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Quantum Symmetry Anomalies

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Abstract

The **axial anomaly** is a quantum term that violates the classical conservation of the axial current.

A mathematical model for physical phenomena may possess a symmetry when its dynamics is analyzed in terms of unquantized, commuting variables, but the symmetry may disappear when the dynamics is quantized and analysis is performed in terms of non-commuting quantum variables. Such a tenuous symmetry is said to be “anomalous,” beset by a “quantum symmetry anomaly.” Correspondingly, constants of motion of the unquantized theory are no longer conserved when quantum effects are taken into account¹. In greater detail, the effect arises for the following reason. Quantized dynamics frequently involves an infinite number of degrees of freedom, even when in the classical, unquantized version there is only a finite number. This infinity leads to various divergences, especially in quantum field theory (but also in some quantum mechanical systems²), and these divergences have to be controlled and “renormalized” in order to well-define the quantum theory. The symmetry anomalies arise when the regularization and renormalization procedures, needed to well-define the theory, do not respect the putative symmetries.

The first instances of quantum symmetry anomalies were identified for models which appear to possess symmetries associated with masslessness: scale symmetry and, for Dirac-Fermions, axial symmetry. We shall here discuss the anomalies in axial symmetries of Dirac Fermions, also called the “Adler-Bell-Jackiw anomalies.”¹

A massless, non-interacting Dirac-Fermi field satisfies the equation

$$i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} \psi(x) = 0. \quad (1)$$

Summation over a repeated index is implied. ψ is a 4-component column spinor; x stands for the space-time variables $x^0 = ct$ and $x^i = r^i$ ($i = 1, 2, 3$); the index μ ranges over temporal (0) and spatial (i) components, and γ^μ ($\mu = 0, 1, 2, 3$) comprise a set of 4×4 Dirac matrices, whose explicit form will not concern us, beyond noting that they satisfy the Clifford algebra.

$$\begin{aligned} \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= 2g^{\mu\nu} I \\ g^{\mu\nu} &= \text{diag} (1, -1, -1, -1) \text{ (Lorentz signature)} \end{aligned} \quad (2)$$

For massive fields the equation reads

$$i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} \psi(x) - m c \psi(x) = 0, \quad (3)$$

where m is the mass. (Henceforth we set Planck’s constant \hbar and the velocity of light c to unity.)

The equations (1) and (3) possess a gauge symmetry.

$$\psi(x) \rightarrow e^{i\theta} \psi(x) \quad (4)$$

¹J.S. Bell and R. Jackiw, “A PCAC Puzzle: $\pi^0 \rightarrow$ gamma gamma in the sigma model,” *Nuovo Cim. A* **51**, 47 (1969); S.L. Adler, “Axial vector vertex in spinor electrodynamics,” *Phys. Rev.* **177**, 2426 (1969).

²B. Holstein, “Anomalies for Pedestrians” *Amer. Jnl. Phys.* **61**, 142 (1993).

If $\psi(x)$ is a solution, so is $e^{i\theta} \psi(x)$ where θ is an arbitrary constant. And this symmetry is present whether ψ is a classical field or a quantum field operator. As a consequence of this symmetry, the charge

$$Q \equiv \int d^3 r \psi^\dagger \psi \quad (5)$$

is time independent, or equivalently a charge current 4-vector J^μ

$$J^\mu \equiv \psi^\dagger \gamma^0 \gamma^\mu \psi \quad (6)$$

satisfies a continuity equation

$$\frac{\partial}{\partial x^\mu} J^\mu(x) = 0. \quad (7)$$

[ψ^\dagger is a 4-component, row spinor, with entries that are complex conjugates of ψ (in the unquantized theory) or Hermitian conjugates of ψ (in the quantized theory).]

It is interesting to delve deeper into the matrix structure of these equations. Upon defining the indempotent and Hermitian γ_5 matrix by

$$\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad \gamma_5 \gamma_5 = I, \quad (8)$$

we verify that γ_5 anti-commutes with the Dirac matrices γ^μ . Next we construct chiral projection matrices

$$P_\pm = \frac{1}{2} (I \pm \gamma_5), \quad P_\pm + P_\mp = I, \quad P_\pm P_\pm = P_\pm, \quad P_\pm P_\mp = 0, \quad (9)$$

which select chiral components of ψ .

$$\psi_\pm \equiv P_\pm \psi, \quad \gamma_5 \psi_\pm = \pm \psi_\pm \quad (10)$$

By action of P_\pm on the equations (1) and (3) we obtain decoupled equations for the chiral components ψ_\pm in the massless case

$$i \gamma^\mu \frac{\partial}{\partial x^\mu} \psi_\pm(x) = 0, \quad (11)$$

but a mixing remains on the massive case,

$$i \gamma^\mu \frac{\partial}{\partial x^\mu} \psi_\pm(x) - m \psi_\mp(x) = 0, \quad (12)$$

while the charge (5) and the current (6) become summed expressions of the (+) variables and the (-) variables.

$$Q = \int d^3 r (\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-) = Q_+ + Q_- \quad (13)$$

$$J^\mu = \psi_+^\dagger \gamma^0 \gamma^\mu \psi_+ + \psi_-^\dagger \gamma^0 \gamma^\mu \psi_- = J_+^\mu + J_-^\mu \quad (14)$$

Since in the massless model there is no mixing between (+) and (-) components, it follows that Q_+ and Q_- are separately conserved, and that J_+^μ and J_-^μ separately obey continuity equations. Alternatively and equivalently one can state that in the massless case the axial vector current

$$J_5^\mu = \psi^\dagger \gamma^0 \gamma^\mu \gamma_5 \psi = J_+^\mu - J_-^\mu \quad (15)$$

satisfies a continuity equation,

$$\frac{\partial}{\partial x^\mu} J_5^\mu(x) = 0 \quad (16)$$

and that the axial charge

$$Q_5 \equiv \int d^3r \psi^\dagger \gamma_5 \psi \quad (17)$$

is time independent. The additional constant of motion arises as a consequence of the axial gauge symmetry. The transformation

$$\psi \rightarrow e^{i\gamma_5\theta} \psi = (\cos\theta + i\gamma_5 \sin\theta) \psi, \quad \psi_\pm \rightarrow e^{\pm i\theta} \psi_\pm \quad (18)$$

maps solutions into solutions of the massless equation, and this is true whether ψ is a classical field or a quantum field operator.

To encounter anomalies, we enlarge the massless model by introducing a coupling to a vector gauge field A_μ , treated for the moment as an externally prescribed quantity, without dynamics. Eq. (1) is now replaced by

$$i\gamma^\mu \left(\frac{\partial}{\partial x^\mu} + iA_\mu(x) \right) \psi(x) = 0. \quad (19)$$

A superficial examination of the system leads to the conclusion that the previous symmetries, (4) and (18) continue to hold; indeed (4) can be generalized to a “local” gauge symmetry with $\theta(x)$ acquiring a space-time dependence, provided A_μ is also transformed.

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{\partial}{\partial x^\mu} \theta(x) \quad (20)$$

[When the transformation parameter θ is position independent, as previously in (4) and (18), the symmetry is a “global” gauge symmetry.]

Correspondingly one would conclude that even in the presence of A_μ that chiral charges Q_\pm remain time-independent and the vector (6) and axial vector currents (15) still satisfy continuity equations (7) and (16).

But these conclusions are valid only if the ψ fields are classical functions and not quantum field operators. For the latter, the problem resides in the fact that the fundamental quantization condition for Dirac-Fermi fields

$$\begin{aligned} \psi_m^\dagger(t, \mathbf{r}) \psi_n(t, \mathbf{r}') + \psi_n(t, \mathbf{r}') \psi_m^\dagger(t, \mathbf{r}) \\ = \delta_{mn} \delta^3(\mathbf{r} - \mathbf{r}') \end{aligned} \quad (21)$$

implies that the product of ψ^\dagger and ψ at the same space-time point is necessarily singular. [In the above (m, n) label the components of ψ^\dagger and ψ .] Since the charges and currents involve bilinears of the Dirac-Fermi fields at the same space-time point, they are necessarily ill-defined in the quantum theory. As mentioned previously, a regularization and renormalization is needed to render the currents well-defined. But it turns out that every regularization/renormalization method in the presence of the vector field A_μ violates the symmetries that are present in the unquantized theory. It is possible to preserve (4) or (18) [or a linear combination of the two] but not both.

Since the preservation of both symmetries is impossible, a choice must be made which one should be preserved. The choice is dictated by the physical context of the theory under examination. Since local gauge symmetries, as in (4) and (20), are frequently needed for consistency of the theory (as in the standard model of particle physics) they are the ones that are preserved, while global axial gauge symmetries as in (18), are abandoned — they become beset by anomalies.

Physical Consequences of Axial Symmetry Anomalies

For the example (19) given above, preserving the local gauge symmetry has the consequence that in the regulated/renormalized quantum field theory the charge (5) remains conserved and the vector current (6) continues to satisfy the continuity equation (7). Correspondingly the axial charge (17) acquires a time dependence and the axial vector current (15) obeys an anomalous continuity equation. Its form is

$$\frac{\partial}{\partial x^\mu} J_5^\mu(x) = \frac{N}{8\pi^2} {}^*F^{\mu\nu}(x) F_{\mu\nu}(x), \quad (22)$$

where $F_{\mu\nu}$ is the field strength constructed from A_μ

$$F_{\mu\nu}(x) \equiv \frac{\partial}{\partial x^\mu} A_\nu(x) - \frac{\partial}{\partial x^\nu} A_\mu(x) \quad (23)$$

and ${}^*F^{\mu\nu}$ is its dual.

$${}^*F^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (24)$$

N is a numerical constant which is determined by the number and strength of Dirac-Fermi fields coupling to A_μ . For the single field of our example, $N = 1$. While we have taken A_μ to be externally prescribed, it has been shown that the result (22) holds with dynamical A_μ . The occurrence of the symmetry anomalies leads to a variety of effects in the standard particle physics model.

On the one hand, the standard model appears to possess symmetries that are not present in Nature, not even approximately. These classical, global gauge symmetries if present in the quantized theory, would forbid the decay of a (massless) neutral pion to

two photons. But the physical pion’s mass can be accurately described as (approximately) vanishing, yet the decay width is not negligible.

$$\Gamma(\pi^0 \rightarrow 2\gamma) \approx 8.4 \text{ eV} \tag{25}$$

Also the same symmetries predict the existence of a neutral pseudo scalar meson, approximately degenerate with the pion. But no such particle has been observed. It is fortunate that the anomalies in the quantized standard model remove the offending global gauge symmetries. Indeed because the strength of the axial anomaly is known, one can calculate the width for neutral pion decay (for massless pions). One finds 7.725 eV, or 8.1 eV, when mass corrections are included. Moreover, this excellent agreement with (25) requires that there be three colors of Fermions. Thus the axial anomaly in the global gauge symmetry not only determines neutral pion decay and cancels the prediction of an unwanted partner meson, but also gives indirect determination of the number of color degrees of freedom. Furthermore, the standard model possess an anomaly in the continuity equation for the fermion number current, thereby allowing proton decay. While this startling result establishes that in our present theory stability of matter is not absolute, there is no practical significance because the predicted decay rate is negligible ³.

On the other hand, local gauge symmetries must be preserved for consistency of the standard model. This is achieved by adjusting the Fermion content (quarks and leptons) so that possible anomalies cancel. This requirement is met if quarks are matched with leptons, and thus the heaviest “top” quark was predicted to exist once the “bottom” quark was discovered, in order that in the third family of Fermions quarks matched the tau leptons. A similar anomaly cancellation requirement was found in string theory and led to the revival of that subject.

These physically important effects vividly demonstrate that quantum symmetry anomalies are not obscure pathologies of the quantum mechanical formalism, but describe in a paradoxical-anomalous fashion aspects of natural phenomena.

Mathematical Connections to Axial Symmetry Anomalies

The discovery of the field theoretic structures associated with axial anomalies seeded an intense interaction between physicists and mathematicians, who for their own purposes had been working with related quantities. The connection arises when the previously described formulas are generalized to incorporate a non-Abelian Lie algebra and group; this is the Yang-Mills theory. To this end, we remain with the massless Dirac equation (19), but replace the function iA_μ by a Lie-algebra, matrix valued quantity $A_\mu \equiv \sum_\alpha A_\mu^\alpha T_\alpha$, where T_α are anti-Hermitian representation matrices satisfying the Lie algebra commutators with

³G. ‘t Hooft, “Symmetry breaking through Bell-Jackiw anomalies,” *Phys. Rev. Lett.* **37**, 8 (1976).

structure constraints f_{ab}^c

$$[T_a, T_b] = \sum_c f_{ab}^c T_c, \quad (26)$$

and are normalized by $\text{tr } T_a T_b = -\delta_{ab}/2$. (For $SU(2)$, $T_a = \sigma_a/2i$, $\sigma \equiv$ Pauli matrix.) The Dirac spinors Ψ acquire components, which are acted upon by the representation matrices.

$$i \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + A_\mu(x) \right) \Psi(x) = 0 \quad (27)$$

The singlet axial vector current J_5^μ obeys the anomalous continuity equation

$$\frac{\partial}{\partial x^\mu} J_5^\mu(x) = \frac{1}{8\pi^2} \text{tr } *F^{\mu\nu}(x) F_{\mu\nu}(x), \quad (28)$$

where $F^{\mu\nu}$ is now the non-Abelian field strength (Yang-Mills curvature).

$$F_{\mu\nu}(x) \equiv \frac{\partial}{\partial x^\mu} A_\nu(x) - \frac{\partial}{\partial x^\nu} A_\mu(x) + [A_\mu(x), A_\nu(x)] \quad (29)$$

(Anomalies also beset non-singlet currents $J_{5a}^\mu = \psi^\dagger \gamma^0 \gamma^\mu \gamma_5 T_a \psi$, but these will not be discussed here.) Also, for the mathematical discussion we pass from Lorentzian to Euclidean signature: $g_{\mu\nu} = \text{diag}(1, 1, 1, 1)$.

The mathematical connection is put into evidence by (28) the generalization of (22), where on the right side occurs the Pontryagin density \mathcal{P} ,

$$\mathcal{P} \equiv -\frac{1}{16\pi^2} \text{tr } *F^{\mu\nu} F_{\mu\nu}, \quad (30)$$

whose 4-dimensional integral measures the topological properties of the Yang-Mills gauge potentials A_μ (connections) and fields $F_{\mu\nu}$ (curvatures) that enter in \mathcal{P} . For the integral to converge, $F_{\mu\nu}$ must tend to zero at infinite argument. This means that A_μ must tend to a pure gauge g , which is group valued,

$$A_\mu(x) \rightarrow g^{-1}(x) \frac{\partial}{\partial x^\mu} g(x) \quad (31)$$

and g is restricted to tend to the identity. Gauge functions g with this restriction fall into equivalence (homotopy) classes labeled by integers, and gauge functions in different classes cannot be deformed into each other. That integer n is given by the Pontryagin number

$$n = \int d^4x \mathcal{P} \quad (32)$$

While \mathcal{P} is gauge invariant, it can also be presented as the divergence of a gauge variant 4-vector K^μ , called the topological current or the Chern-Simons current.

$$\mathcal{P}(x) = \frac{\partial}{\partial x^\mu} K^\mu(x) \quad (33)$$

$$K^\mu(x) = -\frac{1}{16\pi^2} \varepsilon^{\mu\alpha\beta\gamma} \text{tr} \left[\frac{1}{2} A_\alpha(x) \frac{\partial}{\partial x^\beta} A_\gamma(x) + \frac{1}{3} A_\alpha(x) A_\beta(x) A_\gamma(x) \right] \quad (34)$$

Consequently, the 4-dimensional volume integral of $*F^{\mu\nu}F_{\mu\nu}$ in (32) can be written as an integral of K^μ over the 3-dimensional surface (at infinity) bounding the 4-dimensional volume. There the vector potentials in K^μ are replaced by their asymptotic form (31), and the resulting integration gives the integer n that characterizes the winding number, the homotopy class, of g .

The Pontryagin quantity is a topological entity for various reasons. We have seen already that it is determined by the asymptotic behavior of gauge functions, which fall into distinct classes labeled by integers. Also the integral (32) does not require specifying the geometry of the integration volume — even with non-trivial geometries no metric tensor is required in (32). Finally one can check that (32) is invariant against local variations of A_μ .

While gauge field configurations with non-vanishing Pontryagin number are easily constructed, especially interesting is a class of connections that satisfy

$$*F^{\mu\nu} = \pm F^{\mu\nu}. \quad (35)$$

These are called instantons, and by virtue of the Bianchi identity,

$$D_\mu *F^{\mu\nu} = 0, \quad (36)$$

they satisfy the Yang-Mills equation of motion.

$$D_\mu F^{\mu\nu} = 0 \quad (37)$$

$$\left[D_\mu \dots \equiv \frac{\partial}{\partial x^\mu} + [A_\mu, \dots] \right]$$

The physical interpretation of instantons is that they provide a semi-classical signal for the occurrence of quantum tunneling; here it is the tunneling between homotopy classes of gauge fields. Indeed the previously mentioned proton instability is understood as arising from such tunneling; that is why its magnitude is exponentially small and therefore negligible. In detail, the homotopy structure in the gauge theory is analogous to the periodicity of a crystal, and the Yang-Mills theory acquires an unexpected θ - parameter, analogous to the Bloch momentum of a Bloch wave. Equivalently, one recognizes that the quantum Yang-Mills action possess the contribution $\theta \int d^4x \mathcal{P}(x)$. Since \mathcal{P} is a total divergence, this does not affect classical equations of motion, but influences the quantum theory. Since \mathcal{P} is odd under CP transformation, the new term is a source of CP violation, which is only a very weak effect in Nature. This leads to an outstanding puzzle about the standard model: what determines the tiny magnitude of θ ?

The Pontryagin index also carries information about the Dirac equation (27) (in Euclidean space). For generic A_μ , solutions of (27) are not normalizable. However, for particular forms of A_μ , normalizable solutions may exist; they possess definite chirality, say there are $n_+(n_-)$ of positive (negative) chirality. The celebrated Atiyah-Singer index theorem gives a formula for the “index” of the Dirac operator, *i.e.* for $n_+ - n_-$.

$$n_+ - n_- = \int d^4x \mathcal{P} = n \quad (38)$$

We thus recognize that the anomaly equation (28) is a local version for the Atiyah-Singer index theorem.

The topological Chern-Simons current (34) also enjoys a physical role. By selecting a single, definite component to be a contribution to a physical Lagrangian in 4-dimensional space-time, one constructs a theory that violates Lorentz invariance. These days there is great interest in the possibility of (feeble) Lorentz invariance violation, and the topological entities arising from axial anomalies provide an attractive realization of the idea, for which thus far there is no experimental evidence.

For another application of the Chern-Simons term with Lorentzian signature, one chooses a single, definite component, say the third, z , component, and suppresses dependence of the vector potentials on that variable, $x^3 = z$ in the example. One then has in hand a quantity defined on (2+1)-dimensional space-time, which can be used as an addition to any (2+1)-dimensional Lagrange density, describing physics on a plane. The new term is interesting in that it is not gauge invariant, but its variation is gauge covariant. So the equations of motion remain gauge covariant, and the Chern-Simons contribution provides a mass term for the gauge field, while retaining gauge invariance. These structures (mainly in their Abelian version) have been used in analyses of the quantum Hall effect.

The discussion has been concerned with gauge fields and Dirac-Fermi fields. Analogous effects are found with gravitational fields, with the gravitational connection (Christoffel or spin) taking the role of the gauge potential and the Riemann tensor replacing the gauge field strength. Again one finds anomalies involving the gravitational Chern-Pontryagin term. There is a gravitational Chern-Simons current, which may be used to build a Lorentz symmetry violating gravity model, or may be a contribution to a (2+1)-dimensional gravity theory, where the gravitons preserve diffeomorphism invariance, but are massive. (2+1)-dimensional gravity has a physical realization in descriptions of planar motion in the presence of cosmic strings.

The unexpected mathematical properties of the axial anomaly exhibit deep mathematical features in our description of Nature, in its fundamental workings. It is remarkable that these features find their realization in anomalies of the quantum mechanical formalism.

Further Reading

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