Optimal Corporate Investment and Financing Policies with Time-Varying Investment Opportunities

by

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Abstract

Bolton, Chen and Wang (2009) propose a model (the BCW model) of dynamic corporate investment, financing, and risk management for a financially constrained firm. In the BCW model, corporate risk management is a combination of internal liquidity management, financial hedging, investment, and payout decisions. However, Bolton et al. (2009) assume that the firm's investment opportunities are constant over time, which is unrealistic in many situations.

I extend the analytical tractable dynamic framework of Bolton et al. (2009) for firms facing stochastic investment opportunities. My extended model can help financially constrained firms to optimally choose external financing (equity or credit line), internal cash accumulation, corporate investment, risk management and payout policies in an environment subjective to time-varying productivity shocks. The differences of policies from the BCW model and my extended model, as well as the optimal and non-optimal policies are also compared.

Thesis Supervisor: John E. Parsons
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Chapter 1

Introduction

1.1 Overview of the Bolton, Chen, and Wang (2009)

My master's thesis is based on Bolton et al. (2009) — "A Unified Theory of Tobin's q, Corporate Investment, Financing, and Risk Management" (the BCW model), which uses mathematical, optimization and computational methods to analyze corporate investment, financing and risk management policies. Bolton et al. (2009) propose the first elements of a tractable dynamic economic framework — as illustrated in Figure 1-1 — in which corporate investment, asset sales, cash inventory, equity financing, credit line and dynamic hedging policies are characterized analytically for a "financially constrained" firm (that is, a firm facing external financing costs). Their model can help firms facing external financing cost to quantitatively and optimally determine their day-to-day risk management and investment policies, such as when/how the firms should reduce their cash holdings, or when/how they should replenish their dwindling cash inventory, which risks they should hedge and by how much, or to what extent holding cash inventory is a substitute for financial hedging through derivatives.

According to the abstract in Bolton et al. (2009), the BCW model determines the firm's optimal investment, financing, and risk management policies as functions of the following key parameters: (1) the firm's earnings growth and cash flow risk; (2) the external cost of financing; (3) the firm's liquidation value; (4) the opportunity cost of holding cash; (5) investment adjustment and asset sales costs; and (6) the return and covariance characteristics of hedging assets the firm can invest in. The optimal cash inventory policy involves two endogenous barriers and the continuous adjustment in investment and hedging positions in between the barriers. Cash is paid out to shareholders only when the cash-capital ratio hits the upper barrier, and external funds are raised only when

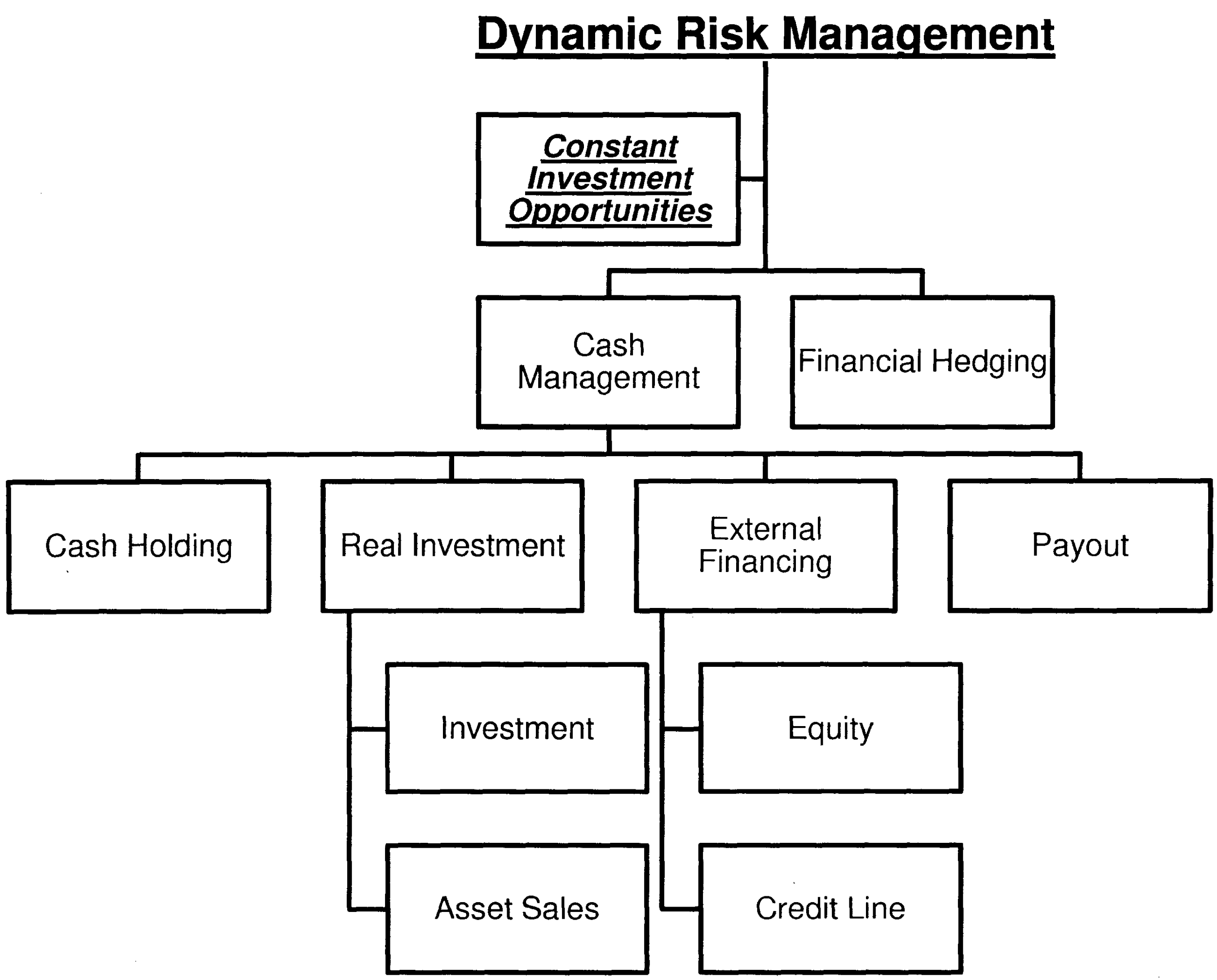

Figure 1-1: A unified framework for risk management according to Bolton et al. (2009).

the firm has depleted its cash. Liquidity management and derivatives hedging are complementary risk-management tools.

1.2 Introducing Time-Varying Investment Opportunities

Bolton et al. (2009) assume constant investment and financing opportunities for the firm, which is unrealistic in many situations. The goal of my thesis is to extend the BCW model by introducing time-varying investment opportunities, which will capture more situations that firms face in practice (at least as a first approximation) and yield a richer set of prescriptions.

For example, suppose the firm is currently in a normal state. The BCW model assumes that it will be in the normal state permanently and derives the corresponding set of investment, financing and risk management policies. However, in practice, the firm can enter a bad state with some probability. The bad state may be a financial crisis, a serious operational accident (e.g., BP oil spill in the Gulf of Mexico in 2010), etc. In the bad state, the firm is likely to face tougher investment opportunities (e.g., lower expected productivity shock or higher volatility). Qualitatively, when the firm is in a normal state, it have to consider the possibility to enter into a bad state in the future, and adjust its policies to prepare for the undesirable situations (e.g., invest less, hold more cash, or delay dividend payout to avoid financial distress if entering a bad state). Thus introducing time-varying investment opportunities to the BCW model can better describe the real-world situations and improve the firm's decisions.

Another example is that the firm's expected productivity shock, i.e. the expected operating cash flow or operating profit per unit of capital stock, is closely related to the price of the firm's output product, e.g. a company producing natural gas has its profitability closely related to the natural gas price. Since most commodity prices is time-varying and usually considered as mean-reverting, it is reasonable to assume that the firm's expected productivity shock is also mean-reverting. The BCW model fails to solve this problem appropriately since it assumes that the productivity shocks are identically and independently distributed (*i.i.d.*). My extension of the BCW model can help to address this problem better since I allow mean-revering expected productivity/profitability for the firm.

To sum up, in Bolton et al. (2009), the only shocks are the firm's idiosyncratic, temporary productivity shocks. In my thesis, I extend the BCW model to allow for persistent productivity shocks to the firm. My goal is to explore the interaction of persistent shocks with the financially

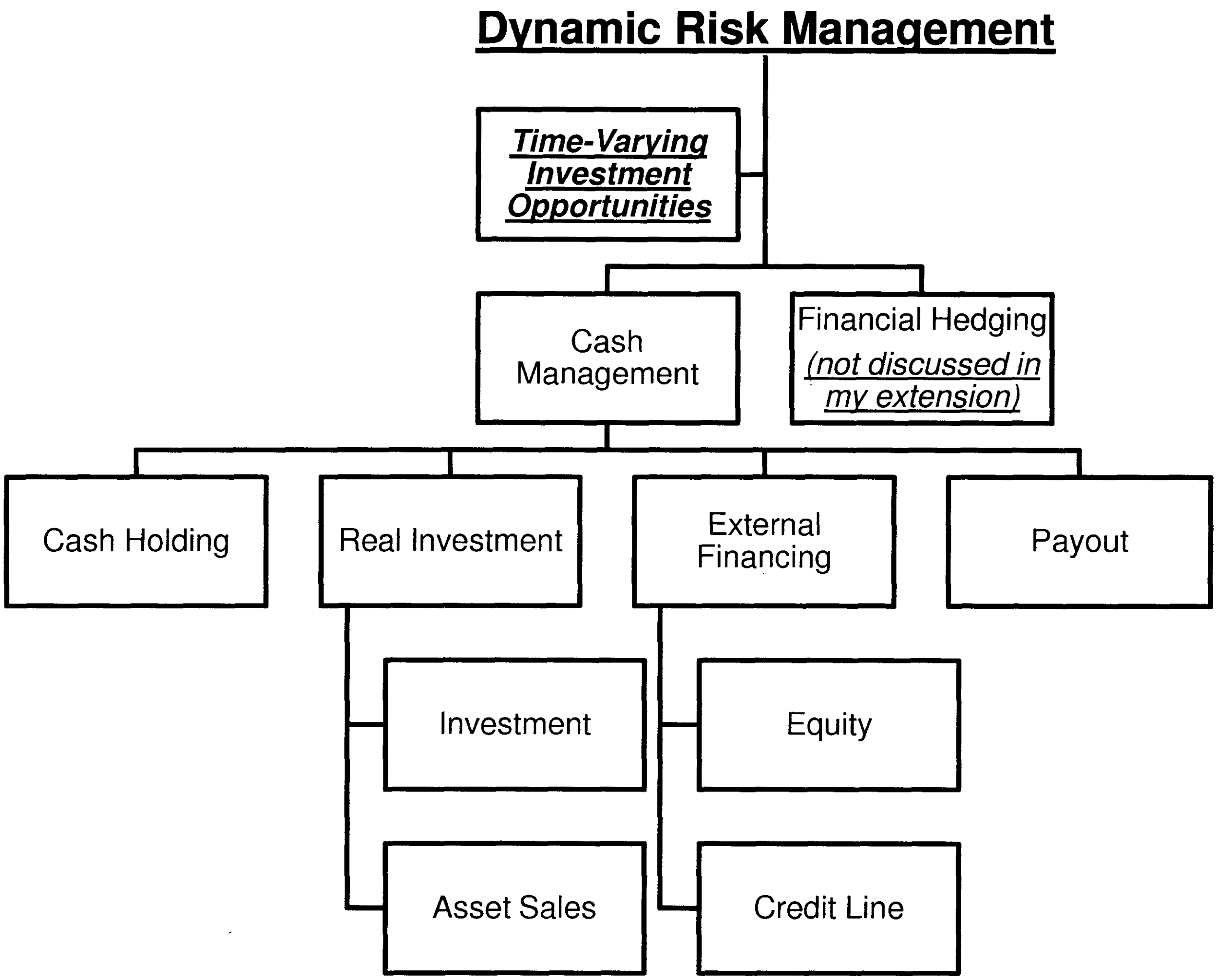

Figure 1-2: A unified framework for risk management according to my extended model with time-varying investment opportunities

constrained firm and the consequences for investment, financing, and risk management. The BCW model will be a benchmark to show how the policies differ between constant investment opportunity setting and time-varying investment opportunity setting.

Figure 1-2 shows the framework of my extended model with time-varying investment opportunities.

1.3 Organization of the Thesis

My thesis is organized as follows: Chapter 2 gives a literature review and discusses how some finance theories are related to the BCW model and my extension. Chapter 3 sets up my extended BCW model with time-varying investment opportunities. The time-varying investment opportunities are captured by two models: a regime-switching model and a mean-reverting model for the firm's productivity shock. Chapter 4 presents the model solution. Chapter 5 continues with quantitative analysis. Chapter 6 concludes.

Chapter 2

Literature Review

2.1 Cash Management

The different theories of cash demand fall into two broad classes: transactions theories, and asset or portfolio theories. The first ones, epitomized in the inventory theoretic models of Baumol (1952) and Tobin (1956) and in the later uncertainty version of Miller and Orr (1966), emphasize cash's role as a medium of exchange. Cash is viewed essentially as an inventory held for transactions purposes. Transfer costs of going between cash and other liquid financial assets justify holding such inventories even though other assets offered higher yields.

Baumol (1952) explicitly models the transactions demand for cash in an inventory theoretic approach. It assumes that the firm's cash manager has to manage an inventory of cash and other interest-earning assets. The cash manager is paid in bonds and spends cash (or makes transactions) at a constant, known rate. The objective is to choose the number of times he transfers between the stock of bonds and cash that maximizes profits, or equivalently minimizes costs. These costs come in two forms: brokerage fees and the interest foregone by holding cash. By making a large number of transfers out of bonds and into cash, he will be able to earn more interest on the bonds that are kept for longer. However, such a strategy will involve large brokerage fees. On the other hand, by making only a few transfers, he will avoid paying frequent brokerage fees but will miss out on the interest the wealth could earn if kept in the form of bonds.

Miller and Orr (1966) is an extension of the Baumol model in which the cash flow is stochastic. The model suggests that cash balance held by firms should depend upon (1) the opportunity cost of holding cash; (2) the cost of making transfers between cash and securities holdings; and (3) the exogenously determined and uncontrollable variability, or lack of synchronization between receipts and payments in the firm's cash flow. The last determining factor is the novel element that distinguishes the Miller-Orr model in structure and predictive content from the Baumol model. The mechanism used to generate the cash transactions flow is a symmetric Bernoulli process, i.e., t times per day a check for exactly m dollars is drawn on or deposited to the firm's cash balance, and the probability of a withdrawal and the probability of a deposit are equal. The cost of cash balance management considered in the Miller-Orr model are identical to the costs employed in the Baumol model.

The cash control policy underlying the Miller-Orr model is a two-parameter control-limit policy. That is, the cash balance will be allowed to wander freely until it reaches either the lower bound, 0, or an upper bound, h, at which times a portfolio transfer will be undertaken to restore the balance to a level of z (z is between 0 and h). Hence, the policy implies that when the upper bound is hit, there will be a lump sum transfer from cash of $(h - z)$ dollars; and when the lower limit is triggered, there will be a transfer to cash of z dollars. The rationality of this approach derives from the optimal (s, S) inventory policy proved by Scarf, Karlin and Arrow (1958). The lower bound is set to 0 because transfers are regarded as taking place instantaneously.

Miller and Orr (1968) summarize that three of the conditions upon which their model is based on stand out as extremely restrictive or "unrealistic": (1) cash flows are symmetric Bernoulli process; (2) the firm operates in a two-asset environment; and (3) the cost of transfer between cash and interest-bearing assets is purely "lumpy". Therefore, a lot of academic papers investigate the effects that each of these "assumptions" has upon the transaction balance relation, and propose improved models to extend these assumptions. These works form three major outgrowthes of the Miller-Orr model.

The connections of the Miller-Orr model to the BCW model and my extension are as follows. According to Bolton et al. (2009) and my thesis, the firm's cash-inventory policy is very rich, as it involves a combination of a double-barrier policy characterized by a single variable, the cash-capital ratio, and the continuous management of cash reserves in between the barriers through adjustments in investment, asset sales, as well as the firm's hedging positions (only discussed in the BCW model). Based on the double-barrier policy from the Miller-Orr model, the BCW model and my extended model provide substantial new insight on how the different boundaries depend on factors such as the growth rate and volatility of earnings, financing costs, cash holding costs, as well as the dynamics of cash holdings in between these boundaries. Besides cash inventory management, both the BCW model and my extended model can also give concrete prescriptions for how a firm choose its investment, financing, hedging, and payout policies, which are all important parts of dynamic corporate financial management.

For example, when the cash-capital ratio is higher, the firm invests more and saves less, as the marginal value of cash is smaller. When the firm is approaching the point where its cash reserves are depleted, it optimally scales back investment and may even engage in asset sales. This way the firm can postpone or avoid raising costly external financing. Since carrying cash is costly, the firm optimally pays out cash at the endogenous upper barrier of the cash-capital ratio. At the lower barrier the firm either raises more external funds or closes down. The firm optimally chooses not to issue equity unless it runs out of cash. Using internal funds (cash) to finance investment defers both the cash-carrying costs and external financing costs. Thus, with a constant investment/financing opportunity set the BCW model generates a dynamic pecking order of financing between internal and external funds. In my extended model, the above set of policies also depends on the time-varying investment opportunities.

2.2 Modigliani-Miller Theorem

Villamil (2008) considers that the Modigliani-Miller Theorem is a cornerstone of modern corporate finance. At its heart, the Modigliani-Miller Theorem is an *irrelevance* proposition: *it provides conditions under which a firm's financial decisions do not affect is value.* Franco (1980) explains the theorem as follows:

> ... with well-functioning markets (and neutral taxes) and rational investors, who can "undo" the corporate financial structure by holding positive or negative amounts of debt, the market value of the firm — debt plus equity — depends only on the income stream generated by its assets. It follows, in particular, that the value of the firm should not be affected by the share of debt in its financial structure or by what will be done with the returns — paid out as dividends or reinvested (profitably).

In fact what is currently understood as the Modigliani-Miller Theorem comprises four distinct results from a series of papers Modigliani and Miller (1958), Miller and Modigliani (1961), and Modigliani and Miller (1963). The first proposition establishes that under certain conditions, a firm's debt-equity ratio does not affect its market value. The second proposition establishes that a firm's leverage has no effect on its weighted average cost of capital (i.e., the cost of equity capital is a linear function of the debt-equity ratio). The third proposition establishes that firm market value is independent of its dividend policy. The fourth proposition establishes that equity-holders are indifferent about the firm's financial policy.

Miller (1991) explains the intuition for the Modigliani-Miller Theorem with a simple analogy. "Think of the firm as a gigantic tub of whole milk. The farmer can sell the whole milk as it is. Or he can separate out the cream, and sell it at a considerably higher price than the whole milk would bring." He continues, "The Modigliani-Miller proposition says that if there were no costs of separation, (and, of course, no government dairy support program), the cream plus the skim milk would bring the same price as the whole milk." The essence of the argument is that increasing the amount of debt (cream) lowers the value of outstanding equity (skim milk) — selling off safe cash flows to debt-holders leaves the firm with more lower valued equity, keeping the total value of the firm unchanged. Put differently, any gain from using more of what might seem to be cheaper debt is offset by the higher cost of now riskier equity. Hence, given a fixed amount of total capital, the allocation of capital between debt and equity is irrelevant because the weighted average of the two costs of capital to the firm is the same for all possible combinations of the two.

The Modigliani-Miller Theorem fundamentally structures the debate on why *irrelevance* fails around the its assumptions including (1) neutral taxes; (2) no capital market frictions (i.e., no transaction costs, asset trade restrictions or bankruptcy costs); (3) symmetric access to credit markets (i.e., firms and investors can borrow or lend at the same rate); and (4) firm financial policy reveals no information. These assumptions are important because they set conditions for effective arbitrage: When a financial market is not distorted by taxes, transaction or bankruptcy costs, imperfect information or any other friction which limits access to credit, then investors can costlessly replicate a firm's financial actions. This gives investors the ability to "undo" firm decisions, if they so desire. *Attempts to overturn the theorem's controversial irrelevance result are a fortiori arguments about which of the assumptions to reject or amend. The systematic analysis of these assumptions lead to an expansion of the frontiers of economics and finance.*

Neoclassical investment models (e.g. Hayashi (1982)) assume that the firm faces frictionless capital markets and that the Modigliani-Miller Theorem holds. However, in reality, firms often face important external financing costs due to asymmetric information and managerial incentive problems. This is mainly why the BCW model is proposed. The model introduces external financing costs, an important friction emphasized in modern corporate finance literature, into the neoclassic theory of investment. Using a tractable and operational dynamic economic framework, it shows how the firm's optimal investment, financing, and risk management policies are interconnected in the presence of external financing costs.

Therefore, the lack of frictionless capital markets (the Modigliani-Miller Theorem dose not hold) will have two significant impacts on the setup of the BCW model: (1) the cost of issuing external equity, and (2) the opportunity cost of holding cash.

Concerning equity financing cost, firstly there are direct issuance costs. For companies either going public or already public, these include the spread paid to the underwriter of an equity offering. They also include other legal and accounting costs associated with an issuance as well as the costs of registration. In addition, the equity issuance cost facing a given firm may depend upon the characteristics of the firm. According to Bolton et al. (2009), Calomiris and Himmelberg (1997) have estimated the direct transactions costs firms face when they issue equity which are substantial. In their sample the mean transactions costs, which include underwriting, management, legal, auditing and registration fees as well as the firm's selling concession, are 9% of an issue for seasoned public offerings and 15.1% for initial public offerings.

Secondly there are indirect cost which are more difficult to nail down and quantify. The primary indirect cost is the underpricing of the new equity sold. According to Bolton, Chen and Wang (2010a), under the broad-term pecking order view, investors are assumed to be fully rational and to fully understand firms' motives for issuing equity. Thus, when equity issuers have better information than investors about the value of their business, investors tend to interpret equity issues as negative signals about the value of the business: they infer that issuers are more likely to issue equity when the stock is overvalued, and therefore lower their assessment of the value of the firm in response to a new equity issue. Many classic writings like Jensen and Meckling (1976), Leland (1994), and Myers, Dill and Bautista (1976) have sought to measure these indirect equity financing costs. For example, Asquith and Mullins (1986) found that the average stock price reaction to the announcement of a common stock issue was -3% and the loss in equity value as a percentage of the size of the new equity issue was -31%. Hennessy and Whited (2007) estimate direct and total costs for a typical firm as a function of the net proceeds of a secondary equity issues. The difference between the two represents the indirect cost. They conclude that indirect cost add between 5 and 15 percentage points of the net proceeds to the marginal cost of equity issuance, and indirect costs add more than 7 percentage points to the average cost of equity issuance.

The BCW model and my extension do not explicitly model information asymmetries and incentive problems. Rather, to be able to work with a model that can be calibrated they directly model the costs arising from information and incentive problems in reduced form. Thus in the BCW model, whenever the firm chooses to issue external equity it summarize the information, incentive, and transactions costs incurred by a fixed cost Φ and a marginal cost γ. Importantly, when firms face fixed costs in raising external equity they will optimally tap equity markets only intermittently and when they do they raise funds in lumps, consistent with observed firm behavior. To preserve the homogeneity of degree one of the model, further assumptions are made. Since these assumptions are not so closely related to the Modigliani-Miller Theorem, we will not explain them here.

Concerning the opportunity cost of holding cash, in the BCW model, the rate of return that the firm earns on its cash inventory is the risk-free rate r minus a carry cost $\lambda > 0$ that captures in a simple way the agency costs that may be associated with free cash in the firm. In the presence of such a cost of holding cash, shareholder value is increased when the firm distributes cash back to shareholders should its cash inventory grows too large. Alternatively, the cost of carrying cash may arise from tax distortions. Cash retentions are tax disadvantaged because the associated tax rates generally exceed those on interest income (Graham (2000) and Faulkender and Wang (2006)).

Since my extension of the BCW focuses on the investment opportunities assumption, the settings related to the market frictions are almost the same as those of the BCW model.

2.3 Investment Theory

As summarized by Dixit and Pindyck (1994) and Hayashi (1982), the literature on investment has been dominated by two theories of investment — the neoclassical theory originated by Jorgenson and the "q" theory suggested by Tobin. One, following Jorgenson (1963), compares the per-period value of an incremental unit of capital (its marginal product) and an "equivalent per-period rental cost" or "user cost" that can be computed from the purchase price, the interest and depreciation rates, and applicable taxes. The firm's desired stock of capital is found by equating the marginal product and the user cost. The actual stock is assumed to adjust to the ideal, either as an ad hoc lag process, or as the optimal response to an explicit cost of adjustment. The book by Nickell (1978) provides a particularly good exposition of developments of this approach.

The other formulation, due to Tobin (1969), compares the capitalized value of the marginal investment to its purchase cost. The value can be observed directly if the ownership of the investment can be traded in a secondary market; otherwise, it is an imputed value computed as the expected present value of the stream of profits it would yield. The ratio of this to the purchase cost (replacement cost) of the unit, called Tobin's q, governs the investment decision. Investment should be undertaken or expanded if q exceeds 1; it should not be undertaken, and existing capital should be reduced, if $q < 1$. The optimal rate of expansion or contraction is found by equating the marginal cost of adjustment to its benefit, which depends on the difference between q and 1. Tax rules can alter this somewhat, but the basic principle is similar. Here, some sort of adjustment cost lie behind the theory. If a firm can freely change its capital stock, then it will continue to increase or decrease its capital stock until q is equal to unity. Also, the role of the production function is never clear in Tobin (1969) exposition.

It is increasingly recognized that the modified neoclassical investment theory with installment costs and the "q" theory are equivalent. Lucas and Prescott (1971) are the first to recognize this, although they never indicate the connection to the "q" theory. Later Abel (1977) shows that the optimal rate of investment is the rate for which $q-1$ is equal to the marginal cost of installment cost. However, his discussion is focused primarily on the Cobb-Douglas technology. Yoshikawa (1980) arrives at the same conclusion as Abel does, but his model is characterized by static expectations. Hayashi (1982) integrates the two theories of investment in a very general model of the firm's present value maximization and derives the optimal rate of investment as a function of q. It turns out that the form of investment function is independent of both the production function and the demand curve for the output.

According to Bolton et al. (2009), the BCW is based on the workhorse neoclassical q model of investment featuring constant investment opportunities as in Hayashi (1982). Bolton et al. (2009) summarize that Tobin (1956) defines the ratio between the firm's market value to the replacement cost of its capital stock, as "Q" and proposes to use this ratio to measure the firm's incentive to invest in capital. This ratio has become known as Tobin's average Q. Hayashi (1982) provides conditions under which average Q is equal to marginal q. In the BCW model, an important result that emerges is that with external financing costs the firm's investment is no longer determined by equating the marginal cost of investing with marginal q, as in the neoclassical Modigliani-Miller (MM) model (with no fixed adjustment costs for investment). Instead, corporate investment is determined by the following first-order condition:

$$\text{marginal cost of investment} = \frac{\text{marginal } q}{\text{marginal cost of financing}}. \tag{2.1}$$

In other words, investment of a financially constrained firm is determined by the ratio of marginal q to the marginal cost of financing. When firms are flush with cash, the marginal cost of financing is approximately one, so that this equation is approximately the same as the one under MM neutrality. But when firms are close to financial distress, the marginal cost of financing may be much larger than one so that optimal investment may be far lower than the level predicted under MM-neutrality.

Chapter 3

Model Setup

In this chapter, I build on the BCW model by introducing time-varying investment opportunities. At the beginning, my motivation is to assume the firm's investment opportunity to be mean-reverting, i.e., the expected productivity per unit of capital stock follows a simple stochastic process such as Ornstein-Uhlenbeck process. Then I find it difficult to solve the model, and use some methods to approximate the mean-reverting model with a regime-switching model.

The regime-switching model turns out to be a more general model, and has better analytical tractability. In this model, a firm can be in one of the n possible regimes or states. In each regime, the firm faces potentially different and exogenously give investment opportunities. When the firm is in regime j ($j = 1, 2, \cdots, n$), it will move from its current regime j to regime k ($k = 1, 2, \cdots, n,\ k \neq j$) with a constant probability $\xi_{jk}\Delta$ over a short period Δ. Finally, I use j_t to denote the regime the firm is in at time t.

Therefore, in my thesis I actually proposed two models to capture the firm's time-varying investment opportunities. The mean-reverting model is approximated with the regime-switching model, and the proposed solution method in the Chapter 4 is to solve the regime-switching model.

Besides the time-varying investment opportunities, I follow Bolton et al. (2009) to set up other parts for the model, such as the financing cost, firm optimality, etc.

3.1 Introducing Time-Varying Investment Opportunities

In the BCW model, the firm's operating revenue at time t is proportional to its capital stock K_t, and is given by $K_t dA_t$, where dA_t is the firm's revenue or productivity shock over time increment dt. Bolton et al. (2009) assume that after accounting for systematic risk the firm's cumulative productivity evolves according to:

$$dA_t = \mu dt + \sigma dZ_t, \qquad t \geq 0, \tag{3.1}$$

where Z_t is a standard Brownian motion under the risk-neutral measure. Thus, productivity shocks are assumed to be *i.i.d.*, and the parameters $\mu > 0$ and $\sigma > 0$ are the mean and volatility of the risk-adjusted productivity shock dA_t. This production specification is often refereed to as the "AK" technology in the macroeconomics literature. The assumption of the productivity shocks implies that investment opportunities are constant over time.

In my thesis, I allow the assumption of the productivity shock to capture time-varying investment opportunities. Although my initial motivation is to assume the expected productivity shock, μ_t, to be stochastic and mean-reverting, here I will firstly describe a more general regime-switching model of μ_t, and then explain how to approximate the mean-reverting μ_t with a regime-switching process.

3.1.1 Regime-Switching Model

Mathematical Formulation

After risk adjustment (i.e. under the risk-neutral probability measure), the firm's revenue dA_t over time period d_t are given by

$$dA_t = \mu(j_t)dt + \sigma(j_t)dZ_t, \tag{3.2}$$

where dZ_t is the dynamic of a standard Brownian motion, which $\mu(j_t)$ and $\sigma(j_t)$ denote the expected return per unit of capital stock and its volatility in regime j_t. With a constant probability $\xi_{jk}\Delta$, the firm will move from regime j to k over a short period Δ.

Why the Model

I primarily have two reasons to use the regime-switching model to capture the time-varying investment opportunities based on the BCW model.

Firstly, using the regime-switching model maintains the analytical tractability of the extended BCW model. For example, I can still exploit the homogeneity of degree one to make a key simplification to solve the extended BCW model, while the simplified solution can generate several novel and economically significant insights about the firm's financing, investment and risk management policies. Also, the mean-reverting introduced Section 3.1.2 can be approximated with the regime-switching model, which I can solve with numerical methods.

Secondly, the regime-switching model can introduce the persistent factor into the productivity shock, and capture more situations than the BCW model that firms face in practice (at least as a first approximation) and yield a richer set of prescriptions. The regime-switching model is widely used to describe the abrupt and sharp changes in the firm's exogenous and endogenous conditions. For example, Piskorski and Tchistyi (2007) consider a model of mortgage design in which they use a Markov switching process to describe interest rates. DeMarzo, Fishman, He and Wang (2009) integrate dynamic moral hazard with the q theory of investment in a continuous-time dynamic optimal contracting framework. This paper is closely related to the BCW model in terms of methodology, but introduces a stochastic price π_t for the firm's output. To keep the analysis simple, π_t is modeled as a two-state Markov regime-switching process. Wang (2009) considers a consumption-saving and portfolio allocation problem, and assumes that the long-run income growth rate follows the continuous-time Markov regime switching model. Most importantly, Bolton et al. (2010a) extend their own BCW model to introduce stochastic financing opportunities, using a regime-switching model. Although my model is similar to Bolton et al. (2010a), I develop the regime-switching independent of their new paper, and focus on the time-varying investment opportunities.

3.1.2 Mean-Reverting Model

Mathematical Formulation

The firm's risk-adjusted productivity shock is assumed to be

$$dA_t = \mu_t dt + \sigma_z dZ_t, \tag{3.3}$$

where μ_t is an Ornstein-Uhlenbeck process following the stochastic differential equation

$$d\mu_t = \kappa(\overline{\mu} - \mu_t)dt + \sigma_w dW_t, \tag{3.4}$$

while the two standard Brownian motions Z_t and W_t can be potentially correlated

$$dZ_t dW_t = \rho_{zw} dt. \tag{3.5}$$

Why the Model

The reason to use the mean-reverting model is that in reality, a firm's expected productivity shock is likely to be closely related to the price of its output. The commodity price is one that we can analyze with some rigor, while a mean-reverting process is widely used to model it. Therefore, it is reasonable to assume that the firm's expected productivity shock also follows a mean-reverting process. For example, consider the company of Chesapeake Energy (CHK), whose major output is natural gas. CHK's financial projections at various natural gas prices in 2010 and 2011 are in Table 3.1 and 3.2 respectively. Figure 3-1 displays CHK's projected operating cash flows versus

Natural Gas Price	\$4.00	\$5.00	\$6.00	\$7.00	\$8.00	\$9.00
Operating cash flow (\$ in millions)	5,020	5,140	5,230	5,310	5,390	5,460

Table 3.1: Part of CHK's 2010 financial projections at different natural gas prices. Source: CHK November 2010 Investor Presentation, slides 28, available on CHK's official website.

Natural Gas Price	\$4.00	\$5.00	\$6.00	\$7.00	\$8.00	\$9.00
Operating cash flow (\$ in millions)	4,760	5,310	5,870	6,420	6,900	7,360

Table 3.2: Part of CHK's 2011 financial projections at different natural gas prices. Source: CHK November 2010 Investor Presentation, slides 29, available on CHK's official website.

natural gas prices in 2010 and 2011, and they turn out to be closely (linearly) correlated. Since the productivity per unit of capital stock can be estimated by

$$\frac{\text{CHK's Operating Cash Flow}}{\text{CHK's Property, Plant and Equipment (PP\&E)}},$$

it is reasonable to assume that the expect productivity per unit of capital stock is also closely related to the natural gas price. Finally, since natural gas price are usually modeled by a mean-reverting process (e.g., Ornstein-Uhlenbeck process), it is reasonable to assume CHK's expected productivity shock μ to be also mean-reverting.

Other than the price factor, CHK's realized productivity per unit of stock can also be affected by uncertainties in market demand, technology process, etc. Therefore, the productivity is also subject to an idiosyncratic shock σ_z that captures all the uncertainties except natural gas price.

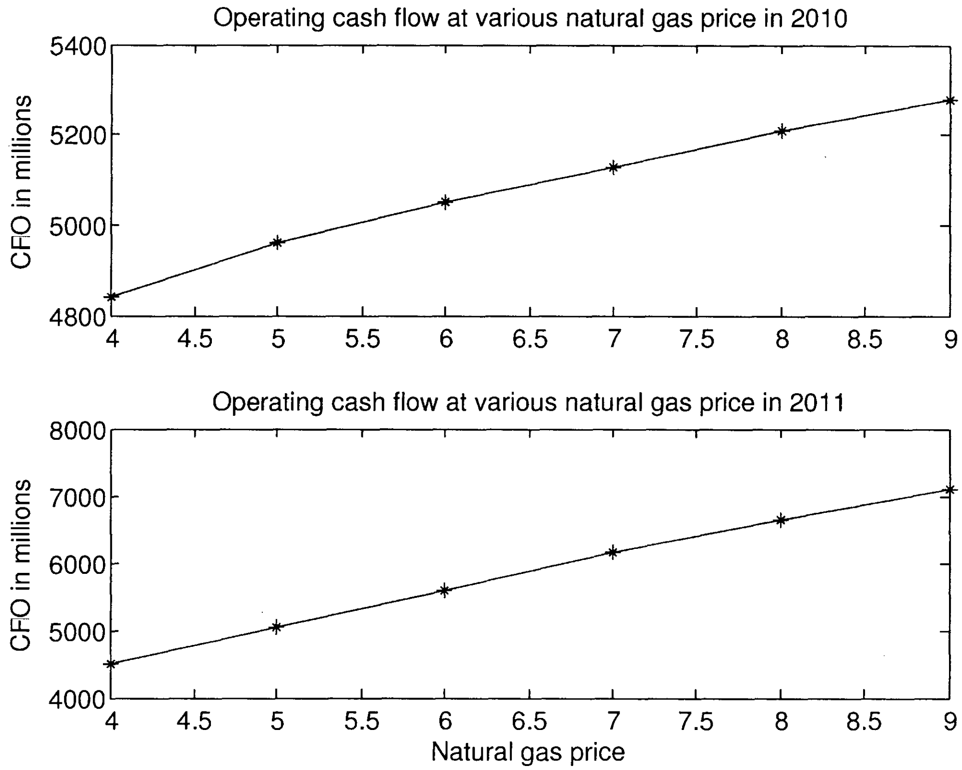

Figure 3-1: CHK's operating cash flow v.s. natural gas price in 2010 and 2011

That's why I model the productivity shock by (3.3) and (3.4).

Approximation with a Regime-Switching Model

Under this assumption, I cannot solve the model. Thus I have to make some assumptions and approximations to simplify it. By assuming that the two standard Brownian motions Z_t and W_t are independent, I can use a continuous-time Markov regime-switching chain with mean-reverting characteristics to approximate the expected productivity shock μ_t, and transform the problem into a regime-switching model discussed in the previous section.

The steps to construct the continuous-time Markov regime-switching chain are as follows:

1. In a discrete-time setting, derive the expected productivity shock μ_t that follows a first-order autoregressive AR(1) process at the annual frequency

$$\mu_t = \eta + \rho\mu_{t-1} + \epsilon_\mu, \qquad \epsilon_\mu \sim \mathcal{N}(0, \sigma_\mu^2). \tag{3.6}$$

Note that Equation (3.6) is the discrete-time counterpart of Equation (3.4) where $\rho =$

$e^{-\kappa}$, $\eta = \overline{\mu}(1 - e^{\kappa})/\kappa$, $\sigma_\mu = \sigma_w\sqrt{(1 - e^{-2\kappa})/(2\kappa)}$. When I calibrate the model, I will directly estimate the parameters for the discrete-time AR(1) process at the annual frequency, rather than firstly estimating the parameters for the continuous-time Ornstein-Uhlenbeck process and then calculating the corresponding parameters for its discrete-time counterpart AR(1) process.

2. Use the quadrature method in Tauchen (1986) to approximate the dynamics of the AR(1) in Equation (3.6), where we assume $\epsilon_\mu \sim \mathcal{N}(0, \sigma_\mu^2)$, with a discrete-time, finite-state, and first-order Markov chain. According to this method, the discrete abscissae of the Markov chain and the transition probabilities are found by a Gauss-Hermite quadrature rule.

   Specifically, we take n discrete abscissae in an interval of semiwidth $I_p = 3\sigma_\mu/\sqrt{1-\rho^2}$, and centered on the long-term mean of process, $\eta/(1-\rho)$. The set of the discretized state variable is $\widetilde{Y} = \{\widetilde{y}(s)|s = 1, \cdots, n\}$, where

$$\widetilde{y}(s) = \frac{\eta}{1-\rho} - \max\left\{\left(\frac{n-1}{2}+1\right) - s, 0\right\}\varphi + \max\left\{s - \left(\frac{n-1}{2}+1\right), 0\right\}\varphi, \tag{3.7}$$

   with $\varphi = 2I_p/n$.

   Next, I define the cells for the state variables as $c(j) = [Y(j), Y(j+1)]$, for $j = 1, \cdots, n$, where

$$\begin{aligned} Y(1) &= -\infty, && (3.8)\\ Y(j) &= \frac{\widetilde{y}(j) + \widetilde{y}(j-1)}{2}, \qquad j = 2, \cdots, n, && (3.9)\\ Y(n+1) &= +\infty. && (3.10) \end{aligned}$$

   To obtain the transition probability matrix under the risk-neutral probability, I have to determine the probability, conditional of the current state y, that the future state is y'. Given the above approximation, this is equivalent to the probability p_{ij} that y' falls into cell $c(j)$, given the current state $y = \widetilde{y}(i)$, for all $j = 1, \cdots, n$ and all $i = 1, \cdots, n$:

$$\begin{aligned} p_{ij} &= \Pr\{y' \in c(j)|y = \widetilde{y}(i)\} \qquad (3.11)\\ &= \Pr(Y(j) \le y' < Y(j+1)|y = \widetilde{y}(i))\\ &= \mathcal{N}\left(\frac{Y(j+1) - \eta - \rho\widetilde{y}(i)}{\sigma_\mu}\right) - \mathcal{N}\left(\frac{Y(j) - \eta - \rho\widetilde{y}(i)}{\sigma_\mu}\right). \end{aligned}$$

The transition probability matrix is $P = [p_{ij}]$ where $i, j = 1, \cdots, n$. The proposed method converges as $n \to \infty$, as shown by Tauchen (1990).

3. Transform the discrete-time transition matrix $P = [p_{ij}]$ into the generator $\Xi = [\xi_{ij}]$ of a continuous-time Markov chain using the method of Jarrow, Lando and Turnbull (1997) (an approximation based on the assumption that the probability of more than one change of state is close to zero within the period Δ). Since the AR(1) process for μ is at the annual frequency, by assuming that the probability of making more than one transition per year is small for the firm's expected productivity shock, the discrete-time Markov chain of μ can be transformed into a continuous-time version using the formula

$$\begin{aligned} \xi_{ii} &= \log(p_{ii}), \\ \xi_{ij} &= \frac{p_{ij}\log(p_{ii})}{p_{ii}-1}, \qquad i \neq j, \; i, j = 1, 2, \cdots, n. \end{aligned}$$

After the above three steps, I manage to approximate the mean-reverting model with a regime-switching model, which I can solve.

3.2 Other Assumptions of Production Technology

Since my thesis is based on Bolton et al. (2009), in the section, the formulations are the almost the same as those in Bolton et al. (2009).

The firm employs only physical capital as an input for production and, the price of physical capital is normalized to unity. I denote by K and I respectively the level of the capital stock and gross investment. As is standard in capital accumulation models, the firm's capital stock K evolves according to

$$dK_t = (I_t - \delta K_t)dt, \qquad t \geq 0, \tag{3.12}$$

where $\delta > 0$ is the rate of depreciation of the capital stock.

The firm's incremental operating profit dY_t over time increment d_t is then given by:

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t, j_t)dt, \qquad t \geq 0, \tag{3.13}$$

where I is the cost of the investment and $G(I, K, j)$ is the additional adjustment cost that the firm incurs in the investment process if it is in regime j. Note that I allow the adjustment costs to be regime dependent. Intuitively, in expansion, the firm may be subject to lower transaction costs. Following the neoclassical investment literature (e.g., Hayashi (1982)), I assume that the firm's adjustment cost is homogeneous of degree one in I and K. In other words, the adjustment cost takes the homogeneous form $G(I, K, j) = g_j(i)K$, where i is the firm's investment capital ratio ($i = I/K$), and $g_j(i)$ is an regime-dependent increasing and convex function. My analysis does not depend on the specific functional form of $g_j(i)$, and to simplify I assume that $g_j(i)$ is quadratic:

$$g_j(i) = \frac{\theta_j(i-\nu_j)^2}{2}. \tag{3.14}$$

Note that the assumption of the investment adjustment cost $g_j(i)$ is different from that in Bolton et al. (2009), where they assume $g(i) = \frac{\theta}{2}i^2$. The reasons to make this change are twofold. First, Equation (3.14) is more general, e.g. if setting $\nu_j = 0$, it will be the same as in the BCW model. Second the new assumption is consistent with the empirical evidences in Eberly et al. (2009). Bolton et al. (2010a) also use the new assumption of Equation (3.14) in the extended BCW model with time-varying financing opportunities.

The firm can liquidate its assets at any time. The liquidation value L_t is proportional to the firm's capital at time t in regime j_t, i.e. $L_t = l_j K_t$, where l_j is the recovery per unit of capital in regime j.

According to Bolton et al. (2009), the homogeneity assumption embedded in the adjustment cost, the constant-return-to-scale production technology, and the liquidation technology allows the model to deliver some key results in a parsimonious and analytically tractable way. Adjustment costs may not always be convex and the production technology may exhibit decreasing returns to scale in practice, but these functional forms substantially complicate the analysis and do not permit a closed-form characterization of investment and financing policies. As will become clear below, the homogeneity assumption helps reduce the problem to one with effectively a single state variable, which is easier to solve. Eberly et al. (2009) provides strong empirical evidence in support of the Hayashi homogeneity assumption for the upper quartile of Compustat firms.

3.3 Information, Incentives and Financing Costs

Firstly I would like the state that the assumptions and formulations in this section is almost the same as those in Bolton et al. (2009) since my extended model is based on the BCW model, and these assumptions are about the financing constraints.

Neoclassical investment models (e.g., Hayashi (1982)) assume that the firm faces frictionless capital markets and that the Modigliani-Miller Theorem holds. However, in reality, firms face important financing frictions for incentive, information asymmetry, and transaction cost reasons. My model incorporates a number of financing costs that firms face in practice and that empirical research has identified, while retaining an analytically tractable setting. The firm may choose to use external financing at any point in time, such as issuing equity or debt.

I assume that the firm incurs a fixed and a variable cost of issuing external equity. The fixed cost is given by $\Phi_j K$, where Φ_j is the fixed cost parameter in regime j. For tractability, I take the fixed cost to be proportional to the firm's capital stock K. Mainly, this assumption ensures that the firm does not grow out of its fixed issuing costs. The firm also incurs a proportional issuance cost γ_j for each unit of external funds it raises, which may also be regime dependent. That is, after paying the fixed cost, the firm pays $\gamma_j > 0$ in regime j for each incremental dollar it raises. We denote by H the process for the firm's cumulative external financing, and hence by dH_t the incremental external financing over time d_t, and by X the firm's cumulative issuance costs.

I also denote by W the process for the firm's cash stock. If the firm runs out of cash ($W_t = 0$), it needs to raise external funds to continue operating or its assets will be liquidated. If the firm chooses to raise new external funds to continue operating, it must pay the financing costs specified above. The firm may prefer liquidation if the cost of financing is too high relative to the continuation value (e.g. when the firm is not productive; that is when μ_j is low). I denote by τ the firm's (stochastic) liquidation time, then $\tau = \infty$ means that the firm never chooses to liquidate.

The rate of return that the firm earns on its cash inventory is the risk-free rate r minus a carry cost $\lambda > 0$ that captures in a simple way the agency costs that may be associated with free cash in the firm. Alternatively, the cost of carrying cash may arise from tax distortions. Cash retentions are tax disadvantaged as interest earned by the corporation on its cash holdings is taxed at the corporate tax rate, which generally exceeds the personal tax rate on interest income. The benefit of a payout is that shareholders can invest at the risk-free rate r, which is higher than $(r - \lambda)$ the net rate of return on cash within the firm. However, paying out cash also reduces the firms cash balance, which potentially exposes the firm to current and future under-investment and future external financing costs. The tradeoff between these two factors determines the optimal payout policy. I denote by U_t the firm's cumulative (non-decreasing) payout to shareholders up to time t, and by dU_t the incremental payout over time interval dt. Distributing cash to shareholders may take the form of a special dividend or a share repurchase.

Combining cash flows from operations dY_t given in Equation (3.13), with the firm's financing policy given by the cumulative payout process U_t, the cumulative external financing process H_t, and the firm's interest earnings minus cash carry cost from its cash inventory, then W_t evolves according to:

$$dW_t = [K_t dA_t - I_t dt - G(I_t, K_t, j_t)]dt + (r(j_t) - \lambda_j)W_t dt + dH_t - dU_t \tag{3.15}$$

where, the second term is the interest income (net of the carry cost λ), the third term dH_t is the cash inflow from external financing, and the last term dU_t is the cash outflow to investors, so that $(dH_t - dU_t)$ is the net cash flow from financing. Note that this is a completely general financial accounting equation, where dH_t and dU_t are endogenously determined by the firm.

3.4 Firm Optimality

According to Bolton et al. (2009), the firm chooses its investment I, its cumulative payout policy U, its cumulative external financing H, and its liquidation time τ to maximize firm value defined below:

$$\mathbb{E}\left[\int_0^{\tau} e^{-\int_0^t r_j dj}(dU_t - dH_t - dX_t) + e^{-\int_0^{\tau} r_j dj}(L_\tau + W_\tau)\right]. \tag{3.16}$$

The expectation is taken under the risk-adjusted probability. The first term is the discounted value of payout to shareholders and the second term is the discounted value upon liquidation. Note that optimality may imply that the firm never liquidates. In that case, I simply have $\tau = \infty$. I impose the usual regularity conditions to ensure that the optimization problem is well posed. The optimization problem is most obviously seen as characterizing the benchmark for the firm's efficient investment, cash-inventory, payout, and external financing policy when the firm faces external financing cost, cash-carrying costs, and time-varying investment opportunities.

Chapter 4

Model Solution

The model solution methods are based on those proposed in Bolton et al. (2009), and is more complicated since I assume time-varying investment opportunities for the firm. Since the mean-reverting model can be approximated by a regime-switching model, here I will only need to discuss the solution methods for the regime-switching model.

When the firm faces costs of raising external funds, it can reduce future financing costs by retaining earnings (i.e. holding cash) to finance its future investments. Also, the firm should adjust it policies to adapt to the persistent and stochastic investment opportunities. Then firm value depends on three natural state variables, it stock of cash W, its capital stock K, and the regime j with different investment opportunities as compared to other regime k. Let $P(K, W, j)$ denote firm value, where $j = 1, 2, \cdots, n$. I am interested in the impact of time-varying investment opportunities on corporate decision making. In particular, I study how firms manage the risks of investment opportunities $\mu(j)$ and volatility $\sigma(j)$. Note that the $\sigma(j)$ in the regime-switching model corresponds to the idiosyncratic shock of σ_z in the mean-revering model. The shock σ_w and mean-reversion rate κ in the mean-reverting μ_t in the continuous-time setting, will jointly determine the parameters of the approximated Markov chain of μ for the regime-switching model.

4.1 Optimal Conditions

According to Bolton et al. (2009), the firm decision-making and firm value depend on which of the following three regions it finds itself in: (1) an external funding/liquidation region, (2) an internal financing region, and (3) a payout region. As will become clear below, the firm is in the external funding/liquidation region when its cash stock W is less than or equal an endogenous lower barrier

$\underline{W}_j$. It is in the payout region when its cash stock W is greater than or equal an endogenous upper barrier $\overline{W}_j$. And it is in the internal financing region when $W \in (\underline{W}_j, \overline{W}_j)$. Note that the following formulations are very similar to those in Bolton et al. (2009) since my model is based on the BCW model.

4.1.1 Internal Financing Region

In this region, the firm use its cash stock for investment at time t, and thus $dH_t = dX_t = 0$. Using the bellman optimality condition for the dynamic programming problem in (3.16), I get the the following equation:

$$rP(K,W,j) = \max\{\mathbb{E}[dP(K,W,j)]\}. \tag{4.1}$$

The left side of (4.1) is the normal return per unit time that the shareholders, using the risk-free rate, would require for holding the firm. The right side is the instantaneous expected changes in the value function. Optimality condition implies that the two sides are equal.

Using Itô Lemma, I have

$$dP(K,W,j) = \frac{\partial P}{\partial K}dK + \frac{\partial P}{\partial W}dW + \frac{\partial^2 P}{\partial W^2}(dW)^2 + \sum_{k=1,k\neq j}^{n} \xi_{j,k}(P(K,W,k) - P(K,W,j)). \tag{4.2}$$

Combining Equation (4.1) and (4.2), I derive the following system of Hamilton-Jacobi-Bellman (HJB) equations when the firm's cash holding is above the financing/liquidation boundary and below the payout boundary for the current state:

$$\begin{aligned} r_j P(K,W,j) \quad = \quad & \max_I [(r_j - \lambda_j)]W + \mu_j K - I - G(I,K,j)]P_W(K,W,j) \\ & + \frac{\sigma_j^2 K^2}{2} P_{WW}(K,W,j) + (I - \delta K)P_K(K,W,j) + \sum_{k=1,k\neq j}^{n} \xi_{jk}(P(K,W,k) - P(K,W,j)), \end{aligned} \tag{4.3}$$

for $\underline{W}_j \leq W \leq \overline{W}_j$. The last term of the right side captures the effect of stochastic transition of the expected growth rate on the expected change in the value function. The value function changes discretely from $P(K,W,j)$ to $P(K,W,k)$ when the expected productivity shock changes from μ_j to μ_k.

According to Bolton et al. (2009), the firm finances its investment out of the cash inventory in this region. The convexity of the physical adjustment cost implies that the investment decision in the model admits an interior solution. The investment-capital ratio $i = I/K$ then satisfies the following first-order condition in regime j:

$$1 + \theta_j i = \frac{P_K(K, W, j)}{P_W(K, W, j)}. \tag{4.4}$$

Following Bolton et al. (2009) and Bolton et al. (2010a), I conjecture that firm value is homogeneous of degree one in W and K in each regime j, so that

$$P(K, W, j) = p_j(w)K, \tag{4.5}$$

where $w = W/K$ is the firm's cash-capital ratio. The dynamics of w is given by

$$dw_t = (r_j - \lambda_j)w_t dt - (i_j(w_t) + g(i_j(w_t)))dt + (\mu_j dt + \sigma_j dZ_t). \tag{4.6}$$

The first term on the right-hand side of Equation (4.6) is the interest income net of cash-carrying costs. The second term is the total flow-cost of (endogenous) investment (capital expenditures plus adjustment costs). While most of the time we have $i_j(w_t) > 0$, the firm may sometimes want to engage in asset sales (i.e. set $i_j(w_t) < 0$) in order to replenish its stock of cash and thus delay incurring external financing costs. Finally, the third term is the realized revenue per unit of capital stock (dA).

Since $P_K(K, W, j) = p_j(w) - wp'_j(w)$, $P_W(K, W, j) = p'_j(w)$ and $P_{WW}(K, W, j) = p''_j(w)/K$, by substituting these items into Equation (4.3) I obtain the following system of ordinary differential equations (ODEs) for $p_j(w)$ where $j = 1, 2, \cdots, n$:

$$\begin{aligned} r_j p_j(w) &= (i_j(w) - \delta_j)(p_j(w) - wp'_j(w)) + ((r_j - \lambda_j)w + \mu_j - i_j(w) - g_j(i_j(w))p'_j(w) \qquad (4.7) \\ &\quad + \frac{\sigma_j^2}{2}p''_j(w) + \sum_{k=1, k\neq j}^{n} \xi_{jk}(p_k(w) - p_j(w)), \qquad w \in [0, \overline{w}_j], \\ g_j(i_j(w)) &= \frac{\theta_j}{2}(i_j(w) - \nu_j)^2. \end{aligned}$$

The first-order (FOC) for the investment-capital ratio $i_j(w)$ is then given by:

$$i_j(w) = \frac{1}{\theta_j}\left(\frac{p_j(w)}{p'_j(w)} - w - 1\right) + \nu_j. \tag{4.8}$$

To completely characterize the solution for $p_j(w)$, I must also determine the boundaries $\underline{w}_j$ at which the firm raises new external funds (or closes down), how much to raise (the target cash-capital ratio after issuance), and $\overline{w}_j$ at which the firm pays out cash to shareholders.

4.1.2 Payout Region

Intuitively, when the cash-capital ratio is very high, the firm is better off paying out the excess cash to shareholders to avoid the carry cost. The natural question is how high the the cash capital ratio needs to be before the firm pays out. Let w denote this endogenous payout boundary. Intuitively, if the firm starts with a large amount of cash ($w > \overline{w}_j$), then it is optimal for the firm to distribute the excess cash as a lump-sum and bring the cash-capital ratio w down to $\overline{w}_j$. Moreover, firm value must be continuous before and after cash distribution. Therefore, for $w > \overline{w}_j$, I have the following equation for $p_j(w)$:

$$p_j(w) = p_j(\overline{w}_j) + (w - \overline{w}_j), \qquad w > \overline{w}_j. \tag{4.9}$$

Since the above equation also holds for w close to $\overline{w}_j$, I may take the limit and obtain the following condition for the endogenous upper boundary $\overline{w}_j$:

$$p_j'(\overline{w}_j) = 1. \tag{4.10}$$

At $\overline{w}_j$ the firm is indifferent between distributing and retaining one dollar, so that the marginal value of cash must equal one, which is the marginal cost of cash to shareholders. Since the payout boundary $\overline{w}_j$ is optimally chosen, I also have the following "super contact" condition (see, e.g. Dumas (1991)):

$$p_j''(\overline{w}_j) = 0. \tag{4.11}$$

4.1.3 External Financing/Liquidation Region

When the firm's cash-capital ratio w is less than or equal to the lower barrier $\underline{w}_j$, the firm either incurs financing costs to raise new funds or liquidates. Depending on parameter values, it may prefer either liquidation or refinancing by issuing new equity. Although the firm can choose to liquidate or raise external funds at any time, I show that it is optimal for the firm to wait until it runs out of cash, i.e. $w = 0$. The reason is explained in Section 5.5.1.

When the expected productivity μ_j is low and/or cost of financing is high, the firm will prefer liquidation to refinancing. In that case, because the optimal liquidation boundary is $\underline{w}_j = 0$, firm value upon liquidation is thus $p_j(0)K = l_jK$. Therefore, I have

$$p_j(0) = l_j. \tag{4.12}$$

If the firm's expected productivity μ_j is high and/or its cost of external financing is low, then it is better off raising costly external financing than liquidating its assets when it runs out of cash. To economize fixed issuance costs ($\Phi_j > 0$), firms issue equity in lumps. With homogeneity, the total equity issue amount is m_jK, where $m_j > 0$ is endogenously determined as follows. First, firm value is continuous before and after equity issuance, which implies the following condition for $p_j(w)$ at the boundary $\underline{w}_j = 0$:

$$p_j(0) = p_j(m_j) - \phi_j - (1 + \gamma_j)m_j. \tag{4.13}$$

The right side represents the firm value-capital ratio $p_j(m_j)$ minus both the fixed and the proportional costs of equity issuance, per unit of capital stock. Second, since m_j is optimally chosen, the marginal value of the last dollar raised must equal one plus the marginal cost of external financing, $1 + \gamma_j$. This gives the following smooth pasting boundary condition at m_j:

$$p_j'(m_j) = 1 + \gamma_j. \tag{4.14}$$

4.1.4 Credit Line Case

The BCW model and my extended model with time-varying investment opportunities can both be extended to allow the firm to access a credit line. This is an important extension to consider, as many firms in practice are able to secure such lines, and for these firms, access to a credit line is an important alternative source of liquidity.

Following Bolton et al. (2009), I model the credit line as a source of funding the firm can draw on at any time it chooses up to a limit. I set the credit limit to a maximum fraction of the firm's capital stock, so that the firm can borrow up to cK, where $c > 0$ is a constant. The logic behind this assumption is that the firm must be able to post collateral to secure a credit line and the highest quality collateral does not exceed the fraction c of the firm's capital stock. I may thus interpret cK to be the firm's short-term debt capacity. For simplicity, I treat c as exogenous in this paper. I also assume that the firm pays a constant spread α over the risk-free rate on the amount of credit it uses. Sufi (2009) shows that a firm on average pays 150 basis points over LIBOR on its credit lines, leading us to set $\alpha = 1.5\%$. Although the credit limit coefficient c is exogenously chosen in the BCW model and my extension, in reality c is very likely to related to the liquidation value of the firm (e.g., the coefficient l), since we assume the firm uses its capital stock as the collateral to secure the credit limit. The analysis of the relations between c and l will be an interesting topic for future research.

Since the firm pays a spread α over the risk-free rate to access credit, it will optimally avoid using its credit line before exhausting its internal funds (cash) to finance investment. As long as the interest rate spread α is not too high, credit line will be less expensive than external equity, so the firm also prefers to first draw on the line before tapping equity markets. My extended model thus generates a pecking order [1] among internal funds, credit lines, and external equity financing.

When credit line is the marginal source of financing ($w < 0$), $p_j(w)$ solves the following system of ODEs for $j = 1, 2, \cdots, n$:

$$\begin{aligned} r_j p_j(w) &= (i_j(w) - \delta_j)(p_j(w) - wp_j'(w)) + ((r_j + \alpha)w + \mu_j - i_j(w) - g_j(i_j(w))p_j'(w) \qquad (4.15) \\ &+ \frac{\sigma_j^2}{2} p_j''(w) + \sum_{k=1, k\neq j}^{n} \xi_{jk}(p_k(w) - p_j(w)), \qquad w \in [-c, 0] \\ g_j(i_j(w)) &= \frac{\theta_j}{2}(i_j(w) - \nu_j)^2. \end{aligned}$$

Next, when cash is the marginal source of financing ($w > 0$), $p_j(w)$ satisfies the system of ODEs (4.3). As for the boundary conditions, when the firm exhausts its credit line before issuing equity, the boundary conditions for the timing and the amount of equity issuance are similar to the ones given in Section 4.1.3. That is, I have $p_j(-c) = p_j(m_j) - \phi_j - (1+\gamma_j)(m_j + c)$, and $p_j'(m_j) = 1 + \gamma_j$. At the payout boundary, the same conditions (4.9-4.11) hold here. Finally, $p_j(w)$ is continuous and smooth everywhere, including at $w = 0$, which gives two additional boundary conditions.

4.2 Numerical Methods

Finally, I need to solve the following system of ODEs. Note that what I need to solve is the firm value capital function $p_j(w)$, the investment ratio $i_j(w)$, the optimal payout boundary $\overline{w}_j$ and the return-to-ration m_j if the firm is to issue equity for regime $j = 1, 2, \ldots, n$.

[1] In very broad terms, equity is the most expensive source of funds and firms therefore issue equity only as a last resort when they have exhausted all other forms of financing.

$$
\begin{aligned}
r_j p_j(w) &= (i_j(w)-\delta_j)(p_j(w)-wp_j'(w)) + ((r_j-\lambda_j)w+\mu_j - i_j(w) - g_j(i_j(w))p_j'(w) \quad (4.16)\\
&\quad + \frac{\sigma_j^2}{2}p_j''(w) + \sum_{k=1,k\neq j}^{n} \xi_{jk}(p_k(w)-p_j(w)), \qquad w\in[0,\overline{w}_j]\\
p_j(w) &= p_j(\overline{w}_j) + (w-\overline{w}_j), \qquad w\in[\overline{w}_j,+\infty]\\
i_j(w) &= \frac{1}{\theta_j}(\frac{p_j(w)}{p_j'(w)} - w - 1) + \nu_j, \qquad j=1,2,\cdots,n\\
g_j(i) &= \frac{\theta_j}{2}(i_j(w)-\nu_j)^2,
\end{aligned}
$$

with the upper boundary condition

$$
p_j'(\overline{w}_j) = 1,\ p_j''(\overline{w}_j) = 0, \qquad j=1,2,\cdots,n. \tag{4.17}
$$

The lower boundary condition is case dependent.

- For the liquidation case, the lower boundary condition is

$$
p_j(0) = l_j, \qquad j=1,2,\cdots,n. \tag{4.18}
$$

- For the equity financing case, the lower boundary condition is

$$
p_j(0) = p_j(m_j) - \phi_j - (1+\gamma_j)m_j,\ p_j'(m_j) = 1+\gamma_j, \qquad j=1,2,\cdots,n. \tag{4.19}
$$

- For the credit line case, the system of ODEs changes to

$$
\begin{aligned}
r_j p_j(w) &= (i_j(w) - \delta_j)(p_j(w) - w p_j'(w)) + ((r_j - \lambda_j)w + \mu_j - i_j(w) - g_j(i_j(w))p_j'(w) \\
&\quad + \frac{\sigma_j^2}{2} p_j''(w) + \sum_{k=1, k\neq j}^{n} \xi_{jk}(p_k(w) - p_j(w)), \qquad w \in [0, \overline{w}_j] \qquad (4.20) \\
r_j p_j(w) &= (i_j(w) - \delta_j)(p_j(w) - w p_j'(w)) + ((r_j + \alpha)w + \mu_j - i_j(w) - g_j(i_j(w))p_j'(w) \\
&\quad + \frac{\sigma_j^2}{2} p_j''(w) + \sum_{k=1, k\neq j}^{n} \xi_{jk}(p_k(w) - p_j(w)), \qquad w \in [-c, 0) \\
p_j(w) &= p_j(\overline{w}_j) + (w - \overline{w}_j), \qquad w \in [\overline{w}_j, +\infty] \\
i_j(w) &= \frac{1}{\theta_j}\left(\frac{p_j(w)}{p_j'(w)} - w - 1\right) + \nu_j, \qquad j = 1, 2, \cdots, n \\
g_j(i) &= \frac{\theta_j}{2}(i_j(w) - \nu_j)^2.
\end{aligned}
$$

with the upper boundary condition

$$p_j'(\overline{w}_j) = 1, \ p_j''(\overline{w}_j) = 0, \qquad j = 1, 2, \cdots, n, \qquad (4.21)$$

and lower boundary condition

$$p_j(-c) = p_j(m_j) - \phi_j - (1 + \gamma_j)(m_j + c), \ p_j'(m_j) = 1 + \gamma_j, \qquad j = 1, 2, \cdots, n. \qquad (4.22)$$

In addition, $p_j(w)$ is continuous and smooth everywhere, including at $w = 0$, which gives two additional boundary conditions.

Firstly I introduce a new notation as $p_j^{(k)}(w)$, where the superscript (k) denotes the number of iterations for the value function $p_j(w)$. Then the numerical methods to solve the system of ODEs are as follows:

1. Initialize $p_j(w)$. In my case, I set all $p_j(w) = 0$ at the beginning.

2. Solve for $p_j^{(0)}$ without μ changes (i.e., setting $\xi_{ij} = 0$) for $j = 1, 2, \cdots, n$. This is essentially solving a nonlinear ODE with a free-boundary condition about $\overline{w}_j^{(0)}$. The steps are

   (1) Postulate the value of the free (upper) boundary $\overline{w}_j^{(0)}$, and use the ODE (4.16) together with the conditions $p_j'(\overline{w}_j^{(0)}) = 1, p_j''(\overline{w}_j^{(0)}) = 0$ to derive the value of $p(\overline{w}_j)$;

   (2) Solve the corresponding initial value problem using the Runge-Kutta method;

(3) Search for the $\overline{w}_j$ that will satisfy the boundary condition for $p_j(w)$ at $w = 0$, e.g., by binary search;

(4) In equity financing case, search for $\overline{w}_j$ jointly with the other free boundaries by imposing additional conditions at the free boundaries. That is, when we finish step (2) using a postulated $\overline{w}_j$, we use the equation $p_j'(m_j) = 1 + \gamma$ to find m_j, then we use the equation $p_j(0) = p_j(m_j) - \phi - (1+\gamma)m_j$ to binary search the correct $\overline{w}_j$ in this iteration.

(5) In credit line case, firstly solve the ODE when $w > 0$ using step (2) with a postulated $\overline{w}_j$, then use the boundary values of $p_j(0)$ and $p_j'(0)$ as the initial values to solve the ODE when $w < 0$, which is similar to the ODE when $w > 0$. Finally at the credit limit $w = -c$, we use step (4) to solve the boundary condition for equity issuance.

3. Given $p_j^{(0)}$ for $j = 1, 2, \cdots, n$, calculate $p_j^{(1)}(w)$ using the actual transition intensity matrix Ξ. The results are the $p_j^{(1)}$ with the free boundary $\overline{w}_j^{(1)}$ for $j = 1, 2, \cdots, n$;

4. Iteratively solving the value functions until converge. The convergency condition is

$$\max_j \left[\sup_w \left(p_j^{(k+1)}(\overline{w}_j^{(k)}) - p_j^{(k)}(\overline{w}_j^{(k)}) \right) \right] < \epsilon = 10^{-3}, \tag{4.23}$$

or the number of iterations $k > k_{\max}$. In my codes, I set $k_{\max} = 100$.

4.3 Parameters Restrictions

Both the model in Bolton et al. (2009) and my extension have parameter values restrictions. The models can generate infinite firm values for many parameter combinations, e.g., the expected productivity shock μ is too high or the risk free rate is too low. I believe the underlying reason is that adjustment costs may not always be convex and the production technology may exhibit decreasing returns to scale in practice, but these functional forms substantially complicate the analysis and do not permit a closed-form characterization of investment and financing policies. The homogeneity assumption (e.g. constant return to scale production technology) helps reduce the problem dimension, which is easier to solve. However, these simplified assumptions are not robust enough and may require high investment adjustment cost parameters θ to make the firm value finite. Eberly et al. (2009) discuss this parameter restriction problem for constant-return-to-scale production technology.

In this section, I discuss the parameter spaces in which the models of Bolton et al. (2009) and my extension with time-varying investment opportunity work. The idea is that there are analytical solutions for firm value and investment policy functions in which the Modigliani-Miller Theorem holds, which is called neoclassical benchmark in Bolton et al. (2009). From the analytical solutions, it is easy to find the parameter restriction to make the solution meaningful. Since in the frictionless market, the firm value is obviously higher than that facing external financing cost, the parameter combinations satisfying the restrictions for the neoclassical benchmark will certainly work for BCW model and my extension. Finally, for the regime-switching model, I can focus on the regime with highest $\mu_{\max}$. If the the BCW model works for the parameter combination with $\mu_{\max}$, the regime-switching model will also work.

Bolton et al. (2009) summarize the solution for the neoclassical q theory of investment, in which the Modigliani-Miller Theorem holds. The firm's first-best investment policy is given by $I^{FB} = i^{FB}K$, where

$$i^{FB} = r + \delta - \sqrt{(r+\delta)^2 - 2(\mu - (r+\delta))/\theta}. \tag{4.24}$$

The value of the firm's capital stock is $q^{FB}K$, where q^{FB} is Tobin's q given by:

$$q^{FB} = 1 + \theta i^{FB}. \tag{4.25}$$

However, in my extended model with time-varying investment opportunities and the model in Bolton et al. (2010a) with time-varying financing opportunities, the investment adjustment cost is assumed to be $G(I,K) = \frac{\theta}{2}(\frac{I}{K} - \nu)^2 K$ rather than $G(I,K) = \frac{\theta}{2}(\frac{I}{K})^2 K$ in Bolton et al. (2009). The reasons to make the changes are discussed in Section 3.2. Under the new formulation of investment adjustment cost, the first-best Tobin's q and investment-capital ratio i^{FB} satisfy the following system of equations according to Bolton et al. (2010a):

$$\begin{aligned} rq^{FB} &= \mu - i^{FB} - \frac{1}{2}\theta(i^{FB} - \nu)^2 + q^{FB}(i^{FB} - \delta), \\ q^{FB} &= 1 + \theta(i^{FB} - \nu). \end{aligned} \tag{4.26}$$

This leads to

$$i^{FB} = r + \delta - \sqrt{(r+\delta)^2 - 2(\mu - (r+\delta))/\theta - (2r\nu + 2\delta\nu - \nu^2)}. \tag{4.27}$$

Therefore, to ensure that the first-best investment policy is well defined, the following parameter restriction has to be imposed for the constant investment opportunity model:

$$(r+\delta)^2 - 2(\mu - (r+\delta))/\theta - (2r\nu + 2\delta\nu - \nu^2) \geq 0. \tag{4.28}$$

Therefore, as long as (4.28) is satisfied for a certain combination of parameters, the regime-switching model will also work. However, this is not a tight bound. Even some parameter combinations violates (4.28), the regime-switching will still work.

I can be simplified (4.28) to

$$\theta \geq \frac{2(\mu - \delta - r)}{(r + \delta - \nu)^2}. \tag{4.29}$$

Intuitively, if θ is large enough, both the models of Bolton et al. (2009) and my extension will work. Among these variables, μ, r, δ can be explicitly estimated, while θ, ν are implicit and created to match the empirical moments in the neoclassical investment models. To illustrate the relations among these parameters to satisfy the restriction, I can fixed one of the two implicit parameters $\nu = 0.12$, and see how the other parameter θ should be with respect to the three explicitly estimated parameters μ, r, and δ, in order to satisfy the parameter restriction of (4.29). The results are shown in Figure 4-1.

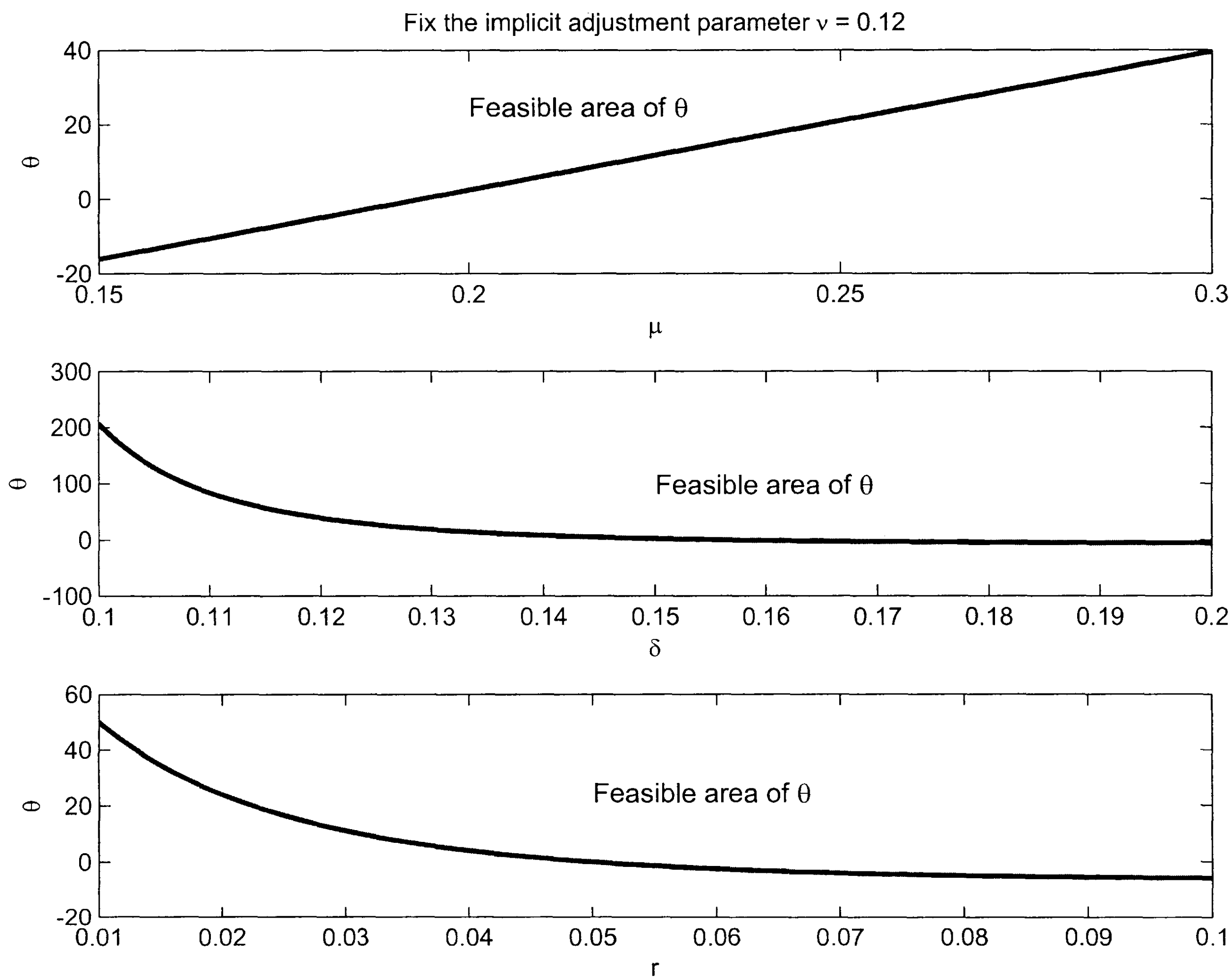

Figure 4-1: The feasible region of θ with respect to one change parameter that can be explicitly estimated. The default values are $r = 0.434$, $\mu = 0.205$, $\delta = 0.15$. The implicit parameter ν is fixed to be 0.12, which is consistent with the estimate in Eberly et al. (2009).

Chapter 5

Quantitative Analysis

Now I turn to quantitative analysis of my extended BCW model with time-varying investment opportunities. In my thesis, I focus on the mean-reverting model, i.e., the expected productivity shock μ_t is mean-reverting. As discussed in Section 3.1, in a continuous-time setting, μ_t follows an Ornstein-Uhlenbeck process. The discrete-time counterpart of the O-U process is an AR(1) process. There are mathematical relations between the parameters for the two process. The approximated regime-switching process should be derived from the discrete-time AR(1) process.

5.1 Parameter Estimates

I set the long-run mean and overall volatility of the risk-adjusted productivity shock to be $\overline{\mu} = 0.203$ and $\sigma = 0.09$. The persistence and volatility of the expected productivity shock μ_t are $\rho = 0.7$ and $\sigma_\mu = 0.1\sigma$, respectively. Consequently, I can also estimate the idiosyncratic volatility $\sigma_z = \sqrt{\sigma^2 - \sigma_\mu^2}$. Note that all these parameters are roughly calibrated from the financial reporting data[1], and annualized in discrete-time setting, so that I can transform them to a regime-switching process of μ_t.

The risk-free rate is $r = 4.34\%$, the rate of depreciation of capital is $\delta = 15\%$, the investment adjustment cost parameters are $\theta = 6.902$ and $\nu = 12\%$. These parameters are chosen by following Eberly et al. (2009). According to Bolton et al. (2010a), I set the cash-carrying cost to be $\lambda = 1.5\%$. The firm's liquidation value is $l = 0.9$ as suggested in Hennessy, Levy and Whited (2007). The proportional financing cost $\gamma = 6\%$ as suggested in Sufi (2009) and the fixed cost of financing is

[1]Source: `http://www.chk.com/Investors/Pages/Reports.aspx`

$\phi = 1\%$, which jointly generate average equity financing costs that are consistent with the data.

Following Bolton, Chen and Wang (2010b), I create the Table 5.1 that recapitulates all the key variables and parameters in the model. As we can see, some parameters are estimated directly, some are estimated indirectly, and the others are from empirical research papers. Developing a more strict and unified method to calibrate the BCW model and my extension is an important future research.

Variable	Symbol	Parameter	Symbol	Value
General				
Capital Stock	K	Risk-free rate	r	4.34%
Cash holding	W	Rate of depreciation	δ	15%
Investment	I	Long-run mean productivity shock	$\overline{\mu}$	0.203
Cumulative productivity shock	A	Overall volatility of productivity shock	σ	0.09
Investment adjustment cost	G	Persistence of expected productivity shock	ρ	0.7
Cumulative operating profit	Y	Volatility of expected productivity shock	σ_μ	0.1σ
Cumulative external financing	H	Idiosyncratic volatility	σ_z	$\sqrt{\sigma^2 - \sigma_\mu^2}$
Cumulative external financing cost	X	Adjustment cost parameter 1	θ	6.902
Cumulative payout	U	Adjustment cost parameter 2	ν	0.12
Firm value	P	Proportional cash-carrying cost	λ	1.5%
Liquidation case				
		Capital liquidation value	l	0.9
Equity case				
		Fixed financing cost	ϕ	1%
		Proportional financing cost	γ	6%
Credit line case				
		Credit line limit	c	20%
		Credit line spread over r	α	1.5%

Table 5.1: This table summarizes the symbols for the key variables used in the model and the parameter values. For each upper-case variable in the left column (except K, A, and F), I use its lower case to denote the ratio of this variable to capital. All the boundary variables are in terms of the cash-capital ratio w_t.

5.2 Approximation of the Mean-Reverting Productivity Shock

As mentioned before, to calibrate the time-varying productivity shock process, I start from a discrete-time version at the annual frequency:

$$dA_t = \mu_t + \epsilon_z, \qquad \epsilon_z \sim \mathcal{N}(0, \sigma_z^2), \tag{5.1}$$

$$\mu_t = \overline{\mu}(1-\rho) + \rho\mu_{t-1} + \epsilon_\mu, \qquad \epsilon_\mu \sim \mathcal{N}(0, \sigma_\mu^2). \tag{5.2}$$

where the parameter values are in Table 5.1. The above assumption shows that the productivity shock is time-varying, since the expected productivity shock μ_t follows an AR(1) process in the discrete-time setting. Using the methods I describe in Section 3.1, the process can be approximated as a continuous-time regime-switching process with arbitrary number of regimes. Intuitively, the more regimes, the more accurate for the results. Also, the model solution methods proposed in Chapter 4 are independent of the number of regimes. However, more regimes means longer time for computation and I have to make tradeoff between the result accuracy and solution times. In the following quantitative analysis, I use five regimes.

Figure 5-1 plots the dynamics of the approximated Markov regime-switching process with five regimes. In Panel A, the process has five possible regimes, each having a different expected productivity shock μ_t. They are labeled from regime 1 to regime 5. Panel B shows the conditional transition probabilities for each of the five regimes. Due to the symmetric property of the normal distribution, the histograms of the conditional transition probabilities also turn out to be symmetric. Panel C shows the long-run stationary distribution of the approximated discrete-time regime-switching process. The distribution is again symmetric and the firm stays in the centered regime more frequently than other regimes. Finally, in Panel D, I transform the discrete-time regime-switching process into a continuous-time version, using the method of Jarrow et al. (1997) (an approximation based on the assumption that the probability of more than one change of regime is close to zero within the period Δ. Here Δ is one year).

5.3 Comparison between the Extended Model and the BCW Model

In this section, I make some comparisons of the policy differences between my extended model with time-varying investment opportunities, and the BCW model with constant investment opportunities. The steps are as follows:

1. In my extended model with time-varying investment opportunities, I assume the expected productivity shock μ_t is mean-reverting with long-run mean $\overline{\mu}$. The parameters values are in Table 5.1. The approximate the mean-reverting process of μ_t with a regime-switching process with five regimes. Each regime have different μ. The dynamics of the regime-switching process are illustrate in Figure 5-1.

2. Solve the regime-switching model for the liquidation, equity financing and credit line cases. The numerical methods introduced in Chapter 4. There will be five curves corresponding to

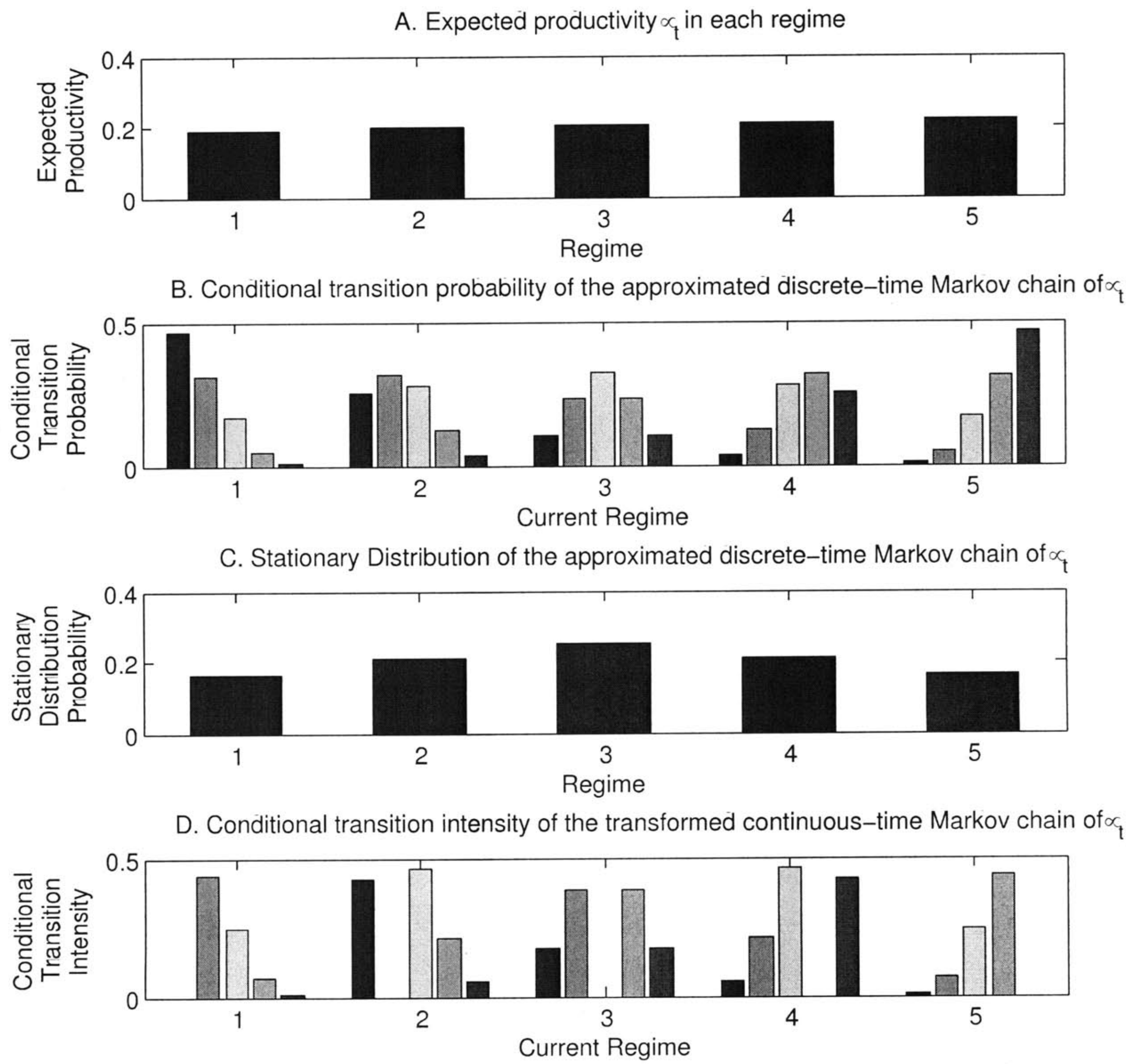

Figure 5-1: Graphical illustration of the mean-reverting productivity shock approximated by a regime-switching process

the five regimes of μ in each case. For the ease of illustration and comparison, I only get the two extreme curves — one corresponding to the highest μ (denoted as μ_H), and the other corresponding to the lowest μ (denoted as μ_L).

3. Derive the policies using the BCW model with constant investment opportunities for the liquidation, equity financing and credit line cases. Here I use $\mu = \overline{\mu}$, and all other parameters are the same as those in my extended model.

4. Compare the policy differences in terms of the firm value function $p(w)$, marginal value of cash $p'(w)$, investment ratio $i(w)$, and the investment sensitivity $i'(w)$ for the paired curves for μ_H and μ_L of my extended model, as well as the curves of the BCW model. The differences will be discussed separately for three cases. Note that there is a mean-preserving relation that $\mu_H + \mu_L = 2\overline{\mu}$.

As we will see soon, the solid curves are derived from my extended model with time-varying investment opportunities. The red and blue colors correspond to the regimes with $\mu = \mu_H$ and $\mu = \mu_L$ respectively. The green and dashed curves represent the policies from the BCW model with $\mu = \overline{\mu}$. Intuitively, the major difference in my extended model is that the policies will not only depend on the cash-capital ratio w, but also depend on the regime j that the firm is currently in. With time-varying investment opportunities incorporated, the firm has to consider the persistent effects of the changing investment opportunities and adjust its investment and financing policies accordingly. More specifically, in my model specification, the firm must be aware that it can switching to another regime with different μ and this process will continue. This feature makes the firm to consider more than only its cash holding before making the decision. For example, when the firm expect better investment opportunities in the future, it is likely to invest more now, etc. The differences are in many aspects, and I will discuss them in detail for each of the liquidation, equity financing and credit line cases.

5.3.1 Liquidation Case

Figure 5-2 plots the firm value-capital ratio $p(w)$, the marginal value of cash $p'(w)$, the investment-capital ratio $i(w)$, and investment-cash sensitivity $i'(w)$ for both the BCW model with constant investment opportunities and my extended model with time-varying investment opportunities, in the liquidation case.

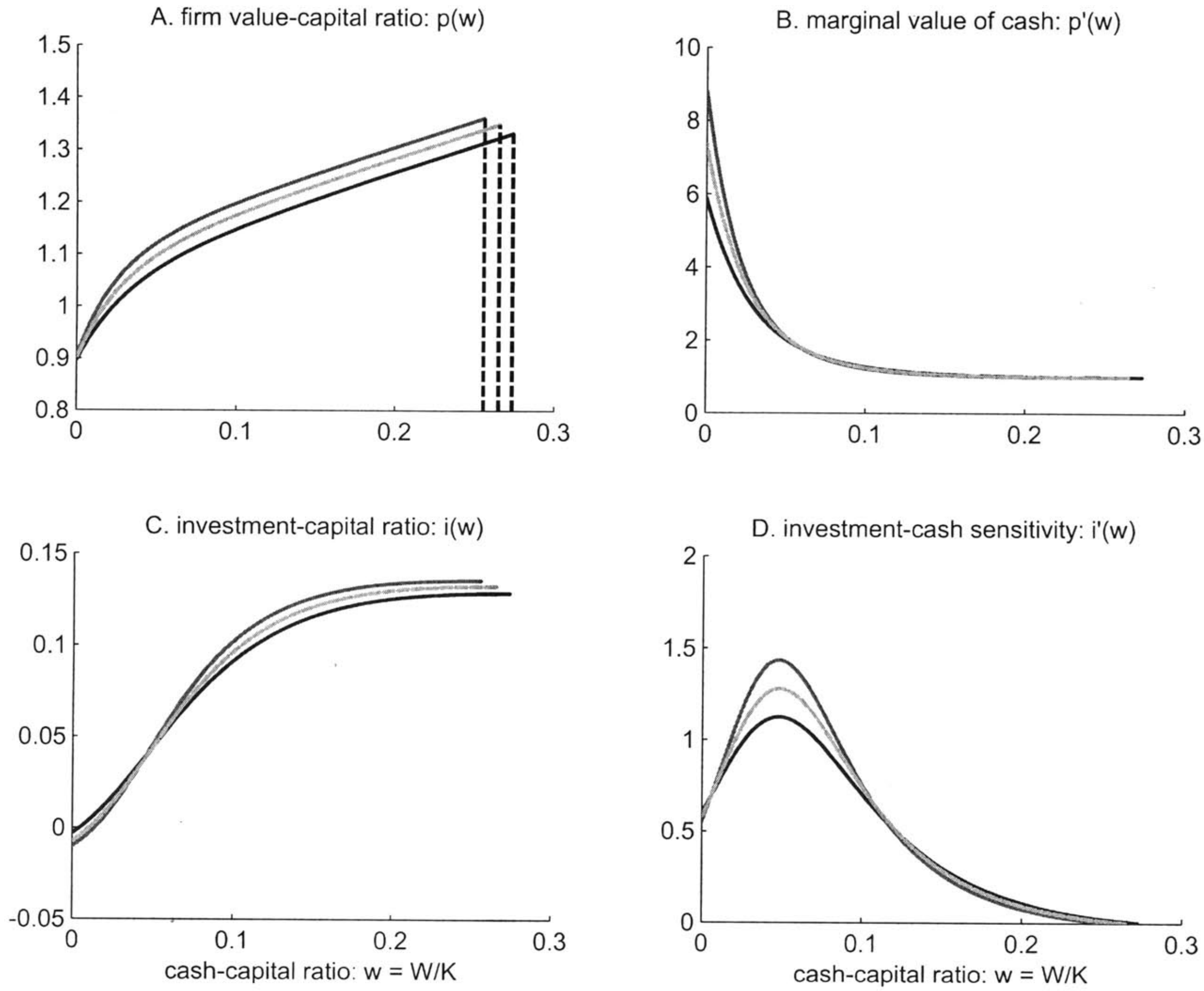

Figure 5-2: Policy comparisons between the BCW model and my extended model in the liquidation case. The solid curves are derived from my extended model with time-varying investment opportunities. The red and blue colors correspond to the regimes with $\mu = \mu_H$ and $\mu = \mu_L$ respectively. The green and dashed curves represent the policies from the BCW model with $\mu = \overline{\mu}$. There is a mean-preserving relation that $2\overline{\mu} = \mu_H + \mu_L$. The black, vertical and dashed lines are the payout boundaries.

In Figure 5-2, for the green dashed curves from the BCW model, as the cash-capital ratio w rises, the financing constraint gets relaxed. As a result, both the the firm value $p(w)$ and investment $i(w)$ rise with w, while the net marginal value of cash and the investment-cash sensitivity fall with w. The transition intensity ξ_{jk} in my extended model naturally has no impact on the results of the BCW model. I consider the dashed curves from the BCW model as the benchmarks.

In Panel A, the BCW model determines the firm value solely with respect to w. However, my extended model provides an additional parameter, the regime or the μ, to jointly determine the firm value with w. This yields a richer prescription since in practice, the investment opportunities are stochastic, and my solutions can provides a good approximation in many situations. Not surprisingly, the firm value increases with μ for every w. That is, when the firm observes that currently it have better investment opportunities than the long-run average ($\mu_H > \overline{\mu}$), it values itself more than using the BCW model. The effect is significant in the valuation process. When w is high enough, the firm value makes roughly parallel shift with μ. However, then the cash approaches zero, the difference of firm values become small with μ. This is because the extreme financing situation (liquidation) reduce the effects of investment opportunities on the firm's value, when the cash position is sufficient low.

In addition, my extended model give a unique payout boundary for each of the regime, rather than a single payout boundary from the BCW model. The payout boundary decrease with μ. When the firm is in Regime H with high μ_H, since its financial constraint is greatly relax, it concerns more about the benefits of reducing cash (due to the cash carrying cost λ), since its high expected productivity shock μ_H can bring itself a better expected future cash-in flow. In addition, it may has worse investment opportunities in the future due to the transition intensity ξ_{jk}. Thus it is more willing to pay out dividend and thus a smaller payout boundary than that in Regime L, and the single payout boundary derived from the BCW model. On the contrary, if the firm is in Regime L with low μ_L, even if it has not much financial constraint, it still wants to hold more cash to prepare for better investment opportunities, since the benefits of this outweighs the benefits of the reducing cash. Of course, the difference of payout boundaries also related to the transition intensity ξ_{jk}. Intuitively, if the ξ_{jk} is small enough, the firm may behave less opportunistically when choosing its payout policies with the hope that better investment opportunities are less likely. Thus it may tend to reduce its payout boundary.

Panel B plots the marginal value of cash $p'(w)$ for regimes H and L from my model, as well as that from the BCW model. Again, my model provides an additional dimension μ to determine the marginal value of cash. The common things between my extended model and the BCW model is that, as w approaches 0, the marginal value of cash rises significantly because an extra dollar of cash can reduce the chance of costly liquidation. As the financial constraint is relaxed (e.g., w is sufficiently large), the marginal values of cash all approach its face value. The difference in my extended model is that, when the firm is in Regime H with high μ_H, even if it is likely to switch into other regimes with low μ, it values more on an additional dollar of cash. This is because the marginal value of cash not only reflects how close the firm is to running out of cash, but also the going concern value for the firm. The firm with better investment opportunities now place higher value on its continuation, and to avoid the costly liquidation. Otherwise, it will permanently lose its future better growth opportunity. The marginal value of cash from the BCW model lie between the curves for Regime L and H from my extended model. Not surprising, the BCW model has limited dimension to determine the marginal value of cash so that it has to act on an average basis. The mean-preserving characteristics of μ_H, μ_L and $\overline{\mu}$ lead to the results.

Panel C and D of Figure 5-2 plot the investment-capital ration $i(w)$ and investment-cash sensitivity $i'(w)$. With sufficiently high w, the firm invest more and is less concerned with preserving cash when it is in Regime H with high μ_H. Now, combining the results from Panel A, I find an interesting things on how the firm responds to the possible worse investment opportunities in my parameter settings when there is high w. If the firm currently have good investment opportunities (in Regime H with high μ_H), but is likely to has worse investment opportunities in the future (switching to other regimes with low μ due to the transition intensity ξ_{jk}), it will (1) invest more than that if it is in Regime L with low μ_L, and also more that that with average $\overline{\mu}$ from the BCW model; (2) reduce dividend payout boundary because of two reasons. First, if it remains in Regime H, it has high μ_H so that it expects are a high cash in-flow for its investment. Second, if it switch to Regime L with low μ_L, it does not need that much cash holding for its investment since the investment opportunities after the switching become worse.

On the other end, as the firm depletes its cash (i.e. $w \to 0$), investment $i(w)$ for firms in Regime H is lower than that in Regime L, or the $i(w)$ derived from the BCW model. This result is closely related to the result on marginal value of cash in Panel B. Underinvestment acts as a risk management device. The value of managing risk in constrained periods (low w) is greater and hence underinvestment is more severe when the going concern value is higher, i.e. when the firm is likely to have better investment opportunities. The difference behavior at the lower and higher ends of w highlights the importance of dynamic risk management. Panel D shows that the investment-cash sensitivity $i'(w)$ is positive but non-monotonic in w for policies in my extended model and the BCW model.

5.3.2 Equity Financing Case

Figure 5-3 plots the firm value-capital ratio $p(w)$, the marginal value of cash $p'(w)$, the investment-capital ratio $i(w)$, and investment-cash sensitivity $i'(w)$ for both the BCW model with constant investment opportunities and my extended model with time-varying investment opportunities, in the equity financing case.

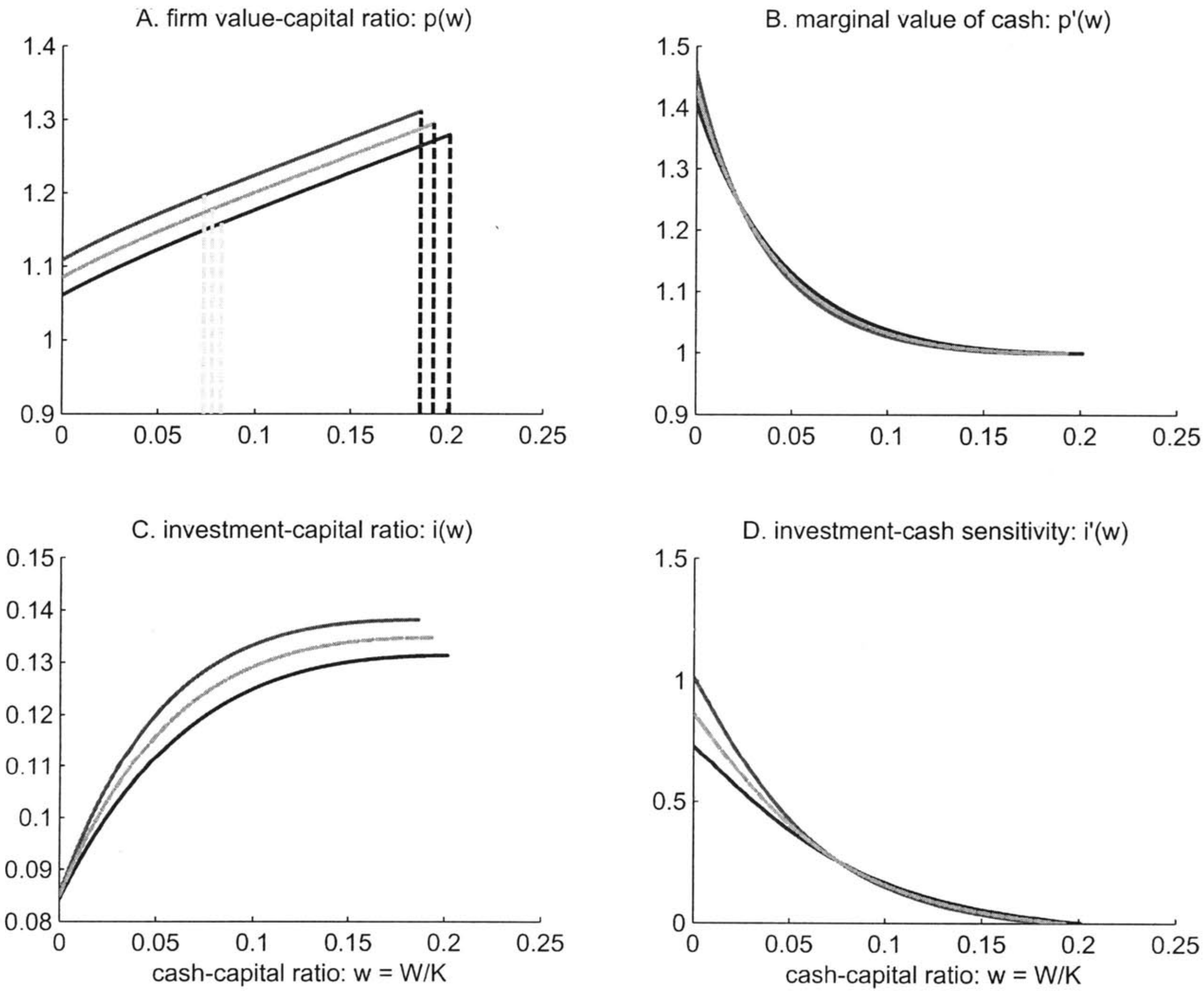

Figure 5-3: Policy comparisons between the BCW model and my extended model in the equity financing case. The red and blue colors correspond to the regimes with $\mu = \mu_H$ and $\mu = \mu_L$ respectively. The green and dashed curves represent the policies from the BCW model with $\mu = \overline{\mu}$. There is a mean-preserving relation that $2\overline{\mu} = \mu_H + \mu_L$. The black, vertical and dashed lines are the payout boundaries. The yellow, vertical and dashed lines are the equity issuance return to target.

I can follow the steps in the liquidation case to make comparisons in the equity financing case. To avoid repeating the same conclusions, I will only focus the different results in the equity financing case. With my parameterization, in Panel A, when the external financing is available, the firm's financing follows pecking order in that it always first uses internal funds before tapping external funds. The green dashed vertical lines are the return target for equity issuance m for each of the regimes from my extended model, as well as the single return to target from the BCW model. Again, the return to target from the BCW model lie between my return to targets for Regime H and L. In addition, due to the availability of external financing, the effect of extreme financing situation (liquidation) on firm value is significantly reduced, as compared to Figure 5-2 in the liquidation case. The effect of investment opportunities dominates so that the firm value makes nearly parallel shift with μ.

In Panel B, although the marginal value of cash from the BCW model still lies between, the marginal value of cash for Regime H is no longer higher than that of Regime L for every w. When w is sufficiently high, the marginal value of cash for Regime H becomes lower than that of Regime L because there is availability of external financing, thus the firm continuation value is reduced for high μ and high w. Also, the magnitude of marginal value of cash is significantly smaller that that in the liquidation case.

In Panel C, $i(w)$ is always higher for higher μ. Again, this is due to the discussion in Panel A that in presence of financial cost, whether the effect of financing constraint or the effect of investment opportunities on investment dominates. In the liquidation case, the financial constraint is so severe that in dominates both the firm value and investment when cash is low enough. Thus both the firm value and investment becomes nearly the same for different μ, either from my extended model or from the BCW model. However, in the equity financing case, the financial cost is greatly reduced. With my parameters values, it have so limited effect on firm value such that the firm value curves make nearly parallel shift with μ, as shown in Panel A. However, for the investment aspect, when w approaches zero, the effect of financial constraint is still significant enough that the difference of $i(w)$ become small with different μ.

5.3.3 Credit Line Case

Figure 5-4 plots the firm value-capital ratio $p(w)$, the marginal value of cash $p'(w)$, the investment-capital ratio $i(w)$, and investment-cash sensitivity $i'(w)$ for both the BCW model with constant investment opportunities and my extended model with time-varying investment opportunities, in the credit line case.

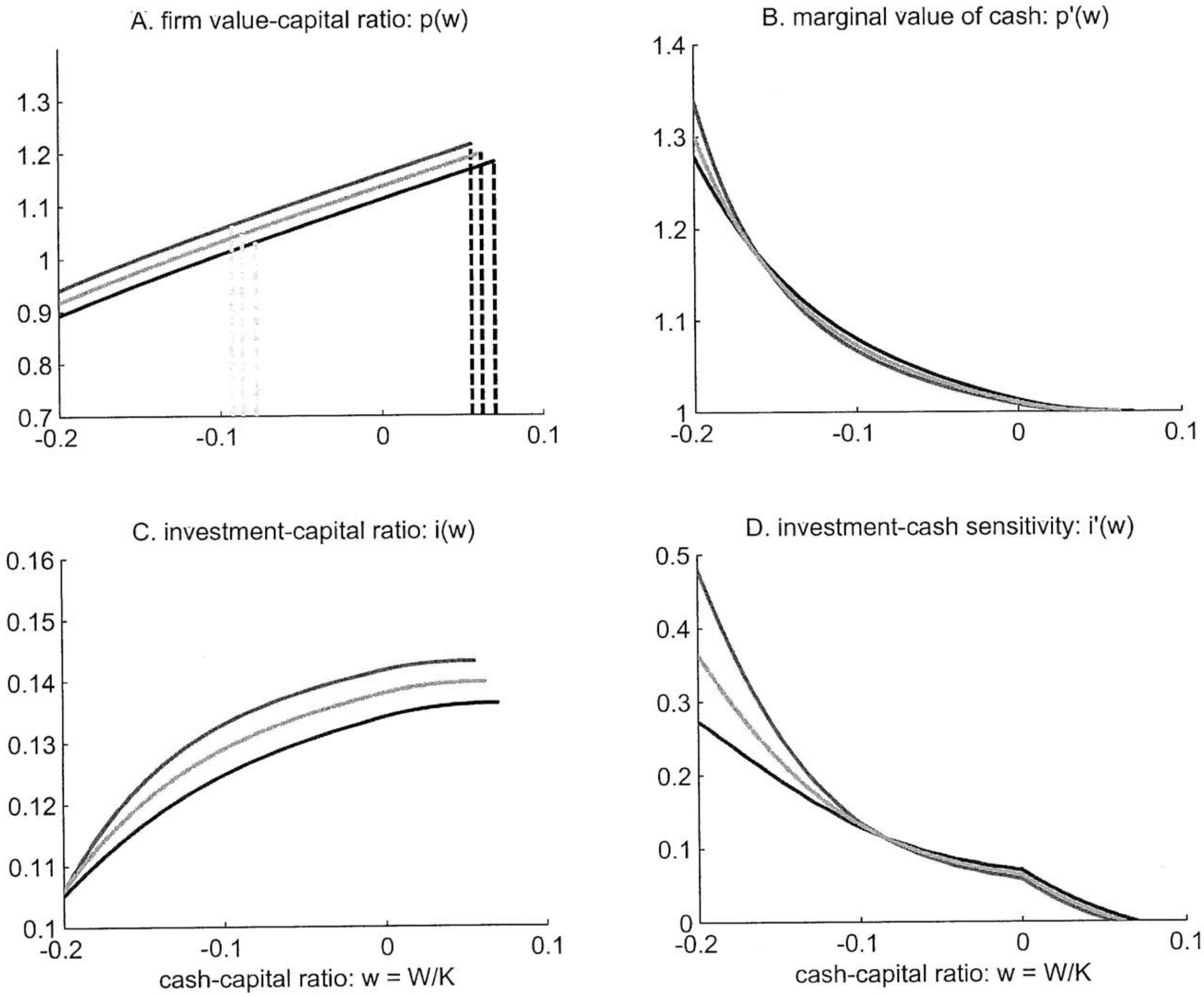

Figure 5-4: Policy comparisons between the BCW model and my extended model in the credit line case. The red and blue colors correspond to the regimes with $\mu = \mu_H$ and $\mu = \mu_L$ respectively. The green and dashed curves represent the policies from the BCW model with $\mu = \overline{\mu}$. There is a mean-preserving relation that $2\overline{\mu} = \mu_H + \mu_L$. The black, vertical and dashed lines are the payout boundaries. The yellow, vertical and dashed lines are the equity issuance return to target.

The credit line case is very similar to the equity financing case, since the firm will issue equity at the credit line limit. Most of the conclusions are the same. The major difference is that the firm's financial constraint is further relaxed due to the availability of borrowing money. Then the firm will follow the pecking order: (1) internal financing (cash holding); (2) credit line (or issuing debt); (3) equity financing. Keeping all other things the same, in the credit line case, the firm has higher firm value and less marginal value of cash, as well as invests more aggressively, than the firm which can only issue equity.

5.4 Optimal Policies

In this section, I will discuss the optimal policies derived from the extended BCW model with time-varying investment opportunities. Since my model is based on the BCW model, many of the discussions here will be similar to those in the Quantitative Analysis section in Bolton et al. (2009).

5.4.1 Liquidation Case

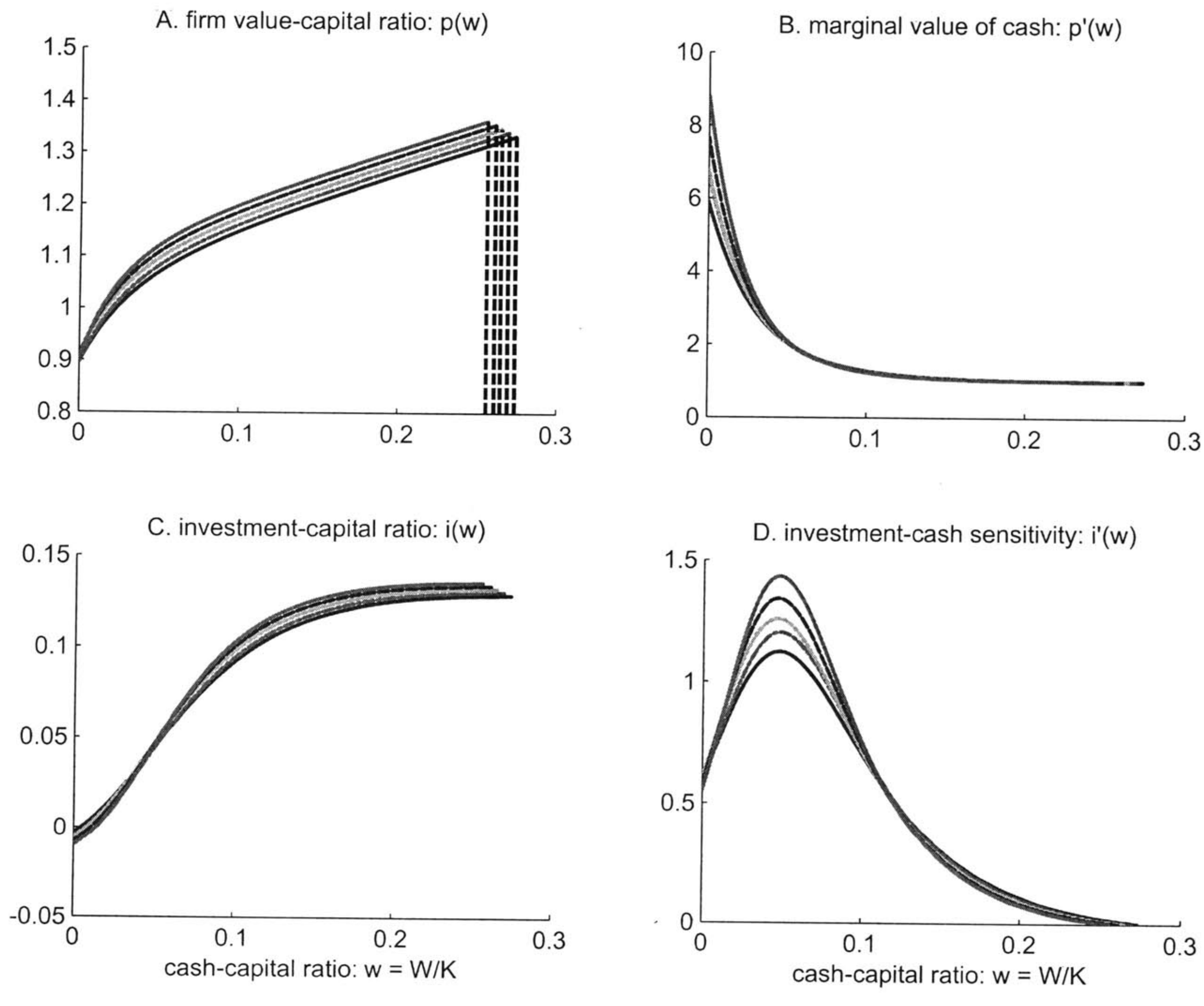

Figure 5-5: Optimal policies for my extended model with time-varying investment opportunities in the liquidation case. The mean-reverting process of the expected productivity shock μ is approximated with a continuous-time regime-switching process with five regimes. The parameter values are shown in Table 5.1. The different combinations of curve color and style represents different regimes. From Panel A, the higher curve corresponds to a regime with higher μ.

Figure 5-5 plots the solution of my extended model in the liquidation case. In Panel A, the firm's value-capital ratio $p_j(w)$ starts at $l = 0.9$ (liquidation value) for all five regimes, when cash balance is equal to 0. $p_j(w)$ for each regime of μ is concave in the region between 0 and the endogenous payout boundary $\overline{w}_j$, and becomes linear with slope 1 beyond the payout boundary ($w > \overline{w}_j$). The payout boundaries $\overline{w}_j$ are decreasing with regimes with higher μ, and I have explain the reasons in Section 5.3, which are related to the transition intensity ξ_{jk}. As I will argue in Section 5.5.1, the firm will never liquidate before its cash balance hits 0.

Panel B of Figure 5-5 plots the marginal value of cash $p'_j(w) = P_W(K, W, j)$. The marginal value of cash increases as the firm becomes more constrained and liquidation becomes more likely. It also confirms that the firm value is concave in the internal financing region ($p''_j(w) < 0$). The external financing constraint makes the firm hold cash today in order to reduce the likelihood that it will be liquidated in the future, which effectively induces "risk aversion" for the firm. Intuitively, the marginal cost from a smaller cash holding is higher than the marginal benefit from a larger cash holding because the increase in the likelihood of liquidation outweighs the benefit from otherwise. In addition, $p'_j(w)$ increases for the firm in regimes with higher μ when the cash inventory is low. This is because the firm has more going concern when it is in regime with higher μ, which are discussed in details in Section 5.3.1.

Panel C and D plot the investment-capital ratio $i(w)$ and sensitivity $i'(w)$, and illustrate under-investment due to the extreme external financing constraints and time-varying investment opportunities. When w is sufficiently low the firm will disinvest by selling assets to raise cash and move away from the liquidation boundary, especially in the regimes with high μ. Note that disinvestment is costly not only because the firm is underinvesting but also because it incurs physical adjustment costs when lowering its capital stock. The firm tries very hard not to be forced into liquidation. At the payout boundary, the firm is trading off the cash-carrying costs, the cost of underinvestment and the probability to enter another regime with different investment opportunities. It will optimally choose to hoard more cash and invest more at the payout boundary when the cash-carrying cost λ is lower. Also, it will invest more when it have higher μ but hold more cash when it has lower μ.

5.4.2 Equity Financing Case

Next I consider the setting where the firm is allowed to issue equity. Figure 5-6 displays the solutions of my extended model with five regimes of μ, fixed financing cost $\phi = 1\%$ and marginal financing cost $\gamma = 6\%$. Observe that at the financing boundary $\underline{w}_j = 0$, the firm's value-capital ratio $p_j(w)$ is

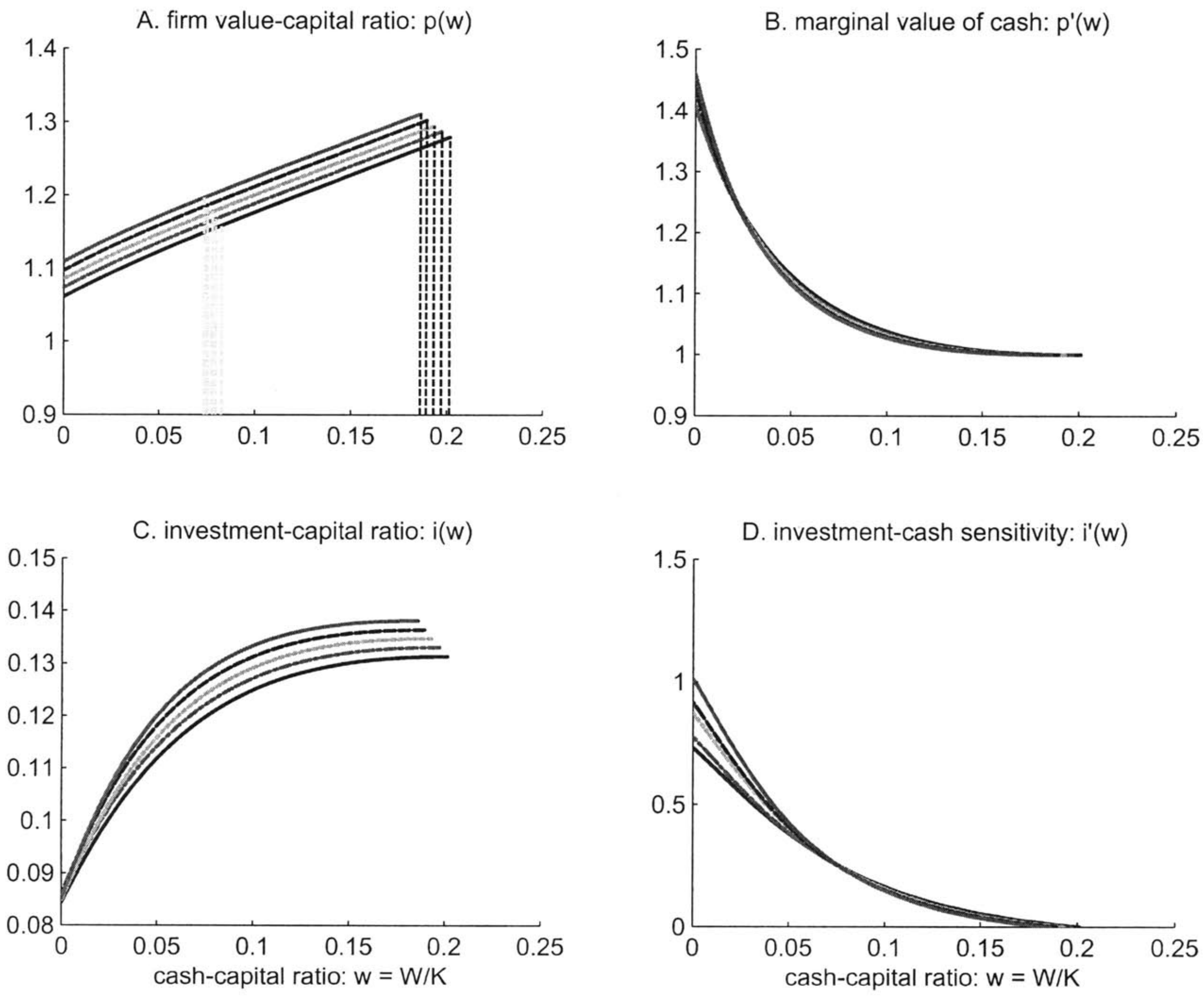

Figure 5-6: Optimal policies for my extended model with time-varying investment opportunities in the equity financing case. The mean-reverting process of the expected productivity shock μ is approximated with a continuous-time regime-switching process with five regimes. The parameter values are shown in Table 5.1. The different combinations of curve color and style represents different regimes. From Panel A, the higher curve corresponds to a regime with higher μ.

strictly higher than l, so that external equity financing is preferred to liquidation under this model model parameterization. Comparing with the liquidation case, I find that the endogenous payout boundary $\overline{w}_j$ are lower. Not surprisingly, firms are more willing to pay out cash when they can raise new funds in the future. The firm's optimal return cash-capital ratio m_j are marked by the vertical, yellow, and dashed lines in Panel A. When the firm is in regimes with lower μ, it will increase both its payout boundary $\overline{w}_j$ and return cash-capital ratio m_j in order to hold more cash for future possibly better investment opportunities.

Panel B plots the marginal value of cash $p_j'(w)$ for different regimes, which are positive and decreasing between 0 and $\overline{w}_j$. Conditional on issuing equity and having paid the fixed financing cost, the firm optimally chooses the return cash-capital ratio m_j such that the marginal value of cash $p_j'(m_j)$ is equal to the marginal cost of financing $1+\gamma$. To the left of the return cash-capital ratio m, the marginal value of cash $p_j'(w)$ lies above $1+\gamma$, reflecting the fact that the fixed cost component in raising equity increases the marginal value of cash. Section 5.5.2 displays the differences for firms using the optimal and non-optimal equity financing policies (due to the return to ratio m_j).

The investment-capital ratio $i_j(w)$ is increasing in w, and reaches the peak at the payout boundary $\overline{w}_j$. Higher fixed cost of financing increases the severity of financing constraints, therefore leading to more underinvestment. This is particularly true in the region to the left of the return cash-capital ratio m_j, where the investment-capital ratio $i_j(w)$ drops rapidly. Another observation is that at low w region, the firm in regimes with higher μ invest more, since there is available equity financing when cash depletes, and the firm worries less about the going concern and losing future better investment opportunities.

5.4.3 Credit Line Case

Figure 5-7 describes the effects of credit line. First, having access to a credit line increases $p_j(w)$ by lowering the cost of financing. Second, the firm hoards significantly less cash when it has access to a credit line: the payout boundary w drops when the credit line increases from $c=0$ to $c=20\%$. Third, for our parameter choices, the firm with a credit line issues more equity. Fourth, with the credit line, the marginal value of cash at $w=0$ substantially lower than that without credit line ($c=0$).

Credit line also substantially mitigates the firms underinvestment problem (see Panel C of Figure 5-7). With the credit line ($c=20\%$), in my parameterization, the firm never engage in

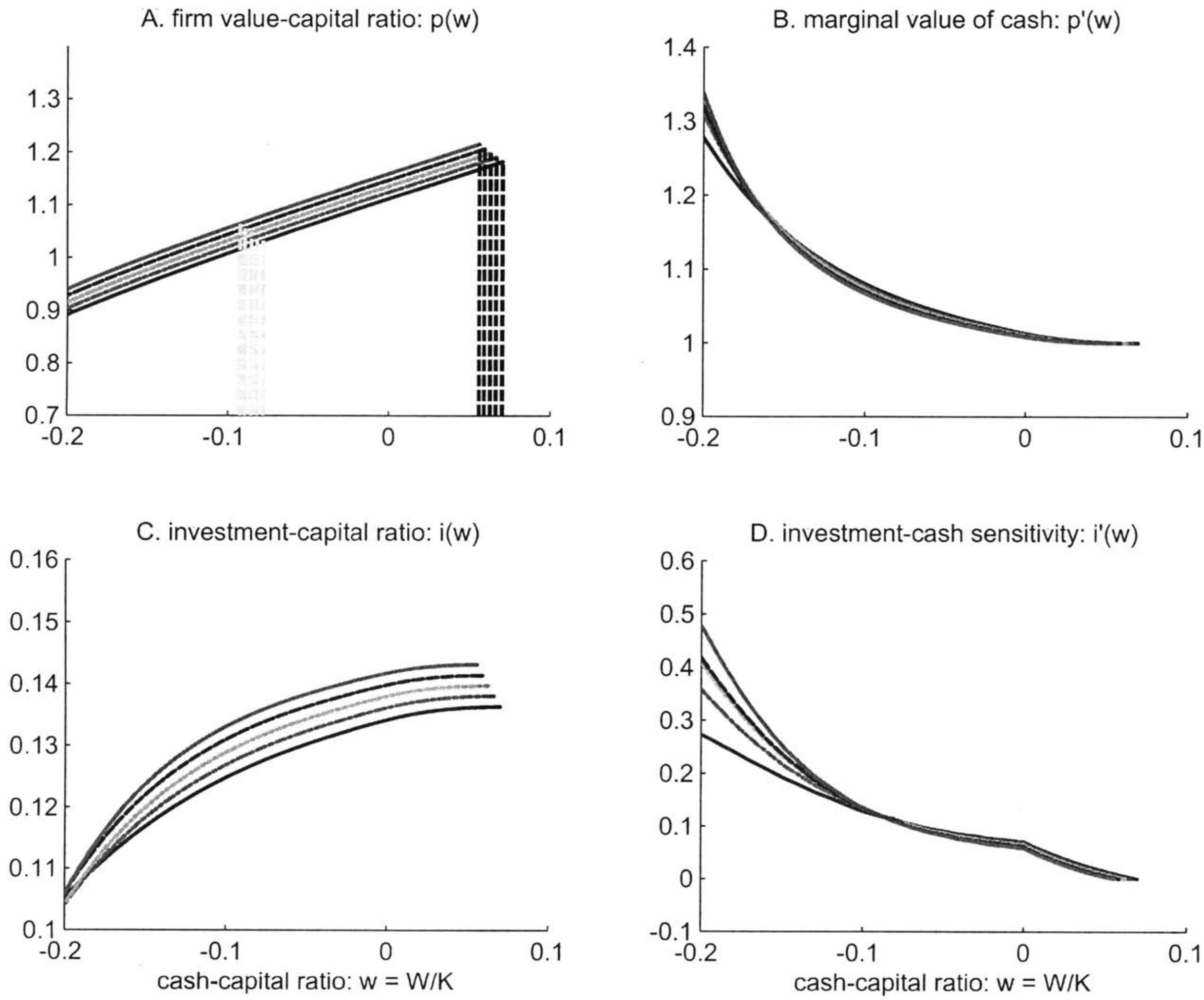

Figure 5-7: Optimal policies for my extended model with time-varying investment opportunities in the credit line case. The mean-reverting process of the expected productivity shock μ is approximated with a continuous-time regime-switching process with five regimes. The parameter values are shown in Table 5.1. The different combinations of curve color and style represents different regimes. From Panel A, the higher curve corresponds to a regime with higher μ.

asset sales. Also, the investment-cash sensitivity $i'_j(w)$ is also substantially lower when the firm has access to a credit line.

5.5 Comparison between Optimal Policies and Non-Optimal Policies for My Extended Model

5.5.1 Liquidation before Cash Depletes

According to Bolton et al. (2009), when the firm's cash-capital ratio w is less than or equal to the lower barrier $\underline{w}$, the firm either incurs financing costs to raise new funds or liquidates. Depending on parameter values, it may prefer either liquidation or refinancing by issuing new equity. Although the firm can choose to liquidate or raise external funds at any time, Bolton et al. (2009) show that it is optimal for the firm to wait until it runs out of cash, i.e. $w = 0$. The intuition is as follows. First, because investment incurs convex adjustment cost and the production is an efficient technology (in the absence of financing costs), the firm does not want to prematurely liquidate. Second, in the case of external financing, cash within the firm earns a below-market interest rate $(r - \lambda)$, while there is also time value for the external financing costs. Since investment is smooth (due to convex adjustment cost), the firm can always pay for any level of investment it desires with internal cash as long as $w > 0$. Thus, without any benefit for early issuance, it is always better to defer external financing as long as possible.

I make some numerical examples to verify that liquidation before the cash becomes zero is obviously not optimal. Figure 5-8 shows the firm value, marginal value of cash, investment, and investment sensitivity for the firm that liquidate when $w = 0.05$ or $w = 0.1$, and compare the differences from those of the firm using optimal policies, i.e. liquidate when $w = 0$.

5.5.2 Non-optimal Equity Issuance Amount

According to Bolton et al. (2009), in the equity financing case, since m_j is optimally chosen, the marginal value of the last dollar raised must equal one plus the marginal cost of external financing, $1 + \gamma$. This gives the following smoothing pasting boundary condition at m_j:

$$p'_j(m_j) = 1 + \gamma. \tag{5.3}$$

If m_j is not optimally chosen, the firm may issue too much ($m_j > m_{j,\text{optimal}}$) or too little

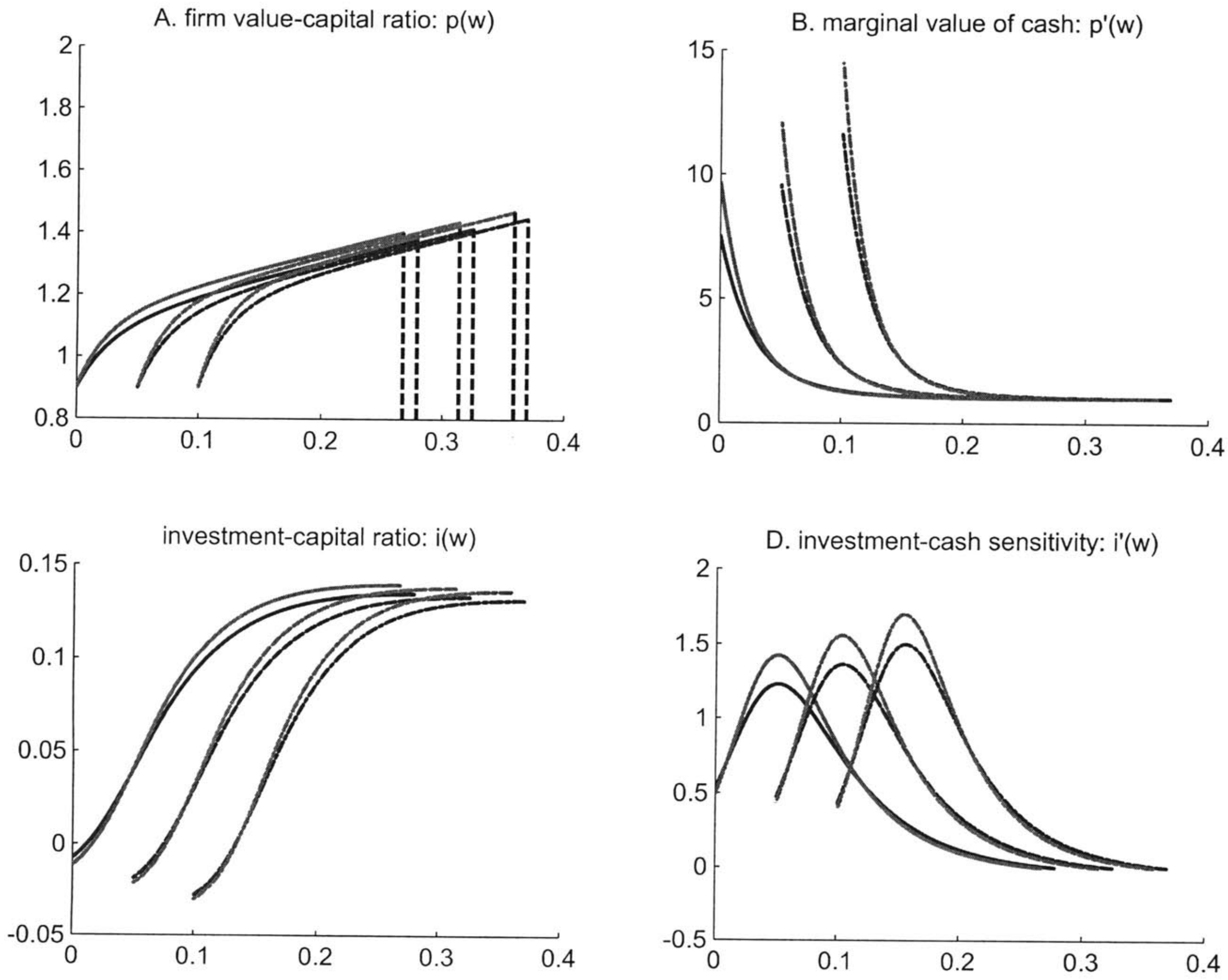

Figure 5-8: Liquidation before Cash Depletes

($m_j < m_{j,\text{optimal}}$) amount of equity. Since $p_j'(w)$ is decreasing with w, in the underfinancing case, $p_j'(m_j) > 1+\gamma$; in the overfinancing case, $p_j'(m_j) < 1+\gamma$. Figure 5-9 shows the firm value, marginal value of cash, investment, and investment sensitivity for the firm that overfinancing ($p_j'(m_j) = 1$) or underfinance $p_j'(m_j) = 1+2\gamma$ by equity, and compare the differences from those of the firm using optimal policies, i.e. satisfying $p_j'(m_j) = 1 + \gamma$.

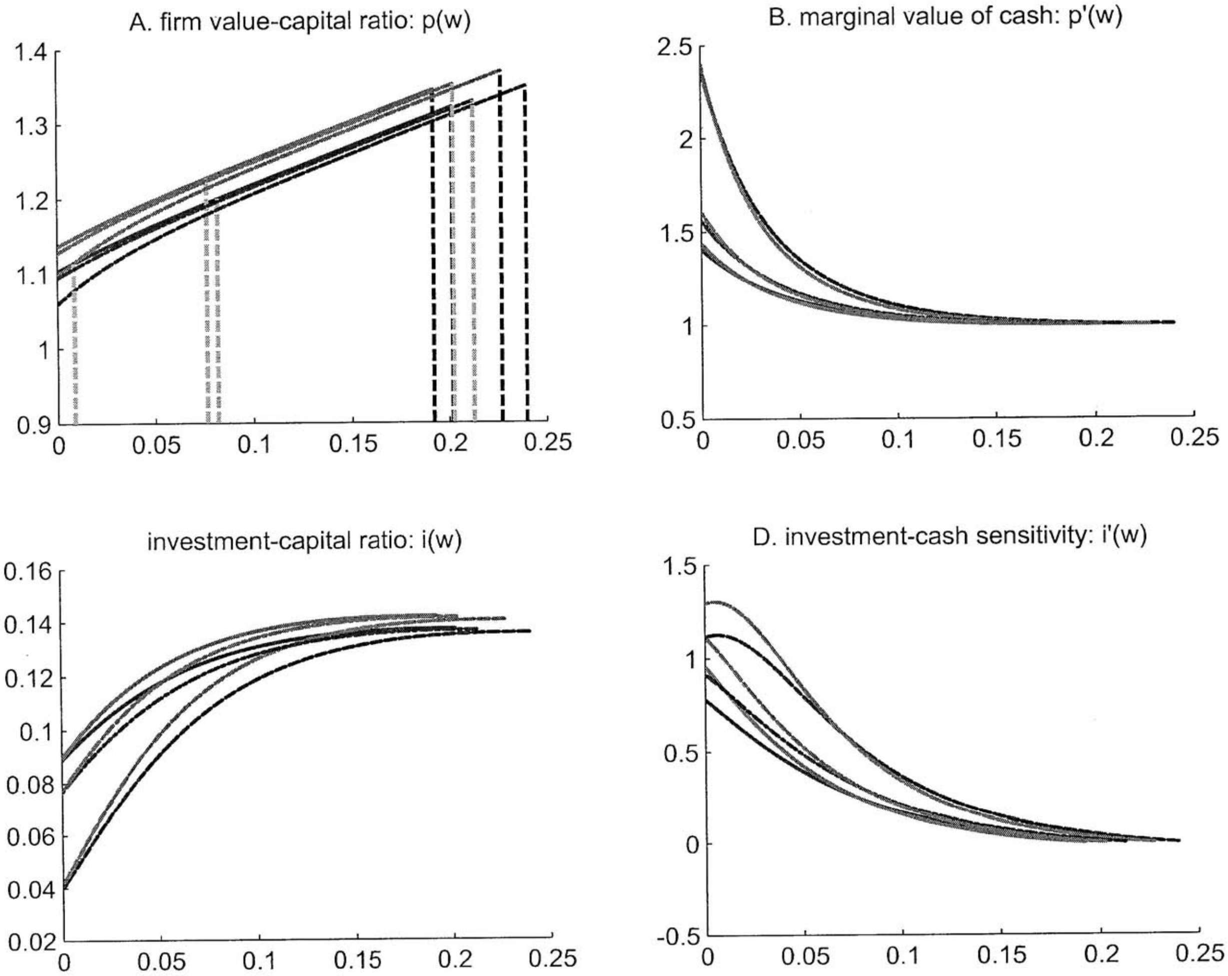

Figure 5-9: Non-optimal Equity Issuance Amount

Chapter 6

Conclusion

Bolton et al. (2009) propose a tractable and operational dynamic economic framework, which shows how the firm's optimal investment, financing, and risk management policies are interconnected in the presence of external financing costs. In the BCW model, corporate risk management involves internal liquidity management, financial hedging, investment/asset sales, and payout. Based on the BCW model, I assume that the firm's investment opportunities change stochastically over time. Firms anticipate the stochastic evolution of these investment opportunities and respond optimally. Then my extended model will study how the firm adjusts its investment and financing policies to time-varying investment opportunities.

My extended model is proposed since in many situations the firm's investment opportunities are not constant over time. For example, the firm currently in a normal state may switch to an abnormal state with lower expected productivity/profitability and higher volatility due to worse market conditions (e.g. financial crisis) or operational accident (e.g. BP oil spill in the Gulf of Mexico in 2010). The events can be described by a regime-switching model for abrupt and big changes in the firm's productivity shock. Also, the firm's expected productivity shock is closely related to the price of the firm's output product, like a company producing natural gas. Since most commodity prices are time-varying and usually considered as mean-reverting, it is reasonable to assume that the firm's expected productivity shock is also mean-reverting. Therefore, I use both the regime-switching model and mean-reverting models to incorporate time-varying investment opportunities into the BCW model. In the solution part, I approximate the mean-reverting model with a regime-switching model, which I can solve.

I use a set of parameters values to solve my extended model and make comparisons of policy differences between my extended model and the BCW model. The major difference in my extended is that the policies will not only depend on the cash-capital ratio w, but also depend on the regime j with different investment opportunities that the firm is currently in. With time-varying investment opportunities incorporated, the firm has to consider the persistent effects of the changing investment opportunities and adjust its investment and financing policies accordingly. More specifically, in my model specification, the firm must be aware that it can switching to another regime with different investment opportunity and this process will continue. This feature makes the firm to consider more than its cash holding before making the decision. For example, when the firm expect better investment opportunities in the future, it is likely to invest more now, etc. The differences are in many aspects, and I have discussed them in detail in my thesis, in terms of firm value, payout boundary, marginal value of cash, investment, etc.

There will be some future research topics. For example, it is interesting to develop a robust method to calibrate the model, e.g. the simulated moment method widely used in related academic papers. The constant-return-to-scale production technology is used in both the BCW model and my extended model for the analytical tractability. However, in practice, the decreasing-return-to-scale production technology works better to match the empirical results. Reformulate the model with the decreasing-return-to-scale production technology will be an important and challenging topic. Other topics may include: study the relationship between the credit limit c and the firm's liquidation value l, introducing the time-varying financing opportunities (e.g. Bolton et al. (2010a)), etc.

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