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UNCERTAINTY, RESOURCE ALLOCATION AND FACTOR SHARES
IN A TWO-SECTOR MODEL

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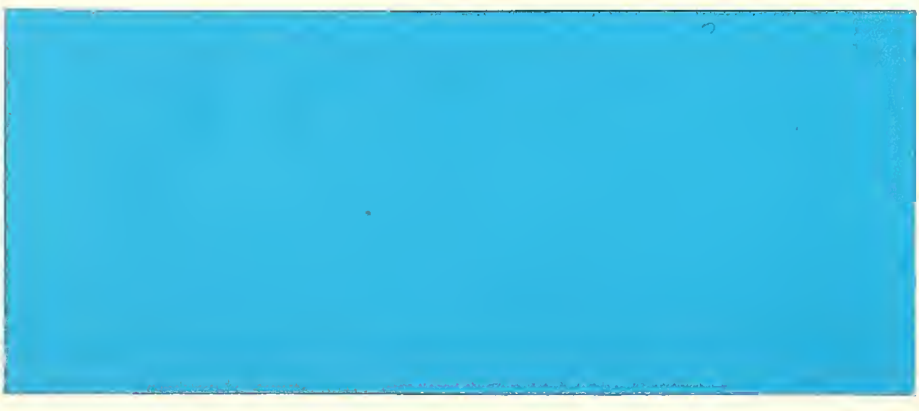

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I. Introduction

The present paper is an exercise in a comparative-static analysis of the impact of uncertainty on choice of techniques and factor shares in a miniature general-equlibrium model. We have two factors of production, labour and land (or capital, which for the present purpose may be assumed to be the same factor as land), and two sectors, one producing land-intensive agricultural goods under some kind of production uncertainty (say, due to vagaries of weather) and the other producing labour-intensive manufactures, afflicted by no such uncertainty. In this model we shall show that under the usual Arrow-type assumptions of risk-aversion,[1] an increase in uncertainty (conversely, output stabilization, say, through irrigation and drainage) reduces (increases) the optimum labour-intensity in techniques of production in both sectors and also raises (lowers) the wage-share in the economy. This is important because quite often we discuss the over-all benefits of a stabilization policy without looking into its impact on income distribution; yet stabilization has significantly differential impact on different factor incomes and it is highly important that the policy-maker be aware of it.[2]

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[1] See Arrow [1].

[2] For a similar analysis of the impact of stabilization of the price of a primary export on factor shares, see Bardhan [2], Section IV.

While in the literature most of the general-equilibrium studies with uncertainty concentrate on questions of welfare economics, the only other paper which asks similar comparative-static questions is that of Rothenberg and Smith [3]. The major differences between our analysis and that of Rothenberg and Smith [3] are as follows:

(a) they consider only the risk-neutral case with expected profit maximization whereas we assume risk-aversion in our maximization of expected utility;

(b) even with their Cobb-Douglas production functions--not assumed here--their results about income distribution are less clear-cut[3] than ours;

(c) in their model with production uncertainty, they assume that decisions regarding use of labour are not made under uncertainty, whereas in our paper decisions about the use of both factors of production are under uncertainty.

II. The Basic Model

Suppose the production function for the agricultural good c (corn) is given by

$$Q_c = AF_c(K_c, L_c) \tag{1}$$

where Q_c is output of c, K_c and L_c are the amounts of land (or capital) and labour respectively used in producing c and A is the random variable reflecting the influence of weather. For characterizing the degree of

[3]For example, their assumption about the relative size of labour force in the two sectors is not needed in our paper.

uncertainty in weather we use the following specific assumptions about the random term A. Taking the approach of Sandmo [4] in a slightly different context, we examine two kinds of shift in the probability distribution of A. One is an additive shift which is equivalent to an increase in the mean with all other moments constant. The other is a multiplicative shift by which the distribution is stretched around zero. A pure increase in dispersion can be defined as a stretching of the distribution around a constant mean. This is equivalent to a combination of additive and multiplicative parameter changes.

Let us write A as

$$A = \lambda u + \rho , \tag{2}$$

where u is the random variable representing some composite index of weather (rainfall, temperature, etc.), λ is the multiplicative shift parameter and ρ the additive one.

With E as the expectation operator,

$$E A = \lambda E u + \rho \tag{3}$$

A multiplicative shift around zero will increase the mean; if the expected value is to be held constant, this should be matched by an additive shift in the negative direction. Taking the differential, this means that

$$d E A = E[u d \lambda + d \rho] = 0 , \tag{4}$$

which implies that

$$\frac{d\rho}{d\lambda} = -E u \tag{5}$$

Thus,

$$\frac{dA}{d\lambda} = u + \frac{d\rho}{d\lambda} = u - E\,u = \frac{1}{\lambda}(A - E\,A) \tag{6}$$

The production function for manufactured goods, m, is given by

$$Q_m = F_m(K_m, L_m), \tag{7}$$

where Q_m is output and K_m and L_m are land and labour respectively used in producing m.

Assuming that both the production functions (1) and (7) are characterized by constant returns to scale in land and labour, one may represent F_i as equal to $L_i\ f_i(k_i)$, $i = c, m$, where k_i is the land-labour ratio used in the i-th sector.

Assuming full-employment of both factors of production, the land-labour ratio of the economy is given by

$$k = k_c \ell_c + k_m(1 - \ell_c) \tag{8}$$

where ℓ_c is the proportion of labour force used in producing c. In our static model both the supplies of land and labour are given; without loss of generality we shall assume the size of the total labour force as unity. The per capita income in the economy is given by

$$y = P\,A\,f_c(k_c)\ell_c + (1 - \ell_c)f_m(k_m), \tag{9}$$

where m is regarded as the numeraire good and P is the given price per unit of c.

Let us suppose that in this economy we maximize $E\,U(y)$, expected utility of per capita income. The utility function is assumed to be strictly concave with a positive marginal utility of income for all y.

It is also assumed that production decision in agricultural sector c involving the use of land and labour is committed before the producer has any knowledge of weather conditions. While this may be realistic for decisions regarding use of labour in soil preparation, sowing, etc., this assumption ignores the fact that the decision on labour use in harvesting is usually taken after the weather uncertainty is over.

Maximizing $EU(y)$ with respect to k_c and ℓ_c and using (8) and (9), we get two necessary conditions:

$$EU'\ell_c[PAf_c'(k_c) - f_m'(k_m)] = 0 \tag{10}$$

$$EU'[PAf_c(k_c) - \{f_m(k_m) + (k_c - k_m)f_m'(k_m)\}] = 0 \tag{11}$$

Since the $f_i(k_i)$ functions do not involve the random term A, it is easy to derive from (10) and (11) that

$$\frac{f_c(k_c)}{f_c'(k_c)} - k_c = \frac{f_m(k_m)}{f_m'(k_m)} - k_m \tag{12}$$

(12) represents the equality of the ratio of marginal products of labour and land in the two sectors. We shall call this ratio w, the wage-rentals ratio. Hence (12) implies the unique relationship between w and land-labour ratio, k_i. In particular,

$$\frac{dk_i}{dw} = \frac{-(f_i)^2}{f_i'' \cdot f_i} > 0 , \qquad i = c,m, \tag{13}$$

where $f_i'' < 0$ with diminishing marginal productivity of land.

III. Comparison with the Risk-neutral Case

Taking eq. (10) we may now contrast the implications of risk-averse behaviour with those of risk-neutral behaviour. Define

$A^* = \frac{f_m'}{Pf_c'}$,

so that from (10),

$$P = \frac{f_m'}{f_c' A^*} = \frac{f_m' E U'}{f_c' E U' A} \tag{14}$$

From (2),

$$A \gtreqless A^* \quad \text{according as } u \gtreqless u^*, \quad \text{where } u^* = \frac{A^* - \rho}{\lambda}$$

Since y is an increasing function of A and hence of u, we may now say that

$$y(u) \gtreqless y(u^*) \quad \text{according as} \quad A \gtreqless A^*$$

Since $U'(y)$ is a decreasing function of y, this implies that

$$U'(y(u))[A - A^*] < U'(y(u^*))[A - A^*] \tag{15}$$

or,

$$E\,U'(y(u))[A - A^*] < U'(y(u^*))E[A - A^*] \tag{16}$$

But from (10), the L.H.S. of (16) is equal to zero. Hence $E\,A > A^*$. Now in the case of risk-neutral behaviour, in (14) $P = \frac{f_m'}{f_c' E\,A}$, since $U'(y)$ is in this case a positive constant. As $E\,A > A^*$, from (14) we can say that the value of $\left(\frac{f_m'}{f_c'}\right)$ is <u>larger</u> in the risk-neutral case. Now $\left(\frac{f_m'}{f_c'}\right)$ can be shown--with the use of (13)--to be a <u>decreasing</u> function of w if the agricultural sector c is always more land-intensive, i.e. $k_c > k_m$. This means that w, the wage-rentals ratio (and hence the land-intensity in the production technique) is <u>higher</u> for the case of risk-averse behaviour compared to that for risk-neutral behaviour.

IV. Effects of Uncertainty

Since k_i is a unique function of w, we can use eqs. (6) and (10) to work out the value of

$$\frac{dw}{d\lambda} = \frac{N}{D} \tag{17}$$

where

$$N = -E[U'' P f_c \ell_c (P A f_c' - f_m') + U' P f_m'](A - E A)\lambda^{-1}$$

$$D = E\left[U'' \frac{dy}{dw}\left(P A f_c' - f_m'\right) + U' \left(P A f_c'' \frac{dk_c}{dw} - f_m'' \frac{dk_m}{dw}\right)\right]$$

We shall first show under what conditions N is positive. From Lemma 1 in the Appendix it is easy to check that $E U'(A - E A)$ is negative, since U' is a decreasing function of A. Lemmas 2 and 3 in the Appendix prove that

$$E U''(P A f_c' - f_m') \geqq 0 \tag{18}$$

and

$$E U'' A(P A f_c' - f_m') < 0 \tag{19}$$

under the following two conditions on the pattern of risk-aversion:

(a) absolute risk-aversion in the Arrow-Pratt sense is non-increasing with increase in income and (b) relative risk-aversion is non-decreasing with increase in income. Both these conditions are familiar from Arrow's portfolio model [1]. Condition (a) implies that the willingness to engage in small bets of a fixed size does not decrease as income increases; and condition (b) implies that if both the size of the bet and income are increased in the same proportion, the willingness to accept the bet does not increase. So if we now rewrite N as

$$-Pf_c'\lambda^{-1}EU'(A-EA)-\lambda^{-1}Pf_c\ell_cEU''A(PAf_c'-f_m')+EA\lambda^{-1}Pf_c\ell_cEU''(PAf_c'-f_m')$$

we can see from the discussion above that it is positive.

Now what about D? Let us first take the expression $EU'\left(PAf_c''\frac{dk_c}{dw}-f_m''\frac{dk_m}{dw}\right)$. Using (13) and the definition of w, this has the same sign as

$$-EU'w(PAf_c'-f_m')-EU'[k_mPAf_c'-k_cf_m']$$

$$=EU'f_m'(k_c-k_m)\ \gtreqless\ 0 \qquad (20)$$

$$\text{as}\quad k_c \gtreqless k_m$$

In deriving (20) use has been made of (10). Since we have assumed c to be the more land-intensive good, (20) is positive.

The rest of D consists of $EU''(PAf_c'-f_m')\frac{dy}{dw}$.

Let us first evaluate $\frac{dy}{dw}$. Since $y=PQ_c+Q_m$, and since it is easy to show that in this model the slope of the production-possibility curve is given by $\frac{dQ_m}{dQ_c}=-\frac{f_m'}{Af_c'}$, we can write

$$\frac{dy}{dw}=\frac{dQ_c}{dw}\frac{(PAf_c'-f_m')}{Af_c'} \qquad (21)$$

Now from (1), (8) and (12) and defining $\sigma_i=\frac{dk_i}{dw}\frac{w}{k_i}$, $i=c,m$, as the elasticity of substitution between land and labour, one can work out after some manipulation and simplification that

$$\frac{dQ_c}{dw}=\frac{1}{w(k_c-k_m)^2}\left[Af_ck_m(k-k_c)\sigma_m+(k_m-k)k_cA\sigma_c\{f_c-f_c'(k_c-k_m)\}\right] \qquad (22)$$

(22) is positive or negative as $k_c \lesseqgtr k_m$. Since we are assuming c to be the more land-intensive good, (22) may be taken as negative.

We may now write, using (21) and (22),

$$EU''(PAf_c' - f_m') \frac{dy}{dw} = EU''(PAf_c' - f_m')^2 \frac{dQ_c}{dw} \frac{1}{Af_c'} \tag{23}$$

Since $U'' < 0$ and $\frac{dQ_c}{dw} < 0$ for $k_c > k_m$, (23) may be taken as positive. So collecting all terms, D in eq. (17) is positive for $k_c > k_m$.

Thus we have proved that under our conditions (a) and (b) on the pattern of risk-aversion $\frac{dw}{d\lambda}$ is positive if c is the more land-intensive good. It is easy to see from our derivation above that D is negative and hence $\frac{dw}{d\lambda}$ is negative if c is the more labour-intensive good. All this means that under Arrow-type assumptions about the pattern of risk-aversion increased production uncertainty raises the relative price of the factor that is used less intensively in the sector afflicted by such uncertainty. In the present case increased uncertainty in producing land-intensive agricultural goods <u>raises</u> the wage-rentals ratio, and since factor endowments are given, <u>raises</u> the wage-share in the economy. Conversely, stabilization of agricultural output (through irrigation, drainage, etc.) will <u>lower</u> the wage-share.

Since from (13) the wage-rentals ratio and the land-intensity of production in both sectors are negatively related, we may say that increased uncertainty (conversely, stabilization) in producing land- intensive agricultural goods will lead to adoption of less (more) labour-intensive techniques of production in <u>both</u> sectors. As for resource allocation, we may see from (22) that a rise in the wage-rentals ratio with increased uncertainty in production of land-intensive agriculture causes resources to shift away from it so that agricultural output declines.

So far we have been concerned with effects of increase or decrease in uncertainty. But before ending we may note a corollary of eq. (10) that suggests itself. Given the random parameter A, eq. (10) is an implicit function in P, w and k. This indicates that even at constant commodity prices the relative factor prices will be affected by factor endowment. In fact, by differentiating in (10), with P constant and A given,

$$\frac{dw}{dk} = \frac{M}{D} \tag{24}$$

where the denominator D, as in eq. (17), is positive or negative as k_c is larger or smaller than k_m, and

$$M = EU'' (PAf'_c - f'_m)(f_m - PAf_c)(k_c-k_m)^{-1}$$

$$= E (k_c-k_m)^{-1} [U''f_m (PAf'_c - f'_m) - Pf_c U''A(PAf'_c - f'_m)]$$

From Lemmas 2 and 3 in the Appendix it is clear that M is positive or negative as k_c is larger or smaller than k_m.

This means that (24) is positive. In other words, given the random parameter A, the wage-rentals ratio is an increasing function of the land-labour endowment ratio even at constant commodity prices under Arrow -type postulates about risk-aversion. Thus unlike in the case of certainty or in that of risk-neutrality in the face of uncertainty, in the case of risk-aversion under uncertainty there is no unique relationship between commodity and factor prices independent of factor endowments under incomplete specialization. Here, therefore, is another exception to the standard factor-price equalization result.

Appendix

Lemma 1

$$E\varphi(A)(A - EA) \gtreqqless 0 , \quad \text{as} \quad \varphi'(A) \gtreqqless 0$$

Proof: We shall prove it only for the case of $\varphi'(A) > 0$; the other two cases may then be worked out easily.

$$\text{If } \varphi'(A) > 0 , \quad \varphi(A) > \varphi(EA) \quad \text{for} \quad A > EA$$

$$\text{and} \quad \varphi(A) < \varphi(EA) \quad \text{for} \quad A < EA ;$$

$$\text{hence} \quad E\varphi(A)(A - EA) > \varphi(EA)E(A - EA) = 0$$

Lemma 2

$$EU''(PAf_c' - f_m') \geqq 0 ,$$

if the degree of absolute risk-aversion, $\frac{-U''}{U'}$, does not increase as income increases.

Proof: Define $\Psi(A) = \frac{-U''(y(A))}{U'(y(A))}$. Differentiating with respect to A, under non-increasing absolute risk-aversion $\Psi'(A) \leqq 0$.

$$\text{So} \quad \Psi(A) \leqq \Psi(A^*) \quad \text{when} \quad A > A^* ,$$

$$\text{where} \quad A^* = \frac{f_m'}{Pf_c'}$$

$$\text{and } \Psi(A) \geqq \Psi(A^*) \quad \text{when} \quad A < A^* ;$$

$$\text{hence } \Psi(A)(A - A^*) \leqq \Psi(A^*)(A - A^*)$$

$$\text{or,} \quad EU''(PAf_c' - f_m') \geqq -\Psi(A^*)EU'(PAf_c' - f_m') = 0$$

Lemma 3

$$E\,U''A(P\,A f_c' - f_m') < 0$$

if the degree of relative risk-aversion, $\frac{-U''y}{U'}$, does not decrease with increase in income.

Proof: Define $\varphi(A) = \frac{-A\,U''(y(A))}{U'(y(A))}$

Under non-decreasing relative risk-aversion, it is easy to check, using (9), that $\varphi'(A) > 0$.

So $\varphi(A) > \varphi(A^*)$ when $A > A^*$

where, as before, $A^* = \frac{f_m'}{Pf_c'}$

and $\varphi(A) < \varphi(A^*)$ when $A < A^*$;

hence $\varphi(A)(A - A^*) > \varphi(A^*)(A - A^*)$

or, $E\,U''A(P\,A f_c' - f_m') < -\varphi(A^*)\,E\,U'(P\,A f_c' - f_m') = 0$.

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