.15}$$ [1]

Thus:

$$\frac{\partial x_k}{\partial(\hat{\mu v}_m^0)} = \frac{\partial^2 y}{\partial p_k\ \partial(\hat{\mu v}_m^0)} = \frac{\partial w_m}{\partial p_k} \quad , \tag{9.16}$$

and the lemma is proved.

Lemma 9.5 can also be proved through direct calculation of the derivatives involved.

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[1] Note that $\partial y/\partial v_m^0$ as evaluated here and in the proof of Theorem 9.3, below, is not the same as that evaluated in Theorem 9.1, because only one optimization problem is here involved.

With this result in hand, it is now easy to prove:

<u>Theorem 9.3</u>: $$\frac{\partial \beta_m}{\partial p_k} = v_m^0 \frac{\partial (x_k/y)}{\partial v_m^0} .$$

<u>Proof</u>:

(9.17) $$\frac{\partial (x_k/y)}{\partial v_m^0} = \frac{y\frac{\partial x_k}{\partial v_m^0} - x_k \frac{\partial y}{\partial v_m^0}}{y^2} = \frac{\hat{\mu}(y\frac{\partial w_m}{\partial p_k} - x_k w_m)}{y^2} ,$$

by Lemma 9.5, the Envelope Theorem applied to (9.5) and the fact that $\pi_m = w_m$. On the other hand,

(9.18) $$\frac{\partial \beta_m}{\partial p_k} = \frac{\hat{\mu}\, v_m^0 (y\frac{\partial w_m}{\partial p_k} - w_m \frac{\partial y}{\partial p_k})}{y^2} = \frac{\hat{\mu}\, v_m^0 (y\frac{\partial w_m}{\partial p_k} - w_m x_k)}{y^2} ,$$

by the Envelope Theorem applied, as before, to (9.5). Comparison of (9.17) and (9.18) yields the statement of the theorem.

The meaning of the theorem in relation to Theorem 9.2 has already been discussed. Before moving on, we might observe that the asymmetrical appearance of v_m^0 in the theorem is due to working with the share of the mth factor, rather than with the ratio of w_m to y, the variable most directly analogous to the ratio of x_k to y. It is obvious from Theorem 9.1 (rather than Theorem 9.1') that this would have suited our purposes equally well, although perhaps it would have seemed less natural.

Returning to the main thread of our discussion, Theorem 9.2, as already indicated, is the parallel of Theorem 7.2 in the present case. Note that whereas in Theorem 7.2, we were concerned with the effect of essentially an output-augmenting technological change in the PPM, the present case can be considered that of a factor-augmenting technological change. It is interesting that where, in the earlier case (see Theorem 7.4) we found the results to turn on the share of the outputs, in the present case, we find that the shares of the factors are involved, although this seems to exhaust the extent of the symmetry.

Unfortunately, Theorem 9.2 is about as far as it seems possible to go in the direction of generalizing the two-sector model's Theorem 8.1 to the multi-sector case. The problem, of course, is that in the present context, unlike the two-sector one, it is not at all straightforward to derive more basic conditions under which a rise in the price of the ith good will lead to an increase in the share of the mth factor. Indeed, this is an old and well-known problem in the analysis of factor price equalization in international trade.[1]

Some idea of the difficulties involved can be obtained by specializing to the case in which $r = m$, so that there are exactly as many factors as commodities. Let F be the factor-intensity matrix, that is:

$$(9.19) \qquad F = \begin{bmatrix} v_{11}/x_1 & \cdots & v_{1r}/x_1 \\ \vdots & & \vdots \\ v_{r1}/x_r & \cdots & v_{rr}/x_r \end{bmatrix}$$

Let w denote the r-component column vector of factor wages and p (as before) the r-component column vector of prices. Then, from constant returns, it is clear that at all points of equilibrium:

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[1] See, for example, Samuelson [6]. We are indebted to Pranab Bardhan for pointing this out to us.

(9.20) $$p = F\,w$$

or

(9.21) $$w = F^{-1}\,p \quad ,$$

assuming (for simplicity) that F is nonsingular. It follows that to trace the effect of a change in p_k on π_m requires knowledge of the inverse of the factor intensity matrix (and how the change in prices affects it). Whereas in the two-sector case, it is possible to state conditions on the factor intensities which determine the signs of the elements of that inverse, no similar simple conditions are known for the more general r-sector case (let alone for the case in which $r \neq m$). Which is in part to say that there is no general definition of relative factor intensity for the case with $r > 2$.

We have thus gone as far as seems possible in the analysis of the present problem. It is clear that the determination in practice of the effect of a change in factor supply on the national-output deflator and the measurement of national output itself requires considerably more detailed information than was the case of a Hicks-neutral change, considered in Section 7, where what was required was knowledge of supply elasticities and output shares.

We now turn to a case more general than that so far considered in this section, and find (not suprisingly) that while rather natural generalizations of the results so far obtained can be readily proved, the results seem further from practical usefulness than those just considered.

As several times mentioned, the case of an increase in the supply of the mth factor can be equivalently considered as a case of mth factor-augmenting technological change proceeding at the same rate in every production function. The problem which we now take up is that of the analysis of the effects on the national output deflator of an mth factor-augmenting technological change in a single production function; the case of such changes proceeding at different rates in different production functions is readily derived from this.

Accordingly, we keep the production functions for goods $2, \ldots, r$ [1] as before, but alter the production function of the first good to be:

$$x_1 = g^1(v_{11}, \ldots, v_{1m-1}, bv_{1m}) \tag{9.22}$$

where $b > 0$ is a parameter, an upward shift in which represents an mth factor-augmenting technological change in the production of the first good. We begin by finding $\partial y/\partial b$.

To do this, we must distinguish between two notations. We shall let g^1_m denote the partial derivative of g^1 with respect to its mth argument, while letting $\partial g^1/\partial v_{1m}$ continue to denote the partial derivative of g^1 with respect to v_{1m} itself. Thus, $\partial g^1/\partial v_{1m} = bg^1_m$. Moreover, it will be important to distinguish between v_{1m}, the amount of the mth factor employed in the production of the first good when (9.5) is involved (the second period) and $\hat{v}_{1m}$, the corresponding employment when (9.4) is involved (the first period). We shall denote with hats derivatives evaluated at the solution of the first-period problem.

Define:

$$\beta_{1m} = \frac{w_m v_{1m}}{y} ; \qquad \hat{\beta}_{1m} = \frac{w_m v_{1m}}{\hat{y}} , \tag{9.23}$$

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[1] We now return to the general case of $r \neq m$.

so that β_{1m} is the share in current national output of the mth factor employed in the production of the first good, and similarly for $\hat{\beta}_{1m}$. We prove the following generalization of Theorem 9.1':

Theorem 9.4: $$\frac{\partial y}{\partial b} = (y/b)(\beta_{1m} - \hat{\beta}_{1m}) \quad .$$

Proof: Applying the Envelope Theorem to (9.5):

$$\partial y/\partial b = \partial \Lambda_2/\partial b = p_1 g^1_m v_{1m} + \left(\sum_{j=1}^{m} \pi_j v^0_j\right)(\partial\hat{\mu}/\partial b) = \frac{w_m v_{1m}}{b} + \frac{y}{\hat{\mu}}(\partial\hat{\mu}/\partial b) \quad , \tag{9.24}$$

where the final equality follows from the first-order conditions and Lemma 9.2. Applying the Envelope Theorem to (9.4):

$$\partial\hat{\mu}/\partial b = \partial \Lambda_1/\partial b = -\lambda \hat{p}_1 \hat{g}^1_m \hat{v}_{1m} = -\lambda \frac{\hat{w}_m \hat{v}_{1m}}{b} = \frac{-\mu \hat{w}_m \hat{v}_{1m}}{b\hat{y}} \quad , \tag{9.25}$$

where the final equality follows from the first-order conditions and Lemma 9.4. Substitution of (9.25) into (9.24) yields the statement of the theorem.

We have called Theorem 9.4 a generalization of Theorem 9.1', and so it is. Suppose that instead of an mth factor-augmenting change in only the first production function, we had such a change in every production function, with the parameter of such change in the ith production function denoted by b_i. Obviously, Theorem 9.4 gives the effects on y of changes in any of the b_i, with obvious notational changes. Now suppose that for some subset of commodities, say, 1,..., h, with $h \leq r$, all the b_i were identical and identically equal to b. Then Theorem 9.4 would yield as an immediate corollary:

<u>Corollary 9.2</u>: If production functions 1,..., h experience <u>mth</u> factor-augmenting technological change at a common rate, the common parameter being denoted by b, then

$$(9.26)\qquad \partial y/\partial b = (y/b)\left(\sum_{i=1}^{h} \beta_{im} - \sum_{i=1}^{h} \hat{\beta}_{im}\right) \quad .$$

In particular, if h = r, so that the change is common to all production functions,

$$(9.27)\qquad \partial y/\partial b = (y/b)(\beta_m - \hat{\beta}_m) \quad .$$

<u>Proof</u>: This is an immediate consequence of Theorem 9.4 (with appropriate notational changes), the fact that $b_i = b$ $(i = 1,..., h)$, and

$$(9.28)\qquad \partial y/\partial b = \sum_{i=1}^{h} (\partial y/\partial b_i)(\partial b_i/\partial b) \quad .$$

The final statement of Corollary 9.2 can be seen to be identical with Theorem 9.1' if we recall that in that theorem, what is being varied is v_m^0. If we examined the derivative of y with respect to the total amount of the <u>mth</u> factor available $(\hat{\mu}\, v_m^0)$, the factor (y/v_m^0) in the statement of Theorem 9.1' would be replaced by $(y/\hat{\mu} v_m^0)$. Similarly, in the case of Corollary 9.2, the <u>effective</u> total supply of the <u>mth</u> factor is $(b\hat{\mu} v_m^0)$. If we evaluated the derivative of y with respect to this rather than with respect to b, then the factor (y/b) in (9.27) would be replaced by $(y/b\hat{\mu} v_m^0)$, or one over the share of an <u>efficiency unit</u> of the <u>mth</u> factor.

We now return to the case of an <u>mth</u> factor-augmenting technological change which occurs only in the production of the first commodity. It is clear that the generalization of Theorem 9.2 is:

Theorem 9.5: If a rise in the price of the kth good (k = 1,..., r) (other prices constant) would increase (decrease) the share in total output of that part of the mth factor employed in the production of the first good (and thus that part directly affected by the technological change), then the mth factor-augmenting technological change in the production of the first commodity leads to an increase (decrease) in the relative importance of the kth price in the national output deflator.

The proof is essentially the same as that of Theorem 9.2.

We can also derive a result generalizing Theorem 9.3 and bearing the same relation to Theorem 9.5 as Theorem 9.3 does to Theorem 9.2. As in deriving Theorem 9.3, it is convenient and also sufficient to work with the maximizing problem whose Lagrangean is given by (9.5). We first replace Lemma 9.5 with:

Lemma 9.6: $$\frac{\partial x_k}{\partial b} = \frac{v_{1m}}{b}\frac{\partial w_m}{\partial p_k} + \frac{w_m}{b}\frac{\partial v_{1m}}{\partial p_k} \qquad (k = 1,\dots, r).$$

Proof: By the Envelope Theorem applied to (9.5),

(9.29) $$\partial y/\partial p_k = x_k$$

and

(9.30) $$\partial y/\partial b = p_1 v_{1m} g^1_m = \frac{v_{1m}p_1 \, \partial g^1/\partial v_{1m}}{b} = \frac{w_m v_{1m}}{b},$$ [1]

[1] Note that $\partial y/\partial b$ as evaluated here and in the proof of Theorem 9.6, below, is not the same as that evaluated in Theorem 9.4 because only one optimization problem is here involved.

using the first-order conditions and the fact that $\pi_m = w_m$. Thus:

$$(9.31) \qquad \frac{\partial x_k}{\partial b} = \frac{\partial^2 y}{\partial p_k \partial b} = \frac{\partial (w_m v_{1m}/b)}{\partial p_k} ,$$

proving the lemma.

Lemma 9.6, like Lemma 9.5, can also be proved by direct (and relatively laborious) calculation of the derivatives involved.

It is now easy to prove:

Theorem 9.6: $$\frac{\partial \beta_{1m}}{\partial p_k} = b \frac{\partial (x_k/y)}{\partial b} \qquad (k = 1,\ldots, r).$$

Proof:

$$(9.32) \qquad \frac{\partial \beta_{1m}}{\partial p_k} = \frac{y\left(v_{1m}\frac{\partial w_m}{\partial p_k} + w_m \frac{\partial v_{1m}}{\partial p_k}\right) - w_m v_{1m}\frac{\partial y}{\partial p_k}}{y^2} = \frac{by\frac{\partial x_k}{\partial b} - w_m v_{1m} x_k}{y^2} ,$$

by Lemma 9.6, and the Envelope Theorem applied to (9.5), On the other hand,

$$(9.33) \qquad \frac{\partial (x_k/y)}{\partial b} = \frac{y\frac{\partial x_k}{\partial b} - x_k \frac{\partial y}{\partial b}}{y^2} = \frac{y\frac{\partial x_k}{\partial b} - x_k p_1 v_{1m} g^1_m}{y^2} = \frac{y\frac{\partial x_k}{\partial b} - \frac{x_k v_{1m} w_m}{b}}{y^2} ,$$

by the Envelope Theorem applied to (9.5) and the first-order conditions. Comparison of (9.32) and (9.33) yields the statement of the theorem.

Unfortunately, while Theorems 9.4 - 9.6 may seem elegant and illuminating, they are even farther from being of practical use than are Theorems 9.1 - 9.3, which they generalize. In order to apply Theorem 9.5, for example, information is required not merely on the share of the mth factor and how it varies with particular prices (as would be required for Theorem 9.2), but on the share of that portion of the <u>mth</u> factor employed in the production of a particular output (the first). At best, this is no easier to obtain than the information required to apply the less general theorem.

This concludes our study of factor-augmenting change in the general multi-sector model, but more can be said about the special two-factor case to which we now return.

10. <u>Factor-Augmenting Technological Change in a Single Sector of the Two-Sector Model</u>

Earlier we were able to exploit the simplicity of the two-sector production model in deriving specific results for the case where technological change is purely factor augmenting at the same rate in each sector. We are now interested in the case of factor-augmenting technological change with differing rates of augmentation in the various sectors. Again, we turn to the two-sector model[1] for concreteness.

For example, assume that only labor employed in the C-sector is subject to labor-augmenting technological change. If b is the technological parameter, then

$$Y_C = F_C(K_C, bL_C) \quad , \tag{10.1}$$

$$Y_I = F_I(K_I, L_I) \quad , \tag{10.2}$$

and

$$Y = Y_C + pY_I \quad . \tag{10.3}$$

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[1] Except for the technological parameter b, our notation agrees with that of Uzawa [7] and our Section 8, above.

If factors are fully employed,

(10.4) $$K_C + K_I = K$$

and

(10.5) $$L_C + L_I = L \quad .$$

Define $X_C \equiv bL_C$, labor in efficiency units employed in the C-sector. If factors are fully mobile and efficiently allocated between sectors, then

(10.6) $$\frac{\partial Y_C}{\partial K_C} = p \frac{\partial Y_I}{\partial K_I} = r$$

and

(10.7) $$\frac{\partial Y_C}{\partial L_C} = b \frac{\partial Y_C}{\partial X_C} = p \frac{\partial Y_I}{\partial L_I} = w \quad ,$$

where w is the wage rate and r the rentals rate.

Given the base-period price ratio $\hat{p}$, the PPM defined by the current endowments K and L, we analyze the effect on the national-output deflator of a change in b.

<u>Lemma 10.1</u>: $$\frac{\partial Y}{\partial b} = \frac{Y}{b}\left(\frac{wL_C}{Y} - \frac{\hat{w}\hat{L}_C}{\hat{Y}}\right) \quad .$$ Furthermore, $(\partial y/\partial b) = 0$ when $p = \hat{p}$.

<u>Proof</u>: The lemma follows from Theorem 9.4 and the fact that when $p = \hat{p}$, we know that $w = \hat{w}$, $L_C = \hat{L}_C$, and $Y = \hat{Y}$.

Next we define z(p) by

$$z(p) = wL_C/Y \quad . \tag{10.8}$$

In order to study the effects of price changes on z(p), we logarithmically differentiate in (10.8), obtaining:

$$\frac{1}{z}\frac{\partial z}{\partial p} = \frac{1}{w}\frac{\partial w}{\partial p} + \frac{1}{L_C}\frac{\partial L_C}{\partial p} - \frac{1}{Y}\frac{\partial Y}{\partial p} \quad . \tag{10.9}$$

We proceed to analyze the right-hand side of (10.9), term by term.

<u>Lemma 10.2</u>: Let $\beta_K = rK/Y$ be capital's share of national income and $\beta_L = wL/Y = 1 - \beta_K$ be labor's share. Then $\frac{1}{Y}\frac{\partial Y}{\partial p} = \frac{\beta_K}{r}\frac{\partial r}{\partial p} + \frac{\beta_L}{w}\frac{\partial w}{\partial p}$.

<u>Proof</u>: By Euler's theorem

$$Y = rK + wL \quad . \tag{10.10}$$

The lemma follows from differentiating (10.10) and substituting β_K and β_L in the result.

<u>Lemma 10.3</u>: Let $\sigma_j = (\omega/k_j)(\partial k_j/\partial\omega)$ be the elasticity of factor substitution in sector j (j = I, C).[1] Then

$$\partial L_C/\partial p = \frac{-(K_C\sigma_C + K_I\sigma_I)}{\omega(k_C - k_I)}\frac{\partial\omega}{\partial p} < 0 \quad .$$

<u>Proof</u>: Holding K, L, and b fixed and differentiating the full-employment condi-

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[1] Recall $k_j \equiv K_j/L_j$ and $\omega \equiv w/r$.

tions yields

(10.11) $$\partial K_C/\partial p + \partial K_I/\partial p = 0$$

and

(10.12) $$\partial L_C/\partial p + \partial L_I/\partial p = 0 \quad .$$

From the definition of the elasticity of substitution, σ_j, we have that

(10.13) $$\frac{1}{K_j}\frac{\partial K_j}{\partial p} - \frac{1}{L_j}\frac{\partial L_j}{\partial p} = \frac{\sigma_j}{\omega}\frac{\partial \omega}{\partial p} \quad ,$$

for j = I, C. Substituting from (10.11) and (10.12) in (10.13) and rearranging yields the system:

(10.14) $$\begin{pmatrix} k_C & k_I \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \partial L_C/\partial p \\ \partial L_I/\partial p \end{pmatrix} = \frac{-1}{\omega}\frac{\partial \omega}{\partial p} \begin{pmatrix} K_C\sigma_C + K_I\sigma_I \\ 0 \end{pmatrix} \quad .$$

The lemma follows after solving (10.14) by Cramer's rule and noting that by Lemma 8.1 sign $(\partial\omega/\partial p)$ = sign $(k_C - k_I)$.

<u>Lemma 10.4</u>: $$\frac{1}{z}\frac{\partial z}{\partial p} = \left(\frac{1}{\omega}\frac{\partial \omega}{\partial p}\right)\left[\beta_K - \frac{K_C\sigma_C + K_I\sigma_I}{L_C(k_C - k_I)}\right] \quad .$$

<u>Proof</u>: Substituting the results of Lemmas 10.2 and 10.3 in equation (10.9) yields

(10.15) $$\frac{1}{z}\frac{\partial z}{\partial p} = \left[\frac{K_C\sigma_C + K_I\sigma_I}{-\omega(k_C - k_I)L_C}\right]\left(\frac{\partial \omega}{\partial p}\right) - \left(\frac{\beta_K}{r}\frac{\partial r}{\partial p} + \frac{\beta_L}{w}\frac{\partial w}{\partial p}\right) + \frac{1}{w}\frac{\partial w}{\partial p} \quad .$$

Substituting $(1 - \beta_K)$ for β_L in (10.15) and noting that $(1/\omega)(\partial\omega/\partial p) = (1/w)(\partial w/\partial p) - (1/r)(\partial r/\partial p)$ establishes the lemma.

<u>Lemma 10.5</u>: If $k_I > k_C$, then $\partial z/\partial p < 0$.

<u>Proof</u>: In this case by Lemma 8.1, $\partial\omega/\partial p < 0$. Noting that when $k_I > k_C$ the term in brackets in the statement of Lemma 10.4 is positive completes the proof.

<u>Lemma 10.6</u>: If $\sigma_I \geq 1$ and $\sigma_C \geq 1$, then $\partial z/\partial p < 0$.

<u>Proof</u>: By Lemma 10.5 we need only study the case in which $k_C > k_I$. (We have excluded the case $k_C = k_I$, since this necessarily involves study of complete specialization, <u>i.e.</u>, corner solutions in production.) Then the term in brackets on the RHS of the statement of Lemma 10.4 is not larger than

$$(10.16) \qquad \frac{rK}{Y} - \frac{K}{L_C(k_C - k_I)}$$

since $\beta_K = rK/Y$ and by hypothesis $K_C\sigma_C + K_I\sigma_I \geq K$. Expression (10.16) is equal to

$$(10.17) \qquad \frac{K[rK_C - Y - rK_IL_C/L_I]}{YL_C(k_C - k_I)} < 0$$

since $Y > rK_C$.

In the next lemma, we weaken the hypothesis of Lemma 10.6.

<u>Lemma 10.7</u>: If $\sigma_C \geq 1$, then $\partial z/\partial p < 0$.

<u>Proof</u>: Again Lemma 10.5 allows us to restrict our attention to the case where

$k_C > k_I$. First, we employ two standard results[1] of two-sector theory:

(10.18) $$\partial Y_C/\partial p < 0$$

and

(10.19) $$\partial Y/\partial p > 0 \quad .$$

[The validity of (10.18) and (10.19) are easily established by drawing a national income line with slope (-p) tangent to the (strictly convex) PPF in the $Y_I - Y_C$ plane.] Combining (10.18) and (10.19) yields

(10.20) $$\frac{\partial(Y_C/Y)}{\partial p} < 0 \quad .$$

Define $\alpha(p)$, the fraction of C-sector income paid in wages, $\alpha(p) = wL_C/Y_C$. (It should be remarked that when $\alpha_C = 1$, labor's share of sector-C income is constant, i.e., $\alpha(p)$ is independent of p.) From Lemma 8.1, when $k_C > k_I$, $(\partial\omega/\partial p) > 0$.

Now, by Euler's theorem,

(10.21) $$Y_C = rK_C + wL_C$$

so that

(10.22) $$\frac{1}{\alpha(p)} = \frac{Y_C}{wL_C} = \frac{k_C}{\omega} + 1 \quad .$$

It is now evident that when $\sigma_C \geq 1$, i.e., when $\partial k_C/\partial\omega \geq k_C/\omega$, $\partial\alpha/\partial\omega \leq 0$, whence $\partial\alpha/\partial p \leq 0$ also.

The lemma now follows from this and (10.20).

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[1] See, e.g., Uzawa [7].

Theorem 10.1: In the two-sector model with labor-augmenting technological change in the C-sector

$$\text{sign } (\partial Y/\partial b) = \text{sign}(\hat{p} - p)$$

if either (A) the I-sector is the more capital-intensive sector ($k_I > k_C$) or (B) the C-sector elasticity of factor substitution is not less than unity ($\sigma_C \geq 1$).

Proof: The theorem follows after applying the results of Lemmas 10.5 and 10.7 in Lemma 10.1.

Corollary 10.1: If either (a) the I-sector is the more capital-intensive sector ($k_I > k_C$) or (b) the C-sector elasticity of factor substitution is not less than unity ($\sigma_C \geq 1$), then the effect of a labor-augmenting technological change in the C-sector is to decrease the relative importance of the price of the investment good in the national output deflator.

Moreover, as we know, this corresponds to

Corollary 10.2: If either (a) or (b) of Theorem 10.1 and Corollary 10.1 holds, $\partial(Y_I/Y)/\partial b < 0$.

Proof: This follows from Theorem 9.6 and Lemmas 10.5 and 10.7.

We now return to the multi-sector case.

11. General Technological Change

It may be worthwhile providing a brief generalization of some of the results of previous sections to the case of a general technological change in the production function of the first commodity.[1] Naturally, we shall not be able to obtain particular results of practical usefulness as long as we maintain a high level of generality, but such a treatment may usefully show what was special and what general about the cases so far considered.

Accordingly, we leave the production functions for goods 2,..., r as before, but alter the production function of the first good to be:

(11.1) $$x_1 = g^1(v_{11}, \ldots, v_{1m}; b)$$

where b is a technological change parameter.[2] We shall denote $\partial g^1/\partial b$ by g^1_b, and, as with other derivatives, shall denote with a hat its value at base-period values of the arguments.

We continue to assume constant returns and prove:

Theorem 11.1: $$\partial y/\partial b = \hat{p}_1 \hat{g}^1_b \left(\frac{p_1 g^1_b}{\hat{p}_1 \hat{g}^1_b} - \frac{y}{\hat{y}}\right) .$$

Proof: Applying the Envelope Theorem to (9.5):

(11.2) $$\partial y/\partial b = \partial \Lambda_2/\partial b = p_1 g^1_b + \left(\sum_{j=1}^{m} \pi_j v^0_j\right)(\partial \hat{\mu}/\partial b) = p_1 g^1_b + (y/\hat{\mu})(\partial \hat{\mu}/\partial b) \quad ,$$

---

[1] Actually, an identical development applies to any change in the PPM, but changes induced by shifts in factor supplies have already been considered.

[2] Note that g^1 is now a function of m+1 variables.

where the final equality follows from Lemma 9.2. Applying the Envelope Theorem to $(\bar{9}.4)$:

$$(11.3) \qquad \partial\hat{\mu}/\partial b = \partial\Lambda_1/\partial b = -\hat{\lambda}p_1\hat{g}_b^1 = -\frac{\hat{\mu}\hat{p}_1\hat{g}_b^1}{\hat{y}} \quad ,$$

where the final equality follows from Lemma 9.4. Substituting from (11.3) into (11.2) and rearranging yields the statement of the theorem.

It is clear that $p_1 g_b^1$ is the first-order marginal effect on the value of output of the technological change -- the marginal revenue product of the technological change, as it were. By an argument identical to that of the proof of Theorem 9.2, Theorem 11.1 is readily seen to imply:

<u>Theorem 11.2</u>: If a rise in the price of the <u>kth</u> good (k = 1,..., r) (other prices constant) would increase (decrease) the ratio of the marginal revenue product of technological change to the total value of output, then the technological change leads to an increase (decrease) in the importance of the <u>kth</u> price in the national output deflator.

We can also derive a result which bears the same relation to Theorem 11.2 as Theorem 9.3 does to Theorem 9.2 or Theorem 9.6 to Theorem 9.5 or, for that matter, Theorem 7.3 to Theorem 7.2. As in the previous section, it is convenient and also sufficient to work with the maximizing problem whose Lagrangean is given in (9.5). We first prove:

<u>Lemma 11.1</u>: $$\frac{\partial x_k}{\partial b} = \frac{\partial(p_1 g_b^1)}{\partial p_k} \qquad (k = 1,\ldots, r).$$

<u>Proof</u>: By the Envelope Theorem applied to (9.5):

(11.4) $$\partial y/\partial p_k = x_k$$

and

(11.5) $$\partial y/\partial b = p_1 g_b^1 \quad .$$[1]

Thus:

(11.6) $$\frac{\partial x_k}{\partial b} = \frac{\partial^2 y}{\partial p_k \partial b} = \frac{\partial(p_1 g_b^1)}{\partial p_k} \quad ,$$

and the lemma is proved.

It is now easy to prove:

<u>Theorem 11.3</u>: $$\frac{\partial(p_1 g_b^1/y)}{\partial p_k} = \frac{\partial(x_k/y)}{\partial b} \qquad (k = 1,\ldots, r).$$

<u>Proof</u>:

(11.7) $$\frac{\partial(p_1 g_b^1/y)}{\partial p_k} = \frac{y\dfrac{\partial(p_1 g_b^1)}{\partial p_k} - (p_1 g_b^1)\dfrac{\partial y}{\partial p_k}}{y^2} = \frac{\dfrac{\partial(p_1 g_b^1)}{\partial p_k} - p_1 g_b^1 x_k}{y^2} \quad ,$$

by the Envelope Theorem applied to (9.5). Similarly,

(11.8) $$\frac{\partial(x_k/y)}{\partial b} = \frac{y\dfrac{\partial x_k}{\partial b} - x_k\dfrac{\partial y}{\partial b}}{y^2} = \frac{y\dfrac{\partial x_k}{\partial b} - x_k p_1 g_b^1}{y^2} \quad .$$

The desired result now follows immediately from Lemma 11.1.

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[1] Note that $\partial y/\partial b$ as evaluated here and in the proof of Theorem 10.3 below is not the same as that evaluated in Theorem 10.1, because only one optimization problem is here involved.

Thus, an increase in the kth price will increase the marginal revenue product of a technological change relative to total money output if and only if, with prices constant, the effect of the technological change will be to increase production of the kth commodity, relative to total money output. This obviously reinforces Theorem 11.2.

12. New Goods, Disappearing Goods, and Corner Solutions

So far we have been assuming that a positive amount of every good is produced in every period. It is obviously important to remove this assumption and to deal with the possibility of corner solutions. Clearly, the market basket of goods produced in the economy does not always contain the same items; new goods are produced and old ones disappear. The question thus naturally arises as to how the prices of such goods ought to be treated in national output deflation.

In the case of new goods, unlike what is ordinarily true of disappearing goods, there are two subcases to consider. The first of these is that which naturally comes to mind when thinking of a new good -- the case of a new invention, of a good which is now produced for the first time because in earlier years the technology for producing it did not exist in some sense. Actually, this is only a limiting case. Most new goods could have been produced at times before they actually were; technical improvements in their manufacture were required to bring the cost down to a profitable level, but earlier production would have been possible at a higher price. This kind of new good introduction shares with that of a pure new invention the property that the good has appeared for the first time as a result of a change in the PPM.

The second kind of new good is one which could perfectly well have been produced earlier, but which was not produced because consumers would not have bought it at a profitable price. Because of a taste change, that is no longer so and the good now appears for the first time because of changes in tastes rather than in the PPM. It is clear that this is also often the case of a disappearing good with the two periods reversed.

Real cases may often be some admixture of these two polar ones, but study of the pure cases will allow us to treat mixed ones. We shall refer to a new good which appears with no taste change but only because of a change in the PPM as a new good for reasons of supply. Similarly, we shall refer to a good which appears with an unchanging PPM because of a taste change as a new good for reasons of demand. We may note that the case of goods which are new for reasons of supply was essentially the only case treated in our study of the cost-of-living index [1]; since this is the case which ordinarily comes to mind, this was perhaps not too great an oversight, particularly as we shall show that the conclusion reached is essentially the same. The isomorphic case for the national output deflator, however, turns out to be that of goods which are new for reasons of demand, and it would obviously be inappropriate to treat only this. We shall, in fact, treat both cases and, as a natural byproduct, easily extend the treatment of the cost-of-living index to cover goods which are new for reasons of demand.

We begin, however, by analyzing the national output deflator when there is a good which is new for reasons of demand. (As already mentioned, this is the case which is isomorphic both to that treated in [1] and to the case of a disappearing good.) Here the PPM is unchanged, but some good, say the first, was not produced in the base period because no one wanted to buy it, at least not at prices which would have made it profitable to produce. Thus, in the base period, the point of actual production lies in the intersection of the PPF with the hyperplane defined by $\hat{x}_1 = 0$. There is no actual price quotation for the first commodity in the first period.

The lack of such a quotation is not a major problem for the national output deflator which can be found (in principle) without reference to the missing price. It is somewhat more convenient (and more in line with general practice), however, to consider such a price explicitly.

In Section 6, we began by locating the crucial value of μ by solving the problem:

$$\text{Minimize } \mu \text{ subject to } \hat{p}'\hat{x} = \hat{y} \quad . \tag{12.1}$$

In the present case, there is no observed value for $\hat{p}_1$. It will obviously suffice to set $\hat{p}_1 = 0$, or indeed, to set it at any price sufficiently low to ensure that $\hat{x}_1$ will be zero when the minimization problem is solved. All such values for $\hat{p}_1$ will lead to the same result. The highest of such prices is what we shall term the <u>supply reservation price</u>; it is the intercept on the (properly interpreted) supply curve for the first commodity.[1]

It is less arbitrary in some sense, however, not to proceed in this way and to replace (12.1) by:

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[1] The supply curve in question is not the usual one with factor prices constant. It is the curve which shows how much of the first commodity would be produced as a function of its price, given fixed prices for the remaining commodities and also given fixed factor <u>supplies</u>.

(12.1') Minimize μ subject to $\hat{p}'\hat{x} = \hat{y}$ and $\hat{x}_1 = 0$.

This obviously leads to the same result.

Note that, with an appropriate adjustment of units, the supply reservation price is the shadow price of the constraint $\hat{x}_1 = 0$. That shadow price is the amount of resource 'dose' μ, which would just need to be used up to produce one unit of the first commodity. The supply reservation price of the first commodity gives the same quantity in terms of dollars -- the price at which the product of one unit of the first commodity would just be worth the value of the resources devoted to it.

With either of these treatments, then, the lack of a price quotation for the first commodity in the base period is not a problem for the construction of the national-output deflator when the first commodity is new for reasons of demand. Neither is this a problem for the construction of a Laspeyres price index in which that commodity will receive zero weight in any case. It is obviously a problem, however, for the construction of a Paasche price index. (Note that the cases would be reversed for a disappearing good.) We now turn to that problem.

We saw in Section 4 that a Paasche deflator bounds the true national output deflator from above when the present problem is not encountered.[1] Obviously, we want to preserve that property. It is not hard to show that this will be accomplished if, in the construction of the Paasche deflator, we use for $\hat{p}_1$ any price at or below the supply reservation price.

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[1] In that section, we observed that the deflator which was bounded was that based on the current period's PPM, which was not the relevant deflator when the PPM changed. In the present case, however, we are dealing with an unchanging PPM. In any event, given that a Paasche index is to be constructed, as is the current actual practice, the present problem must be met; further, as already remarked, the same problem arises for a Laspeyres deflator in the case of disappearing goods.

To see this, observe that in both the Paasche deflator and the national output deflator which it bounds (the one using today's PPM and PPF), the numerator will be the same, namely the value of today's output at current prices, $p'x$. Given a choice of $\hat{p}_1$, the denominator of the Paasche deflator will be simply $\hat{p}'x$, while the denominator of the national output deflator, essentially as we have seen,[1] will be the largest value of output that could have been obtained with prices $\hat{p}$, on the same PPF as x, and with zero production of the first commodity. If $\hat{p}_1$ is set at or below the supply reservation price, however, the constraint that the first commodity not be produced will not be binding, so the denominator of the national output deflator is the unconstrained maximum value which could have been achieved at prices $\hat{p}$ on the same PPF as x. It is obvious that this is not less than $\hat{p}'x$ and that it will be greater if the PPF is strictly convex.

This obviously leads to the conclusion that <u>the value used for $\hat{p}_1$ in the construction of the Paasche index should be the supply reservation price or below</u>, just as in the parallel case for the cost-of-living index, a similar argument leads to the use of a price at the demand reservation price or above. Moreover, since of all Paasche indices which bound the national output deflator from above, it is obviously desirable to use the one providing the least upper bound, we see immediately that the proper price to use in this situation is the supply reservation price itself, just as we found in the parallel case for the cost-of-living index that the proper price to use was the demand reservation price itself. This is a natural result in the light of the shadow price interpretation of the supply reservation price.

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[1] Note that because the Paasche index bounds the national output deflator based on today's PPM, the deflator in question is described in a way opposite to that which we have generally employed.

Note that the supply reservation price here involved is the one with factor supplies fixed and the remaining prices fixed at their base period values. In the case of a disappearing good, the remaining prices would be fixed at their current period values.

We now turn to the case of a good which is new for reasons of supply. Here the problem is somewhat different. In the case of a good which is new for reasons of demand, the construction of the national output deflator could proceed by restricting $\hat{x}_1$ to be zero because that was a case in which there was no demand for the first commodity at profitable prices. Obviously the same thing is true here when we consider the national output deflator based on yesterday's PPM; no really new problem arises when that is the deflator to be used, and we have already suggested that it is the more appropriate of the two deflators. The numerator of the deflator based on yesterday's PPM is, as before, the maximum value that could have been produced by the economy with yesterday's resources at today's prices. If the first good could have been produced yesterday at today's prices, such production will properly enter the numerator; if the first good was a pure new invention (or would have a very high supply reservation price at today's prices), then in that maximum, x_1 will be zero and the entire money value of production of the first good will be treated as a contribution to real output, other things equal. In either case, the correct procedure is clear.

When we consider the national output deflator based on today's PPM, however, the matter is a bit more complicated. (Note that it is this deflator which is bounded by a Paasche index.) In constructing this deflator, we must maximize the value of production which would have been achieved with today's PPF at yesterday's prices. In the pure case of a good new for reasons of demand, we could do this constraining the production of the first good to be zero, considering that no demand for the good existed at profitable prices. In the present case, such a procedure might well be inappropriate. Had we had today's technology and yesterday's demand conditions, it is entirely possible that the first good would have been produced and sold. A price quotation for that good might perfectly well have existed and been higher than the supply reservation price. All we can say in this case is that such a price would certainly not have been higher than the _demand_ reservation price, or else the good would not have been produced essentially for reasons of demand. To put it another way, it is possible that with today's technology the first good could have been profitably sold yesterday. Such sales would have been profitable if they could have been made above the supply reservation price and they would have been non-zero had they been made below the demand reservation price. What in fact the price would have been, we do not know without examination of the general equilibrium of all markets.

Fortunately, such ambiguity does not prevent us from reaching a definite conclusion as to what price should be used in the construction of the Paasche deflator. There are two cases to consider.

First, suppose that the demand reservation price for the good in question would have lain above the supply reservation price. Then the good would have been produced and the national output deflator should use the price at which it would have been produced -- some price in the interval between the two reservation prices. A Paasche deflator which used the same price as the national output deflator would certainly be one, as we know, which bounded the latter deflator from above. _A fortiori_, a Paasche deflator which uses the (in this case) lower supply reservation price will certainly also bound the national output deflator from above.

The second case is that in which the demand reservation price would still have lain below the supply reservation price. In this case, the first good would not have been produced yesterday even with today's technology, and the restriction that the first good would not have been produced will be an appropriate one in the construction of the national output deflator. Indeed, we are now back in the case of a good new for reasons of demand, despite the change in the PPM, and our previous analysis applies. The supply reservation price should be used in the construction of the Paasche deflator. (Note that this time it is the supply reservation price which would have been obtained with *today's* PPM.)

Thus in every case we find that the supply reservation price should be used in the construction of a Paasche deflator. Only in the case of a good wholly new for reasons of supply (in the sense that it could have been profitably produced *and sold* last period had current technology and resources been available) does this fail to put the most efficient upper bound on the national output deflator. Even in this case, such use does provide an upper bound; discovery of the most efficient upper bound, like discovery of the value of the national output deflator itself, would require knowledge of the price at which transactions would actually have taken place.

Before closing, we may briefly extend our earlier treatment of new goods in the cost-of-living index to cover goods new for reasons of *demand*. This is the case in which, had today's tastes ruled yesterday, the first good, say, would have been produced and *bought*. That transaction would have taken place at a price below the demand reservation price (but above the supply reservation price), and the true cost-of-living index ought to be calculated using that price. Since, in practice, the correct price will be unknown, the best we can do is to use a Paasche cost-of-living index which provides a lower bound on the true one, and we shall certainly achieve this by using a Paasche index which employs the demand reservation price.[1] So in any case, the demand reservation price should be used, although only in the case of a good new for reasons of supply will it provide the most efficient lower bound on the true cost-of-living index.

13. Quality Changes

In this section, we take up the question of how quality change in one of the goods ought properly to be treated in the construction of the national output deflator. This is a difficult problem, largely because it is so difficult to model quality changes in an adequate way, and we are unable to reach any positive recommendation. All that we are actually able to do is to show the circumstances under which a treatment of quality change along fairly standard lines will in fact be appropriate. Not too suprisingly, those circumstances turn out to be quite restrictive.

Suppose that there is a quality change in the first good. The old variety of the good (for simplicity) ceases to be produced and a new one is instead. Obviously, in principle, this can be treated as a combination of the disappearance of an old good and the appearance of a new one; in practice, this is not done and it would be cumbersome to do so. We continue to treat automobiles today as in some sense the same commodity as automobiles of a few years ago, even though there

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[1] Of course, this is the demand reservation price which would have obtained yesterday had today's tastes been in effect. In the case of a good which disappears for reasons of demand, a similar recommendation applies to the construction of a Laspeyres cost-of-living index bounding from above the true index constructed with yesterday's indifference map.

are many differences of greater or less importance. It is clear that one may wish to make an adjustment to the national output deflator in order to take such quality changes into account. Assuming, for the sake of definiteness, that the new quality of the good is somehow an improvement over the old one, then, in the case of the cost-of-living index, if prices have not changed one wants to say that the cost-of-living has gone down because consumers are better off being able to buy the new variety than they would have been had they only been able to purchase the old one.

Similarly, in the present case, even if all outputs, as measured, are unchanged, we may wish to regard real output as having risen because the new variety of good may embody more resources than did the old one. To put it differently, with the same resour es and technology, had the old variety been produced, other outputs and all prices remaining fixed, it is possible that more of it would have been produced than was produced of the new variety. In such a case, we would not want the shift from the old to the new variety to disguise the fact that the capacity of the economy to produce real output has increased. If more steel, labor, and other inputs are embodied in new cars than in old ones, then the production of a given number of cars represents a bigger output when new cars are involved than when old ones are. Moreover, this is true regardless of how consumers view the change. Their views and tastes are relevant for deciding whether the cost of living has decreased and their real income risen, but not for deciding whether real output in terms of the production system has risen (except insofar as tastes affect prices).

Accordingly, in such a case, we would want to reduce the national output deflator so that an unchanged money output will correspond to an increased real output. It is natural to seek to do this by a downward adjustment in the price of automobiles.

Formally, let b be a parameter indexing the quality change. Given $\hat{y}$, $\hat{p}$, and p, y, the numerator of the national output deflator will depend on b. Since $\hat{y}$ and $\hat{p}$ will remain fixed throughout this discussion, we may suppress them and write $y = y(p_1, \ldots, p_r, b)$. Taking a value of b equal to unity for the case of no quality change, adjustment of the price of the first commodity to take account of the effects of the quality change amounts to finding a p_1^* such that

$$y(p_1^*, p_2, \ldots, p_r, 1) = y(p_1, p_2, \ldots, p_r, b) \quad . \tag{13.1}$$

In general, it will be possible to find such a p_1^*. The problem arises when we require (as is natural to do in practice) that the appropriate price adjustment be made knowing only the physical characteristics of the good involved (here summarized by b) and possibly the amount of its production, or, in general, knowing only the production function for the new and old quality of the good. One would expect to use this information to determine, as it were, the relative quantities of resources embodied in each quality. Ordinarily such adjustments would not be allowed to depend, in particular, on the outputs of the other commodities not affected by the quality change. Unfortunately, such a natural-appearing requirement leads to very restrictive conditions on the kinds of changes in resource use which quality changes so treated can represent.

Since the problem is fully isomorphic to that treated in Section 5 of [1] for the case of quality adjustments to the cost-of-living index, we shall simply state and interpret the crucial result without repeating the proof. It is:

Theorem 13.1:

(a) A necessary and sufficient condition for p_1^* to be independent of $x_2, \ldots, x_r$ is that the shift in the PPF resulting from the quality change be representable as:

$$\hat{\mu} = \phi(g^*(x_1, b)x_1, x_2, \ldots, x_r) \quad . \tag{13.2}$$

(b) If, in addition, p_1^* is to be independent of x_1, or if there are everywhere constant returns, then t e function $g^*(x_1, b)$ is independent of x_1 and the shift in the PPF can be represented as

$$\hat{\mu} = \phi(bx_1, x_2, \ldots, x_r) \tag{13.3}$$

by an appropriate choice of units for b. In the constant returns case, this will be true of the entire PPM, and not merely of the PPF corresponding to $\hat{\mu}$.

If we write $b = 1/a$ and compare (13.3) with the discussion at the beginning of Section 7, we see immediately that the condition of the theorem is that from the point of view of resource use, the shift from the old to new quality of the good must appear as equivalent to a Hicks-neutral technological change in the production function of the first good.[1] If constant returns are not imposed and the price adjustment allowed to depend on the amount of the first commodity produced, then the extent of the Hicks-neutral change can also depend on the amount of that production, but it will be Hicks-neutral, nevertheless. The new variety must embody more of every kind of factor than the old variety and the percentage change

[1] Note that a quality improvement -- in the sense of more resources embodied in the new variety than in the old -- corresponds to a decrease in b or an increase in a. This is as it should be. Greater quantities of the old variety than of the new could be produced, other things equal.

in factor usage must be the same for all factors.[1] This is obviously an extremely restrictive condition.

Unfortunately, this rather negative conclusion is as far as we have been able to take the analysis. What adjustments should be made in the case of more general and realistic quality changes remains to be studied.[2]

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[1] A similar remark obviously holds if the new variety embodies fewer resources than the old and the price is to be adjusted upward.

[2] The cases for the cost-of-living index studied in [1], unrealistic as they were there, are hopelessly so in the present context. They would involve a quality change in the first good which, from the point of view of resource utilization, were equivalent to a Hicks-neutral technological change in the production function of some other good. Further, if constant returns were imposed, that change would have to be independent of the amount of production of the first good, which is ridiculous. A wholly different approach is clearly required.

R E F E R E N C E S

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