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Susan Athey

No. 97-11 June, 1997

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Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information*

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Abstract: This paper derives sufficient conditions for a class of games of incomplete information, such as first price auctions, to have pure strategy Nash equilibria (PSNE). The paper treats games between two or more heterogeneous agents, each with private information about his own type (for example, a bidder's value for an object or a firm's marginal cost of production), and the types are drawn from an atomless joint probability distribution which potentially allows for correlation between types. Agents' utility may depend directly on the realizations of other agents' types, as in Milgrom and Weber's (1982) formulation of the "mineral rights" auction. The restriction we consider is that each player's expected payoffs satisfy the following *single crossing condition*: whenever each opponent uses a nondecreasing strategy (that is, an opponent who has a higher type chooses a higher action), then a player's best response strategy is also nondecreasing in her type.

The paper has two main results. The first result shows that, when players are restricted to choose among a finite set of actions (for example, bidding or pricing where the smallest unit is a penny), games where players' objective functions satisfy this single crossing condition will have PSNE. The second result demonstrates that when players' utility functions are continuous, as well as in mineral rights auction games and other games where "winning" creates a discontinuity in payoffs, the existence result can be extended to the case where players choose from a continuum of actions.

The paper then applies the theory to several classes of games, providing conditions on utility functions and joint distributions over types under which each class of games satisfies the single crossing condition. In particular, the single crossing condition is shown to hold in all first-price, private value auctions with potentially heterogeneous, risk-averse bidders, with either independent or affiliated values, and with reserve prices which may differ across bidders; mineral rights auctions with two heterogeneous bidders and affiliated values; a class of pricing games with incomplete information about costs; a class of all-pay auction games; and a class of noisy signaling games. Finally, the formulation of the problem introduced in this paper suggests a straightforward algorithm for numerically computing equilibrium bidding strategies in games such as first price auctions, and we present numerical analyses of several auctions under alternative assumptions about the joint distribution of types.

Keywords: Games of incomplete information, pure strategy Nash equilibrium, auctions, pricing games, signaling games, supermodularity, log-supermodularity, single crossing, affiliation.

JEL Classifications: C62, C63, D44, D82

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1. Introduction

This paper derives sufficient conditions for a class of games of incomplete information, such as first price auction games, to have pure strategy Nash equilibria (PSNE). The class of games is described as follows: there are I agents with private information about their types, and the types are drawn from a joint distribution which allows for correlation between types. Types are drawn from a convex subset of $\Re$, and the joint distribution of types is atomless. We allow for heterogeneity across players in the distribution over types as well as the players' utility functions, and the utility functions may depend directly on other players' types. Thus, the formulation includes the "mineral rights" auction (Milgrom and Weber (1982)), where bidders receive a signal about the underlying value of the object, and signals and values may be correlated across players. The existence result does not make any assumptions about quasi-concavity or differentiability of the underlying utility functions of the agents, nor does it require that each agent has a unique optimal action for any type.

The main restriction studied in this paper is what we call the *single crossing condition* for games of incomplete information: for every player i, whenever each of player i's opponents uses a pure strategy such that higher types choose (weakly) higher actions, player i's expected payoffs satisfy Milgrom and Shannon's (1994) single crossing property. In particular, when choosing between a low action and a high action, if a low type of agent i weakly (strictly) prefers the higher action, then all higher types of agent i will weakly (strictly) prefer the higher action as well. This condition implies that in response to nondecreasing strategies by opponents, each player will have a best response strategy where higher types choose higher actions. The single crossing condition contrasts with that studied by Vives (1990), who shows that a sufficient condition for existence of PSNE is that the game is supermodular in the strategies, in the following sense: if one player's strategy increases pointwise (almost everywhere), the best response strategies of all opponents must increase pointwise (a.e). However, in Vives' analysis, the strategies themselves need not be monotone in types. Vives' condition is applicable games where players have supermodular utility functions, but not in the auctions and log-supermodular pricing games highlighted in this paper.

The paper has four parts. The first part shows that when a game of incomplete information satisfies the single crossing condition described above, but when the players are restricted to choose from a finite action space, a PSNE exists. Next, we show that under the further assumption that players' utility functions are continuous, or in a class of "winner-take-more" games such as first price auctions, we can find a sequence of equilibria of finite-action games which converges to an equilibrium in a game where actions are chosen from a continuum. The third part of the paper builds on Athey (1995, 1996) to study conditions on utility functions and type distributions under which commonly studied games satisfy the single crossing condition. Games which satisfy the single crossing condition include any first price auction, where values are private, bidders are (weakly) risk averse, and the types are independent or affiliated. In the class of "mineral rights" auctions, the conditions are satisfied when there are two heterogeneous bidders whose types are affiliated. Other applications which satisfy the conditions include all-pay auctions and multi-unit auctions with heterogeneous bidders and independent private values, noisy signaling games (such as limit pricing with demand shocks), and a class of supermodular and log-supermodular quantity and pricing games with incomplete information about costs. The final part of the paper focuses on numerical computation of equilibria, showing that equilibria to first price auctions may be easily computed for games with a finite number of potential bids. Several examples of auctions are provided, considering alternative scenarios for heterogeneity, private versus common values, and correlated versus independent values.

The existence result for finite action spaces analyzed in the first part of this paper is straightforward to prove using a reformulation of the problem, which allows us to simplify the game to a finite-dimensional problem and then apply standard fixed point theorems. The existence result proceeds in two steps. First, we observe that if a player uses a nondecreasing strategy which maps types into actions, the strategy will be a nondecreasing step function. Thus, we can restate the player's problem as determining at which realizations of his type the strategy will "step" to the next highest action, as well as what action is taken at the "step point." Once the problem has been reformulated in this way, we can view a nondecreasing strategy as a subset of finite-dimensional Euclidean space, and the existence result then relies on Kakutani's fixed point theorem. The single crossing condition plays two roles in this analysis. First, it simplifies the strategies enough such that they can be represented with vectors of "step points." Second, we show that it implies that the set of vectors which represent optimal actions is convex. The paper also demonstrates that the logic of the argument can be extended to games with nonmonotonic strategies. For example, a PSNE will exist in games where every player's best response to a U-shaped strategy by the opponents is U-shaped. However, this result requires an additional assumption to guarantee that the best response correspondence is convex: we assume that players are never indifferent between two actions over an open interval of types.

Thus, the first part of this paper shows that with finite actions, PSNE exist quite generally. The second part of the paper proceeds to derive conditions under which these results extend to continuous action spaces. We show that when there exists an equilibrium in nondecreasing strategies for every finite game, and when the players' objectives are continuous, there exists a limit of a sequence of equilibrium strategies for finite games which is an equilibrium in a game with continuous actions. It is important to know that the strategies are monotonic (or satisfy related properties such as a U-shape) because when the strategies are of bounded variation, a sequence of equilibrium strategies for finite games has an almost-everywhere convergent subsequence (by

Helly's selection theorem (Billingsley, 1968)). However, in auctions and related games where winning gives a discrete change in payoffs, the players' objectives are not in general continuous in their own action, and thus establishing existence in such games requires additional work. Using properties of the equilibria to finite-action games, we show that if an auction game satisfies the single crossing property described above, so that best responses to nondecreasing strategies are nondecreasing, plus some additional regularity conditions, then existence result from the finite action auction game will extend to auctions with continuous action spaces.

Analyzing existence in games with a continuum of types and a continuum of actions is difficult because the strategy space is not a finite subset of Euclidean space. Many previous results about existence of pure strategy equilibria are concerned with issues of topology and continuity in the relevant strategy spaces. For example, Milgrom and Weber (1985) show that pure strategy equilibria exist when type spaces are atomless and players choose from a finite set of actions, types are independent conditional on some common state variable (which is finite-valued), and each player's utility function depends only on his own type, the other players' actions, and the common state variable (the utility cannot depend on the other players' types directly). They also study a condition which they call "continuity of information." Similarly, Radner and Rosenthal (1982) show that players choose from a finite set of actions, types are independent,[1] and each player's utility function depends only on his own type, but the type distributions are atomless, then a pure strategy equilibrium will exist. The authors then provide several counter-examples of games which fail to have pure strategy equilibria, in particular games where players' types are correlated. The counter-examples of Radner and Rosenthal fail our sufficient conditions for existence in games with finite actions a different reason: the best response of one player is always a little bit more complicated than the strategy of his opponent.

In contrast, our analysis allows arbitrary correlation between types, to the extent that the joint distributions over types lead to expected payoff functions which satisfy the single crossing property of incremental returns. Thus, any required restrictions on the distribution (such as affiliation) have economic interpretations. Weber (1994) studies mixed strategy equilibria in auction a class of auction games where the affiliation inequality fails; this paper shows that affiliation is not essential except to the extent that it implies monotonicity properties of the players' objectives.

Now consider the special case of first price auctions. The issue of the existence of pure strategy equilibria in first price auctions with heterogeneous agents (and continuous actions spaces)

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[1] Radner and Rosenthal (1982) also treat the case where each player can observe a finite-valued "statistic" about a random variable which affects the payoffs of all agents.

has challenged economists for many years. Recently, several authors have made substantial progress, establishing existence and sometimes uniqueness for asymmetric independent private values auctions (Maskin and Riley (1993, 1996); Lebrun (1995, 1996); Bajari (1996a)), as well as affiliated private values or common value auctions with conditionally independent signals (Maskin and Riley, 1996)). Lizzeri and Persico (1997) have independently shown that a condition closely related to the single crossing condition is sufficient for existence and uniqueness of equilibrium in two-player mineral rights auction games with heterogeneous bidders, but their approach does not extend to more than two bidders without symmetry assumptions. Many interesting classes of auctions with heterogeneous bidders are not treated by the existing analysis, and even for the auctions where existence is known, computation of equilibrium (which involves numerically computing the solution to a system of nonlinear differential equations with two boundary points) can be difficult due to pathological behavior of the system.[2]

Two main approaches to existence have been used in the case of first price auctions: (i) establishing that a solution exists to a set of differential equations (Lebrun (1995), Bajari (1996a), Lizzeri and Persico (1997)), and (ii) establishing that an equilibrium exists when either types or actions are drawn from finite sets, and then invoking limiting arguments (Lebrun (1996), Maskin and Riley (1992)). Since there exist games (with discontinuous payoffs) which have pure strategy equilibria for every finite action set, but where there is no pure strategy equilibrium in the infinite case (for example, see Fullerton and McAfee (1996)), these limiting arguments generally involve more work and use the special structure of the game at hand.[3] The strategy taken in this paper is different from that of the existing literature, in that we treat the issue of existence of equilibrium separately from the issue of monotonicity of strategies in different classes of auctions. However, we do rely on limiting arguments, and use the special structure of a "winner-take-more" game to prove that discontinuities do not in fact arise in the limit.

The third part of the paper derives conditions under which the single crossing condition holds for different classes of auctions. This distinction is useful because in some classes of auctions (i.e. private value auctions), it is relatively easy to establish that the single crossing condition holds, while it can be challenging in other classes (for example, mineral rights auctions with more than two bidders, where players form expectations about their own value from winning based on their signal and the other players' actions). Knowing that the single crossing condition implies existence can also be helpful if it is possible to demonstrate that the single crossing condition holds

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[2] Marshall et al (1994) summarize some of the problems; they provide and implement an algorithm which alleviates these difficulties for a simple class of distributions, $F(x|a)= x^a$. See Section 5 of this paper for more discussion.

[3] Simon and Zame (1990) provide an elegant treatment of limiting arguments in the context of mixed strategy equilibria.

for a range of parameter values, or if it is possible to establish that it holds in the relevant region using numerical analysis.

The final part of this paper analyzes the computation of equilibria to auctions. The theoretical and computational difficulties in analyzing auctions with heterogeneous bidders have confounded attempts to apply and test auction theory in real-world problems,[4] where heterogeneity and correlation between types are the rule rather than the exception, and as well the private values assumption may be tenuous. Further, even if bidders are ex ante symmetric, if they collude or engage in joint bidding, asymmetries will arise (Marshall et al, 1994). Since very little is known about the theoretical properties of general auction games with heterogeneous bidders, numerical computation can also play an important role in suggesting avenues for future theorizing. Further, the growing literature on structural empirical analyses of auctions (i.e. Laffont, Ossard, and Vuong (1995), Bajari (1996b)) requires that equilibria to auctions be computed in each iteration of an econometric procedure.

This paper proceeds by observing that the representation of nondecreasing strategies with finite actions as a vector of "step points" implies that equilibria to auction games can be computed with very simple algorithms. Our analysis shows that the equilibria to such finite-action games get "close" to equilibria of continuous-action games as the number of actions increases. There are well-developed numerical techniques for approximating a fixed point in finite-dimensional problems (Judd, forthcoming). We provide a numerical analysis of several examples of first price auctions with alternative assumptions about heterogeneity and the type distribution.

2. Existence of Equilibrium with Finite Actions

This section derives sufficient conditions for the existence of a PSNE in a game of incomplete information, where the types of the players are drawn from an atomless distribution and the players are restricted to choose from a finite set of actions. Consider a game of incomplete information between I players, $i=1,..,I$, where each player first observes their own type $t_i \in T_i \equiv [\underline{t}_i, \bar{t}_i] \subseteq \Re$ and then takes an action a_i from an action space $\mathcal{A}^i \subset \Re$. Each player's utility, $u_i(\mathbf{a},\mathbf{t})$, may depend on the actions taken as well as the types directly. The joint density over player types is $f(\mathbf{t})$, with conditional densities $f(\mathbf{t}_{-i}|t_i)$. The objective function for player i is then specified as follows. Given any set of strategies for the opponents, $\alpha_j:[\underline{t}_j, \bar{t}_j] \rightarrow \mathcal{A}^j$, $j \neq i$, we can write player i's

[4] Only a few studies exist; Hendricks and Porter (1988) analyze the case of informed versus uninformed bidders in mineral rights auctions, while Pesendorfer (1995) uses a model of an asymmetric private value auction to analyze asymmetric auctions for school milk contracts. Bajari (1996b) uses a structural econometric approach to study asymmetric construction auctions.

objective function as follows (using the notational convention $(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}))\equiv$ $(..,\alpha_{i-1}(t_{i-1}),a_i,\alpha_{i+1}(t_{i+1}),..)$):

$$U_i(a_i,\boldsymbol{\alpha}_{-i}(\cdot),t_i)\equiv\int_{t_{-i}}u_i((a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i})),\mathbf{t})f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$$

The following basic assumptions are maintained throughout the paper:

> The types have joint density with respect to Lebesque measure, $f(\mathbf{t})$, which is bounded and has no mass points.[5] Further, $\int_{\mathbf{t}_{-i}\in S}u_i((a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i})),\mathbf{t})f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$ exists and is finite for all S and all $\alpha_j:[\underline{t}_j,\bar{t}_j]\rightarrow\mathcal{A}^j$, $j\neq i$. (2.1)

We proceed by proving two results. The first looks for pure strategy equilibria in nondecreasing strategies, while the second potentially allows for more complicated strategies. However, the second result requires that players are never indifferent between two strategies over an open interval of types (this assumption would not be satisfied for an auction without additional structure, since two actions might both win with probability zero against a more aggressive opponent).

2.1. *Pure Strategy Equilibrium in Nondecreasing Strategies*

This section studies games with finite action spaces which satisfy the single crossing condition. The single crossing will play two roles in the analysis. First, it will guarantee that we can represent each agent's strategy with a vector of finite dimension. Second, it will be used to guarantee that each player's best response correspondence is convex (recall that we have not made assumptions which could be used to guarantee a unique optimum). These two properties of games with the single crossing condition allow us to use Kakutani's fixed point theorem to guarantee existence of a PSNE.

Before beginning our analysis of equilibria in nondecreasing strategies, we introduce the definition of Milgrom and Shannon's (1994) single crossing property of incremental returns (SCP-IR), as well as the corresponding theorem which states that SCP-IR is sufficient for a monotone comparative statics conclusion to hold in problems which may be non-differentiable, non-concave, or have multiple optima. Consider the following definition:

Definition 2.1 *$h(x,\theta)$ satisfies the (Milgrom-Shannon) single crossing property of incremental returns (SCP-IR) in $(x;\theta)$ if, for all $x_H>x_L$, $g(\theta)=h(x_H,\theta)-h(x_L,\theta)$ satisfies the*

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5 In games with finite actions, condition (2.1) can be relaxed to allow for mass points at the lower end of the distribution, so long as for each player, there exists a $k>\underline{t}_j$ such that the lowest action chosen by player j is chosen throughout the region $[\underline{t}_j,k)$.

following conditions for all $\theta_H > \theta_L$: (a) $g(\theta_L) \geq 0$ implies $g(\theta_H) \geq 0$, and (b) $g(\theta_L) > 0$ implies $g(\theta_H) > 0$.

The definition requires that if x_H is (strictly) preferred to x_L for θ_L, then x_H must still be (strictly) preferred to x_L if θ increases from θ_L to θ_H. In other words, the incremental returns to x cross zero at most once, from below, as a function of θ (Figure 1). Note the relationship is *not* symmetric between x and θ. Milgrom and Shannon show that SCP-IR implies that the set of optimizers is nondecreasing in the Strong Set Order (SSO), defined as follows:

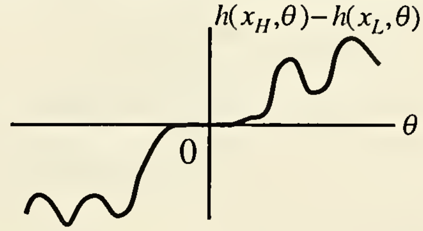

Figure 1: h satisfies SCP-IR if, for all $x_H \geq x_L$, the function $h(x_H,\theta)-h(x_L,\theta)$ crosses zero at most once, from below, as a function of θ.

Definition 2.2 *A set $A \subseteq \Re$ is greater than a set $B \subseteq \Re$ in the strong set order (SSO), written $A \geq_S B$, if, for any $a \in A$ and any $b \in B$, $\max(a,b) \in A$ and $\min(a,b) \in B$. A set-valued function $A(\tau)$ is nondecreasing in the strong set order (SSO) if for any $\tau_H > \tau_L$, $A(\tau_H) \geq_S A(\tau_L)$.*

Lemma 2.1 *(Milgrom and Shannon, 1994) Let $h: \Re^2 \rightarrow \Re$. Then h satisfies SCP-IR if and only if $x^*(\theta) \equiv \arg\max_{x \in B} h(x,\theta)$ is nondecreasing in the strong set order for all B.*

Under SCP-IR, there might be a $x' \in x^*(\theta_L)$ and a $x'' \in x^*(\theta_H)$ such that $x' > x''$, so that some selection of optimizers is decreasing on a region; however, if this is true, then $x' \in x^*(\theta_H)$ as well. Using Definition 2.1, we can state the sufficient condition for existence of a pure strategy Nash equilibria in nondecreasing strategies.

> We say that a game satisfies the ***Single Crossing Condition*** **(SCC)** *for games of incomplete information* if, for $i=1,..,I$, given any set of $I-1$ nondecreasing functions $\alpha_j:[\underline{t}_j, \bar{t}_j] \rightarrow \mathcal{A}^j$, $j \neq i$, player i's objective function, $U_i(a_i, \alpha_{-i}(\cdot), t_i)$, satisfies single crossing of incremental returns (SCP-IR) in $(a_i;t_i)$. (2.2)

The first result is then stated as follows:

Theorem 2.1 *Consider the game of incomplete information described above, where (2.1) and the Single Crossing Condition (2.2) hold. If $\mathcal{A}^i$ is finite for all i, this game has a PSNE, where each player's equilibrium strategy, $\beta_i(t_i)$, is a nondecreasing function of t_i.*

The proof of Theorem 2.1 relies on the fact that, when players choose from a finite set, any nondecreasing strategy $\alpha_i(t_i)$ is simply a nondecreasing step function. Then, the strategy can be described simply by naming the values of the player's type t_i at which the player "jumps" from one action to the next higher action. To formalize this idea, consider the following representation of nondecreasing strategies. For simplicity, we treat the case where each player has the same feasible action space, but this assumption is made purely to conserve notation. Let $\mathcal{A}=\{A_0,A_1,..,A_M\}$ be the set of potential actions, in ascending order, where $M+1$ is the number of potential actions. Let $T_i^M \equiv \bigtimes_{m=1}^{M} [\underline{t}_i, \bar{t}_i]$, and define the set of nondecreasing vectors in T_i^M as follows: $\Sigma_i^M \equiv \{\mathbf{x} \in T_i^{M+2} | x_0 = \underline{t}_i, x_1 \leq x_2 \leq \cdot\cdot \leq x_M, x_{M+1} = \bar{t}_i\}$. Further, let $\Sigma \equiv \Sigma_1^M \times \cdot\cdot \times \Sigma_I^M$.

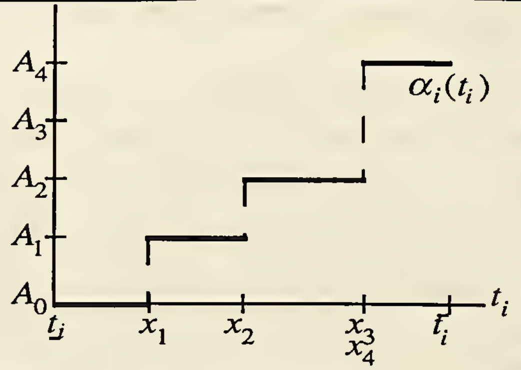

Figure 2: The elements of the vector **x** specify when the agent "jumps" to a higher action.

Consider a candidate nondecreasing strategy for player i, α_i:$[\underline{t}_i, \bar{t}_i] \rightarrow A$. For any such function α_i, we can assign a corresponding vector $\mathbf{x} \in \Sigma_i^M$ according to the following algorithm (illustrated in Figure 2).

Definition 2.3 (i) Given a nondecreasing strategy $\alpha_i(t_i)$, we say that "the vector $\mathbf{x} \in \Sigma_i^M$ *represents $\alpha_i(t_i)$" if $x_m = \inf\{t_i | \alpha_i(t_i) \geq A_m\}$ whenever there is some $n \geq m$ such that $\alpha_i(t_i) = A_n$ on an open interval of T_i, and $x_m = \bar{t}_i$ otherwise.*

(ii) Given $\mathbf{x} \in \Sigma_i^M$*, let* $\{\mathbf{x}\}$ *denote the set* $\{\underline{t}_i, x_1, .., x_M, \bar{t}_i\}$*, and let* $m^*(t,\mathbf{x}) \equiv \max\{m | x_m < t\}$. *Then, we say a nondecreasing "strategy $\alpha_i(t_i)$ is consistent with* $\mathbf{x}$*" if $\alpha_i(t_i) = A_{m^*(t_i,\mathbf{x})}$ for all t_i* $\in T_i \backslash \{\mathbf{x}\}$.

Each component of $\mathbf{x}$ is therefore a point of discontinuity, or a "jump point," of the step function described by α_i. Simply by naming the $\mathbf{x}$ which corresponds to α_i, we have specified what action the player takes everywhere except at the jump points and endpoints, that is, the actions are specified on $T_i \backslash \{\mathbf{x}\}$.

To interpret part (ii), note that $\alpha_i(t_i)$ is consistent with $\mathbf{x}$ if action A_m is used between x_m and x_{m+1}. Since $\mathbf{x}$ does not specify behavior for $t_i \in \{\mathbf{x}\}$, a given $\mathbf{x} \in \Sigma_i$ might correspond to more than one nondecreasing strategy. However, because there are no atoms in the distributions of types, a player's behavior on the set $\{\mathbf{x}\}$ (which has measure zero) will not affect the best responses of other players. The proof makes use of this fact, proceeding by finding a fixed point in the space of nondecreasing best responses which can be described by elements of Σ, and then filling in optimal behavior for each player at the jump points. Behavior at the jump points can be assigned arbitrarily since it won't affect the best responses of the other players; thus, it can be assigned to be optimal without loss of generality, making the resulting strategies best responses for all types.

Now consider notation for the objective function faced by a given player. Let $\mathbf{X}$ denote the vector $(\mathbf{x}^1, .., \mathbf{x}^I)$, which describes a set of step functions for each player. Consider player 1 with type t_1, and let $V_1(a_1;\mathbf{X},t_1)$ denote the expected payoffs to player 1 when player 1 chooses $a_1 \in \mathcal{A}$

and players 2,..,I use strategies which are consistent with $(\mathbf{x}^2,..,\mathbf{x}^I)$. Then $V_1(a_1;\mathbf{X},t_1)$ can be written as follows:

$$V_1(a_1;\mathbf{X},t_1) \equiv \sum_{m_2=0}^{M} \cdots \sum_{m_I=0}^{M} \int_{\mathbf{t}_{-1}} u_1(a_1,A_{m_2},...,A_{m_I},\mathbf{t}) \cdot \mathbf{1}_{[x^2_{m_2},x^2_{m_2+1})\times\cdots\times[x^I_{m_I},x^I_{m_I+1})}(\mathbf{t}_{-1}) \cdot f(\mathbf{t}_{-1}|t_1)d\mathbf{t}_{-1}$$

$$= \sum_{m_2=0}^{M} \cdots \sum_{m_I=0}^{M} \int_{t_2=x^2_{m_2}}^{x^2_{m_2+1}} \cdots \int_{t_I=x^I_{m_I}}^{x^I_{m_I+1}} u_1(a_1,A_{m_2},...,A_{m_I},\mathbf{t}) \cdot f(\mathbf{t}_{-1}|t_1)d\mathbf{t}_{-1} \qquad (2.3)$$

where $\mathbf{1}_S(\mathbf{y})$ is an indicator function which takes the value 1 if $\mathbf{y} \in S$ and 0 otherwise.

By (2.1), the behavior of opponents on sets of measure zero do not affect player i, so player i views each opponent as using a nondecreasing strategy whenever they use actions consistent with $\mathbf{x}^i$ on $T_i \backslash \{\mathbf{x}^i\}$. Then, by (2.2), $V_i(a_i;\mathbf{X},t_i)$ satisfies the SCP-IR in $(a_i;t_i)$. Let $a_i^{BR}(t_i|\mathbf{X}) = \arg\max_{a_i \in \mathcal{A}^i} V_i(a_i;\mathbf{X},t_i)$; this is nonempty for all t_i by finiteness of $\mathcal{A}^i$. By Lemma 2.1, there exists a selection, $\gamma_i(t_i) \in a_i^{BR}(t_i|\mathbf{X})$, from the set which is nondecreasing in t_i (in particular, the lowest and highest members of this set are nondecreasing; see Milgrom and Shannon (1994)). As we argued above, there exists a $\mathbf{y} \in \Sigma_i$ which represents $\gamma_i(t_i)$, so that $\gamma_i(t_i)$ is consistent with $\mathbf{y}$. Thus, there is at least one best response vector $\mathbf{y}$ which represents an optimal strategy. Now define the set of all such vectors as follows:

$$\Gamma_i(\mathbf{X}) = \{\mathbf{y}: \exists\ \alpha_i(t_i) \text{ which is consistent with } \mathbf{y} \text{ such that } \alpha_i(t_i) \in a_i^{BR}(t_i|\mathbf{X})\}.$$

The existence proof proceeds by showing that a fixed point exists for this correspondence. Once the problem is formulated in this way, it is straightforward to verify that the correspondence $\Gamma(\mathbf{X})$ is nonempty and has a closed graph. However, convexity of the correspondence requires additional work. Observe that even if the player's payoff function is strictly quasi-concave, and even if a change in the player's type changes payoffs everywhere, the player still might be

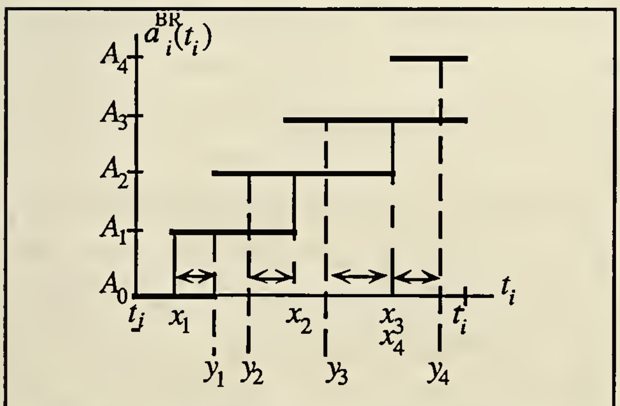

Figure 3: The set $a_i^{BR}(t_i)$ is nondecreasing in the Strong Set Order. The vectors **x** and **y** represent "jump points" corresponding to optimal strategies. The arrows indicate convex combinations of **x** and **y**.

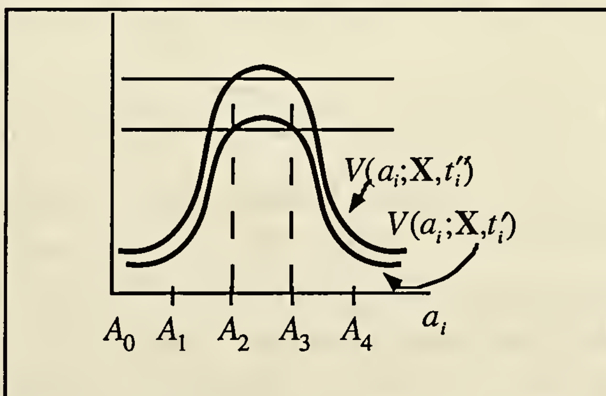

Figure 4: Even if V is strictly quasi-concave in a_i, the agent may be indifferent between two actions for a range of types.

indifferent between two actions over a set of types, as shown in Figure 4. Thus, it is important to address the issue of multiple optimal actions. The proof makes use of an important consequence of the SCP-IR: it implies that the set of best response actions is increasing in the strong set order. In Figure 3, notice that as t_i increases, higher actions come into the set of optimizers, but once a lower action leaves the set of optimizers, it never reappears. Further, once a given action has entered the set of optimizers, no lower actions enter the set for the first time. When this property is satisfied, it is straightforward to show that the set of vectors which represent optimal behavior will be convex. In the figure, **x** and **y** are both vectors of jump points representing optimal behavior; the arrows in the figure show convex combinations of x_m and y_m for $m=1,..,4$, and any such convex combination also represents optimal behavior.

> *Lemma 2.2: Define* Γ_i *as above,* $i=1,..,I$. *Then there exists a fixed point of the correspondence* $(\Gamma_1(\mathbf{X}),.., \Gamma_I(\mathbf{X})):\Sigma\to\Sigma$.
>
> **Proof of Lemma:** The proof proceeds by checking that the correspondence $\Gamma\equiv(\Gamma_1(\mathbf{X}),..,\Gamma_I(\mathbf{X}))$ is nonempty, has a closed graph, and is convex-valued. Then Kakutani's fixed point theorem will give the result. The details are in the Appendix.

With this result in place, all that remains in proof of Theorem 2.1 is to assign strategies to players which are consistent with the fixed point of $\Gamma(\mathbf{X})=\mathbf{X}$. By definition, $\Gamma(\mathbf{X})$ describes strategies for each player which are optimal almost everywhere in response to behavior by the other players which is consistent with **X**, and that each player does not care what the other players do on a set of measure 0. Thus, for each player i we can assign any behavior we like to the "jump points," $\{\mathbf{x}^i\}$, without affecting the best responses of the opponents. Then, all that remains is to fill in the behavior of each player at the "jump points." Consider an **X** such that $\mathbf{X}\in\Gamma(\mathbf{X})$. The correspondence Γ was constructed so that each $\mathbf{x}^i$ represents a nondecreasing, optimal strategy for player i given **X**, call it $\beta_i(t_i)$. Then the vector of nondecreasing strategies $(\beta_1(t_1),..,\beta_I(t_I))$ is a pure strategy Nash Equilibrium of the original game, since $\beta_i(t_i)\in a_i^{BR}(t_i|\mathbf{X})$ for $i=1,..,I$ and $t_i\in T_i$.

2.2. *An Existence Theorem for Strategies of Limited Complexity*

Now, we turn to a generalization of Theorem 2.1 beyond games with the single crossing condition (2.2). The basic idea is that an equilibrium will exist if we can find bounds on the "complexity" of each player's strategy such that each player's best response stays within those bounds when the opponents use strategies which can be represented within those bounds. The formalization we use for representing strategies builds directly on our representation of nondecreasing strategies (there are, of course, alternative representations). We will need a definition, as follows:

Definition 2.4 *The strategy* $\alpha_i(t_i)$ *has at most* K *direction changes if there exists a nondecreasing vector* $\mathbf{z} \in \Sigma_I^K$, *such that for all* $0 \leq k \leq K$ *the following holds:*

(i) If k is even, then for all $z_k \leq t_i' \leq t_i'' \leq z_{k+1}$, $\alpha_i(t_i') \leq \alpha_i(t_i'')$.

(ii) If k is odd, then for all $z_k \leq t_i' \leq t_i'' \leq z_{k+1}$, $\alpha_i(t_i') \geq \alpha_i(t_i'')$.

We say that such a $\mathbf{z}$ *represents the direction changes of* $\alpha_i(t_i)$.

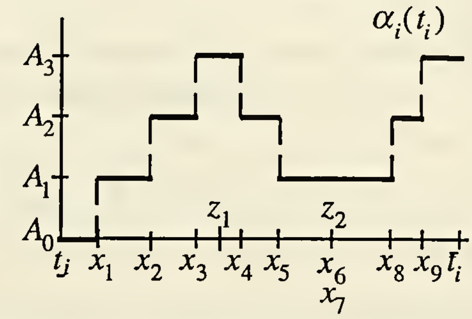

Figure 5: A representation of a strategy with two direction changes and four actions (M=3).

This definition, illustrated in Figure 5, merely formalizes the idea that a strategy changes from nondecreasing to nonincreasing, or vice versa, at most K times. Thus, a nondecreasing strategy has at most 0 direction changes. An important feature of strategies with at most K direction changes is that they are functions of bounded variation; this will imply that our existence results extend to games with continuous action spaces in Section 3.

We can then generalize condition (2.2) as follows:

> A game of incomplete information satisfies the ***Limited Complexity Condition*** if there exists a vector of nonnegative integers $\mathbf{K}=(K_1,..,K_I)$, such that, for all i, if each of player i's opponents $j \neq i$ use strategies which have at most K_j direction changes, and $\mathcal{A}$ is finite, then there exists a best response for player i which has at most K_i direction changes. Further, when opponents use such strategies, then for all $a_i \neq a_i'$, there is no open interval $T_i' \subseteq T_i$ such that $U_i(a_i, \boldsymbol{\alpha}_{-i}(\cdot), t_i) = U_i(a_i', \boldsymbol{\alpha}_{-i}(\cdot), t_i)$ for all $t_i \in T_i'$. (2.4)

Unlike Theorem 2.1, where we could prove convexity of the best response correspondence from the SCP-IR, here we require an additional assumption which implies that the best response action is unique for almost all types. This in turn implies that there will be a unique vector of jump points representing the best response strategy. To see why this assumption is required, consider Figure 6, where a player is indifferent between two actions over two regions of types. There are two vectors, $\mathbf{x}$ and $\mathbf{y}$, both of which represent optimal behavior. However, the convex combination of $\mathbf{x}$ and $\mathbf{y}$ would assign the player to use action A_1 in

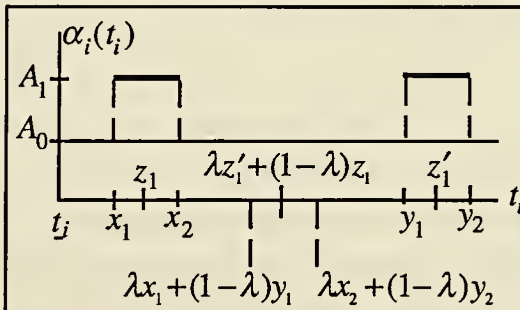

Figure 6: Player i is indifferent between actions A_0 and A_1 over an open intervals of types, violating (2.4), and the set of vectors representing optimal behavior is not convex.

the region $(\lambda x_1+(1-\lambda)y_1, \lambda x_2+(1-\lambda)y_2)$, which is not optimal. Condition (2.4) rules this out.

While the Limited Complexity Condition may arise naturally in some problems (the case of U-shaped strategies seems especially promising), another possible motivation for the condition could be bounded rationality on the part of the players.

Under condition (2.4), we can extend the logic of Theorem 2.1 in a straight-forward way. The representation is a natural extension of the one developed for nondecreasing strategies; the vector which represents our strategy will have K subvectors, each of which describes behavior on a monotonic portion of the strategy. This is formalized in the appendix, as part of the proof of the following theorem.

Theorem 2.2 Consider a game of incomplete information, where (2.1) and the Limited Complexity Condition (2.4) hold for some **K**. *Then this game has a PSNE where each player's strategy,* $\beta_i(\cdot)$, *has at most* K_i *direction changes.*

It is interesting to discuss the relationship between this result and the results from the existing literature, especially Radner and Rosenthal (1982). On the one hand, condition (2.4) seems quite general: we do not need it to hold for all **K**, but rather just for some **K**. What it rules out is games where a given player's response to a "simple" strategy becomes ever more complicated. Radner and Rosenthal maintain assumption (2.1) and finite actions, and further they require independence of types. Under those conditions, they find existence of a pure strategy equilibrium. They then present counterexamples, such as an incomplete-information variant of matching pennies, where correlated information leads to nonexistence of a pure strategy equilibrium.

Since our results do not place ex ante restrictions on the joint distribution over types, it is interesting to revisit their example. The setup is as follows: the game is zero-sum, and each player can choose actions A_0 or A_1. When the players match their actions, player 2 pays \$1 to player 1, while if they do not match, the players each receive zero. The types do not directly affect payoffs, and are uniformly distributed on the triangle $0 \le t_1 \le t_2 \le 1$. We now argue that this game fails condition (2.4). Since player i is only indifferent between the actions if $\Pr(a_j=A_0|t_i)=.5$, and since $\Pr(a_j=A_0|t_i)$ cannot be constant in t_i when player j uses a finite number of direction changes, the there will be no open interval on which player i is indifferent between the two actions. The issue is that whenever player 1 uses a strategy with K direction changes (i.e. alternates between A_0 and A_1 K times, starting with A_0), player 2 potentially has the incentive to switch between A_0 and A_1 K times as well, but starting with A_1 due to the incentive of player 1 to avoid a match. This can be represented as $K+1$ direction changes using Definition 2.4, with the first change degenerate. However, in response to an arbitrary strategy by player 2 with $K+1$ direction changes, player 1 wishes to use a strategy with $K+1$ direction changes. In turn, such a strategy by player 1 will induce a response by player 2 represented by $K+2$ direction changes. Notice that this logic does not change if we reorder the action space for player 2; the fact that player 2 tries to "run away" from player 1's strategy, but then player 1 tries to "catch" player 2, leads to a situation where strategies can potentially get progressively more complex, and thus our condition (2.4) fails.

3. Existence of Equilibrium in Games with a Continuum of Actions

This section shows that the results about existence in games with a finite number of actions can be used to construct equilibria of games with a continuum of actions. As discussed in the introduction, the assumption about finite actions plays a dual role in our analysis. First, it guarantees existence of an optimal action for every type. The second role is to simplify the description of strategies so that they can be represented with finite-dimensional vectors. In games where payoff functions are continuous in actions, we no longer rely on the assumption of finite actions for the first purpose. The arguments in this section then show how the existence of equilibrium in a sequence of finite games will imply existence in a game with a continuum of actions. We also extend the results to classes of games, such as first price auctions, which potentially have discontinuities in payoffs when an increase in a player's action causes a discrete change in the probability of "winning."

The properties of the equilibrium strategies implied by the Single Crossing Condition or the Limited Complexity Condition play a special role in this section. While arbitrary sequences of functions need not have convergent subsequences, sequences of nondecreasing functions (or more generally, functions of bounded variation, which can be expressed as the difference between two nondecreasing functions (Billingsley, 1986, p. 435)) do have almost-everywhere convergent subsequences by Helly's Theorem. Thus, our restrictions play two roles in the existence results: first, they guarantees that the strategies in finite games can be represented with a finite vector so that Kakutani's fixed point theorem can be applied, and second, they guarantee that sequences of equilibria to finite games have almost-everywhere convergent subsequences. All that remains to show is that the limits of these sequences are in fact equilibria to the continuous action game.

3.1. Games with Continuous Payoff Functions

This section extends the results of Section 2 to games with a continuum of actions. The following theorem shows that in a game of incomplete information, if payoffs are continuous and all finite-action games have equilibria in functions of bounded variation, then the continuum-action game will have an equilibrium as well.

Theorem 3.1 *Consider a game of incomplete information which satisfies (2.1). Suppose that (i)* $\mathcal{A} \equiv (\mathcal{A}^1,..,\mathcal{A}^I)$, *where* $\mathcal{A}^i = [\underline{a}_i, \overline{a}_i]$, *(ii) for all i,* $u_i(\mathbf{a},\mathbf{t})$ *is continuous in* $\mathbf{a}$ *on* $[\underline{a}_i, \overline{a}_i]$, *and (iii) for any finite* $\mathcal{A}' \subset \mathcal{A}$, *a PSNE exists in the game where the players choose actions from* $\mathcal{A}'$, *where the equilibrium strategies* $\boldsymbol{\beta}_n(\mathbf{t})$ *are functions of bounded variation.*

Then a PSNE exists in the game where players choose actions from $\mathcal{A}$.

Corollary 3.1.1 *Consider a game of incomplete information which satisfies (2.1). Suppose that (i)* $\mathcal{A} \equiv (\mathcal{A}^1,..,\mathcal{A}^I)$, *where* $\mathcal{A}^i = [\underline{a}_i, \overline{a}_i]$, *and (ii) for all i,* $u_i(\mathbf{a},\mathbf{t})$ *is continuous in* $\mathbf{a}$ *on* $[\underline{a}_i, \overline{a}_i]$. *Then:*

(i) If the Single Crossing Condition (2.2) holds, then there exists a PSNE, $\boldsymbol{\beta}^*(\mathbf{t})$, *where each player's strategy is nondecreasing in her type.*

(ii) If the Limited Complexity Condition (2.4) holds for some $\mathbf{K}$, *then there exists a PSNE,* $\boldsymbol{\beta}^*(\mathbf{t})$, *where each player i's strategy has at most* K_i *direction changes.*

While Corollary 3.1.1 does require the regularity assumption that u_i is continuous and integrable, it does *not* require differentiability or quasi-concavity, two assumptions which often arise in alternative approaches. Furthermore, it does not place any additional restrictions on the correlation structure between types above what is required for the (economically interpretable) condition that best responses are nondecreasing.

Despite the generality of Corollary 3.1.1, the restriction that payoffs be continuous still rules out many interesting games. The next section extends this result to games with discontinuous payoffs such as auctions.

3.2. *Games with Discontinuities: Auctions and Pricing Games*

This section studies games whose payoffs exhibit a particular type of discontinuity. A player sees a discrete change in her payoffs depending on whether she is a "winner" (i.e., the high bidder in a single-unit auction or the low price in a pricing game), or a "loser." Examples include auctions (first price or all-pay), pricing games, and more general mechanism design problems where the goal is to allocate resources between multiple players. Winners receive payoffs $\overline{v}_i(a_i,\mathbf{t})$, while losers receive payoffs $\underline{v}_i(a_i,\mathbf{t})$. The allocation rule $\varphi_i(\mathbf{a})$ specifies the probability that the player receives the winner's payoffs as a function of the actions taken by all players. We will restrict attention to games with only two outcomes. Under our assumptions, then, a player's payoffs given a realization of types and actions is given as follows:

$$u_i(\mathbf{a},\mathbf{t}) = \varphi_i(\mathbf{a}) \cdot \overline{v}_i(a_i,\mathbf{t}) + (1-\varphi_i(\mathbf{a})) \cdot \underline{v}_i(a_i,\mathbf{t}) \qquad (3.1)$$

This formulation highlights the second assumption implicit in this formulation: payoffs depend on the other players actions only through the allocation, not through payoffs directly. This assumption can be relaxed, but it sufficiently complicates the analysis that we will not consider it here.

Each player chooses from a set of actions, $\mathcal{A}=Q\cup[\underline{a}_i,\overline{a}_i]$. The action $Q<\min_i\{\underline{a}_i\}$ guarantees each player zero expected payoffs. We will maintain the following assumption about payoffs:

$$\overline{v}_i(Q,\mathbf{t})=\underline{v}_i(Q,\mathbf{t})=0 \text{ for all } \mathbf{t}. \tag{3.2}$$

For all $a_i\in[\underline{a}_i,\overline{a}_i]$ and all $\mathbf{t}$,

$$\text{(i) } \overline{v}_i(a_i,\mathbf{t})>0 \text{ implies } \overline{v}_i(a_i,\mathbf{t}) > \underline{v}_i(a_i,\mathbf{t}), \text{ and (ii) } \overline{v}_i(a_i,\mathbf{t})\leq 0 \text{ implies } \underline{v}_i(a_i,\mathbf{t})\leq 0. \tag{3.3}$$

There are several classes of examples which have this structure. In a first-price auction, the winner receives the object and pays her bid, while losers get payoffs of $\underline{v}_i(a_i,\mathbf{t})=0$. In Milgrom and Weber's (1982) formulation of the mineral rights auction, $\overline{v}_i(a_i,\mathbf{t})$ represents the expected payoffs to the bidder conditional on the vector of type realizations, and the vector $\mathbf{t}$ is interpreted as a vector of signals about each player's true value for the object (where signals and values may be correlated). In an all-pay auction, the player pays her bid no matter what, but the winner receives the object. In some pricing games, the lowest price (the highest action) implies that the firm captures a segment of price-sensitive consumers, while having a higher price implies that the firm only serves a set of local customers.

We will restrict attention to allocation rules which take the following form:

$$\varphi_i(\mathbf{a})=\sum_{\substack{\{\sigma_L,\sigma_T\}\subseteq\{1,\ldots,I\}\setminus i\\ \text{s.t. } \sigma_L\cap\sigma_T=\varnothing}}\left[\mathbf{1}_{\{|\sigma_L|\geq I-k\}}+\mathbf{1}_{\{|\sigma_L|+|\sigma_T|\geq I-k>|\sigma_L|\}}\rho(\sigma_T)\right]\cdot\prod_{j\in\sigma_L}\mathbf{1}_{\{m_j(a_j)<m_i(a_i)\}}\cdot\prod_{j\in\sigma_T}\mathbf{1}_{\{m_j(a_j)=m_i(a_i)\}} \tag{3.4}$$

In this expression, $m_i(\cdot)$ is a strictly increasing function. Player i receives the object with probability zero if k or more opponents choose actions such that $m_j(a_j)>m_i(a_i)$, and with probability 1 if $I-k$ or more opponents choose actions such that $m_j(a_j)<m_i(a_i)$. The remaining events are "ties," in which case $\rho:2^{\{1,\ldots,I\}}\rightarrow[0,1]$ is the probability that player i wins. We further assume that:

$$\text{If } |\sigma_T|>0, \text{ then } \rho(\sigma_T)<1. \tag{3.5}$$

This assumption requires that if player i ties for winner with a non-empty subset of players, no player wins with probability 1. This last assumption simplifies the proof, but can be relaxed.

To interpret expression (3.4), consider the example of a first price auction for a single object. In this case, k=1 and $m_j(a_j)=a_j$. That is, the player wins with probability zero if 1 or more opponents place a higher bid. If ties are broken randomly, the probability of winning given that no opponents place a higher bid and further the player ties with the subset of opponents σ_T is given by $\rho(\sigma_T)=\frac{1}{|\sigma_T|+1}$. More general mechanism design problems also fall into this framework. When a player's payoffs satisfy the single crossing property, only direct mechanisms in which the allocation rule is monotonic will be incentive compatible; thus, any incentive-compatible rule for allocating k winners from I players will take the form of (3.4). Further, consider an optimal mechanism to allocate a single object between risk neutral agents whose (independently distributed) types represent their valuations for the object. In this case, the actions are reports of types, and the allocation rule determines the winner according to the agent who has the highest "virtual type" (Myerson, 1981).

Thus, the players' expected payoffs can be written as follows:

$$
\begin{aligned}
&U_i(a_i, \boldsymbol{\alpha}_{-i}(\cdot), t_i) \\
&= \int u_i(a_i, \boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}), \mathbf{t}) f(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i} \\
&= \int \left[\varphi_i(a_i, \boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i})) \cdot \overline{v}_i(a_i, \mathbf{t}) + (1 - \varphi_i(a_i, \boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}))) \cdot \underline{v}_i(a_i, \mathbf{t})\right] \cdot f(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i} \\
&= \int \underline{v}_i(a_i, \mathbf{t}) \cdot f(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i} + \int \left[\overline{v}_i(a_i, \mathbf{t}) - \underline{v}_i(a_i, \mathbf{t})\right] \cdot \varphi_i(a_i, \boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i})) \cdot f(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i}
\end{aligned}
$$

We maintain the following additional assumptions:

For all i and $a_i \in [\underline{a}_i, \overline{a}_i]$, $\overline{v}_i(a_i,\mathbf{t})$ and $\underline{v}_i(a_i,\mathbf{t})$ are bounded and continuous in $(a_i,\mathbf{t})$, and $\Delta v_i(a_i,\mathbf{t}) \equiv \overline{v}_i(a_i,\mathbf{t}) - \underline{v}_i(a_i,\mathbf{t})$ is nondecreasing in $\mathbf{t}$ and strictly increasing in $(-a_i, t_i)$. (3.6)

For all $a_i \in [\underline{a}_i, \overline{a}_i]$, all $\mathbf{k}_1 \leq \mathbf{k}_2$ such that $\mathbf{k}_1, \mathbf{k}_2 \in \text{supp}(f(_i(\mathbf{t}_{-i}))$, $E\left[\Delta v_i(a_i,\mathbf{t}) | t_i, \mathbf{k}_1 \leq \mathbf{t}_{-i} \leq \mathbf{k}_2\right]$ is strictly increasing in t_i and nondecreasing in $\mathbf{k}_1, \mathbf{k}_2$. (3.7)

Since (3.7) is the most restrictive our these assumptions, it is worth pausing to note that it has in fact been characterized by Milgrom and Weber (1982) for the case where Δv_i is nondecreasing in $\mathbf{t}$, as assumed in (3.6) (log-supermodularity of densities is discussed in more detail in Section 4 and in the Appendix).

Lemma 3.2.1 *Consider a conditional density $f(\mathbf{t}_{-i}|t_i)$ which is log-supermodular a.e. Then $E\left[g_i(\mathbf{t}) | t_i, \mathbf{k}_1 \leq \mathbf{t}_{-i} \leq \mathbf{k}_2\right]$ is nondecreasing in $(t_i, \mathbf{k}_1, \mathbf{k}_2)$ for all g_i nondecreasing if and only if $f(\mathbf{t}_{-i}|t_i)$ is log-supermodular a.e. (equivalently, $\mathbf{t}$ is affiliated).*

With these assumptions in place, we can state our existence result, proved in the Appendix:

Theorem 3.2 *Consider a game which satisfies (2.1). Let the action space for each player be $\mathcal{A} = Q \cup [\underline{a}_i, \overline{a}_i]$, assume that the support of the distribution $F(\mathbf{t})$ is a product set, and assume (3.1)-(3.7). Suppose further that the Single Crossing Condition, (2.2), holds for $a_i \in [\underline{a}_i, \overline{a}_i]$. Then, there exists a PSNE, $\boldsymbol{\beta}^*(\mathbf{t})$, where each player's strategy is nondecreasing in her type.*

Once existence is established for the continuous-action game, standard arguments can be used to verify the usual regularity properties. For example, strategies are strictly increasing on the interior of the set of actions played with positive probability, and no player sees a gap in the set of actions played with positive probability by opponents. Further, with appropriate differentiability assumptions, we can use a differential equations approach to characterize the equilibrium.

The intuition behind the proof of Theorem 3.2 can be summarized as follows. The only reason that Theorem 3.1 cannot be applied directly is that the game has a potential discontinuity in a player's own action. If the opponents use a particular action a with positive probability, then increasing one's own action to be just higher than a will lead to a discontinuous increase in the probability of winning. However, observe that if each $\beta_i^*(t_i)$ is strictly increasing on T_i, expected payoffs will be continuous in a_i, since the Lebesque measure of the sets $\left\{t_j \middle| \beta_j^*(t_j) \leq k\right\}$ and $\left\{t_j \middle| \beta_j^*(t_j) < k\right\}$ changes continuously in k, and since we have assumed that the type distributions are atomless. If, in contrast, $\beta_i^*(t_i)$ is constant at b on a subinterval S of T_i, then each player $j \neq i$ sees a discontinuity in their expected payoffs at $a_j = b$.

Let $\boldsymbol{\beta}^*(\mathbf{t})$ denote the equilibrium strategies. Our argument establishes that for each player i, almost every type t_i sees zero probability of a tie at her optimal action $\beta_i^*(t_i)$. Ruling out the possibility of such mass points in the limit involves showing that mass cannot be "adding up" close to a particular action as the action space gets finer. Recall that a sequence of nondecreasing functions has an almost-everywhere convergent subsequence, and further this subsequence converges uniformly except on a set of arbitrarily small measure. Thus, if a player's limiting strategy involves a mass point at some action b, then given a $d>0$, a positive mass of players must be using a strategy on $[b-d, b+d]$ as the action grid gets fine. But then, other players have an incentive to "jump over" that player's action b rather than using an action less than $a'-d$: with an action only slightly higher than $b+d$, the other players can beat all of the types playing on $[b-d, b+d]$. But this will in turn undermine the incentives of the first player to choose an action on $[b-d, b+d]$.

This result then generalizes the best available existence results about first price auctions. Previous studies (Maskin and Riley (1993, 1996); Lebrun (1995, 1996), Bajari (1996a)) have analyzed independent private values auctions, as well as affiliated private values auctions and common value auctions with conditionally independent signals about the object's value (Maskin and Riley (1993, 1996). The work closest to ours is Lizzeri and Persico (1997), who have independently established existence and uniqueness of equilibria in a class of games similar to the one studied above, but with the restriction to two bidders (further, their approach is based on differential equations, while differentiability of utility functions is not assumed here). The approach taken in this paper is different from those of the existing literature, in that it separates out the issue of monotonicity of strategies and existence, showing that monotonicity implies existence. Thus, the only role played by assumptions about the joint distribution over types is to guarantee that the single crossing property holds.

The next section analyzes applications, including first price auctions. To preview, however, the weakest general conditions for private value auctions requires (i) log-supermodular utility functions, which amounts to log-concave utility functions if utility takes the form $V_i(t_i-a_i)$, and (ii) affiliated types. In the more general "mineral rights" auctions, which allow for utility functions of the form $v_i(a_i,\mathbf{t})$, for two bidders we require that $v_i(a_i,\mathbf{t})$ be supermodular in (a_i,t_1) and (a_i,t_2), strictly increasing in $(-a_i,t_i)$, and that values are affiliated. We are not aware of general conditions for monotonicity in the mineral rights auction with more than two bidders; however, our numerical results indicate that monotonicity holds in the relevant range (i.e., near equilibrium) for several examples with log-normally distributed values and signals.

4. Characterizing the Single Crossing Condition in Applications

This section characterizes the single crossing condition in several classes of games of incomplete information. It applies results from Athey (1995, 1996) to describe conditions on the primitives of a game, that is, the utility functions of the players and the joint distribution of types, which are sufficient for the expected value of the utility function to satisfy SCP-IR when all other players use nondecreasing strategies. Thus, applying Theorems 2.1, 3.1, and 3.2, we are able to characterize classes of games which have PSNE.

The results in this section are grouped according to the structure of the problem: additively separable problems (such as investment games), multiplicatively separable problems (such as private value auctions), and non-separable problems (such as mineral rights auctions). Table 7.1 in the Appendix summarize our analysis from this section, stating the conditions to check for games of incomplete information which take a variety of structures. The results are applications of theorems about comparative statics in stochastic problems from Athey (1995, 1996). All of these results can be applied to derive sufficient, and sometimes necessary, conditions on payoff functions and type distributions so that the game satisfies the Single Crossing Condition.

The results make use of the properties *supermodularity* and *log-supermodularity*. Since we will be interested in product sets, we will state the definitions for that case. Let $X=\underset{n=1,..,N}{\times} X_n$ and consider an order for each X_n, which will be denoted $\geq$. Suppose that each set X_n is a totally ordered set. An example of such a product set is $\Re^N$ with the usual order, where $\mathbf{x}\geq\mathbf{y}$ if $x_n\geq y_n$ for $n=1,..,N$. We will use the operations "meet" ($\vee$) and "join" ($\wedge$), defined for product sets as follows: $\mathbf{x}\vee\mathbf{y}=(\max(x_1,y_1),..,\max(x_n,y_n))$ and $\mathbf{x}\wedge\mathbf{y}=(\min(x_1,y_1),..,\min(x_n,y_n))$.

Definition 4.1 *A function h:$X\to\Re$ is* ***supermodular*** *if, for all* $\mathbf{x},\mathbf{y}\in X$, $h(\mathbf{x}\vee\mathbf{y})+h(\mathbf{x}\wedge\mathbf{y})\geq h(\mathbf{x})+h(\mathbf{y})$. *A non-negative function h:$X\to\Re$ is* ***log-supermodular***[6] *if, for all* $\mathbf{x},\mathbf{y}\in X$, $h(\mathbf{x}\vee\mathbf{y})\cdot h(\mathbf{x}\wedge\mathbf{y})\geq h(\mathbf{x})\cdot h(\mathbf{y})$.

[6] Karlin and Rinott (1980) called log-supermodularity multivariate total positivity of order 2.

Clearly, a non-negative function $h(\mathbf{x})$ is log-supermodular if $\ln(h(\mathbf{x}))$ is supermodular. When $h: \Re^n \to \Re$, and we order vectors in the usual way, Topkis (1978) proves that if h is twice differentiable, h is supermodular if and only if $\frac{\partial^2}{\partial x_i \partial x_j} h(\mathbf{x}) \geq 0$ for all $i \neq j$. For the moment, four additional facts about these properties are important: (1) if $h(x,t)$ is supermodular or log-supermodular, then $h(x,t)$ satisfies SCP-IR; (2) sums of supermodular functions are supermodular, while products of log-supermodular functions are log-supermodular; (3) if $h(\mathbf{x})$ is supermodular (resp. log-supermodular), then so is $h(\alpha_1(x_1),..,\alpha_n(x_n))$, where $\alpha_i(\cdot)$ is nondecreasing; (4) a density is log-supermodular if and only if the random variables are *affiliated* (as defined in Milgrom and Weber, 1982).[7]

4.1. General Characterizations

4.1.1. Additively separable expected payoffs

First consider games where payoffs are given by $u_i(\mathbf{a},\mathbf{t}) = g_i(a_i,t_i) + h_i(\mathbf{a},\mathbf{t})$, and $U_i(a_i,\alpha_{-i}(\cdot),t_i) = g_i(a_i,t_i) + H_i(a_i,t_i)$. A game which fits into this framework is an all-pay auction, where $g_i(a_i,t_i)$ is the expected utility from losing the auction and having to pay a_i, while $H_i(a_i,t_i)$ gives the expected returns to winning as opposed to losing the auction, weighted by the probability of winning.

Since the sum of supermodular functions is supermodular, and since supermodularity implies the SCP-IR, supermodularity of g_i and H_i will imply the SCP-IR. The following result, due to Milgrom and Shannon (1994), shows function that supermodularity is the "right" property for additive problems.

Lemma 4.1: Consider g_i, H_i:$\Re^2 \to \Re$. $g_i(a_i,t_i)+H_i(a_i,t_i)$ satisfies single crossing of incremental returns (SCP-IR) in $(a_i;t_i)$ for all H_i which are supermodular in a_i, if and only if g_i is supermodular.

For example, in an all-pay auction, we might wish to characterize conditions on g_i which are sufficient for the SCP-IR to hold without specifying any additional structure on opponent strategies besides monotonicity, and allowing for general type distributions. Supermodularity is the weakest condition on g_i which guarantees SCP-IR of payoffs across an unrestricted class of functions H_i.

The next step is to characterize when expected values of payoff functions are supermodular. The following Lemma applies results from Athey (1995, 1996) to this problem:

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[7] See Whitt (1982) and Karlin and Rinott (1980) for related discussions of this property in statistics. Log-supermodularity of a density is also equivalent to the as *monotone likelihood ratio property* (MLRP) (Milgrom, 1981). A probability density $f(s;\theta)$ satisfies the MLRP if the likelihood ratio $f(t;\theta_h)/f(t;\theta_L)$ is nondecreasing in t for all $\theta_h > \theta_L$.

Lemma 4.2 (i) $\int_{\mathbf{t}_{-i}} k_i(a_i,\mathbf{t}) f_i(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$ *is supermodular in* (a_i,t_i) *for all* k_i *which are supermodular in* (a_i,t_j), $j=1,..,I$, *if and only if* $\int_{\mathbf{t}_{-i}\in S} f_i(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$ *is nondecreasing in* t_i *for all S such that* $\mathbf{1}_S(\mathbf{t}_{-i})$ *is nondecreasing.*

(ii) If $f(\mathbf{t})$ *is log-supermodular, then* $\int_{\mathbf{t}_{-i}\in S} f_i(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$ *is nondecreasing in* t_i *for all S such that* $\mathbf{1}_S(\mathbf{t}_{-i})$ *is nondecreasing* $\mathbf{t}_{-i}$.

Athey (1995) provides a more thorough characterization of supermodularity in stochastic problems based on alternative assumptions about the payoff functions; Lemma 4.2 is most applicable for games of incomplete information, and it does not rely on higher order derivatives of the payoffs.

Together, the above results can be used to characterize the single crossing condition in a class of games with additively separable payoffs.

Theorem 4.3 Consider a game of incomplete information which satisfies (2.1). Suppose that payoffs are given by $u_i(\mathbf{a},\mathbf{t}) = g_i(a_i,t_i)+h_i(\mathbf{a},\mathbf{t})$. *Then if (i) for all i,* g_i *and* h_i *are supermodular in* (a_i,t_i), *(ii)* h_i *is supermodular in* (a_i,a_j) *and* (a_i,t_j), $i\neq j$, *and (iii) the joint distribution over types* $f(\mathbf{t})$ *is log-supermodular, then the game satisfies the Single Crossing Condition (2.2).*

> **Proof:** Apply Lemmas 4.1 and 4.2, letting $k(a_i,\mathbf{t}) = h_i(a_i,\alpha_{-i}(\mathbf{t}_{-i}),\mathbf{t})$, recalling Fact (3) from above, which guarantees that for nondecreasing $\alpha_{-i}(\mathbf{t}_{-i})$, supermodularity of h_i implies supermodularity of k_i.

While Theorem 4.3 can be used to establish existence of equilibrium, the games studied in Theorem 4.3 also satisfy Vives' (1990) sufficient condition for existence of PSNE. When payoffs are supermodular, a pointwise increase in $\alpha_j(t_j)$ increases the returns to a_i for all t_i, and thus the game is supermodular in strategies under the pointwise order. Under those conditions, Vives applies theorems about existence of equilibria in supermodular games *without* relying on the Single Crossing Condition, and thus without reference to assumptions about the joint distribution over types. However, Vives' result relies crucially on the assumption that payoffs are supermodular, and it will not be applicable to the other classes of games, including games with log-supermodular or single-crossing payoff functions, as well as the additively separable all-pay auction with independent private values (which relies on Lemma 4.1 directly). Further, because monotonicity of strategies is of independent interest in many games, we will at times be interested in applying Lemma 4.2 and Theorem 4.3 even in games with supermodular payoffs.

In terms of applications, many of the supermodular games with complete information which have been studied by economists (see Topkis (1979), Vives (1990), and Milgrom and Roberts (1990) for examples) can also be studied as games of incomplete information. For example, pricing or quantity games have variations where firms have incomplete information about their rivals production costs or information about demand. Games between two players whose choices are strategic substitutes can also be considered, such as a Cournot quantity game between two firms whose quantities decrease the marginal revenue of the opponents, but where the firms have incomplete information about rivals costs.

4.1.2. Multiplicatively separable expected payoffs

The next class of games we consider is a class where player i's payoffs can be written as the product of two nonnegative terms, so that $u_i(\mathbf{a},\mathbf{t})=g_i(a_i,t_i)\cdot h_i(\mathbf{a},\mathbf{t})$ and $U_i(a_i,\alpha_{-i}(\cdot),t_i) = g_i(a_i,t_i)\cdot H_i(a_i,t_i)$. A game which fits into this framework is a private-value first price auction, where $H_i(a_i,t_i) = \Pr\{\text{bid } a_i \text{ wins the auction given } t_i\}$; other examples include pricing games where firms' products are imperfect substitutes.

The theory of comparative statics for additively separable problems can be applied to multiplicatively separable problems merely by taking logarithms, i.e., $\ln(U_i(a_i,\alpha_{-i}(\cdot),t_i)) = \ln(g_i(a_i,t_i))+\ln(H_i(a_i,t_i))$, and applying Lemma 4.1. Thus, for multiplicatively separable problems, log-supermodularity is the right property to require in order to guarantee the SCP-IR.

The methods for analyzing log-supermodularity in stochastic problems are different from those for supermodular functions; however, the sufficient conditions are in the end quite similar. The following characterization theorem follows from Athey (1996):

Lemma 4.4: Let $k:\Re^I\rightarrow\Re_+$, where $f(\mathbf{t})$ is a probability density. For $i=1,..,I$, let $k_i:\Re^{I+1}\rightarrow\Re_+$ and let $f_i(\mathbf{t}_{-i}|t_i)$ be the conditional density of $\mathbf{t}_{-i}$ given t_i. Then the following conditions hold:

(i) $\int_{\mathbf{t}_{-i}} k_i(a_i,\mathbf{t})f_i(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$ is log-supermodular in (a_i,t_i) for all $i=1,..,I$ and all k_i log-supermodular, if and only if $f(\mathbf{t})$ is log-supermodular.

(ii) $\int_{\mathbf{t}_{-i}} k_i(a_i,\mathbf{t})f_i(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$ is log-supermodular in (a_i,t_i) for all $f(\mathbf{t})$ log-supermodular, if and only if $k(a_i,\mathbf{t})$ is log-supermodular.

Log-supermodularity is an especially convenient property for working with expectations because it is preserved by multiplication; thus, multiplying the integrand by an indicator function $\mathbf{1}_S(\mathbf{t}_{-i})$ preserves log-supermodularity so long as the set S is a *sublattice* (see Topkis, 1978); a common example of a sublattice is a cube in $\Re^n$. For more discussion see Athey (1996).

We can pull these results together into the following set of sufficient conditions for the Single Crossing Condition to hold, in a manner similar to Theorem 4.3.

Theorem 4.5 Consider a game of incomplete information which satisfies (2.1). Suppose that payoffs are given by $u_i(\mathbf{a},\mathbf{t}) = g_i(a_i,t_i)\cdot h_i(\mathbf{a},\mathbf{t})$. Then if (i) for all i, g_i and h_i are nonnegative and log-supermodular, and (ii) the joint distribution over types $f(\mathbf{t})$ is log-supermodular, then the game satisfies the Single Crossing Condition (2.2).

We will apply this result to private values auctions and pricing games in Section 4.2.

4.1.3. Nonseparable Expected Payoffs

This section analyzes games which do not fit into the classes of additively or multiplicatively separable games analyzed above. Of course, we can always apply the above results about payoffs of the form $u_i(\mathbf{a},\mathbf{t})=g_i(a_i,t_i)+h_i(\mathbf{a},\mathbf{t})$ or $u_i(\mathbf{a},\mathbf{t})=g_i(a_i,t_i)\cdot h_i(\mathbf{a},\mathbf{t})$, letting $g_i(a_i,t_i)\equiv 1$. However, the requirements on h_i which must be satisfied to apply Theorems 4.3 and 4.5 are stronger than necessary if $g_i(a_i,t_i)$ is constant. Further, some games (such as the mineral rights auction) do not satisfy the conditions of Theorems 4.3 and 4.5. This section shows that weaker conditions on the utility function suffice, if either (i) there are only two players, as in the noisy signaling game studied in Section 4.2.3, or if (ii) $u_i(\mathbf{a},\mathbf{t})$ takes a very special form. In particular, in many-player games, u_i must depend on the opponents' types and actions through a single index, denoted s_i. That is, $u_i(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}),\mathbf{t}) = k_i(a_i,s_i,t_i;\boldsymbol{\alpha}_{-i})$. For example, in a first price mineral rights auction with identical bidders using symmetric strategies, $\alpha_j(\cdot)=\alpha_l(\cdot)$ for all $j,l\neq i$. Then, $k_i(a_i,s_i,t_i;\boldsymbol{\alpha}_{-i}) = E_{\mathbf{t}_{-i}}[v_i(a_i,\mathbf{t})|\max\{t_j:j\neq i\}=s_i]\cdot \mathbf{1}_{[a_i\geq\alpha_j(s_i)]}(s_i)$. That is, s_i is the value of the highest opponent type, and payoffs depend on opponent types only through the realization of this type and the associated action. In a multi-unit auction, s_i might be a different order statistic of the distribution. In other applications, s_i might be a sufficient statistic for $\mathbf{t}_{-i}$.

Our theorem makes use of a weak version of the SCP-IR. Formally:

Definition 4.2 *$h(x,\theta)$ satisfies (Milgrom and Shannon's) weak single crossing property of incremental returns (WSCP-IR) in $(x;\theta)$ if, for all $x_H>x_L$, $g(\theta)=h(x_H,\theta)-h(x_L,\theta)$ satisfies the following condition for all $\theta_H>\theta_L$: $g(\theta_L)>0$ implies $g(\theta_H)\geq 0$.*

The following Lemma is proved in Athey (1996).

Lemma 4.6: *Consider a function $k_i:\Re^3\rightarrow\Re$ and a conditional density $f_{s_i}(s_i|t_i)$ whose support does not change with t_i, and suppose $f_{s_i}(s_i|t_i)$ is log-supermodular. Suppose further that $k_i(a_i,s_i,t_i)$ is supermodular in (a_i,t_i) and satisfies WSCP-IR in $(a_i;s_i)$. Then $\int k_i(a_i,s_i,t_i)f_{s_i}(s_i|t_i)dt_i$ satisfies SCP-IR in $(a_i;t_i)$.*[8]

Lemma 4.6 can be applied to games of incomplete information, and the following theorem will be used in our study of mineral rights auctions as well as noisy signaling games.

Theorem 4.7 *Consider a game of incomplete information which satisfies (2.1). Suppose that for all $i=1,..I$, there exists a random variable s_i and a family of functions $k_i(\cdot;\boldsymbol{\alpha}_{-i}):\Re^3\rightarrow\Re$ indexed by opponent strategies, such that (i) $U_i(a_i,\boldsymbol{\alpha}_{-i}(\cdot),t_i)=E_{s_i}[k(a_i,s_i,t_i;\boldsymbol{\alpha}_{-i})|t_i]$; (ii) when $\boldsymbol{\alpha}_{-i}(\cdot)$ is nondecreasing, $k(a_i,s_i,t_i;\boldsymbol{\alpha}_{-i})$ is supermodular in $(a_i;t_i)$ and satisfies WSCP-IR in $(a_i;s_i)$; and (iii) the conditional density $f_{s_i}(s_i|t_i)$ is log-supermodular and the support does not change with t_i. Then the game satisfies the Single Crossing Condition (2.2).*

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8 Athey (1996) further shows that none of the hypotheses can be weakened without strengthening the others; see Table 7.1 in the Appendix.

4.2. Applications

4.2.1. Auctions

4.2.1.1. Private Values Auctions

Consider a first-price, private value auction, where each player i (i=1,..,I) observes his own value t_i, and values are drawn from a distribution satisfying (2.1). Each player's utility function is given by $V_i(a_i,t_i)$. Suppose players $j\neq i$ use nondecreasing strategies. Writing the objective function when ties are resolved randomly is cumbersome, so for the moment consider the game where the auctioneer keeps the object in the case of ties. Then, let $H_i(a_i|t_i) = \Pr(a_i>\alpha_j(t_j)$ for $j\neq i|t_i)$.

To see when the conditions of Theorem 4.5 are satisfied, consider first the utility function. If utility takes the form $V_i(a_i,t_i)=V_i(t_i-a_i)$, where $V_i(0)\geq 0$ and $\ln(V_i)$ is concave and strictly increasing, it is straightforward to verify that V_i will be log-supermodular in (t_i,a_i). Next, consider the issue of when $H(a_1|t_1)$ is log-supermodular. Since the strategies are nondecreasing (and recalling Fact 3, that log-supermodularity is preserved by monotone transformations of the variables), $H(a_1|t_1)$ will be log-supermodular if the joint distribution of types, $F(\mathbf{t})$, is log-supermodular. A sufficient (but not necessary) condition for this is that $f(\mathbf{t})$ is log-supermodular, i.e., types are affiliated (this can be shown using Lemma 4.4, letting $g_i(\mathbf{a},\mathbf{t})=\mathbf{1}_{[\mathbf{t}\leq\mathbf{a}]}(\mathbf{t})$).

Now, consider the case where ties are resolved uniformly. Then define $h_1(\mathbf{a},\mathbf{t})$ as follows (and likewise for players 2,..,I):

$$h_1(\mathbf{a})=\sum_{j_2=0}^{1}\cdots\sum_{j_I=0}^{1}\frac{1}{1+\sum_{k=2}^{I}j_k}\cdot\prod_{k=2}^{I}\Big((1-j_k)\cdot\mathbf{1}_{[a_k<a_1]}+j_k\cdot\mathbf{1}_{[a_k=a_1]}\Big)$$

Thus, the probability that player i wins the auction when she has type t_i and chooses an action a_i is given as $\Pr\{a_1 \text{ wins}|t_1\}=H(a_1|t_1)=E_{\mathbf{t}_{-1}|t_1}\big[h(a_1,\boldsymbol{\alpha}_{-1}(\mathbf{t}_{-1}))\big]$. If the types are affiliated, then Lemma 4.4 implies that this probability will be log-supermodular in (a_1,t_1) if the integrand is log-supermodular in $(a_1,\mathbf{t})$. Log-supermodularity can be verified pairwise, so that (using the symmetry) we need only to check log-supermodularity of $h(a_1,\boldsymbol{\alpha}_{-1}(\mathbf{t}_{-1}))$ in (a_1,t_j) and (t_j,t_l) for $1\neq j\neq l$. This can be verified directly.

Thus, we have the following proposition (the assumption of the proposition that $V_i(a_i,t_i)$ is nonnegative is innocuous since the agent knows $V_i(a_i,t_i)$ after observing her type and thus can always choose an action a_i which yields nonnegative payoffs).

Proposition 4.8 *Consider a private values, first price auction, where (2.1) holds. Suppose further that (i) for all i, the utility functions $V_i(a_i,t_i)$ are non-negative and log-supermodular, (ii) types are affiliated ($f(\mathbf{t})$ is log-supermodular), and (iii) ties are broken uniformly. Then:*

(1) The game satisfies the Single Crossing Condition (2.2), and a PSNE exists in all finite-

action games.

(2) If, in addition, for all i (iv) V_i is strictly increasing in $(-a_i,t_i)$, (v) V_i is bounded and continuous, and (vi) the support of $f(\mathbf{t})$ is a product set, then (3.1)-(3.7) are satisfied and there exists a PSNE in continuous-action games.

Now consider a generalization of Proposition 4.3 to multi-unit auctions, where each agent demands a single unit. For example, in a 2-unit auction, the players with the highest two bids win an object. Unfortunately, this complicates the analysis of log-supermodularity of the function $\Pr(a_i \text{ wins}|t_i)$. However, if the types are drawn independently, then $\Pr(a_i \text{ wins}|t_i)$ does not depend on t_i, and the expected payoff function reduces to $V_i(a_i,t_i)\cdot\Pr(a_i \text{ wins})$. This is always log-supermodular if $V_i(a_i,t_i)$ is log-supermodular; thus, for the case of independent types, the extension to multi-unit auctions is immediate.

4.2.1.2. All-Pay Auction

In this section, we consider an alternative auction format, the all-pay auction. In this auction, the highest bidder receives the object, but all bidders pay their bids. This game has been used to model activities such as lobbying. To keep the analysis simple, we will use an independent private values formulation. Let $V_i(a_i,t_i)$ take the form $V_i(t_i-a_i)$. Then a player's expected payoffs from action a_i can be written as follows:

$$V_i(-a_i) + V_i(t_i-a_i)\cdot\Pr(a_i \text{ wins})$$

This game has an additively separable form, and thus by Lemma 4.1, we look for the components of the objective function to be supermodular. Since $\Pr(a_i \text{ wins})$ is nonnegative and nondecreasing in a_i, it is straightforward to verify that $V_i(t_i-a_i)\cdot\Pr(a_i \text{ wins})$ is supermodular if $V_i(t_i-a_i)$ is increasing in t_i and supermodular in (a_i,t_i). In turn, $V_i(t_i-a_i)$ is supermodular if and only if it is concave, that is, the bidder is risk averse.

Observe that we have not discussed the interactions between the players' strategies in determining the function $\Pr(a_i \text{ wins})$; since types are independent and bidder valuations are private, this is not important for establishing the single crossing conclusion, in contrast to Theorem 4.3. However, it is *not* true that a pointwise increase in player *j*'s strategy leads to a pointwise increase in player *i*'s best response, and thus the game is not supermodular in strategies.

Summarizing, we have the following proposition:

Proposition 4.9 *Consider a private values, all-pay auction, where (2.1) holds. Suppose further that (i) for all i, the utility functions $V_i(t_i-a_i)$ are concave, and $V_i(0)\geq 0$, (ii) types are independent, and (iii) ties are broken uniformly. Then:*

(1) The game satisfies the Single Crossing Condition (2.2), and a PSNE exists in all finite-action games.

(2) If, in addition, for all i (iv) V_i is strictly increasing, (v) V_i is bounded and continuous, and

(vi) the support of $f(\mathbf{t})$ is a product set, then (3.1)-(3.7) are satisfied and there exists a PSNE in continuous-action games.

4.2.1.3. First Price Mineral Rights Auctions

We now generalize our analysis of auctions to consider Milgrom and Weber's (1982) model of a mineral rights auction (their general existence and characterization results treat the I-bidder, symmetric case and several other special cases), allowing for risk averse, asymmetric bidders whose utility functions are not necessarily differentiable.

We begin with the two bidder case. Suppose each agent's utility (written $V_i(a_i,t_1,t_2)$) is supermodular in (a_i,t_1) and (a_i,t_2); this can be interpreted as the player's expected payoffs conditional on both players' signals.[9] When player two uses the bidding function $\alpha_2(t_2)$, player 1's expected payoffs given her signal can be written as follows (assuming ties are broken randomly):

$$U_1(a_1,\alpha_2(\cdot),t_1) \equiv \int_{t_2} V_1(a_1,t_1,t_2)\cdot I_{a_1>\alpha_2(t_2)}(t_2)\cdot f_2(t_2|t_1)dt_2$$
$$+\tfrac{1}{2}\int_{t_2} V_1(a_1,t_1,t_2)\cdot I_{a_1=\alpha(t_2)}(t_2)\cdot f_2(t_2|t_1)dt_2$$

To keep things simple, consider the finite-action game. Figure 7 illustrates how player 1's incremental returns to increasing her bid from A_{m_L} to A_{m_H} (where $m_L < m_H$) change with the signal of the opponent, given that the opponent's strategy is consistent with $\mathbf{x}^2$. There are several regions to consider. When the opponent bids less than A_{m_L}, the outcome of the auction is unchanged by the increase in bid, but player 1 simply pays more. For a risk neutral bidder, this would be a constant.

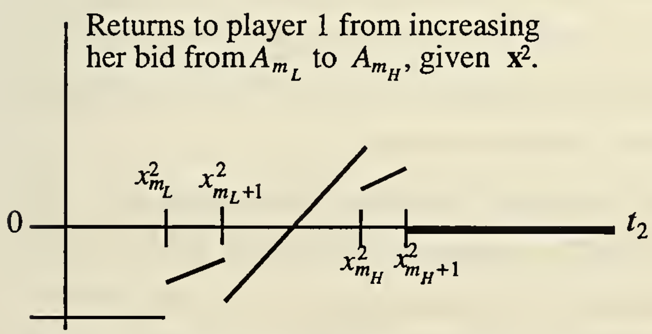

Figure 7: Player 1's payoff function satisfies WSCP-IR in $(a_1;t_2)$.

When the opponent bids more than A_{m_H}, the outcome of the auction is also not changed by the bid increase: player 1 loses in both cases and pays nothing. In the intermediate cases, there are the regions which involve ties at either the low or the high bid, and the region where

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[9] To see an example where these assumptions on the payoff function are satisfied, let $V_i(a_i,\mathbf{t}) = \int \hat{v}_i(z-a_i)g(z|t_1,t_2)dz$, where z is affiliated with t_1 and t_2, and $\hat{v}_i$ is nondecreasing and concave. To see why, note that t_1 and t_2 each induce a first order stochastic dominance shift on G, and $\hat{v}_i$ is supermodular in (a_i,z). By Athey (1995), supermodularity of the expectation in (a_i,t_1) and (a_i,t_2) follows.

the bid increase causes player 1 to change from losing for certain, to winning for certain. Within each of these regions, expected payoffs are nondecreasing in the opponent's type since this increases the expected value of the object to bidder 1.

When bidder two plays a nondecreasing strategy, bidder one's payoff function given a realization of t_2 satisfies WSCP-IR in $(a_1;t_2)$, and it is supermodular in $(a_1;t_1)$. We can further apply Lemma 3.2.1 to show that the conditional expected payoffs for a bidder are always strictly increasing in his type so long as $v_i(a_i,\mathbf{t})$ is nondecreasing in $\mathbf{t}$, thus satisfying (3.7). This gives the following result:

Proposition 4.10 *Consider a 2-bidder first price "mineral rights" auction, where (2.1) holds. Suppose further that (i) for all i, the utility functions $V_i(a_i,\mathbf{t})$ are supermodular in (a_i,t_j), j=1,2, and nondecreasing in* $\mathbf{t}$, *(ii) types are affiliated ($f(\mathbf{t})$ is log-supermodular), and (iii) ties are broken uniformly. Then:*

(1) The game satisfies the Single Crossing Condition (2.2), and a PSNE exists in all finite-action games.

(2) If, in addition, for all i (iv) V_i is strictly increasing in $(-a_i,t_i)$, (v) V_i is bounded and continuous, and (vi) the support of $f(\mathbf{t})$ is a product set, then (3.1)-(3.7) are satisfied and there exists a PSNE in continuous-action games.

What happens when we try to extend this model to more than two bidders? If the bidders face a symmetric distribution, and all opponents use the same bidding function, then only the maximum signal of all of the opponents will be relevant to bidder one. Define $s_1 = \max(t_2,..,t_n)$. Milgrom and Weber (1982) show that (t_1,s_1) are affiliated when the distribution is symmetric. Further, if the opponents are using the same strategies, whichever opponent has the highest signal will necessarily have the highest bid. It is straightforward to extend Milgrom and Weber's arguments to show that $\hat{V}_1(a_1,t_1,s_1) \equiv E\left[V_1(a_1,t_1,..,t_n)\middle|s_1\right]$ is increasing in (t_1,s_1) and supermodular in (a_1,t_1) whenever V_1 is nondecreasing in $\mathbf{t}$ and is supermodular in (a_1,t_i), i=1,..,n. Then we can apply Proposition 4.9 to this problem exactly as if there were only two bidders.

Unfortunately, when players use asymmetric strategies, affiliation of the signals is not sufficient to guarantee that (t_1,s_1) are affiliated, nor is their joint distribution function log-supermodular. Thus, in some regions, a small increase in the signal received by a given player might increase the likelihood that a given bid is higher than all opponents. However, this potential competing effect may not be the dominant one in particular examples; thus, one can proceed by positing that the single crossing condition holds, characterize the equilibrium under that hypothesis, and then verify that the single crossing condition is in fact satisfied for the functional forms and ranges of parameters of interest. Thus, separating the single crossing condition from the existence theorem allows us to proceed even in the absence of general sufficient conditions for the single crossing condition.

In the case where players are risk neutral, we rewrite the n-bidder problem another way in order to highlight the difficulty. Let $V_1(a_1,\mathbf{t}) \equiv E[z_1|\mathbf{t}] - a_1$, where z_1 is player 1's true value for the object. This is nondecreasing in $\mathbf{t}$ by Lemma 3.2.1. Then, we can rewrite player 1's expected payoffs as follows (ignoring ties):

$$\left(E\left[E[z_1|\mathbf{t}]\middle|t_1,\alpha_j(t_j)<a_1\right]-a_1\right)\cdot \Pr\left(\alpha_j(t_j)<a_1\middle|t_1\right). \tag{4.2}$$

Applying Lemma 4.4 to this problem in a manner analogous to the private value auction, we know that $\Pr\left(\alpha_j(t_j)<a_1\middle|t_1\right)$ is log-supermodular when the density is affiliated and the strategies are nondecreasing. To isolate the issue with the first term of (4.2), we draw a distinction between the two ways that a_1 affects this term. We can restrict attention to actions by player 1 less than or equal to the player's conditional expected payoff. Note that $E\left[E[z_1|\mathbf{t}]\middle|t_1,\alpha_j(t_j)<a_1\right]-c$ is log-supermodular in (t_1,c), since by Lemma 3.2.1, the first term is nondecreasing in t_1. However, our assumptions do *not* imply that $E\left[E[z_1|\mathbf{t}]\middle|t_1,\alpha_j(t_j)<a_1\right]-c$ is log-supermodular in (a_1,t_1). Thus, the single crossing property will require that the interactions between a_1 and t_1 which we know work in the right direction are strong enough to outweigh any competing effects.

4.2.2. Pricing Games

This section studies pricing games with incomplete information, where constant marginal costs are the private information of the players of the game. Spulber (1995) recently analyzed how incomplete information about a firms' cost parameters alters the results of a Bertrand pricing model, showing that firms price above marginal cost and have positive expected profits. Spulber's model assumes that costs are independently and identically distributed, and that values are private; now we show that this model can be easily generalized to asymmetric, affiliated signals.

Let t_i represent the marginal cost of firm i. Consider a general demand function for firm i, $D^i(p_1,..,p_I)$, where the firms produce goods which are only imperfect substitutes. When the opponents use price functions $\rho_j(t_j)$, firm 1's problem can now be written as follows:

$$\max_{p_1\in P}\,[p_1-t_1]\cdot\int\cdots\int D^1\left(p_1,\rho_2(t_2),\ldots,\rho_I(t_I)\right)f(t_2,\ldots,t_I|t_1)d\mathbf{t}_{-1}$$

By Lemma 4.4, the expected demand function is log-supermodular if the cost parameters are affiliated and $D^1(p_1,..,p_I)$ is log-supermodular. The interpretation of the latter condition is that the elasticity of demand is a non-increasing function of the other firms' prices. As discussed in Milgrom and Roberts (1990b), demand functions which satisfy this criteria include logit, CES, transcendental logarithmic, and a set of linear demand functions (see Topkis (1979)). Another special case is the case of perfect substitutes, where demand to the lowest-price firm is given by

$D(P)$, where P is the lowest price offered in the market, and all other firms get zero demand.

Then, since $[p_i - t_i]$ is log-supermodular, we have the following result (noting that the case with perfect substitutes is formally like a first-price auction):

Proposition 4.11 *Consider a pricing game as described above, where (2.1) holds. Suppose further that (i) for all i, the demand functions $D^i(p_1,..,p_I)$ are non-negative and log-supermodular, (ii) types are affiliated ($f(\mathbf{t})$ is log-supermodular), (iii) in the case of perfect substitutes, ties are broken uniformly. Then:*

(1) The game satisfies the Single Crossing Condition (2.2), and a PSNE equilibrium exists when either D^i is continuous or when the action set is finite.

(2) For the case where goods are perfect substitutes, if in addition, for all i, (iv) $D(P)$ is strictly decreasing in P, (v) $D(P)$ is bounded and continuous, and (vi) the support of $f(\mathbf{t})$ is a product set, then (3.1)-(3.7) are satisfied and there exists a PSNE in continuous-action games.

This example can also be extended to problems with incomplete information about demand elasticities.

4.2.3. Noisy Signaling Games

Consider a signaling game between two players, the sender (player 1) and the receiver (player 2), where player 2 observes the signal of player 1 only with noise. Examples of noisy signaling games include games of limit pricing (Matthews and Mirman (1983)), where an entrant does not know the cost of the incumbent, but can draw inferences about the incumbent's cost by observing a noisy signal of the incumbent's product market decision (the noise might be due to demand shocks). In another example, Maggi (1996) studies the value of commitment in a game where one player gets to move first, committing herself to an action, but the move is only imperfectly observed by the opponent. This model is used to show that incomplete information about the first mover's type (i.e. production cost) can restore some value to commitment, in contrast to Bagwell (1995)'s result that commitment has no value in a game of complete information but imperfectly observed actions.

Consider a game where Player 1 observes his own type and chooses an action. Player 2 observes a noisy signal of player 1's action and then chooses an action in response. Player 1 receives payoffs $u_1(a_1,a_2,t_1)$, while player 2 receives payoffs $u_2(a_1,a_2,t_1)$ (in the limit pricing game, the player's action is not important, but the type t_1 represents the unknown production cost; in the commitment game, player 1's type is not important to player 2, but the action (i.e. output) is). Player 2's "type," t_2, is simply a signal about a_1, which does not directly affect payoffs. Let $f_1(t_2|a_1)$ be the density of the signal t_2 conditional on the action a_1. Thus the expected payoff to player 1 from choosing action a_1 when player 2 uses strategy $\alpha_2(t_2)$ is as follows:

$\int u_1(a_1,\alpha_2(t_2),t_1)f_1(t_2|a_1)dt_2$. Let $f_2(t_1|t_2,\alpha_1(\cdot))$ be the conditional density of t_1 given the signal t_2 and the belief that player 1 uses strategy $\alpha_1(\cdot)$, calculated using Bayes' rule. Then, the expected payoffs to player 2 from choosing action a_2 when player 1 uses strategy $\alpha_1(t_1)$ is as follows: $\int u_2(\alpha_1(t_1),a_2,t_1)f_2(t_1|t_2,\alpha_1(\cdot))dt_1$.

Suppose that $\alpha_1(t_1)$ is nondecreasing. Further, suppose that $f_1(t_2|a_1)$ is log-supermodular (equivalent to the MLRP). Then, the induced density $f_2(t_1|t_2,\alpha_1(\cdot))$ will also be log-supermodular. Sufficient conditions for the single crossing property in this scenario are then given as follows:

Proposition 4.12 *Consider a noisy signaling game as described above, where (2.1) holds. Suppose further that (i) $u_2(a_1,a_2,t_1)$ satisfies WSCP-IR in $(a_2;a_1)$ and $(a_2;t_1)$, (ii) $f_1(t_2|a_1)$ is log-supermodular and the support does not move with a_1, and either (iii)(a) $u_1(a_1,a_2,t_1)$ is supermodular in (a_2,t_1) and in (a_1,t_1), or else (iii)(b) $u_1(a_1,a_2,t_1)$ is nonnegative and log-supermodular. Then the game satisfies the Single Crossing Condition (2.2).*

Proof: Lemma 4.6 implies that under our assumption that u_2 satisfies WSCP-IR in $(a_2;a_1)$ and $(a_2;t_1)$, $\int u_2(\alpha_1(t_1),t_1,a_2)f_2(t_1|t_2,\alpha_1(\cdot))dt_1$ satisfies SCP-IR in $(a_2;t_2)$ when $\alpha_1(t_1)$ is nondecreasing.

It remains to establish that player 1's expected payoffs satisfy (2.2). Under assumption (iii)(a), we can check that $\int u_1(x,\alpha_2(t_2),t_1)f_1(t_2|y)dt_2$ is supermodular in (x,t_1) and (y,t_1). Since supermodularity is preserved by arbitrary sums, u_1 supermodular in (a_1,t_1) implies supermodularity of $\int u_1(x,\alpha_2(t_2),t_1)f_1(t_2|y)dt_2$ in (x,t_1). If $\alpha_2(t_2)$ is nondecreasing, the assumption that u_1 is supermodular in (a_2,t_1) implies that $u_1(a_1,\alpha_2(t_2),t_1)$ is supermodular in (t_2,t_1). But Lemma 4.2 implies that $\int u_1(x,\alpha_2(t_2),t_1)f_1(t_2|y)dt_2$ is supermodular in (t_1,y).

Under assumption (iii)(b), we can apply Lemma 4.4 to establish that $\int u_1(a_1,\alpha_2(t_2),t_1)f_1(t_2|a_1)dt_2$ will be log-supermodular when u_1 and f_1 are log-supermodular and $\alpha_2(\cdot)$ is nondecreasing.

5. Numerical Computation of Equilibria: First Price Auctions

In this section, we consider the issue of computation of equilibria of first price auctions. Since the existence result considered in this paper applies to a wider class of economic environments (arbitrary asymmetries, correlated values) than had been previously analyzed theoretically or numerically, we can take a few first steps towards numerically characterizing the equilibria to such auction games. This numerical exploration can potentially suggest avenues for future theorizing about characterizations of equilibria, and we give some examples of suggestive numerical results about affiliated private values and common value auctions with asymmetries in the covariance structure of types.

Prior to Marshall et al (1994), there were no general numerical algorithms available for computing equilibria to asymmetric first price auctions with continuous bidding units, and in fact existence was not known for some of the kinds of asymmetries considered in their paper. Marshall et al (1994) argue that numerical computation of equilibria in asymmetric first price auctions in the independent private values case is difficult, but can be done. They summarize some of the difficulties as follows:

> Although these solutions belong to a class of 'two-point boundary problems' for which their exist efficient numerical solution techniques, they all suffer from major pathologies at the origin. First, forward extrapolation produces 'nuisance' solutions (linear in our case) that do not satisfy the terminal conditions and act as 'attractors' on the algorithm. Second, and not unrelated, backward solutions are well-behaved except in neighborhoods of the origin where they become highly unstable with the consequence that standard (backwards) "shooting" by interpolation does not work.

Marshall et al (1994) then devise a technique which makes use of "backward series expansions" and a transformation of the problem which is more numerically stable than the original problem. Their algorithm requires analytical input to transform the problem appropriately. Generalizing their algorithm to the case of more than two type distributions, or to correlated values, would require non-trivial extensions of their numerical and analytic algorithms.

In contrast to the problem of computing the solution to a set of differential equations, the algorithm for computing the equilibria to the auction game with finite actions constructed in this paper involves few conceptual subtleties. We simply want to find the matrix **X** so that **X**=BR(**X**), where the calculation of BR(**X**) is a simple exercise (both in terms of programming time and computation time); it can be broken down into a sequence of single variable optimization problems, one for each jump point of each player. That is, for each player i, let x_1^i be the smallest value of t_i at which the player prefers action A_1 to A_0 (computing expected payoffs to each action according to $V_i(a_i;\mathbf{X},t_i)$), and then proceed to find x_2^i, searching over $t_i \geq x_1^i$.[10] In mineral rights auctions, where players must compute the expected value of the object conditional on winning against the opponents' current strategies, the conditional expectations must be numerically computed; we looked at signals and values which had a log-normal distribution.[11]

The more difficult part of the problem is solving the nonlinear set of equations **X**=BR(**X**). The theory of numerical analysis[12] suggests a number of standard ways to solving this problem. Thus, here we will only sketch a few of the numerical issues which arise. First, since we have no global

[10] The algorithm also checks for the cases where a particular action is never used for any type.

[11] To approximate the relevant conditional expectations and probabilities, we made use of the library of routines made available by Vassilis Hajivassiliou as a companion to Hajivassiliou, McFadden, and Ruud (1996).

[12] See Judd (forthcoming) for an excellent treatment of numerical analysis in economics.

"contraction mapping" theorem, the algorithm $\mathbf{X}^{k+1}=BR(\mathbf{X}^k)$ is not guaranteed to converge for any starting value, and indeed it does not appear to in numerical trials. However, variations on the algorithm $\mathbf{X}^{k+1}= \lambda\cdot BR(\mathbf{X}^k) + (1-\lambda)\cdot\mathbf{X}^k)$) were effective at generating starting values for other algorithms. Methods based on quasi-Newton approaches can also be used. There are potentially large computational benefits to using an analytic Jacobian since the Jacobian is sparse. In particular, the point at which player i jumps to action A_m, denoted x^i_m, affects only the following elements of the best response of opponent $j\neq i$: x^j_{m-1}, x^j_m, and x^j_{m+1}. Thus, to determine the effect of changing each component of $\mathbf{x}^i$ on the equation $\mathbf{X}-BR(\mathbf{X})$, it is necessary to call the function $BR(\cdot)$ only three times. This allows us to compute a Jacobian of dimension $M\cdot I \times M\cdot I$ with only $I\cdot 3$ function calls. The number of bidding increments can thus be increased without a affecting the number of function calls required by the nonlinear equation solver, while the computation time for the function BR(X) will increase linearly in M. However, even with this modification, computation could be slow for the mineral rights examples considered below. A final alternative is a simplex method, which was reliable in trials.

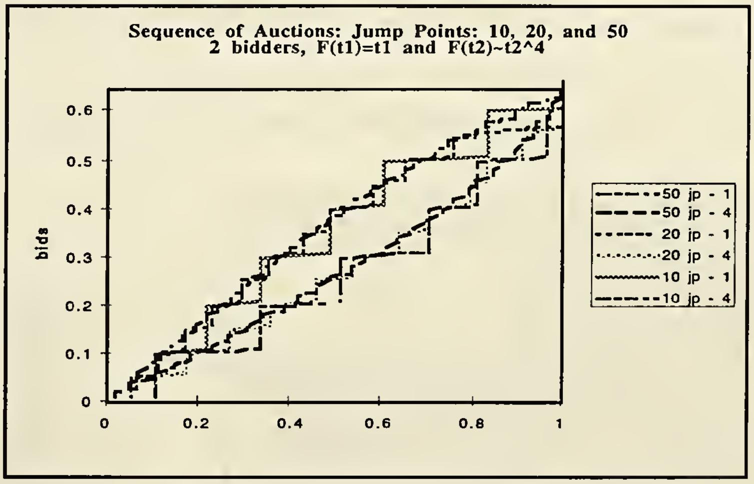


Figure 8: First Price Auction games between two coalitions.

The first set of approximations we report were motivated by Marshall et al (1994). We computed the same set of auctions studied in their paper, and compared the calculations for equilibrium bid functions and expected revenue. Marshall et al (1994) chose a set of distributional assumptions motivated by the case where five bidders draw values from uniform [0,1], and a subset of the bidders collude, bidding as a group with the group's value being the maximum of the values of the participants. A comparison of the expected

Coal-ition $F(x)=x^a$	n Indiv. Bidders $F(x)=x$	Our results, M=100: Expected Revenue Auction-eer	Coal-ition	Indiv. bidders	Marshall et al Results: Expected Revenue Auction-eer	Coal-ition	Indiv bidders
$a=1$	n=4	.6664	.0334	.0334	.6668	.0335	.0333
$a=2$	n=3	.6508	.0344	.0374	.6510	.0352	.0371
$a=3$	n=2	.6085	.0405	.0487	.6089	.0406	.0488
$a=4$	n=1	.5055	.0574	.0840	.5057	.0567	.0860

revenue calculations for several auctions is presented in Figure 8. The table indicates that the expected revenue from the auction with discrete and continuous bidding units is within .001 in all cases. Thus, in these auctions, the difference between the continuous and discrete games is small. Figure 6 shows how the equilibria to the discrete game change as more bidding units are allowed.

Now consider an auction with affiliated *private* values, where the types are distributed $ln(\mathbf{t})\sim N(\mu,\Sigma)$.[13] There are several potentially interesting asymmetries: those arising from differences in means, and those arising from differences in variances (which also affect the mean of **t**), and those arising from differences in covariances. The numerical results for differences in means and variances are not surprising: in the trials we conducted, types with higher means always bid less aggressively and get higher expected revenue. Changes in variances affect the shape of the distribution, so the higher variance types may be less aggressive in some regions and more in others, but they tend to do better overall. Differences in covariances are perhaps slightly more subtle, but no less intuitive. We will illustrate this case with an example. Suppose that two bidders have types which are highly correlated, while a third bidder has a type which is less correlated with the other two. Then the two bidders with highly correlated types bid more aggressively and win more often than the third bidder, who also gets higher expected revenue. The intuition is simple: the two bidders whose types are more highly correlated are always concerned with competing with one another, even when their values are high.

Variable	Mean	Variance Matrix		
$ln(t_1)$	0	1	.5	.25
$ln(t_2)$	0	.5	1	.25
$ln(t_3)$	0	.25	.25	1

	Player 1	Player 2	Player 3	Auctioneer
Expected Revenue	0.4691	0.4691	0.5008	1.4190
Prob. of Winning	0.3283	0.3283	0.3434	

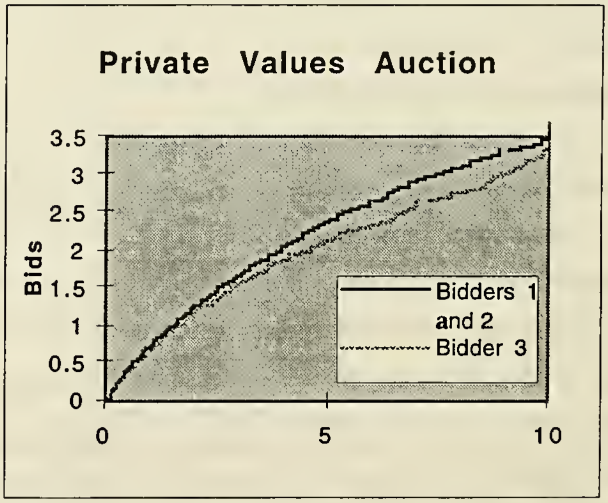


Figure 9: Private values first price auction where the bidders have positively correlated values.

Next, we can consider pure "common values" auctions, where each player sees a private signal about the common value. Although we do not have a theorem guaranteeing that the single crossing property holds, we can numerically verify that it holds in the neighborhood of the equilibrium we

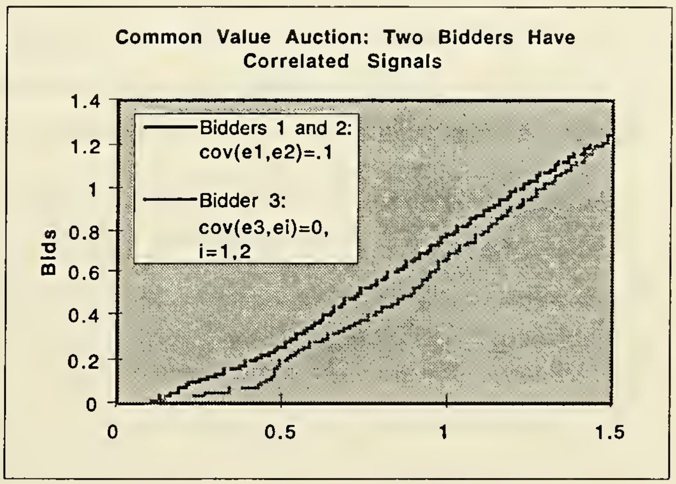
[13] See Wilson (1996) for a theoretical study of equilibrium in oral auctions with log-normal values.

compute for a particular game; the numerical analysis indicates that it does hold in all of the auctions we consider here. Let z be the common value. Though our numerical algorithms allow for general variance-covariance structures, it is perhaps more intuitive to consider a particular example of a signal structure. Suppose each player i receives a signal $\ln(s_i) = \ln(z)+\ln(e_i)$, where $cov(z,e_i)=0$. The assumption of a common value implies that signals will be positively correlated, and further, the errors e_i may also be correlated. We can vary the informativeness of each player's signal as well as the correlation of the signals. An interesting observation is that our result of private values auctions, that the "more independent" bidder (bidder 3) does better, is no longer true. While the "more correlated" bidders (bidders 1 and 2) still compete more aggressively, this aggressiveness has a new effect for bidder 3 which does not arise in the private value auction. In the private value auction, the competition between bidders 1 and 2 was most disadvantageous to those two bidders; bidder 3 won less often, but received higher expected payoffs upon winning due to his less aggressive bidding. But in the common value auction, having two more aggressive opponents is especially bad news due to the winner's curse. Across a range of parameter values, the expected revenue for the bidder with an independent signal was close to that of the two bidders with correlated signals. Following is an example where bidder 3 does worse.

Variable	Mean	Variance Matrix			
$ln(e_1)$	0	.75	.1	0	0
$ln(e_2)$	0	.1	.75	0	0
$ln(e_3)$	0	0	0	.75	0
$ln(z)$	0	0	0	0	.25

	Player 1	Player 2	Player 3	Auctioneer
Expected Revenue	0.0315	0.0315	0.019	1.005
Prob. of Winning	0.378	0.378	0.242	

Figure 10: Common values first price auction where two bidders received positively correlated signals of the true value, and a third bidder receives an independent signal.



Another set of experiments we conducted concerned the effect of the precision of each player's signal about the true value of the object. We first examined the case of two bidders, one of whom receives a signal which is more precise than the others; we then looked at the effect of a third bidder on equilibrium strategies on expected revenue.

Notice in this auction that the player with the more precise signal, player 1, bids slightly less aggressively. However, since her beliefs are more sensitive to her signals, her conditional expected value is more variable than that of player 2, and so she is more likely to see high expected values. Thus, she wins just over half of the auctions. Despite the fact that she wins just over half, she achieves much higher revenue in each auction. It is perhaps surprising that player 2, despite having less precise information, still wins so frequently.[14]

Variable	Mean	Variance Matrix		
$ln(e_1)$	0	.1	0	0
$ln(e_2)$	0	0	.3	0
$ln(z)$	0	0	0	1
	Player 1	Player 2	Auctioneer	
Expected Revenue	0.333	.166	1.150	
Prob. of Winning	0.528	.471		

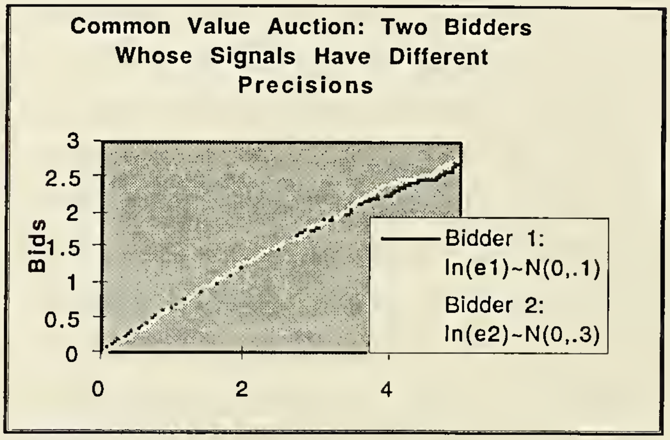


Figure 11: Common values auction where one bidder receives a signal which is more precise than the other bidder's signal.

Returning to the pure common values model, we consider a final experiment, where we add a third player with a more precise signal (var(e_3))=.1). The strategies of the players look very different than in the two-bidder model. Now, the two well-informed players bid more aggressively than the player with the noisier signal; this exacerbates the winner's curse for player 2. In addition, the bidder with the less precise signal wins less often and receives very little expected revenue relative to the two-bidder example.

In general, we can perform numerical computations for a variety of auctions. By changing the covariance structure between signals and values, we can vary the importance of the winner's curse, the informativeness of the signals, and the mean values. Further characterizations of mineral rights auctions with heterogeneous bidders await future research.

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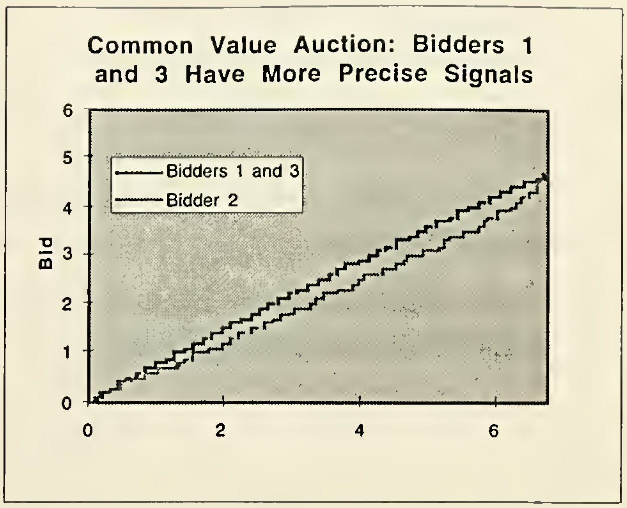
[14] We also examined the effect of lessening the importance of the winner's curse in this example. We allowed the bidders to have different values for the object, z_1 and z_2, where cov(ln(z_1),ln(z_2))=.5, and all other parameters of the model are the same as the latter example. The qualitative nature of the equilibrium remained unchanged, with player 1 bidding less aggressively and winning just over half the auctions. However, expected revenue went up for each bidder. Player 1 has expected revenue of .65, while player 2 has expected revenue of .56. There are two effects here: the bidders' values are less correlated with one another, so they do not expect as much competition. Second, the winner's curse has less bite. For the auctioneer, the first effect is most important: the auctioneer receives expected revenue of .99, less than in the pure common values auction.

Variable	Mean	Variance	Matrix		
$ln(e_1)$	0	.1	0	0	0
$ln(e_2)$	0	0	.3	0	0
$ln(e_3)$	0	0	0	.1	0
$ln(z)$	0	0	0	0	1
	Player 1	Player 2	Player 3	Auctioneer	
Expected Revenue	0.130	0.030	0.130	1.335	
Prob. of Winning	0.369	0.260	0.369		

Figure 12: Common values auction where two bidders receive more precise signals than the third bidder.



6. Conclusions

This paper has introduced a restriction on a class of games called the Single Crossing Condition for games of incomplete information, where in response to nondecreasing strategies by opponents, players choose their actions as nondecreasing functions of their types. We have shown that pure strategy Nash equilibria will exist in such games when the set of available actions is finite, and further, with appropriate continuity or in auction games, a sequence of equilibria of finite-action games will have a subsequence which converges to an equilibrium. We have further established that similar results can be obtained for games which satisfy our "Limited Complexity Condition." The formulation of games of incomplete information developed in this paper has the following advantages: (1) existence of pure strategy Nash equilibria can be verified by checking general and economically interpretable conditions, (2) the results for finite-action games require very few regularity assumptions, and (3) the equilibria are straightforward to numerically calculate for finite-action games, and these approximate the continuous equilibria for continuous games and auctions. The application of these results to first price auction games led to a generalization of the existing literature on the existence of equilibria in auctions with heterogeneous bidders with correlated signals and/or common values. The condition for monotonicity of strategies, the single crossing property, can be characterized for many commonly studied games using the results of Athey (1995, 1996). Finally, numerical analysis can be used to analyze behavior in auction games whose properties have not been fully characterized in the existing literature.

7 Appendix

7.1 Single Crossing in Games of Incomplete Information

The following table summarizes the results of Section 4.1, which provides characterizations of the single crossing condition games of incomplete information, identifying conditions on utility functions and probability distributions which are necessary and sufficient for the single crossing property of incremental returns. The results are applications of theorems from Athey (1995, 1996) to games of incomplete information. In order to conveniently state the set of theorems about the SCP-IR in stochastic problems, it will be useful to state the following definition.

Definition 7.1 *Two hypotheses H1 and H2 are a **minimal pair of sufficient conditions** relative to the conclusion C if the following statements are true:*

(i) H1 and H2 imply C.

(ii) If H2 fails, then C must fail somewhere that H1 is satisfied.

(iii) If H1 fails, then C must fail somewhere that H2 is satisfied.

This definition summarizes a strong relationship between two hypotheses and a conclusion; not only are the hypotheses sufficient for the conclusion, but further neither can be relaxed in the context of the other.

To read the following table:

In each row: H1 and H2 are a *minimal pair of sufficient conditions* (Definition 7.1) relative to the conclusion C; further, C is equivalent to the single crossing result in column 4.

Notation and Definitions: Bold variables are vectors in $\Re^n$; italicized variables are real numbers; f is non-negative; a.e. indicates "for almost all (Lebesque) $\mathbf{t}$"; spm. indicates supermodular, and log-spm. indicates log-supermodular (Definition 4.1); sets are increasing in the strong set order (Definition 2.2); SCP-IR (WSCP-IR) indicates (weak) single crossing of incremental returns to a_i (Definitions 2.1 and 4.2); arrows indicate weak monotonicity.

Observe: Suppose $\alpha_j(t_j)$ is nondecreasing for all $j \neq i$. Then if $u_i(a_i,a_j,t_i,t_j)$ satisfies SCP-IR (WSCP-IR) in $(a_i;a_j)$ and $(a_i;t_j)$, then $k_i(a_i,t_i,t_j)=u_i(a_i,\alpha_j(t_j),t_i,t_j)$ satisfies SCP-IR (WSCP-IR) in $(a_i;t_j)$. If $u_i(\mathbf{a},\mathbf{t})$ is log-spm., then $k_i(a_i,\mathbf{t})=u_i(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}),\mathbf{t})$ is log-spm. If $u_i(\mathbf{a},\mathbf{t})$ is spm. in (a_i,a_j) for $j \neq i$ and (a_i,t_j) for $j=1,..,I$, then $k_i(a_i,\mathbf{t})=u_i(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}),\mathbf{t})$ is spm. in (a_i,t_j), $j=1,..,I$.

Table 7.1: The Single Crossing Condition in Games of Incomplete Information

Assume: $f(\mathbf{t})\geq 0$ and $\int_S f(\mathbf{t})dt$ finite for all S; for 1.3 and 2.2, F is a probability distribution.

	H1: Hypothesis must hold a.e., i=1,..,I.	H2 Hypothesis on f (a.e.)	C: Conclusion Holds for i=1,..,I, whenever $\alpha_j(\cdot)$ nondecreasing, $j\neq i$.	Equivalent Single Crossing Conclusion: i's objective is SCP-IR in $(a_i;t_i)$, i=1,..,I.
Two-Player Games				
1.	Assume: $h_i(a_i,a_j,t_i,t_j)$ is spm. in (a_i,t_i).			
	$h_i(a_i,a_j,t_i,t_j)$ is SCP-IR in $(a_i;a_j)$, $(a_i;t_j)$.	$f(\mathbf{t})$ is log-spm.	$H_i(a_i,t_i)=\int h_i(a_i,\alpha_j(t_j),t_i,t_j)f(t_j\|t_i)dt_j$ is SCP-IR in $(a_i;t_i)$.	same.
2.	$h_i(a_i,a_j,t_i,t_j)\geq 0$ is log-spm.	$f(\mathbf{t})$ is log-spm.	$H_i(a_i,t_i)=\int h_i(a_i,\alpha_j(t_j),t_i,t_j)f(t_j\|t_i)dt_j$ is log-spm. in $(a_i;t_i)$.	$H_i(a_i,t_i)\cdot g_i(a_i,t_i)$ is SCP-IR for all $g_i:\Re\rightarrow\Re_+$ log-spm.
3.	$h_i(a_i,a_j,t_i,t_j)$ is spm. in (a_i,a_j), (a_i,t_l), l=1,2.	$F_j(t_j\|t_i)\downarrow t_i$. (implied by f log-spm.)	$H_i(a_i,t_i)=\int h_i(a_i,\alpha_j(t_j),t_i,t_j)f(t_j\|t_i)dt_j$ is spm. in $(a_i;t_i)$.	$H_i(a_i,t_i)+g_i(a_i,t_i)$ is SCP-IR for all $g_i:\Re\rightarrow\Re$ spm.
Games Where Opponent Behavior is Summarized with Sufficient Statistic, $s_i\in\Re$				
4.	Assume: $k_i(a_i,t_i,s_i)$ is spm. in (a_i,t_i).			
	$k_i(a_i,t_i,s_i)$ is SCP-IR in $(a_i;s_i)$.	$f_s(s_i\|t_i)$ is log-spm.	$\int k_i(a_i,t_i,s_i)f_s(s_i\|t_i)ds_i$ is SCP-IR in $(a_i;t_i)$.	same.
5.	Assume: supp$[F_s(s_i\|t_i)]$ is constant in t_i; $k_i(a_i,t_i,s_i)$ is spm. in (a_i,t_i).			
	$k_i(a_i,t_i,s_i)$ is WSCP-IR in $(a_i;s_i)$.	$f_s(s_i\|t_i)$ is log-spm.	$\int k_i(a_i,t_i,s_i)f_s(s_i\|t_i)ds_i$ is SCP-IR in $(a_i;t_i)$.	same.
Multiplayer-Player Games				
6.	$h_i(\mathbf{a},\mathbf{t})\geq 0$ is log-spm.	$f(\mathbf{t})$ is log-spm.	$H_i(a_i,t_i)=$ $\int h_i(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}),\mathbf{t})f_i(\mathbf{t}_{-i}\|t_i)d\mathbf{t}_{-i}$ is log-spm. in (a_i,t_i).	$H_i(a_i,t_i)\cdot g_i(a_i,t_i)$ is SCP-IR for all $g_i:\Re\rightarrow\Re_+$ log-spm.
7.	$h_i(\mathbf{a},\mathbf{t})$ is spm. in (a_i,a_j), $j\neq i$, & in $(a_i;t_j)$, j=1,..,I.	$\int_S dF(\mathbf{t}_{-i};t_i)\uparrow t_i$ for all S s.t. $1_S(\mathbf{t}_{-i})\uparrow\mathbf{t}_{-i}$. (implied by $f(\mathbf{t})$ log-spm.)	$H_i(a_i,t_i)=$ $\int h_i(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}),\mathbf{t})f_i(\mathbf{t}_{-i}\|t_i)d\mathbf{t}_{-i}$ is spm. in (a_i,t_i).	$H_i(a_i,t_i)+g_i(a_i,t_i)$ is SCP-IR for all $g_i:\Re\rightarrow\Re$ spm.
8.	$h_i(\mathbf{a},\mathbf{t})$ is spm. in (a_i,a_j), $j\neq i$, & in $(a_i;t_j)$, j=1,..,I.	$f(\mathbf{t})$ is log-spm.	$H_i(a_i,t_i)=$ $\int_S h_i(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}),\mathbf{t})\frac{f_i(\mathbf{t}_{-i}\|t_i)}{\int_S f_i(\mathbf{t}_{-i}\|t_i)d\mathbf{t}_{-i}}d\mathbf{t}_{-i}$ is spm. in (a_i,t_i) $\forall$ S sublattice.	$H_i(a_i,t_i)+g_i(a_i,t_i)$ is SCP-IR for all $g_i:\Re\rightarrow\Re$ spm.

Proof of Lemma 2.2

The proof continues from the text, showing that the conditions for Kakutani are satisfied:

Nonempty. Established in our above definition of $\Gamma(\mathbf{X})$.

Closed graph. It is clear from the definition (2.3) that $V_i(a_i;\mathbf{X},t_i)$ is continuous in the elements of $\mathbf{X}$ under our assumption (2.1), which guarantees that there are no mass points in the type distribution. Consider a sequence $(\mathbf{X}^k,\mathbf{Y}^k)$ which converges to $(\mathbf{X},\mathbf{Y})$, such that $\mathbf{Y}^k\in\Gamma(\mathbf{X}^k)$ for all k. We wish to show that $\mathbf{Y}\in\Gamma(\mathbf{X})$. To do this, we show that for each player and almost every type, the action assigned by $\mathbf{Y}$ is a best response to $\mathbf{X}$. Consider player i, and a type $t_i\in T_i\backslash\{\mathbf{y}^i\}$. Then there exists an $m\in\{0,..,M\}$ such that $y^i_m<t_i<y^i_{m+1}$. Since $\mathbf{y}^{i,k}$ converges to $\mathbf{y}^i$, there must exist an K such that, for all $k>K$, $y^{i,k}_m<t_i<y^{i,k}_{m+1}$, and thus A_m is one of t_i's best responses to $\mathbf{X}^k$ since $\mathbf{Y}^k\in\Gamma(\mathbf{X}^k)$. But, since $V_i(a_i;\mathbf{X},t_i)$ is continuous in $\mathbf{X}$, if $V_i(A_m;\mathbf{X}^k,t_i)\geq V_i(A_{m'};\mathbf{X}^k,t_i)$ for all $k>K$ and all m', then $V_i(A_m;\mathbf{X},t_i)\geq V_i(A_{m'};\mathbf{X},t_i)$.

Γ is convex-valued. Here we show that the fact that $a_i^{BR}(t_i|\mathbf{X})$ is nondecreasing in the strong set order implies that $\Gamma_i(\mathbf{X})$ is convex. Fix $\mathbf{X}$ and suppose that $\mathbf{w},\mathbf{y}\in\Gamma_i(\mathbf{X})$. Let $\mathbf{z}=\lambda\,\mathbf{w}+(1-\lambda)\,\mathbf{y}$ for $\lambda\in(0,1)$. Observe that since convex combinations of nondecreasing vectors are nondecreasing vectors, $\mathbf{z}\in\Sigma_i$. Now, for $m=0,..,M$, we show that A_m is an optimal action on (z_m,z_{m+1}). This will imply that there exists an optimal strategy which is consistent with $\mathbf{z}$, and thus by definition $\mathbf{z}\in\Gamma_i(\mathbf{X})$. A consequence of the strong set order for subsets of $\Re$ which will be important for our analysis is that, if $A\leq_S C\leq_S D$, then $A\cap D\subseteq C$.

There are four cases to consider. (i) If $w_m=w_{m+1}$ and $y_m=y_{m+1}$, then $z_m=z_{m+1}$ and there is nothing to show. (ii) Suppose that $w_m<w_{m+1}$ and $y_m<y_{m+1}$. This implies that A_m is optimal on (w_m,w_{m+1}) and (y_m,y_{m+1}). Then, since $a_i^{BR}(t_i|\mathbf{X})$ is nondecreasing in the strong set order in t_i, it must be that A_m is optimal on $(\min(w_m,y_m),\max(x_{m+1},y_{m+1}))$. Thus, A_m is optimal on (z_m,z_{m+1}). (iii) Suppose that $w_m=w_{m+1}$ and $y_m<y_{m+1}$. If $y_m\leq w_{m+1}\leq y_{m+1}$, then A_m must be optimal on (z_m,z_{m+1}). If $w_{m+1}<y_m<y_{m+1}$, find a $k>m$ such that an optimal action for some $t_i\in(w_{m+1},y_m)$ is A_k. But since $a_i^{BR}(t_i|\mathbf{X})$ is nondecreasing in the strong set order in t_i, we must have A_k optimal on (y_m,y_{m+1}) and A_m optimal at w_{m+1}. But this implies that A_m is optimal on (w_{m+1},y_{m+1}) and thus on (z_m,z_{m+1}). (iv) If $y_m<y_{m+1}<w_{m+1}$, there is some $k<m$ such that A_k is optimal for some $t_i\in(y_{m+1},w_m)$, which implies by the strong set order that A_m is optimal on (y_m,w_{m+1}), and thus on (z_m,z_{m+1}).

Thus, $\beta_i(t_i,\mathbf{z})=A_{m^*(t_i,\mathbf{z})}$ is a nondecreasing strategy consistent with $\mathbf{z}$ which assigns optimal actions to almost every type; we can fill in optimal behavior at the jump points, implying that $\mathbf{z}\in\Gamma_i(\mathbf{X})$.

Apply Kakutani. Since Σ^M is a compact, convex subset of $(M+2)\cdot I$ –dimensional Euclidean space, and since the correspondence is nonempty, has a closed graph, and is convex-valued, then we can apply Kakutani's fixed point theorem to guarantee that a fixed point exists.

Proof of Theorem 2.2

First, consider a strategy $\alpha_i(t_i)$ and the corresponding vector $\mathbf{z}\in\Sigma_I^K$ which represents the direction changes of $\alpha_i(t_i)$ (Definition 2.4), which describes the points where the strategy experiences a direction change. For $0\leq k\leq K$ such that k is even, we know that the player's strategy will be nondecreasing on $[z_k,z_{k+1}]$. Then, for each $m=1,..,M$, if there is some $n\geq m$ such that A_n is played on $[z_k,z_{k+1}]$, let $x_{kM+m}=\inf\{z_k\leq t_i\leq z_{k+1}|\alpha_i(t_i)\geq A_m\}$. Otherwise, let $x_{kM+m}=z_{k+1}$. Now consider k odd, $k\leq K$. Then, for each $m=1,..,M$, if there is some $n\leq M-m$ such that A_n is played on $[z_k,z_{k+1}]$, let $x_{kM+m}=\inf\{z_k\leq t_i\leq z_{k+1}|\alpha_i(t_i)\leq A_{M-m}\}$. Otherwise, let $x_{kM+m}=z_{k+1}$.

Since $\mathbf{x}$ does not specify behavior for $t_i\in\{\mathbf{x}\}$, a given $\mathbf{x}\in\Sigma_I^{M\cdot(K+1)}$ corresponds to more than one strategy. As above, we can define behavior consistent with $\mathbf{x}$. Let $l^*(t,\mathbf{x})\equiv\max\{l|x_l<t\}$. Then, for all $t_i\in T_i\backslash\{\mathbf{x}\}$, find the integers $0\leq\hat{k}\leq K$ and $1\leq\hat{m}\leq M$ such that $l^*(t_i,\mathbf{x})=\hat{k}\cdot M+\hat{m}$. If this $\hat{k}$ is even, let $\beta_i(t_i,\mathbf{x})=A_{\hat{m}}$, while if $\hat{k}$ is odd, let $\beta_i(t_i,\mathbf{x})=A_{M-\hat{m}}$. Any function $\beta_i(t_i,\mathbf{x})$ which assigns this behavior on $T_i\backslash\{\mathbf{x}\}$ is said to be consistent with $\mathbf{x}$.

Using the representation just developed, let $\mathbf{\Sigma(K)}\equiv\Sigma_1^{M\cdot(K_1+1)}\times\cdot\cdot\times\Sigma_I^{M\cdot(K_I+1)}$. Since convex combinations of nondecreasing vectors are nondecreasing, $\mathbf{\Sigma(K)}$ is a convex subset of Euclidean space. Define Γ: $\mathbf{\Sigma(K)}\rightarrow\mathbf{\Sigma(K)}$ as follows. For each player i, if $\mathbf{X}$ represents strategies for all $j\neq i$ that have at most K_j direction changes, then $\Gamma^i(\mathbf{X})$ contains the representation of all best responses for player i with no more than K_i direction changes (this vector exists and is unique by (2.4)). The arguments of Lemma 2.2 can be followed exactly to show that the correspondence has a closed graph; convexity follows by uniqueness. Then, we can follow the proof of Lemma 2.2 exactly to guarantee existence of a fixed point for the correspondence Γ, and then to construct the corresponding pure strategy Nash equilibrium of the original game.

Proof of Theorem 3.1

Proof: For each player i, consider a sequence of action sets $\{\mathcal{A}^{i,n}\}$, where $\mathcal{A}^{i,n}=\{\underline{a}_i+\frac{m}{10^n}(\overline{a}_i-\underline{a}_i):m=0,..,10^n\}$. Let $\mathcal{A}^n=(\mathcal{A}^{1,n},..,\mathcal{A}^{I,n})$. For each n, the function $\beta_{i,n}:T_i\rightarrow\mathcal{A}^{1,n}\subset[\underline{a}_i,\overline{a}_i]$ is of bounded variation by assumption; this is equivalent to the statement that there exist two nondecreasing functions, $\underline{\beta}_{i,n}$ and $\overline{\beta}_{i,n}$, such that $\beta_{i,n}=\overline{\beta}_{i,n}-\underline{\beta}_{i,n}$. Helly's Selection Theorem (Billingsley (1968), p. 227) guarantees that a sequence of nondecreasing, bounded functions on $T_i\subseteq\Re$ has a subsequence which converges almost everywhere to a nondecreasing function (and in particular, it converges at continuity points of the limiting function). Thus, we can find a subsequence of $\{1,2,..\}$ and a function of bounded variation β^* such that $\{\beta_{i,n}\}$ converges to β^* except at points of discontinuity of β^*. Since there are a finite number of players, there must exist a subsequence $\{n\}$ such that $\{\beta_{1,n}(t_1),..,\beta_{I,n}(t_I)\}$ converges almost everywhere to $\boldsymbol{\beta}^*(\mathbf{t})$. Consider this subsequence.

We wish to show that for each player and almost every type, the action assigned by $\boldsymbol{\beta}^*(\mathbf{t})$ is a best response to $\beta^*(\mathbf{t})$. Let $\hat{Z}^i=\{t_i|\exists n \text{ s.t. } t_i \in \{\mathbf{x}^{i,n}\}\}$, and note that $\hat{Z}$ is countable (it is a subset of the rationals) and thus has measure zero. Let D^i be the set of all actions which are in $\mathcal{A}^{i,n}$ for some finite n. Consider player i, and a type $t_i \in T_i \setminus \hat{Z}^i$, such that $\beta_i^*(t_i)$ is continuous at t_i (recall that discontinuities occur only on a set of measure 0). Let $b=\beta_i^*(t_i)$; Helly's Selection Theorem implies $\beta_{i,n}(t_i)$ converges to b.

Since $\beta_{i,n}(t_i)$ is an equilibrium strategy for any n, $U_i(\beta_{i,n}(t_i),\boldsymbol{\beta}_{-i,n}(\mathbf{t}_{-i}),t_i) \geq U_i(a',\boldsymbol{\beta}_{-i,n}(\mathbf{t}_{-i}),t_i)$ for every $a' \in \mathcal{A}^{i,n}$. Because payoffs are continuous and since $\boldsymbol{\beta}_{-i,n}(\mathbf{t}_{-i})$ converges to $\boldsymbol{\beta}^*_{-i}(\mathbf{t}_{-i})$ for almost all $\mathbf{t}_{-i}$, $u_i(\beta_{i,n}(t_i),\boldsymbol{\beta}_{-i,n}(\mathbf{t}_{-i}),t_i)$ converges to $u_i(b,\boldsymbol{\beta}^*_{-i}(\mathbf{t}_{-i}),t_i)$ for almost all $\mathbf{t}_{-i}$. Since the type distribution is atomless, this in turn implies that $U_i(b,\boldsymbol{\beta}^*_{-i}(\cdot),t_i) \geq U_i(a',\boldsymbol{\beta}^*_{-i}(\cdot),t_i)$ for all $a' \in D^i$. Now consider an action $a'' \notin D^i$. Note that there exists a sequence $\{a^k\}$, $a^k \in D^i$, which converges to a''. But $U_i(b,\boldsymbol{\beta}^*_{-i}(\cdot),t_i) \geq U_i(a^k,\boldsymbol{\beta}^*_{-i}(\cdot),t_i)$ for all k by our previous arguments. Thus, by continuity of U_i in a_i, $U_i(b,\boldsymbol{\beta}^*_{-i}(\cdot),t_i) \geq U_i(a'',\boldsymbol{\beta}^*_{-i}(\cdot),t_i)$. Q.E.D.

Proof of Theorem 3.2

Following the proof of Theorem 3.1, we will consider a sequence of games with successively finer action spaces, indexed by $\{n\}$, where $\mathcal{A}^{i,n}$ is the action space for player i in game n, and a PSNE of this game in nondecreasing strategies is described by $\boldsymbol{\beta}_n(\mathbf{t})$. The sequence $\{n\}$ is chosen so that it converges almost everywhere to a set of strategies denoted $\boldsymbol{\beta}^*(\mathbf{t})$.

To simplify the exposition of the proof, we make an initial transformation of the problem, so that the allocation rule given in equation (3.4) is restricted to the rule $m_i(a_i) = a_i$. Since m_i is assumed to be strictly increasing, and since all properties we have required of our primitives in (3.1)-(3.7) are robust to (strictly) monotone transformations of the action, we can simply rescale the action space and redefine our functions $\bar{v}_i$ and $\underline{v}_i$ appropriately; for simplicity, we will not adjust the notation for $\bar{v}_i$ and $\underline{v}_i$.

To begin, we introduce some notation. Define:

$$w_{i,n}(a_i,\mathbf{t}_{-i}) \equiv \varphi_i(a_i,\boldsymbol{\beta}_{-i,n}(\mathbf{t}_{-i})) \qquad w_i^*(a_i,\mathbf{t}_{-i}) \equiv \varphi_i(a_i,\boldsymbol{\beta}^*_{-i}(\mathbf{t}_{-i}))$$

$$\Delta v_i(a_i,\mathbf{t}) = \bar{v}_i(a_i,\mathbf{t}) - \underline{v}_i(a_i,\mathbf{t}) \qquad \hat{\underline{v}}(a_i,t_i) = \int \underline{v}_i(a_i,\mathbf{t}) \cdot f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$$

Further, we define several events, taken from the perspective of player i, playing against opponents' equilibrium strategies. For each of the following, we introduce notation for the event that, when players $\{I\}\backslash i$ use strategies $\boldsymbol{\beta}_{-i,n}(\cdot)$, the realization of $\mathbf{t}_{-i}$ and the outcome of the tie-breaking mechanism (described by $\rho(\sigma_T)$ in (3.4)) are such that the action a_i produces the stated outcome:

$W_{i,n}(a_i)$: Player i wins using a_i (either by winning a tie or winning outright).

$\tau^W_{i,n}(a_i)$: Player i ties for winner at a_i and player i wins the tie.

$\tau^{L}_{i,n}(a_i)$: Player i ties for winner at a_i and player i loses the tie.

Thus, if ε_n is the minimum bidding increment, $W_{i,n}(a_i) = \{W_{i,n}(a_i-\varepsilon_n) \cup \tau^{L}_{i,n}(a_i-\varepsilon_n) \cup \tau^{W}_{i,n}(a_i)\}$. Let $W^*_i(a_i)$, $\tau^{W*}_i(a_i)$, and $\tau^{L*}_i(a_i)$ represent the corresponding events when i plays against the opponents' limiting strategies.

In our analysis, it will sometimes be useful to define expected payoffs conditional on zero-probability events. We will use the following convention:

$$E\left[\Delta v_i(a_i,\mathbf{t})\middle|t_i, W_i(a_i)\right] \equiv \lim_{\varepsilon\to 0} E\left[\Delta v_i(a_i,\mathbf{t})\middle|t_i, W_i(a_i) \cup \{\mathbf{t}_{-i} \in [\underline{\mathbf{t}}_{-i}, \underline{\mathbf{t}}_{-i}+\varepsilon)\}\right].$$

In order to establish that "mass points" are inconsistent with equilibrium, we make use of revealed preference to show that each player could do better than use actions which support the mass point. In particular, each type who uses an action which is a mass point in the opponents' limiting strategies might also see nonnegative expected payoffs from increasing her action a small amount, "jumping over" another player's mass point. A difficulty may arise if there is a set of actions which win with probability 0 for a given player i in a given finite game, which might happen when playing against a more aggressive opponent. In such a case, player i's lowest types may be indifferent between a set of actions, all of which generate zero expected payoffs, even though action a_i which wins with probability zero might also satisfy $E\left[\Delta v_i(a_i,\mathbf{t})\middle|t_i, W_{i,n}(a_i)\right]<0$; such a player might not wish to increase her action a little bit to guarantee a win, since increasing the probability of winning above zero may be undesirable nearby action a_i. The following Lemma shows that there exist equilibria to the finite-action games where all actions would give positive returns to winning, even when winning is a zero-probability event.

> **Lemma 3.2.2** If $\mathcal{A}^{i,n}$ is finite for all i, we can find a PSNE of this game (described by strategies $\boldsymbol{\beta}_n(\mathbf{t})$) which satisfies the following: (i) each player's equilibrium strategy, $\beta_{i,n}(t_i)$, is a nondecreasing function of t_i. (ii) For all i and almost all t_i, $E\left[\Delta v_i(\beta_{i,n}(t_i),\mathbf{t})\middle|t_i, W_{i,n}(\beta_{i,n}(t_i))\right] \geq 0$.

Now restrict attention to sequences $\{\boldsymbol{\beta}_n(\mathbf{t})\}$ which satisfy condition (ii) of Lemma 3.2.2. The next Lemma will show that the following "no mass point" condition is satisfied:

For each player i, for all $a_i \in \mathcal{A}$, $\Pr(\beta^*_i(t_i) = a_i)\cdot\Pr(\tau^{L*}_i(a_i)) = 0$. (NMP)

This condition requires that for every possible action of player i, either player i uses the action with probability zero, or else the probability of a tie for winner with an opponent at the action is zero. If ties occur at a_i with probability zero, then $\Pr(W^*_i(a_i))$ will be continuous at a_i. The proof of Lemma 3.2.3 will make use of the fact that if a mass point occurs in the limiting strategies, there must be something "close" to a mass point in finite games with a large enough number of actions.

> **Lemma 3.2.3:** Construct $\boldsymbol{\beta}^*(\mathbf{t})$ as the limit of equilibrium strategies to finite games,

$\boldsymbol{\beta}_n(\mathbf{t})$, which satisfy conditions (i) and (ii) from Lemma 3.2.2. Then $\boldsymbol{\beta}^*(\mathbf{t})$ satisfies (NMP).

The following Lemmas establish that under (NMP), $\boldsymbol{\beta}^*(\mathbf{t})$ describe equilibrium behavior for almost all types.

Lemma 3.2.4: Under (NMP), for all i and almost all t_i such that $\beta_{i,n}(t_i)$ converges to $\beta_i^*(t_i)$, the following conditions hold: (i) $U_i(a_i, \boldsymbol{\beta}^*_{-i}(\cdot), t_i)$ is continuous in a_i at $a_i = \beta_i^*(t_i)$, and (ii) $U_i(\beta_{i,n}(t_i), \boldsymbol{\beta}_{-i,n}(\cdot), t_i)$ converges to $U_i(\beta_i^*(t_i), \boldsymbol{\beta}^*_{-i}(\cdot), t_i)$.

Lemma 3.2.5: Under (NMP), $\beta_i^*(t_i)$ is a best response to $\boldsymbol{\beta}^*_{-i}(\cdot)$ for almost all t_i.

Together with our previous arguments, Lemma 3.2.5 implies that a game satisfying the hypotheses of this theorem will have a set of strategies, $\boldsymbol{\beta}^*(\mathbf{t})$, which assign optimal actions to almost every type of each bidder. Since behavior on a set of measure zero does not matter to opponents, we complete the argument by showing that these strategies in fact assign optimal actions to every type t_i of each bidder i, such that $t_i > \underline{t}_i$. Standard arguments similar to those in the proof of Lemma 3.2.3 can be used to establish that, for each i, $\Pr(\beta_i^*(t_i)=b)=0$ for all $b \in \operatorname{int}\{a_i\text{: } \Pr(W_i^*(a_i))>0\}$ (otherwise, types who chose actions just below b would prefer an action just above b, as in Lemma 3.2.3, but there must be some types playing actions just below b, otherwise b would not be optimal for player i). Now consider $c = \inf\{a_i\text{: } \Pr(W_i^*(a_i))>0\}$. Condition (NMP) guarantees that if $\Pr(\beta_i^*(t_i) = c)>0$, then $\Pr(W_i^*(c))=0$. But this in turn implies that at most the lowest type of each of player i's opponents use c, so the opponents see discontinuities in their payoffs only for their lowest types.

Proofs of Lemmas:

Proof of Lemma 3.2.2

We proceed by constructing such an equilibrium from the limit of a sequence of equilibria of constrained games, where in each constrained game, the lowest types of each player must choose action Q. We first show that, using minor modifications of our Theorem 2.1, each constrained game has a fixed point; it then follows from continuity of the payoffs in opponent strategies that the limit of the sequence of constrained games is an equilibrium in the unconstrained games.

Observe that the constrained strategies of the players will by definition be nondecreasing, since action Q is the lowest available action and the constraint requires that the lowest types use Q; thus we do not need to modify our notation from Section 2 for representing nondecreasing strategies with finite vectors. We then define the constrained best response of player i to an arbitrary (constrained or unconstrained) strategy by opponents represented by $\mathbf{X}$:

$$\hat{a}_i^{BR}(t_i|\mathbf{X},\delta) = \begin{cases} a_i^{BR}(t_i|\mathbf{X}) & t_i > \underline{t}_i + \delta \\ Q \cup a_i^{BR}(t_i|\mathbf{X}) & t_i = \underline{t}_i + \delta \\ Q & t_i < \underline{t}_i + \delta \end{cases} \tag{7.1}$$

Using this notation, we modify our correspondence, as follows:

$$\hat{\Gamma}_i(\mathbf{X},\delta)=\{\mathbf{y}: \exists\ \alpha_i(t_i) \text{ which is consistent with } \mathbf{y} \text{ such that } \alpha_i(t_i)\in\ \hat{a}_i^{BR}(t_i|\mathbf{X},\delta)\}. \qquad (7.2)$$

It is straightforward using our definitions to show that an equivalent, second definition is:

$$\hat{\Gamma}_i(\mathbf{X},\delta)=\{\mathbf{y}: \exists\ \mathbf{z}\in\Gamma_i(\mathbf{X}) \text{ such that } y_m=\max(\underline{t}_i+\delta,z_m)\}. \qquad (7.3)$$

We now show that the conditions for Kakutani are satisfied for each $\hat{\Gamma}(\mathbf{X},\delta)$.

Nonempty: We know from Theorem 2.1 that $\Gamma_i(\mathbf{X})$ is nonempty. Thus, we can construct elements of $\hat{\Gamma}_i(\mathbf{X},\delta)$, as can be easily seen in (7.3).

Closed graph: Recall that $\Gamma_i(\mathbf{X})$ has a closed graph. Take a sequence such that $\lim_k(\mathbf{X}^k,\mathbf{Y}^k)=(\mathbf{X},\mathbf{Y})$, such that $\mathbf{Y}^k\in\hat{\Gamma}(\mathbf{X}^k,\delta)$ for all k. We argued in Theorem 2.1 that $V_i(a_i;\mathbf{X},t_i)$ is continuous in the elements of $\mathbf{X}$. We need to show that for each player and almost every type, the action assigned by $\mathbf{Y}$ is in $\hat{a}_i^{BR}(t_i|\mathbf{X},\delta)$. Consider player i. Let $\alpha_i(t_i;\mathbf{y}^i)$ be a nondecreasing strategy consistent with $\mathbf{y}^i$. Suppose first that $t_i<\underline{t}_i+\delta$. Then since $\mathbf{y}^{i,k}\in\hat{\Gamma}_i(\mathbf{X}^k,\delta)$ for all k, the definition of the correspondence implies that that $y_1^{i,k}>t_i$ for all k, and thus $\alpha_i(t_i;\mathbf{y}^i)=Q\in\ \hat{a}_i^{BR}(t_i|\mathbf{X},\delta)$. Now consider $t_i>\underline{t}_i+\delta$ such that $t_i\in T_i\backslash\{\mathbf{y}^i\}$. Then there exists an $m\in\{0,..,M\}$ such that $y_m^i<t_i<y_{m+1}^i$. Since $\mathbf{y}^{i,k}$ converges to $\mathbf{y}^i$, there must exist an K such that, for all $k>K$, $y_m^{i,k}<t_i<y_{m+1}^{i,k}$, and thus $A_m\in\ \hat{a}_i^{BR}(t_i|\mathbf{X},\delta)$ since $\mathbf{Y}^k\in\Gamma(\mathbf{X}^k)$. Since $t_i>\underline{t}_i+\delta$, this in turn implies $a_i^{BR}(t_i|\mathbf{X}^k)$. But, since $V_i(a_i;\mathbf{X},t_i)$ is continuous in $\mathbf{X}$, if $V_i(A_m;\mathbf{X}^k,t_i)\geq V_i(A_{m'};\mathbf{X}^k,t_i)$ for all $k>K$ and all m', then $V_i(A_m;\mathbf{X},t_i)\geq V_i(A_{m'};\mathbf{X},t_i)$. This implies $A_m\in\ a_i^{BR}(t_i|\mathbf{X})$ and thus $A_m\in\ \hat{a}_i^{BR}(t_i|\mathbf{X},\delta)$, as desired.

Convexity: The proof of Theorem 2.1 showed that the fact that $a_i^{BR}(t_i|\mathbf{X})$ is nondecreasing in the strong set order implies that the correspondence Γ is convex-valued. Since the definition of $\hat{\Gamma}_i(\mathbf{X},\delta)$ is identical to that of $\Gamma(\mathbf{X})$ except that $\hat{a}_i^{BR}(t_i|\mathbf{X},\delta)$ is used instead of $a_i^{BR}(t_i|\mathbf{X})$, all we need to show is that $\hat{a}_i^{BR}(t_i|\mathbf{X},\delta)$ is nondecreasing in the strong set order. Consider $t_i'<t_i''$. If $\underline{t}_i+\delta<t_i'<t_i''$, then $\hat{a}_i^{BR}(t_i|\mathbf{X},\delta)=a_i^{BR}(t_i|\mathbf{X})$ for $t_i=t_i',t_i''$, and thus $\hat{a}_i^{BR}(t_i'|\mathbf{X},\delta)\leq_S\hat{a}_i^{BR}(t_i''|\mathbf{X},\delta)$. If $\underline{t}_i+\delta>t_i'$, then $\hat{a}_i^{BR}(t_i'|\mathbf{X},\delta)=Q$. The definition of the strong set order asks us to verify that if $c>b$, then whenever $c\in\ \hat{a}_i^{BR}(t_i'|\mathbf{X},\delta)$ and $b\in\ \hat{a}_i^{BR}(t_i''|\mathbf{X},\delta)$, then we also must have $b\in\ \hat{a}_i^{BR}(t_i'|\mathbf{X},\delta)$ and $c\in\ \hat{a}_i^{BR}(t_i''|\mathbf{X},\delta)$. But, since the only element of $\hat{a}_i^{BR}(t_i'|\mathbf{X},\delta)$ is the lowest available action, Q, this is satisfied trivially. Thus, $\hat{a}_i^{BR}(t_i|\mathbf{X},\delta)$ is nondecreasing in the strong set order, and our arguments from Theorem 2.1 show that $\hat{\Gamma}_i(\mathbf{X},\delta)$ must then be convex.

Fixed point exists: By Kakutani's fixed point theorem, $\hat{\Gamma}(\mathbf{X},\delta)$ has a fixed point.

Construction of Equilibrium: Consider a sequence $\mathbf{X}^k$ such that $\mathbf{X}^k\in\hat{\Gamma}(\mathbf{X}^k,1/k)$ for each k. (Thus, if each player i uses actions consistent with $\mathbf{X}^k$, he will use action Q for types less than $\underline{t}_i+1/k$.) Such an $\mathbf{X}^k$ is guaranteed to exist for each k since $\hat{\Gamma}(\mathbf{X},\delta)$ has a fixed point for all $\delta>0$. Since each $\mathbf{X}^k$ is an element of a compact subset of finite-dimensional Euclidean space, we can find a subsequence of $\{k\}$ such that $\{\mathbf{X}^k\}$ converges to a matrix $\mathbf{X}$, and we simply need to establish that $\mathbf{X}\in\Gamma(\mathbf{X})$. Without loss of generality, let $\{k\}$ denote that subsequence.

We show that, for each player and almost every type, actions consistent with $\mathbf{x}^i$ are in $a_i^{BR}(t_i|\mathbf{X})$. Consider t_i such that $t_i \in T_i \backslash \{\mathbf{x}^i\}$. Then there exists an $m \in \{0,..,M\}$ such that $x_m^i < t_i < x_{m+1}^i$. Since $\mathbf{x}^{i,k}$ converges to $\mathbf{x}^i$, there must exist an K such that, for all $k>K$, $x_m^{i,k} < t_i < x_{m+1}^{i,k}$ and $t_i > \underline{t}_i + 1/k$. Find such a $k>K$. Then $A_m \in \hat{a}_i^{BR}(t_i|\mathbf{X}^k, 1/k)$ since $\mathbf{X}^k \in \Gamma(\mathbf{X}^k)$. By definition and since $t_i > \underline{t}_i + 1/k$, $A_m \in a_i^{BR}(t_i|\mathbf{X}^k)$. But, since $V_i(a_i;\mathbf{X},t_i)$ is continuous in $\mathbf{X}$, if $V_i(A_m;\mathbf{X}^k,t_i) \geq V_i(A_{m'};\mathbf{X}^k,t_i)$ for all $k>K$ and all m', then $V_i(A_m;\mathbf{X},t_i) \geq V_i(A_{m'};\mathbf{X},t_i)$. This implies $A_m \in a_i^{BR}(t_i|\mathbf{X})$, as desired.

Finally, we show that the equilibrium we have constructed satisfies the desired property, that conditional expected payoffs always converge to a nonnegative number. In our sequence above, $V_i(A_m;\mathbf{X}^k,t_i) \geq V_i(A_{m'};\mathbf{X}^k,t_i)$ for all $k>K$, and in particular, $V_i(A_m;\mathbf{X}^k,t_i) \geq V_i(Q;\mathbf{X}^k,t_i)$. We know that for each k, $A_m > Q$ wins with positive probability. Thus by (3.3) and revealed preference, $E\left[\Delta v_i(A_m,\mathbf{t})|t_i, W_i(A_m;\mathbf{X}^k)\right] \geq 0$ for all k, which in turn implies $\lim_k E\left[\Delta v_i(A_m,\mathbf{t})|t_i, W_i(A_m;\mathbf{X}^k)\right] \geq 0$, as desired.

Proof of Lemma 3.2.3:

Consider player 1, and suppose that $\Pr(\beta_1^*(t_1) = b) \cdot \Pr(\tau_1^{L*}(b)) > 0$. This implies that there is a second player, let this be player 2, such that $\Pr(\beta_2^*(t_2) = b) > 0$. For simplicity, suppose all other players use b with probability 0, though the argument can be easily extended.

Observe that since each $\beta_{i,n}(t_i)$ is measurable and converges almost everywhere to $\beta_i^*(t_i)$, the sequence converges uniformly to $\beta_i^*(t_i)$ except on a set of arbitrarily small measure (Royden, 1988, p. 73). Thus, if $\beta_i^*(t_i) = b$ on an open interval S_i, for every $\eta>0$ there exists an E_i with measure less than η such that, for all $d>0$, there exists an N_d such that, for all $n>N_d$, $|\beta_{i,n}(t_i) - b| < d$ on $S_i \backslash E_i$. Further, since each $\beta_{i,n}(t_i)$ is nondecreasing in t_i, we know that there is an open interval $S_i' \subset S_i$ such that $|\beta_{i,n}(t_i) - b| < d$ on S_i'.

A little notation, for i=1,2:

$$\hat{a}_i(d,n) = \min\{a_i : a_i \in \mathcal{A}_n^i \text{ and } a_i > b - d\}$$

Find a set of types $\mathbf{E}$ such that $\boldsymbol{\beta}_n(\cdot)$ converges uniformly to $\boldsymbol{\beta}^*(\cdot)$ except for types $\mathbf{t} \in \mathbf{E}$, and such that $\sup_{t_i} \{\Pr(\mathbf{t}_{-i} \in \mathbf{E}_{-i}|t_i)\} < \frac{1}{10}\Pr(\tau_i^{L*}(b))$. Define $\zeta_i \equiv \sup_{t_i} \{\Pr(\mathbf{t}_{-i} \in \mathbf{E}_{-i}|t_i)\}$.

$$S_i = \{t_i : \beta_i^*(t_i) = b\}$$

$$\tilde{S}_i = \text{int}\{t_i \in S_i \backslash E_i\}.$$

$$\hat{\mathbf{t}}_{-i}(t_i,N) = \inf_{n>N}\left\{\sup\left\{\mathbf{t}_{-i} : \beta_{j,n}(t_j) \leq \beta_{i,n}(t_i)\ \forall j \neq i\right\}\right\}$$

Now, pick a $t_i \in \tilde{S}_i$, and find a $t_i' < t_i$ such that $t_i' \in \tilde{S}_i$. Pick any d, choose N_d such that, for all j=1,..,I, all $n>N_d$, and all $t_j \notin E_j$, $|\beta_{j,n}(t_j) - \beta_j^*(t_j)| < d$, and further, $\Pr(\beta_{j,n}(t_j) \in (b-d, b+d)\ \forall j=3,..,I) < \frac{1}{10}\Pr(\tau_i^{L*}(b))$. (Recall that players 3,..,I play action b with probability zero in the limit). Then, the following series of inequalities holds for all $n>N_d$:

$$
\begin{aligned}
0 &\le \inf_{n>N_d}\left\{E\left[\Delta v_i(\beta_{i,n}(t_i'),\mathbf{t}_{-i},t_i')\middle|t_i',W_{i,n}(\beta_{i,n}(t_i'))\right]\right\} \\
&< \inf_{n>N_d}\left\{E\left[\Delta v_i(b-d,\mathbf{t}_{-i},t_i')\middle|t_i',W_{i,n}(\beta_{i,n}(t_i'))\right]\right\} \\
&\le E\left[\Delta v_i(b-d,\mathbf{t}_{-i},t_i')\middle|t_i',\mathbf{t}_{-i}\le\hat{\mathbf{t}}_{-i}(t_i',N_d)\right] \\
&< E\left[\Delta v_i(b-d,\mathbf{t}_{-i},t_i)\middle|t_i,\mathbf{t}_{-i}\le\hat{\mathbf{t}}_{-i}(t_i',N_d)\right]
\end{aligned} \tag{7.4}
$$

The first inequality holds since by Lemma 3.2.2, each member of the set is nonnegative; the second inequality holds since Δv is nonincreasing and $|\beta_{i,n}(t_i')-b|<d$; the third inequality holds because by (3.6) and (3.7), expected payoffs are continuous and nondecreasing in the set of types defeated, and so the smallest expected payoffs must correspond to the smallest set of types potentially defeated, $\left\{\mathbf{t}_{-i}\le\hat{\mathbf{t}}_{-i}(t_i,N_d)\right\}$. The relation is an inequality rather than an equality since the player might tie rather than win against these types, but winning against the highest of these rather than having a tie will increase expected payoffs under assumption (3.7). The last inequality follows by (3.7), since expected returns to winning are strictly increasing in t_i.

It follows from its definition that $\hat{\mathbf{t}}_{-i}(t_i,N)$ is nondecreasing in N; but then, by continuity of Δv and since N_d is decreasing with d, we can find a d small enough such that $E\left[\Delta v_i(b+d,\mathbf{t}_{-i},t_i)\middle|t_i,\mathbf{t}_{-i}\le\hat{\mathbf{t}}_{-i}(t_i',N_d)\right]>0$ as well. Thus, define the following constant, which based on the preceding analysis is strictly positive for $t_i\in\tilde{S}_i$ and d small enough:

$$
\kappa(t_i,d)=E\left[\Delta v_i(b+d,\mathbf{t}_{-i},t_i)\middle|t_i,\mathbf{t}_{-i}\le\hat{\mathbf{t}}_{-i}(t_i',N_d)\right]
$$

Since Δv is nonincreasing in actions and following the arguments from the previous paragraph, $\kappa(t_i,d)$ is nonincreasing in d. Then, define the following positive constant, which is also nonincreasing in d (recalling that by definition $\zeta_i<\frac{1}{10}\Pr(\tau_i^{L*}(b))$):

$$
\gamma_i(t_i,d) = \tfrac{1}{10}\,\kappa(t_i,d)\cdot\left(\Pr\left(\tau_i^{L*}(b)\middle|t_i\right)-\zeta_i\right) \tag{7.5}
$$

Finally, pick a $d>0$ small enough such that, for $t_i\in\tilde{S}_i$, and $n>N_d$, the following inequalities hold:

$$
\gamma_i(t_i,d) > \underline{\hat{v}}_i(a',t_i)-\underline{\hat{v}}_i(b+d,t_i)+\int\left[\Delta v_i(a',\mathbf{t})-\Delta v_i(b+d,\mathbf{t})\right]f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}
$$
$$
\text{for all } a'\in[b-d,b+d]. \tag{7.6}
$$

$$
\begin{aligned}
&E\left[\Delta v_i(b+d,\mathbf{t}_{-i},t_i)\middle|t_i,W_{i,n}(b+d)\setminus W_{i,n}(\hat{a}_i(d,n))\right] \\
&\quad > \tfrac{1}{2}\,E\left[\Delta v_i(b+d,\mathbf{t}_{-i},t_i)\middle|t_i,\mathbf{t}_{-i}\le\hat{\mathbf{t}}_{-i}(t_i',N_d)\right]=\tfrac{1}{2}\,\kappa(t_i,d)
\end{aligned} \tag{7.7}
$$

It is possible to satisfy (7.6) since $\gamma_i(t_i,d)$ is nonincreasing in d, and since $b+d-a'$ approaches zero as d gets small and Δv is continuous in actions. The inequality in (7.7) can be satisfied by assumption (3.7), which states that the expected value of payoffs is nondecreasing in the set of types over which we condition, since $\hat{\mathbf{t}}_{-i}(t_i', N_d)$ must by definition be less than the highest vector of opponent types who use action $b+d$.

Pick an $n>N_d$. The action $b+d$ will be preferred to $a' \in (b-d,b+d)$ if:

$$\int \underline{v}_i(b+d,\mathbf{t}) \cdot f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} + \int \Delta v_i(b+d,\mathbf{t}) \cdot w_{i,n}(b+d,\mathbf{t}_{-i}) \cdot f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$$
$$\geq \int \underline{v}_i(a',\mathbf{t}) \cdot f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} + \int \Delta v_i(a',\mathbf{t}) \cdot w_{i,n}(a',\mathbf{t}_{-i}) \cdot f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$$

This can be rewritten as follows:

$$\begin{aligned} &E\left[\Delta v_i(b+d,\mathbf{t})\middle|t_i, W_{i,n}(b+d) \setminus W_{i,n}(a')\right] \cdot \Pr\left(W_{i,n}(b+d) \setminus W_{i,n}(a')\middle|t_i\right) \\ &\quad > \hat{\underline{v}}_i(a',t_i) - \hat{\underline{v}}_i(b+d,t_i) \qquad (7.8) \\ &\quad + E\left[\Delta v_i(a',\mathbf{t}) - \Delta v_i(b+d,\mathbf{t})\middle|t_i, W_{i,n}(a')\right] \cdot \Pr\left(W_{i,n}(a')\middle|t_i\right) \end{aligned}$$

Consider first $a' = \hat{a}_i(d,n)$, the lowest feasible action chosen by a type in S_i. Observe that $\Pr\left(W_{i,n}(b+d) \setminus W_{i,n}(\hat{a}_i(d,n))\middle|t_i\right) > \frac{1}{2}(\Pr(\tau_i^{L*}(b)) - \zeta_i)$ for $n>N_d$, since all opponent types who choose action b in the limit must choose actions on $[\hat{a}_i(d,n), b+d)$ for $n>N_d$. Thus, by choosing action $b+d$ rather than $\hat{a}_i(d,n)$, player i chooses a strictly higher action than those types she would lose to or tie with using action $\hat{a}_i(d,n)$; at worst, all of the types who choose action b in the limit choose action $\hat{a}_i(d,n)$ with grid n, and player i defeats those players she would have otherwise tied with by increasing her action to $b+d$.

Combining the latter argument with equations (7.7) and (7.5) allows us to conclude that the LHS of (7.8) is strictly greater than $\gamma_i(t_i,d)$ for d small enough. On the other hand, by (7.6), the RHS of (7.8) is strictly less than $\gamma_i(t_i,d)$ for d small enough. Thus, for the chosen d and for n large enough, type t_i will not choose action $\hat{a}_i(d,n)$. But this then implies that all $t_i \in \tilde{S}_i$ will choose actions on $(\hat{a}_i(d,n), b+d)$ for i=1,2. That will in turn imply that (recalling that d was chosen so that $\Pr(\beta_j(t_j) \in (b-d,b+d)\ \forall j=3,..,I) < \frac{1}{10} \cdot \Pr(\tau_i^{L*}(b))$, and letting ε_n be the minimum action increment for the game with action space indexed by n):

$$\Pr\left(W_{i,n}(b+d) \setminus W_{i,n}(a'+\varepsilon_n)\middle|t_i\right) > \tfrac{1}{2}(\Pr(\tau_i^{L*}(b)) - \zeta_i).$$

Thus, the arguments of the previous paragraph apply again: no types $t_i \in \tilde{S}_i$ will choose action $a'+\varepsilon_n$. The argument can be repeated for both players, "unraveling" back to the point where for i=1,2, no types $t_i \in \tilde{S}_i$ choose actions on $(b-d,b+d)$, a contradiction.

Proof of Lemma 3.2.4: Part (i): Consider $a_i \in \mathcal{A}_i$. Recall that:

$$U_i(a_i, \beta^*_{-i}(\cdot), t_i) = \hat{\underline{v}}_i(a_i,t_i) + E\left[\Delta v_i(a_i,\mathbf{t})\middle|t_i, W_i^*(a_i)\right] \cdot \Pr\left(W_i^*(a_i)\middle|t_i\right) \qquad (7.9)$$

Suppose first that $\Pr(\tau_i^{L*}(b))=0$. Then, by our assumption (3.5), $\Pr(\tau_i^{W*}(b))=0$ as well, that is, the probability of a tie for winner at b is zero for player i. This will in turn imply that $\Pr(W_i^*(a_i))$ is continuous at $a_i=b$. Since $\bar{v}_i$ and $\underline{v}_i$ are continuous, (7.9) will be continuous as well.

Now suppose that $\Pr(\tau_i^{L*}(b))>0$. But then, by (NMP), $\Pr(\beta_i^*(t_i))=b)=0$, and the set of types for whom payoffs are discontinuous at b has measure zero. Since the number of opponents is finite, the set of actions b such that $\Pr(\tau_i^{L*}(b))>0$ is also countable, and so the set of types for whom payoffs are discontinuous has measure zero.

Part (ii): Consider t_i such that $\beta_{i,n}(t_i)$ converges to $\beta_i^*(t_i)$ and $\Pr(W_i^*(a_i))$ is continuous at $a_i=\beta_i^*(t_i)$ (the latter condition is true for almost every t_i by part (i)). First, note that

$$\begin{aligned} &U_i(\beta_i^*(t_i),\boldsymbol{\beta}^*_{-i}(\cdot),t_i)-U_i(\beta_{i,n}(t_i),\boldsymbol{\beta}_{-i,n}(\cdot),t_i) \\ &\quad= [U_i(\beta_i^*(t_i),\boldsymbol{\beta}^*_{-i}(\cdot),t_i)-U_i(\beta_{i,n}(t_i),\boldsymbol{\beta}^*_{-i}(\cdot),t_i)] \\ &\quad+[U_i(\beta_{i,n}(t_i),\boldsymbol{\beta}^*_{-i}(\cdot),t_i)-U_i(\beta_{i,n}(t_i),\boldsymbol{\beta}_{-i,n}(\cdot),t_i)]. \end{aligned} \tag{7.10}$$

The first term of the RHS of (7.10) goes to zero as n gets large by continuity of $U_i(a_i,\boldsymbol{\beta}^*_{-i}(\cdot),t_i)$ in a_i in the relevant region (part (i)). So it remains to consider the second term of the RHS of (7.10). This term will converge to zero if $U_i(a_i,\boldsymbol{\beta}_{-i,n}(\cdot),t_i)$ converges uniformly (across a_i in a neighborhood of $\beta_i^*(t_i)$) to $U_i(a_i,\boldsymbol{\beta}^*_{-i}(\cdot),t_i)$. To see that uniform convergence holds, pick a η, $d>0$, and find an N_d and a set $\mathbf{E}$ of measure less than η such that for all $n>N_d$, $|\beta_j^*(t_j)-\beta_{j,n}(t_j)|<d$, $j=1,..,I$, except for $\mathbf{t}$ in set $\mathbf{E}$. Then, for such $n>N_d$, the following inequalities hold:

$$\begin{aligned} K_{in}(d,t_i) &\equiv \int_{\mathbf{t}_{-i}\notin \mathbf{E}_{-i}} \max\left(|\Delta v_i(\underline{a}_i,\mathbf{t})|,|\Delta v_i(\bar{a}_i,\mathbf{t})|\right)\mathbf{1}_{\beta_i^*(t_i)-2d\le\boldsymbol{\beta}^*_{-i}(\mathbf{t}_{-i})\le\beta_i^*(t_i)+2d}(\mathbf{t}_{-i})f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} \\ &\quad+\int_{\mathbf{t}_{-i}\in \mathbf{E}_{-i}} \max\left(|\Delta v_i(\underline{a}_i,\mathbf{t})|,|\Delta v_i(\bar{a}_i,\mathbf{t})|\right)f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} \\ &\ge \max_{a_i\in\left[\beta_i^*(t_i)-d,\ \beta_i^*(t_i)+d\right]} \left\{ \begin{aligned} &\int_{\mathbf{t}_{-i}\notin \mathbf{E}_{-i}} |\Delta v_i(a_i,\mathbf{t})|\mathbf{1}_{a_i-d\le\boldsymbol{\beta}^*_{-i}(\mathbf{t}_{-i})\le a_i+d}(\mathbf{t}_{-i})f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} \\ &+\int_{\mathbf{t}_{-i}\in \mathbf{E}_{-i}} |\Delta v_i(a_i,\mathbf{t})|f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} \end{aligned} \right\} \\ &\ge \max_{a_i\in\left[\beta_i^*(t_i)-d,\ \beta_i^*(t_i)+d\right]} \left|\int_{\mathbf{t}_{-i}} \Delta v_i(a_i,\mathbf{t})\left[w_{i,n}(a_i,\mathbf{t})-w_i^*(a_i,\mathbf{t})\right]f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}\right| \\ &= \max_{a_i\in\left[\beta_i^*(t_i)-d,\ \beta_i^*(t_i)+d\right]} \left|U_i(a_i,\boldsymbol{\beta}_{-i,n}(\cdot),t_i)-U_i(a_i,\boldsymbol{\beta}^*_{-i}(\cdot),t_i)\right| \end{aligned} \tag{7.11}$$

The first term of $K_{in}(d,t_i)$ can be made arbitrarily small by decreasing d, since f and Δv_i are bounded and since $\Pr(W_i^*(a_i))$ is continuous at $a_i=\beta_i^*(t_i)$. The second term can be made arbitrarily small by choosing η small enough. This establishes that there is a uniform bound on $|U_i(a_i,\boldsymbol{\beta}_{-i,n}(\cdot),t_i)-U_i(a_i,\boldsymbol{\beta}^*_{-i}(\cdot),t_i)|$ for $a_i\in[\beta_i^*(t_i)-d,\beta_i^*(t_i)+d]$.

Proof of Lemma 3.2.5: We know that for every n and for almost all t_i, $U_i(\beta_{i,n}(t_i),\boldsymbol{\beta}_{-i,n}(\cdot),t_i)\ge U_i(a',\boldsymbol{\beta}_{-i,n}(\cdot),t_i)$ for every $a'\in\mathcal{A}^{i,n}$. Thus, if $a'\in D^i$ (Where D^i is the set of all actions a' so that for large enough N, a' is an available action for $n>N$), then Lemma 3.2.4 (ii) implies that $U_i(\beta_i^*(t_i), \boldsymbol{\beta}_{-i}^*(\cdot),t_i) \geq U_i(a', \boldsymbol{\beta}_{-i}^*(\cdot),t_i)$.

Now consider an action $a' \notin D^i$. If $U_i(a', \boldsymbol{\beta}_{-i}^*(\cdot),t_i)=0$, then it suffices to compare $\beta_i^*(t_i)$ to the action Q, which is always available by assumption. If $U_i(a', \boldsymbol{\beta}_{-i}^*(\cdot),t_i)$ is continuous at a', and we can find a sequence $\{a^k\}$, $a^k \in D^i$, which converges to a', so that $U_i(a^k, \boldsymbol{\beta}_{-i,n}(\cdot),t_i)$ converges to $U_i(a', \boldsymbol{\beta}_{-i}^*(\cdot),t_i)$, as desired.

Suppose that $\Pr(\beta_j^*(t_j)=a')>0$ (a mass point exists at a') for some $j \neq i$, and $U_i(a', \boldsymbol{\beta}_{-i}^*(\cdot),t_i)>0$. This can be true only if $E\left[\Delta v_i(a', \mathbf{t}_{-i}, t_i) \middle| t_i, W_i^*(a')\right]>0$. Then, by continuity of Δv_i and by (3.7), there exists an $\delta>0$ such that $E\left[\Delta v_i(a'+\delta, \mathbf{t}_{-i}, t_i) \middle| t_i, W_i^*(a'+\delta)\right]>0$, where δ is chosen such that $a'+\delta$ is used on a set of measure zero by the other players in the limit and such that $a'+\delta \in D^i$. But, $a'+\delta$ wins against, instead of ties with, player j at a'. This leads to a discrete increase in the probability that player i wins. Thus, for small enough δ, $U_i(a'+\delta, \boldsymbol{\beta}_{-i}^*(\cdot),t_i)>U_i(a', \boldsymbol{\beta}_{-i}^*(\cdot),t_i)$. Since $a'+\delta \in D^i$, by our earlier arguments we know that $U_i(\beta_i^*(t_i), \boldsymbol{\beta}_{-i}^*(\cdot),t_i) \geq U_i(a'+\delta, \boldsymbol{\beta}_{-i}^*(\cdot),t_i)$.

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