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The Theory of Wage Differentials: Production Response and Factor Price Equalisation*

by

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*The views expressed in this paper are the authors' responsibility and do not reflect those of the Department of Economics nor of the Massachusetts Institute of Technology.

The Theory of Wage Differentials: Production Response and Factor Price Equalisation*

Previous analyses of the case where there is a distortionary wage differential between different activities, by writers such as Fishlow and David [2], Hagen [3], Bhagwati and Ramaswami [1], and Johnson [4], have led to the discovery of the following pathologies:

(i) the production possibility curve may become convex to the origin, instead of concave [1][2][4];

(ii) the feasible production possibility curve will shrink inside the "best" production possibility curve [3]; and

(iii) the commodity price-ratio will not be tangential (at points of incomplete specialisation in production) to the feasible production possibility curve.

However, this is not the end of the story. It can be further shown that (i) the shift in the production of a commodity, as its relative price changes, may be either positive or negative; and that the shift is not necessarily predictable from the convexity or concavity of the feasible production possibility curve; and (ii) given the commodity price-ratio, we cannot necessarily have unique capital-labour ratios in the two activities or unique factor price-ratios: an important and interesting implication of which result is that the factor price-equilisation theorem breaks down, despite all the Samuelson conditions being met, even if there is an identical wage differential in the same sector in both countries.

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*We would like to note that research in this field is independently being conducted by Steve Magee, P. J. Lloyd, and by Murray Kemp and Horst Herberg.

$$Q_1 = Lf^1(R_1) \quad (1)$$

$$Q_2 = (1-L)f^2(R_2) \quad (2)$$

$$LR_1 + (1-L)R_2 = R \quad (3)$$

$$f_1^1(R_1) = p\ f_1^2(R_2) \quad (4)$$

$$\{f^1 - R_1\ f_1^1(R_1)\}\gamma = p\ \{f^2 - R_2\ f_1^2(R_2)\} \quad (5)$$

Equations (1), (2), and (3) represent the production functions and factor allocations. Equation (4) states that the reward of the first factor (i.e. its marginal value product) is the same in the production of either commodity. Equation (5) states that the reward of the second factor in the production of the second commodity is γ times its reward in the production of the first.

II: The Comparative Statics of Equilibrium Outputs

In order now to investigate the response of output of either of the two commodities as the commodity price-ratio changes, it is convenient to work in terms of the variable w representing the ratio of the reward of the second factor to that of the first in the production of the first commodity. Then we can write:

$$\frac{f^1-R_1\ f_1^1}{f_1^1} = w \quad (6)$$

$$\frac{f^2-R_2\ f_1^2}{f_1^2} = \gamma w \quad (7)$$

Given our concavity assumptions we can solve (6) and (7) uniquely[1] to obtain R_1 and R_2 as functions $R_1(w)$ and $R_2(w)$ of w. It is easily seen that $R_i(w)$ is an increasing function of w. Given R, let $w_i(R)$ be the unique solution of $R_i(w) = R$. Then the relevant range of values for w is the interval $[\underline{w}, \overline{w}]$ where $\underline{w}(\overline{w})$ is the smaller (larger) of $w_1(R)$ and $w_2(R)$ (see Figure 1). The value of L corresponding to any given w in this interval is obtained from equation (3). The equilibrium value (or values) of w corresponding to a given p is (are) obtained from equation (4).

Let us examine this equation more closely. Let us first rewrite it as:

$$\frac{f_1^1(R_1)}{f_1^2(R_2)} = p \tag{8}$$

The left-hand side of equation (8) is a function of w alone. Denoting this function by p(w) we get:

$$\frac{p'(w)}{p(w)} = \frac{1}{p(w)}\frac{dp(w)}{dw} = \frac{f_{11}^1(R_1)}{f_1^1(R_1)}\frac{dR_1(w)}{dw} - \frac{f_{11}^2(R_2)}{f_1^2(R_2)}\cdot\frac{dR_2(w)}{dw}$$

where $f_{11}^i(R_i)$ is $\frac{d^2 f^i(R_i)}{dR_i^2}$. From (6) and (7) we get:

$$-\frac{f_{11}^1(R_1)\ f^1(R_1)}{\left\{f_1^1(R_1)\right\}^2}\cdot\frac{dR_1(w)}{dw} = 1 \tag{9}$$

$$-\frac{f_{11}^2(R_2)\ f^2(R_2)}{\left\{f_1^2(R_2)\right\}^2}\cdot\frac{dR_2(w)}{dw} = \gamma \tag{10}$$

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[1] If we wish to ensure that a solution exists for all non-negative values of w, we have to assume the Inada conditions:

$$\lim_{R_i \to 0(\infty)} \frac{f^i - R_i\ f_1^i(R_i)}{f_1^i(R_i)} = 0(\infty)$$

Hence $$\frac{p'(w)}{p(w)} = -\frac{f_1^1(R_1)}{f^1(R_1)} + \frac{f_1^2(R_2)}{f^2(R_2)}$$

$$= -\frac{1}{w+R_1} + \frac{\gamma}{\gamma w+R_2} \quad \text{[using (6) and (7)]}$$

$$= \frac{(\gamma R_1 - R_2)}{(w+R_1)(\gamma w+R_2)} \tag{11}$$

Equation (11) points at once to an interesting set of possibilities.

First, if in the relevant interval $[\underline{w}, \overline{w}]$ of values of w, $[\gamma R_1(w) - R_2(w)]$ changes sign, then $p'(w)$ also changes sign, since $w+R_1(w)$ and $\gamma w+R_2(w)$ are both non-negative. In other words, p(w) is not a montonic function of w. Thus equation (8) can have more than one value of w in $[\underline{w}, \overline{w}]$ as a solution. This means that the same commodity price ratio p can be consistent with more than one equilibrium combination of the outputs of the two commodities.

Second, consider two countries with identical production functions and the same type and degree of distortion (i.e. the second factor in the production of the second commodity receives γ times its reward in the production of the first commodity in both countries). Suppose they face the same commodity price-ratio p. If $\gamma R_1(w) - R_2(w)$ changes sign in both countries within the respective interval of values of w, then one country's equilibrium value w could be different from that of the other. In other words, factor price equalisation will fail to take place. It is important to note that this failure could take place, even though there is no factor-intensity reversal in the usual sense: even though $[R_1(w) - R_2(w)]$ has the same sign for all relevant values of w, for both countries, $[\gamma R_1(w) - R_2(w)]$ can still change its sign.[2]

[2] It should of course be kept in mind that $R_2(w)$ is the factor intensity in the production of the second commodity when the factor price-ratio faced by producers of this commodity is γw.

The precise conditions under which such multiple-equilibria will arise can be readily derived and related to the conditions defining the nature of output response to price change. To do this, we proceed now to derive first the slope of the production possibility curve, given γ. Using (1), (2), and (3), we get:

$$\frac{dQ_1}{dw} = \frac{dL}{dw}f^1 + Lf_1^1 \frac{dR_1}{dw}$$

$$\frac{dQ_2}{dw} = -\frac{dL}{dw}f^2 + (1-L)f_1^2 \frac{dR_2}{dw}$$

$$\frac{dL}{dw} = \frac{(R_2-R)\frac{dR_1}{dw} + (R-R_1)\frac{dR_1}{dw}}{(R_2-R_1)^2}$$

Hence

$$\frac{dQ_1}{dw} = \frac{f^1\{(R_2-R)\frac{dR_1}{dw} + (R-R_1)\frac{dR_2}{dw}\} + f_1^1(R_2-R)(R_2-R_1)\frac{dR_1}{dw}}{(R_2-R_1)^2}$$

$$= \frac{\{(f^1-R_1f_1^1) + R_2f_1^1\}(R_2-R)\frac{dR_1}{dw} + f^1(R-R_1)\frac{dR_2}{dw}}{(R_2-R_1)^2}$$

$$= \frac{f_1^1\left[\{(\frac{f^1-R_1f_1^1}{f_1^1}) + R_2\}(R_2-R)\frac{dR_1}{dw} + \frac{f^1}{f_1^1}(R-R_1)\frac{dR_2}{dw}\right]}{(R_2-R_1)^2}$$

$$= \frac{f_1^1\{(w+R_2)(R_2-R)\frac{dR_1}{dw} + (w+R_1)(R-R_1)\frac{dR_2}{dw}\}}{(R_2-R_1)^2} \qquad \text{[using (6)]}$$

and

$$\frac{dQ_2}{dw} = \frac{f_1^2\{(\gamma w+R_2)(R_2-R)\frac{dR_1}{dw} + (\gamma w+R_1)(R-R_1)\frac{dR_2}{dw}\}}{(R_2-R_1)^2}$$

Now:

$$\frac{dR_1}{dw} = \frac{-(f_1^1)^2}{f_{11}^1 \; f^1} \quad \text{and} \quad \frac{dR_2}{dw} = \frac{-(f_1^2)^2 \gamma}{f_{11}^2 \; f^2}$$

Next, let $\sigma(R_i)$ denote the elasticity of substitution of the factors in the production of the ith commodity. It is well known that

$\sigma(R_i) = - \dfrac{f_1^i(f^i - R_i \; f_1^i)}{R_i \; f^i \; f_{11}^i}$. Using (6) and (7) therefore we can write:

$$\frac{dQ_1}{dw} = \frac{f_1^1 \; \{(w+R_2)(R_2-R) \; \sigma_1 R_1 + (w+R_1)(R-R_1) \; \sigma_2 R_2\}}{w(R_2 - R_1)^2} \tag{12}$$

$$\frac{dQ_2}{dw} = \frac{-f_1^2 \; \{(\gamma w+R_2)(R_2-R) \; \sigma_1 R_1 + (\gamma w+R_1)(R-R_1) \; \sigma_2 R_2\}}{w(R_2-R_1)^2} \tag{13}$$

Hence

$$\frac{dQ_1}{dQ_2} = - \frac{f_1^1}{f_1^2} \left\{ \frac{(w+R_2)(R_2-R) \; \sigma_1 R_1 \quad + \quad (w+R_1)(R-R_1) \; \sigma_2 R_2}{(\gamma w+R_2)(R_2-R) \; \sigma_1 R_1 + (\gamma w+R_1)(R-R_1) \; \sigma_2 R_2} \right\}$$

$$= - p \left\{ \frac{(w+R_2)(R_2-R) \; \sigma_1 R_1 + (w+R_1)(R-R_1) \; \sigma_2 R_2}{(\gamma w+R_2)(R_2-R) \; \sigma_1 R_1 + (\gamma w+R_1)(R-R_1)\sigma_2 R_2} \right\} \tag{14}$$

It is seen from (14) that if there is no distortion, i.e. if $\gamma = 1$, then $\dfrac{dQ_1}{dQ_2} = - p$, showing that the domestic rate of transformation $(\dfrac{-dQ_1}{dQ_2})$ equals the commodity price ratio p. However, if $\gamma \neq 1$, we see from (14) that:

$$p - (- \frac{-dQ_1}{dQ_2}) = p(\gamma-1)w \left\{ \frac{(R_2-R) \; \sigma_1 R_1 + (R-R_1) \; \sigma_2 R_2}{(\gamma w+R_2)(R_2-R)\sigma_1 R_1 + (\gamma w+R_1)(R-R_1) \; \sigma_2 R_2} \right\} \tag{14a}$$

The expression in the square parenthesis is always positive; hence $p \gtrless - \dfrac{dQ_1}{dQ_2}$ according as $\gamma \gtrless 1$. This means that the commodity price-ratio, p, will not equal and will indeed exceed (fall short of) the domestic rate of transformation $(\frac{-dQ_1}{dQ_2})$ according as the degree of distortion in the factor price ratio faced by the producers of the second commodity as compared to those of the first, i.e. γ, exceeds (falls short of) unity.

Note further that, in general, the degree of divergence between the commodity price-ratio and the marginal rate of transformation can be expected to vary with the equilibrium point on the production possibility curve at which this divergence is being measured (although this is not inevitable).[3] It is also clear that this variation in the degree of divergence can obtain under CES production functions (where σ_1 and σ_2 are constant) and even under Cobb-Douglas production functions (where $\sigma_1 = \sigma_2 = 1$). Furthermore, the possibility of multiple equilibria, which we have already noted, also implies that, corresponding to the <u>same</u> commodity price-ratio, there could be different divergences between the price-ratio and the marginal rate of transformation at alternative equilibrium points on the production possibility curve.

We are now in a position to examine the response of the equilibrium output Q_2 of the second commodity to change in the international price-ratio p. From (13), it is clear that $\frac{dQ_2}{dw} \gtreqless 0$ according as $R_2(w) \gtrless R_1(w)$. By definition (except in the trivial case where $\underline{w} = \overline{w}$, either $R_2(w) > R_1(w)$ or $R_2(w) < R_1(w)$ for all w in $[\underline{w}, \overline{w}]$. Since the interval $[\underline{w}, \overline{w}]$ is determined uniquely once the aggregate factor endowment is known, the sign of $\frac{dQ_2}{dw}$ is also uniquely determined. To get the response of Q_2 with respect to p we

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[3]This can be seen readily by dividing (14a) on both sides by 'p', which yields the formula for $-p/dQ_1/dQ_2$, the relative degree of divergence. Note also that, except for the multiple-equilibrium possibility discussed in the text, any movement along the production possibility curve in equilibrium will require a change in the commodity price-ratio.

have to evaluate $\frac{dQ_2}{dp} = \frac{dQ_2}{dw} \cdot \frac{dw}{dp}$. Remember also that $\frac{dw}{dp}$ will have the same sign as $p'(w)$ in equation (11); and that $p'(w) \gtreqless 0$ according as $[\gamma R_1 - R_2] \gtreqless 0$. Using these arguments, we can now proceed to analyse the following six cases (excluding the degenerate case of $\underline{w} = \overline{w}$):

<u>Case I</u>: $R_1(w) > R_2(w)$ and $\gamma R_1(w) > R_2(w)$ for all w in $[\underline{w}, \overline{w}]$. Given $R_1(w) > R_2(w)$, this case will arise when either $\gamma \geq 1$ or when γ is less than unity but not sufficiently less than unity to make $\gamma R_1(w)$ less than $R_2(w)$ for some w in $[\underline{w}, \overline{w}]$. In this case, $\frac{dQ_2}{dw} > 0$ and assuming incomplete specialisation $\frac{dw}{dp} > 0$. Hence $\frac{dQ_2}{dp} > 0$. Thus, if we compare the equilibrium output Q_2 corresponding to two different international price ratios, the one associated with the higher price of the second commodity in terms of the first will be larger. Thus the (comparative static) response of equilibrium output to a price change is 'normal'.

<u>Case II</u>: $R_1(w) < R_2(w)$ <u>and</u> $\gamma R_1(w) < R_2(w)$ for all w in $[\underline{w}, \overline{w}]$. Given $R_1(w) < R_2(w)$, this case will arise when either $\gamma \leq 1$ or when γ is greater than unity but not sufficiently greater to make $\gamma R_1(w)$ exceed $R_2(w)$ for some w in $[\underline{w}, \overline{w}]$. In this case $\frac{dQ_2}{dw} < 0$ and again assuming incomplete specialisation $\frac{dw}{dp} < 0$. Hence $\frac{dQ_2}{dp} > 0$. Thus the output response is again 'normal'.

<u>Case III</u>: $R_1(w) > R_2(w)$ but $\gamma R_1(w) < R_2(w)$ for all w in $[\underline{w}, \overline{w}]$. Given $R_1(w) > R_2(w)$ this can happen only when γ is sufficiently less than unity. In this case $\frac{dQ_2}{dw} > 0$ and, assuming incomplete specialisation, $\frac{dw}{dp} < 0$. Hence $\frac{dQ_2}{dp} < 0$. Thus if we compare the equilibrium outputs corresponding to two different international prices for the second commodity in terms of the

first, then the output corresponding to a higher price will be smaller. This is a case of 'perverse' comparative-static response.

Case IV: $R_1(w) < R_2(w)$ but $\gamma R_1(w) > R_2(w)$ for all w in $[\underline{w}, \overline{w}]$. Given $R_1(w) < R_2(w)$, this can arise only when γ is sufficiently greater than unity. In this case also the output response in a comparative-static sense is 'perverse'.

Case V: $R_1(w) > R_2(w)$ but $\gamma R_1(w) - R_2(w)$ changes sign at one or more w in $[\underline{w}, \overline{w}]$. We saw earlier than when $\gamma R_1(w) - R_2(w)$ changes sign in $[\underline{w}, \overline{w}]$, the same international price ratio p may correspond to more than one equilibrium value for w and hence for the outputs Q_1 and Q_2. Thus the derivative $\frac{dw}{dp}$ will be different depending on the particular equilibrium value of w at which it is evaluated. Hence the sign of $\frac{dQ_2}{dp}$ will depend on the particular equilibrium point at which it is evaluated. It is easy to see that if we order the equilibrium points in increasing order of the value of Q_2 then $\frac{dQ_2}{dp}$ will alternate in sign as we move from one equilibrium point to the next. Thus if at one equilibrium point the comparative static response is 'normal', then at the next it will be 'perverse'.

Case VI: $R_1(w) < R_2(w)$ but $\gamma R_1(w) - R_2(w)$ changes sign at one or more w in $[\underline{w}, \overline{w}]$. Here again the possibility of multiple equilibria arises and the conclusions in Case V apply to this case also.[4]

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[4] We may note that the possibility of multiple w-values corresponding to a single p-value can be readily illustrated, using the well-known Lerner technique. Figure 2 shows how, consistent with commodity 1 remaining intensive in the use of factor 2 in two alternative equilibria (i.e. OM^2 is steeper than OM^1, and so is ON^2 steeper than ON^1), two alternative values of the factor-price ratio are possible (at AC and DF respectively for commodity 1 and at AB and DE for commodity 2 which has to pay more for its use of factor 2 than commodity 1 has to) when the commodity price-ratio involves exchange of $\overline{1}$ for $\overline{2}$ units of the two commodities.

III: Relationship of Output Response to Shape of the Production Possibility Curve

The possibility of "perverse" production response to change in the commodity price-ratio, in the presence of the wage differential, raises in turn the question as to whether the "perverse" response will arise if and only if the production possibility curve is convex to the origin.

Such an inference is implicit in the earlier literature [1][4], in the way the diagrams are drawn, for example, to show that the output of a commodity increases with its relative price when the production possibility curve is concave to the origin. However, such an inference is logically valid only when there is no wage differential. In the absence of such a differential the commodity price-ratio will be tangential to the production possibility curve and hence the output response to price change depends entirely on the curvature of this curve. But, once the differential is present, the commodity price-ratio no longer equals the domestic rate of transformation and hence there is no a priori reason to expect any necessary connection between output response and the curvature of the production possibility curve. Our numerical example in Section IV does in fact show that there is no such connection. However it is nevertheless of interest to derive analytically the curvature of the production possibility curve. To this we now turn.

We showed [equation (14)] that:

$$\frac{dQ_1}{dQ_2} = -\frac{f_1^1}{f_1^2}\left[\frac{(w+R_2)(R_2-R)\ \sigma_1 R_1 + (w+R_1)(R-R_1)\ \sigma_2 R_2}{(\gamma w+R_2)(R_2-R)\ \sigma_1 R_1 + (\gamma w+R_1)(R-R_1)\ \sigma_2 R_2}\right].$$

We can then derive $\frac{d^2Q_1}{dQ_2^{\,2}}$ by using the relation $\frac{d^2Q_1}{dQ_2^{\,2}} = \frac{d}{dw}\left(\frac{dQ_1}{dQ_2}\right) \circ \frac{dw}{dQ_2}$ and $\frac{dw}{dQ_2} = 1/\frac{dQ_2}{dw}$. We have already derived the expression for $\frac{dQ_2}{dw}$ in equation (13). Let us denote the numerator of the detailed expression for $\frac{dQ_1}{dQ_2}$, with the negative sign, by N(w) and the denominator by D(w). Then

$$\frac{d}{dw}\ \frac{dQ_1}{dQ_2} = \frac{D(w)\ \frac{dN}{dw} - N(w)\ \frac{dD}{dw}}{D^2}$$

Now

$$\frac{dN}{dw} = -\ f^1_{11} \cdot \frac{dR_1}{dw}\ \left[(w+R_2)(R_2-R)\ \sigma_1 R_1 + (w+R_1)(R-R_1)\ \sigma_2 R_2\right]$$

$$-\ f^1_1\ \left[(1 + \frac{dR_2}{dw})(R_2-R)\ \sigma_1 R_1 + (w+R_2)\ \frac{dR_2}{dw}\ \sigma_1 R_1 + (w+R_2)(R_2-R)\right.$$

$$(R_1\ \frac{d\sigma_1}{dw} + \sigma_1\ \frac{dR_1}{dw}) + (1 + \frac{dR_1}{dw})(R-R_1)\ \sigma_2 R_2$$

$$\left. + (w+R_1)(\frac{-dR_1}{dw})\ \sigma_2 R_2 + (w+R_1)(R-R_1)(R_2 \frac{d\sigma_2}{dw} + \sigma_2\ \frac{dR_2}{dw})\right]$$

Using the relations $\frac{dR_i}{dw} = \frac{\sigma_i R_i}{w}$ we get:

$$\frac{dN}{dw} = -\ \frac{\sigma_1 R_1}{w}\ f^1_{11}\ \left[(w+R_2)(R_2-R)\ \sigma_1 R_1 + (w+R_1)(R-R_1)\ \sigma_2 R_2\right]$$

$$-\ f^1_1\ \left[(R_2-R)\ \sigma_1 R_1 + (R-R_1)\ \sigma_2 R_2 + \frac{\sigma_2 R_2}{w}\ \left\{(R_2-R)\ \sigma_1 R_1 + (w+R_2)\sigma_1 R_1\right.\right.$$

$$\left. + \sigma_2 (w+R_1)(R-R_1)\right\} + \frac{\sigma_1 R_1}{w}\left\{\sigma_1 (w+R_2)(R_2-R) + (R-R_1)\ \sigma_2 R_2\right.$$

$$\left.\left. - (w+R_1)\ (\sigma_2 R_2)\right\} + (w+R_2)(R_2-R)\ R_1 \frac{d\sigma_1}{dw} + (w+R_1)(R-R_1)\ \frac{d\sigma_2}{dw}\right]$$

Therefore

$$\frac{dN}{dw} = -\ \frac{\sigma_1 R_1 f^1_{11}}{w}\ \left[(w+R_2)(R_2-R)\ \sigma_1 R_1 + (w+R_1)(R-R_1)\ \sigma_2 R_2\right]$$

$$-\ f^1_1\ \left[(R_2-R)\ \sigma_1 R_1 + (R-R_1)\ \sigma_2 R_2 + (\frac{\sigma_1 + \sigma_2}{w})\right.$$

(continued)

$$\left\{(w+R_2)(R_2-R)\ \sigma_1R_1 + (w+R_1)(R-R_1)\ \sigma_2R_2\right\}$$

$$+ \frac{\sigma_1\sigma_2}{w}\ (R_2-R_1)(R_1R_2-wR) + R_1(w+R_2)(R_2-R)\ \frac{d\sigma_1}{dw}$$

$$\left. + R_2(w+R_1)(R-R_1)\ \frac{d\sigma_2}{dw}\right]$$

Therefore

$$\frac{dN}{dw} = -\left[(w+R_2)(R_2-R)\ \sigma_1R_1 + (w+R_1)(R-R_1)\ \sigma_2R_2\right] \frac{f_1^1}{w} \left[\frac{\sigma_1R_1f_{11}^1}{f_1^1} + (\sigma_1 + \sigma_2)\right]$$

$$- f_1^1 \left[(R_2-R)\ \sigma_1R_1 + (R-R_1)\ \sigma_2R_2 + \frac{\sigma_1\sigma_2}{w}\ (R_2-R_1)(R_1R_2-wR)\right.$$

$$\left. + R_1(w+R_2)(R_2-R)\ \frac{d\sigma_1}{dw} + R_2(w+R_1)(R-R_1)\ \frac{d\sigma_2}{dw}\right]$$

Next,

$$\frac{dD}{dw} = f_{11}^2\ \frac{dR_2}{dw}\ \left[(\gamma w+R_2)(R_2-R)\ \sigma_1R_1 + (\gamma w+R_1)(R-R_1)\ \sigma_2R_2\right]$$

$$+ f_1^2\ \left[(\gamma + \frac{dR_2}{dw})(R_2-R)\ \sigma_1R_1 + (\gamma w+R_2)\ \frac{dR_2}{dw}\ \sigma_1R_1\right.$$

$$+ (\gamma w+R_2)(R_2-R)\ \left\{\sigma_1\frac{dR_1}{dw} + R_1\ \frac{d\sigma_1}{dw}\right\}$$

$$+ (\gamma + \frac{dR_1}{dw})(R-R_1)\ \sigma_2R_2 + (\gamma w+R_1)(\frac{-dR_1}{dw})\ \sigma_2R_2$$

$$\left. + (\gamma w+R_1)(R-R_1)\ \left\{\sigma_2\ \frac{dR_2}{dw} + R_2\ \frac{d\sigma_2}{dw}\right\}\right]$$

Therefore

$$\frac{dD}{dw} = \frac{\sigma_2R_2f_{11}^1}{w}\ \left[(\gamma w+R_2)(R_2-R)\ \sigma_1R_1 + (\gamma w+R_1)(R-R_1)\ \sigma_2R_2\right]$$

$$+ f_1^2\ \left[\gamma\ \left\{(R_2-R)\ \sigma_1R_1 + (R-R_1)\ \sigma_2R_2\right\}\right.$$

(continued)

$$+ \frac{\sigma_2 R_2}{w} \left\{(R_2-R)\ \sigma_1 R_1 + (\gamma w+R_2)\ \sigma_1 R_1 + (\gamma w+R_1)(R-R_1)\ \sigma_2\right\}$$

$$+ \frac{\sigma_1 R_1}{w} \left\{(\gamma w+R_2)(R_2-R)\ \sigma_1 + (R-R_1)\ \sigma_2 R_2 - (\gamma w+R_1)\ \sigma_2 R_2\right\}$$

$$\left. + R_1(\gamma w+R_2)(R_2-R)\ \frac{d\sigma_1}{dw} + R_2(\gamma w+R_1)(R-R_1)\frac{d\sigma_2}{dw}\ \right]$$

Therefore

$$\frac{dD}{dw} = \frac{f_{11}^2 \sigma_2 R_2}{w} \left[(\gamma w+R_2)(R_2-R)\ \sigma_1 R_1 + (\gamma w+R_1)(R-R_1)\ \sigma_2 R_2\right]$$

$$+ f_1^2 \left[\gamma \left\{(R_2-R)\ \sigma_1 R_1 + (R-R_1)\ \sigma_2 R_2\right\}\right.$$

$$+ \left(\frac{\sigma_1+\sigma_2}{w}\right) \left\{(\gamma w+R_2)(R_2-R)\ \sigma_1 R_1 + (\gamma w+R_1)(R-R_1)\ \sigma_2 R_2\right\}$$

$$+ \frac{\sigma_1 \sigma_2}{w}\ (R_2-R_1)(R_1 R_2-\gamma wR) + R_1(\gamma w+R_2)(R_2-R)\ \frac{d\sigma_1}{dw}$$

$$\left. + R_2(\gamma w+R_1)(R-R_1)\ \frac{d\sigma_2}{dw}\right]$$

Therefore

$$\frac{dD}{dw} = \left[(\gamma w+R_2)(R_2-R)\ \sigma_1 R_1 + (\gamma w+R_1)(R-R_1)\ \sigma_2 R_2\right] \left(\frac{f_1^2}{w}\right) \left[\frac{f_{11}^2 \sigma_2 R_2}{f_1^2} + \sigma_1+\sigma_2\right]$$

$$+ f_1^2 \left[\gamma \left\{(R_2-R)\ \sigma_1 R_1 + (R-R_1)\ \sigma_2 R_2\right\} + \frac{\sigma_1 \sigma_2 (R_2-R_1)}{w}\ \ (R_1 R_2-wR)\right.$$

$$\left. + R_1(\gamma w+R_2)(R_2-R)\ \frac{d\sigma_1}{dw} + R_2(\gamma w+R_1)(R-R_1)\ \frac{d\sigma_2}{dw}\ \right]$$

Hence:

$$D\ \frac{dN}{dw} - N\ \frac{dD}{dw} = \frac{DN}{w} \left[\frac{\sigma_1 R_1 f_{11}^1}{f_1^1} - \frac{\sigma_2 R_2 f_{11}^2}{f_1^2}\right]$$

(continued)

$$- \left\{ Df_1^1 + \gamma N f_1^2 \right\} \left\{ (R_2-R)\,\sigma_1 R_1 + (R-R_1)\,\sigma_2 R_2 - R\sigma_1\sigma_2(R_2-R_1) \right\}$$

$$- \frac{\sigma_1\sigma_2 R_1 R_2 (R_2-R_1)}{w} (Df_1^1 + Nf_1^2)$$

$$- R_1(R_2-R)\,\frac{d\sigma_1}{dw} \left\{ (w+R_2)Df_1^1 + (\gamma w+R_2)Nf_1^2 \right\}$$

$$- R_2(R-R_1)\,\frac{d\sigma_2}{dw} \left\{ (w+R_1)Df_1^1 + (\gamma w+R_1)Nf_1^2 \right\}$$

$$= \frac{DN}{w} \left[\frac{-w}{w+R_1} + \frac{\gamma w}{\gamma w+R_2} \right]$$

$$+ (\gamma-1)R_1R_2f_1^1f_1^2 \left\{ \sigma_1(R_2-R) + \sigma_2(R-R_1) \right\}$$

$$\left\{ (R_2-R)\,\sigma_1R_1 + (R-R_1)\,\sigma_2R_2 - R\sigma_1\sigma_2(R_2-R_1) \right\}$$

$$- (\gamma-1)f_1^1f_1^2 \left\{ (R_2-R)\,\sigma_1R_1 + (R-R_1)\,\sigma_2R_2 \right\} \sigma_1\sigma_2R_1R_2(R_2-R_1)$$

$$- (\gamma-1)(R_2-R_1)w(R-R_1)(R_2-R)R_1R_2f_1^2f_1^1 \left\{ \sigma_2\frac{d\sigma_1}{dw} - \sigma_1\frac{d\sigma_2}{dw} \right\}$$

Hence $\left[\text{using } \frac{dQ_2}{dw} = \frac{-D}{w(R_2-R_1)^2} \right]$ we get:

$$\frac{d^2Q_1}{dQ_2^{\,2}} = \frac{-w(R_2-R_1)^2}{D^2}$$

$$\left[\frac{N(\gamma R_1-R_2)}{(w+R_1)(\gamma w+R_2)} + \frac{(\gamma-1)R_1R_2f_1^1f_1^2}{D} \left\{ \left\{ (R_2-R)\,\sigma_1R_1 + (R-R_1)\,\sigma_2R_2 \right. \right. \right.$$

$$\left. - R\,\sigma_1\sigma_2(R_2-R_1) \right\} \left\{ \sigma_1(R_2-R) + \sigma_2(R-R_1) \right\}$$

(continued)

$$- \sigma_1\sigma_2(R_2-R_1) \left\{(R_2-R)\ \sigma_1 R_1 + (R-R_1)\ \sigma_2 R_2\right\}$$

$$\left. - w\ (R_2-R_1)(R_2-R)(R-R_1) \left\{\sigma_2 \frac{d\sigma_1}{dw} - \sigma_1 \frac{d\sigma_2}{dw}\right\}\right\}\Bigg] \qquad (15)$$

It should be obvious from (15) that it is difficult in general to determine the sign of $\frac{d^2Q_1}{dQ_2^{\ 2}}$. One has therefore to consider special cases.

(1) If there is no differential (i.e. $\gamma = 1$), then $\frac{d^2Q_1}{dQ_2^{\ 2}} < 0$ since the lengthy right-hand-side term in the bracket in (15) cancels out and $(R_1-R_2)N$ is always positive. Thus we get the standard result that the production possibility curve is concave to the origin.

(2) Where, however, $\gamma \neq 1$, we can show that the production possibility curve may have <u>both</u> convex and concave stretches. We can do this by evaluating $\frac{d^2Q_1}{dQ_2^{\ 2}}$ at two extreme points: complete specialisation in Q_1 (so that $R_1 = R$) and in Q_2 (so that $R_2 = R$), and showing that the production possibility curve is concave at one end and convex at the other. Assuming that $\frac{d\sigma_i}{dw}$ is well-behaved for $i = 1, 2$, we can see from (15) that, quite generally:

$$\frac{d^2Q_1}{dQ_2^{\ 2}}\ (\text{given } R_1 = R) = \frac{-w(R_2-R)^2}{D^2}$$

$$\left[\frac{N(\gamma R-R_2)}{(w+R)(\gamma w+R_2)} + \frac{(\gamma-1)R_2 f_1^1 f_1^2 (R_2-R)^2\ \sigma_1^2 R^2 (1-2\sigma_2)}{D}\right]$$

$$\frac{d^2Q_1}{dQ_2^{\ 2}}\ (\text{given } R_2 = R) = \frac{-w(R-R_1)^2}{D^2}$$

$$\left[\frac{N(\gamma R_1-R)}{(w+R_1)(\gamma w+R_1)} + \frac{(\gamma-1)R_1 f_1^1 f_1^2 (R-R_1)^2\ \sigma_2^2 R^2 (1-2\ \sigma_1)}{D}\right]$$

Suppose now that $R_1(w) > R_2(w)$ for all w in $[\underline{w}\ ,\ \overline{w}]$. Then $R_1(\underline{w})=R=R_2(\overline{w})$ and $R_1(w) > R > R_2(w)$ for all w in $(\underline{w}\ ,\ \overline{w})$. In this case, it follows that $N > 0$ and $D < 0$. Assume further the specific values: $\gamma > 1$ and $\sigma_1(\overline{w}) \geq \frac{1}{2}$. Then clearly $\frac{d^2Q_1}{dQ_2^{\ 2}} < 0$ when $R_2 = R$ (i.e. $w = \overline{w}$). If it so happens that when $R_1 = R$ (i.e. $w = \underline{w}$), $\gamma R < R_2(\overline{w})$ and $\sigma_2(\underline{w}) \leq \frac{1}{2}$, then $\frac{d^2Q_1}{dQ_2^{\ 2}} > 0$. Thus the production possibility curve will be convex in the neighbourhood of one specialisation point and concave in the neighbourhood of the other.

(3) We may finally consider the case where $\sigma_1 = \sigma_2$ equals a constant, showing that this can lead to a production possibility curve which is smoothly convex (to the origin) throughout.[5] In this case of CES production functions, with identical elasticities for both industries, (15) reduces to:

$$\frac{d^2Q_1}{dQ_2^{\ 2}} = \frac{-w(R_2-R_1)^2}{D^2}\left[\frac{N(\gamma R_1-R_2)}{(w+R_1)(\gamma w+R_2)} + \frac{\sigma^2(\gamma-1)R_1R_2R(R_2-R_1)^2 f_1^1 f_1^2(1-2\sigma)}{D}\right]$$

If we write $f^i = [\alpha_i R_i^{\ -\varepsilon} + (1-\alpha_i)]^{-1/\varepsilon}$ (where $\sigma = \frac{1}{\varepsilon+1}$) then we get $R_1(w) = \left(\frac{\alpha_1}{1-\alpha_1}\right) w^\sigma$ and $R_2(w) = \left(\frac{\gamma\alpha_2}{1-\alpha_2}\right)^\sigma w^\sigma$. This means that $R_2 = \eta R_1$ where $\eta = (\beta\gamma)^\sigma$, and $\beta = \frac{\alpha_2}{1-\alpha_2}\Big/\frac{\alpha_1}{1-\alpha_1}$. Since $\sigma > 0$, $\eta \gtrless 1$ according as $\beta\gamma \gtrless 1$. It can be shown that $N = -f_1^1(\eta-1)\ \sigma R_1(wR+\eta R_1^{\ 2})$ and $D = f_1^2(\eta-1)\ \sigma R_1(\gamma wR+\eta R_1^{\ 2})$. Substituting these into the expression for $\frac{d^2Q_1}{dQ_2^{\ 2}}$ we get:

$$\frac{d^2Q_1}{dQ_2^{\ 2}} = \frac{-wR_1^{\ 4}(\eta-1)^3\ f_1^1}{D^2}\left[\frac{(\eta-\gamma)(wR+\eta\ R_1^{\ 2})}{(w+R_1)(\gamma w+\eta\ R_1)} + \frac{(\gamma-1)(1-2\sigma)\eta RR_1}{(\gamma wR+\eta\ R_1^{\ 2})}\right] \qquad (16)$$

---

[5] Kemp and Herberg have independently arrived at this conclusion for CES production functions.

It is clear from (16) that if $\alpha_1 = \alpha_2$ and $\frac{1}{2} \leq \sigma \leq 1$ (i.e. when $\beta = 1$), $\frac{d^2Q_1}{dQ_2^2} > 0$ for all $\gamma \neq 1$. The reason is that, in this case, either $1 > \eta > \gamma$ or $\gamma > \eta > 1$. Thus the production possibility curve is convex throughout.

IV: A Numerical Example

The following numerical example demonstrates the possibilities of (a) multiple equilibria corresponding to a given commodity price-ratio, (b) perverse comparative-static response to changes in this price ratio and (c) 'normal' response being associated with 'perverse' curvature of the production possibility curve.

Let $f^1 = \left[\frac{1}{2} R_1^{-1} + \frac{1}{2}\right]^{-1}$ and $f^2 = \left[\frac{1}{9} R_2^{1/2} + \frac{8}{9}\right]^2$. Let $\gamma = 8$ and $R = 4$. It is easy to deduce,using equations (6) and (7), that $R_1(w) = w^{1/2}$, $R_2(w) = w^2$, $\underline{w} = 2$ and $\overline{w} = 16$. Hence $\left[\gamma R_1(w) - R_2(w)\right] = 8w^{1/2} - w^2$ and thus is positive in $\underline{w} = 2 \leq w < 4$, zero when $w = 4$ and negative in $4 < w \leq 16 = \overline{w}$. Of course $R_2(w) > R_1(w)$ for all w in (2, 16). The resulting production possibility curve is presented in Figure 3. This curve is convex throughout, as can be verified also by algebraic analysis.

In Figure 4, the function $p(w)$ is plotted. As was proved earlier (recall equation (11)) and as is evident from the figure, $p(w)$ increases with w when $\gamma R_1(w) > R_2(w)$, i.e. when w is in the interval (2, 4) and decreases as w increases when $\gamma R_1(w) < R_2(w)$, i.e. when w is in the interval (4, 16). Thus $p(w)$ starts from a value of about 5.56 when $w = 2$, reaches a maximum of 6 when $w = 4$, and declines steadily to a value of 4.32 when $w = 16$. Thus it follows that if the commodity price-ratio happens to be anywhere in the range $5.56 \leq p < 6$, there are <u>two</u> equilibrium values of w corresponding to <u>each</u>

such p, one in the interval (2, 4) and the other in the interval (4, 7.81). If either $4.32 \leq p < 5.56$ or $p = 6$, there is one and only one equilibrium value of w.

Next, note that with $R_2(w) > R_1(w)$ for all w in (2, 16), $\frac{dQ_2}{dw} < 0$. Suppose we now increase p from $p = 4.32$ upto $p = 5.56$. Then the equilibrium value of w decreases steadily from $w = 16$ to $w = 7.81$ (approximately) and Q_2 increases steadily from $Q_2 = 0$ to $Q_2 = 0.66$ (approximately). Thus the response is 'normal,' i.e. the equilibrium output of the second commodity is larger when its relative price is higher, even though the production possibility curve is convex to the origin.

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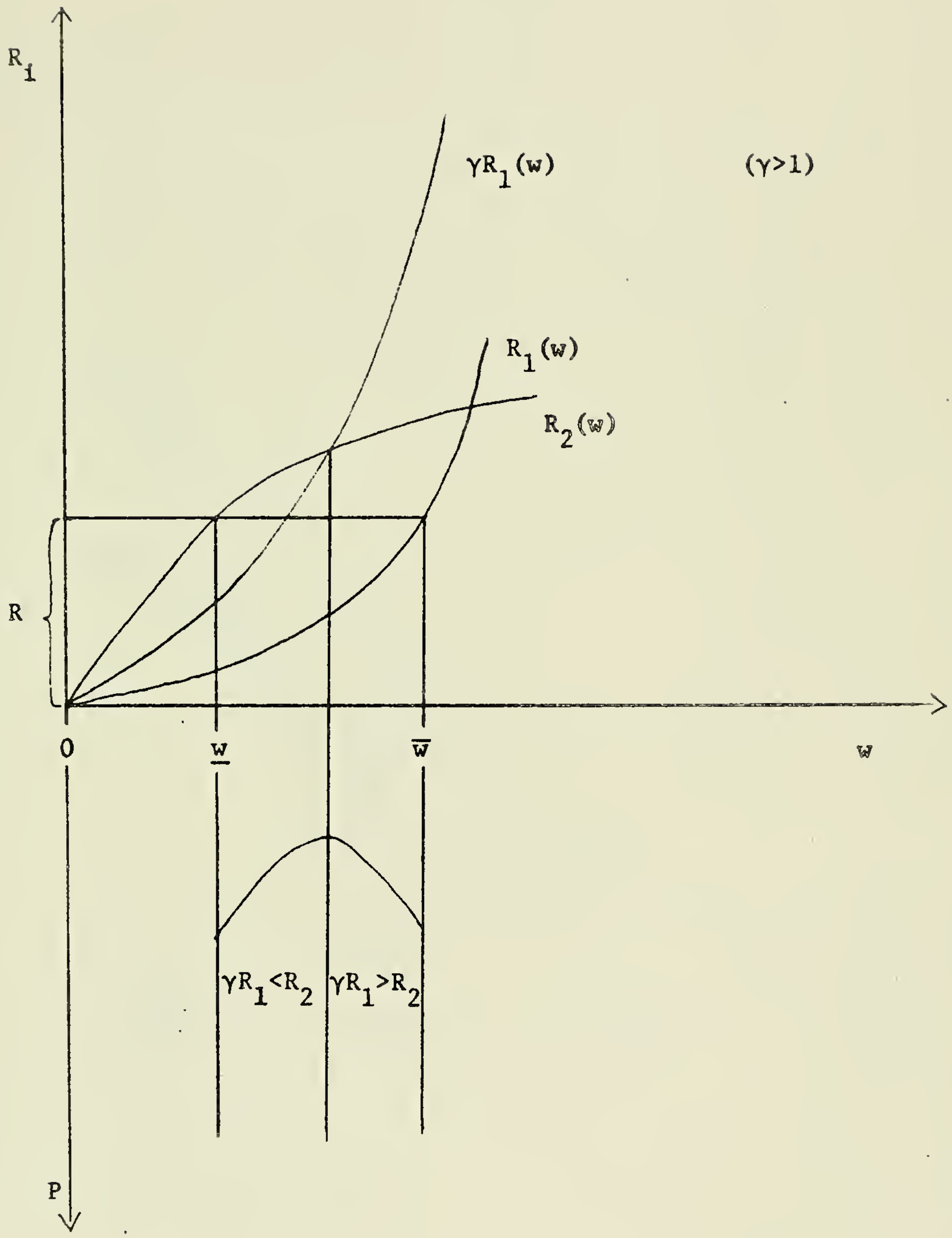
R_i
$\gamma R_1(w)$
$(\gamma>1)$
$R_1(w)$
$R_2(w)$
R
0
$\underline{w}$
$\overline{w}$
w
$\gamma R_1<R_2$
$\gamma R_1>R_2$
P

FIGURE 1


Factor 1
A
D
M^2
M^1
N^2
N^1
Comm. $\overline{1}$
Comm. $\overline{2}$
0
B
E C
F
Factor 2

FIGURE 2

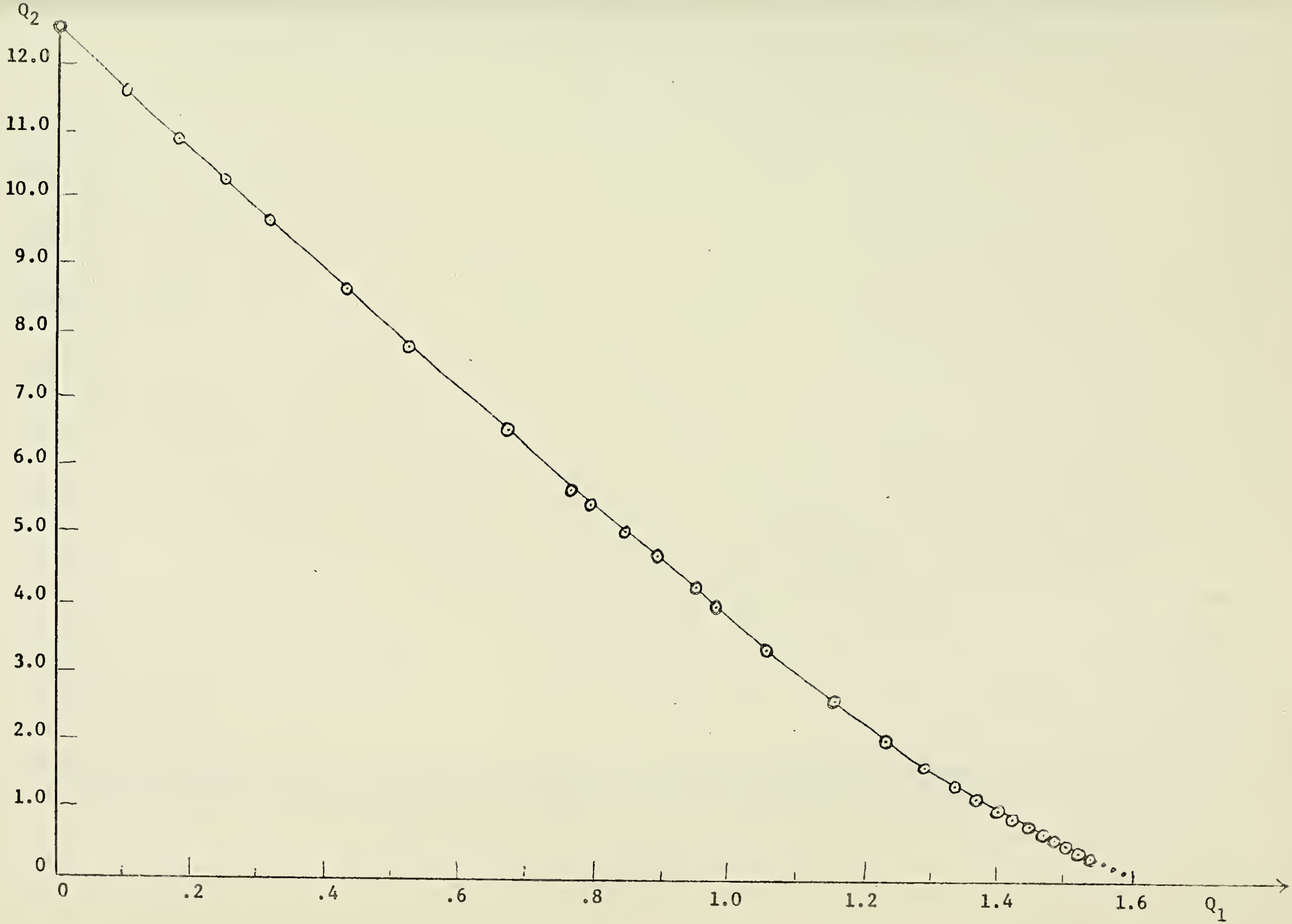

FIGURE 3

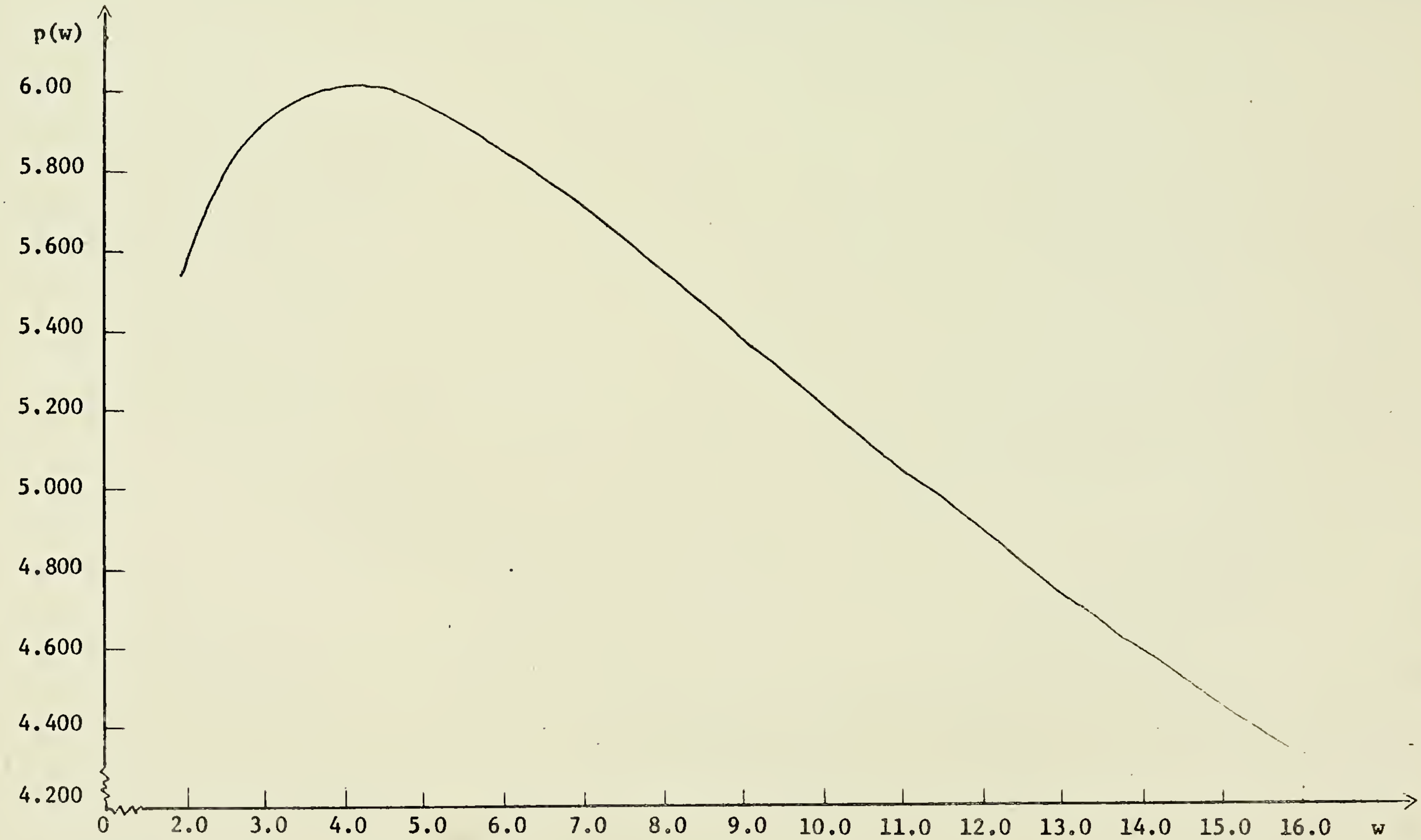
p(w)
6.00
5.800
5.600
5.400
5.200
5.000
4.800
4.600
4.400
4.200
0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
11.0
12.0
13.0
14.0
15.0
16.0
w

FIGURE 4

Date Due

DEC 29 '76	JAN 04 '82	
MAY 10 '76	APR 17 '82	
SEP 24 '76	JUL 25 '83	
~~NOV 10 '76~~	DEC 5 1991	
~~MAR 31 '77~~		
Apr 21 '77		
JUN 1 '77	JAN 21 1992	
DEC 27 '77	MAR. 31 1993	
~~MAY 8 '78~~		
JUN 9 '78		
MAY 29 '79		
~~MAY 31 '79~~		
MAY 8 '79		