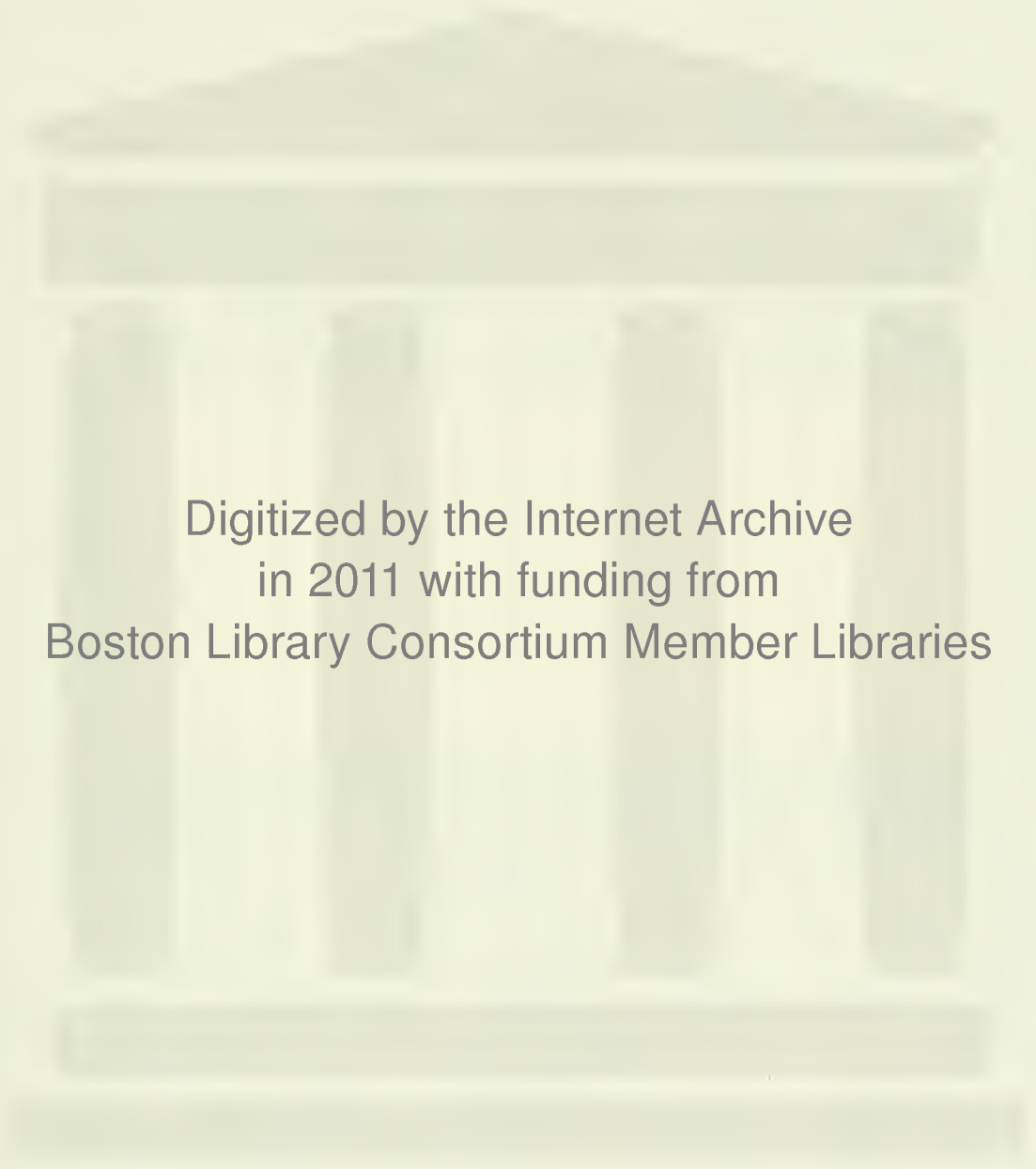


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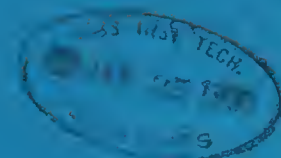




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A THEORY OF DYNAMIC OLIGOPOLY, II:

PRICE COMPETITION

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Jean Tirole

No. 373

April 1985

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A Theory of Dynamic Oligopoly, II:

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by

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April 1985

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Introduction

Modeling price competition has posed a major challenge for economic research ever since Bertrand (1883). Bertrand showed that, in a market for a homogeneous good where two or more firms sell at constant cost and set prices simultaneously, the equilibrium price is competitive, i.e., equal to marginal cost. This classic result seems to contradict observation in two ways. First, in markets with few sellers, firms apparently do not typically sell at marginal cost. Second, even in periods of technological and demand stability, oligopolistic markets are not always stable. Prices may fluctuate, sometimes wildly.

Of course, one reason for these discrepancies between theory and fact is that the Bertrand model is static, whereas dynamics may be an important ingredient of actual price competition.¹ Indeed, two classic concepts in the industrial organization literature, the Edgeworth cycle and the kinked demand curve equilibrium, offer dynamic alternatives to the Bertrand model.

The Edgeworth Cycle : According to this story, firms undercut each other successively to increase their market share (price war phase) until the war becomes too costly, at which point some firm increases its price. The other firms then follow suit (relenting phase), after which price cutting begins again. The market price thus evolves in cycles. The idea of price movements is due to Edgeworth (1925), who, in his criticism of Bertrand, showed that a static price equilibrium does not in general exist when firms face capacity

1. Another possible explanation for lack of perfect competition - indeed, the most common theoretical one - is that products of different firms are not perfect substitutes. Alternatively, as Edgeworth (1925) suggested, firms may be capacity-constrained.

constraints. His resolution of this nonexistence problem was the cycle.

The Kinked Demand Curve (Hall and Hitch (1939), Sweezy (1939)): In contrast to the Edgeworth cycle, the market price is stable in the long run. A price (the "focal" price) above marginal cost is sustained by each firm's fear that, if it undercuts, the other firms will follow suit. A firm has no incentive to charge more than the focal price because, in that case, it believes that the other firms will not follow.

Despite their long history, the Edgeworth cycle and kinked demand curve have received for the most part only informal treatments. The primary purpose of this paper is to provide equilibrium foundations for these two types of dynamics.

The basis of our analysis is a model of duopoly with alternating moves described in Section 2. In this model firms take turns choosing prices. Thus, when a firm picks its price, it has perfect information about the current price of its rival. The fact that, after choosing a price, a firm cannot change it for two periods is meant to capture the idea of short-run commitment (see our companion piece, Maskin-Tirole (1982), for a detailed motivation of the model).

A firm maximizes the present discounted value of its profit. Its strategy is assumed to depend only on the physical state of the system (i.e., to be Markov). In our model, the state is simply the other firms's current price.

We first show through examples that an equilibrium of this model may be a price cycle² or a kinked demand curve (Section 3). In Section 4 we extend the

2. Unlike in Edgeworth's treatment, the existence of price cycles in our model does not rely on capacity constraints.

model to allow for capacity as well as price competition. We provide, also by way of examples, theoretical explanations of two other prominent market phenomena: excess capacity and market sharing.

Excess Capacity : Firms in equilibrium may hold more capacity than they actually need, even in the absence of uncertainty or varying demand. In our example, a firm accumulates capacity that, in equilibrium, it never uses so that it could react effectively should the other firm ever lower its price. Only then will the other firm be deterred from price cutting.

Market Sharing : Firms may choose not to supply the entire demand they face even if their price exceeds marginal cost. The explanation for this behavior is again to be found in dynamics. We give an example with a high-cost and a low-cost firm. In equilibrium the high-cost firm "bribes" the low-cost firm into accepting a high price by allowing it a high market share. This apparent sacrifice on the part of the higher cost firm deters its rival from undercutting. Furthermore, by contrast with conventional market-sharing agreements, it requires no monetary transfers and is purely non-cooperative.

Section 5 examines the general nature of equilibrium in our model. In particular, it establishes that any equilibrium must be either of the kinked demand type (where the market price converges in finite time to a unique focal price) or the Edgeworth cycle variety (in which the market price never settles down).

Section 6 proves that there exists a multiplicity of kinked demand curve equilibria. Specifically, it exhibits the exact range of possible equilibrium focal prices when the discount factor is near 1. This range - a closed interval containing the monopoly price - lies well above the competitive price. We go on, in Section 7, to offer an explanation for this multiplicity, as well as

to suggest an additional criterion that serves to reduce the equilibrium set dramatically. Using this criterion, "renegotiation-proofness," we investigate firms' adjustment to stochastic shifts in demand.

Section 8, which treats Edgeworth cycles, is the counterpart of Section 6 on kinked demand curves. It demonstrates, by construction, the existence of Edgeworth cycle equilibria with high discount factors and proves that, in any such equilibrium, average profit must be no less than half the monopoly level.

One restrictive feature of the analysis through Section 8 is that it relies on a model where firms' relative timing is exogenous. Accordingly, in Section 9 we attempt to endogenize the timing structure. Thus, rather than insisting that firms alternate, we suppose that they can move in any period. Once a firm selects a price, however, it remains committed to that price for two periods, reflecting the possibility, say, that it takes time to change price lists. Our main result is the observation that, although in principle firms could move simultaneously, they will turn out in equilibrium to move alternately. Thus this result provides some basis for the attention we pay to the fixed-timing model.

The reader may be disturbed by our ostensibly arbitrary assumption in the endogenous-timing model that commitments last for two periods - why not, say, three or four? One possible justification is to imagine a continuous time model in which, whenever a firm sets a price, it is committed for an uncertain length of time determined by a Poisson process. As we show in Section 9, an equilibrium of this model is formally identical to one in the discrete time, two-period commitment framework.

We conclude in Section 10 by discussing some of the outstanding open questions in our model. To avoid interfering with the exposition, we relegate proofs to the Appendix. We attempt, however, to provide informal explanations of our results in the text.

2. The Model

In this section we describe the main features of the exogenous-timing duopoly model. For further discussion of this model, we refer the reader to our earlier paper, Maskin-Tirole (1982), which applies the model to capacity competition. Competition between the two firms ($i=1,2$) takes place in discrete time with an infinite horizon. Time periods are indexed by t ($t=0,1,2,\dots$). The time between consecutive periods is T . At time t , firm i 's instantaneous profit π^i is a function of the two firms' current prices p_t^1 and p_t^2 , but not of time: $\pi^i = \pi^i(p_t^1, p_t^2)$. We will assume that the goods produced by the two firms are perfect substitutes, and that firms share the market equally when they charge the same price. The price space is discrete, i.e., firms cannot offer prices in units smaller than, say, a penny.³ In most of the paper we assume that firms have the same unit cost c . Letting $D(\cdot)$ denote the market demand function, define

$$(1) \quad \pi(p) \equiv (p-c)D(p) \quad .$$

The total profit function $\pi(p)$ is assumed to be strictly concave. Let p^m denote the monopoly price, i.e., the value of p maximizing (1). From our

3. The reason for this restriction is to ensure that optimal reactions exist. In a static, Bertrand model, best responses to prices above marginal cost are not defined when the price space is a continuum.

assumptions

$$\pi^i(p_t^1, p_t^2) = \begin{cases} \pi(p_t^i), & \text{if } p_t^i < p_t^j \\ \pi(p_t^i)/2, & \text{if } p_t^i = p_t^j \\ 0, & \text{if } p_t^i > p_t^j. \end{cases}$$

Firms discount the future with the same interest rate r ; thus their discount factor is $\delta \equiv \exp(-rT)$. Because one expects that ordinarily firms can change prices fairly quickly, we will often think of T as being small and, therefore, of δ as being close to one. Firm i 's intertemporal profit at time t is

$$\sum_{s=0}^{\infty} \delta^s \pi^i(p_{t+s}^1, p_{t+s}^2) .$$

Let us now consider the timing of price setting. In odd-numbered periods t , firm one chooses its price, which remains unchanged until period $t+2$. That is, $p_{t+1}^1 = p_t^1$ if t is odd. Similarly, firm two chooses prices only in even-numbered periods, so that $p_{t+1}^2 = p_t^2$ if t is even. Firms' strategies are assumed to only depend on the payoff-relevant state, those variables that directly enter its payoff function. In period $2t+1$, when it is firm one's turn to pick a price, the payoff relevant state is simply firm two's current price $p_{2t+1}^2 = p_{2t}^2$. Firm one's choice of price is, therefore, contingent only on p_{2t}^2 . That is, his reaction function takes the form $p_{2t+1}^1 = R^1(p_{2t}^2)$. Similarly, firm two reacts to firm one's prices according to a reaction function $R^2(\cdot)$, where $p_{2t+2}^2 = R^2(p_{2t+1}^1)$. These functions, R^1 and R^2 , are (first-order) Markov strategies and are called dynamic reaction functions. We shall allow them to be random functions.

We are interested in pairs of dynamic reaction functions that form perfect equilibria. Perfection requires that, starting in any state, a firm's dynamic reaction function maximize its present discounted profit given the other firm's reaction function. We call such a pair of strategies a Markov Perfect Equilibrium (MPE). From dynamic programming we know that to test whether strategies form a MPE it suffices to check that in each payoff-relevant state the price prescribed a firm's strategy maximizes its present discount profit assuming that, thereafter both firms adhere to their strategies. That is, it is enough to rule out profitable one-shot deviations.

Hence, $\{R^1, R^2\}$ is a MPE if, for all prices \hat{p} ,

$$(2) \quad V^1(\hat{p}) = \max_p [\pi^1(p, \hat{p}) + \delta W^1(p)],$$

$$(3) \quad W^1(\hat{p}) = E_p [\pi^1(\hat{p}, p) + \delta V^1(p)],$$

where the expectation in (3) is taken with respect to the distribution of $R^2(\hat{p})$ and the analogs to (2) and (3) hold for firm 2. The expression, $V^i(p)$ is firm i 's valuation (present discounted profit) if (a) it is about to move, (b) the other firm's current price is p , and (c) firms henceforth play according to $\{R^1, R^2\}$. The expression $W^i(p)$ is firm i 's valuation if last period it played p , the other firm is about to move and firms use $\{R^1, R^2\}$ forever more. For further discussion of Markov Perfect Equilibria and valuation functions, we refer the reader to Maskin-Tirole (1982).

Most of our results will be demonstrated for discount factors close to one, which, as we already suggested, is often a reasonable assumption for price competition. Thus, a typical proposition will hold for all δ greater than a given $\bar{\delta} < 1$. We sometimes also require the set of possible prices to be sufficiently "fine."

3. Edgeworth Cycles and Kinked Demand Curves: Examples

This section exhibits two examples of Markov Perfect Equilibria, one an "Edgeworth cycle," the other a "kinked demand curve." In both examples the market demand curve is given by $D(p) = 1-p$, and production is costless. Firms can charge any of seven prices: $p(i) = \frac{i}{6}$ for $i = 0, 1, \dots, 6$. The corresponding profits, $\pi(p(i)) = p(i)(1-p(i))$ are proportional to 0, 5, 8, 9, 8, 5, 0. The monopoly price is $p^m = p(3) = \frac{1}{2}$.

Consider the dynamic reaction function given by Table 1.

Table 1: An Edgeworth Cycle

$\pi(p)$	p	$R(p)$	
0	$p(6)$	$p(4)$	
5	$p(5)$	$p(4)$	
8	$p(4)$	$p(3)$	
9	$p(3)$	$p(2)$	
8	$p(2)$	$p(1)$	
5	$p(1)$	$p(0)$	
0	$p(0)$	$p(0)$	with probability $\alpha(\delta)$
		$p(5)$	with probability $1-\alpha(\delta)$

$$\text{where } \alpha(\delta) \equiv \frac{(3\delta^2 - 1)(1 + \delta^2 + \delta^4)}{8 + 7\delta^2 + 2\delta^4 + 3\delta^6}.$$

Claim 1 : The pair of strategies $\{R, R\}$, where R is given by Table 1, forms a MPE for discount factors close to one.

We will not prove Claim 1. To do so, it suffices to check that the strategies satisfy the dynamic programming equations (2) and (3) when the discount factor is high.

In the equilibrium of Table 1, firms undercut each other successively until the price reaches the competitive level, $p(0)$, at which point some firm eventually returns to a high price $p(5)$. Market dynamics thus consist of a price war followed by a relenting phase. This second phase is a "war of attrition" at $p(0)$ in which each firm waits for the other to raise its price (relent). One may wonder why firms attach positive probability to maintaining the competitive price, where they make no profit. The explanation is that relenting is a public good from the firms' point of view. Both firms wish to raise their prices, but each would like the other to raise its price first so as to be able to undercut it. Therefore, mixed strategies, where each firm relents with probability less than one, are quite natural as a resolution to this free-rider problem.

Notice that during the price war phase, a firm undercuts not simply to increase market share (as we will see in the next example, such a motivation may not be sufficient) but because, with good reason, it does not trust its rival. That is, it anticipates that maintaining its price will not prevent the other firm from being aggressive. In that sense, mistrust is a self-justifying attitude.

Table 1 implies that a market onlooker would observe a path of market prices resembling that in Figure 1. We should emphasize that, unlike Edgeworth, we do not require capacity constraints to obtain this cycle. Nevertheless we call this kind of price path an Edgeworth cycle.

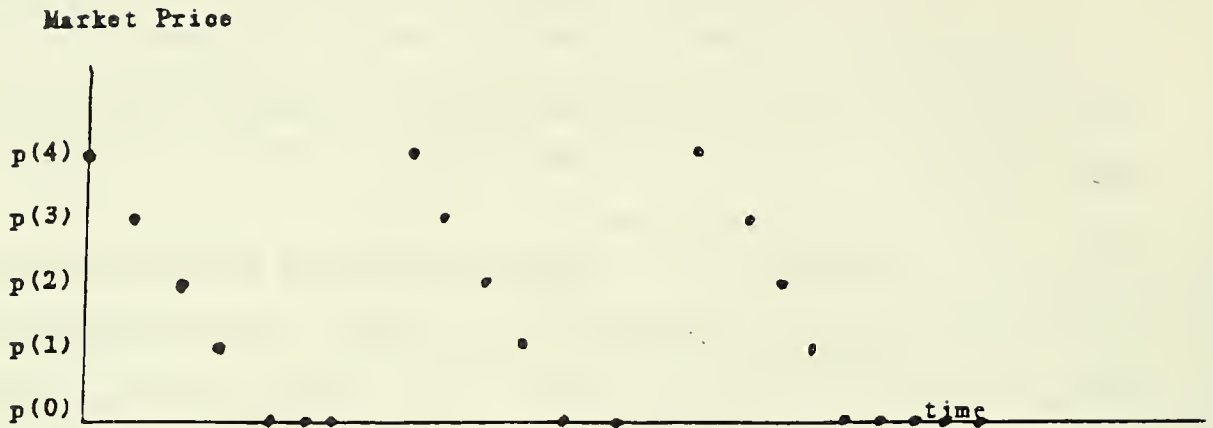


Figure 1

Suppose next that dynamic reaction functions are described by Table 2.

Table 2: A Kinked Demand Curve

p	R(p)
p(6)	p(3)
p(5)	p(3)
p(4)	p(3)
p(3)	p(3)
p(2)	p(1)
p(1)	p(1) with probability $\beta(\delta)$
	p(3) with probability $1-\beta(\delta)$
p(0)	p(3)

where $\beta(\delta) \equiv (5+\delta)/(5\delta+9\delta^2)$

Claim 2 : The pair of strategies, $\{R,R\}$, where R is given by Table 2, forms a MPE for discount factors close to one.⁴

4. To prove Claim 2, like Claim 1, it suffices to check that (2) and (3) are satisfied.

Notice that ultimately the market price reaches $p(3)$, the monopoly price, and thereafter remains there. To see why this equilibrium resembles that of the traditional kinked demand curve, suppose that the market price were $p(3)$ and that firm 1 contemplated charging a higher price. Firm 1 would predict that firm 2 would not follow suit - i.e., would keep its price at $p(3)$. Firm 1 would thus anticipate losing all its customers by raising its price and so would find such a move undesirable. Alternatively, suppose that firm 1 contemplated undercutting to $p(2)$. In that first period, its market share would rise, and its profit would increase from 4.5 to 8. However, this action would trigger a price war: firm 2, in turn, would undercut to $p(1)$. At $p(1)$ a war of attrition would begin. As in Example 1, each firm would like the other to relent (to return to $p(3)$)-first. Unlike in Example 1, however, this free-rider problem is not due to the firms' desire to undercut each other; the price will end up being $p(3)$ in any case. Rather, each firm would prefer to earn positive profit in the short run by charging $p(1)$ rather than earning zero short run profit by raising its price to $p(3)$.

Because price falls significantly in a price war, long run profits are lower than had the price remained at $p(3)$, even for firm 1, who triggered the war. Hence it is not in the long run interest of a firm to undercut the monopoly price. Because of our perfection requirement, the length of a price war must strike a balance. On the one hand, it must be long enough to deter price cutting. On the other hand, it must not be so costly that, when one firm cuts its price, the other firm is unwilling to carry on with the war and instead prefers to relent immediately. Despite these conflicting requirements, we shall see below that kinked demand curve equilibria always exist, at least for discount rates that are not too low.

Examples 1 and 2 together demonstrate that an Edgeworth cycle and kinked demand curve can coexist for the same parameter values. As we shall see below, this is quite a general phenomenon.

4. Excess Capacity and Market Sharing: Examples

Although important, price is only one dimension in which oligopolists compete. In particular, firms also make quantity decisions. In this section we illustrate two ways that quantity/capacity choice may be coupled with price competition. Because our model becomes very complex once additional choice variable are introduced, our examples are drastic simplifications.

Excess Capacity

Several explanations of excess capacity have been suggested in the literature (see, e.g., Scherer (1980)). One view proposes that excess capacity is insurance against cyclic fluctuations or uncertainty about demand. Another suggests that such capacity creates a barrier to entry.⁵ Here we suggest still another possibility, which is connected with price wars.

In the kinked demand curve equilibrium of Example 2, undercutting the monopoly price is deterred by the threat of a price war. Recall, however, that in this example firms are not capacity constrained. Once we introduce such constraints, it is easy to see that the monopoly price may not be sustainable if firm 2 has only enough capacity to supply half the demand at the monopoly price. Indeed, firm 1 will wish to undercut if it has more than this capacity, and firm 2 will not be able to retaliate effectively because it

5. This is the view advanced by Spence (1977). Dixit (1979) argues, however, that in models more plausible than that used by Spence, capacity functions as a deterrent to entry (i.e., an incumbent firm will install more capacity if threatened by entry) but is nonetheless fully used.

cannot expand output at the lower prices to reduce the first firm's market share. Thus the threat of a price war is a significant deterrent to price cutting only if firms have more capacity than they will use when price is at the monopoly level.

The example we develop illustrates this idea. First, firms choose capacities simultaneously and once-and-for-all. Firms then compete through prices as in Section 3. One technical issue, which we ignore, is specifying which firm chooses its price first. But for discount factors near one, this issue is minor.

Firms can charge two prices: the monopoly price, p^m , and a price, \hat{p} , satisfying $p^m/2 < \hat{p} < p^m$. The demand function is $D(p) = 1-p$, and production is costless. Firms can choose from two capacity levels: $q^m = \frac{1-p^m}{2}$ (the capacity necessary to supply half of the market at the monopoly price), and $\hat{q} = 1-\hat{p}$ the capacity necessary to supply the whole market at price \hat{p} (hence, $\hat{q} > 2q^m$). The unit cost of capacity is z .

If one firm charges \hat{p} and has capacity q^m whereas the other charges p^m , the former firm cannot meet all its demand, and so consumers must be rationed. For simplicity, we will assume that the residual demand for the other firm is $(1-p^m) - q^m = q^m$. This amounts to supposing that the most eager buyers pay the lowest price. If both firms charge the same price, then they both supply half the demand at this price, capacities permitting. Notice that a firm's demand does not depend on its capacity (otherwise, it would be all too easy to generate "excess capacity").

This rationing scheme, the capacities q^1 and q^2 , and the demand and cost functions define firm i 's instantaneous profit function $\pi^i(p_t^1, p_t^2, q^1, q^2)$. Firm

i's objective function can be written

$$-zq^i + \sum_{t=1}^{\infty} \delta^t \pi^i(p_t^1, p_t^2, q^1, q^2).$$

Consider the following symmetric strategies:

(4) Both firms choose capacity \hat{q} .

(5) If either firm has chosen capacity q^m , $R^i(p^m) = R^i(\hat{p}) = \hat{p}$, $i=1,2$

If both firms have chosen capacity \hat{q} , $R^i(p^m) = p^m$ and

$$R^i(\hat{p}) = \begin{cases} \hat{p} & \text{with probability } \gamma(\delta) \\ p^m & \text{with probability } 1-\gamma(\delta), \end{cases}$$

where $\gamma(\delta) \equiv ((1+2\delta)\pi(p) - \delta\pi(p^m))/(\delta\pi(p) + \delta\pi(p^m))$.

Claim 3 : For small capacity cost z and high discount factor δ , the symmetric strategies described by (4) and (5) form a MPE. In this equilibrium, moreover, firms build capacity that they never use.

The proof of Claim 3 is given in the Appendix. In equilibrium firms choose capacity \hat{q} and always charge p^m . Thus they have excess capacity $(\hat{q}-q^m)$. This example is clearly extreme. Aside from the fact that firms are limited to very few prices and capacity levels, capacities are chosen once and for all. In practice, of course, firms adjust capacities over time. Our model gets at the idea, however, that capacities are adjusted more slowly than prices. Thus we can think of short-run price competition and longer-run capacity competition. A more thorough going treatment of capacity competition (where price competition is captured by reduced-form profit functions) is considered in Maskin-Tirole (1982).

Market Sharing

We now show that it may be in the interest of a firm to ration its customers. In a static model, of course, a firm will always supply the demand it faces as long as price exceeds its marginal cost. In a dynamic framework, however, a firm may temper its rival's aggression by voluntarily giving up some of its market share.

We content ourselves with an extremely stylized example of market sharing. Firm 1 has unit cost c^1 , and firm 2 has a cost advantage: $c^1 < c^2$. The monopoly price corresponding to c^1 is p^m (which is lower than firm 2's monopoly price). Timing is the same as before: firms choose prices alternately. They can choose between two prices, p^m and p^* , where $p^* > p^m$. When a firm chooses a price, it also chooses a selling constraint, s , which is fixed for two periods. A firm's selling constraint is the maximum quantity it will be ready to supply at its chosen price (one can think of s as, say, a short-run inventory). To set a selling constraint is costless. If both firms charge the same price and are unconstrained, they each supply half the market at that price. If, however, one cannot supply half the market (because of a self-imposed constraint), the residual demand spills over to the other firm. To simplify computations we assume that, when firms offer different prices, consumers do not buy from the high price firm even if they are rationed by the low price firm. This assumption, however, is not essential to our conclusions.

Consider the following strategies (if no selling constraint is specified, a firm is unconstrained):

$$(6) \quad R^1(p^m) = p^m$$

$$R^1(p^*) = \begin{cases} p^*, & \text{if } s^2 \leq \bar{s}^2 \\ p^m, & \text{if } s^2 > \bar{s}^2 \end{cases}$$

$$(7) \quad R^2(p^m) = p^m$$

$$R^2(p^*) = p^* \text{ and } s^2 = \bar{s}^2,$$

where \bar{s}^2 is defined as follows. Letting $\bar{\theta}^2 = \bar{s}^2 / (1-p^*)$, choose $\bar{\theta}^2$ to satisfy

$$1 - \bar{\theta}^2 = (1 - \frac{\delta}{2}) \frac{\pi^1(p^m)}{\pi^1(p^*)}$$

where $\pi^1(p) = (p-c^1)(1-p)$. Notice that $\bar{\theta}^2 < 1/2$.

We can now state:

Claim 4: The strategies described by (6)-(8) form a MPE for appropriate values of the parameters.

The proof of Claim 4 is provided in the Appendix. In the equilibrium described by (6)-(8) the price p^* can be sustained even though firm 1 prefers the lower price p^m . Firm 2 can "convince" 1 to accept p^* by committing itself to a relatively small market share.⁶ We conclude that keeping one's market share comparatively low may be rational in a dynamic setting.

5. Equilibrium Price Competition

Henceforth, we revert to the symmetric model of Section 2, where price is the only choice variable. Firms can charge any of n prices, the price grid. To simplify notation we will assume that the monopoly price p^m belongs to the

6. This is an example of a "puppy dog" strategy: remain small so as not to trigger aggressive behavior by one's rival (see Rudeberg and Tirole (1984)).

price grid, and that the grid is subdivided in equal intervals of size k (this is not essential). Taking a finer grid consists of shrinking k . Some of our results will depend on the grid being 'fine,' i.e., on k being 'small enough.'

Consider (possibly mixed) strategies R^1 and R^2 . In any period the market can be in any of $2n$ states. A state specifies (a) the firm that is currently committed to a price and (b) the price to which it is committed. The Markov strategies induce a Markov chain in this set of states. Let $x_{hg}(t)$ denote the t -step transition probability between states h and g for this Markov chain. The states h and g (with h possibly equal to g) communicate if there exist positive t_1 and t_2 such that $x_{hg}(t_1) > 0$ and $x_{gh}(t_2) > 0$. An ergodic class is a maximal set of states each pair of which communicate (see, e.g., Derman (1970)). A recurrent state is a member of some ergodic class.

Rather than considering states, we focus on the market price, the minimum of the two prices in a given period. The market price does not form a Markov chain, but, abusing terminology, we shall refer nonetheless to recurrent market prices and ergodic classes of market prices. A set of prices forms an ergodic class of market prices if it is derived from an ergodic class of states.⁷ A recurrent market price is a member of an ergodic class.

We are interested in long-run properties of Markov Perfect Equilibria, i.e., in their ergodic classes. A MPE is a kinked demand curve equilibrium if it has an ergodic class consisting of a single price (a 'focal ergodic class'); it is an Edgeworth cycle equilibrium if it has an ergodic class of

7. Formally, let $P(h)$ denote the set of potential market prices when the state is h (remember that mixed strategies are allowed). A set P of prices is an ergodic set of market prices if and only if there exists a set of states H such that (i) H is an ergodic set of states and (ii) $P = \bigcup_{h \in H} P(h)$.

market prices that is not a singleton ('Edgeworth ergodic class').

A natural first question is whether a MPE can have several ergodic classes. This question is partially answered by Propositions 1 and 2.

Proposition 1 For a given price grid, a MPE cannot have two focal ergodic classes if the discount factor is close enough to 1.

Proposition 2 A MPE can not possess both a focal and an Edgeworth ergodic class.

We have not yet been able to prove that a MPE cannot possess two Edgeworth ergodic classes. But Proposition 1 and 2 show that Markov perfect equilibria can indeed be subdivided into two categories that are independent of initial conditions. In one category, the market price converges in finite time to a focal price. In the other, the market price never settles down.

We now turn to a general study of kinked demand curves and Edgeworth cycles.

6. Kinked Demand Curves: General Results

In this section we completely characterize kinked demand curve equilibria for fine grids and high discount factors. We first define two prices x and y ($x < p^m < y$) that will play a crucial role in this characterization. We choose x and y so that

$$\pi(x) > \frac{4}{7}\pi(p^m) \geq \pi(x-k) \text{ and } x < p^m$$

and

$$\pi(y) > \frac{2}{3}\pi(p^m) \geq \pi(y+k) \text{ and } y > p^m.$$

Thus profits at x and y are approximately four sevenths and two thirds of monopoly profit. We now study the set of prices that are focal prices of some MPE. This set is characterized in two steps.

Proposition 3 (necessary conditions) If p is a focal price of some MPE,

(i) $p \leq y$

(ii) for a sufficiently fine grid and a high discount factor, $p \geq x$.

Proposition 4 (sufficient conditions) For a given (sufficiently fine) grid and a price p belonging to this grid and to the interval $[x, y]$, p is the focal price of some MPE for a discount factor near one.

Propositions 3 and 4 completely characterize the set of possible focal prices for fine grids when firms place sufficient weight on the future. We should emphasize two aspects of this characterization. First, focal prices are bounded away from the competitive price (zero profit level); firms must make at least four-sevenths of the monopoly profit in equilibrium. Second, there is a nondegenerate interval of prices that can correspond to a kinked demand curve equilibrium. This multiplicity accords well with the informal story behind the kinked demand curve. As this story is usually told, if other firms imitate price cuts but do not imitate price rises, a firm's marginal revenue curve will have a discontinuity at the current price. As long as the marginal cost curve passes through the interval of discontinuity, the current price can be an equilibrium (see Scherer (1980)).

7. Multiplicity, Renegotiation, and Demand Shifts

The technical reason for the multiplicity of equilibria is that the cross partial derivative of the profit function, $\frac{\partial^2 \pi}{\partial p^1 \partial p^2}$, is not single-signed. In Maskin-Tirole (1982) we showed (in a model where firms compete in capacities) that when this cross partial is negative - so that a firm's marginal profit is declining in the action of the firm - dynamic reaction functions are negatively shaped. The explanation for this negative slope is much the same as that for the downward sloping reaction functions in the static Cournot model: if marginal profit decreases as the other firm increases its action, then the action satisfying the first order conditions for profit-maximization also decreases. As in the Cournot model, moreover, downward sloping reaction functions make the possibility of a continuum of equilibrium a nonrobust pathology.

In contrast with the Cournot model, the cross partial in our price model changes sign: when the other firm's price is sufficiently low (i.e., lower than its own price), a firm's marginal profit is zero; when the two prices are equal, marginal profit is negative (since raising one's price drives away all customers); finally, when the other firm's price is higher, a firm's marginal profit is positive if, its price is below the monopoly level. This nonmonotonicity of marginal profit gives rise to dynamic reaction functions that are decidedly nonmonotonic. In a kinked demand curve equilibrium, a firm will respond to a price cut above relenting price \underline{p} by lowering its own price. But below \underline{p} , a price cut induces it to raise its price to p^f . This nonmonotonicity is also responsible for the multiplicity of possible focal prices.

Although this multiplicity has good economic and technical explanations, it may nonetheless be disturbing to those who insist that a theory should make a precise behavioral prediction. We would like to point out, however, that there are quite reasonable properties one might require of an equilibrium (in addition to Markov perfection), which may considerably narrow the range of equilibria. One such property is "renegotiation-proofness."

If a firm undercuts the focal price in a kinked demand curve equilibrium it precipitates a price war. Now, firms' strategies in a price war form a MPE but suppose that, after the initial price cut occurred, firms could "talk things over." If, at that point, there existed an alternative equilibrium in which both firms did better than in the price war, the firms might well agree to move to that equilibrium instead. But this renegotiation could destroy the deterrent to cut prices in the first place; for, if a firm realized that lowering its price would not touch off a price war but instead would lead to an equilibrium that it found better than a war, it might find such a price cut advantageous. Hence, our focal price equilibrium would collapse.

Accordingly, we will define a MPE to be renegotiation-proof if, at any price, p , there exists no alternative, MPE that Pareto-dominates it.⁸ The requirement of renegotiation-proofness drastically reduces the equilibrium set.

Proposition 5: For a sufficiently fine grid there exists $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$ there exists a symmetric renegotiation-proof MPE when firms have discount factor δ . This MPE is a kinked demand curve equilibrium with focal price p^m . Moreover, for any $\epsilon > 0$, any symmetric renegotiation-proof MPE

8. Essentially the same criteria has been studied in the repeated games literature by Farrell (1983)

(\hat{R}, \hat{R}) must satisfy $\Pr\{\hat{R}(p^m) = p^m\} > 1-\epsilon$ for δ close enough to 1.

As the proof of Proposition 5 shows, one renegotiation-proof equilibrium is a kinked demand curve equilibrium with a monopoly focal price in which, at a price (a) greater than or equal to p^m , firms undercut to p^m , (b) between p^m and some price \underline{p} (see the appendix for the definition of \underline{p}), firms undercut to \underline{p} , and (c) less than or equal to \underline{p} , firms raise their price to p^m . The proposition asserts, moreover, that not only do renegotiation-proof equilibria exist but that all symmetric ones must be approximately kinked demand curve equilibria with monopoly focal prices when δ is near 1.

The proposition has implications for the way we might expect firms to react to shifts in demand. Suppose that the current profit function is $\pi(p)$, but that in the future, the profit function might permanently shift to $(1+\gamma)\pi(p)$ or $(1-\gamma)\pi(p)$. Let us suppose that the probability of either such change in any given period is ρ . If ρ is small enough, it will not affect current behavior at all. Thus, if (K, R) is the renegotiation-proof MPE of Proposition 5, such behavior remains in equilibrium even with the prospect of a shift in demand (but before the shift actually occurs) as long as ρ is sufficiently small (alternatively, we could simply suppose that future shifts in demand are completely unforeseen). We will suppose that after a shift occurs, firms move to the renegotiation-proof equilibria (K_-, R_-) if the shift is downward, and to (K_+, R_+) if the shift is upward. Imagine that firms begin by behaving according to (K, R) and that at some point there is a downward shift in profit to $(1-\gamma)\pi(p)$. The monopoly price, p^m before the shift exceeds that, \underline{p}^m , after the shift. Hence, if firms were at the steady-state price, i.e., at \underline{p}^m , beforehand, they can move directly to the new renegotiation-proof steady state afterwards. Thus price will fall from p^m to \underline{p}^m once and for all.

If, instead, there is an upward shift, the new monopoly price p_+^m exceeds p^m . If the shift is large, so that p^m is less than the new "relenting" price p_+ (the price below which firms return to focal price p_+^m), then firms simply raise their prices directly to p_+^m , and that is the end of the story. If, however, the shift is smaller, so that p^m exceeds p_+ , the first firm to respond will cut its price (to gain a larger market share). This will be followed by an ultimate price rise to p_+^m . Thus, comparatively small increase in demand temporarily lower prices (i.e., induce price wars) as firms scramble to take advantage of the larger demand. In the end, however, the higher demand induces a higher price.

8. Edgeworth cycles: General results

We now turn to Edgeworth cycles. We start by proving existence of an Edgeworth cycle in a general framework.

Proposition 6 Assume that the profit function $\pi(p)$ is strictly concave. For a fine grid and a discount factor near 1, there exists an Edgeworth cycle.

It may be instructive to consider the equilibrium strategies used in the proof, which is provided in the Appendix. In this equilibrium there exist two prices \underline{p} and \bar{p} satisfying $\underline{p} < p^m < \bar{p}$ and such that the optimal symmetric strategies are given by (9).

$$\begin{aligned}
 & \bar{p} && \text{for } p > \bar{p} \\
 & p-k && \text{for } \bar{p} \geq p > \underline{p} \\
 & c && \text{for } p \geq \underline{p} > c \\
 (9) \quad R(p) = & c \text{ with probability } \mu(\delta) && \text{for } p=c \\
 & \bar{p} + k \text{ with probability } 1-\mu(\delta) \\
 & c, && \text{for } p < c
 \end{aligned}$$

Thus, beginning at \bar{p} , the equilibrium involves a gradual price war until an intermediate price, p , is reached at which point the firms undercut to the competitive price where each firm tries to "induce" the other firm to relent first.

We now examine to the question of how low profits can be in an Edgeworth cycle. For symmetric equilibria we have the following result.

Proposition 7 For a discount factor near 1, at least one firm earns average profit no less than a quarter of monopoly profit, $\pi(p^m)$, in a MPE. Hence, in a symmetric equilibrium, this must be true of both firms.

Thus, regardless of the equilibrium, the average market price must be bounded away from the competitive price. We showed above that for a kinked demand curve equilibrium, aggregate profits per period must exceed four sevenths of the monopoly profit. This result and proposition 7 show that one should not expect low prices in equilibrium if firms place enough weight on future profit. This conclusion contrasts with the properties of a MPE in the simultaneous-move price-setting game, where profits are very close or equal to zero (Bertrand equilibrium).⁹

9. Assume that both firms are forced to play simultaneously (in odd periods, say). Then there is no payoff-relevant variable at the time firms make their decisions. Assume that the profit function is strictly concave in the firm's own price. If S^2 is the mixed strategy of firm 2, firm 1's

profit can be written $\sum_p \Pr\{S^2=p\} \pi^1(p^1, p)$. This function has a unique

maximum or possibly two consecutive optima p^* and p^*+k . The same holds for firm 2. We then conclude that the unique equilibrium is always to set $p=c+k$.

9. Endogeneous timing

We now abandon the assumption that firms move alternately. For the moment we will continue to suppose that time is measured discretely, and so the intertemporal profit functions are the same as before. Firm 1 (firm 2) is no longer constrained to choose prices only in odd (even) periods. Nonetheless, when a firm selects a price, it remains committed to that price for two periods.¹⁰ If in any period a firm does not have a commitment pending, it is free to choose a new price. Failure to do so amounts to being out of the market for one period. Thus, in any period where it has no commitment, a firm can choose any of the n prices in the price grid or no price at all (the null action).

From the point of view of a firm that is about to move, the payoff-relevant information is whether (a) the other firm is currently committed to a price and (b) if so, which price. We continue to require that strategies be Markov, i.e., dependent only on payoff-relevant information. Thus a Markov strategy for firm i can be described by the pair $\{R^i(\cdot), S^i\}$, where $R^i(p)$ is, as before, firm i 's reaction to the price p and S^i is its action when the other firm is not currently committed to a price. Both $R^i(p)$ and S^i are random variables taking their values in the union of the price grid and the null action.

Notice that if, along the equilibrium path, a firm chooses prices according to $R^i(\cdot)$, the firms alternate in their price selections (alternating

10. More generally, we might imagine that the firm is committed for m periods. What is important is that m be greater than 1, i.e., the period of commitment should exceed the basic decision period. Our reason for concentrating on $m=2$ will emerge below, where we consider a continuous time model.

mode). By contrast, if S^i dictates i 's equilibrium behavior, firms move at the same time (simultaneously mode).

We are interested in whether the alternating structure we imposed in Sections 2-8 emerges as equilibrium behavior in our expanded model. Accordingly, we will say that a MPE (R^1, R^2) of the fixed-timing (alternating move) game is robust to endogenous timing if there exist strategies S^1 and S^2 such that

(i) $(\{R^1, S^1\}, \{R^2, S^2\})$ is a MPE of the endogenous timing game;

(ii) starting from the simultaneous mode, firms switch to the alternating mode in finite time with probability one.

Notice that because R^1 and R^2 are MPE's of the fixed timing game, they never entail choice of the null strategy. Hence, once firms reach the alternating mode, they stay there forever.

Analogously, if (S^1, S^2) is a MPE of the game in which firms are constrained to move simultaneously, we shall say that is robust to endogenous timing if there exist reaction functions R^1 and R^2 such that

(iii) $(\{R^1, S^1\}, \{R^2, S^2\})$ is a MPE of the endogeneous-timing game,

(iv) starting from the alternating mode, firms switch to the simultaneous mode in finite time with probability one.

Our principal result of this section is the observation that symmetric alternating-move but not simultaneous-move MPE's are robust.

Proposition 8: For a sufficiently fine grid and a discount factor near enough to 1, any symmetric alternating-move MPE (R, R) but no simultaneous-move MPE (S^1, S^2) is robust to endogenous timing.

In the endogenous-timing model we have been discussing, only two possible relative timings are possible: simultaneous moves or equal-spaced alternation. This is, of course, a consequence of our two-period commitment assumption. With longer commitments, asymmetric spacing would be possible. We would like to argue, however, that it is quite natural to concentrate on the two-period case when there is uncertainty about how long commitments will last.

We shall now measure time continuously and suppose that firms discount it at rate r . The instantaneous profit function $\pi^i(p^1, p^2)$ represents firm i 's flow of profit per unit time. We suppose that, when a firm chooses a price, then in any small interval Δt , the probability that its commitment will lapse is $\lambda \Delta t$, where λ is a constant. That is, commitment lengths are described by a Poisson process with parameter λ .

Notice that a Markov strategy in this model is exactly the same as in the discrete framework. The only payoff-relevant information for a firm that is about to move is whether or not the other firm is currently committed to a price and, if so, what that price is. In particular, the length of time the other firm has so far been committed is irrelevant given our Poisson assumption. Thus, as before, we can describe firm i 's behavior by the pair $(R^i(\cdot), S^i)$, where $R^i(p)$ is the action that i takes (possibly random) when the other firm is currently committed to price p , and S^i is its action when the other firm is currently uncommitted.

Suppose that the strategies $\{(R^1, S^1), (R^2, S^2)\}$ form a MPE. If firm 1 is about to choose a price (for convenience, we will implicitly assume in the following discussion that R^1 and R^2 place zero probability on the null action)

and firm 2's current price is \hat{p} , then firm 1's present discounted profit is

$$(10) \quad V^1(\hat{p}) = \max_p \{ \pi^1(p, \hat{p})\Delta t + \lambda \Delta t W^1(p) + (1 - \lambda \Delta t) e^{-r\Delta t} V^1(\hat{p}) \}.$$

Moreover, if firm 1's current price is \hat{p} and firm 2 has just reacted to \hat{p} , then firm 1's present discounted profit is

$$(11) \quad W^1(\hat{p}) = E_p \{ \pi^1(\hat{p}, p)\Delta t + \lambda \Delta t V^1(p) + (1 - \lambda \Delta t) W^1(\hat{p}) e^{-r\Delta t} \},$$

where p is distributed according to $K^2(\hat{p})$. Equations (10) and (11) can be rewritten as

$$(12) \quad V^1(\hat{p}) = \max_p \{ \pi^1(p, \hat{p}) / (\lambda + r) + \lambda W^1(p) / (\lambda + r) \}$$

$$(13) \quad W^1(\hat{p}) = E_p \{ \pi^1(\hat{p}, p) / (\lambda + r) + \lambda V^1(p) / (\lambda + r) \}$$

But notice that if we replace $\pi^1(p^1, p^2) / (\lambda + r)$ by $\bar{\pi}^1(p^1, p^2)$ and $\lambda / (\lambda + r)$ by $\bar{\delta}$ in (12) and (13), these equations have exactly the same form as (2) and (3).

Hence our continuous-time model with uncertain commitment lengths formally reduces to the discrete-time, two-period commitment model, and all the analysis for the latter model carries over to the former.

10. Open Questions

Even though Propositions 1 through 8 tell us quite a bit about the nature of equilibrium in our model, several important questions remain outstanding. First, as mentioned after Proposition 2, we do not yet know whether an equilibrium can have more than one ergodic class, although we strongly suspect that it cannot.

Second, we believe that the structure of Edgeworth cycle equilibria can be made more precise. As currently defined, an Edgeworth cycle is simply an equilibrium without a focal price. We conjecture that in any Edgeworth cycle with a sufficiently fine grid, there exist prices \bar{p} and \underline{p} ($\underline{p} < \bar{p}$) such that,

for any $p > \bar{p}$, $\underline{p} \leq R^i(p) \leq p$ and $R^i(p) \leq \bar{p}$; and, for any $p \leq \underline{p}$, $R^i(p) \geq \bar{p}$.

Similarly, in any kinked demand curve equilibrium, there ought to be a price $\underline{p} < p^f$ such that, for all $p \in (\underline{p}, p^f)$, $\underline{p} \leq R^i(p) \leq p$, and for $p \leq \underline{p}$, $R^i(p) \geq p^f$.

To date we have examined kinked demand curve equilibria only in the extreme cases where $\delta=0$ and $\delta=1$. In the former case, where there is effectively no future, equilibrium reduces to Bertrand perfect competition. In the latter case, as we have seen, the focal prices are bounded well away from marginal cost. Intuitively, it seems plausible that the band of possible focal prices should be monotonically increasing in δ , but this intuition remains to be confirmed.

Finally, we have supposed in our endogenous timing model that, when a firm's price commitment expires, it is out of the market if it fails to set a new price (which could be equal to the old one). In some circumstances, it may be more realistic to suppose that if a firm sets a price, that price remains in effect until explicitly changed. That is, a firm, as before, remains committed to a price for a certain length of time, but when that commitment elapses, continues to charge that price until it selects a new one. Actually, we suspect that this modification will not affect the set of equilibrium possibilities (at least, the kinked demand curves), but this must still be investigated.

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Appendix

Claim 3: For small capacity cost z and high discount factor δ , the symmetric strategies described by (4) and (5) form a MPE. In this equilibrium, moreover, firms build capacity that they never use.

Proof: First assume that both firms have capacity q^m . Then from our assumption on rationing each firm faces demand of at least q^m , regardless of prices. Thus a firm will charge p^m . Although profits depend on which firm moves first, each firm's profit is approximately $p^m q^m / 2(1-\delta)$ when δ is close to 1.

Assume next that firm 1 has capacity \hat{q} and firm 2 has capacity q^m . We must check that always playing \hat{p} is an optimal strategy for both firms. Clearly firm 2 could not do better by raising its price to p^m , since it would lose market share completely if firm 1 charged \hat{p} . If firm 1 charged p^m some period, firm 2 would again be better off charging \hat{p} rather than p^m : its payoff in the two periods before it moved again would be $\Pi(\hat{p})(1+\delta/2)$ rather than $\Pi(p^m)/2$, and the former is higher. As for firm 1, if firm 2 charged p^m , 1's gain from charging \hat{p} rather than p^m would be $\hat{p}(\hat{q} + \delta(\hat{q} - q^m)) - (\Pi(p^m)/2)[1+\delta] > (3/16)[1+\delta/2 - (1/8)[1+\delta]] > 0$. If firm 2 charged \hat{p} , firm 1 would also be better off playing \hat{p} ; by doing so rather than charging p^m , it would gain

$(\hat{q}-q^m)\hat{p}-q^m p^m = (1-\hat{p}-1/4)\hat{p}-1/8 > 0$, since $(1/2)\hat{p} > 1/8$. Profit is approximately (for δ close to 1) $(\hat{q}-q^m)\hat{p}/(1-\delta)$ for firm 1 and $q^m \hat{p}/(1-\delta)$ for firm 2.

Assume, finally, that both firms have capacity \hat{q} . Then, regardless of the prices they choose, they will never be capacity constrained. Checking that p^m is a focal price is then trivial. For δ close to 1, each firm's profit is (approximately) $q^m \hat{p}/(1-\delta)$.

To see that choosing capacity \hat{q} is an equilibrium strategy for small values of z , note that deviating from \hat{q} to q^m changes a firm's net profit by

$$z[\hat{q}-q^m] - \frac{q^m [p^m - \hat{p}]}{(1-\delta)},$$

which, indeed, is negative for small z .

Q.E.D.

Claim 4: The strategies described by (6)-(8) form a MPE for appropriate values of the parameters.

Proof: We first verify that firm 1's specified strategy is optimal, given that of firm 2. Notice first that it cannot be optimal for firm 1 to set a supply constraint because such a

constraint would not influence firm 2's behavior and would only reduce firm 1's profit. Assume that firm 2 has chosen p^* and supply constraint z^2 (which is equivalent to restricting itself to a fraction (i.e., market share), $\theta^2 = z^2/(1-p^*)$, of $\Pi^2(p^*)$). If firm 1 undercuts to p^m it gets

$$\Pi^1(p^m) + \delta \frac{\Pi^1(p^m)}{2(1-\delta)},$$

whereas by choosing p^* , it gets

$$(1-\theta^2)\Pi^1(p^*) + (1-\theta^2) \frac{\delta\Pi^1(p^*)}{(1-\delta)},$$

since, according to (4), firm 2 will choose price p^* and market share θ^2 in response. From equation (4), firm 1 should undercut if and only if $\theta^2 > \theta^2$, as specified. Now assume that firm 2 has chosen p^m . If it conforms to (3), firm 1's payoff is then $\Pi^1(p^m)/2(1-\delta)$. If instead of playing p^m it chooses p^* , it obtains (using (4)):

$$\begin{aligned} \frac{\delta(1-\theta^2)\Pi^1(p^*)}{1-\delta} &= \frac{\delta}{1-\delta} \left(1 - \frac{\delta}{2}\right) \Pi^1(p^m) \\ &< \frac{\Pi^1(p^m)}{2(1-\delta)}. \end{aligned}$$

Hence, it is optimal for firm 1 to play according to (3).

Consider firm 2's behavior. If firm 1 has charged p^m , then if 2 conforms to (4), its payoff is $\Pi^2(p^m)/2(1-\delta)$. If instead it chooses p^* and market share θ^2 (we shall see below that

θ^{-2} is optimal given p^*), it obtains

$$\delta \frac{\theta^{-2} \Pi^2(p^*)}{(1-\delta)} .$$

Thus we must find parameter values such that

$$(a1) \quad \frac{\Pi^2(p^m)}{2(1-\delta)} \geq \frac{\delta \theta^{-2} \Pi^2(p^*)}{1-\delta} .$$

Assume now that firm 1 has played p^* . If firm 2 plays p^m , its profit is

$$\Pi^2(p^m) + \delta \frac{\Pi^2(p^m)}{2(1-\delta)} .$$

If it plays p^* , it should either constrain itself to market share θ^{-2} (to prevent firm 1 from undercutting) or not constrain itself at all. If θ^1 is firm 1's market share (in equilibrium $\theta^1=1/2$, but we must also consider the possibility that firm 1 constrains itself), firm 2's payoff when it does not constrain itself is

$$(1-\theta^1) \Pi^2(p^*) + \delta^2 \frac{\Pi^2(p^m)}{2(1-\delta)} .$$

If it constrains itself to θ^{-2} , it obtains

$$\theta^{-2} \frac{\Pi^2(p^*)}{1-\delta} .$$

Thus we need to find parameter values such that

$$\frac{-2}{\Theta} \frac{\Pi^2(p^*)}{1-\delta} \geq \max \left\{ \Pi^2(p^m) \left(1 + \frac{\delta}{2(1-\delta)}\right), \Pi^2(p^*) + \delta^2 \frac{\Pi^2(p^m)}{2(1-\delta)} \right\}.$$

Notice that if p^* is close to p^m and δ is not too small, then

$$\Pi^2(p^m) \left(1 + \frac{\delta}{2(1-\delta)}\right) > \Pi^2(p^*) + \delta^2 \frac{\Pi^2(p^m)}{2(1-\delta)}.$$

Therefore, it suffices to require that

$$\frac{-2}{\Theta} \frac{\Pi^2(p^*)}{1-\delta} \geq \Pi^2(p^m) \left(1 + \frac{\delta}{2(1-\delta)}\right),$$

i.e.,

$$(a2) \quad \frac{-2}{\Theta} \Pi^2(p^*) \geq \frac{2-\delta}{2} \Pi^2(p^m).$$

It remains to show that we can find p^* near p^m and δ near 1 satisfying conditions (5), (a1), and (a2). First, remember that p^m maximizes Π^1 . Thus, if p^* is near p^m , (5) implies that:

$$(a3) \quad \frac{-2}{\Theta} \sim \frac{\delta}{2}.$$

For p^* near p^m we can approximate (a1) by

$$\Pi^2(p^m) \geq 2 \frac{-2}{\Theta} \delta \Pi^2(p^*) \sim 2 \frac{-2}{\Theta} \delta \Pi^2(p^m) \left[1 + \frac{d\Pi^2(p^m)/dp}{\Pi^2(p^m)} (p^* - p^m) \right].$$

Letting $\epsilon \equiv \frac{d\Pi^2(p^m)/dp}{\Pi^2(p^m)} (p^* - p^m)$, we require that:

$$(a4) \quad 1 > 2\theta^{-2}\delta(1+\epsilon) .$$

Using ϵ , we can rewrite (a2) approximately as

$$(a5) \quad \theta^{-2}(1+\epsilon) > \frac{2-\delta}{2} .$$

We must verify that (a3)-(a5) can be satisfied simultaneously. Replacing θ^{-2} in (a4) and (a5) using (a3) we obtain

$$(a6) \quad 1 > \delta^2(1+\epsilon)$$

and

$$(a7) \quad \delta(1+\epsilon) > 2 - \delta .$$

Now choose a small value of ϵ (by taking p^* near p^m). If we then choose slightly less than $1/\sqrt{1+\epsilon}$, both (a6) and (a7) are satisfied. Thus we can find parameter values for (3)-(5) which describes a MPE.

Q.E.D.

Proposition 1: For a given price space, a MPE cannot have two focal ergodic classes if the discount factor is

close enough to 1.

To demonstrate Proposition 1, we first establish two lemmas that will prove useful below.

Lemma A: If p^f is a focal price, then $\Pi(p^f) > 0$.

Proof of Lemma A: Suppose that, starting from p^f , firm 1, say, raises its price to $p > p^f$, where $\Pi(p) > 0$ (we will handle the case where no such p exists in a moment). There exists a price $\hat{p} > p^f$ with $\Pi(\hat{p}) > 0$ such that with positive probability firm 2 reacts to p with \hat{p} . But then firm 1 can earn positive profit by also playing \hat{p} , and so raising its price to p guarantees it positive expected profit. Thus $\Pi(p^f) > 0$.

If the firm cannot raise its price to p where $\Pi(p) > 0$, then $p^f > p^m$ and $\Pi(p^f) \leq 0$. In this case, however, the firm can always undercut and make a positive profit.

Q.E.D.

For an equilibrium pair of dynamic reaction functions (R^1, R^2) , a semi-focal price is a price p such that p is in the support of both $R^1(p)$ and $R^2(p)$.

Lemma B: A firm never reacts to a price above a focal or semi-focal price, p^f , by undercutting to a price $\hat{p} < p^f$ if $\Pi(p^f) > 0$.

Proof of Lemma B: Let p^f be a (semi-)focal price. Assume that

firm i reacts to $p > p^f$ by charging $\hat{p} < p^f$. Letting $W^i(p)$ denote the valuation of firm i when it has just played p , i.e., its present discounted profit when firms play their equilibrium strategies (see Maskin-Tirole (1982)), we must have

$$(a8) \quad \Pi(\hat{p}) + \delta W^i(\hat{p}) \geq \Pi(p^f) + \delta W^i(p^f),$$

since firm i could have undercut to p^f . But p^f is a semi-focal price. Thus, firm i does not gain by undercutting to \hat{p} when the other firm charges p :

$$(a9) \quad \Pi(\hat{p}) + \delta W^i(\hat{p}) \leq \frac{\Pi(p^f)}{2} + \delta W^i(p^f),$$

But (a8) and (a9) are inconsistent if $\Pi(p^f) > 0$.

Q.E.D.

Proof of Proposition 1:

Consider a fixed price space. First note that if $\Pi(p_1) \neq \Pi(p_2)$, p_1 and p_2 cannot both be focal prices for a given MPE if δ is sufficiently close to 1, as it would be in either firm's interest to jump to the high profit from the low profit focal price. Assume therefore that p_1 and p_2 are focal prices for which $\Pi(p_1) = \Pi(p_2)$. If $p_1 < p_2$, strict quasi-concavity of the profit function implies that $p_1 < p^m < p_2$.

If, beginning from p_2 , firm i , say, undercuts to p^m , Lemma B implies that the other firm will react by choosing a price greater than or equal to p_1 . Thus firm i 's present discounted profit starting two periods hence is at least $\delta^2 \Pi(p_1)/2(1-\delta)$ (since present discounted profit is nondecreasing in the other firm's current price). Therefore, firm i can guarantee itself

$$\Pi(p^m) + \delta^2 \frac{\Pi(p_1)}{2(1-\delta)}$$

by undercutting to p^m from p_2 . But

$$\Pi(p^m) > \frac{\Pi(p_2)}{2}(1+\delta) .$$

Therefore p_2 cannot be a focal price.

Q.E.D.

Proposition 2: A MPE cannot possess both a focal and an Edgeworth ergodic class.

Proof: Assume that a MPE has a focal price p^f and an ergodic Edgeworth class P . Notice that $p^f \notin P$. Let \bar{p} be the greatest element in P . Because \bar{p} is recurrent, there exists some price $p \in P$ with $p < \bar{p}$ such that some firm reacts to p by playing \bar{p} with positive probability.

First assume that $\bar{p} > p^f$. If there exists a price in P less than p^f , then starting above p^f , it must be optimal for some firm to undercut below p^f , contradicting Lemma B. Therefore the set P is entirely above p^f . Now assume that firm i , say, reacts to $p \in P$ by playing \bar{p} with positive probability. Because it could have instead undercut to p^f , we have

$$(a10) \quad \delta W^i(\bar{p}) \geq \Pi(p^f) + \delta \frac{\Pi(p^f)}{2(1-\delta)},$$

where $W^i(\bar{p})$ denotes firm i 's valuation (present discounted profit) when it has just played \bar{p} . By Lemma A, (a10) implies

$$(all) \quad \delta W^i(\bar{p}) > \frac{\Pi(p^f)}{2(1-\delta)}.$$

But (all) implies that p^f is not a focal price since it would be in the interest of firm i to raise its price from p^f to \bar{p} .

Assume, therefore, that $\bar{p} < p^f$. Assume that firm i reacts

to $p \in P$ by playing \bar{p} with positive probability. Since it could have reacted by playing p^f instead, we have

$$(a12) \quad \delta W^i(\bar{p}) \geq \frac{\delta \Pi(p^f)}{2(1-\delta)} .$$

But because p^f is a focal price, firm i does not gain by lowering its price from p^f to \bar{p} , i.e.,

$$(a13) \quad \frac{\Pi(p^f)}{2(1-\delta)} \geq \Pi(\bar{p}) + \delta W^i(\bar{p}) .$$

Inequalities (a12) and (a13) imply that

$$(a14) \quad \Pi(p^f) \geq 2\Pi(\bar{p}) ,$$

which in turn implies that \bar{p} is lower than p^m (otherwise, from the strict quasi-concavity of Π , \bar{p} would exceed p^f). Now, in the Edgeworth ergodic class P , price is never above \bar{p} . Therefore, starting from a price in P , a firm's present discounted profit is no higher than $\Pi(\bar{p})/2(1-\delta)$. Letting $V^i(\bar{p})$ denote the valuation of firm i when the other firm has just played \bar{p} (i.e., firm i 's present discounted profit when the current market price is \bar{p}), we have, using (a14),

$$(a15) \quad V^i(\bar{p}) + W^j(\bar{p}) < \frac{\Pi(\bar{p})}{1-\delta} \leq \frac{\Pi(p^f)}{2(1-\delta)} .$$

When firm j has just charged \bar{p} , firm i can guarantee itself $\delta \Pi(p^f)/2(1-\delta)$ by charging p^f . Therefore,

$$(a16) \quad v^i(\bar{p}) \geq \frac{\delta \Pi(p^f)}{2(1-\delta)} .$$

Similarly firm j , rather than charging \bar{p} , could have guaranteed itself $\delta \Pi(p^f)/2(1-\delta)$ by raising its price to p^f . Therefore,

$$(a17) \quad \delta w^j(\bar{p}) \geq \frac{\delta \Pi(p^f)}{2(1-\delta)} .$$

Adding (a16) and (a17), we obtain

$$v^i(\bar{p}) + w^j(\bar{p}) \geq \frac{\Pi(p^f)}{(1-\delta)} ,$$

contradicting (a15).

Q.E.D.

Proposition 3 (Necessary Conditions): If p is a focal price of some MPE, then

$$(i) \quad p \leq y$$

(ii) for a fine grid and a high discount factor,

$$p \geq x,$$

where $y > p^m$, $\Pi(y) = (2/3)\Pi(p^m)$; and $x < p^m$, $\Pi(x) = (4/7)\Pi(p^m)$.

To prove Proposition 3, we make use of the following lemma, which holds for any semi-focal price.

Lemma C: If p^f is a semi-focal price and $p > p^f$, then for $i = 1, 2$, the support of $R^i(p)$ lies in the interval $[p^f, p]$.

Proof: We already know from Lemma B that firms never undercut from above to below a semi-focal price. We now show that starting from above p^f a firm does not want to raise its price. Imagine, however, that for some i , there exists $\hat{p} \in R^i(p)$ with $\hat{p} > p > p^f$. Since firm i could undercut from p to p^f , we have:

$$\delta W^i(\hat{p}) \geq \Pi(p^f) + \delta W^i(p^f),$$

which implies that

$$(a18) \quad \delta W^i(\hat{p}) > \frac{\Pi(p^f)}{2} + \delta W^i(p^f).$$

But (a18) implies that p^f is not a semi-focal price, since it tells us that at p^f it is in firm i 's interest to raise the price to \hat{p} .

Q.E.D. :

Proof of Proposition 3:

(i) We first show that, if p^f is a focal price, $p^f \leq y$. Assume $p^f > y$. At p^f each firm has payoff $\Pi(p^f)/2(1-\delta)$. This payoff

must exceed that obtained by undercutting. Each firm can always guarantee itself

$$\Pi(p^m) + \delta^3 \frac{\Pi(p^f)}{2(1-\delta)}$$

by undercutting to p^m and returning to p^f thereafter. Therefore

$$\Pi(p^m) \leq \frac{\Pi(p^f)}{2} [1 + \delta + \delta^2] < \frac{3\Pi(p^f)}{2},$$

a contradiction.

(ii) Let us now show that for fine grids and high discount factors, $p^f \geq x$.

Assume to the contrary that $\Pi(p^f) < (4/7)\Pi(p^m)$. Define \bar{p} in (p^f, p^m) to satisfy

(a19) \bar{p} belongs to the price grid

(a20) $\Pi(\bar{p}) > \Pi(p^f)(1 + \delta/2)$

(a21) $\Pi(\bar{p}-k) < \Pi(p^f)(1 + \delta/2)$,

where k is the interval between prices. It is easy to see that for sufficiently fine grids, \bar{p} exists and is unique. From Lemma C we know that firms always weakly undercut prices above p^f . We shall consider two cases that depend on whether or not firms

strictly undercut all prices between p^f and \bar{p} .

Case (a): Both firms strictly undercut all prices $p \in (p^f, \bar{p}]$ (i.e., for $i = 1, 2$ and $p \in (p^f, \bar{p}]$, the support of $R^i(p)$ is contained in $[p^f, p)$).

By induction starting with price $p^f + k$, one can show using (a21) that at any $p \in (p^f, \bar{p}]$ each firm undercuts to p^f directly (otherwise, its payoff is at most $\Pi(p-k) + \delta^2 \Pi(p^f) / 2(1-\delta)$, which is less than $\Pi(p^f) + \delta \Pi(p^f) / 2(1-\delta)$ by definition of \bar{p}).

Consider a firm's behavior when the current price is $\bar{p} + k$. By (a20) a firm will not undercut directly to p^f . Thus there are four possibilities:

$$(1) \quad R^i(\bar{p} + k) = \bar{p} + k \text{ for some } i.$$

In this case, firm $j (\neq i)$ can ensure that the price remains at $\bar{p} + k$ forever, guaranteeing it a profit higher than at p^f . Hence, at p^f , firm j would gain by raising the price to $\bar{p} + k$, a contradiction. Thus we can rule out this case.

(2) $R^i(\bar{p} + k) = \bar{p}$ and the support of $R^j(\bar{p} + k) = \{\bar{p}, \bar{p} + k\}$, i.e., firm j randomizes between \bar{p} and $\bar{p} + k$.

Firm j must be indifferent between playing \bar{p} and $\bar{p} + k$:

$$\Pi(\bar{p}) + \frac{\delta^2 \Pi(p^f)}{2(1-\delta)} = \frac{\Pi(\bar{p} + k)}{2} + \delta^2 \left(\Pi(p^f) + \frac{\delta \Pi(p^f)}{2(1-\delta)} \right).$$

But $\Pi(\bar{p}) > \Pi(\bar{p} + k) / 2 + \delta^2 \Pi(p^f) / 2$, at least if the grid is not too coarse. Therefore, this case can also be ruled out.

$$(3) \quad R^1(\bar{p} + k) = R^2(\bar{p} + k) = \bar{p}$$

(4) The support of $R^i(\bar{p}+k) = \{\bar{p}, \bar{p}+k\}$ for $i=1,2$.

Notice that in this last case the two firms must randomize between \bar{p} and $\bar{p}+k$ with the same probability distribution. Otherwise, they could not both be indifferent between \bar{p} and $\bar{p}+k$ (as we saw, at and below \bar{p} the equilibrium strategies are symmetric).

Now consider the firms' behavior when the current price is $\bar{p}+2k$. We first show that neither firm wants to undercut to \bar{p} . If it does so undercut, it gets

$$(a22) \quad \Pi(\bar{p}) + \delta^2 \frac{\Pi(p^f)}{2(1-\delta)} .$$

If, however, it undercuts to $\bar{p}+k$, its opponent will play \bar{p} or $\bar{p}+k$, the following period, and the first firm gets at least

$$\Pi(\bar{p}+k) + \delta^2 \left(\Pi(p^f) + \frac{\delta \Pi(p^f)}{2(1-\delta)} \right) ,$$

which is greater than (a22). Thus a firm's reaction to $\bar{p}+2k$ must be $\bar{p}+k$ or $\bar{p}+2k$. Consider case (3) above. There are four subcases:

(3-1) $R^i(\bar{p}+2k) = \bar{p}+2k$ for some i .

This can be ruled out the same way we eliminated case 1.

(3-2) $R^i(\bar{p}+2k) = \bar{p}+k$ for some i and the support of $R^j(\bar{p}+2k) = \{\bar{p}+k, \bar{p}+2k\}$.

For firm j to be indifferent between $\bar{p}+k$ and $\bar{p}+2k$, we need

$$\Pi(\bar{p}+k) + \delta^2 (\Pi(p^f) + \frac{\delta \Pi(p^f)}{2(1-\delta)}) = \frac{\Pi(\bar{p}+2k)}{2} + \delta^2 \Pi(\bar{p}) + \frac{\delta^4 \Pi(p^f)}{2(1-\delta)},$$

that is,

$$(a23) \quad \Pi(\bar{p}+k) + \Pi(p^f) \left(\delta^2 + \frac{\delta^3}{2} \right) = \frac{\Pi(\bar{p}+2k)}{2} + \delta^2 \Pi(\bar{p}).$$

Assuming that the grid is not too coarse and using the definition of \bar{p} , however, we may conclude that (a23) cannot hold. Therefore subcase (3-2) can be ruled out.

$$(3-3) \quad R^1(\bar{p}+2k) = R^2(\bar{p}+2k) = \bar{p}+k$$

$$(3-4) \quad \text{The support of } R^i(\bar{p}+2k) = \{\bar{p}+k, \bar{p}+2k\}.$$

In case (3-4), the two firms must again randomize with the same probability distribution.

Consider now case (4) and its corresponding four subcases:

$$(4-1) \quad R^i(\bar{p}+2k) = \bar{p}+2k.$$

Like (3-1), this case can be ruled out.

$$(4-2) \quad R^i(\bar{p}+2k) = \bar{p}+k \text{ for some } i \text{ and the support of } R^j(\bar{p}+k) = \{\bar{p}+k, \bar{p}+2k\}.$$

Firm j 's randomizing behavior in (4-2) requires that

$$(a24) \quad \Pi(\bar{p}+k) + \delta W^j(\bar{p}+k) = \frac{\Pi(\bar{p}+2k)}{2} + \delta^2 V^j(\bar{p}+k).$$

But recall that, in case (4), firm j is indifferent at $\bar{p}+k$ between playing \bar{p} and $\bar{p}+k$. Therefore,

$$(a25) \quad v^j(\bar{p}+k) = \Pi(\bar{p}) + \delta \frac{2\Pi(p^f)}{2(1-\delta)} = \frac{\Pi(\bar{p}+k)}{2} + \delta W^j(\bar{p}+k) .$$

Substituting into (a24) the values of $v^j(\bar{p}+k)$ and $W^j(\bar{p})$ obtained from (a25), one easily obtains a contradiction, viz., that firm j is strictly better off undercutting to $\bar{p}+k$.

$$(4-3) \quad R^1(\bar{p}+2k) = R^2(\bar{p}+2k) = \bar{p}+k$$

(4-4) The support of $R^i(\bar{p}+2k) = \{\bar{p}+k, \bar{p}+2k\}$ for $i=1,2$, and firms randomize with the same probability distribution.

Finally, consider a firm's behavior when its opponent's price is $\bar{p}+3k$. We will show that in cases (3-3), (3-4), and (4-4) a firm will not undercut to a price below $\bar{p}+2k$.

Case (3-3): If a firm undercuts to $\bar{p}+k$ (it does not want to undercut to less than $\bar{p}+k$ since at $\bar{p}+2k$ it strictly prefers to undercut to $\bar{p}+k$ rather than undercutting to less than $\bar{p}+k$), it gets

$$\Pi(\bar{p}+k) + \frac{\delta^2 \Pi(p^f)}{2(1-\delta)} .$$

If it undercuts to $\bar{p}+2k$, it obtains

$$\Pi(\bar{p}+2k) + \delta^2 \Pi(\bar{p}) + \delta^4 \frac{\Pi(p^f)}{2(1-\delta)} .$$

But $\Pi(\bar{p}+2k) > \Pi(\bar{p}+k)$ and $\Pi(\bar{p}) > \Pi(p^f)(1+\delta)/2$. Therefore, firms never want to undercut to less than $\bar{p}+2k$.

Cases (3-4) and (4-4): In these two cases, $\bar{p}+2k$ is a semi-focal price. Therefore, for $i=1,2$,

$$\Pi(\bar{p}+k) + \delta W^i(\bar{p}+k) = \frac{\Pi(\bar{p}+2k)}{2} + \delta W^i(\bar{p}+2k) ,$$

which implies that

$$\Pi(\bar{p}+k) + \delta W^i(\bar{p}+k) < \Pi(\bar{p}+2k) + \delta W^i(\bar{p}+2k) .$$

Therefore, firms do not want to undercut from $\bar{p}+3k$ to $\bar{p}+k$.

We can now establish that, in cases (3-3), (3-4), and (4-4), p^f cannot be a focal price after all. Assume that firm j has just played p^f . Firm i 's present discounted profit from that point on is $\Pi(p^f)/2(1-\delta)$. But if firm i raises its price to $(\bar{p}+3k)$, it can get at least

$$\delta^2 \Pi(\bar{p}+k) + \delta^4 \left[\Pi(p^f) + \frac{\delta \Pi(p^f)}{2(1-\delta)} \right] ,$$

since (a) firm j will respond with a price no lower than $\bar{p}+2k$, after which (b) firm i can lower its price to $\bar{p}+k$, which, in turn, will (c) induce firm j to choose a price no lower than \bar{p} . The difference between these two payoffs is

$$D = \delta^2 \Pi(\bar{p}+k) + \delta^4 \frac{\Pi(p^f)}{2} - (1+\delta+\delta^2+\delta^3) \frac{\Pi(p^f)}{2} .$$

Note that, although \bar{p} depends on δ , $\Pi(\bar{p}+k) - \Pi(\bar{p})$ is bounded below by some $a > 0$ for a given k . Thus, from the definition of \bar{p} , we obtain

$$D \geq \delta^2 a + \Pi(p^f) \left[\frac{\delta^4}{2} - \frac{1}{2} - \frac{\delta}{2} + \frac{\delta^2}{2} \right] ,$$

which is positive for δ close to 1. Thus p^f is not a focal price, and so cases (3-3), (3-4), and (4-4) are impossible.

Next consider case (4-3). Suppose that, say, firm i contemplates raising its price from the focal price to above the monopoly price. We claim that firm j will not undercut to $\bar{p}+k$ (or a fortiori to a price below $\bar{p}+k$): If firm j undercuts to $\bar{p}+k$, it gets

$$(a26) \quad \Pi(\bar{p}+k) + \delta W^j(\bar{p}+k) = \frac{\Pi(\bar{p}+k)}{2} + V^j(\bar{p}+k),$$

where we have used the fact that $\bar{p}+k$ is a semi-focal price. If firm j instead undercuts to the monopoly price, it gets at least

$$(a27) \quad \Pi(p^m) + \delta^2 V^j(\bar{p}+k),$$

since, from Lemma B, firm i will respond by choosing a price no lower than $\bar{p}+k$. Using

$$V^j(\bar{p}+k) = \Pi(\bar{p}) + \delta^2 \frac{\Pi(p^f)}{2(1-\delta)},$$

we can express the difference between (a26) and (a27) as

$$\hat{D} = \Pi(p^m) - \frac{\Pi(\bar{p}+k)}{2} - (1-\delta^2)\Pi(\bar{p}) - \frac{(1+\delta)\delta^2}{2} \Pi(p^f).$$

For k small and δ near 1,

$$\hat{D} \approx \Pi(p^m) - \frac{\Pi(\bar{p})}{2} - \Pi(p^f) \approx \Pi(p^m) - \frac{7}{4} \Pi(p^f),$$

which is positive since $\Pi(p^f) < (4/7)\Pi(p^m)$. We thus conclude

firm j will not undercut to a price below $\bar{p}+2k$. Hence, firm i gains by raising its price from p^f , which implies that case (a) is impossible.

Case (b): For some $p \in (p^f, \bar{p}]$ there is firm i such that p is the support of $R^i(p)$.

Let p_1 be the lowest such price. Notice first that p must be a semi-focal price, i.e., p_1 is in the support of $R^j(p_1)$ as well. If not, at any $p \in (p^f, p_1)$, firm j undercuts to p^f (this can be proved by induction starting from p^f+k), and therefore firm i obtains the payoff $\Pi(p_1)/2 + \delta^2 \Pi(p^f)/2(1-\delta)$ if it plays p_1 . If instead it undercuts to p^f , it gets

$$\Pi(p^f) + \frac{\delta \Pi(p^f)}{2(1-\delta)},$$

which is clearly higher ($\Pi(p_1) \leq \Pi(\bar{p}) < (3/2)\Pi(p^f)$). Moreover, at p_1 both firms randomize between p_1 and p^f with the same probability distribution (otherwise, they would not both be indifferent between these two choices).

Let us first suppose that at prices $p \in (p_1, p^m)$ both firms react by strictly lowering their prices (we shall take up the possibility of weak undercutting below).

Consider price p_1+k . Because at p_1 a firm is indifferent between charging p_1 and p^f , it must strictly prefer to charge p_1 at p_1+k . Hence, given our strict undercutting assumption, both firms play p_1 in response to p_1+k . By induction, the reaction is

the same to any price not exceeding \bar{p}_1 , where \bar{p}_1 is defined relatively to p_1 in the same way \bar{p} is derived from p . Namely, letting $V(p_1) = V^1(p_1) = V^2(p_1) = \Pi(p^f) + \delta\Pi(p^f)/2(1-\delta) = \Pi(p_1)/2 + \delta W(p_1)$, where $W(p_1) = W^1(p_1) = W^2(p_1)$, we define \bar{p}_1 so that

(a28) \bar{p}_1 belongs to the price grid

$$(a29) \quad \Pi(\bar{p}_1) + \delta^2 V(p_1) > \Pi(p_1) + \delta W(p_1)$$

$$(a30) \quad \Pi(\bar{p}_1 - k) + \delta^2 V(p_1) < \Pi(p_1) + \delta W(p_1).$$

It is easy to show that \bar{p}_1 exists, is unique, and is lower than the monopoly price if the grid is fine and the discount factor is close to one.*

Between p_1+k and \bar{p}_1 , both firms undercut to p_1 . We claim that, at any price above \bar{p} but below p^m , firms undercut by k . From (a29) this is clear for p_1+k . We now prove it by induction for higher prices. \bar{p}_1 . Consider $p = \bar{p}_1 + 2nk$. If a firm undercuts to $p-k$, it gets

$$A = \Pi(\bar{p}_1 + (2n-1)k) + \delta^2 \Pi(\bar{p}_1 + (2n-3)k) + \dots \\ + \delta^{2n-2} (\Pi(\bar{p}_1 + k) + \delta^2 (\Pi(p_1) + \delta W(p_1))).$$

* $\Pi(-p_1) \sim \Pi(p_1)/2 + \Pi(p^f) \leq (3/4)\Pi(p^f) + \Pi(p^f) = (7/4)\Pi(p^f)$ and we assumed that $\Pi(p^f) < (4/7)\Pi(p^m)$.

If instead it undercuts to $p-2k$ (it will not undercut to less than $p-2k$ since by induction it reacts to $p-k$ by playing $p-2k$), it gets

$$B \equiv \Pi(\bar{p}_1 + (2n-2)k) + \delta^2 \Pi(\bar{p}_1 + (2n-4)k) \\ + \dots + \delta^{2n-2} (\Pi(\bar{p}_1) + \delta^2 V(p_1)) .$$

But we know that, for any q less than $2n$,

$$\Pi(\bar{p}_1 + (2n-q)k) > \Pi(\bar{p}_1 + (2n-q-1)k) ,$$

and that

$$V(p_1) = \Pi(p_1)/2 + \delta W(p_1) < \Pi(p_1) + \delta W(p_1) .$$

Thus $A > B$.

Now consider $p = \bar{p}_1 + (2n+1)k$. If a firm undercuts to $\bar{p}_1 + 2nk$ it gets

$$C \equiv \Pi(\bar{p}_1 + 2nk) + \delta^2 \Pi(\bar{p}_1 + (2n-2)k) + \delta^{2n} [\Pi(\bar{p}_1) + \delta^2 V(p_1)] .$$

If instead it undercuts to $\bar{p} + (2n-1)k$, it gets A . But again,

$$\Pi(\bar{p}_1 + (2n-q)k) > \Pi(\bar{p}_1 + (2n-q-1)k) ,$$

and, by definition of \bar{p}_1 ,

$$\Pi(\bar{p}_1) + \delta^2 v(p_1) \geq \Pi(p_1) + \delta W(p_1) .$$

Therefore $C > A$.

We conclude that if firms strictly undercut between p_1 and p^m , the price trickles down from p^m to \bar{p}_1 . In particular, if a firm raises its price from the focal price to the monopoly price, it will earn a profit near $\Pi(p^m)$ every other period for quite a long time if the grid size is small. Since half the monopoly profit greatly exceeds half the focal price profit, it therefore pays a firm to raise its price, a contradiction.

We now consider the possibility of weak undercutting. Let p_2 be the lowest price above p_1 such that, for some firm i , p_2 is in the support of $R^i(p_2)$. Then,

$$v^i(p_2) = \frac{\Pi(p_2)}{2} + \delta W^i(p_2) \geq \Pi(p_1) + \delta W(p_1) ,$$

so that

$$(a31) \quad \delta W^i(p_2) \geq \left[-\frac{\Pi(p_2)}{2} + \frac{\Pi(p_1)}{2} + \frac{\Pi(p^f)}{2} \right] + \frac{\Pi(p^f)}{2(1-\delta)} .$$

For p^f to be a focal point, firm i cannot gain by raising its price to p_2 . We therefore require $\delta W^i(p_2) \leq \Pi(p^f)/2(1-\delta)$, so that the bracketed expression in (a31) must be nonpositive:

$$(a32) \quad \Pi(p_2) \geq \Pi(p_1) + \Pi(p^f).$$

But from the definition of \bar{p}_1 , for fine grids and high discount factors,

$$(a33) \quad \Pi(\bar{p}_1) \sim \Pi(p_1)/2 + \Pi(p^f).$$

Formulas (a32) and (a33) imply that p_2 must be well above \bar{p}_1 . But we have shown that equilibrium between p_1 and p_2 consists of trickling down to \bar{p}_1 and then undercutting to p_1 . Thus, again if, at the focal price, a firm raises its price to p_2 , the price will long remain above \bar{p}_1 if the grid is fine. Hence, raising the price will be worthwhile if the discount factor is near one. This contradicts the fact that p^f is a focal price.

We conclude that p^f cannot be a focal price unless $\Pi(p^f) \geq (4/7)\Pi(p^m)$ if the grid is fine and the discount factor is close to one.

Q.E.D.

Proposition 4: (Sufficient Conditions) For a given (fine) grid and a price p belonging to this grid and to the interval $[x,y]$, p is the focal price of some MPE for a discount factor close to one.

Proof: We must show that any price in $[x,y]$ is a focal price for sufficiently high discount factors. To do this, we will consider three cases depending on the relative magnitudes of $\Pi(p^f)$ and $(2/3)\Pi(p^m)$ and of p^f and p^m , where p^f is the focal price candidate in $[x,y]$.

Case (a): $\Pi(p^f) > (2/3)\Pi(p^m)$ and $p^f \leq p^m$.

Consider the following strategy.

$$(a34) \quad R(p) = \begin{cases} p^f & \text{for } p \geq p^f \\ \underline{p} & \text{for } p^f > p > \underline{p} \\ p^f & \text{for } \underline{p} \geq p, \end{cases}$$

where $\underline{p} < p^f$ is defined by

$$(a35) \quad (1+\delta)\Pi(\underline{p}) \geq \frac{\delta\Pi(p^f)}{2} > (1+\delta)\Pi(\underline{p}-k).$$

Notice that, for a fine grid and high discount factor, profit at \underline{p} is approximately one fourth that at p^f .

We claim that the strategy pair (R,R) is an equilibrium. Note first that if undercutting p^f is worthwhile, then a firm must gain by undercutting either to p^f-k or to \underline{p} . The requirement that the latter kind of undercutting be unprofitable

can be expressed as

$$\frac{\Pi(p^f)}{2(1-\delta)} \geq \Pi(\underline{p})(1+\delta) + \frac{\delta^2 \Pi(p^f)}{2(1-\delta)},$$

which reduces to

$$\frac{\Pi(p^f)}{2}(1+\delta) \geq \Pi(\underline{p})(1+\delta),$$

and, from (a35), is clearly satisfied for δ near 1. That the former kind of undercutting is worthwhile amounts to

$$\frac{\Pi(p^f)}{2(1-\delta)} \geq \Pi(p^{f-k}) + \frac{\delta^3 \Pi(p^f)}{2(1-\delta)},$$

which simplifies to

$$\frac{\Pi(p^f)}{2} (1+\delta+\delta+\delta^2) \geq \Pi(p^{f-k}),$$

and is also obviously satisfied. If at p^f a firm raised its price, its present discounted profit would be $(\delta^2/2(1-\delta))\Pi(p^f)$, which is less than the payoff from sticking to p^f . Thus, at p^f , firms will adhere to the prescribed strategy.

At $p > p^f$ a firm does not want to undercut to a price below p^f for the same reason that undercutting at p^f is unprofitable. If it chooses a price in the interval $[p^f, p]$ its present discounted profit is at most

$$\Pi(p^m) + \frac{\delta^2 \Pi(p^f)}{2(1-\delta)},$$

whereas if it follows the prescribed strategy its profit is

$$\Pi(p^f) + \frac{\delta \Pi(p^f)}{2(1-\delta)}.$$

But the latter is bigger than the former for δ near 1 since $\Pi(p^f) > (2/3)\Pi(p^m)$.

Next, consider $p \in (p, p^f)$. The prescribed reaction to p is p , and it is clear that the optimal deviations from p can only be $p-k$ or p^f . Hence, for equilibrium, we require that

$$(a36) \quad \Pi(p)(1+\delta) + \frac{\delta^2 \Pi(p^f)}{2(1-\delta)} \geq \Pi(p-k) + \frac{\delta^3 \Pi(p^f)}{2(1-\delta)}$$

and

$$(a37) \quad \Pi(p)(1+\delta) + \frac{\delta^2 \Pi(p^f)}{2(1-\delta)} \geq \frac{\delta \Pi(p^f)}{2(1-\delta)}.$$

From the definition of p , (a37) holds. But

$$\frac{\delta \Pi(p^f)}{2(1-\delta)} = \frac{\delta}{2}(1+\delta)\Pi(p^f) + \frac{\delta^3 \Pi(p^f)}{2(1-\delta)} > \Pi(p-k) + \frac{\delta^3 \Pi(p^f)}{2(1-\delta)},$$

for δ sufficiently large, which implies that (a36) is also satisfied.

Finally, at $p \leq p$, a firm is best off returning to p^f , by definition of p . We have thus established that (R,R) is an

equilibrium.

For the other two cases, we shall merely exhibit equilibrium strategies that support the p^f in question as a focal price. The formal proofs that these strategies form equilibria are very much like that in case (a).

Case b: $\Pi(p^f) > (2/3)\Pi(p^m)$ and $p^f > p^m$.

Let

$$R(p) = \left\{ \begin{array}{l} p^f, \text{ for } p \geq p^f \\ p_1, \text{ for } p^f > p > p_1 \\ p_1, \text{ with probability } \alpha \\ \underline{p}, \text{ with probability } (1-\alpha) \\ \underline{p}, \text{ for } p_1 > p > \underline{p} \\ p^f, \text{ for } \underline{p} \geq p, \end{array} \right\} \text{ for } p = p_1$$

where $p^f > p^m > p_1 > \underline{p}$

and

$$\Pi(\underline{p})(1+\delta) \geq \delta \frac{\Pi(p^f)}{2} > \Pi(\underline{p}-k)(1+\delta)$$

$$\Pi(\underline{p})(1+\delta) + \alpha \left(\frac{\delta^2 + \delta^3}{2} \right) \Pi(p^f) = \frac{\Pi(p_1)}{2} + \alpha \left(\frac{\Pi(p_1)}{2} + \frac{\delta^2 + \delta^3}{2} \Pi(\underline{p}) \right).$$

$$\Pi(p_1) = \Pi(p^f) - \epsilon,$$

for ϵ small. We claim that (R,R) forms an equilibrium for a fine grid and high discount factor.

Case (c): $\Pi(p^f) \leq (2/3)\Pi(p^m)$ and $p^m > p^f \geq x$

Let

$$R(p) = \begin{cases} \bar{p}, & \text{if } p > \bar{p} \\ \bar{p}, & \text{with probability } \alpha \\ p^f, & \text{with probability } 1-\alpha \end{cases} \text{ if } p = \bar{p}$$

$$R(p) = \begin{cases} p^f, & \text{if } \bar{p} > p \geq p^f \\ \underline{p} & \text{if } p^f > p > \underline{p} \\ \bar{p}, & \text{if } \underline{p} \geq p \end{cases}$$

where $p^m > \bar{p} > p^f > \underline{p}$

and

$$(a38) \quad \Pi(\bar{p}) \geq \Pi(p^f)(1 + \frac{\delta}{2}) > \Pi(\bar{p}-k)$$

$$(a39) \quad V(\bar{p}) = \frac{\Pi(\bar{p})}{2} + \delta W(\bar{p}) = \Pi(p^f) + \delta \frac{\Pi(p^f)}{2(1-\delta)}$$

$$(a40) \quad \Pi(\underline{p})(1+\delta) + \delta^2 V(\bar{p}) \geq \delta W(\bar{p}) > \Pi(\underline{p}-k)(1+\delta) + \delta^2 V(\bar{p})$$

$$(a41) \quad \alpha = \frac{(2+\delta)\Pi(p^f) - \Pi(\bar{p})}{\delta\Pi(\bar{p}) + \delta\Pi(p^f)}$$

Here $V(p)$ is the valuation of a firm when its rival has just played p , and $W(p)$ is the firm's valuation when it itself has just played p .

Notice that \bar{p} as defined by (a38) exists and is unique, since $\Pi(p^f) \leq (2/3)\Pi(p^m)$. For δ large and k small, α is approximately equal to one fifth. The reader can check that for

fine grids and high discount factors the pair of strategies (R,R) forms a MPE.

Proposition 5: For a sufficiently fine grid there exists $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$ there exists a symmetric renegotiation-proof MPE when firms have discount factor δ . This MPE is a kinked demand curve equilibrium with focal price p^m . Moreover, for any $\epsilon > 0$, any symmetric renegotiation-proof MPE $\{\hat{R}, \hat{R}\}$ must satisfy $\Pr\{\hat{R}(p^m) = p^m\} > 1 - \epsilon$ for δ close enough 1.

Proof: Consider the following strategy R:

$$R(p) = \begin{cases} p^m, & p \geq p^m \\ \underline{p}, & p \in (\underline{p}, p^m) \\ p^m, & p \leq \underline{p} \end{cases}$$

where \underline{p} satisfies

$$(i) \quad \frac{\Pi(p^m)}{2(1-\delta)} > \Pi(\underline{p})(1+\delta) + \frac{\delta^2}{2(1-\delta)}\Pi(p^m) \geq \frac{\delta\Pi(p^m)}{2(1-\delta)}$$

and

$$(ii) \quad \frac{\delta\Pi(p^m)}{2(1-\delta)} > \Pi(\underline{p}-k)(1+\delta) + \frac{\delta^2}{2(1-\delta)}\Pi(p^m) .$$

From the argument in the proof of Proposition 4, \underline{p} exists and (R,R) forms a MPE for k small enough and δ near 1. For $\hat{p} \geq p^m$ and $\hat{p} \leq \underline{p}$, aggregate present discounted profit is maximized when both

firms follow R stating at price \hat{p} . Hence at such a price there is no MPE that Pareto dominates (R,R). Consider $\hat{p} \in (\underline{p}, p^m)$. By construction of R,

$$V(\hat{p}) = \Pi(\underline{p})(1+\delta) + \delta^2 \Pi(p^m)/2(1-\delta)$$

and

$$W(\hat{p}) = \delta^2 \Pi(p^m)/2(1-\delta) .$$

Suppose that at \hat{p} there exists a Pareto-dominating MPE (\hat{R}^1, \hat{R}^2) in which, say, firm 1 moves first. Then $\hat{V}^1(\hat{p}) \geq V(\hat{p})$ and $\hat{W}^2(\hat{p}) \geq W(\hat{p})$ with at least one strict inequality. Suppose, for the moment, that (\hat{R}^1, \hat{R}^2) is a kinked demand curve equilibrium (we will consider the possibility of an Edgeworth cycle below). For δ close enough to 1, the focal price must be p^m . Now, for some i and p , suppose that \tilde{p} is in the support of $\hat{R}^i(p)$, where $\tilde{p} > p$. From Lemma C, we know that $p < p^m$. Now

$$(a42) \quad \delta \hat{W}^i(\tilde{p}) \geq \frac{\delta \Pi(p^m)}{2(1-\delta)} ,$$

because firm i could choose p^m rather than \tilde{p} . Similarly

$$(a42i) \quad \hat{V}^j(\tilde{p}) \geq \frac{\Pi(p^m)}{2(1-\delta)} \quad \text{if } \tilde{p} \geq p^m$$

and

$$(a42ii) \quad \hat{V}^j(\tilde{p}) \geq \frac{\delta \Pi(p^m)}{2(1-\delta)} \quad \text{if } \tilde{p} < p^m .$$

Now,

$$\hat{W}^i(\tilde{p}) + \hat{V}^j(\tilde{p}) \leq \frac{\Pi(p^m)}{1-\delta} ,$$

and so if $\tilde{p} > p^m$, (a42) and (a42i) imply that $\hat{V}^j(\tilde{p}) + \hat{W}^i(\tilde{p}) = \Pi(p^m)/(1-\delta)$, that is, $\tilde{R}^j(\tilde{p}) = p^m$. But if $\tilde{R}^j(\tilde{p}) = p^m$, then $\delta \hat{W}^i(\tilde{p}) = \varepsilon^2 \Pi(p^m)/2(1-\delta)$, a contradiction of (a42). Hence $\tilde{p} \leq p^m$. Now, if $\tilde{p} < p^m$, (a42) and (a42ii) imply that

$$(a43) \quad \hat{W}^i(\tilde{p}) + \hat{V}^j(\tilde{p}) \geq \frac{\Pi(p^m)}{2} + \frac{\delta \Pi(p^m)}{1-\delta} .$$

However,

$$(a44) \quad \Pi(\tilde{p}) + \frac{\delta \Pi(p^m)}{1-\delta} \geq \hat{W}^i(\tilde{p}) \geq \hat{W}^i(\tilde{p}) + \hat{V}^j(\tilde{p}) ,$$

since, starting at \tilde{p} , the market price can be no greater than \tilde{p} for at least one period. Formulas (a43) and (a44) imply that

$$(a45) \quad \Pi(\tilde{p}) > \Pi(p^m)/2$$

But then (a42) and (a43) imply that

$$\Pi(\tilde{p}) + \delta \hat{W}^i(\tilde{p}) > \frac{\Pi(p^m)}{2} + \frac{\delta \Pi(p^m)}{2(1-\delta)} ,$$

which contradicts that fact that p^m is a focal price. Hence, for

$$p < p^m,$$

$$(a46) \quad \text{if } \tilde{p} \text{ is in the support of } \hat{R}^i(p) \text{ and } \tilde{p} > p, \text{ then } \tilde{p} = p^m.$$

Moreover, from (i) and (ii) of the statement of the proposition

$$(a47) \quad \hat{R}^1(p) = \hat{R}^2(p) = p^m \text{ for all } p \leq \underline{p}.$$

Now, if $\hat{R}^1(\hat{p}) \leq \underline{p}$, then (\hat{R}^1, \hat{R}^2) cannot Pareto-dominate (R, R) at \hat{p} from (a47). From (i) and (a47), $\hat{R}^1(\hat{p}) = p^m$ (otherwise, the new equilibrium would leave firm 1 worse off). Hence (a46) implies

$$(a48) \quad \text{supp } \hat{R}^1(\hat{p}) \subseteq (\underline{p}, \hat{p}).$$

Hence, because, starting from \hat{p} , the market price must first fall to \underline{p} before returning to p^f ,

$$\hat{V}^1(\hat{p}) + \hat{W}^2(\hat{p}) \leq \Pi(\hat{p}) + \delta \Pi(\underline{p})(1+\delta) + \frac{\delta^3 \Pi(p^m)}{1-\delta}.$$

Now by definition of R ,

$$V(\hat{p}) + W(\hat{p}) = \Pi(\underline{p})(1+\delta) + \frac{\delta^2 \Pi(p^m)}{1-\delta}.$$

Thus, since $\hat{V}^1(\hat{p}) + \hat{W}^2(\hat{p}) > V(\hat{p}) + W(\hat{p})$, we have

$$\Pi(\hat{p}) + \delta \Pi(\underline{p})(1+\delta) + \frac{\delta^3 \Pi(p^m)}{1-\delta} > \Pi(\underline{p})(1+\delta) + \frac{\delta^2 \Pi(p^m)}{1-\delta},$$

that is,

$$\Pi(\hat{p}) > (1-\delta)(1+\delta)\Pi(\underline{p}) + \delta^2 \Pi(p^m),$$

which, for δ close enough to 1, is impossible. We conclude that (\hat{R}^1, \hat{R}^2) cannot Pareto-dominate after all.

Next, suppose that (\hat{R}^1, \hat{R}^2) is an Edgeworth cycle. Because $(1-\delta)(\hat{V}^1(\hat{p}) + \hat{W}^2(\hat{p})) > (1-\delta)(V(\hat{p}) + W(\hat{p}))$, and the right hand side of this inequality goes to $\Pi(p^m)$ as $\delta \rightarrow 1$, we know that, for all $\epsilon > 0$,

$$(a49) \quad \Pr\{\min\{\hat{R}^1(p^m), \hat{R}^2(p^m)\} \geq p^m\} > 1-\epsilon,$$

if δ is close enough to 1. Let $\epsilon^1 = \Pr\{\hat{R}^1(p^m) > p^m\}$. Then, if firm 2 always plays p^m , its average payoff, starting at p^m , is bounded from below by

$$(a50) \quad (1-\epsilon)(\epsilon^1 \Pi(p^m) + (1-\epsilon^1) \Pi(p^m)/2).$$

Now, if as $\delta \rightarrow 1$ and $\epsilon \rightarrow 0$, ϵ^1 remains bounded away from zero, then there exists $b > 0$ such that eventually (a50) must exceed $\Pi(p^m)/2 + b$. That is,

$$(a51) \quad (1-\delta)\hat{W}^2(p^m) > \frac{\Pi(p^m)}{2} + b.$$

But (a51) implies that

$$(a52) \quad \hat{V}^1(p^m) < \frac{\Pi(p^m)}{2(1-\delta)} - \frac{b}{1-\delta}$$

which contradicts the fact that

$$\hat{V}^1(p^m) \geq v^1(\hat{p}) = \Pi(\underline{p})(1+\delta) + \frac{\delta^2 \Pi(p^m)}{2(1-\delta)}.$$

Hence, ϵ^1 must go to zero with ϵ , and we conclude that for all $\gamma^1 > 0$

$$(a53) \quad \Pr \{\hat{R}^1(p^m) = p^m\} > 1-\gamma^1 \quad \text{for } \delta \text{ close enough to } 1.$$

Similarly, for any $\gamma^2 > 0$

$$(a54) \quad \Pr \{\hat{R}^2(p^m) = p^m\} > 1-\gamma^2 \quad \text{for } \delta \text{ close enough to } 1.$$

Thus, for δ close enough to 1, (\hat{R}^1, \hat{R}^2) is practically a kinked demand curve equilibrium with focal price p^m . In particular, (a46)-(a48) all hold and we can derive the same contradiction as before. This establishes the first assertion of the proposition.

Next, suppose that (\hat{R}, \hat{R}) is a symmetric renegotiation-proof MPE. Now, as $\delta \rightarrow 0$, $\hat{V}(p)(1-\delta) \rightarrow \hat{W}(p)(1-\delta)$ for any p . Moreover,

$$(\hat{V}(p) + \hat{W}(p))(1-\delta) \leq (1-\delta)\Pi(p^m). \quad \text{Hence,}$$

$$(a55) \quad \hat{V}(p)(1-\delta) \text{ and } \hat{W}(p)(1-\delta) \text{ converge, to } \Pi(p^m)/2,$$

otherwise, for δ near 1, firms could do better by moving to equilibrium (R, R) . But (a55) implies that for any $\epsilon > 0$ and δ close enough to 1,

$$(a55) \quad \Pr \{ R(p^m) \geq p^m \} > 1 - \epsilon .$$

Given (a56) we can argue exactly as in the preceding paragraph to conclude that (\hat{R}, \hat{R}) must be nearly a kinked demand curve equilibrium with focal price p^m .

Q.E.D.

Proposition 6: Assume that the profit function $\Pi(p)$ is strictly concave. For a fine grid and a discount factor near 1, there exists an Edgeworth cycle.

Proof: Let c denote the unit cost of production. Consider a pair of prices $\{\underline{p}, \bar{p}\}$ such that $c < \underline{p} < p^m < \bar{p}$ and the reaction function R , where

$$R(p) = \begin{cases} \bar{p}, & \text{if } p > \bar{p} \\ \underline{p} - k, & \text{if } \bar{p} \geq p > \underline{p} \\ c, & \text{if } \underline{p} \geq p > c \\ \left. \begin{array}{l} c, \text{ with probability } \nu \\ \bar{p} + k, \text{ with probability } 1 - \nu \end{array} \right\} & \text{if } p = c \\ c, & \text{if } p < c . \end{cases}$$

We will show that \underline{p} , \bar{p} , and ν can be chosen so that $\{R, R\}$ is a MPE. We first study the conditions that these parameters must satisfy for equilibrium. We consider two cases:

Case (1): $\bar{p} - \underline{p} = (2t+1)k$

Let $V(p)$ denote the valuation of a firm when the other firm has just played p ; take $W(p)$ to be the valuation of a firm that has just played p ; and define $\bar{V} \equiv V(\bar{p})$. We have

$$(a42) \quad \bar{V} = \Pi(\bar{p}-k) + \delta^2 \Pi(\bar{p}-3k) + \dots + \delta^{2t} \Pi(\underline{p}) + \delta^{2t+4} \bar{V}$$

where we have used

$$(a43) \quad V(c) = \delta^2 \bar{V}.$$

Case (2): $\bar{p} - \underline{p} = 2tk$

In this case, we have

$$(a44) \quad \bar{V} = \Pi(\bar{p}-k) + \delta^2 \Pi(\bar{p}-3k) + \dots + \delta^{2t-2} \Pi(\underline{p}+k) + \delta^{2t} V(\underline{p}).$$

But

$$(a45) \quad V(\underline{p}) = \delta W(c) = V(c) = \delta^2 \bar{V},$$

where we used the fact that a firm is willing to stay at c when its rival's price is c . Therefore (a44) can be written as

$$(a46) \quad \bar{V} = \Pi(\bar{p}-k) + \delta^2 \Pi(\bar{p}-3k) + \dots + \delta^{2t-2} \Pi(\underline{p}+k) + \delta^{2t+2} \bar{V}.$$

Given \bar{p} and \bar{V} , we now investigate the conditions that \underline{p}

must satisfy. Namely, at $\underline{p}+k$, a firm must wish to undercut by k , whereas at \underline{p} it must prefer to lower its price to c (in which case, its valuation is $\delta^{2\bar{v}}$ (see (a43)):

$$(a47) \quad \Pi(\underline{p}) + \delta^{4\bar{v}} \geq \delta^{2\bar{v}} > \Pi(\underline{p}-k) + \delta^{4\bar{v}} .$$

For given \bar{p} and \bar{v} , μ must be such that a firm is indifferent between staying at price c and returning to $\bar{p}+k$. That is,

$$(a48) \quad \delta^2(1-\mu)(\Pi(\bar{p}) + \delta\bar{W}) + \mu\delta^{4\bar{v}} = \delta^{2\bar{v}} ,$$

where \bar{W} is given by

$$(a49) \quad \bar{W} = \delta\Pi(\bar{p}-2k) + \delta^3\Pi(\bar{p}-4k) + \dots + \delta^{2t-1}\Pi(\underline{p}+k) + \delta^{2t+3\bar{v}}$$

if $\bar{p}-\underline{p} = (2t+1)k$

and

$$(a50) \quad \bar{W} = \delta\Pi(\bar{p}-2k) + \delta^3\Pi(\bar{p}-4k) + \dots + \delta^{2t-1}\Pi(\underline{p}) + \delta^{2t+3\bar{v}}$$

if $\bar{p}-\underline{p} = 2tk$. Notice that $0 < \mu < 1$ since

$$\delta^{2\bar{v}} > \delta^{4\bar{v}}$$

and

$$\delta^2 \bar{V} < \delta^2 (\Pi(\bar{p}) + \delta \bar{W})$$

(the latter inequality follows from the fact that above \bar{p} , a firm prefers to lower its price to \bar{p} and get $\Pi(\bar{p}) + \delta \bar{W}$ rather than reducing to $\bar{p} - k$ and getting \bar{V}).

We have exhibited the properties that, given \bar{p} , \underline{p} and μ must satisfy. The price \bar{p} itself must be such that above $\bar{p} + k$ a firm prefers to reduce its price to \bar{p} rather than to $\bar{p} + k$. Choose \bar{p} to be the lowest price above \underline{p}^m such that at $\bar{p} + 2k$ firms strictly prefer to lower their price to \bar{p} rather than to $\bar{p} + k$ (the existence of such a price will be proved below). Thus \bar{p} is the lowest price above \underline{p}^m such that

$$(a51) \quad \Pi(\bar{p}) + \delta \bar{W} > \Pi(\bar{p} + k) + \delta^2 \bar{V},$$

where \bar{W} and \bar{V} are defined by {(a42), (a49)} or {(a44), (a50)}.

We complete the proof as follows. We first prove that (a) if (a42) and (a46) through (a51) are satisfied, the proposed strategies form a MPE. We then show (b) that there exist \bar{p} , \underline{p} and μ satisfying these relationships.

(a) Demonstration of Equilibrium: Notice first that a firm will never raise its price above that of the other firm to a point strictly between $\bar{p} + k$ and c (If a firm raises its price to $p \in (c, \bar{p} + k)$, its discounted profit is $\delta^2 V(R(p))$, where $R(\cdot)$

and $V(\bullet)$ are nondecreasing. We claim that, at $p < c$, raising its price to c is an optimal reaction for a firm. By doing so, the firm gets

$$\delta W(c) = V(c) = \delta^2 \bar{V},$$

which is the same payoff as when the firm raises its price to $\bar{p}+k$. On the other hand if the firm reacts by playing $\hat{p} < c$, its payoff is less than $\delta^2 V(c) = \delta^4 \bar{V}$. Hence, playing c (or $\bar{p}+k$) is optimal.

From (a47) and the parenthetical argument of the preceding paragraph, a firm will never play a price between c and \underline{p} ; it would be better off playing c or $\bar{p}+k$.

Let us now show by induction that, for $p \in (\underline{p}, \bar{p}^m]$ reducing one's price by k is optimal. We have to show that the firm does not prefer to play p or $p-2k$. These two conditions suffice since (i) a firm would not want to lower its price below $p-2k$ given that, by induction, it does not want to do so at $\bar{p}-k$, and (ii) it does not want to raise its price since, again by induction, it does not want to do so when the price is lower.

Let us consider the two cases.

Case (1): $p - \underline{p} = (2t+1)k$

We want to show that

$$\begin{aligned} & \Pi(p-k) + \delta^2 \Pi(p-3k) + \dots + \delta^{2t} \Pi(p) + \delta^{2t+4\bar{v}} \\ & \geq \max \{ \Pi(p)/2 + \delta^2 \Pi(p-2k) + \dots + \delta^{2t} \Pi(p+k) + \delta^{2t+4\bar{v}}, \\ & \quad \Pi(p-2k) + \delta^2 \Pi(p-4k) + \dots + \delta^{2t-2} \Pi(p+k) + \delta^{2t+2\bar{v}} \} . \end{aligned}$$

The second inequality is follows from

$$\Pi(p-qk) > \Pi(p-(q+1)k) \text{ for } q > 0$$

and

$$(a47) \quad \Pi(p) + \delta^{4\bar{v}} \geq \delta^{2\bar{v}} .$$

We must show that the first inequality also holds, i.e.,

$$\begin{aligned} \Pi(p-k) - \Pi(p)/2 & \geq \delta^2 [\Pi(p-2k) - \Pi(p-3k)] + \dots \\ & \quad + \delta^{2t} [\Pi(p+k) - \Pi(p)] . \end{aligned}$$

From the concavity of Π ,

$$\begin{aligned} & \delta^2 [\Pi(p-2k) - \Pi(p-3k)] + \dots + \delta^{2t} [\Pi(p+k) - \Pi(p)] \\ & \leq (\delta^2/2)(\Pi(p-2k) - \Pi(p-k)) . \end{aligned}$$

Therefore, it remains to show that

$$\Pi(p-k) \geq (1/2)[\Pi(p) + \delta^2 \Pi(p-2k)] - \delta^2 \Pi(\underline{p}-k)/2 .$$

But $\Pi(p-k) \geq (1/2)[\Pi(p) + \Pi(p-2k)]$ and $\Pi(\underline{p}-k) > 0$. Thus the firm prefers to undercut by k after all.

Case (2): $p-\underline{p} = 2tk$

Analogously with case (1) we need to show that

$$\begin{aligned} & \Pi(p-k) + \delta^2 \Pi(p-3k) + \dots + \delta^{2t-2} \Pi(\underline{p}+k) + \delta^{2t+2} \bar{v} \\ & \geq \max \{ \Pi(p)/2 + \delta^2 \Pi(p-2k) + \dots + \delta^{2t-2} \Pi(\underline{p}+2k) + \\ & \quad \delta^{2t} \Pi(\underline{p}) + \delta^{2t+4} \bar{v} , \\ & \quad \Pi(p-2k) + \delta^2 \Pi(p-4k) + \dots + \delta^{2t-2} \Pi(\underline{p}) + \delta^{2t+2} \bar{v} \} . \end{aligned}$$

The second inequality is again immediate. To demonstrate the first one, recall that

$$\Pi(\underline{p}) + \delta^{2\bar{v}} < \delta^{2\bar{v}} + [\Pi(\underline{p}) - \Pi(\underline{p}-k)] .$$

From concavity it suffices to show that

$$\begin{aligned} \Pi(p-k) - \Pi(p)/2 & \geq (\delta^2/2)[\Pi(p-2k) - \Pi(\underline{p})] + \\ & \quad \delta^{2t} [\Pi(\underline{p}) - \Pi(\underline{p}-k)] . \end{aligned}$$

An even stronger sufficient condition is

$$(a52) \quad \Pi(p-k) \geq (1/2)[\Pi(p) + \Pi(p-2k)] + [\Pi(\underline{p})/2 - \Pi(\underline{p}-k)]$$

But again,

$$\Pi(p-k) \geq (1/2)[\Pi(p) + \Pi(p-2k)]$$

and

$$\Pi(p)/2 - \Pi(p-k) < 0$$

(if the grid is not too coarse). Thus (a52) holds.

Next consider p in $(p^m, \bar{p}]$. The previous proof that at p a firm prefers to undercut to $p-k$ rather than staying at p relied only on the concavity of Π , not on the fact that p was lower than p^m . Recall that \bar{p} was defined as the lowest price above p^m such that at $\bar{p}+2k$ a firm prefers to undercut to \bar{p} rather than to $\bar{p}+k$. Therefore, for p between p^m and \bar{p} , a firm at $p+k$ prefers to undercut to p rather than to $p-k$. Thus again the two necessary inequalities are satisfied.

Finally, we consider $p > \bar{p}$. We must show that, at p , firms undercut to \bar{p} . The most a firm can get by choosing a price above \bar{p} given that the other firm reacts by undercutting to \bar{p} is

$$\Pi(\bar{p}+k) + \delta^2 \bar{v}.$$

But by definition of \bar{p} we know that a firm strictly prefers to undercut to \bar{p} rather than to $p+k$. At p a firm will not undercut to a price below \bar{p} since at $\bar{p}+k$ it will not do so. Therefore at

p a firm will undercut to \bar{p} after all.

Thus if conditions (a42) and (a46) through (a51) are satisfied the strategies $\{R,R\}$ form a MPE.

(b) Existence of \bar{p}, p , and μ : We will define \underline{p} , \bar{p} and μ as functions of \bar{V} and apply a fixed point argument to prove that there exists a \bar{V} such that the previous necessary and sufficient conditions are satisfied.

Consider an arbitrary \bar{V} in $[0, \frac{\Pi(p^m)}{2(1-\delta)}]$. From the strict concavity of Π , formula (a47) defines a unique $\underline{p}(\bar{V})$ in $[0, p^m]$ (since $(\delta^2 - \delta^4)\bar{V} \leq \delta^2(1+\delta)\Pi(p^m)/2 < \Pi(p^m)$). Now given \bar{V} and $\underline{p} = \underline{p}(\bar{V})$, define the function

$$U(p, \bar{V}) = \begin{cases} \Pi(p-k) + \delta^{2t+4} \frac{\delta^2}{\bar{V}} \Pi(p-3k) + \dots + \delta^{2t} \Pi(\underline{p}) \\ \quad + \delta^{2t+4} \bar{V}, \text{ if } p - \underline{p} = (2t+1)k \\ \Pi(p-k) + \delta^2 \Pi(p-3k) + \dots + \delta^{2t-2} \Pi(\underline{p}+k) \\ \quad + \delta^{2k+2} \bar{V}, \text{ if } p - \underline{p} = 2tk. \end{cases}$$

This function will represent the fictitious payoff of a firm who undercuts the price p by k . It is defined exactly like $V(p)$. We note first that U is continuous in \bar{V} although (because \underline{p} must belong to the grid) $\underline{p}(\bar{V})$ is not. To see this, observe that if \underline{p} is locally constant in \bar{V} , U is linear in \bar{V} . Moreover, when \underline{p} jumps, U does not. Note also that U increases in \bar{V} .

Next notice that at a price $p \in (\underline{p}, \underline{p}^m + k]$ a firm with fictitious payoff U (which is defined exactly like V) prefers to undercut by k and obtain payoff $U(p, \bar{V})$ rather than to undercut by

$2k$ and obtain payoff $U(\bar{p}-k, \bar{V})$. The demonstration is the same as that in part (a) of this proof. Define $\bar{p}(\bar{V})$ as the lowest price above \bar{p}^m such that

$$U(\bar{p}+k, \bar{V}) > U(\bar{p}+2k, \bar{V}) ,$$

which means that at $\bar{p}+2k$ a firm would prefer to undercut to \bar{p} rather than to $\bar{p}+k$, given these fictitious payoffs. Finally, define $\mu(\bar{V})$ by (a48) where $\bar{p} = \bar{p}(\bar{V})$ and

$$\bar{W}(\bar{V}) = \begin{cases} \delta \Pi(\bar{p}(\bar{V})-2k) + \dots + \delta^{2t-1} \Pi(\underline{p}(\bar{V})+k) + \delta^{2k+3} \bar{V} , \\ \quad \text{if } \bar{p}(\bar{V}) - \underline{p}(\bar{V}) = (2t+1)k \\ \\ \delta \Pi(\bar{p}(\bar{V})-2k) + \dots + \delta^{2k-1} \Pi(\underline{p}(\bar{V})) + \delta^{2k+3} \bar{V} , \\ \quad \text{if } \bar{p}(\bar{V}) - \underline{p}(\bar{V}) = 2tk . \end{cases}$$

As before, one can check that $0 < \mu(\bar{V}) \leq 1$.

To complete the proof, we must show that there exists \bar{V} such that

$$\bar{V} = U(\bar{p}(\bar{V}), \bar{V}) \equiv \tilde{U}(\bar{V}) .$$

Like U , $\tilde{U}(\bar{V})$ is continuous in \bar{V} , even though $\bar{p}(\bar{V})$ is not. This is because $\bar{p}(\bar{V})$ jumps only when $U(\bar{p}(\bar{V})+k, \bar{V}) = U(\bar{p}(\bar{V})+2k, \bar{V})$, where \tilde{U} is continuous by definition. Thus to establish that \tilde{U} has a fixed point, it remains only to show that \tilde{U} maps $[0, \Pi(\bar{p}^m)/2(1-\delta)]$ into itself. Clearly $\tilde{U}(\bar{V}) \geq 0$ if $\bar{V} \geq 0$.

We now show that

$$\tilde{U}(\bar{V}) \leq \frac{\Pi(p^m)}{2(1-\delta)} \text{ if } \bar{V} \leq \frac{\Pi(p^m)}{2(1-\delta)} .$$

Define $\bar{Z} \equiv \delta U(\bar{p}(\bar{V})-k, \bar{V})$. \bar{Z} is the present discounted profit of a firm that played \bar{p} last period (and is undercut to $\bar{p}-k$). Thus $\bar{Z} + \tilde{U}(\bar{V})$ represents the aggregate payoff when the price is $\bar{p}(\bar{V})$. Now the average aggregate payoff (i.e., $\bar{Z} + \tilde{U}(\bar{V})$) multiplied by $(1-\delta)$ is no greater than $\Pi(p^m)$. Indeed, it must be strictly less than $\Pi(p^m)$ because, starting from \bar{p} , the market price cannot always be p^m . Thus there exists $\alpha > 0$ such that

$$(a53) \quad \Pi(p^m) \geq (\bar{Z} + \tilde{U}(\bar{V}))(1-\delta) + \alpha .$$

On the other hand,

$$(a54) \quad \tilde{U}(\bar{V}) \leq \Pi(\bar{p}(\bar{V})-k) + \delta \bar{Z} ,$$

since the firm that is undercut to $\bar{p}(\bar{V})-k$ and has valuation \bar{Z} could undercut to $\bar{p}(\bar{V})-3k$ instead of $\bar{p}(\bar{V})-2k$ but does not choose to. Using (a53) and (a54), we get

$$(a55) \quad \Pi(p^m) \geq (1-\delta) \left[\tilde{U}(\bar{V}) + \frac{\tilde{U}(\bar{V}) - \Pi(\bar{p}(\bar{V})-k)}{\delta} \right] + \alpha .$$

If δ is sufficiently close to 1, (a55) implies that

$$\tilde{U}(\bar{V}) < \frac{\Pi(p^m)}{2(1-\delta)}.$$

Therefore, \tilde{U} has a fixed point.

Q.E.D.

Proposition 7: For a high discount factor, the average aggregate profit per period in a symmetric MPE exceeds half the monopoly profit.

Proof: Consider a symmetric MPE $\{R,R\}$. Assume that firm 1, say, chooses a price $\bar{p} > p^m$. Firm 2 then reacts by choosing a price that solves

$$(a56) \quad \max \left\{ \max_{p < \bar{p}} (\Pi(p) + \delta W(p)), \frac{\Pi(\bar{p})}{2} + \delta W(\bar{p}), \max_{p > \bar{p}} \delta W(p) \right\}.$$

Let \underline{p} be the smallest price that solves (a56). There are two cases.

Case (a): $\underline{p} \geq p^m$

Then the equilibrium payoff of firm 1 is at least

$$(a57) \quad \delta^2 [\Pi(p^m) + \delta W(p^m)]$$

since, after firm 2's reaction, it could undercut to p^m . Now because a firm could always raise its price to \bar{p} and two periods

later undercut to p^m , we have

$$(a58) \quad W(p^m) \geq \delta^3 (\Pi(p^m) + \delta W(p^m)) .$$

From (a58), (a57) is at least

$$\frac{\delta^2 \Pi(p^m)}{2} + \frac{\delta^4 \Pi(p^m)}{1-\delta^4} = \frac{(1/2) \delta^2 (1+\delta^4)}{1+\delta+\delta^2+\delta^3} \left[\frac{\Pi(p^m)}{(1-\delta)} \right] .$$

Notice that $\lim_{\delta \rightarrow 1} \frac{(1/2) \delta^2 (1+\delta^4)}{1+\delta+\delta^2+\delta^3} = \frac{1}{4}$. Thus firm 1's payoff

per period is at least $\Pi^4/4$ (minus ϵ) if δ is sufficiently close to one. By symmetry, the same is true of firm 2.

Case (b): $p \leq p^m$

In this case, for all $p < \underline{p}$,

$$\Pi(\underline{p}) + \delta W(\underline{p}) > \Pi(p) + \delta W(p)$$

and

$$(a58) \quad \Pi(\underline{p}) + \delta W(\underline{p}) \geq \Pi(p^m) + \delta W(p^m) .$$

The first inequality implies that at a price above \underline{p} , a firm will never undercut to below \underline{p} . Therefore, we have

$$(a59) \quad W(p^m) \geq \delta V(\underline{p}) .$$

But at \underline{p} a firm can always charge \underline{p} . Hence,

$$(a60) \quad V(\underline{p}) \geq \Pi(\underline{p})/2 + \delta W(\underline{p}) .$$

Using (a58)-(a60) we obtain

$$\Pi(\underline{p}) + \delta W(\underline{p}) \geq \Pi(p^m) + \delta^2 \Pi(\underline{p})/2 + \delta^3 W(\underline{p}) ,$$

or

$$(1-\delta)W(\underline{p}) \geq \frac{\Pi(p^m) - \Pi(\underline{p})(1-\delta^2/2)}{\delta(1+\delta)}$$

Since $\Pi(\underline{p}) < \Pi(p^m)$

$$(a61) \quad (1-\delta)W(\underline{p}) > \frac{\Pi(p^m)\delta}{4} .$$

But the right hand side of (a61) converges to $\Pi(p^m)/4$ when δ converges to one. Thus for a high discount factors, each firm's profit per period is at least $\Pi(p^m)/4$ (minus ϵ).

Q.E.D

Proposition 8: For a sufficiently fine grid and a discount factor near enough to 1, any symmetric alternating-move MPE (R,R) but no simultaneous-move MPE (S^1, S^2) is robust to endogenous timing.

Proof: Consider a symmetric MPE (R,R) of the alternating-move model, and let $V(p)$ and $W(p)$ be the associated valuation functions. Let \hat{p} denote the smallest price that maximizes $\Pi(p) + \delta W(p)$. We will construct a strategy S^* for the simultaneous mode such that $(\{R, S^*\}, \{R, S^*\})$ forms a MPE for the endogeneous-timing game.

For the moment, suppose that firms, when in the simultaneous mode, can choose either (a) null action or (b) a price no greater than \hat{p} (we will admit the possibility of firms' choosing prices greater than \hat{p} later on). Once we specify firms' behavior in this mode, then their payoffs are completely determined, assuming they play according to R in the alternating mode. Thus we can think of firms in the simultaneous mode as playing a one-shot game in which they choose mixed strategies S^1 and S^2 and their payoffs are determined by $(\{R, S^1\}, (R, S^2))$. Because this is a symmetric game there exists a symmetric equilibrium (S^*, S^*) . Let U^* be a firm's corresponding present discounted profit. We claim that $(\{R, S^*\}, \{R, S^*\})$ is a MPE for the endogenous-timing game.

We first note that S^* must place positive probability on the null action. If this were not the case, then firms would remain in the simultaneous mode forever. But, as we argued in footnote 8, any simultaneous-mode equilibrium must entail (essentially) zero profit. By contrast, if a firm played the null action and thereby moved the firms into the alternating mode, Proposition 7 would guarantee it of at least a quarter of monopoly profit,

which is clearly preferable. Thus S^* must indeed assign the null action positive probability.

We next observe that, in the simultaneous mode, firm i cannot gain from choosing a price p greater than \hat{p} , given that firm j sticks to S^* . If firm j does not choose the null action - i.e., it selects a price - then firm i sells nothing with a price greater than \hat{p} . If firm j does select the null action, then the firms move into the alternating mode, and firm i 's payoff is $\Pi(p) + \delta W(p)$, which, by definition of \hat{p} , is no greater than that from choosing \hat{p} . Hence a firm has no incentive to choose prices greater than \hat{p} in the simultaneous mode.

It remains only to show that, in the alternating mode, a firm has no incentive to play the null action. If it did so, its payoff would be δU , since the firms would then be in the simultaneous mode. Now, because, as we have noted, it is optimal for a firm to play the null action in the simultaneous mode,

$$U < \delta V(\hat{p}).$$

Hence, by playing the null action in the alternating mode, a firm obtains a payoff less than $\delta^2 V(p)$. If instead it chooses a price $p > \hat{p}$, the other firm will react with a price no lower than \hat{p} , and so its payoff is at least $\delta^2 V(\hat{p})$. Hence the null action is not preferable.

To see that a simultaneous-move MPE (S^1, S^2) cannot be robust

to endogenous timing, recall that in such an equilibrium, $S^i = c+k$ for $i=1,2$. Now if (S^1, S^2) were robust, there would exist reaction functions R^1 and R^2 such that, starting from the alternating mode, firms switch to the simultaneous mode in finite time with probability one. But given that firms are in the alternating mode, a firm is always strictly better off choosing $p=c+k$ than the null action. Hence, once the alternating mode is reached firms stay there, a contradiction.

Q.E.D.

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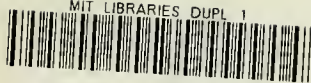
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