The moduli space of hypersurfaces whose singular locus has high dimension

by

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Abstract

Fix integers n and b with $n \geq 3$ and $1 \leq b \leq n-1$. Let k be an algebraically closed field. Consider the moduli space X of hypersurfaces in $\mathbb{P}^n_k$ of fixed degree l whose singular locus is at least b-dimensional. We prove that for large l, X has a unique irreducible component of maximal dimension, consisting of the hypersurfaces singular along a linear b-dimensional subspace of $\mathbb{P}^n$.

Thesis Supervisor: Bjorn Poonen
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Since my first day in graduate school, Dennis Gaitsgory has been my inspiration for mathematics and for algebraic geometry in particular. Already the first few weeks of his commutative algebra class made me decide to do algebraic geometry in graduate school. I admire his precision of language, fluidity of thought, and efforts to teach us how he thinks about algebraic geometry. Full of energy, he always manages to give beautiful expositions that satisfy his high aesthetic requirements, and at the same time gives the full story and generality. I feel I began to think somehow differently as a result of Dennis's exciting classes.

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Contents

1 Introduction **9**

2 Notation **13**

3 The incidence correspondence **15**

4 Specialization arguments **19**

4.1 Specialization of a closed subscheme to a union of linear subspaces . . . 19

4.2 An upper bound on the dimension of the space of F such that $C \subset V(F)_{\text{sing}}$, for a fixed C of small degree 23

4.3 An upper bound on the dimension of the space of F such that $C \subset V(F)$, for a fixed C of small degree 25

5 The case of small degree d **29**

5.1 The component corresponding to $d = 1$ 29

5.2 The result of Eisenbud and Harris 30

5.3 The case $d \leq \frac{l+1}{2}$ (small degree) 31

5.4 Preparations for the computation of the second largest component . . 34

6 The case of large degree d, when $k = \overline{\mathbb{F}_p}$ **37**

6.1 Reduction to a problem over finite fields 38

6.2 Final preparations 39

6.3 The key step (large degree d) 40

7 Proof of the main theorem **43**

7.1 Restatement of the problem and the end of the proof 43

7.2 Uniqueness of the largest component (in characteristic 0) 45

7.3 The second largest component 49

8 The space of F such that $V(F)_{\text{sing}}$ contains a fixed C (case $b = 1$) **57**

8.1 Understanding the condition $C \subset V(F)_{\text{sing}}$ for a fixed C 58

8.2 Computing the Hilbert polynomial of $\mathcal{J}$ 60

9 Alternative argument for the case $b = 1$ **65**

9.1 Mumford regularity 65

9.2 Uniform bound on $\mu(C)$ 66

9.3 The curves of small degree 71

9.4 The case of large degree $d \geq \sqrt[M]{l}$ 73

Chapter 1

Introduction

Let n and b be fixed integers with $n \geq 3$ and $1 \leq b \leq n-1$, and let k be an algebraically closed field of characteristic $p \geq 0$. Fix a positive integer l. Inside the projective space of all hypersurfaces in $\mathbb{P}^n$ of degree l, consider the ones which are singular along some b-dimensional closed subscheme,

$$X = \{[F] \in \mathbb{P}(k[x_0, ..., x_n]_l) \mid \dim V(F)_{\text{sing}} \geq b\}$$

(this is a closed subset).

A simple argument (Lemma 5.1) will show that

$$X^1 := \{[F] \in X \mid L \subset V(F)_{\text{sing}} \text{ for some linear } b\text{-dimensional } L \subset \mathbb{P}^n\}$$

is an irreducible closed subset of X of dimension $\binom{l+n}{n} - a_{n,b}(l)$, where

$$\begin{aligned} a_{n,b}(l) &:= \binom{l+b}{b} + (n-b)\binom{l-1+b}{b} + 1 - (b+1)(n-b) \\ &= \frac{n-b+1}{b!} l^b + \dots . \end{aligned}$$

Theorem 1.1. *There exists an integer $l_0 = l_0(n, b, p)$, such that for all $l \geq l_0$, X^1 is the unique irreducible component of X of maximal dimension.*

In fact, the proof of the theorem will give a simple procedure to compute a possible value of l_0, given n, b, p (assuming a conjecture of Eisenbud and Harris when $b \geq 2$). In addition, again for large l, we find the second largest component of X, at least when $\operatorname{char} k \neq 0$: it comes from the hypersurfaces singular along an integral closed subscheme of degree 2 (Corollary 7.16).

We now sketch the main idea of the proof. Let Hilb^d denote the disjoint union of the finitely many Hilbert schemes $\mathrm{Hilb}_{\mathbb{P}^n}^{P_\alpha}$, where P_α ranges over the Hilbert polynomials of integral b-dimensional closed subschemes $C \subset \mathbb{P}^n$ of degree d, and define the restricted Hilbert scheme $\widetilde{\mathrm{Hilb}}^d$ as the closure in Hilb^d of the set of points corresponding to integral subschemes. Let $V = k[x_0, ..., x_n]_l$. Consider the incidence correspondence

$$\widetilde{\Omega}^d = \{(C, [F]) \in \widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V) \mid C \subset V(F)_{\mathrm{sing}}\}.$$

The first step[1] is to show that for $2 \leq d \leq \frac{l+1}{2}$ ("small" degree), any irreducible component of $\widetilde{\Omega}^d$ has dimension less than $\dim X^1$. For this, we apply the theorem on dimension of fibers to the map $\pi \colon \widetilde{\Omega}^d \longrightarrow \widetilde{\mathrm{Hilb}}^d$. A result of Eisenbud and Harris gives $\dim \widetilde{\mathrm{Hilb}}^d$ when $b = 1$; for $b > 1$, they state a conjecture for the corresponding result. (We assume this conjecture but also note that our proof can be modified to give an alternative unconditional — but ineffective — proof of Theorem 1.1.) So it remains to give an upper bound for the dimension of the fiber of π over an integral C of degree d. For this, we specialize C to a union of d b-dimensional linear subspaces that contain a common $(b-1)$-dimensional linear subspace.

The second step is to handle the case $d \geq \frac{l+1}{2}$ ("large" degree). For this, the first main observation is that it suffices to assume that $k = \overline{\mathbb{F}_p}$ in the statement of the main theorem. The reason is that the variety X is the basechange by $\operatorname{Spec} k \longrightarrow \operatorname{Spec} \mathbb{Z}$ of a projective variety $X^{\mathrm{univ}} \longrightarrow \operatorname{Spec} \mathbb{Z}$, and in order to give an upper bound for $\dim X^{\mathrm{univ}} \times \overline{\mathbb{Q}}$, by upper-semicontinuity, it suffices to give an upper bound for $\dim X^{\mathrm{univ}} \times \overline{\mathbb{F}_p}$ for a single prime p (we will take $p = 2$).

[1] We are going to be slightly imprecise here; see Section 5.3 for the exact statement.

So let $k = \overline{\mathbb{F}_p}$ and $d \geq \frac{l+1}{2}$. We have to give an upper bound for the dimension of

$$T^d = \{[F] \in \mathbb{P}(k[x_0, ..., x_n]_l) \mid V(F)_{\text{sing}} \text{ contains} \\ \text{a subscheme with Hilbert polynomial among } \{P_\alpha\}\}.$$

Any variety T over $\overline{\mathbb{F}_p}$ comes from a variety T_0 defined over some finite field $\mathbb{F}_{q_0}$; in order to give an upper bound $\dim T \leq A$, it suffices to prove that $\#T_0(\mathbb{F}_q) = O(q^A)$ as $q \to \infty$, by the result of Lang-Weil [8]. So we reduce the problem to giving an upper bound on the number of hypersurfaces $F \in \mathbb{F}_q[x_0, ..., x_n]_l$ such that $V(F)_{\text{sing}}$ contains an integral closed subscheme of large degree d.

For this, we mimic the main argument in [10]. We sketch it here in the case $b = 1$ and $l \equiv 1 \pmod p$ to simplify notation. Write F in the form

$$F = F_0 + \sum_{i=0}^{n} G_i^p x_i,$$

where F_0 has degree l, each G_i has degree $\tau = \frac{l-1}{p}$, and note that

$$\frac{\partial F}{\partial x_i} = \frac{\partial F_0}{\partial x_i} + G_i^p.$$

Fix F_0. We exhibit a large supply of $(G_0, ..., G_n)$ such that the F constructed in this way has the property that $V(F)_{\text{sing}}$ contains no integral curves of degree d. To do this, we first give a large supply of $(G_0, ..., G_{n-2})$ such that $V(\frac{\partial F}{\partial x_0}, ..., \frac{\partial F}{\partial x_{n-2}})$ has all components of dimension 1. The number of such components is bounded by Bézout's theorem. It remains to give a large supply of G_{n-1} such that no irreducible component C of $V(\frac{\partial F}{\partial x_0}, ..., \frac{\partial F}{\partial x_{n-2}})$ of degree d is contained in $V(\frac{\partial F_0}{\partial x_0} + G_{n-1}^p)$. We accomplish this by specializing C to a union of d lines again, and giving an upper bound on the number of G_{n-1} with $C \subset V(\frac{\partial F_0}{\partial x_0} + G_{n-1}^p)$. With some technical details concerning the uniqueness of the largest-dimensional component in characteristic 0, this completes the proof of Theorem 1.1. The discussion of the second largest component is along the same lines.

We also give an alternative approach for the case of small degree d (when $b = 1$).

Namely, in Chapter 8, we fix an integral curve $C \hookrightarrow \mathbb{P}^n$ with ideal sheaf $\mathcal{I}$, and associate to it an ideal sheaf $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^n}$ (with $\mathcal{J} \supset \mathcal{I}^2$) such that for $F \in k[x_0, ..., x_n]_l$, we have $C \subset V(F)_{\text{sing}}$ if and only if $F \in \Gamma(\mathbb{P}^n, \mathcal{J}(l))$. Next, we compute the Hilbert polynomial of $\mathcal{J}$, and hence the dimension of $W_C := \{F \in k[x_0, ..., x_n]_l \mid C \subset V(F)_{\text{sing}}\}$ (in terms of invariants of C) for $l \gg 0$. We use Mumford regularity to find a polynomial $P_2(d)$ such that this formula for $\dim W_C$ is valid for all integral curves C of degree d and for all $l \geq P_2(d)$.

Chapter 2

Notation

For a field k, the graded ring $k[x_0, ..., x_n]$ will be denoted by S. For a graded S-module M (in particular, for a homogeneous ideal), M_l will denote the l-th graded piece of M. When $I \subset S$ is a homogeneous ideal, $(I^2)_l$ is denoted simply by I_l^2. Also, $k[x_0, ..., x_n]_{\leq l}$ denotes the vector space of (inhomogeneous) polynomials whose total degree is at most l. When the field k and the integer l are fixed, V will denote the vector space $V = k[x_0, ..., x_n]_l$.

For a finite-dimensional k-vector space V, $\mathbb{P}(V)$ denotes the projective space parametrizing lines in V, so for a k-scheme S, $\mathrm{Hom}_{\mathrm{Sch}/k}(S, \mathbb{P}(V))$ consists of a line bundle $\mathcal{L}$ on S, together with an injective bundle map (i.e., with locally free cokernel) $\mathcal{L} \hookrightarrow V \otimes_k \mathcal{O}_S$. Given a homogeneous ideal $I \subset k[x_0, ..., x_n]$, $V(I)$ denotes the closed subscheme $\mathrm{Proj}(k[x_0, ..., x_n]/I) \hookrightarrow \mathbb{P}_k^n$, and for $i = 0, ..., n$, $D_+(x_i)$ is the complement of $V(x_i)$. We often abbreviate $V(\{G_i\}_{i \in I}) \subset \mathbb{P}^n$ as $V(G_i)$, when the index set I is irrelevant or understood.

For $F \in S_l$, $V(F)_{\mathrm{sing}} \subset \mathbb{P}^n$ is the closed subscheme $V(F, \frac{\partial F}{\partial x_i}) = V(F, \frac{\partial F}{\partial x_0}, ..., \frac{\partial F}{\partial x_n})$ of $\mathbb{P}^n$, so when $F \neq 0$, the underlying topological space of $V(F)_{\mathrm{sing}}$ is the singular locus of $V(F)$.

If $C \hookrightarrow \mathbb{P}^n$ is a closed subscheme of dimension b and Hilbert polynomial $P_C(z) = \frac{d}{b!} z^b + \dots$, we say that C has degree d.

We will reuse l_0 for different bounds as we go along, in order to avoid unnecessary notation; however, it will be clear that we are actually referring to different values of

l_0 even though we use the same symbol. Also, it will be understood that sometimes the value of l_0 is the maximum of a finite set of previously defined bounds, each of them still denoted by l_0.

When X is a scheme of finite type over an algebraically closed field, we often identify X with its set of closed points, since most of our arguments will be just on the level of closed points. So when we say "$x \in X$," we usually refer to a closed point $x \in X$ (this will be clear from the context).

For integers b and n with $1 \leq b \leq n-1$, we denote by $\mathbb{G}(b, n)$ the Grassmanian of b-dimensional projective linear subspaces of $\mathbb{P}^n$.

For a scheme X, let $\mathrm{QCoh}(X)$ and $\mathrm{Coh}(X)$ denote the categories of quasi-coherent and coherent sheaves on X, respectively.

Chapter 3

The incidence correspondence

The goal of this chapter is twofold: first, to prove that the incidence correspondence is a closed subset of the product $\mathrm{Hilb}^P \times \mathbb{P}(k[x_0, ..., x_n]_l)$ (Corollary 3.2). and second, to show that we can define a universal incidence correspondence Ω^P over $\operatorname{Spec}\mathbb{Z}$ and to introduce the universal moduli spaces $T^P \longrightarrow \operatorname{Spec}\mathbb{Z}$ (defined at the end of the section). The reason we want to work over $\operatorname{Spec}\mathbb{Z}$ is that later we will use upper-semicontinuity to compare $\dim T^P_{\overline{\mathbb{Q}}}$ with $\dim T^P_{\overline{\mathbb{F}_p}}$.

Recall that if Y_0 is a scheme and $\alpha\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ is a map of vector bundles on Y_0, the functor $\mathrm{Van.Loc.}\,\alpha\colon \mathrm{Sch}^{op} \longrightarrow \mathrm{Sets}$ given by

$$\mathrm{Van.Loc.}\,\alpha(S) = \{t\colon S \longrightarrow Y_0 \mid t^*\alpha = 0\}$$

is representable, by a closed subscheme of Y_0. If $U = \operatorname{Spec} A$ is an affine open $U \subset Y_0$ on which $\mathcal{E}_1, \mathcal{E}_2$ are trivial, so the map $\alpha\colon A^{r_1} \longrightarrow A^{r_2}$ on U is given by an $r_2 \times r_1$ matrix (f_{ij}) with entries in A, then $(\mathrm{Van.Loc.}\,\alpha) \cap U \hookrightarrow U$ is given by the closed embedding $\operatorname{Spec}(A/(f_{ij})) \hookrightarrow \operatorname{Spec}(A)$. If $F \in \mathbb{Z}[x_0, ..., x_n]_l$ is a homogeneous polynomial of degree l, it gives rise to a map $\beta\colon \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}} \longrightarrow \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(l)$; then the functor $\mathrm{Van.Loc.}\,\beta$ is represented by the closed subscheme $V(F) \subset \mathbb{P}^n_{\mathbb{Z}}$.

Let $l \geq 1$ be an integer, and let $V = \mathbb{Z}[x_0, ..., x_n]_l$. For $F \in V$, we can describe the map β above as the composition

$$\mathcal{O}_{\mathbb{P}^n} \longrightarrow V \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(l),$$

where the first map is given by $F \in V = \Gamma(\mathbb{P}^n_{\mathbb{Z}}, V \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n})$ and the second one is the canonical map.

Let $V' = \mathbb{Z}[x_0, ..., x_n]_{l-1}$. Consider the linear maps $D_i \colon V \longrightarrow V', F \mapsto \frac{\partial F}{\partial x_i}$ for $i = 0, ..., n$, and fix a nonzero polynomial $P \in \mathbb{Q}[z]$. The functor $\mathrm{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(V) \colon \mathrm{Sch}^{op} \longrightarrow \mathrm{Sets}$ is given as follows: an element of $\mathrm{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(V)(S)$ consists of a closed subscheme $X \hookrightarrow \mathbb{P}^n_S$ such that the composition $X \hookrightarrow \mathbb{P}^n_S \longrightarrow S$ is flat and each fiber has Hilbert polynomial equal to P, together with a line bundle $\mathcal{L}$ on S and an injective bundle map $\alpha \colon \mathcal{L} \hookrightarrow V \otimes_{\mathbb{Z}} \mathcal{O}_S$.

A map $\alpha \colon \mathcal{L} \longrightarrow V \otimes_{\mathbb{Z}} \mathcal{O}_S$ induces maps $\alpha_i \colon \mathcal{L} \longrightarrow V \otimes_{\mathbb{Z}} \mathcal{O}_S \xrightarrow{D_i \otimes \mathrm{id}} V' \otimes_{\mathbb{Z}} \mathcal{O}_S$, for $i = 0, ..., n$. Let $\gamma \colon V \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}} \longrightarrow \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(l)$ and $\gamma' \colon V' \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}} \longrightarrow \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(l-1)$ be the canonical maps. Since the pullback to $\mathbb{P}^n_S$ of the target of α coincides with the pullback of the source of γ (similarly for α_i and γ'),

$$\begin{array}{ccccc} X & \hookrightarrow & \mathbb{P}^n_S & \xrightarrow{r} & \mathbb{P}^n_{\mathbb{Z}} \\ & \searrow & \downarrow{\scriptstyle \pi} & & \\ & & S & & \end{array}$$

we can form the compositions

$$\begin{aligned} \varepsilon \colon \pi^*\mathcal{L} &\xrightarrow{\pi^*\alpha_0} V \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n_S} \xrightarrow{r^*\gamma} \mathcal{O}_{\mathbb{P}^n_S}(l) \\ \varepsilon_i \colon \pi^*\mathcal{L} &\xrightarrow{\pi^*\alpha_i} V' \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n_S} \xrightarrow{r^*\gamma'} \mathcal{O}_{\mathbb{P}^n_S}(l-1), \end{aligned}$$

which are maps of line bundles on $\mathbb{P}^n_S$. Thus, for any $(X \hookrightarrow \mathbb{P}^n_S, \mathcal{L}, \alpha \colon \mathcal{L} \hookrightarrow V \otimes_{\mathbb{Z}} \mathcal{O}_S) \in \mathrm{Hilb}^P \times \mathbb{P}(V)(S)$, we have attached maps $\varepsilon, \varepsilon_i, i = 0, ..., n$ of line bundles on $\mathbb{P}^n_S$.

Consider the subfunctor $\mathcal{F} \colon \mathrm{Sch}^{op} \longrightarrow \mathrm{Sets}$ of the (representabe) functor $\mathrm{Hilb}^P \times \mathbb{P}(V)$, given as follows: $\mathcal{F}(S)$ is the set of all $(X \hookrightarrow \mathbb{P}^n_S, \mathcal{L}, \mathcal{L} \hookrightarrow V \otimes_{\mathbb{Z}} \mathcal{O}_S) \in \mathrm{Hilb}^P \times \mathbb{P}(V)(S)$

such that the pullback of ε and each ε_i (for $i = 0, ..., n$) to X vanishes.

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^n_S \\ & & \downarrow \\ & & S \end{array}$$

Proposition 3.1. *The functor $\mathcal{F}$ is representable by a closed subscheme Ω^P of* $\mathrm{Hilb}^P \times \mathbb{P}(V)$.

Proof. Consider the scheme $Y = \mathrm{Hilb}^P \times \mathbb{P}(V)$, and let $(X \hookrightarrow \mathbb{P}^n_Y, \mathcal{L}, \alpha \colon \mathcal{L} \hookrightarrow V \otimes_k \mathcal{O}_Y)$ be the tautological element of $\mathrm{Hilb}^P \times \mathbb{P}(V)(Y)$. This gives rise to maps $\varepsilon, \varepsilon_i$ of line bundles on $\mathbb{P}^n_Y$. Let $\tilde{\varepsilon}, \tilde{\varepsilon_i}$ be the pullbacks of $\varepsilon, \varepsilon_i$ to X.

For a scheme S, $\mathcal{F}(S)$ consists of all maps $S \to Y$ such that the maps of line bundles $\tilde{\varepsilon}, \tilde{\varepsilon_i}$ on X pull back to zero on $X \times_Y S$. Since Y is noetherian and the morphism $X \to Y$ is flat and projective, this functor is representable, by a closed subscheme of Y (see Theorem 5.8 and Remark 5.9 in [5]). □

If k is an algebraically closed field and Ω^P_k denotes the basechange $\Omega^P \times \operatorname{Spec} k$, we know the set of closed points of Ω^P_k:

$$\mathrm{Hom}_{\mathsf{Sch}/k}(\operatorname{Spec} k, \Omega^P_k) = \mathcal{F}(\operatorname{Spec} k).$$

From the definitions, this is just

$$\left\{(C, [F]) \in \mathrm{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(k[x_0, ..., x_n]_l) \mid C \subset V\left(F, \frac{\partial F}{\partial x_i}\right)\right\}$$

(inclusion above denotes scheme-theoretic inclusion).

Corollary 3.2. *Let k be an algebraically closed field, $l \geq 1$ an integer, and $P \in \mathbb{Q}[z]$ a polynomial. The set*

$$\left\{(C, [F]) \in \mathrm{Hilb}^P_{\mathbb{P}^n_k} \times \mathbb{P}(k[x_0, ..., x_n]_l) \mid C \subset V\left(F, \frac{\partial F}{\partial x_i}\right)\right\}$$

is a closed subset of (the set of closed points of) $\mathrm{Hilb}^P_{\mathbb{P}^n_k} \times \mathbb{P}(k[x_0, ..., x_n]_l)$.

Let T^P denote the scheme-theoretic image of $\Omega^P \to \mathbb{P}(V)$, so we have a diagram

$$\begin{array}{ccc} \Omega^P & \hookrightarrow & \mathrm{Hilb}^P \times \mathbb{P}(V) \\ \downarrow & & \downarrow \\ T^P & \hookrightarrow & \mathbb{P}(V). \end{array}$$

Since surjections and closed embeddings are stable under base-change, for any algebraically closed field k, we have a corresponding diagram

$$\begin{array}{ccc} \Omega_k^P & \hookrightarrow & \mathrm{Hilb}^P_{\mathbb{P}^n_k} \times \mathbb{P}(V_k) \\ \downarrow & & \downarrow \\ T_k^P & \hookrightarrow & \mathbb{P}(V_k) \end{array}$$

(where $V_k = V \otimes_{\mathbb{Z}} k = k[x_0, ..., x_n]_l$) and by looking at closed points, it follows that

$$T_k^P = \{[F] \in \mathbb{P}(V_k) \mid V(F)_{\mathrm{sing}} \text{ contains a subscheme with Hilbert polynomial } P\}.$$

Chapter 4

Specialization arguments

The first main technique that we use in the proof of Theorem 1.1 is a specialization argument, that allows us to bound $\dim\{F \in k[x_0, ..., x_n]_l \mid C \subset V(F)_{\text{sing}}\}$ from above for a fixed C, by degenerating C to a union of linear spaces. In Section 4.1, we prove (for lack of reference) that we can specialize a b-dimensional integral closed subscheme C of $\mathbb{P}^n$ to a union of d b-dimensional linear spaces containing a common $(b-1)$-dimensional linear space. Next, the bound we obtain in Section 4.2 will be the main ingredient for the discussion of the cases of small degree $2 \leq d \leq \frac{l+1}{2}$ in Chapter 5. Finally, Section 4.3 is a preparation for the discussion of the case of large degree $d \geq \frac{l+1}{2}$, which will be treated in Chapter 6. The main result of Section 4.3 is stated in Corollaries 4.9 and 4.11 in a form that is most convenient for later purposes.

In this chapter, k is a fixed algebraically closed field.

4.1 Specialization of a closed subscheme to a union of linear subspaces

The result of this section is known, but we were unable to find a reference, so we include it here.

Let $C \subset \mathbb{P}^n$ be an integral b-dimensional closed subscheme of degree d. Let $P = V(x_0, ..., x_{n-b})$ be the $(b-1)$-dimensional "linear subspace at infinity." Suppose that the linear subspace $H = V(x_{n-b+1}, ..., x_n)$ intersects C in d distinct points Q_i. Let L_i be the unique b-dimensional linear space through P and Q_i. The L_i are distinct because if $L_i = L_j$ for some $i \neq j$, the line through Q_i and Q_j would be contained in H but would have to intersect P; this is impossible, since $P \cap H = \emptyset$. Consider the projective linear transformations

$$A_a = \left(\begin{array}{ccc|ccc} a & & & & & \\ & \ddots & & & & \\ & & a & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right)$$

(where the bottom block has size $b \times b$) and let $C_a = A_a C$.

Proposition 4.1. *The underlying topological space of the flat limit* $C_0 = \lim_{a \to 0} C_a$ *is* $\bigcup_{i=1}^{d} L_i$.

Proof. Let $C = V(\{G_s\}) \subset \mathbb{P}^n$ (as a scheme), where $G_s \in k[x_0, ..., x_n]$ are homogeneous. Consider the map

$$\sigma \colon \mathbb{P}^n \times (\mathbb{A}^1 - \{0\}) \to \mathbb{P}^n, \quad ([x_0, ..., x_n], a) \mapsto (x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n),$$

and define the closed subscheme $X \subset \mathbb{P}^n \times (A^1 - \{0\})$ as the fiber product

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^n \times (\mathbb{A}^1 - \{0\}) \\ \downarrow & & \downarrow \sigma \\ C & \hookrightarrow & \mathbb{P}^n. \end{array}$$

In other words,

$$X = V(G_s(x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n)) \subset \mathbb{P}^n_{\mathbb{A}^1 - \{0\}},$$

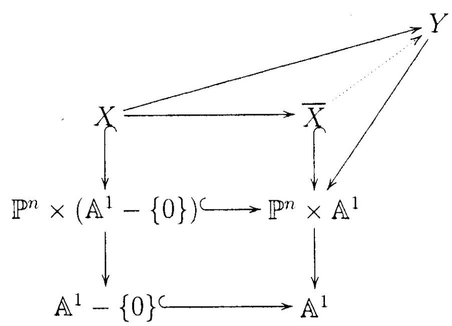
where we regard $G_s(x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n) \in k[a, a^{-1}][x_0, ..., x_n]$. This is a flat family $X \to \mathbb{A}^1 - \{0\}$, whose fiber over $a \neq 0$ is C_a (as a subscheme of $\mathbb{P}^n$).

Let $\overline{X}$ be the scheme-theoretic closure of X in $\mathbb{P}^n \times \mathbb{A}^1$. By the proof of Proposition III.9.8 in [7], the flat limit of the family (C_a) is the scheme-theoretic fiber $\overline{X}_0$.

Consider

$$Y = V(G_s(x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n)) \subset \mathbb{P}^n \times \mathbb{A}^1.$$

Then Y is a closed subscheme of $\mathbb{P}^n \times \mathbb{A}^1$ containing X_0 (scheme-theoretically), so Y contains $\overline{X}$. Thus, $\overline{X}_0 \subset Y_0$ is a closed subscheme.



We have

$$Y_0 = V(G_s(x_0, ..., x_{n-b}, 0, ..., 0)) \subset \mathbb{P}^n.$$

Thus, as a set, Y_0 is $\bigcup_{i=1}^d L_i$.

We claim that Y_0 is reduced away from P. Equivalently, for $i = 0, .., n-b$, we have to check that $Y_0 \cap D_+(x_i)$ is reduced. To simplify notation, suppose that $i = 0$. Then

$$\begin{aligned} Y_0 \cap D_+(x_0) &= \operatorname{Spec} \frac{k[x_1, ..., x_n]}{(G_s(1, x_1, ..., x_{n-b}, 0, ..., 0))} \\ &= \operatorname{Spec} \left(\frac{k[x_1, ..., x_n]}{(G_s(1, x_1, ..., x_n), x_{n-b+1}, ..., x_n)} \right) [x'_{n-b+1}, ..., x'_n]. \end{aligned}$$

So we have to show that the 0-dimensional ring

$$\frac{k[x_1, ..., x_n]}{(G_s(1, x_1, ..., x_n), x_{n-b+1}, ..., x_n)}$$

is reduced. We have assumed that C intersects $V(x_{n-b+1}, ..., x_n)$ transversely, so

$$\operatorname{Proj} \frac{k[x_0, ..., x_n]}{(G_s(x_0, ..., x_n), x_{n-b+1}, ..., x_n)}$$

is a reduced 0-dimensional scheme; looking at its intersection with $D_+(x_0)$, we obtain the desired conclusion.

Now that Y_0 is reduced away from a subscheme of smaller dimension, it follows that the Hilbert polynomial of Y_0 has the same degree and leading coefficient (namely, b and $d/b!$, respectively) as the Hilbert polynomial of $(Y_0)_{\text{red}}$. The Hilbert polynomial of the flat limit $\overline{X}_0$ also has degree b and leading coefficient $d/b!$. Moreover, Y_0 is equidimensional, so the inclusion $\overline{X}_0 \hookrightarrow Y_0$ must be a homeomorphism. □

Remark 4.2. The proof above does not imply that Y_0 is reduced everywhere. Let us look at Y_0 in the chart $D_+(x_n)$, so

$$\begin{aligned} Y_0 \cap D_+(x_n) &= \operatorname{Spec} \frac{k[x_0, ..., x_{n-1}]}{(G_s(x_0, ..., x_{n-b}, 0, ..., 0))} \\ &= \operatorname{Spec} \left(\frac{k[x_0, ..., x_n]}{(G_s(x_0, ..., x_n), x_{n-b+1}, ..., x_n)} \right) [x'_{n-b+1}, ..., x'_{n-1}]. \end{aligned}$$

Let $S = k[x_0, ..., x_n]/(G_s(x_0, ..., x_n), x_{n-b+1}, ..., x_n)$. We know that $\operatorname{Proj} S$ is reduced as a scheme by the transversality assumption on $C \cap H$; however, this does not in general imply that S itself is reduced as a ring.

Let $V = k[x_0, ..., x_n]_l$. For each closed subscheme $C \subset \mathbb{P}^n$, define the k-vector space

$$W_C = \{F \in V \mid C \subset V(F)_{\text{sing}}\}.$$

Corollary 4.3. *Let $C \hookrightarrow \mathbb{P}^n$ be an integral closed subscheme of dimension b and degree d. There exist d b-dimensional linear subspaces $L_1, ..., L_d$ of $\mathbb{P}^n$ containing a common $(b-1)$-dimensional linear subspace, such that*

$$\dim W_C \leq \dim W_{\cup L_i},$$

where $\cup L_i$ is given the reduced induced structure.

Proof. Let P be the Hilbert polynomial of C. Recall the incidence correspondence from Corollary 3.2 and apply the upper semicontinuity theorem (see Section 14.3 in [2]) to the map

$$\{(C,[F]) \in \mathrm{Hilb}^P \times \mathbb{P}(V) \mid C \subset V(F)_{\mathrm{sing}}\} \xrightarrow{\pi} \mathrm{Hilb}^P .$$

By Proposition 4.1, $\cup L_i$ (with some scheme structure) is the flat limit C_0 of a family (C_a), with each C_a $(a \neq 0)$ being projectively equivalent to $C = C_1$, and hence $\pi^{-1}(C_a) \simeq \pi^{-1}(C)$ for each $a \neq 0$. Therefore,

$$\dim \mathbb{P}(W_C) = \dim \pi^{-1}(C) \leq \dim \pi^{-1}(C_0) = \dim \mathbb{P}(W_{C_0}) \leq \dim \mathbb{P}(W_{\cup L_i}). \qquad \square$$

4.2 An upper bound on the dimension of the space of F such that $C \subset V(F)_{\mathrm{sing}}$, for a fixed C of small degree

Fix a positive integer l. Recall the notation $V = k[x_0, ..., x_n]_l$.

Lemma 4.4. *Let $L \subset \mathbb{P}^n$ be a b-dimensional linear subspace. Then for $F \in V$, we have $L \subset V(F)_{\mathrm{sing}}$ if and only if $F \in I_L^2$. Moreover,*

$$\mathrm{codim}_V\{F \in V \mid L \subset V(F)_{\mathrm{sing}}\} = \binom{l+b}{b} + (n-b)\binom{l-1+b}{b}.$$

Proof. Without loss of generality, $L = V(I)$ with $I = (x_{b+1}, ..., x_n)$. For $F \in V$, we claim that $(F, \frac{\partial F}{\partial x_i}) \subset I$ if and only if $F \in I^2$. Suppose that $(F, \frac{\partial F}{\partial x_i}) \subset I$. Write $F = F_0 + \sum_{i=b+1}^n F_i x_i + T$, where $F_0, F_i \in k[x_0, ..., x_b]$ are homogeneous of degrees $l, l-1$ respectively, and $T \in I_l^2$. Since $\frac{\partial T}{\partial x_i} \in I$ for all i, we can assume without loss of generality that $T = 0$. Now, the condition $\frac{\partial F}{\partial x_i} \in I$ for $i = b+1, ..., n$ implies $F_i \in I \cap k[x_0, ..., x_b] = 0$, so $F_i = 0$. Then $F = F_0 \in I \cap k[x_0, ..., x_b] = 0$, so $F = 0$

overall, as desired. Clearly, $(S/I^2)_l \simeq k[x_0, ..., x_b]_l \oplus (\bigoplus_{i=b+1}^n k[x_0, ..., x_b]_{l-1}x_i)$ has dimension as in the statement. □

Lemma 4.5. *Let $L_1, ..., L_d$ be d b-dimensional linear subspaces of $\mathbb{P}^n$ containing a common $(b-1)$-dimensional linear subspace. Then for $d \leq \frac{l+1}{2}$, we have*

$$\operatorname{codim}_V(W_{\cup L_i}) \geq \binom{l+b}{b} + (n-b)\sum_{e=1}^{d}\binom{l-2e+1+b}{b}.$$

Proof. We induct on d. For $d = 1$, we have equality. Assume $2 \leq d \leq \frac{l+1}{2}$. Assume that the b-dimensional linear subspaces $L_1, ..., L_d$ all contain $P = [0, \underbrace{*, ..., *}_{b}, 0, ..., 0]$ and that none of them is contained in the hyperplane $x_0 = 0$, so the ideal of each of them is of the form $(x_{b+1} - p_{b+1}x_0, ..., x_n - p_n x_0)$ for a uniquely determined tuple $(p_{b+1}, ..., p_n) \in k^{n-b}$. Let

$$I_i = (x_{b+1} - p_{b+1}^{(i)}x_0, x_{b+2} - p_{b+2}^{(i)}x_0, ..., x_n - p_n^{(i)}x_0) \quad \text{for } i = 1, ..., d-1,$$

and without loss of generality

$$I_d = (x_{b+1}, ..., x_n).$$

By Lemma 4.4, $W_{\cup L_i} = (I_1^2 \cap \cdots \cap I_d^2)_l$, so we have to give a lower bound for $\dim(S/I_1^2 \cap \cdots \cap I_d^2)_l$. For $e \in \{d-1, d\}$, let $\mu_e = \dim(S/I_1^2 \cap \cdots \cap I_e^2)_l$. There is a short exact sequence

$$0 \to \left(\frac{I_1^2 \cap \cdots \cap I_{d-1}^2}{I_1^2 \cap \cdots \cap I_d^2}\right)_l \to \left(\frac{S}{I_1^2 \cap \cdots \cap I_d^2}\right)_l \to \left(\frac{S}{I_1^2 \cap \cdots \cap I_{d-1}^2}\right)_l \to 0.$$

So we have to write down enough linearly independent elements in $(I_1^2 \cap \cdots \cap I_{d-1}^2 / I_1^2 \cap \cdots \cap I_d^2)_l$.

For each $i = 1, ..., d-1$, there exists $m_i \in \{b+1, ..., n\}$ such that $p_{m_i}^{(i)} \neq 0$. Let

$F = \prod_{i=1}^{d-1}(x_{m_i} - p^{(i)}_{m_i}x_0)^2$. Consider all elements

$$Fx_jP(x_0, ..., x_b) \in \left(\frac{I_1^2 \cap \cdots \cap I_{d-1}^2}{I_1^2 \cap \cdots \cap I_d^2}\right)_l,$$

where $j \in \{b+1, ..., n\}$ and $P(x_0, ..., x_b)$ runs through a basis of $k[x_0, ..., x_b]_{l-2d+1}$. Their number is $(n-b)\binom{l-2d+1+b}{b}$ and we claim that they are all linearly independent. Indeed, it suffices to check that their images under the injection $(I_1^2 \cap \cdots \cap I_{d-1}^2 / I_1^2 \cap \cdots \cap I_d^2)_l \hookrightarrow (S/I_d^2)_l$ become linearly independent. This is evident, however, since $(S/I_d^2)_l \simeq k[x_0, ..., x_b]_l \oplus k[x_0, ..., x_b]_{l-1}x_{b+1} \oplus \cdots \oplus k[x_0, ..., x_b]_{l-1}x_n$ as k-vector spaces, and the images of the elements under consideration are

$$(p^{(1)}_{m_1})^2 ... (p^{(d-1)}_{m_{d-1}})^2 x_0^{2(d-1)} x_j P(x_0, ..., x_b).$$

Therefore

$$\mu_d \geq \mu_{d-1} + (n-b)\binom{l-2d+1+b}{b},$$

and the statement follows by induction. □

4.3 An upper bound on the dimension of the space of F such that $C \subset V(F)$, for a fixed C of small degree

Lemma 4.6. *Fix positive integers l, m, with $m \leq l+1$. For any integral closed subscheme $C \subset \mathbb{P}^n$ of dimension b and degree $d \geq m$, we have*

$$\operatorname{codim}_V\{G \in V \mid C \subset V(G)\} \geq \sum_{e=1}^{m} \binom{l-e+1+b}{b} =: A_b(l, m).$$

Proof. As above, we specialize C to a union of d b-dimensional linear spaces containing P (notation as in the previous lemma). Throwing away some of these linear spaces if necessary, we may assume $d = m$. So we induct on $m = 1, ..., l+1$ to give a lower bound for $\dim(S/I_1 \cap \cdots \cap I_m)_l$. We follow the notation and proof of Lemma 4.5, except that this time, we consider $F = \prod_{i=1}^{m-1}(x_{m_i} - p_{m_i}^{(i)} x_0)$ and the linearly independent elements $FP \in (I_1 \cap \cdots \cap I_{m-1}/I_1 \cap \cdots \cap I_m)_l$, where P runs through a basis of $k[x_0, ..., x_b]_{l-m+1}$. Thus,

$$\mu_m \geq \mu_{m-1} + \binom{l-m+1+b}{b},$$

which proves the statement by induction. □

Remark 4.7. Note that

$$A_b(l, l+1) \geq \sum_{e=1}^{\lceil \frac{l+1}{2} \rceil} \binom{l-e+1+b}{b} \geq \frac{l+1}{2} \binom{\frac{l}{2}+b}{b},$$

so $A_b(l, l+1)$ dominates a polynomial in l of degree $b+1$.

Corollary 4.8. *Let k be an algebraically closed field, and $k_0 \subset k$ a subfield. Again, let m, l be fixed integers, with $m \leq l+1$. Let $C \subset \mathbb{P}_k^n$ be a b-dimensional integral closed subscheme (not necessarily defined over k_0) of degree $d \geq m$. Then*

$$\operatorname{codim}_{k_0[x_0,...,x_n]_l}\{G \in k_0[x_0, ..., x_n]_l \mid C \subset V(G)\} \geq A_b(l, m).$$

Here, the condition $C \subset V(G)$ (inclusion of closed subschemes of $\mathbb{P}_k^n$) makes sense when we regard $G \in k[x_0, ..., x_n]_l$ first.

Proof. It suffices to prove that

$$\dim_{k_0}\{G \in k_0[x_0, ..., x_n]_l \mid C \subset V(G)\} \leq \dim_k\{G \in k[x_0, ..., x_n]_l \mid C \subset V(G)\}.$$

This is automatic, since any k_0-linearly independent elements in $k_0[x_0, ..., x_n]_l$ are k-linearly independent in $k[x_0, ..., x_n]_l$. □

Corollary 4.9. *Let $k_0 = \mathbb{F}_q$ now. Let $C \subset \mathbb{P}_{\overline{\mathbb{F}_p}}^n$ be an integral b-dimensional closed subscheme of degree $d \geq m$ (again, $m \leq l+1$ is fixed). For G chosen randomly from*

$\mathbb{F}_q[x_0,...,x_n]_l$, *we have*

$$\mathrm{Prob}(C \subset V(G)) \leq q^{-A_b(l,m)}.$$

Proof. This is just a restatement of Corollary 4.8, since

$$\#\{G \in \mathbb{F}_q[x_0,...,x_n]_l \mid C \subset V(G)\} = q^{\dim\{G|C\subset V(G)\}}. \qquad \square$$

Lemma 4.10. *Let k be an algebraically closed field, and $S \subset \mathbb{P}^n_k$ an integral closed subscheme of dimension at least $b+1$. Then*

$$\mathrm{codim}_{k[x_0,...,x_n]_l}\{G \in k[x_0,...,x_n]_l \mid S \subset V(G)\} \geq \binom{l+b+1}{b+1}.$$

Proof. We can assume that $\dim S = b+1$. This is a particular case of Lemma 4.6; just note that $A_{b+1}(l,1) = \binom{l+b+1}{b+1}$. $\square$

The same argument leading from Lemma 4.6 to Corollary 4.9 leads from Lemma 4.10 to the following

Corollary 4.11. *Let $k_0 = \mathbb{F}_q$ now. Let $S \subset \mathbb{P}^n_{\overline{\mathbb{F}}_p}$ be an integral closed subscheme of dimension at least $b+1$. For G chosen randomly from $\mathbb{F}_q[x_0,...,x_n]_l$, we have*

$$\mathrm{Prob}(S \subset V(G)) \leq q^{-\binom{l+b+1}{b+1}}.$$

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Chapter 5

The case of small degree d

With the preparations from the previous chapter, it is now easy to handle the cases of small degree $2 \leq d \leq \frac{l+1}{2}$. The new ingredient here is a result of Eisenbud and Harris (conjectural for $b \geq 2$), which gives the dimension of the restricted Hilbert scheme. So we can treat the cases of small degree d by applying the theorem on the dimension of fibers to the map $\widetilde{\Omega}^d \longrightarrow \widetilde{\text{Hilb}}^d$ (Section 5.3). The main result of this chapter is Corollary 5.6. Finally, in Section 5.4, we perform the analogous calculation for the second largest component of X.

Again, k is a fixed algebraically closed field.

5.1 The component corresponding to $d = 1$

The lemma below is simple, since any two linear b-dimensional subspaces of $\mathbb{P}^n$ are projectively equivalent. Recall the definitions of X^1 and $a_{n,b}(l)$ from the introduction. Let $\mathbb{G}(b, n)$ be the Grassmanian of projective linear b-dimensional subspaces of $\mathbb{P}^n$.

Lemma 5.1. *The set X^1 is an irreducible closed subset of X of dimension equal to $A := \binom{l+n}{n} - a_{n,b}(l)$.*

Proof. Consider

$$\Omega^1 = \{(L, [F]) \in \mathbb{G}(b, n) \times \mathbb{P}(V) \mid L \subset V(F)_{\text{sing}}\} \subset \mathbb{G}(b, n) \times \mathbb{P}(V).$$

By Corollary 3.2, this is a closed subset of the product, since $\Omega^1 = \Omega^P$ with $P(z) = \binom{z+b}{b}$.

Let $\pi\colon \Omega^1 \to \mathbb{G}(b,n)$ and $\rho\colon \Omega^1 \to \mathbb{P}(V)$ denote the two projections. The fiber of π over any linear b-dimensional L is $\mathbb{P}(W_L)$. So Ω^1 is irreducible, and has dimension $\dim \mathbb{P}(W_L) + \dim \mathbb{G}(b,n) = A$ (use Lemma 4.4).

Consider now $\rho\colon \Omega^1 \twoheadrightarrow X^1$. To prove that Ω^1 and X^1 have the same dimension, it suffices to show that some fiber of ρ is 0-dimensional. This is easy (we prove a more general statement later; see Lemma 7.4). □

5.2 The result of Eisenbud and Harris

We first recall (see [1], p. 3) the following classical result.

Theorem 5.2 (Chow's finiteness theorem). *Fix positive integers n, b, d. There are only finitely many Hilbert polynomials P_α of integral b-dimensional closed subschemes of $\mathbb{P}^n_k$ of degree d. The algebraically closed field k varies as well in this statement.*

Fix k. For an integer $d \geq 1$, let $\mathrm{Hilb}^{b,d}_{\mathbb{P}^n}$ be the disjoint union of the Hilbert schemes $\mathrm{Hilb}^{P_\alpha}_{\mathbb{P}^n}$ for all the finitely many possible Hilbert polynomials P_α of an integral b-dimensional closed subscheme $C \subset \mathbb{P}^n$ of degree d. Define the restricted Hilbert scheme $\widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n}$ to be the Zariski closure in $\mathrm{Hilb}^{b,d}_{\mathbb{P}^n}$ of the set of integral subschemes, with reduced subscheme structure. Eisenbud and Harris [4] prove the following result for the dimension of $\widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n}$ in the case $b = 1$.

Theorem 5.3. *Let $b = 1$. For $d \geq 2$, the largest irreducible component of $\widetilde{\mathrm{Hilb}}^{1,d}_{\mathbb{P}^n}$ is the one corresponding to the family of plane curves of degree d; in particular, $\dim \widetilde{\mathrm{Hilb}}^{1,d}_{\mathbb{P}^n} = 3(n-2) + \frac{d(d+3)}{2}$.*

In analogy, for $b \geq 2$, Eisenbud and Harris state the following conjecture:

Conjecture 5.4. For $d \geq 2$, the largest irreducible component of $\widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n}$ is the one corresponding to the family of degree-d hypersurfaces contained in linear $(b+1)$-dimensional subspaces of $\mathbb{P}^n$; in particular, $\dim \widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n} = (b+2)(n-b-1) - 1 + \binom{d+b+1}{b+1}$.

From now on, we will be assuming that this conjecture holds, so the results we obtain will depend on it, except in the case $b = 1$. However, we also give an unconditional (but ineffective) proof of Theorem 1.1 (see Remark 7.3).

From now on, we fix b and n, and abbreviate $\widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n}$ as $\widetilde{\mathrm{Hilb}}^d$.

Let Ω^d be the disjoint union of the finitely many Ω^{P_α} (notation as in Proposition 3.1). Also, define T^d as the scheme-theoretic image of $\Omega^d \to \mathbb{P}(\mathbb{Z}[x_0, ..., x_n]_l)$, so we have a diagram

$$\begin{array}{ccc} \Omega^d & \longrightarrow & \mathrm{Hilb}^d \times \mathbb{P}(\mathbb{Z}[x_0, ..., x_n]_l) \\ \downarrow & & \downarrow \\ T^d & \hookrightarrow & \mathbb{P}(\mathbb{Z}[x_0, ..., x_n]_l). \end{array}$$

For any algebraically closed field k, we have

$$T_k^d = \bigcup T_k^{P_\alpha} = \{[F] \in \mathbb{P}(V_k) \mid V(F)_{\mathrm{sing}} \text{ contains}$$
$$\text{a subscheme with Hilbert polynomial among } \{P_\alpha\}\}.$$

Any integral closed subscheme of degree 1 is linear, so $X^1 = T_k^1$. We will use X^1 and T_k^1 interchangeably.

5.3 The case $d \leq \frac{l+1}{2}$ (small degree)

Fix an integer l as usual, and fix an integer $d > 1$. As usual, let $V = k[x_0, ..., x_n]_l$. Recall that

$$\widetilde{\Omega}^d = \{(C, [F]) \in \widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V) \mid C \subset V(F)_{\mathrm{sing}}\}.$$

Define

$$R^d = \{(C, [F]) \in \widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V) \mid C \text{ is integral, } C \subset V(F)_{\mathrm{sing}}\} \subset \widetilde{\Omega}^d.$$

Let $\overline{R^d}$ be the closure of R^d inside $\widetilde{\Omega}^d$ (or inside $\widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V)$). Let $\pi\colon \widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V) \to \widetilde{\mathrm{Hilb}}^d$ and $\rho\colon \widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V) \to \mathbb{P}(V)$ denote the first and second projections.

Lemma 5.5. *There exists l_0 (easily computable) such that for all pairs (d, l) with*

$2 \le d \le \frac{l+1}{2}$ *and* $l \ge l_0$, *we have*

$$\dim \overline{R^d} < \dim X^1.$$

It follows that $\dim \rho(\overline{R^d}) < \dim X^1$.

Proof. Let Z be an irreducible component of $\overline{R^d}$. Certainly, $Z \cap R^d \neq \emptyset$, so $\pi(Z)$ contains an integral subscheme $C \subset \mathbb{P}^n$. Degenerate C to a union $\bigcup_{i=1}^d L_i$ of d b-dimensional linear spaces, as in Section 4.1. Let L_0 be any linear b-dimensional subspace of $\mathbb{P}^n$. By abuse of notation, let $\pi \colon Z \twoheadrightarrow \pi(Z) \subset \widetilde{\mathrm{Hilb}}^d$. By the theorem on the dimension of fibers, we have

$$\begin{aligned} \dim Z &\le \dim \pi^{-1}(C) + \dim \pi(Z) \\ &\le \dim \mathbb{P}(W_C) + \dim \pi(Z) \\ &\le \dim \mathbb{P}(W_{\cup L_i}) + \dim \widetilde{\mathrm{Hilb}}^d. \end{aligned} \tag{5.1}$$

Thus, it suffices to check that

$$\dim \mathbb{P}(W_{\cup L_i}) + \dim \widetilde{\mathrm{Hilb}}^d < \dim \mathbb{P}(W_{L_0}) + (b+1)(n-b)$$

(recall Lemma 5.1), or, equivalently, that

$$\operatorname{codim}_V W_{L_0} + \dim \widetilde{\mathrm{Hilb}}^d < \operatorname{codim}_V W_{\cup L_i} + (b+1)(n-b).$$

By Lemmas 4.4 and 4.5, it suffices to prove the inequality

$$\begin{aligned} &\binom{l+b}{b} + (n-b)\binom{l-1+b}{b} + \dim \widetilde{\mathrm{Hilb}}^d \\ < &\binom{l+b}{b} + (n-b)\sum_{e=1}^{d}\binom{l-2e+1+b}{b} + (b+1)(n-b), \end{aligned}$$

or, equivalently,

$$\dim \widetilde{\text{Hilb}}^d - (b+1)(n-b) < (n-b)\sum_{e=2}^{d}\binom{l-2e+1+b}{b}, \tag{5.2}$$

for all $2 \le d \le \frac{l+1}{2}$ and $l \ge l_0$. Let $c = (b+2)(n-b-1) - 1 - (b+1)(n-b)$. Assume Conjecture 5.4; then 5.2 is equivalent to

$$c + \binom{d+b+1}{b+1} < (n-b)\sum_{e=2}^{d}\binom{l-2e+1+b}{b} \tag{5.3}$$

for all $2 \le d \le \frac{l+1}{2}$ and $l \ge l_0$.

For $l \ge 2d-1$, the right hand side of (5.3) is at least

$$\begin{aligned}(n-b)\sum_{e=2}^{d}\binom{2d-2e+b}{b} &= (n-b)\sum_{k=0}^{d-2}\binom{2k+b}{b} \qquad (\text{where } k = d-e)\\ &= (n-b)\sum_{k=0}^{d-2}\frac{(2k+b)(2k+b-1)...(2k+1)}{b!}\\ &= (n-b)\sum_{k=0}^{d-2}\left(\frac{2^b k^b}{b!} + \dots\right).\end{aligned}$$

Recall that $\sum_{k=0}^{d} k^b$ is a polynomial in d of degree $b+1$ and leading coefficient $\frac{1}{b+1}$; so the right hand side of (5.3) dominates a polynomial in d of degree $b+1$ and leading coefficient $(n-b)\frac{2^b}{b!}\frac{1}{b+1} = \frac{(n-b)2^b}{(b+1)!}$. Since $\binom{d+b+1}{b+1}$ is a polynomial in d of the same degree $b+1$, but smaller leading coefficient $\frac{1}{(b+1)!}$, the inequality (5.3) holds for all $l \ge 2d-1$ and all $d > d_0$ for some d_0 (which is easy to calculate algorithmically, for fixed n, b).

On the other hand, for each fixed value $d = 2, ..., d_0$, the right hand side of (5.3) is a polynomial in l of degree b and positive leading coefficient $\frac{(n-b)(d-1)}{b!}$, while the left hand side is a constant. So there is l_0 (easily computable for given b, n, d_0) such that for all $d = 2, ..., d_0$ and $l \ge l_0$, the inequality (5.3) holds true. Therefore, for all $2 \le d \le \frac{l+1}{2}$ and $l \ge l_0$, the inequality from the statement of the lemma holds, as well. □

Let l_0 be as in Lemma 5.5.

Corollary 5.6. *Let $2 \le d \le \frac{l+1}{2}$ and $l \ge l_0$. If $Z \subset T_k^d$ is an irreducible component, then either $Z = X^1$, or $\dim Z < \dim X^1$.*

Proof. We claim that if $[F] \in T_k^d - \left(T_k^d \cap (\cup_{d'=1}^{d-1} T_k^{d'})\right)$, then $V(F)_{\text{sing}}$ contains an integral b-dimensional subscheme of degree d. Indeed, $V(F)_{\text{sing}}$ contains some integral b-dimensional closed subscheme of degree $\tilde{d} \in \{1, ..., d\}$; if $[F] \notin \cup_{d'=1}^{d-1} T_k^{d'}$, then necessarily $\tilde{d} = d$.

Now, we can induct on d, so assume that $Z \nsubseteq \cup_{d'=1}^{d-1} T_k^{d'}$. Note that $Z - \left(Z \cap (\cup_{d'=1}^{d-1} T^{d'})\right) \subset Z$ is a dense open subset of Z, which therefore has the same dimension as Z, but is contained in $T_k^d - \left(T_k^d \cap (\cup_{d'=1}^{d-1} T_k^{d'})\right) \subset \rho(R^d) \subset \rho(\overline{R^d})$. Thus $\dim Z \le \dim \rho(\overline{R^d}) < \dim X^1$, by Lemma 5.5 □

In Remark 7.3, we will give a (non-effective) proof of Theorem 1.1 that does not rely on Conjecture 5.4; for this, we will need the following preparation.

Lemma 5.7. *Fix an integer B. There exists l_0 such that for all $2 \le d \le B$ and $l \ge l_0$, for any irreducible component Z of T_k^d, either $Z = X^1$, or $\dim Z < \dim X^1$.*

Proof. Just note that inequality (5.2) in the proof of the previous lemma is satisfied when $d \in \{2, ..., B\}$ is fixed and $l \gg 0$. □

5.4 Preparations for the computation of the second largest component

Here we discuss a calculation similar to the one in the previous section, which will later be used for the computation of the dimension of the second largest component of X. Define

$$\beta_2(l) = \binom{l+b+1}{b+1} - \binom{l+b-3}{b+1} + (n-b-1)\left(\binom{l+b}{b+1} - \binom{l+b-2}{b+1}\right)$$

and set $\gamma_2(l) = \beta_2(l) + 1 - (b+2)n + \frac{b(b+1)}{2}$. We will later see that $\binom{l+n}{n} - \gamma_2(l)$ is the dimension of the second largest component of X, at least when $\operatorname{char} k \neq 0$. We are still assuming Conjecture 5.4.

Lemma 5.8. *There exists l_0 (easily computable) such that for all pairs (d, l) with $3 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$ (if $b = n-1$, assume $d \geq 4$), and any irreducible component Z of T_k^d, either $Z \subset T_k^1 \cup T_k^2$, or*

$$\dim Z < \binom{l+n}{n} - \gamma_2(l).$$

(In the case $b = n-1, d = 3$, we will prove a slightly weaker but sufficient statement in Remark 7.15.)

Proof. Precisely as in Lemma 5.5, because of inequality (5.1), it suffices to establish the inequality

$$\dim \mathbb{P}(W_{\cup L_i}) + \dim \widetilde{\mathrm{Hilb}}^d < \binom{l+n}{n} - \gamma_2(l), \qquad \text{i.e.,}$$

$$\gamma_2(l) - 1 + \dim \widetilde{\mathrm{Hilb}}^d < \mathrm{codim}_V(W_{\cup L_i}).$$

Set $c = -\frac{(b+1)(b+4)}{2} - 1$. By Lemma 4.5 and Conjecture 5.4, we are reduced to proving that

$$c + \beta_2(l) + \binom{d+b+1}{b+1} < \binom{l+b}{b} + (n-b) \sum_{e=1}^{d} \binom{l-2e+1+b}{b},$$

or, equivalently, that

$$c + (n-b)\binom{l+b-3}{b-1} + \binom{l+b-3}{b} + \binom{d+b+1}{b+1} < (n-b) \sum_{e=3}^{d} \binom{l-2e+1+b}{b}. \tag{5.4}$$

Suppose first that $n - b > 1$. If $d = 3$, this inequality is certainly satisfied for $l \gg 0$ (look at the leading terms of both sides). Consider now $d \geq 4$. Since $n - b > 1$, we can find l' such that for all $l \geq l'$,

$$c + (n-b)\binom{l+b-3}{b-1} + \binom{l+b-3}{b} < (n-b)\binom{l-5+b}{b}.$$

What is left now is to prove that there exists l'' such that for $l \geq l''$ and $4 \leq d \leq \frac{l+1}{2}$, we have

$$\binom{d+b+1}{b+1} < (n-b)\sum_{e=4}^{d}\binom{l-2e+1+b}{b}.$$

This is analogous to (5.3) and follows exactly as in the proof of Lemma 5.5. Now we just take $l_0 = \max(l', l'')$.

Suppose now that $n - b = 1$ and $d \geq 4$. If $d = 4$, inequality (5.4) certainly holds for large l (the leading term of the right hand side is $\frac{2l^b}{b!}$). Consider $d \geq 5$. We can find l' such that for all $l \geq l'$,

$$c + \binom{l+b-3}{b-1} + \binom{l+b-3}{b} < \binom{l-5+b}{b} + \binom{l-7+b}{b}.$$

Finally, we have to show that that there exists l'' such that for $5 \leq d \leq \frac{l+1}{2}$ and $l \geq l''$, we have

$$\binom{d+b+1}{b+1} < \sum_{e=5}^{d}\binom{l-2e+1+b}{b}.$$

Again, this is analogous to inequality (5.3). □

Chapter 6

The case of large degree d, when $k = \overline{\mathbb{F}_p}$

Fix n and $1 \leq b \leq n-1$ as usual, and fix a prime p. Let $k = \overline{\mathbb{F}_p}$. Recall the definition of $A_b(l,m)$ from Section 4.3.

The goal of this chapter is to handle the case of large d when $k = \overline{\mathbb{F}_p}$. Specifically, we prove the following

Proposition 6.1. *Fix a triple of positive integers (l,m,a). Set $\tau = \lfloor \frac{l-1}{p} \rfloor$ and $m' = \min(m, \tau+1)$. Suppose that*

$$\binom{\tau+b+1}{b+1} > a-1 \quad \text{and} \quad A_b(\tau, m') > a-1.$$

Let $d \geq m$. If Z is an irreducible component of T_k^d, then either $Z \subset T_k^{d'}$ for some $1 \leq d' < d$, or

$$\dim Z \leq \binom{l+n}{n} - a.$$

Let $Z \subset T_k^d$ be an irreducible component (notation and assumptions as above). Suppose that $Z \not\subseteq \bigcup_{d'=1}^{d-1} T_k^{d'}$. Then $Z - \left(Z \cap \left(\bigcup_{d'=1}^{d-1} T_k^{d'}\right)\right) \subset Z$ is a dense open subset, and therefore is of the same dimension as Z. It is contained in

$$\hat{T}^d := T_k^d - \left(T_k^d \cap \left(\bigcup_{d'=1}^{d-1} T_k^{d'} \right) \right).$$

So the goal is now to prove that $\dim \hat{T}^d \leq \binom{l+n}{n} - a$.

6.1 Reduction to a problem over finite fields

We begin with a general discussion, which applies to any (quasiprojective) variety over $\overline{\mathbb{F}_p}$. Let $T = \cap V(G_i) - \cap V(G'_j) \subset \mathbb{P}^M_{\overline{\mathbb{F}_p}}$ be a quasiprojective variety over $\overline{\mathbb{F}_p}$, where $G_i, G'_j \in \overline{\mathbb{F}_p}[y_0, ..., y_M]$. Let A be an integer, and suppose we want to prove that $\dim T \leq A$. There is a finite field $\mathbb{F}_{q_0}$ such that $G_i, G'_j \in \mathbb{F}_{q_0}[y_0, ..., y_M]$, so T comes from $T_0 := \cap V(G_i) - \cap V(G'_j) \subset \mathbb{P}^M_{\mathbb{F}_{q_0}}$, which is now a variety over $\mathbb{F}_{q_0}$. We know that $\dim T = \dim T_0$, so suffices to prove that $\dim T_0 \leq A$. For this, by the result of Lang-Weil [8], it suffices to prove that $\#T_0(\mathbb{F}_q) = O(q^A)$ as $q \to \infty$ (through powers of q_0 of course).

Consider now $T = \hat{T}^d \subset \mathbb{P}(\overline{\mathbb{F}_p}[x_0, ..., x_n]_l)$, and let $\hat{T}^d_0$ (a variety over a finite field $\mathbb{F}_{q_0}$) be as in the previous paragraph. In particular, $\hat{T}^d_0(\mathbb{F}_q)$ consists of all $[F] \in (\mathbb{F}_q[x_0, ..., x_n]_l - \{0\})/\mathbb{F}_q^*$ such that when we regard $[F]$ in $(\overline{\mathbb{F}_p}[x_0, ..., x_n]_l - \{0\})/\overline{\mathbb{F}_p}^*$, we have that $[F] \in \hat{T}^d \subset \mathbb{P}(\overline{\mathbb{F}_p}[x_0, ..., x_n]_l)$.

Remark 6.2. Even if F has coefficients in $\mathbb{F}_q$, we always consider $V(F)$ and $V(F)_{\text{sing}}$ as subschemes of $\mathbb{P}^n_{\overline{\mathbb{F}_p}}$ by first regarding F in $\overline{\mathbb{F}_p}[x_0, ..., x_n]$.

By the argument in the proof of Corollay 5.6, the set $\hat{T}^d_0(\mathbb{F}_q)$ is a subset of

$$\widetilde{T}^d := \{[F] \in (\mathbb{F}_q[x_0, ..., x_n]_l - \{0\})/\mathbb{F}_q^* \mid V(F)_{\text{sing}} \subset \mathbb{P}^n_{\overline{\mathbb{F}_p}} \text{ contains an integral } b\text{-dimensional subscheme (over } \overline{\mathbb{F}_p}) \text{ of degree } d\}.$$

So our goal now is to prove that $\#\widetilde{T}^d = O(q^{\binom{l+n}{n}-a})$ as $q \to \infty$ (through powers of q_0).

As F is chosen randomly from $\mathbb{F}_q[x_0, ..., x_n]_l$, let Λ be the event that $V(F)_{\text{sing}}$

contains an *integral* b-dimensional subscheme of degree d. Thus, our task is to prove that $\text{Prob}(\Lambda)q^{\binom{l+n}{n}} = O(q^{\binom{l+n}{n}-a+1})$, or equivalently, that $\text{Prob}(\Lambda) = O(q^{-a+1})$ as $q \to \infty$ (through powers of q_0).

6.2 Final preparations

Consider the natural homogenization map $\sim \colon \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq l} \xrightarrow{\simeq} \mathbb{F}_q[x_0, ..., x_n]_l$ with respect to the variable x_n. We have to be slightly careful because this is not the usual homogenization map (which takes a polynomial and homogenizes it to the smallest possible degree); we think of $\sim$ as "homogenization-to-degree-l" map. Recall that $\tau = \lfloor \frac{l-1}{p} \rfloor$.

Lemma 6.3. *Let $Z \subset \mathbb{P}^n_{\overline{\mathbb{F}_p}}$ be an integral closed subscheme not contained in the hyperplane $V(x_n)$. Let $F_0 \in \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq l-1}$ be a fixed polynomial. Then, as G is chosen randomly from $\mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq \tau}$, we have*

$$\text{Prob}(Z \subset V((F_0 + G^p)^{\sim})) \leq \text{Prob}(Z \subset V(G^{\sim})).$$

Here, the first $\sim$ is homogenization to degree $l-1$, and the second one is homogenization to degree τ.

Proof. Let $I \subset \overline{\mathbb{F}_p}[x_0, ..., x_{n-1}]$ be the (radical) ideal of $Z \cap D_+(x_n) \subset D_+(x_n)$. We claim that for an inhomogeneous polynomial $H \in \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq l-1}$, we have $Z \subset V(H^{\sim})$ if and only if $H \in I$. For this, first notice that $V(H^{\sim})$ is either $V(H)^-$ or $V(H)^- \cup V(x_n)$ (where $V(H)^-$ is the topological closure of $V(H) \subset D_+(x_n)$ in $\mathbb{P}^n_{\overline{\mathbb{F}_p}}$), depending on whether or not the degree of H is equal to the degree of homogenization of the map $\sim$. Since Z is irreducible and not contained in $V(x_n)$, we have $Z \subset V(H^{\sim})$ if and only if $Z \subset V(H)^-$. In turn, since $Z \cap D_+(x_n) \neq \emptyset$, this condition is equivalent to $Z \cap D_+(x_n) \subset V(H)$, which is precisely the condition $H \in I$.

Therefore, $Z \subset V((F_0 + G^p)^{\sim})$ if and only if $F_0 + G^p \in I$. If $F_0 + G^p \in I$ and $F_0 + G_1^p \in I$, then $(G - G_1)^p \in I$, and hence $G' := G - G_1 \in I$. So the number of G with $F_0 + G^p \in I$ is either zero, or is equal to the number of elements $G' \in I$ with

$G' \in \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq \tau}$. This is precisely the number of $G' \in \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq \tau}$ such that $Z \subset V((G')^{\sim})$. $\square$

Corollary 6.4. *Keep the notation of Lemma 6.3.*

a) If $\dim Z \geq b+1$, *then*

$$\mathrm{Prob}(Z \subset V((F_0 + G^p)^{\sim})) \leq q^{-\binom{\tau+b+1}{b+1}}.$$

b) If $\dim Z = b$ *and* $\deg Z = d \geq m$, *then*

$$\mathrm{Prob}(Z \subset V((F_0 + G^p)^{\sim})) \leq q^{-A_b(\tau, m')},$$

where $m' = \min(m, \tau + 1)$.

Proof. Combine Lemma 6.3 with Corollaries 4.9 and 4.11. $\square$

6.3 The key step (large degree d)

Fix a triple (l, m, a) of positive integers. Recall that $\tau = \lfloor \frac{l-1}{p} \rfloor$ and $m' = \min(m, \tau+1)$. Let $d \geq m$.

As $F^{\sim}$ is chosen randomly from $\mathbb{F}_q[x_0, ..., x_n]_l$, or, equivalently, as F is chosen randomly from $\mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq l}$, let E_n be the event that the following two conditions are satisfied:

- For each $i = 0, ..., n-b-1$, the variety $V(\frac{\partial F^{\sim}}{\partial x_0}, ..., \frac{\partial F^{\sim}}{\partial x_i})$ has all irreducible components of dimension $n - i - 1$, except possibly for components contained in the hyperplane $V(x_n)$.

- If $C \subset V(\frac{\partial F^{\sim}}{\partial x_0}, ..., \frac{\partial F^{\sim}}{\partial x_{n-b-1}})$ is a b-dimensional integral closed subscheme of degree d, then either $C \subset V(x_n)$, or $C \not\subseteq V(\frac{\partial F^{\sim}}{\partial x_{n-1}})$.

We now proceed to bound $\mathrm{Prob}(E_n)$ from below (this is the hard part).

Lemma 6.5.

$$\operatorname{Prob}(E_n) \geq \left(\prod_{i=0}^{n-b-1} \left(1 - \frac{(l-1)^i}{q^{\binom{\tau+b+1}{b+1}}} \right) \right) \left(1 - \frac{(l-1)^{n-b}}{q^{A_b(\tau,m')}} \right). \tag{6.1}$$

Proof. We now mimic the main argument in [10, Section 2.3]. We will generate a random F by choosing $F_0 \in \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq l}, G_i \in \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq \tau}$ randomly, in turn, and then setting

$$F := F_0 + G_0^p x_0 + \cdots + G_{n-1}^p x_{n-1}. \tag{6.2}$$

For $F \in \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq l-1}$, the number of tuples $(F_0, G_0, ..., G_{n-1})$ for which (6.2) holds is independent of F. We have

$$\frac{\partial F}{\partial x_i} = \frac{\partial F_0}{\partial x_i} + G_i^p.$$

Moreover, the homogenization map $\sim$ commutes with differentiation, so

$$\frac{\partial F^{\sim}}{\partial x_i} = \left(\frac{\partial F_0}{\partial x_i} + G_i^p \right)^{\sim}$$

(again, the two uses of $\sim$ here refer to homogenizations to different degrees, l and $l-1$, respectively).

Let $i \in \{0, ..., n-b-1\}$. Suppose that $F_0, G_0, ..., G_{i-1}$ are fixed such that $V(\frac{\partial F^{\sim}}{\partial x_0}, ..., \frac{\partial F^{\sim}}{\partial x_{i-1}})$ has only $(n-i)$-dimensional components, except possibly for components contained in the hyperplane $V(x_n)$. By Bézout's theorem (see p. 10 in [5] for the version we are using here), $V(\frac{\partial F^{\sim}}{\partial x_0}, ..., \frac{\partial F^{\sim}}{\partial x_{i-1}})$ has at most $(l-1)^i$ irreducible components. Let Z be one of them, and suppose that $Z \not\subseteq V(x_n)$. As G_i is chosen randomly from $\mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq \tau}$, we claim that

$$\operatorname{Prob}\left(Z \subset V\left(\frac{\partial F^{\sim}}{\partial x_i} \right) \right) \leq q^{-\binom{\tau+b+1}{b+1}}.$$

This follows from Corollary 6.4a, since $\dim Z = n - i \geq b + 1$.

For the final step, conditioned on a choice of $F_0, G_0, ..., G_{n-b-1}$ such that $V(\frac{\partial F^{\sim}}{\partial x_0}, ..., \frac{\partial F^{\sim}}{\partial x_{n-b-1}})$ has only b-dimensional components, except possibly for components contained in $V(x_n)$, we claim that the probability, as $G_{n-1} \in \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq \tau}$, that some b-dimensional component C of $V(\frac{\partial F^{\sim}}{\partial x_0}, ..., \frac{\partial F^{\sim}}{\partial x_{n-b-1}})$ of degree d and not contained in $V(x_n)$, is contained in $V(\frac{\partial F^{\sim}}{\partial x_{n-1}})$, is at most $(l-1)^{n-b} q^{-A_b(\tau, m')}$.

Indeed, the number of b-dimensional components C of $V(\frac{\partial F^{\sim}}{\partial x_0}, ..., \frac{\partial F^{\sim}}{\partial x_{n-b-1}})$ of degree d is at most $(l-1)^{n-b}$, by Bézout's theorem again (this is a bound on the total number of components of all dimensions). If we fix a b-dimensional component C of degree d and not contained in $V(x_n)$, for fixed $F_0, G_0, ..., G_{n-b-1}$, the probability (as G_{n-1} is chosen randomly from $\mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq \tau}$) that $C \subset V\left((\frac{\partial F_0}{\partial x_{n-1}} + G_{n-1}^p)^{\sim}\right)$, is at most $q^{-A_b(\tau, m')}$, by Corollary 6.4b. □

Proof of Proposition 6.1. By the hypothesis of Proposition 6.1, each of the exponents on the right hand side of (6.1) is greater than $a-1$. By virtue of the inequality $\prod(1-\varepsilon_i) \geq 1 - \sum \varepsilon_i$, Lemma 6.5 implies that $\text{Prob}(E_n) \geq 1 - \frac{1}{q^{a-1}}$ for large q. Therefore,

$$1 - \text{Prob}(E_n) = O\left(\frac{1}{q^{a-1}}\right) \quad \text{as } q \to \infty.$$

As $F \in \mathbb{F}_q[x_0, ..., x_{n-1}]_{\leq l}$, let E_n' be the event that any integral b-dimensional closed subscheme $C \subset V(F)_{\text{sing}}$ of degree d is contained in $V(x_n)$. Then E_n implies E_n'. For each $i = 0, ..., n-1$, define E_i, E_i' in analogy with E_n, E_n', except with de-homogenization with respect to the variable x_i (and any ordering of the remaining variables). The same conclusion $1 - \text{Prob}(E_i) = O(\frac{1}{q^{a-1}})$ holds for all $i = 0, ..., n$. Note that Λ (defined at the end of Section 6.1) implies $\bigcup_{i=0}^{n} \overline{E_i'}$, where $\overline{E_i'}$ denotes the event opposite to E_i'. Indeed, $\bigcap V(x_i) = \emptyset$, so we cannot have $C \subset V(F)_{\text{sing}}$ contained in all the coordinate hyperplanes. Therefore,

$$\text{Prob}(\Lambda) \leq \sum_{i=0}^{n}(1 - \text{Prob}(E_i')) \leq \sum_{i=0}^{n}(1 - \text{Prob}(E_i)) = O\left(\frac{1}{q^{a-1}}\right) \text{ as } q \to \infty,$$

as desired. □

Chapter 7

Proof of the main theorem

We now put together the main results Corollary 5.6 and Proposition 6.1 and finish the proof of Theorem 1.1. Namely, Theorem 1.1 follows immediately from our previous work when $k = \overline{\mathbb{F}_p}$, and we use upper-semicontinuity applied to $T^d \to \operatorname{Spec}\mathbb{Z}$ to prove the case $\operatorname{char} k = 0$ (Section 7.1). However, there are technicalities (Corollary 7.7) concerning the uniqueness of the largest component in characteristic 0, which we discuss in Section 7.2. Finally, in Section 7.3, we finish the discussion of the second largest component, but only when $\operatorname{char} k \neq 0$ (Corollary 7.16).

7.1 Restatement of the problem and the end of the proof

Lemma 7.1. *Let* $[F] \in \mathbb{P}(V)$ *be such that* $\dim V(F)_{\text{sing}} \geq b$. *Then there is an integral* b*-dimensional closed subscheme* $C \hookrightarrow \mathbb{P}^n$ *of degree at most* $l(l-1)^{n+1}$ *such that* $C \subset V(F)_{\text{sing}}$.

Proof. Let $Z_1, \ldots, Z_s$ be the irreducible components of $V(F)_{\text{sing}} = V(F, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n})$. Then by Bézout's theorem ([5], p. 10),

$$\sum_{i=1}^{s} \deg(Z_i) \leq \deg(F) \prod_{\substack{0 \leq j \leq n \\ \partial F/\partial x_j \neq 0}} \deg\left(\frac{\partial F}{\partial x_j}\right) \leq l(l-1)^{n+1}.$$

But some component Z_i has dimension at least b, so, intersecting with hyperplanes if necessary, this component will contain an integral b-dimensional closed subscheme of degree at most $\deg(Z_i) \leq l(l-1)^{n+1}$. $\square$

Proof of Theorem 1.1 assuming Conjecture 5.4. By Lemma 7.1, X is a finite union:

$$X = \bigcup_{d=1}^{l(l-1)^{n+1}} T_k^d. \tag{7.1}$$

In particular, X is a closed subset of $\mathbb{P}(V)$. The statement of Theorem 1.1 is now equivalent to the following one: for any $d \geq 2$, we have $\dim(T_k^d - T_k^1) < \dim X^1$. But $T_k^d - T_k^1 = (T^d - T^1)_k$, and if $k_0 \subset k$ is a subfield, then $\dim(T^d - T^1)_k = \dim(T^d - T^1)_{k_0}$. So it suffices to assume that $k = \overline{\mathbb{F}_p}$ or $k = \overline{\mathbb{Q}}$.

First, suppose that $k = \overline{\mathbb{F}_p}$. Let $\tau(l) = \lfloor \frac{l-1}{p} \rfloor$ and $m(l) = \lceil \frac{l+1}{2} \rceil$. Notice that $m(l) \geq \tau(l)+1$, so $m' = \tau(l)+1$ in Proposition 6.1. There exists $l_0 = l_0(n, b, p)$ (easily computable) such that for all $l \geq l_0$, we have $\binom{\tau(l)+b+1}{b+1} > a_{n,b}(l)$ and $A_b(\tau(l), \tau(l)+1) > a_{n,b}(l)$, by Remark 4.7 and the fact that $a_{n,b}(l)$ is a polynomial in l of degree b. We can assume in addition that l_0 satisfies Corollary 5.6. We claim that for any $l \geq l_0(n, b, p)$, the statement of Theorem 1.1 holds.

In fact, we prove by induction on $d \geq 2$ that for any irreducible component Z of T_k^d, either $Z = X^1$ or $\dim Z < \dim X^1$. For $2 \leq d \leq \frac{l+1}{2}$ this follows from Corollary 5.6. Let $d \geq \frac{l+1}{2}$. Assume that the statement holds for all $2 \leq d' \leq d-1$. Then it also holds for d, by Proposition 6.1, applied to the triple $(l, m(l), a_{n,b}(l)+1)$.

Now, let $k = \overline{\mathbb{Q}}$. Let p be any prime, and consider $l \geq l_0(n, b, p)$ as above. By the previous paragraph, for any $d \geq 2$, $\dim T_{\mathbb{F}_p}^d = \dim T_{\overline{\mathbb{F}_p}}^d \leq \dim X^1$. But, since $T^d \to \operatorname{Spec} \mathbb{Z}$ is projective, by the upper semicontinuity theorem, we know

$$\dim T_{\overline{\mathbb{Q}}}^d = \dim T_{\mathbb{Q}}^d \leq \dim T_{\mathbb{F}_p}^d \leq \dim X^1.$$

Therefore, as long as $l \geq l_0(n, b, p)$ for some p (take $p = 2$ to obtain the best value of l_0 here), we know that X^1 (over $\overline{\mathbb{Q}}$) is *an* irreducible component of X (over $\overline{\mathbb{Q}}$) of maximal dimension.

We now address the question of uniqueness of X^1 as a largest component. In Section 7.2 we will show that it is possible to choose p such that $X^1 \not\subseteq T^d_{\overline{\mathbb{F}_p}}$ for any $d \geq 2$. For such p, and for $d \geq 2$, the conclusion from two paragraphs ago implies

$$\dim T^d_{\overline{\mathbb{F}_p}} < \dim X^1.$$

So

$$\dim T^d_{\overline{\mathbb{Q}}} \leq \dim T^d_{\overline{\mathbb{F}_p}} < \dim X^1.$$

By (7.1), any irredcible component of X is either X^1 or is contained in Td_k for some $d \geq 2$. This completes the proof. □

Remark 7.2. We postpone for the next section the fact that over $\overline{\mathbb{F}_p}$, we have $X^1 \not\subseteq T^d_{\overline{\mathbb{F}_p}}$, provided that $p \neq 2$ or $n - b$ is even. So for $l \geq l_0(n, b, 2)$, we know that X^1 is an irreducible component of X of largest dimension; for $l \geq l_0(n, b, 2)$ when $n - b$ is even, and for $l \geq l_0(n, b, 3)$ when $n - b$ is odd, we also know that X^1 is the unique largest-dimensional component of X.

Remark 7.3. We now give a (non-effective) proof of Theorem 1.1 without using Conjecture 5.4. It is the same as before, except that we use Lemma 5.7 in place of Corollary 5.6, and use a different value of m in Proposition 6.1. Set $B = p^b(n - b + 1)$ in Lemma 5.7, and set $m = p^b(n - b + 1) + 1$ in Proposition 6.1. By the definition in Lemma 4.6 and by the definition of $\tau(l)$, we have that $A_b(\tau, m)$ grows as a polynomial in l of degree b and leading coefficient $\frac{m}{p^b b!} > \frac{n-b+1}{b!}$, so $A_b(\tau, m) > a_{n,b}(l)$ for sufficiently large l (recall the definition of $a_{n,b}(l)$ from the intoduction). Thus, the hypothesis of Proposition 6.1 is satisfied again.

7.2 Uniqueness of the largest component (in characteristic 0)

We set the following notation for this section. Consider a b-dimensional closed subscheme $C = V(f, x_{b+2}, ..., x_n)$ of $\mathbb{P}^n$, where $f \in k[x_0, ..., x_{b+1}]_d - \{0\}$, and set

$W = (f, x_{b+2}, ..., x_n)_l^2$. In order to finish the proof of Theorem 1.1, it will be sufficient to consider the case when C is a linear b-dimensional subspace in the next lemma; however, we will use the more general statement (when $d = 2$) in Section 7.3.

Lemma 7.4. *Assume $l \geq 2d+1$. There is a dense open subset $U_1 \subset \mathbb{P}(W)$ such that for all $[F] \in U_1$, $V(F)_{\text{sing}} = C$ (set-theoretically).*

Proof. Consider the incidence correspondence

$$Y_1 = \{([F], P) \in \mathbb{P}(W) \times (\mathbb{P}^n - C) \mid P \in V(F)_{\text{sing}}\} \subset \mathbb{P}(W) \times (\mathbb{P}^n - C)$$

(it is a closed subset of this product, and hence a quasiprojective variety). We are going to show that $\dim Y_1 < \dim \mathbb{P}(W)$; this will imply that the closure $\overline{Y_1}$ of Y_1 in $\mathbb{P}(W) \times \mathbb{P}^n$ also has dimension smaller than that of $\mathbb{P}(W)$, and thus the image of this closure under the projection to $\mathbb{P}(W)$ will be a proper closed subset of $\mathbb{P}(W)$. Its complement U_1 will satisfy the condition of the lemma.

Consider the second projection $\tau \colon Y_1 \to \mathbb{P}^n - C$, and let $P \in \mathbb{P}^n - C$. We claim the fiber $\tau^{-1}(P)$ is a projective linear subspace of $\mathbb{P}(W)$ of codimension $n+1$. This will imply that Y_1 is irreducible, of dimension $\dim Y_1 = \dim \mathbb{P}(W) - 1$.

Suppose first that $P \in \cup_{i=b+2}^n D_+(x_i)$. Without loss of generality, assume that $P = [a_0, ..., a_{n-1}, 1]$. Notice that $\tau^{-1}(P)$ is just

$$\mathbb{P}\left(((x_0 - a_0 x_n, ..., x_{n-1} - a_{n-1}x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2)_l\right) \subset \mathbb{P}(W),$$

so it remains to show that

$$\dim \left(\frac{W}{(x_0 - a_0 x_n, ..., x_{n-1} - a_{n-1}x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2} \right)_l = n+1,$$

i.e., that the map

$$\left(\frac{(f, x_{b+2}, ..., x_n)^2}{(x_0 - a_0x_n, ..., x_{n-1} - a_{n-1}x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2}\right)_l \hookrightarrow$$
$$\left(\frac{S}{(x_0 - a_0x_n, ..., x_{n-1} - a_{n-1}x_n)^2}\right)_l \simeq k[x_n]_l \oplus \left(\bigoplus_{i=0}^{n-1} k[x_n]_{l-1}(x_i - a_ix_n)\right)$$

is an isomorphism. The images of x_n^l and $x_n^{l-1}(x_i - a_ix_n)$ for $i = 0, ..., n-1$ give a basis of the target.

Suppose now that $P \in V(x_{b+2}, ..., x_n)$, without loss of generality $P = [1, a_1, ..., a_{b+1}, 0, ..., 0]$. As above, we have to prove that the following map is an isomorphism:

$$\left(\frac{(f, x_{b+2}, ..., x_n)^2}{(x_1 - a_1x_0, ..., x_{b+1} - a_{b+1}x_0, x_{b+2}, ..., x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2}\right)_l \hookrightarrow$$
$$\left(\frac{S}{(x_1 - a_1x_0, ..., x_{b+1} - a_{b+1}x_0, x_{b+2}, ..., x_n)^2}\right)_l \simeq$$
$$k[x_0]_l \oplus \left(\bigoplus_{i=1}^{b+1} k[x_0]_{l-1}(x_i - a_ix_0)\right) \oplus \left(\bigoplus_{i=b+2}^{n} k[x_0]_{l-1}x_i\right).$$

Now, dehomogenize f with respect to x_0, consider a Taylor expansion at $(a_1, ..., a_{b+1})$, and homogenize to degree l again, so $f \equiv ax_0^d \pmod{(x_1 - a_1x_0, ..., x_{b+1} - a_{b+1}x_0)}$ with $a \neq 0$. So $f^2 \equiv a^2x_0^{2d} \pmod{(x_1 - a_1x_0, ..., x_{b+1} - a_{b+1}x_0)}$. Now, the elements $f^2x_0^{l-2d-1}(x_i - a_ix_0)$ (for $i = 1, ..., b+1$), $f^2x_0^{l-2d-1}x_i$ (for $i = b+2, ..., n$), and $f^2x_0^{l-2d}$ map to a basis of the target. □

We will use the lemma below only when C is linear, but we prove it here for a more general C for the purposes of the later discussion in Remark 7.17.

Lemma 7.5. *Suppose that $l \geq 2d$. If char $k \neq 2$, then there exists a dense open subset $U_2 \subset \mathbb{P}(W)$ such that for all $[F] \in U_2$, we have*

$$\dim\{P \in C \mid \dim T_PV(F)_{\text{sing}} \geq b+1\} \leq b-1.$$

If char $k = 2$ and C is a b-dimensional linear subspace, and $n - b$ is even, then the same conclusion holds.

Proof. Consider the incidence correspondence

$$Y_2 = \{([F], P) \in \mathbb{P}(W) \times C \mid \dim T_P V(F)_{\text{sing}} \geq b+1\} \subset \mathbb{P}(W) \times C$$

(this is a closed subset). We will show that $Y_2 \neq \mathbb{P}(W) \times C$, i.e., $\dim Y_2 \leq \dim \mathbb{P}(W) + b - 1$. Once this is done, the map $Y_2 \to \mathbb{P}(W)$ will give a dense open $U_2 \subset \mathbb{P}(W)$ such that the fiber over any $[F] \in U_2$ has dimension at most $b-1$.

Suppose that $\operatorname{char} k \neq 2$. Fix a point $P = [p_0, ..., p_{b+1}, 0, ..., 0] \in C$ with at least 2 nonzero coordinates such that $V(f) \subset \mathbb{P}^{b+1} = V(x_{b+2}, ..., x_n)$ is smooth at P. Without loss of generality, $\frac{\partial f}{\partial x_{b+1}}(P) \neq 0$ and $p_0 \neq 0$. We claim that there exists $[F] \in \mathbb{P}(W)$ with $\dim T_P V(F)_{\text{sing}} \leq b$.

For $[F] \in \mathbb{P}(W)$, we have $V(F)_{\text{sing}} = V(F, \frac{\partial F}{\partial x_0}, ..., \frac{\partial F}{\partial x_n})$, so we have to look at the Jacobian

$$J(P) = \begin{pmatrix} \frac{\partial F}{\partial x_0}(P) & \frac{\partial F}{\partial x_1}(P) & \cdots & \frac{\partial F}{\partial x_n}(P) \\ \frac{\partial^2 F}{\partial x_0^2}(P) & \frac{\partial^2 F}{\partial x_0 \partial x_1}(P) & \cdots & \frac{\partial^2 F}{\partial x_0 \partial x_n}(P) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_0}(P) & \frac{\partial^2 F}{\partial x_n \partial x_1}(P) & \cdots & \frac{\partial^2 F}{\partial x_n^2}(P) \end{pmatrix}.$$

We know that $\dim T_P V(F)_{\text{sing}} = n - \operatorname{rk} J(P)$, so $\dim T_P V(F)_{\text{sing}} \leq b$ if and only if $\operatorname{rk} J(P) \geq n - b$. In other words, we have to give some $[F] \in \mathbb{P}(W)$ such that some $(n-b) \times (n-b)$ minor of the Jacobian is nonzero. Consider

$$F = x_0^{l-2d} f^2 + \sum_{i=b+2}^{n} x_0^{l-2} x_i^2.$$

We claim that the bottom right $(n-b) \times (n-b)$ minor of $J(P)$ is nonzero. Since $p_0 \neq 0$ and $\frac{\partial f}{\partial x_{b+1}}(P) \neq 0$,

$$\frac{\partial^2 F}{\partial x_{b+1}^2}(P) = 2x_0^{l-2d} \left(\left(\frac{\partial f}{\partial x_{b+1}} \right)^2 + f \frac{\partial^2 f}{\partial x_{b+1}^2} \right) (P) \neq 0,$$

so the minor

$$\left(\frac{\partial^2 F}{\partial x_i \partial x_j}(P) \right)_{b+1 \leq i,j \leq n} \tag{7.2}$$

is a diagonal matrix with nonzero diagonal entries.

Now suppose that char $k = 2$ but $n - b$ is even and $C = V(x_{b+1}, ..., x_n)$. Let $P = [1, 0, ..., 0]$. Consider $F = \sum_{i=1}^{\frac{n-b}{2}} x_{b+2i-1}x_{b+2i}x_0^{l-2}$. Then the minor (7.2) is nonzero again. □

Remark 7.6. This lemma fails when char $k = 2$, C is linear, and $n - b$ is odd.

Corollary 7.7. *Suppose that char $k \neq 2$ or char $k = 2$ but $n - b$ is even. Then $X^1 \not\subseteq T^d_{\overline{\mathbb{F}}_p}$ for any $d \geq 2$.*

Proof. Let $C = V(x_{b+1}, ..., x_n)$. Let U_1 and U_2 be as given by Lemmas 7.4 and 7.5. Let $U = U_1 \cap U_2$. So U is a dense open subset of $P(W)$ such that for all $[F] \in U$, $V(F)_{\text{sing}} = L$ set-theoretically, and in addition, the closed embedding $L \hookrightarrow V(F)_{\text{sing}}$ is an isomorphism over the complement of a closed subset of smaller dimension. Thus the Hilbert polynomial of $V(F)_{\text{sing}}$ has degree b and leading term $1/b!$, so $V(F)_{\text{sing}}$ does not contain any closed subscheme of dimension b and degree $d \geq 2$. In other words, $[F] \in X^1 - T^d_{\overline{\mathbb{F}}_p}$. □

Similarly, we can apply Lemmas 7.4 and 7.5 to an integral $C = V(f, x_{b+2}, ..., x_n)$ of degree 2 and obtain the following

Corollary 7.8. *Suppose that* char $k \neq 2$. *There exists $[F] \in \mathbb{P}(V)$ such that $V(F)_{\text{sing}}$ is a b-dimensional integral closed subscheme of degree* 2 *(as a set), and such that $V(F)_{\text{sing}}$ does not contain any b-dimensional closed subscheme of degree $d \geq 3$.*

7.3 The second largest component

In contrast to the treatment of the largest component of X, the existence of a component of the expected second-largest dimension is a little more subtle, so there will be an extra twist in the argument. We will determine the second largest component of X when char $k \neq 0$ (Corollary 7.16). Unfortunately, a technical problem will prevent us from deducing the corresponding statement when char $k = 0$ (see Remark 7.17).

For now, k is again any algebraically closed field.

Fix n, b as usual, and let $d \geq 1$. Define

$$\beta_d(l) = \binom{l+b+1}{b+1} - \binom{l-2d+b+1}{b+1} + (n-b-1)\left(\binom{l+b}{b+1} - \binom{l-d+b}{b+1}\right)$$
$$= \frac{(n-b+1)d}{b!} l^b + \dots$$

Let $I = (f, x_{b+2}, \dots, x_n) \subset S = k[x_0, \dots, x_n]$, where $f \in k[x_0, \dots, x_{b+1}]_d - \{0\}$. Consider the composition

$$\Phi\colon k[x_0, \dots, x_{b+1}]_l \oplus \left(\bigoplus_{i=b+2}^{n} k[x_0, \dots, x_{b+1}]_{l-1} x_i\right) \hookrightarrow S_l \twoheadrightarrow S_l/(I^2 \cap S_l).$$

Note that Φ is surjective.

Lemma 7.9. *We have that*

$$\ker(\Phi) = \{P + \sum_{i=b+2}^{n} P_i x_i \ : \ f^2|P, f|P_i \text{ for } i = b+2, \dots, n\}.$$

For $l \geq 2d$, the codimension of I_l^2 in S_l equals $\beta_d(l)$.

Proof. If $P + \sum P_i x_i \in \ker(\Phi)$, then we can write $P + \sum P_i x_i = T \in I^2$. Expand both sides as polynomials in $x_{b+2}, \dots, x_n$ and just compare the two expressions. The second part is an immediate consequence. □

Lemma 7.10. *Let $C \hookrightarrow \mathbb{P}^n$ be any integral b-dimensional closed subscheme of degree 2, with (saturated) ideal I. If $F \in k[x_0, \dots, x_n]_l$ satisfies $C \subset V(F)_{\text{sing}}$, then $F \in I_l^2$.*

Proof. Projection from a point on C shows that C is contained in a linear $(b+1)$-dimensional subspace of $\mathbb{P}^n$. So we can assume that $C = V(I)$, with $I = (f, x_{b+2}, \dots, x_n)$, where $f \in k[x_0, \dots, x_{b+1}]_2 - \{0\}$ is irreducible. We claim that the ideal I^2 is saturted. Indeed, let $F \in S$ be homogeneous, and suppose that $x_j^M F \in I^2$ for all $j = 0, \dots, n$ (and for some M). Write $F = P + \sum_{i=b+2}^{n} P_i x_i + T$, where $P, P_i \in k[x_0, \dots, x_{b+1}]$ are homogeneous of the appropriate degrees, and $T \in (x_{b+2}, \dots, x_n)^2$. Since $x_0^M F \in I^2$, Lemma 7.9 implies that $f^2 | x_0^M P$ and $f | x_0^M P_i$ for each $i = b+2, \dots, n$. Since f and x_0 are relatively prime, it follows that $f^2 | P$ and $f | P_i$ for each i, and hence $F \in I^2$.

Since C is a local complete intersection and the ideal I^2 is saturated, the conclusion now follows from Proposition 8.2, where we prove a more general result. $\square$

Let $P = \binom{z+b+1}{b+1} - \binom{z-1+b}{b+1}$ (this is the Hilbert polynomial of a degree-2 hypersurface in $\mathbb{P}^{b+1}$). Recall that $\widetilde{\text{Hilb}}^P$ denotes the closure in Hilb^P of the set of integral b-dimensional closed subschemes of degree 2; in this case, a point in $\widetilde{\text{Hilb}}^P$ is, up to a change of coordinates, a closed subscheme of the form $V(f, x_{b+2}, ..., x_n) \subset \mathbb{P}^n$, where $f \in k[x_0, .., x_{b+1}]_2 - \{0\}$ (not necessarily irreducible of course). Note that

$$\begin{aligned}\dim \widetilde{\text{Hilb}}^P &= \dim \mathbb{G}(b+1, n) + \dim \mathbb{P}(k[x_0, ..., x_{b+1}]_2) \\ &= (b+2)n - \frac{b(b+1)}{2}.\end{aligned} \tag{7.3}$$

By Lemma 7.9, if $f \in k[x_0, ..., x_{b+1}]_2 - \{0\}$, then

$$\dim \mathbb{P}\left((f, x_{b+2}, ..., x_n)_l^2\right) = \binom{l+n}{n} - \beta_2(l) - 1. \tag{7.4}$$

Recall the usual incidence correspondence (where inclusion is scheme-theoretic)

$$\widetilde{\Omega}^P = \{(C, [F]) \in \widetilde{\text{Hilb}}^P \times \mathbb{P}(V) \mid C \subset V(F)_{\text{sing}}\} \subset \widetilde{\text{Hilb}}^P \times \mathbb{P}(V).$$

Recall that π and ρ denote the projections to $\widetilde{\text{Hilb}}^P$ and $\mathbb{P}(V)$, respectively. For $C \subset \mathbb{P}^n$ a closed subscheme, let I_C denote its (saturated) ideal. Consider the subset

$$Z' = \{(C, [F]) \in \widetilde{\text{Hilb}}^P \times \mathbb{P}(V) \mid F \in I_C^2\} \subset \widetilde{\Omega}^P.$$

Lemma 7.11. *The subset Z' of $\widetilde{\Omega}^P$ is irreducible.*

Proof. By Lemma 7.9, for a fixed $f \in k[x_0, ..., x_{b+1}]_2 - \{0\}$ and given $F = F_0 + \sum_{i=b+2}^n F_i x_i + T \in k[x_0, ..., x_n]_l$, where $F_0 \in k[x_0, ..., x_{b+1}]_l$, $F_i \in k[x_0, ..., x_{b+1}]_{l-1}$, and $T \in (x_{b+2}, ..., x_n)_l^2$, we have that $F \in (f, x_{b+2}, ..., x_n)_l^2$ if and only if $f^2 | F_0$ and $f | F_i$ for each $i = b+2, ..., n$.

Let $V' = k[x_0, ..., x_{b+1}]_{l-4} \oplus \left(\bigoplus_{i=b+2}^n k[x_0, ..., x_{b+1}]_{l-3}\right) \oplus (x_{b+2}, ..., x_n)_l^2$. Denote by

$\mathbb{A}(k[x_0, ..., x_{b+1}]_2)$ the affine space parametrizing points in $k[x_0, ..., x_{b+1}]_2$. Consider the composition

$$\begin{array}{c} \mathrm{Aut}(\mathbb{P}^n) \times (\mathbb{A}(k[x_0, ..., x_{b+1}]_2) - \{0\}) \times \mathbb{P}(V') \\ \downarrow \\ \mathrm{Aut}(\mathbb{P}^n) \times \mathbb{P}(k[x_0, ..., x_{b+1}]_2) \times \mathbb{P}(V) \\ \downarrow \\ \widetilde{\mathrm{Hilb}}^P \times \mathbb{P}(V) \end{array}$$

where the first map is given by

$$(\sigma, f, [Q, R_{b+2}, ..., R_n, T]) \longmapsto (\sigma, [f], [f^2 Q + \sum_{i=b+2}^{n} f R_i x_i + T])$$

and the second map is given by

$$(\sigma, [f], [F]) \longmapsto (V(f^\sigma, x_{b+2}^\sigma, ..., x_n^\sigma), [F]^\sigma).$$

By construction, Z' is precisely the image of the composition, hence is irreducible. $\square$

Remark 7.12. It is not true that the fibers of $\widetilde{\Omega}^P \xrightarrow{\pi} \widetilde{\mathrm{Hilb}}^P$ are all of the same dimension. For example, let $b = 1, n = 3$, and look at $C = V(x_2^2, x_3) \in \widetilde{\mathrm{Hilb}}^P$. Let $F = x_2^3 x_0^{l-3}$. Then $(C, [F]) \in \pi^{-1}(C)$, but $F \notin (x_2^2, x_3)^2$. This is why we have to study the auxiliary Z'.

Let Z be the closure of Z' in $\widetilde{\Omega}^P$.

Lemma 7.13. *We have that*

$$\dim Z = \binom{l+n}{n} - \beta_2(l) - 1 + (b+2)n - \frac{b(b+1)}{2}.$$

Proof. First, $\pi(Z') = \widetilde{\mathrm{Hilb}}^P$, since given any $C \in \widetilde{\mathrm{Hilb}}^P$, the ideal I_C^2 contains forms of degree 4 already, so we can certainly find $F \in (I_C^2)_l$. Thus, $\pi\colon Z \twoheadrightarrow \widetilde{\mathrm{Hilb}}^P$ is onto. A generic $C \in \widetilde{\mathrm{Hilb}}^P$ is an integral b-dimensional closed subscheme of degree

2; for such a C, by Lemma 7.10, we know $Z'_C = \widetilde{\Omega}^P_C$ and hence also $Z_C = Z'_C$. This allows us to compute $\dim Z_C = \dim Z'_C = \binom{l+n}{n} - \beta_2(l) - 1$. This computes $\dim Z = \dim \widetilde{\mathrm{Hilb}}^P + \dim Z_C$ and gives the desired result, by virtue of (7.3) and (7.4). ☐

Lemma 7.14. *$X^2 := \rho(Z)$ is an irreducible closed subset of X of dimension $\binom{l+n}{n} - \beta_2(l) - 1 + (b+2)n - \frac{b(b+1)}{2}$. If $[F] \in X$ contains an integral closed subscheme of dimension b and degree 2 in its singular locus, then $[F] \in X^2$.*

Proof. It is clear that $\rho(Z)$ is an irreducible closed subset of X, since Z is irreducible and closed in $\widetilde{\Omega}^P$. Choose any integral b-dimensional C of degree 2. Apply Lemma 7.4 to C to find $[F] \in \mathbb{P}(V)$ such that we have a homeomorphism $C \hookrightarrow V(F)_{\text{sing}}$. If $\hat{C} \in \widetilde{\mathrm{Hilb}}^P$ is another closed subscheme contained in $V(F)_{\text{sing}}$, then necessarily we have $C \hookrightarrow \hat{C}$, since C is reduced. Hence $C = \hat{C}$, since C and $\hat{C}$ have the same Hilbert polynomial. Therefore, the map $Z \to \rho(Z)$ has a 0-dimensional fiber, so $\dim \rho(Z) = \dim Z$.

Let $[F] \in X$ be such that $V(F)_{\text{sing}}$ contains an integral b-dimensional closed subscheme C of $\mathbb{P}^n$ of degree 2. Then we know that $F \in I_C^2$ by Lemma 7.10, so $(C, [F]) \in Z'$, and hence in fact $[F] \in \rho(Z') \subset \rho(Z) = X^2$. ☐

Remark 7.15. Lemma 5.8 did not treat the case $b = n-1, d = 3$. We discuss this now. When $b = n-1$, we can describe X explicitly. Indeed, if $V(G)$ is an integral $(n-1)$-dimensional closed subscheme of $\mathbb{P}^n_k$ (here k has any characteristic) with $V(G) \subset V(F)_{\text{sing}}$, then necessarily $F = G^2H$ for some H (since $V(G)$ is a complete intersection and the ideal (G^2) is saturated; see Proposition 8.2). For $d = 1, \ldots, \lfloor \frac{l}{2} \rfloor$, consider the map

$$\begin{aligned} \varphi_d \colon \mathbb{P}(k[x_0, \ldots, x_n]_d) \times \mathbb{P}(k[x_0, \ldots, x_n]_{l-2d}) &\longrightarrow \mathbb{P}(k[x_0, \ldots, x_n]_l) \\ (G, H) &\longmapsto G^2H. \end{aligned}$$

Certainly, $\mathrm{im}(\varphi_d) \subset T_k^d \subset X$ and $X = \bigcup_{d=1}^{\lfloor \frac{l}{2} \rfloor} \mathrm{im}(\varphi_d)$, so

$$X = X^1 \cup \mathrm{im}(\varphi_2) \cup \mathrm{im}(\varphi_3) \cup \left(\bigcup_{d=4}^{\lfloor \frac{l}{2} \rfloor} T^d \right).$$

Since any point in the image of φ_d has only finitely many preimages, it follows that

$$\dim \mathrm{im}(\varphi_d) = \binom{d+n}{n} + \binom{l-2d+n}{n} - 2.$$

So $\dim \mathrm{im}(\varphi_3) < \dim \mathrm{im}(\varphi_2) = \dim X^2$ for $l \geq l_0$ (where l_0 is effectively computable) and hence when $b = n-1$, it suffices bound $\dim T_k^d$ only for $d \geq 4$, which was handled by Lemma 5.8.

Corollary 7.16. *Suppose that* $\mathrm{char}\, k = p > 0$. *There exists (again, effectively computable)* $l_0 = l_0(n, b, p)$ *such that for all* $l \geq l_0$, X^2 *is the unique irreducible component of* X *of second largest dimension.*

Proof. Let $k = \overline{\mathbb{F}_p}$. With the above preparations, the proof is now analogous to that of Theorem 1.1. We use Lemma 5.8 (with Remark 7.15 if $b = n-1$) and Proposition 6.1 to argue that if $Z \subset T_k^d$ is an irreducible component of T_k^d (where $d \geq 3$), then either $Z \subset T_k^1 \cup T_k^2$, or $\dim Z < \dim X^2$ (as long as $l \geq l_0$, for some effectively computable l_0).

We have

$$X = \bigcup_{d=1}^{N} T_k^d \qquad \text{for } N = l(l-1)^{N+1}..$$

If Z is an irreducible component of X with $\dim Z \geq \dim X^2$, then $Z \subset T_k^d$ for some d. If $d \geq 3$, then by the previous paragraph, we have $Z \subset T^1 \cup T^2$. So in any case, $Z \subset T^1 \cup T^2 = X^1 \cup X^2$. Hence $Z = X^1$ or $Z = X^2$. □

Remark 7.17. Let $p \neq 2$. If we could prove that $\dim T_{\overline{\mathbb{F}_p}}^d < \dim X^2$ for all $d \geq 3$, we would be able to deduce that for $d \geq 3$,

$$\dim T_{\overline{\mathbb{Q}}}^d \leq \dim T_{\overline{\mathbb{F}_p}}^d < \dim X^2.$$

Suppose instead that $\dim T_k^d \geq \dim X^2$ for some $d \geq 3$ and $k = \overline{\mathbb{F}_p}$. Let Z be an irreducible component of T_k^d with $\dim Z \geq \dim X^2$. We have $Z \subset X^1 \cup X^2$ by the proof of Corollary 7.16. Moreover, $Z \not\subseteq X^2$ (since $X^2 \not\subseteq T_k^d$ by Corollary 7.8), so $Z \subset X^1$. So it would suffice to prove that $\dim(T_k^d \cap X^1) < \dim X^2$ for $d \geq 3$ (this inequality fails when $d = 2$). This is the technical problem that unfortunately does not allow us to remove the assumption $\operatorname{char} k \neq 0$ from Corollary 7.16.

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Chapter 8

The space of F such that $V(F)_{\text{sing}}$ contains a fixed C (case $b = 1$)

In the last two chapters, we approach Theorem 1.1 from a different point of view (when $b = 1$), which will result in an alternative argument for the case when d is small. This second approach gives a proof without explicit bounds, but addresses some questions that appear naturally, and which are interesting on their own right. Namely, for a fixed (reduced) $C \subset \mathbb{P}^n$, we study the linear space $(W_C)_l := \{F \in S_l \mid C \subset V(F)_{\text{sing}}\}$ (Section 8.1). It turns out that C gives rise to a certain ideal sheaf $\mathcal{J}$ (coming from exact sequences involving Kähler differentials), such that $(W_C)_l = \Gamma(\mathbb{P}^n, \mathcal{J}(l))$ (Proposition 8.2). In turn, one naturally asks for the Hilbert polynomial of the sheaf $\mathcal{J}$; this question is answered in Section 8.2 in terms of invariants of C (Proposition 8.5).

Here, k is any algebraically closed field.

Let $S = k[x_0, ..., x_n]$ with the usual grading. Recall the functor

$$\widetilde{\Gamma}\colon \operatorname{QCoh}(\mathbb{P}^n) \to \text{Graded } S\text{-modules},$$
$$\mathcal{F} \mapsto \bigoplus_{l \in \mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{F}(l)).$$

We denote its left adjoint by $\widetilde{\text{Loc}}$ (denoted $\sim$ on p. 116 in [7]). Recall that $\widetilde{\text{Loc}}\widetilde{\Gamma} \xrightarrow{\simeq} \text{id}$

and that if M is a finitely-generated graded S-module, then $M \to \widetilde{\Gamma \mathrm{Loc}}(M)$ is an isomorphism in large degrees.

8.1 Understanding the condition $C \subset V(F)_{\mathrm{sing}}$ for a fixed C

Let $I \subset A = k[x_1, ..., x_n]$ be a radical ideal, and $C = \mathrm{Spec}(A/I)$. For $f \in A, f \neq 0$, we have $C \subset V(f)_{\mathrm{sing}}$ if and only if $f \in \mathfrak{m}^2$ for any maximal $\mathfrak{m} \supset I$ (here, $V(f)_{\mathrm{sing}}$ is the set of singular points of $V(f) = \mathrm{Spec}(A/f)$). We now have to understand this condition.

Lemma 8.1. *For $f \in I$, we have $f \in \mathfrak{m}^2$ for all $\mathfrak{m} \supset I$ if and only if f belongs to the kernel of*

$$I \to \Omega_{A/k}/I\Omega_{A/k}.$$

Proof. Suppose that f satisfies $f \in \mathfrak{m}^2$ for all $\mathfrak{m} \supset I$. Let $B = A/I$. We claim that f belongs to the kernel of the map $I \to \Omega_{A/k}/I\Omega_{A/k}$; i.e., we claim

$$df \in I\Omega_{A/k},$$

where $d\colon A \to \Omega_{A/k}$ is the canonical derivation. (In this way, we linearize our unhandy condition that $f \in \mathfrak{m}^2$). We know that for each maximal $\mathfrak{m} \supset I$, we have $\Omega_{A/k}/\mathfrak{m}\Omega_{A/k} = \Omega_{A/k} \otimes_A A/\mathfrak{m} \simeq \mathfrak{m}/\mathfrak{m}^2$ as $A/\mathfrak{m}$-vector spaces, and

$$\begin{array}{ccc} I/I^2 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \\ \downarrow d & & \downarrow \simeq \\ \Omega_{A/k}/I\Omega_{A/k} & \longrightarrow & \Omega_{A/k}/\mathfrak{m}\Omega_{A/k} \end{array}$$

commutes, so the condition $f \in \mathfrak{m}^2$ is equivalent to $df \in \mathfrak{m}\Omega_{A/k}$.

Since $\Omega_{A/k}$ is a *free* A-module, we conclude that

$$df \in \bigcap_{\mathfrak{m} \supset I} (\mathfrak{m}\Omega_{A/k}) = \left(\bigcap_{\mathfrak{m} \supset I} \mathfrak{m} \right) \Omega_{A/k} = I\Omega_{A/k}.$$

The converse is obvious from the commutative diagram above. □

Let $i\colon C \hookrightarrow \mathbb{P}^n$ be any reduced closed subscheme (not necessarily integral or 1-dimensional), and let $\mathcal{I}$ be its ideal sheaf. Define $\mathcal{G} \in \mathrm{Coh}(C)$ as the kernel of the first map in the second fundamental exact sequence,

$$0 \to \mathcal{G} \to i^*\mathcal{I} \to i^*\Omega_{\mathbb{P}^n} \to \Omega_C \to 0,$$

and $\mathcal{H} \in \mathrm{Coh}(C)$ by the exactness of

$$0 \to \mathcal{H} \to i^*\Omega_{\mathbb{P}^n} \to \Omega_C \to 0,$$

so we have a short exact sequence

$$0 \to \mathcal{G} \to i^*\mathcal{I} \to \mathcal{H} \to 0.$$

Since i_* is exact (as i is a closed embedding, hence affine), it follows that $i_*i^*\mathcal{I} \twoheadrightarrow i_*\mathcal{H}$ is surjective, and hence so is the composition $\mathcal{I} \twoheadrightarrow \mathcal{I}/\mathcal{I}^2 = i_*i^*\mathcal{I} \twoheadrightarrow i_*\mathcal{H}$. Let $\mathcal{J}$ denote its kernel, so we have a short exact sequence

$$0 \to \mathcal{J} \to \mathcal{I} \to i_*\mathcal{H} \to 0.$$

In other words, $\mathcal{J}$ is defined by the exactness of

$$0 \to \mathcal{J} \to \mathcal{I} \to \Omega_{\mathbb{P}^n}/I\Omega_{\mathbb{P}^n} \to i_*\Omega_C \to 0.$$

Note that $\mathcal{I}^2 \subset \mathcal{J}$, and we have an equality if C is a local complete intersection, since in this case, $\mathcal{G} = 0$ (see Exercise 16.17 in [2]).

Proposition 8.2. *With notation as above, for $F \in S_{\text{homog}}$, we have $C \subset V(F)_{\text{sing}}$ if and only if $F \in \widetilde{\Gamma}(\mathcal{J})$.*

Proof. For $i = 0, ..., n$, let $f_i = F(x_0, ..., 1, ..., x_n)$ be the i-th dehomogenization of F. Assume that $F \neq 0$, so also $f_i \neq 0$. The condition $F \in \widetilde{\Gamma}(\mathcal{J})$ is equivalent to $f_i \in \Gamma(D_+(x_i), \mathcal{J})$ for all $i = 0, ..., n$. On the other hand, $C \subset V(F)_{\text{sing}}$ is equivalent to $C \cap D_+(x_i) \subset V(f_i)_{\text{sing}}$, and hence the statement of the proposition reduces to the following affine statement.

Let $A = k[x_1, ..., x_n]$ and $C = \operatorname{Spec}(A/I)$, where $I \subset A$ is radical. Let $B = A/I$. Define an A-module J by the exactness of

$$0 \to J \to I \to \Omega_A/I\Omega_A \to \Omega_B \to 0.$$

Then for a nonzero polynomial $f \in A$, we have $C \subset V(f)_{\text{sing}}$ if and only if $f \in J$. This follows from Lemma 8.1. □

8.2 Computing the Hilbert polynomial of $\mathcal{J}$

Lemma 8.3. *Let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a short exact sequence in $\operatorname{QCoh}(\mathbb{P}^n)$, *with $\mathcal{F}'$ coherent. Then for large l,*

$$0 \to \widetilde{\Gamma}(\mathcal{F}')_l \to \widetilde{\Gamma}(\mathcal{F})_l \to \widetilde{\Gamma}(\mathcal{F}'')_l \to 0$$

is a short exact sequence of k-vector spaces.

Proof. Apply Theorem III.5.2(b) in [7] to $\mathcal{F}'$. □

Let C be any integral curve over k (not necessarily projective for now), and let $p\colon \widetilde{C} \to C$ be its normalization. Consider the canonical map $\alpha\colon \Omega_C \to p_*\Omega_{\widetilde{C}}$. Let $\mathcal{R}_1, \mathcal{R}_2 \in \operatorname{Coh}(C)$ denote its kernel and cokernel:

$$0 \to \mathcal{R}_1 \to \Omega_C \xrightarrow{\alpha} p_*\Omega_{\widetilde{C}} \to \mathcal{R}_2 \to 0.$$

Since p is an isomorphism over a dense open $U \subset C$, so is α, and hence $\mathcal{R}_1$ and $\mathcal{R}_2$ have finite support, contained in C_{sing}. For each $P \in C_{\text{sing}}$, the stalks $(\mathcal{R}_1)_P$ and $(\mathcal{R}_2)_P$ are finite-dimensional k-vector spaces.

Define

$$\mu(C) := \sum_{P \in C_{\text{sing}}} (\dim_k(\mathcal{R}_1)_P - \dim_k(\mathcal{R}_2)_P).$$

For the rest of this chapter, let $i\colon C \hookrightarrow \mathbb{P}^n$ be an integral curve of degree d, and let $\tilde{g}$ be the genus of its normalization.

Lemma 8.4. *For large l,*

$$\dim_k \Gamma(C, \Omega_C(l)) = dl + \tilde{g} - 1 + \mu(C).$$

Proof. Consider the exact sequence

$$0 \to \mathcal{R}_1 \to \Omega_C \xrightarrow{\alpha} p_*\Omega_{\tilde{C}} \to \mathcal{R}_2 \to 0.$$

For large l, the sequence

$$0 \to \Gamma(C, \mathcal{R}_1(l)) \to \Gamma(C, \Omega_C(l)) \to \Gamma(C, (p_*\Omega_{\tilde{C}})(l)) \to \Gamma(C, \mathcal{R}_2(l)) \to 0 \tag{8.1}$$

is exact.

Note that

$$\Gamma(C, \mathcal{R}_1(l)) \simeq \Gamma(C, \mathcal{R}_1) = \bigoplus_{P \in C_{\text{sing}}} (\mathcal{R}_1)_P,$$

and similarly for $\mathcal{R}_2$.

Now, we look at the term $\Gamma(C, (p_*\Omega_{\tilde{C}})(l))$. By the projection formula, we know

$$(p_*\Omega_{\tilde{C}})(l) \simeq p_*(\Omega_{\tilde{C}} \otimes_{\mathcal{O}_{\tilde{C}}} p^*\mathcal{O}_C(l)).$$

Since C has degree d, $p^*\mathcal{O}_C(l)$ is a line bundle on $\widetilde{C}$ of degree dl (see Corollary 5.8 on

p. 306 in [9]). By the Riemann-Roch theorem applied to $\widetilde{C}$, it follows that for large l,

$$\dim_k \Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes p^*\mathcal{O}_C(l)) = dl + \tilde{g} - 1.$$

Take the alternating sum of dimensions in (8.1). $\square$

For an integral curve $i\colon C \hookrightarrow \mathbb{P}^n$ with ideal sheaf $\mathcal{I}$ and $I = \widetilde{\Gamma}(\mathcal{I})$, we let d be its degree and p_a be its arithmetic genus, so for large l, we have

$$\dim_k(S/I)_l = dl + 1 - p_a.$$

For $l \geq 1$, let

$$(W_C)_l = \{F \in S_l \mid C \subset V(F)_{\text{sing}}\}.$$

Proposition 8.5. *Let $C, i, \tilde{g}$ be as in Lemma 8.4. For $l \gg 0$,*

$$\dim_k(S_l/(W_C)_l) = ndl + 1 + (n+1)(1 - d - p_a) - \tilde{g} - \mu(C).$$

Proof. Recall the short exact sequences

$$0 \to \mathcal{H} \to i^*\Omega_{\mathbb{P}^n} \to \Omega_C \to 0 \quad \text{(definition of } \mathcal{H}) \tag{8.2}$$

and

$$0 \to \mathcal{J} \to \mathcal{I} \to i_*\mathcal{H} \to 0 \quad \text{(definition of } \mathcal{J}).$$

We will be using Lemma 8.3 continuously without explicit notice. By Proposition 8.2, for all l, we have $(W_C)_l = \widetilde{\Gamma}(\mathcal{J})_l$, so we have to compute $\dim_k \widetilde{\Gamma}(\mathcal{O}_{\mathbb{P}^n}/\mathcal{J})_l$ for large l. From the short exact sequence

$$0 \to i_*\mathcal{H} \to \mathcal{O}_{\mathbb{P}^n}/\mathcal{J} \to \mathcal{O}_{\mathbb{P}^n}/\mathcal{I} \to 0,$$

we obtain a short exact sequence

$$0 \to \widetilde{\Gamma}(i_*\mathcal{H})_l \to \widetilde{\Gamma}(\mathcal{O}_{\mathbb{P}^n}/\mathcal{J})_l \to \widetilde{\Gamma}(\mathcal{O}_{\mathbb{P}^n}/\mathcal{I})_l \to 0$$

for large l. Since the last term is $(\widetilde{\Gamma}(\mathcal{O}_{\mathbb{P}^n})/\widetilde{\Gamma}(\mathcal{I}))_l = (S/I)_l$ for large l, and hence of dimension $dl + 1 - p_a$, it suffices to compute the dimension of the first term.

Applying the exact functor i_* to (8.2), we obtain short exact

$$0 \to i_*\mathcal{H} \to \Omega_{\mathbb{P}^n}/\mathcal{I}\Omega_{\mathbb{P}^n} \to i_*\Omega_C \to 0,$$

which for large l gives short exact

$$0 \to \widetilde{\Gamma}(i_*\mathcal{H})_l \to \widetilde{\Gamma}(\Omega_{\mathbb{P}^n}/\mathcal{I}\Omega_{\mathbb{P}^n})_l \to \Gamma(C, \Omega_C(l)) \to 0.$$

We know the last term has dimension $dl + \tilde{g} - 1 + \mu(C)$.

Finally, we have to compute $\dim_k \widetilde{\Gamma}(\Omega_{\mathbb{P}^n}/\mathcal{I}\Omega_{\mathbb{P}^n})_l$ for large l. Recall from Theorem II.8.13 in [7] the short exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Since $\mathcal{O}_{\mathbb{P}^n}$ is locally free, applying $- \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}$ to this short exact sequence yields a short exact sequence

$$0 \to \frac{\Omega_{\mathbb{P}^n}}{\mathcal{I}\Omega_{\mathbb{P}^n}} \to \left(\frac{\mathcal{O}_{\mathbb{P}^n}(-1)}{\mathcal{I}\mathcal{O}_{\mathbb{P}^n}(-1)}\right)^{\oplus(n+1)} \to \frac{\mathcal{O}_{\mathbb{P}^n}}{\mathcal{I}} \to 0.$$

Once again by Lemma 8.3, we know that

$$0 \to \widetilde{\Gamma}\left(\frac{\Omega_{\mathbb{P}^n}}{\mathcal{I}\Omega_{\mathbb{P}^n}}\right)_l \to \widetilde{\Gamma}\left(\frac{\mathcal{O}_{\mathbb{P}^n}(-1)}{\mathcal{I}\mathcal{O}_{\mathbb{P}^n}(-1)}\right)_l^{\oplus(n+1)} \to \widetilde{\Gamma}\left(\frac{\mathcal{O}_{\mathbb{P}^n}}{\mathcal{I}}\right)_l \to 0$$

is exact for large l. Again, we have to compute the dimensions of the second and third terms. For large l, the third term has dimension $dl + 1 - p_a$, as before.

We are left to compute $\dim_k \widetilde{\Gamma}(\mathcal{O}(-1)/\mathcal{I}\mathcal{O}(-1))_l$ for large l. Notice that $\mathcal{I}\mathcal{O}(-1) \simeq$

$\mathcal{I}(-1)$ and that for large l, $\widetilde{\Gamma}(\mathcal{O}(-1)/\mathcal{I}(-1))_l$ is $\left(\widetilde{\Gamma}(\mathcal{O}(-1))/\widetilde{\Gamma}(\mathcal{I}(-1))\right)_l$, which is $(S/I)_{l-1}$ for large l. This is a shift of S/I and thus dimension equal to $d(l-1)+1-p_a$.

Going back through the exact sequences, we complete the calculation. □

Remark 8.6. Comparing Proposition 8.5 with Lemma 7.9 (combined with Proposition 8.2) shows that for an integral *plane* curve C, we have $\mu(C) = p_a - \tilde{g}$. However, this fails for a general integral curve C.

Remark 8.7. Lemma 8.4 and the previous remark imply that for any integral plane curve $C \hookrightarrow \mathbb{P}^2 \subset \mathbb{P}^n$, the Hilbert polynomial of the sheaf Ω_C of Kähler differentials is

$$\chi(\Omega_C(l)) = dl + p_a - 1.$$

Chapter 9

Alternative argument for the case $b = 1$

The goal is now to make the conclusion of Proposition 8.5 uniform over integral curves of given degree; in fact, we will prove that there exists a polynomial $P_2(D) \in \mathbb{Z}[D]$ such that the formula for $\dim(W_C)_l$ holds for all integral C of degree d and all $l \geq P_2(d)$ (Lemma 9.8). The technique we use throughout this chapter is Mumford regularity (see Section 9.1). Section 9.2 is technical; the goal there is to give a uniform bound on the invariant $\mu(C)$ just in terms of the degree of C. Once this is done, the second proof of Theorem 1.1 is easily finished in Section 9.3 (small degree) and Section 9.4 (large degree).

Again, k is an algebraically closed field.

9.1 Mumford regularity

Definition 9.1. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular if for any $i \geq 1$,

$$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0.$$

One can show (see Chapter 5.2 in [5]) that if $\mathcal{F}$ is m-regular, then $H^i(\mathbb{P}^n, \mathcal{F}(l)) = 0$ for all $i \geq 1, l \geq m - i$ (in other words, $\mathcal{F}$ is also m'-regular, for any $m' \geq m$).

Moreover, if $\mathcal{F}$ is m-regular, then for any $l \geq m$, the sheaf $\mathcal{F}(l)$ is generated by global sections.

Theorem 9.2 (Mumford). *There exists a polynomial $F_n \in \mathbb{Z}[x_0, ..., x_n]$ with the following property. Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^n}$ be any ideal sheaf, and let the Hilbert polynomial of $\mathcal{I}$ be*

$$\chi(\mathcal{I}(l)) = \sum_{i=0}^{n} a_i \binom{l}{i}.$$

Then $\mathcal{I}$ is m-regular, where $m = F_n(a_0, ..., a_n)$.

9.2 Uniform bound on $\mu(C)$

The goal of this technical section is to prove the following

Lemma 9.3. *There are polynomials $P_0, P_1 \in \mathbb{Z}[D]$ such that if $C \hookrightarrow \mathbb{P}^n$ is any integral curve of degree d, then*

$$P_0(d) \leq \mu(C) \leq P_1(d).$$

The proof requires some preparation.

Lemma 9.4. *There exists a polynomial $Q \in \mathbb{Z}[D, G]$ such that for any integral curve $C \hookrightarrow \mathbb{P}^n$ of degree d and arithmetic genus g, and for any $l \geq Q(d, g)$, we have*

$$H^1(C, \Omega_C(l)) = 0.$$

Proof. Fix a surjection

$$\bigoplus \mathcal{O}_{\mathbb{P}^n}(q_i) \to \Omega_{\mathbb{P}^n} \to 0.$$

Consider a polynomial $F_n \in \mathbb{Z}[x_0, ..., x_n]$ from Mumford's theorem. Write $\binom{l+n}{n} - (Dl + 1 - G) = \sum_{i=0}^{n} a_i \binom{l}{i}$, where $a_i \in \mathbb{Z}[D, G]$. If $C \hookrightarrow \mathbb{P}^n$ is any integral curve of degree d and arithmetic genus g, so that its ideal sheaf $\mathcal{I}$ has Hilbert polynomial

$\binom{l+n}{n} - (dl+1-g)$, we know that $\mathcal{I}$ is m-regular, for any $m \geq F_n(a_0, ..., a_n)|_{(D,G)=(d,g)} \in \mathbb{Z}$. Set $Q(D,G) = F_n(a_0, ..., a_n) + \max(-q_i) \in \mathbb{Z}[D,G]$.

Now, let $i \colon C \hookrightarrow \mathbb{P}^n$ be any integral curve of degree d and arithmetic genus g, and suppose that $l \geq Q(d,g)$. Since we have a surjection $i^*\Omega_{\mathbb{P}^n} \twoheadrightarrow \Omega_C$, in order to prove that $H^1(C, \Omega_C(l)) = 0$, it suffices to prove that $H^1(C, i^*\Omega_{\mathbb{P}^n}(l)) = 0$. In turn, we have a surjection

$$\bigoplus \mathcal{O}_C(q_i + l) \longrightarrow i^*\Omega_{\mathbb{P}^n}(l) \longrightarrow 0,$$

and so it suffices to prove that $H^1(C, \mathcal{O}_C(q_i + l)) = 0$ for all i. Twist the short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow i_*\mathcal{O}_C \longrightarrow 0$$

by $q_i + l$ and note that since $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q_i + l)) = 0$, it suffices to prove that $H^2(\mathbb{P}^n, \mathcal{I}(q_i + l)) = 0$. This holds since $\mathcal{I}$ is $(q_i + l)$-regular by our choice of l. □

Lemma 9.5. *There is a polynomial $Q_1(D,G) \in \mathbb{Z}[D,G]$ such that for any integral curve $i \colon C \hookrightarrow \mathbb{P}^n$ of degree d and arithmetic genus g, if we define a sheaf $\mathcal{H}$ by exactness of*

$$0 \longrightarrow \mathcal{H} \longrightarrow i^*\Omega_{\mathbb{P}^n} \longrightarrow \Omega_C \longrightarrow 0,$$

then $H^1(C, \mathcal{H}(l)) = 0$ for all $l \geq Q_1(d,g)$.

Proof. Let $F_n \in \mathbb{Z}[x_0, ..., x_n]$ be a polynomial from Mumford's theorem. Write $\binom{l+n}{n} - (Dl + 1 - G) = \sum_{i=0}^{n} a_i \binom{l}{i}$ and consider $F_n(a_0, ..., a_n) \in \mathbb{Z}[D,G]$. Set $Q_1(D,G) = 2F_n(a_0, ..., a_n) \in \mathbb{Z}[D,G]$. Let $i \colon C \hookrightarrow \mathbb{P}^n$ be any integral curve of degree d, arithmetic genus g, and ideal sheaf $\mathcal{I}$; let $\mathcal{H}$ be defined as in the statement of the lemma, and let $l \geq Q_1(d,g)$. We claim that $H^1(C, \mathcal{H}(l)) = 0$.

Indeed, if $l_0 = F_n(a_0, ..., a_n)|_{(D,G)=(d,g)} \in \mathbb{Z}$, we know that $\mathcal{I}$ is l_0-regular. In particular, $\mathcal{I}(l_0)$ is generated by global sections, and so there is a surjection

$$\bigoplus \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{I}(l_0) \longrightarrow 0.$$

Twist this surjection by $l - l_0 \geq l_0$ to obtain a surjection

$$\bigoplus \mathcal{O}_{\mathbb{P}^n}(l_0 + l') \to \mathcal{I}(l) \to 0,$$

where $l' \geq 0$. Apply i^* to obtain a surjection

$$\bigoplus \mathcal{O}_C(l_0 + l') \to i^*\mathcal{I}(l) \to 0.$$

Now, $H^1(C, \mathcal{O}_C(l_0 + l')) = 0$: twist the short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_C \to 0$ by $l_0 + l'$ and note that $H^2(\mathbb{P}^n, \mathcal{I}(l_0 + l')) = 0$, since $\mathcal{I}$ is l_0-regular. The surjection above implies that $H^1(C, i^*\mathcal{I}(l)) = 0$. Since we know that there is a surjection

$$i^*\mathcal{I} \to \mathcal{H} \to 0,$$

twisting by l and taking cohomology implies $H^1(C, \mathcal{H}(l)) = 0$, as desired. □

Setting $Q_2(D, G) = \binom{l+n}{n} - (Dl + 1 - G)$, we know that for any integral curve $C \hookrightarrow \mathbb{P}^n$ of degree d, arithmetic genus g, and ideal sheaf $\mathcal{I}$, and any $l \geq Q_2(d, g)$, we have $H^1(\mathbb{P}^n, \mathcal{I}(l)) = 0$. Combining this with the lemma above, we conclude that there exists a polynomial $Q_0(D, G) \in \mathbb{Z}[D, G]$ with the following property. Let $i \colon C \hookrightarrow \mathbb{P}^n$ be any integral curve, and let $d, g, \mathcal{I}$ denote its degree, arithmetic genus, and ideal sheaf, respectively. Define the sheaf $\mathcal{H}$ by exactness of $0 \to \mathcal{H} \to i^*\Omega_{\mathbb{P}^n} \to \Omega_C \to 0$. Let $l \geq Q_0(d, g)$ be any integer. Then $H^1(\mathbb{P}^n, \mathcal{I}(l-1)) = 0$ and $H^1(C, \mathcal{H}(l)) = 0$. Fix such a polynomial $Q_0(D, G)$.

Lemma 9.6. *There exists a polynomial $\tilde{Q}(D, G) \in \mathbb{Z}[D, G]$ such that for any integral curve $i \colon C \hookrightarrow \mathbb{P}^n$ of degree d and arithmetic genus g, we have*

$$\dim_k \Gamma(C, i^*\Omega_{\mathbb{P}^n}(l)) \leq \tilde{Q}(d, g).$$

for $l = Q_0(d, g)$.

Proof. Set $\tilde{Q}(D, G) = (n + 1)\binom{Q_0(D,G)-1+n}{n}$ and let $i \colon C \hookrightarrow \mathbb{P}^n$ be an integral curve as in the statement.

Start with the short exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0,$$

apply i^*, twist by $l = Q_0(d,g)$, and take global sections, to obtain an injection

$$0 \to \Gamma(C, i^*\Omega_{\mathbb{P}^n}(l)) \to \Gamma(C, \mathcal{O}_C(l-1))^{\oplus(n+1)}.$$

On the other hand, twisting the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_C \to 0$$

by $l-1$ and taking global sections, we obtain a surjection

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l-1)) \to \Gamma(C, \mathcal{O}_C(l-1)) \to 0,$$

because $H^1(\mathbb{P}^n, \mathcal{I}(l-1)) = 0$ (since $l = Q_0(d,g)$). Thus,

$$\dim_k \Gamma(C, i^*\Omega_{\mathbb{P}^n}(l)) \leq (n+1)\dim_k \Gamma(C, \mathcal{O}_C(l-1)) \leq \tilde{Q}(d,g),$$

as required. □

Fix a polynomial $\tilde{Q}$ as in the statement of Lemma 9.6.

Corollary 9.7. *For any integral curve $C \hookrightarrow \mathbb{P}^n$ of degree d and arithmetic genus g, we have*

$$\dim_k \Gamma(C, \Omega_C(l)) \leq \tilde{Q}(d,g),$$

for $l = Q_0(d,g)$.

Proof. Twist the short exact sequence

$$0 \to \mathcal{H} \to i^*\Omega_{\mathbb{P}^n} \to \Omega_C \to 0,$$

by $l = Q_0(d,g)$, and take global sections. By choice of $Q_0(d,g)$, we have $H^1(C, \mathcal{H}(l)) =$

0, so $\dim \Gamma(C, \Omega_C(l)) \leq \dim \Gamma(C, i^*\Omega_{\mathbb{P}^n}(l))$. Now apply Lemma 9.6. $\square$

Proof of Lemma 9.3. Castelnuovo's theorem (see Theorem 3.7 in [3]) gives a polynomial bound on the arithmetic genus in terms of d. So it suffices to find polynomials $P_0, P_1 \in \mathbb{Z}[D, G]$ in two variables such that for any integral curve C of degree d and arithmetic genus g, we have

$$P_0(d, g) \leq \mu(C) \leq P_1(d, g).$$

Define $P_0(D, G) = -DQ(D, G) - G + 1$ and $P_1(D, G) = \tilde{Q}(D, G) - DQ_0(D, G) + 1$.

Let $C \hookrightarrow \mathbb{P}^n$ be any integral curve, let d, g be its degree and arithmetic genus, and let $p\colon \widetilde{C} \to C$ be the normalization map. Recall that $\mu(C)$ is defined as $\mu(C) = \chi(\mathcal{R}_1) - \chi(\mathcal{R}_2)$, where $\mathcal{R}_1, \mathcal{R}_2$ denote the torsion sheaves in the exact sequence below

$$0 \to \mathcal{R}_1 \to \Omega_C \to p_*\Omega_{\widetilde{C}} \to \mathcal{R}_2 \to 0.$$

The Hilbert polynomials of $\mathcal{R}_1, \mathcal{R}_2$ are of degree 0. Thus, twisting the above exact sequence by an integer l, using that p is an affine map, and applying Riemann-Roch to $\widetilde{C}$, we obtain

$$\mu(C) = \chi(\Omega_C(l)) - \chi(p_*\Omega_{\widetilde{C}}(l)) = \chi(\Omega_C(l)) - \chi(\Omega_{\widetilde{C}}(l)) = \chi(\Omega_C(l)) - (dl + \tilde{g} - 1)$$

for *any* integer l.

Set $l = Q(d, g)$ in the above expression (see Lemma 9.4). For this choice of l, we know that $H^1(C, \Omega_C(l)) = 0$, and so $\chi(\Omega_C(l)) = \dim_k \Gamma(C, \Omega_C(l)) \geq 0$. Consequently, $\mu(C) \geq -(dl + \tilde{g} - 1) = -dQ(d, g) - \tilde{g} + 1 \geq -dQ(d, g) - g + 1$, which gives the lower bound.

As for the upper bound, set $l = Q_0(d, g)$ in the expression for $\mu(C)$. We obtain

$$\mu(C) \leq \dim_k \Gamma(C, \Omega_C(l)) - (dl + \tilde{g} - 1) \leq \tilde{Q}(d, g) - dQ_0(d, g) + 1,$$

by Corollary 9.7. $\square$

9.3 The curves of small degree

Lemma 9.8. *There exists a polynomial $P_2 \in \mathbb{Z}[D]$ such that if $C \hookrightarrow \mathbb{P}^n$ is any integral curve of degree d and $l \geq P_2(d)$, then*

$$\mathrm{codim}_{Vk[x_0,\ldots,x_n]_l}\{F \in k[x_0,\ldots,x_n]_l \mid C \subset V(F)_{\mathrm{sing}}\} = ndl+1+(n+1)(1-d-p_a)-\tilde{g}-\mu(C)$$

(where $p_a, \tilde{g}, \mu(C)$ are as in Section 8.1).

Proof. Recall that we have attached a certain ideal sheaf $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^n}$ to the curve C, such that for $F \in k[x_0,\ldots,x_n]_l$, $C \subset V(F)_{\mathrm{sing}}$ if and only if $F \in \Gamma(\mathbb{P}^n, \mathcal{J}(l))$. Also, by Proposition 8.5, for $l \gg 0$,

$$\dim_k \Gamma(\mathbb{P}^n, \mathcal{J}(l)) = \binom{l+n}{n} - [ndl + 1 + (n+1)(1-d-p_a) - \tilde{g} - \mu(C)] = \sum_{i=0}^{n} a_i \binom{l}{i},$$

where a_i are polynomial expressions in $d, p_a, \tilde{g}, \mu(C)$. Consider $F_n(a_0,\ldots,a_n)$ (where F_n is from Mumford's theorem); this is likewise a polynomial in $d, p_a, \tilde{g}, \mu(C)$. Since we have a lower and upper bound for each of these quantities in terms of d, there is a polynomial $P_2 \in \mathbb{Z}[D]$, independent of C, such that $F_n(a_0,\ldots,a_n) \leq P_2(d)$ for any such curve C.

Thus, for $l \geq P_2(d)$, we have $l \geq F_n(a_0,\ldots,a_n)$, and hence $\mathcal{J}$ is l-regular. Thus $H^i(\mathbb{P}^n, \mathcal{J}(l)) = 0$ for all $i \geq 1$, so $\dim_k \Gamma(\mathbb{P}^n, \mathcal{J}(l))$ agrees with the value at l of the Hilbert polynomial $\chi(\mathcal{J}(l))$. □

Corollary 9.9. *Let P_2 be as above. There is a polynomial $P_3 \in \mathbb{Z}[D]$ such that if $C \hookrightarrow \mathbb{P}^n$ is any integral curve of degree d and $l \geq P_2(d)$, then*

$$\mathrm{codim}_V(\{F \in V \mid C \subset V(F)_{\mathrm{sing}}\}) \geq ndl + P_3(d).$$

Proof. Lemma 9.3 yields a polynomial bound on $\mu(C)$. By Castelnuovo's theorem and the fact that $\tilde{g} \leq p_a$, polynomial bounds exist for p_a and $\tilde{g}$, too. □

Remark 9.10. Even if are interested in proving the inequality from the corollary, the proof goes through the equality in Lemma 9.8 first, in order to give bounds for the coefficients of the Hilbert polynomial $\chi(\mathcal{J}(l))$ in terms of d and to be able to apply Mumford's theorem. In particular, we had to bound $\mu(C)$ from both sides.

Recall by Theorem 5.3 that $\dim \widetilde{\mathrm{Hilb}}^d \leq P_4(d)$, where $P_4(d) = 3(n-2)+d(d+3)/2$. Let $L \subset \mathbb{P}^n$ be a line.

Corollary 9.11. *There exists a polynomial $\tilde{P} \in \mathbb{Z}[D]$ such that for $l \geq \tilde{P}(d)$, we have*

$$\dim(W_C)_l + \dim \widetilde{\mathrm{Hilb}}^d < \dim(W_L)_l + \dim \mathbb{G}(1,n),$$

for any integral curve $C \hookrightarrow \mathbb{P}^n$ of degree $d \geq 2$.

Proof. Rewrite the inequality as

$$\mathrm{codim}_{S_l}(W_L)_l + \dim \widetilde{\mathrm{Hilb}}^d < \mathrm{codim}_{S_l}(W_C)_l + 2(n-1). \tag{9.1}$$

The left hand side is bounded above by $nl + 1 + P_4(d)$, and the right hand side, for $l \geq P_2(d)$, is at least $ndl + P_3(d) + 2(n-1)$ by Corollary 9.9. Thus (9.1) will hold as long as $l \geq P_2(d)$ and $nl + 1 + P_4(d) < ndl + P_3(d) + 2(n-1)$. There is a polynomial $\tilde{P} \in \mathbb{Z}[d]$ such that these two conditions are satisfied whenever $d \geq 2$ and $l \geq \tilde{P}(d)$. ☐

So replacing $\tilde{P}$ by d^M if necessary for some M (take $d = \deg \tilde{P} + 1$ and treat the finitely many $d \geq 2$ for which $\tilde{P}(d) \geq d^M$ separately as we had done before in Lemma 5.7), we have handled now the cases of small degree $d \leq \lfloor \sqrt[M]{l} \rfloor$.

Corollary 9.12. *There exists M such that for all pairs (d,l) with $2 \leq d \leq \sqrt[M]{l}$, and any irreducible component $Z \subset T^d$, either $Z = X^1$, or $\dim Z < \dim X^1$.*

(This is weaker than Corollary 5.6, but sufficient.)

9.4 The case of large degree $d \geq \sqrt[M]{l}$

Let $m(l) = \lceil \sqrt[M]{l} \rceil$, $m'(l) = \min(m(l), \tau(l) + 1)$, where $\tau(l) = \lfloor \frac{l-1}{p} \rfloor$ as in Proposition 6.1 (so $m'(l) = m(l)$ for large l). Notice that

$$\begin{aligned} A_1(l, m) &= lm - \frac{m(m+1)}{2} + 2m \\ &= l\lceil \sqrt[M]{l} \rceil - \frac{\lceil \sqrt[M]{l} \rceil (\lceil \sqrt[M]{l} \rceil + 1)}{2} + 2\lceil \sqrt[M]{l} \rceil \end{aligned}$$

is of order $l \sqrt[M]{l}$ and is therefore greater than $a_{n,1}(l)$ (which is linear in l), as long as l is large enough. So there exists l_0 such that for $l \geq l_0$, we have $A_1(l, m(l)) > a_{n,1}(l)$ and $\binom{\tau(l)+2}{2} > a_{n,1}(l)$. Apply Proposition 6.1 to the triple $(l, m(l), a_{n,1}(l) + 1)$ and combine it with Corollary 9.12 to finish this alternative proof of Theorem 1.1.

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