THIRD-ORDER THEORY OF PUMP-DRIVEN PLASMA

INSTABILITIES-----LASER-PELLET INTERACTIONS

by

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ABSTRACT

A general theory of coherent wave-wave coupling is first
set up. This is then used to furnish a systematic description
of laser-driven plasma instabilities. This description is
physically motivated, and generalizes in a straightforward
manner to magnetized plasmas, depleted pump-waves, plasmas
containing beams, and so on.

First a generalized-coupling-of-modes formalism is con-
structed in terms of the nonlinear conductivity of a medium.
This formalism accommodates arbitrarily many waves, with arbi-
trary propagation and polarization vectors. Generalized-
coupling-of-modes equations are derived, describing how each
wave-envelope varies in space and time due to its being driven
by the other waves to which it is coupled by the nonlinear con-
ductivity. The formalism can describe pump-driven instabili-
ties including the effects of pump depletion, attenuation and
evolution.

The formalism is then specialized to deal with linear
perturbations coupled together by an undepleted, unattenuated
pump-wave. The self-consistent harmonic structure, which the
pump-wave in a nonlinear medium must have, is computed and
included. The coupling coefficients are worked out for a
plasma with arbitrarily many species each described by a warm-
fluid model.

The work is then further specialized to recover systemat-
ically those laser-driven instabilities occurring in unmagnetized
plasma. The effects of third-order conductivity are that the
pump-driven instabilities of simple parametric type become
modified, and that additional instabilities appear. Three-
dimensional dispersion relations are derived for the various

instabilities, and one-dimensional cross-sections of the time-asymptotic unstable pulse-responses are found.

This description of laser-driven instabilities requires no assumptions of frequency disparity or phase-velocity disparity between decay products. The theory therefore generalizes to both weakly and strongly magnetized plasmas. Also the coupling coefficients are additive over particle populations, so that there is no conceptual difficulty in generalizing to a plasma containing several beams or temperature components.

Thesis Supervisor: Professor Abraham Bers

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CHAPTER 1

INTRODUCTION

1.1 Motivation for Study

1.1.1 IMPORTANCE OF ANALYTIC STUDIES OF LASER-DRIVEN INSTABILITIES

A major difficulty in the energetics of laser-pellet fusion is the competition[1,2] between anomalous scattering[2-4] and anomalous absorption[5-8]. The outcome of this competition in a typical reactor scheme[9-11] cannot be predicted unambiguously from experimental observations[12-19] made to date. Anomalous scattering and anomalous absorption are due to pump-driven instabilities, the laser light constituting the pump wave. Computer simulations[20-23] of these instabilities performed on present-day machines perforce omit vital aspects of the physical problem. Therefore they must be complemented by analytic studies of laser-driven reflective and absorptive instabilities.

To make a start on understanding laser-pellet interaction, one neglects asymmetries in illumination, heating, and blowoff, together with their associated magnetic fields[24-25]. One further replaces the resulting spherically-symmetric pellet by a plane-parallel model; an unmagnetised plasma with a density-gradient along z is irradiated by a laser-beam propagating along z and polarized along x.

One-dimensional simulation[17,18,20,21] of the above model depicts growth and saturation of backscatter instabilities.

Separate two-dimensional numerical simulations in perpendicular planes depict the development of sidescatter[22] and absorption[23]. To study the simultaneous nonlinear development of all these instabilities would require a three-dimensional simulation. However a full three-dimensional simulation with a useful number of particles is beyond the capability of present day computers. Analytic studies of the three-dimensional problem are therefore vital, even if initially confined to the linear phase of unstable growth.

At realistic laser power intensities the linear growth rates of all the important laser-driven instabilities are comparable[26]. Thus a correct analysis of the nonlinear saturation of these instabilities must treat them all on the same footing. This requires that the problem of linear growth and unstable propagation be first solved in three dimensions for all these instabilities individually. This can be done piecemeal, as discussed in Section 1.1.2, or systematically, as discussed in Section 1.1.3 and as carried out in this thesis.

At laser power intensities of 10^{15}-10^{16} watts/cm^2, simple parametric instabilities become modified[6,27], thus their growth rates are reduced. Further, other instabilities[6,7,28] which are intrinsically higher-order in the pump-field become important. These effects account for the above-mentioned convergence of the growth rates of the important instabilities. These effects can be described systematically in terms of the third-order conductivity of the plasma, and are so described in this thesis.

At realistic laser power intensities the linear growth rates are high enough that the effect of plasma inhomogeneity[29,30] may be neglected. This point is further discussed in Section 1.2.1. The plasma is driven strongly enough by the laser pump-wave that only coherent interaction need be considered. Thus we are motivated to set up a general theory of coherent wave-wave interaction via arbitrarily high-order conductivity in a homogeneous medium. We are further motivated to apply this theory to linear laser-driven instabilities in the plasma sur-rounding the pellet. This general theory and its application to laser-driven instabilities form the main part of this thesis.

1.1.2 PREVIOUS PIECEWISE TREATMENTS OF LASER-DRIVEN INSTABILITIES

Theoretical developments in the field of linear laser-driven instabilities may be divided historically into two phases. The first phase lasted from the middle sixties until recently and comprised piecewise treatments of specific instabilities. This first phase is outlined in this section. The second phase is current and comprises attempts to construct general schemes for dealing with all laser-driven instabilities. This second phase will be outlined in 1.1.3.

In 1962, Askar-yan[31] investigated filamentation in plasmas from a macroscopic viewpoint. In 1963, Dawson and Oberman[32] showed that coherent, nonthermal ion fluctuations could greatly enhance plasma resistivity at frequencies around w_p. In 1964, Dubois and Gilinsky[33] showed that this

could also happen at frequencies around $2w_p$, and interpreted their own result as parametric downconversion from the applied field to two electron-plasma waves. In 1965, Dubois and Goldman[5] described "anomalous" absorption at w_p in terms of parametric downconversion into an electron-plasma wave and an ion-acoustic wave. Also in 1965 Silin[6] used the fluid equations for electrons and ions to derive not only the aforesaid "parametric" or "decay" instability but also the purely-growing "nonoscillatory" instability. Still in 1965 Silin and Gorbunov[4] treated stimulated Raman scattering in plasma. In 1966 Comisar[2], and a little later Montgomery and Alexeff[3], derived stimulated Raman scattering from a simpler coupled-mode treatment. The "nonoscillatory" instability was rederived from the fluid model via coupled differential equations by Nishikawa[7] in 1968. This was extended to Vlasov plasmas by Sanmartin[34] in 1970.

1.1.3 CURRENT EFFORTS TOWARDS A UNIFIED THEORY OF LASER-DRIVEN INSTABILITIES

Current theoretical developments in the field of linear laser-driven instabilities are directed towards general schemes which include all such instabilities. We outline five such schemes including the one set forth in this thesis.

The first general theory is that of Drake, Kaw, et.al.[35] These workers describe all instabilities in terms of the successive operation of two physical mechanisms. The first physical mechanism is that high-frequency oscillations beat with the pump-field to produce a low-frequency radiation-pressure

pattern which forces a bunching of the electrons. The second

mechanism is that the low-frequency electron-bunches quiver in

the pump field to form a high-frequency current which regenerates

the high-frequency oscillations. This theory has the virtues

of providing a concrete and conceptually simple physical expla-

nation for the instabilities that it covers, and of reducing

the complexity of the calculations. This reduction in complex-

ity is accomplished by considering only the first harmonic of

the pump wave, and considering only those instabilities with a

frequency disparity (more strictly, a phase-velocity disparity)

between the decay products. The theory requires modification[31]

in order to deal with the two-plasmon instability. It is not

immediately clear how to extend the theory to strongly-magnetized

and multiply-irradiated plasmas where the above reduction in

complexity cannot occur.

The second general theory is that of Ott and Manheimer[37].

This resembles that of Kaw, et.al. in that the physical mechan-

isms are again radiation pressure leading to low-frequency

electron bunching, and quivering of the bunched electrons

leading to a high-frequency current. It also resembles that of

Kaw, et.al. in that the physical explanations and the mathemat-

ical calculations are simple and straightforward. Again the

theory needs modification to cover the two-plasmon instability.

The theory extends to plasma which is weakly magnetized in the

sense that the electron and ion cyclotron-frequencies lie well

below the high and low perturbation-frequencies, respectively.

Again the feasibility of broader extensions is not immediately clear.

A review of the interaction of high-frequency electromagnetic wave with plasma has been carried out by Silin[38]. This pump wave is allowed to be strong in the sense that the electron quivering velocity it induces can be much greater than the thermal velocity. Accordingly the nonlinear dispersion relation for perturbations coupled by the pump-wave is worked out exactly to all orders in the pump-field. In these respects the review is more complete than this thesis. However the approximation is made that the wavevector of the pump is zero. Consider an instability occurring in unmagnetized plasma and having an electromagnetic decay-product. For such an instability, neglecting the k-vector of the electromagnetic pump would completely alter the three-dimensional kinematics and would completely alter the three-dimensional dependence of the growth rate upon the directions of propagation of the decay-products. Thus the review perforce refrains from analyzing scattering instabilities in unmagnetized plasma. The review does however consider instabilities occurring in a magnetized plasma and having mixed electrostatic and electromagnetic decay-products. These instabilities are implicitly restricted to those for which neither the wavevector-matching conditions nor the strength of the nonlinear coupling are appreciably affected by the magnetitude of the pump-wavevector.

A general scheme for obtaining laser-driven instabilities
and carrying out stability analyses on them has been put forward
by Jorna[39]. This scheme has the defect that it characterizes
each particle species by a field damping rate, whereas in fact
a damping rate is a characteristic of a particular wave and not
of a particular species. The Bers-Briggs stability criteria[40]
are appealed to. However a full three-dimensional stability
analysis[41] is not carried out. A series of one-dimensional
stability analyses is carried out with the wavevector in the
perpendicular direction ranging over values that are purely
real. These results do not necessarily furnish a one-dimen-
sional cross-section of an unstable pulse.

Jorna's theory derives dispersion relations by relating
electric-field perturbations at selected frequencies to
particle-density perturbations of certain species at other
selected frequencies. It does this in a way which is not
physically motivated, unlike the theories of Kaw, et.al., and
Manheimer, et.al. This derivation of dispersion relations is
not easily generalized to magnetic fields, multiple pumps,
particle beams and so on.

This thesis generates all laser-driven instabilities
from a generalized-coupling-of-modes formalism. This formalism
is first specialized to consider coupled perturbations on a
self-consistent pump-equilibrium, the latter consisting of an
undepleted pump-wave together with its harmonics. The formalism
is then further specialized to an electromagnetic pump in an

unmagnetized plasma. One or both of these successive special-
izations can be revoked in the near future; thus the broader
usefulness of the theory is guaranteed.

1.2 Physical Parameters of Problem

1.2.1 EXPECTED SPACETIME DEVELOPMENT OF MACROSCOPIC PLASMA PARAMETERS

In this section we examine the spatial and temporal
scales of the pellet plasma with a view to justifying the
homogeneous-medium approximation used in this thesis. We
also consider the validity of approximating the spherical
geometry by a planar geometry. The question of physical
departures from the spherical geometry will be considered in
Section 1.2.2.

First consider the temporal scale of the evolution of
the pellet blowoff plasma. This is of the order of the laser
pulse length, which in current experiments[17-19] is in the
region of 100 picoseconds to a nanosecond. This temporal scale
is much greater than the growth times of the laser-driven
instabilities, which for power-intensities of 10^{15} watts/cm^2
are in the region of 0.5-5.0 picoseconds. This inequality of
temporal scales is unaltered or strenghened by increasing the
pulse length for a given total energy, since linear instability
growth rates vary as the power intensity or as some fractional
power thereof. However, a stretched-out laser-pulse leads to
problems in the spatial scale-ordering, as will now be described.

The fusion-reactor scheme[6-9] put forward by Nuckolls, et.al. differs greatly in its proposed pellet size from experimental spherical-pellet schemes currently operating[51-53] or under construction. Nevertheless the sizes and temperatures of the plasma atmospheres produced by laser prepulsing in the latter are comparable to those postulated in the former. Plasma atmosphere dimensions are of the order of hundreds of micrometers, and this provides one natural spatial scale for characterizing plasma inhomogeneity. In considering instabilities of parametric type, however, an important scale-length is the length over which the decay-products of the instability become appreciably dephased from the pump. This length is much shorter than the overall scale-length of the inhomogeneity and so becomes the important parameter. For the simplest coupled-mode instabilities the dephasing causes the growth to saturate, as shown by Rosenbluth, et. al.[29,30], after a number of e-foldings given by

$$\log(\text{gain}) = 2\pi\gamma_o^2 (\ v_{g1z}v_{g2z} \ dk/dz)^{-1} \qquad 1.2-(1)$$

Here γ_o is the homogeneous growth rate, $k(z)$ is the wavevector mismatch, v_{g1}, v_{g2} are the group velocities, and z is the (radial) direction of inhomogeneity. For the calculations in this thesis the laser is taken to have a wavelength of 1.06 μm and a power intensity of 10^{15} watts/cm^{-2}, and the plasma a

temperature of 1 KeV. The use of (1) then predicts that all
laser-driven instabilities except the Raman will execute
hundreds of e-foldings from the thermal noise which constitutes
their starting amplitudes, and thus will be effectively unchanged
by the spatial inhomogeneity. The Raman instability is stabilized
in its backscatter form but not in its sidescatter form. The
calculations in this thesis based on the homogeneous model thus
retain their usefulness in the great majority of laser-driven
instabilities.

The spatial ordering described here involves the insta-
bility growth rates, and is thus invalidated if relatively low
laser power intensities are maintained for relatively long
periods as required in early versions of laser-fusion proposals[9,10].
However later versions[11] of laser-fusion proposals feature
nanosecond overall pulse-length and less emphasis on time-tail-
oring. Also current experiments[17-19] on pellets certainly have
nanosecond or subnanosecond pulse lengths and pulse-shapes with
no time-tailoring. Thus the calculations in this thesis are
expected to retain physical applicability.

The calculations in this thesis are carried out for
simplicity for the case of a drift-free plasma. The outward
drift-velocity of the actual blowoff plasma is limited to the
ion-sound-speed c_s. The propagation velocities appearing in
the pulse-response diagrams of Chapter 4 are the group-velocities
of the decay-products of the corresponding instabilities. For
each instability it is true that at least one decay-product has

a group-velocity much greater than c_s. Thus the form of the growth of the unstable pulse is not vitally affected by the relatively low blowoff drift-velocity.

The propriety of approximating a spherical laser-wavefront of vacuum wavelength 1.06μ by a plane wavefront is discussed in Section 3.2. There, one takes into account both the increased wavelength in the plasma and the behavior as described by Ginsburg[35] near the critical surface

$$w_p = w_l$$

which forms the classical turning point. The effect of spherical geometry needs to be considered separately for the instability decay-products only when these have characteristic wavelengths longer than that of the laser. This is true for the filamentation and modulation instabilities. For these the optimum k-values for fastest growth are greater at higher laser power intensities. For a power intensity of 10^{15} watts/cm^2 the fastest-growing filaments are only about 10 microns across near the critical surface and 20 microns across at the quarter-critical surface. Comparison with critical-surface radius of say 600μ justifies the use of the planar approximation for the problem. Again the ordering is invalidated by excessively long, relatively low-power laser pulses or early sections of time-tailored pulses as proposed in early versions[9,10] of fusion schemes.

1.2.2 STABILITY OF SPHERICAL GEOMETRY AGAINST ASYMMETRIC
       DEFORMATION

The previous section considered the validity of the

plane-parallel laser-beam, homogeneous-plasma model, as used

to approximate the inhomogeneous spherically-symmetric laser-

pellet plasma.

This section considers the evidence for and against an

even greater departure from the plane-parallel homogeneous

laser-plasma geometry. This is the possibility of asymmetric

deformation of the pellet. This deformation may arise spon-

taneously as a Rayleigh-Taylor instability[43] or be driven by

asymmetry in the illuminating beams. Departures from spherical

symmetry tend to be ironed out by the thermal conductivity of

the electrons in the plasma atmosphere[44]. However thermal

transport by electrons may be inhibited by magnetic fields[45],

or by anomalously low conductivity[46]. Indeed, highly uneven

illumination as of a focal spot on a plane target, creates, via

the thermoelectric effect, self-magnetic fields in the target

plasma which may be of the order of megagauss and which inhibit

transverse thermal transport, thus preserving the temperature

inequality[47]. Such megagauss fields have been observed[25].

The Rayleigh-Taylor instability gives rise to deforma-

tions which have growth times of the order of a nanosecond for

realistic laser-pellet parameters[43]. Thus if the incident

illumination is approximately uniform in intensity over the

critical surface and if the laser-pulse length is kept markedly

shorter than in the original fusion scheme put forward by

Nuckolls, et.al.[9], the asymmetry should not interfere with the
implosion. This combination of approximately uniform illumina-
tion and short pulse length is used in many current experiments.
In particular, experiments using four-beam irradiation and nano-
second or sub-nanosecond pulse length have been reported on by
groups at Sandia[17], Rochester[18] and KMS[19]. Thus the assumption
of a spherical geometry without deformations is a reasonable
one for computational purposes. The further simplification to
a plane-parallel homogeneous model is discussed in 1.2.1 and
in 3.2.

1.3 Outline of Study

1.3.1 DEFINITION OF PROBLEM

To gain insight and simplify calculations, the actual
inhomogeneous, spherical, convergently-illuminated plasma is
replaced by a simplified physical model. The simplified phys-
ical model is that of a plane-polarized, undepleted, unattenuated
laser-pump-wave, permeating an unmagnetized homogeneous plasma
and giving rise to various coupled-mode instabilities. The
theoretical problem is to set up a formalism which is physically
appropriate for the straightforward derivation of the instabili-
ties in the simplified model, and yet generalizes without dif-
ficulty to magnetized plasma, depleted pumps, differential
drift velocities and so on.

1.3.2 DERIVATION OF THEORY

The general theory is one which describes coherent wave-wave interaction. It is a generalized-coupling-of-modes formalism. It describes a medium by a constitutive relation, namely a nonlocal conductivity which expresses the electric current at a space-time point in terms of the electric field at other points. This conductivity is not only nonlocal, thus leading to dispersion, but also nonlinear, thus leading to wave-wave interaction. More precisely, the electric current has a functional Taylor expansion in terms of the electric field. The kernel of the linear term is the linear conductivity in the space-time domain, the kernel of the second-order nonlinear term is the second-order nonlinear conductivity and so on. One treats the nonlinear conductivity as a small perturbation on the linear system defined by the linear conductivity together with Maxwell's equations. The normal modes of that linear system are coupled by the nonlinearity and their amplitudes suffer a slow spacetime variation. It is this slow spacetime variation which is described by the generalized-coupling-of-modes equations which are the heart of the theory.

1.3.3 AREAS OF APPLICATION

The generalized-coupling-of-modes equations are first applied to the case of a pump-wave whose spatial attenuation and temporal depletion are negligible because the other modes to which it couples are all of sufficiently small amplitude.

This case describes parametric instabilities during the time
of their exponential growth from thermal noise and before their
nonlinear saturation. Equations are derived which describe the
exponential growth of the small-amplitude pump-coupled modes in
terms of coupling coefficients. These coupling coefficients
are evaluated explicitly for a plasma containing arbitrarily
many particle populations, each characterized by species,
temperature and drift velocity, and each described by a warm
fluid model.

The work is then further specialized to the case in which
the pump-wave is a laser beam and the medium is an unmagnetized
drift-free plasma. The three-dimensional dispersion relations
for laser-driven instabilities are derived systematically and
the corresponding one-dimensional stability analyses are car-
ried out for propagation in the direction of greatest growth.

1.3.4 CONCLUSIONS

The successful application of the generalized-coupling-
of-modes formalism to the particular case of laser-driven in-
stabilities in unmagnetized plasma reinforces our belief that
it is the most physically appropriate and computationally
straightforward formalism to use in attacking more general
problems. These more general problems include strong electro-
static pumps, magnetized plasmas, temporally depleted pump-
waves,and spatially-attenuated pump-waves. Such wider applica-
tions of the generalized-coupling-of-modes formalism are dis-
cussed in greater detail in Chapter 5.

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CHAPTER 2

THE GENERAL THEORY OF COHERENT WAVE-WAVE COUPLING

2.1   Genesis of Theory

2.2   Scope of the Theory

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2.4   Simple Coupling-of-Modes via **Second-Order Nonlinear**
        Currents

2.5   Generalized-Coupling-of-Modes via Arbitrarily-
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CHAPTER 2

THE GENERAL THEORY OF COHERENT WAVE-WAVE COUPLING

2.1 Genesis of Theory

By the conductivity of a medium we mean the functional
dependence of electric current on electric field within that
medium. The requirement that the field and the dependent
current obey Maxwell's equations yields an electrodynamic theory
governing electromagnetic fields within that medium. A cur-
rent linearly dependent on electric field gives rise to the
usual theory of linear electromagnetic waves in a conductive
medium. An admixture of quadratic dependence on electric
field allows coherent wave-wave interaction which is described
by the well-known coupling-of-modes theory. An electric cur-
rent expressible as an expansion in arbitrarily high powers of
electric field allows coherent wave-wave interaction of
arbitrarily high order, which is described by the generalized-
coupling-of-modes theory presented in this chapter.

The usual coupling-of-modes theory is briefly sketched in
section 2.4, preparing the reader for the detailed presentation
of the generalized-coupling-of-modes theory in section 2.5.

2.2 Scope of the Theory

In this section we compare and contrast the generalized-
coupling-of-modes theory both with the simple coupling-of-modes
theory of which it is an outgrowth and also with the various
special-purpose theories which have been developed to deal with

specific instabilities.

The simple coupling-of-modes theory[1] does allow gradual spatio-temporal variation in the envelopes of electromagnetic waves, and so does allow for spatial attenuation and temporal depletion of pump-waves due to interactions with other waves in a plasma. However, the effect of such pump-wave evolution on the interactions themselves is not included. The simple coupling-of-modes theory describes 3-wave interactions in which two waves combine to produce a 2nd-order nonlinear current which drives a third wave. However, higher-order wave-wave interactions are not included.

The generalized-coupling-of-modes theory also allows for gradual spatio-temporal variation - gradual in a sense to be defined precisely later - in the envelopes of electromagnetic waves. Further, the wave-wave interactions are described in a manner which correctly incorporates the effect of this variation to all orders. Thus an instability, driven by a pump, and depleting that pump, may be described taking into account not only the instantaneous amplitude of the pump but also its space-time derivatives. In this way the effect of pump evolution on pump-driven instabilities may be determined. The generalized-coupling-of-modes theory describes general $(n + 1)$-wave interactions in which n waves combine to produce an nth-order nonlinear current which drives an $(n + 1)^{th}$ wave. Thus wave-wave interactions of arbitrarily high order are included.

Various special-purpose theories have been developed to deal with specific instabilities. All assume undepleted, unattenuated pump-waves. These theories fall into three main groups.

The first group contains theories which treat all orders of wave-wave interactions correctly, and thus can treat pump-waves of arbitrarily high intensity. However, they place severe restrictions on the directions of propagation and polarization of the waves considered. One-dimensional electrostatic theories[2] of pump-driven instability, including oscillating-frame analyses,[3,4] fall into this group.

The second group contains theories[5,6] which construct four-wave interactions from successive three-wave interactions, in a way which depends crucially on disparities between phase-velocities of particular interacting waves. This reliance on phase-velocity disparities between particular interacting waves limits the number of instabilities that can be considered, especially in magnetized plasma. Also, interactions are formulated as being between certain waves defined by their electric fields and certain other waves defined by the density of one plasma component. This lack of parallelism makes it hard to deal with additional components, whether these be additional ion species or additional electron population components with differing drift velocities and temperatures.

The third group of theories restrict themselves to three-wave interaction between two pump-waves and a single product-

wave which therefore grows linearly. This restriction makes
these theories relatively simple and they will not be discussed
further.

The propagation and polarization vectors of all waves in
G-C-O-M may lie at arbitrary angles to each other and to any
magnetic field in the plasma. The G-C-O-M theory does not
require phase-velocity disparities between particular interact-
ing waves. All waves are defined by their electric fields,
and their interaction is described in terms of nonlinear con-
ductivity of the plasma. The total conductivity of the plasma
is just the sum of the conductivities of the species it con-
tains. Therefore there is within G-C-O-M theory no methodo-
logical difficulty in introducing additional ion species, or
in introducing electron population components with differing
drift velocities and temperatures and treating them as multiple
species. Further, there is no methodological difficulty in
using any model whatever to describe the plasma, whether cold
fluid, warm fluid, or kinetic.

The G-C-O-M equations as presented in this thesis do not
contain terms describing the effect of inhomogeneity of the
medium. Thus these equations could be used to describe in-
stabilities in inhomogeneous plasmas only in the W.K.B. sense.
The computation and employment of terms describing the effect
of medium inhomogeneity on wave-wave interactions lies beyond
the scope of this work.

2.3 Present Extent of Utilization

The applications of the generalized-coupling-of-modes theory which are set out in chapters 3 and 4 do not utilize the full capability of that theory. To facilitate comparison with certain other treatments of laser-driven instabilities, the theory is there applied to an unmagnetized plasma containing a single unattenuated, undepleted pump-wave with a unique frequency, wavevector and polarization. The pump-wave is sufficiently strong that instabilities due to 3rd-order non-linear currents become important, and instabilities driven by 2nd-order nonlinear currents are modified. Wave-wave interactions of order higher than those caused by 3rd-order non-linear currents are not considered.

The full utility of the generalized-coupling-of-modes theory will only be realized in future applications. One such application is to magnetized plasma, where the richer mode structure as compared with the unmagnetized case allows a richer variety of mode-couplings and hence of laser-driven instabilities. A second such application is to a laser-irradiated plasma in which one or more decay-products of laser-driven instabilities have, without losing their coherence, grown sufficiently to act as additional pumps. A third such application is to a plasma irradiated by more than one laser beam. A fourth such application is to plasma containing attenuated and depleted pump-waves. We hope to bring the

generalized-coupling-of-modes theory to bear on these problems immediately upon completion of this thesis.

2.4 Simple Coupling-of-Modes via 2nd-Order Nonlinear Currents

In this section we outline the simplest possible non-trivial theory of nonlinear coherent wave-wave coupling, namely the simple coupling-of-modes theory. Only the basic formulation will be given here. This formulation was used by Bers in deriving a rich body of results to which the interested reader is referred.

We consider a homogeneous medium in which the dependence of electric current on electric field is as follows. There is a part of the current which is 1st-order in electric field and which arises from the familiar linear conductivity of the medium. The other part of the current is 2nd-order in electric field, and constitutes the simplest possible type of nonlinear conductivity in the medium.

We derive equations which strongly resemble the usual equations describing linear electromagnetic waves in a conducting medium. Indeed the equations describe how the linear system, defined by Maxwell's equations and the linear part of the conductivity, is driven by the nonlinear currents which are external to it. Each electromagnetic wave of that linear system is driven by nonlinear current due to other electromagnetic waves. The physical motivation for writing the coupling-of-modes equations in this fashion is to relate as many as possible of the interacting waves back to the well-understood

normal modes of the linear system.

Let us now perform the mathematical derivation explicitly.
Consider a medium in which the electric current at any one
point in space and time depends not only upon the electric
field at that point, but also upon the values of electric field
at other space-time points. Then the medium is temporally and
spatially dispersive. Further, let the dependence have a lin-
ear part and a quadratic part. Then the medium has a linear
conductivity and also has the simplest non-trivial type of
nonlinear conductivity. The functional dependence of electric
current on electric field then assumes the following general
form:

$$\vec{J}(\vec{x},t) = \iint d\vec{x}' \, dt' \, \overset{\Rightarrow}{G}^{LIN}(\vec{x},t,\vec{x}',t')\vec{E}(\vec{x}',t')$$

$$+ \frac{1}{2!} \iiiint d\vec{x}' \, dt' \, d\vec{x}'' \, dt'' \, \overset{\Rightarrow}{G}^{NL(2)}(\vec{x},t,\vec{x}',t',\vec{x}'',t'')$$

$$\vec{E}(\vec{x}',t') \, \vec{E}(\vec{x}'',t'')$$

2.4-(1)

Here the first and second terms on the right-hand-side are the
linear and nonlinear currents respectively. The linear con-
ductivity in the space-time domain, $\vec{G}^{LIN}$, has a tensor char-
acter and one of its indices is contracted with that of $\vec{E}$.
The 2nd-order nonlinear conductivity in the space-time domain,

$\vec{\vec{G}}^{NL(2)}$, has a tensor character and two of its indices are contracted each with the index of the corresponding $\vec{E}$. The ranges over which the space-time integrations are carried out must, from causality, be such that the 4-vectors $(\vec{x}' - \vec{x},\ t' - t)$, $(\vec{x}'' - \vec{x},\ t'' - t)$ lie within the backward light-cone.

Let the medium be spatially and temporally homogeneous. Then the linear conductivity depends on its space and time arguments $(\vec{x}, t, \vec{x}', t')$ only through their differences. One may write

$$\vec{\vec{G}}^{LIN}\left(\vec{x}, t, \vec{x}', t\right) \equiv \vec{\vec{G}}^{LIN}\left(\vec{x}' - \vec{x},\ t' - t\right)$$

$$\equiv \vec{\vec{G}}^{LIN}\left(-\vec{\xi},\ -\tau\right) \quad \text{say}.$$

$$2.4\text{-}(2)$$

Similarly the nonlinear conductivity depends on its space and time arguments $(\vec{x}, t, \vec{x}', t', \vec{x}'', t'')$ only through their differences, so

$$\vec{\vec{G}}^{NL(2)}\left(\vec{x}, t, \vec{x}', t', \vec{x}'', t''\right) \equiv \vec{\vec{G}}^{NL(2)}\left(\vec{x}' - \vec{x},\ t' - t,\ \vec{x}'' - \vec{x},\ t'' - t\right)$$

$$\equiv \vec{\vec{G}}^{NL(2)}\left(-\vec{\xi}',\ -\tau',\ -\vec{\xi}'',\ -\tau''\right) \quad \text{say}. \qquad 2.4\text{-}(3)$$

Equation (1) now acquires the form

$$\vec{J}(\vec{x},t) = \iint d\vec{\xi} \, d\tau \, \vec{\vec{G}}^{LIN}(-\vec{\xi},-\tau)\vec{E}(\vec{x}+\vec{\xi},\, t+\tau)$$

$$+ \frac{1}{2!} \iiiint d\vec{\xi}' \, d\tau' \, d\vec{\xi}'' \, d\tau'' \, \vec{\vec{\vec{G}}}^{NL(2)}(-\vec{\xi}',-\tau',-\vec{\xi}'',-\tau'')$$

$$\vec{E}(\vec{x}+\vec{\xi}',t+\tau') \, \vec{E}(\vec{x}+\vec{\xi}'',t+\tau'')$$

2.4-(4)

This expression for the current must be substituted into
Maxwell's equations to obtain the equations governing the
electromagnetic field in the medium. Before doing this, let
us express $\vec{E}(\vec{x},t)$ in a form which uses our knowledge of linear
media.

The most useful form for $\vec{E}(\vec{x},t)$ is motivated by treating
the nonlinear conductivity as a small perturbation on that lin-
ear system which is defined by Maxwell's equations and the
linear conductivity. The possible electromagnetic fields in
that system are just linear superpositions of its normal modes.
Each such normal mode has, of course, a space-time dependence
which is just a complex exponential. On introducing nonlinear
conductivity, one again allows the electromagnetic field to have
the form of a superposition of waves. Now, however, not all

the waves need to be normal modes of the linear system. Further-more, each wave is allowed to have a space-time dependence in the form of a complex exponential modulated by a slowly-varying envelope:

$$\vec{E}(\vec{x},t) = \sum_{a} \vec{E}_a(\vec{x},t) \, e^{i\vec{k}_a \cdot \vec{x} - i w_a t}$$

2.4-(5)

Here one has adopted the convention that the range of the suffix a includes complex conjugate fields, so that for every value a' in the range there exists another value a" with $\vec{E}_{a'} = \vec{E}_{a''}^*$, $\vec{k}_{a'} = -\vec{k}_{a''}^*$, $w_{a'} = w_{a''}^*$.

The next step is to insert the form (5) for the electric field into the expression (4) for the electric current. The evaluation of the first term on the right-hand-side of (4) is carried out using to advantage the slowness of the variation of the electric field envelopes. The values of the envelopes $\vec{E}_a(\vec{x} + \vec{\xi}, t + \tau)$ are approximated by their 1st-order Taylor expansions about the point $(\vec{x}, t)$ before the integration is carried out. The evaluation of the second term on the right-hand-side of (4) is carried out using to advantage the fact that this second term is to be thought of as the perturbation which induces the slow envelope variations. Thus these slow variations need not be taken into account in evaluating the perturbation itself. The values of the envelopes

$\vec{E}(\vec{x} + \vec{\xi}', t + \tau')$, $\vec{E}(\vec{x} + \vec{\xi}'', t + \tau'')$ are approximated by their values at the point $(\vec{x},t)$ before the integration is carried out. The evaluation yields an expression for the electric current in terms of the electric field envelopes, the space-time derivatives of those envelopes, and the conductivities in the wavevector-frequency domain (see Appendix A1):

$$\vec{J}(\vec{x},t) = \sum_a e^{i\vec{k}_a\cdot\vec{x} - i w_a t} \left\{ \vec{\vec{G}}^{LIN}(\vec{k}_a, w_a)\vec{E}_a(\vec{x},t) \right.$$
$$\left. - i \frac{\partial \vec{\vec{G}}^{LIN}}{\partial \vec{k}} \frac{\partial \vec{E}_a}{\partial \vec{x}} + i \frac{\partial \vec{\vec{G}}^{LIN}}{\partial w} \frac{\partial \vec{E}_a}{\partial t} \right\}$$
$$+ \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c)\cdot x - i(w_b + w_c)t}$$
$$\left\{ \vec{\vec{G}}^{NL(2)}(\vec{k}_b, w_b, \vec{k}_c, w_c)\vec{E}_b(\vec{x},t)\vec{E}_c(\vec{x},t) \right\}$$

2.4-(6)

Here, as before, the first and second terms on the right-hand-side are the linear and nonlinear currents respectively. The same symbol has been used for conductivity in the wavevector-frequency domain as for conductivity in the space-time domain; this will not cause confusion. As before, each linear conductivity, $\vec{\vec{G}}^{LIN}(\vec{k}_a, w_a)$, has a tensor character and one of its indices is contracted with that of $\vec{E}_a$. This holds true also in the derivative terms, where in addition the wavevector-derivative $\partial/\partial\vec{k}$ is contracted with the space-derivative $\partial/\partial\vec{x}$.

As before, each 2nd-order nonlinear conductivity, $\vec{\vec{G}}^{NL}(2)$
$(\vec{k}_b, w_b, \vec{k}_c, w_c)$ has a tensor character and two of its indices
are contracted with those of $\vec{E}_b$ and $\vec{E}_c$ respectively.

The forms (5) and (6) for the electric field and electric
current are now substituted into Maxwell's equations to obtain
the equations governing the behavior of the electric-field-
envelopes in the medium. It is convenient to use Maxwell's
equations in the form from which $\vec{B}$ has been eliminated:

$$\left\{ \frac{\partial}{\partial \vec{x}}\frac{\partial}{\partial \vec{x}} \cdot - \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{x}} + \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right\} \vec{E} + \left\{ \mu_0 \frac{\partial}{\partial t} \right\} \vec{J} = \vec{0}$$

$$2.4-(7)$$

After substitution, the slow space-time variations of the
field envelopes are retained to first order, except in terms
involving the nonlinear current where they are neglected.
Bringing the nonlinear current terms to the right-hand-side,
(7) assumes the following form, which describes the behavior
of the electric-field-envelopes in the medium (see Appendix A1):

$$\sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \left\{ \vec{\vec{L}}^{LIN}(\vec{k}_a, w_a) \vec{E}_a(\vec{x}, t) \right.$$

$$\left. - i \frac{\partial \vec{\vec{L}}^{LIN}}{\partial \vec{k}} \frac{\partial \vec{E}_a}{\partial \vec{x}} + i \frac{\partial \vec{\vec{L}}^{LIN}}{\partial w} \frac{\partial \vec{E}_a}{\partial t} \right\} =$$

$$\frac{1}{2!} \sum_b \sum_c e^{i(\vec{K}_b + \vec{K}_c) \cdot \vec{x} - i(w_b + w_c)t} (-i\mu_o)(w_b + w_c)$$

$$\vec{\vec{G}}^{NL(2)}(\vec{K}_b, w_b, \vec{K}_c, w_c) \vec{E}_b(\vec{x}, t) \vec{E}_c(\vec{x}, t)$$

<div align="right">2.4-(8)</div>

Here the linear dispersion tensor

$$\vec{\vec{L}}^{LIN}(\vec{K}_a, w_a) \equiv k_a k_a - k_a^2 + \frac{w_a^2}{c^2} + i\mu_o w_a \vec{\vec{G}}^{LIN}(\vec{K}_a, w_a)$$

<div align="right">2.4-(9)</div>

is contracted with $\vec{E}_a$ wherever it appears on the left-hand-side of (8). The wavevector-derivative $\partial/\partial \vec{k}$ is contracted with the space-derivative $\partial/\partial \vec{x}$.

Assume that the effect of the interactions is small enough so that the modes considered are still separable in the wavevector-frequency domain. Multiplication by the relevant exponentials then isolates from (10) the following set of equations, each equation describing how a particular mode-envelope $\vec{E}_a$ is driven;

$$\vec{\vec{L}}^{LIN}(\vec{K}_a, w_a) \vec{E}_a(\vec{x}, t) - i \frac{\partial \vec{\vec{L}}^{LIN}}{\partial \vec{k}} \frac{\partial \vec{E}_a}{\partial \vec{x}} + i \frac{\partial \vec{\vec{L}}^{LIN}}{\partial w} \frac{\partial \vec{E}_a}{\partial t}$$

$$= \sum_{\substack{those\ pairs\ (b,c) \\ which\ drive\ \vec{E}_a}} \left\{ e^{i(-\vec{K}_a + \vec{K}_b + \vec{K}_c) \cdot \vec{x} - i(-w_a + w_b + w_c)t} \right.$$

$$\left(-i\mu_o\right)\left(\omega_b + \omega_c\right)\overline{\overline{G}}^{NL(2)}\left(\vec{k}_b, \omega_b, \vec{k}_c, \omega_c\right)$$
$$\left. \vec{E}_b(\vec{x},t)\,\vec{E}_c(\vec{x},t) \right\}$$
   2.4-(10)

Any mode appearing in (5) has a wavevector and frequency which lie at or near the sum-wavevector and sum-frequency of any pair of modes which drive it. This ensures that the summation on the right-hand-side of (10) includes only those pairs $(\vec{E}_b, \vec{E}_c)$ of modes such that the wavevector and frequency mismatches

$$\delta\vec{k} \equiv -\vec{k}_a + \vec{k}_b + \vec{k}_c$$

$$\delta\omega \equiv -\omega_a + \omega_b + \omega_c$$

   2.4-(11)

are small.

The family of equations (10) are the coupling-of-modes equations. They describe the behavior of the electric field in the nonlinear medium specified by (1). They do this, not directly in terms of $\vec{E}(\vec{x},t)$, but rather in terms of a family of wave-envelopes $\vec{E}_a(\vec{x},t)$. The waves which these envelopes modulate are chosen having regard to the physical problem to be solved and the properties of the medium in the absence of nonlinearity. The coupling-of-modes equations (10) are a

family of coupled differential equations. Each such equation
describes the space-time variation of one wave-envelope brought
about by the nonlinear coupling to the other waves.

2.5 Generalized Coupling-of-Modes via Arbitrarily-High-Order Nonlinear Currents

In this section we outline a generalization of the simple
coupling-of-modes theory set out in the previous section. The
generalized-coupling-of-modes theory is again a theory of non-
linear coherent wave-wave coupling. The physical motivation
for setting up the theory and the line of argument used to
arrive at the final mode-coupling equations are almost the
same as in the previous section. This will enable us to pre-
sent the formulation fairly concisely.

We consider a homogeneous medium in which the dependence
of electric current on electric field is as follows. There is
a part of the current which is 1st-order in electric field and
which arises from the familiar linear conductivity of the
medium. There are also other parts of the current which are
respectively 2nd-order in electric field, 3rd-order in
electric field, and so on. Thus the nonlinear conductivity
of the medium has a well-behaved expansion in powers of the
electric field.

Again we derive equations which describe how the linear
system, defined by Maxwell's equations and the linear part of
the conductivity, is driven by the nonlinear currents which

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are external to it. Again the physical motivation for writing
the mode-coupling equations in this fashion is to relate as
many as possible of the interacting waves back to the well-
understood normal modes of the linear system.

The explicit mathematical derivation follows. Again
consider a medium with a non-local dependence of electric
current on electric field, a dependence which must therefore
be expressed by space-time integrals. Let the electric current
have a well-behaved expansion in arbitrarily high integral
powers of the electric field. Then the medium sustains a
linear current, a 2nd-order nonlinear current, a 3rd-order
nonlinear current, and so on, given by the successive terms in
(cf 2.4-(1)):

$$
\vec{J}(\vec{x},t) = \iint d\vec{x}'\,dt'\, \overleftrightarrow{G}^{LIN}(\vec{x},t,\vec{x}',t')\vec{E}(x',t')
$$

$$
+ \frac{1}{2!} \iiiint d\vec{x}'\,dt'\,d\vec{x}''\,dt''\, \overleftrightarrow{G}^{NL(2)}(\vec{x},t,\vec{x}',t',\vec{x}'',t'')
$$

$$
\vec{E}(\vec{x}',t')\,\vec{E}(\vec{x}'',t'')
$$

$$
+ \frac{1}{3!} \iiiiiint d\vec{x}'\,dt'\,d\vec{x}''\,dt''\,d\vec{x}'''\,dt'''
$$

$$
\overleftrightarrow{G}^{NL(3)}(\vec{x},t,\vec{x}',t',\vec{x}'',t'',\vec{x}''',t''')
$$

$$
\vec{E}(\vec{x}',t')\,\vec{E}(\vec{x}'',t'')\,\vec{E}(\vec{x}''',t''')
$$

$$
+ \ldots\ldots
$$

2.5-(1)

The conductivities in the space-time domain, $\vec{G}^{LIN}$, $\vec{G}^{NL(2)}$, $\vec{G}^{NL(3)}$ and so on, have a tensor character. Each of these conductivities has all but one of its indices contracted with the succeeding $\vec{E}$-vectors.

Further, let the medium be spatially and temporally homogeneous. Then (1) acquires the form (cf 2.4-(4)):

$$\vec{J}(\vec{x},t) = \iint d\vec{\xi}\, d\tau\; \vec{G}^{LIN}(-\vec{\xi},-\tau)\,\vec{E}(\vec{x}+\vec{\xi},t+\tau)$$

$$+ \frac{1}{2!} \iiint d\vec{\xi}'\, d\tau'\, d\vec{\xi}''\, d\tau''\; \vec{\vec{G}}^{NL(2)}(-\vec{\xi}',-\tau',-\vec{\xi}'',-\tau'')$$
$$\vec{E}(\vec{x}+\vec{\xi}',t+\tau')\,\vec{E}(\vec{x}+\vec{\xi}'',t+\tau'')$$

$$+ \frac{1}{3!} \iiiint d\vec{\xi}'\, d\tau'\, d\vec{\xi}''\, d\tau''\, d\vec{\xi}'''\, d\tau'''$$
$$\vec{\vec{G}}^{NL(3)}(-\vec{\xi}',-\tau',-\vec{\xi}'',-\tau'',-\vec{\xi}''',-\tau''')$$
$$\vec{E}(\vec{x}+\vec{\xi}',t+\tau')\,\vec{E}(\vec{x}+\vec{\xi}'',t+\tau'')\,\vec{E}(\vec{x}+\vec{\xi}''',t+\tau''')$$

$$+ \ldots\ldots\ldots$$

2.5-(2)

Before substituting this rather general form for the
electric current into Maxwell's equations, one uses one's
physical knowledge of the characteristics of the medium and of
the interactions to be studied, in order to restrict the form
of the electric field. It is assumed that the electrodynamics
of the linear system, defined by Maxwell's equations and the
linear part of the conductivity, are already fairly well
understood in terms of (possibly weakly damped or growing)
sinusoidal waves. The nonlinear conductivity perturbs this
linear system by allowing interaction between waves with dif-
ferent wavevectors and frequencies, provided these satisfy one
of the approximate sum rules:

$$\left(\vec{k}_a, w_a\right) \;\cong\; \left(\vec{k}_b, w_b\right) + \left(\vec{k}_c, w_c\right)$$

$$\text{or} \quad \left(\vec{k}_a, w_a\right) \;\cong\; \left(\vec{k}_b, w_b\right) + \left(\vec{k}_c, w_c\right) + \left(\vec{k}_d, w_d\right)$$

or $\quad$ . . . .

$$2.5\text{-}(3)$$

In a physically interesting problem, the waves in question will
comprise specific normal modes of the linear system, possibly
some pump-wave or pump-waves, and possibly some harmonics or
sidebands of these. The interactions between all these waves
will cause each sinusoid to be modulated by a slowly-varying

wave-envelope, so that the total electric field assumes the form

$$\vec{E}(\vec{x},t) = \sum_a \vec{E}_a(\vec{x},t) e^{i\vec{k}_a \cdot \vec{x} - i\omega_a t}$$

2.5-(4)

By saying that a particular $\vec{E}_a(\vec{x},t)$ is slowly-varying, we mean that its space-time derivatives can be neglected in the final mode-coupling equations beyond some finite order. Since the space-time variations of the wave-envelopes are not known until the mode-coupling e ations are solved, the assumption that these variations are slow can only be justified a posteriori. The space-time derivatives will be retained to all orders, at least formally, for the sake of flexibility. This is done by retaining all orders in the Taylor expansion of $\vec{E}_a(\vec{x} + \vec{\xi}, t + \tau)$ about $(\vec{x},t)$. Carrying out the space-time integrations in (2) then yields (see Appendix A2) an expression for the electric current, in terms of electric-field wave-envelopes and their derivatives in the space-time domain, and also of conductivities and their derivatives in the wavevector-frequency domain. This expression for the electric current can be written in various ways. Here we present a form for the current similar to 2.4-( ), augmented by enough additional terms to indicate how the expansion proceeds:

$$\vec{J}(\vec{x},t) = \sum_a e^{i\vec{k}_a \cdot \vec{x} - i w_a t} \left\{ \overset{\Rightarrow}{G}^{LIN}(\vec{k}_a, w_a) \vec{E}_a(\vec{x},t) \right.$$

$$- i \frac{\partial \overset{\Rightarrow}{G}^{LIN}}{\partial \vec{k}} \frac{\partial \vec{E}_a}{\partial \vec{x}} + i \frac{\partial \overset{\Rightarrow}{G}^{LIN}}{\partial w} \frac{\partial \vec{E}_a}{\partial t}$$

$$- \frac{1}{2!} \frac{\partial^2 \overset{\Rightarrow}{G}^{LIN}}{\partial \vec{k} \partial \vec{k}} \frac{\partial^2 \vec{E}_a}{\partial \vec{x} \partial \vec{x}} + \frac{\partial^2 \overset{\Rightarrow}{G}^{LIN}}{\partial \vec{k} \partial w} \frac{\partial^2 \vec{E}_a}{\partial \vec{x} \partial t}$$

$$- \frac{1}{2!} \frac{\partial^2 \overset{\Rightarrow}{G}^{LIN}}{\partial w^2} \frac{\partial^2 \vec{E}_a}{\partial t^2} + \cdots \cdots \right\}$$

$$+ \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c) \cdot \vec{x} - i(w_b + w_c) t}$$

$$\left\{ \overset{\Rightarrow}{G}^{NL(2)}(\vec{k}_b, w_b, \vec{k}_c, w_c) \vec{E}_b(\vec{x},t) \vec{E}_c(\vec{x},t) \right.$$

$$- i \frac{\partial \overset{\Rightarrow}{G}^{NL(2)}}{\partial \vec{k}_b} \frac{\partial \vec{E}_b}{\partial \vec{x}} \vec{E}_c + i \frac{\partial \overset{\Rightarrow}{G}^{NL(2)}}{\partial w_b} \frac{\partial \vec{E}_b}{\partial t} \vec{E}_c$$

$$- i \frac{\partial \overset{\Rightarrow}{G}^{NL(2)}}{\partial \vec{k}_c} \vec{E}_b \frac{\partial \vec{E}_c}{\partial \vec{x}} + i \frac{\partial \overset{\Rightarrow}{G}^{NL(2)}}{\partial w_c} \vec{E}_b \frac{\partial \vec{E}_c}{\partial t} + \cdots \right\}$$

$$+ \frac{1}{3!} \sum_b \sum_c \sum_d e^{i(\vec{K}_b + \vec{K}_c + \vec{K}_d)\cdot\vec{x} - i(w_b + w_c + w_d)t}$$

$$\left\{ \stackrel{\Rrightarrow}{G}^{NL(3)} (\vec{K}_b, w_b, \vec{K}_c, w_c, \vec{K}_d, w_d) \right.$$

$$\vec{E}_b(\vec{x}, t) \, \vec{E}_c(\vec{x}, t) \, \vec{E}_d(\vec{x}, t)$$

$$\left. + \cdots \right\}$$

$$+ \cdots\cdots\cdots$$

$$2.5 - (5)$$

Here the single summation on the right-hand-side is recognizable as the linear current of 2.4-(6), augmented by higher space-time derivatives of the wave-envelopes and corresponding higher wavevector-frequency derivatives of the linear conductivity. The double summation on the right-hand-side is recognizable as the 2nd-order nonlinear current of 2.4-(6), augmented by space-time derivatives of the wave-envelopes and corresponding wavevector-frequency derivatives of the 2nd-order nonlinear conductivity. The triple summation on the right-hand-side is recognizable as the 3rd-order nonlinear current on comparing its structure with the structure of the previous terms. The tensor and vector indices in the above are contracted as follows. As before, the conductivities are contracted with the corresponding electric-field wave-envelopes and the wavevector-derivatives are contracted with the corresponding space-derivatives.

The expression (4) for the electric field and the expression (5) for the electric current are substituted into Maxwell's wave-equation:

$$\left\{ \frac{\partial}{\partial \vec{x}} \frac{\partial}{\partial \vec{x}} \cdot - \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{x}} + \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right\} \vec{E} + \left\{ \mu_0 \frac{\partial}{\partial t} \right\} \vec{J} = \vec{0}$$

The result is an equation governing the behavior of the electric-field wave-envelopes in the nonlinear medium. This will be written in a form similar to 2.4-(8), augmented by enough additional terms to indicate how the expansion proceeds:

$$\sum_a e^{i\vec{k}_a \cdot x - i w_a t} \left\{ \overset{\Rightarrow}{L}^{LIN}(\vec{k}_a, w_a) \vec{E}_a(\vec{x}, t) \right.$$

$$- i \frac{\partial \overset{\Rightarrow}{L}^{LIN}}{\partial \vec{k}} \frac{\partial \vec{E}_a}{\partial \vec{x}} + i \frac{\partial \overset{\Rightarrow}{L}^{LIN}}{\partial w} \frac{\partial \vec{E}_a}{\partial t}$$

$$- \frac{1}{2!} \frac{\partial^2 \overset{\Rightarrow}{L}^{LIN}}{\partial \vec{k} \partial \vec{k}} \frac{\partial^2 \vec{E}_a}{\partial \vec{x} \partial \vec{x}} + \frac{\partial^2 \overset{\Rightarrow}{L}^{LIN}}{\partial \vec{k} \partial w} \frac{\partial^2 \vec{E}_a}{\partial \vec{x} \partial t} - \frac{1}{2!} \frac{\partial^2 \overset{\Rightarrow}{L}^{LIN}}{\partial w^2} \frac{\partial^2 \vec{E}_a}{\partial t^2}$$

$$\left. + \cdots \cdots \right\} \qquad =$$

$$\frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c)\cdot\vec{x} - i(\omega_b + \omega_c)t} \left(-i\mu_0 \right)\left(\omega_b + \omega_c + i\frac{\partial}{\partial t} \right)$$

$$\left\{ \bar{\bar{G}}^{NL(2)}(\vec{k}_b, \omega_b, \vec{k}_c, \omega_c) \vec{E}_b(\vec{x}, t)\vec{E}_c(\vec{x}, t) \right.$$

$$- i \frac{\partial \bar{\bar{G}}^{NL(2)}}{\partial \vec{k}_b} \frac{\partial \vec{E}_b}{\partial \vec{x}} \vec{E}_c + i \frac{\partial \bar{\bar{G}}^{NL(2)}}{\partial \omega_b} \frac{\partial \vec{E}_b}{\partial t} \vec{E}_c$$

$$\left. - i \frac{\partial \bar{\bar{G}}^{NL(2)}}{\partial \vec{k}_c} \vec{E}_b \frac{\partial \vec{E}_c}{\partial \vec{x}} + i \frac{\partial \bar{\bar{G}}^{NL(2)}}{\partial \omega_c} \vec{E}_b \frac{\partial \vec{E}_c}{\partial t} + \dots \right\}$$

$$+ \frac{1}{3!} \sum_b \sum_c \sum_d e^{i(\vec{k}_b + \vec{k}_c + \vec{k}_d)\cdot\vec{x} - i(\omega_b + \omega_c + \omega_d)t}$$

$$(-i\mu_0)\left(\omega_b + \omega_c + \omega_d + i\frac{\partial}{\partial t} \right)$$

$$\left\{ \bar{\bar{G}}^{NL(3)}(\vec{k}_b, \omega_b, \vec{k}_c, \omega_c, \vec{k}_d, \omega_d) \right.$$

$$\vec{E}_b(\vec{x}, t)\vec{E}_c(\vec{x}, t)\vec{E}_d(\vec{x}, t)$$

$$\left. + \dots\dots \right\}$$

$$+ \dots\dots$$

<div align="right">2.5-(6)</div>

Here $\vec{L}^{LIN}$ is the linear dispersion tensor as displayed in 2.4-(9).

Equation (6) is an expansion in orders of nonlinear conductivity, within each term of which lies an expansion in orders of the slow space-time derivatives. As discussed previously, the ordering and truncation of the combined expansion must be carried out with regard to the physical magnitudes of the terms in question , some of which may only be known a posteriori.

Again assume that the effect of the interaction is small enough so that the modes considered are still separable in the wavevector-frequency domain. Multiplication by the relevant exponentials then separates from (6) the following set of equations, each equation describing how a particular mode-envelope $\vec{E}_a$ is driven;

$$\vec{\vec{L}}^{LIN}(\vec{k}_a, w_a)\, \vec{E}_a(\vec{x}, t)$$

$$-\, i\, \frac{\partial \vec{\vec{L}}^{LIN}}{\partial \vec{k}}\, \frac{\partial \vec{E}_a}{\partial \vec{x}}\, +\, i\, \frac{\partial \vec{\vec{L}}^{LIN}}{\partial w}\, \frac{\partial \vec{E}_a}{\partial t}$$

$$-\, \frac{1}{2!}\, \frac{\partial^2 \vec{\vec{L}}^{LIN}}{\partial \vec{k}\, \partial \vec{k}}\, \frac{\partial^2 \vec{E}_a}{\partial \vec{x}\, \partial \vec{x}}\, +\, \frac{\partial^2 \vec{\vec{L}}^{LIN}}{\partial \vec{k}\, \partial w}\, \frac{\partial^2 \vec{E}_a}{\partial \vec{x}\, \partial t}\, -\, \frac{1}{2!}\, \frac{\partial^2 \vec{\vec{L}}^{LIN}}{\partial w^2}\, \frac{\partial^2 \vec{E}_a}{\partial t^2}$$

$$+\, \cdots \cdots \qquad\qquad =$$

$$\sum_{\substack{\text{those pairs } (b,c)\\ \text{of waves which}\\ \text{drive wave } a}} e^{\, i\left(-\vec{k}_a + \vec{k}_b + \vec{k}_c\right)\cdot \vec{x}\, -\, i\left(-w_a + w_b + w_c\right) t}$$

$$(-i\mu_0)\left(w_b + w_c + i\frac{\partial}{\partial t}\right)$$

$$\left\{ \vec{\vec{G}}^{NL(2)}\left(\vec{k}_b, w_b, \vec{k}_c, w_c\right) \vec{E}_b(\vec{x}, t)\, \vec{E}_c(\vec{x}, t)\right.$$

$$-\, i\, \frac{\partial \vec{\vec{G}}^{NL(2)}}{\partial \vec{k}_b}\, \frac{\partial \vec{E}_b}{\partial \vec{x}}\, \vec{E}_c\, +\, i\, \frac{\partial \vec{\vec{G}}^{NL(2)}}{\partial w_b}\, \frac{\partial \vec{E}_b}{\partial t}\, \vec{E}_c$$

$$\left.-\, i\, \frac{\partial \vec{\vec{G}}^{NL(2)}}{\partial \vec{k}_c}\, \vec{E}_b\, \frac{\partial \vec{E}_c}{\partial \vec{x}}\, +\, i\, \frac{\partial \vec{\vec{G}}^{NL(2)}}{\partial w_c}\, \vec{E}_b\, \frac{\partial \vec{E}_c}{\partial t}\, +\, \cdots \right\}$$

$$+ \sum_{\substack{\text{those triplets } (b,c,d) \\ \text{of waves which} \\ \text{drive wave } a}} e^{i\left(-\vec{k}_a + \vec{k}_b + \vec{k}_c + \vec{k}_d\right)\cdot \vec{x} - i\left(-w_a + w_b + w_c + w_d\right)t}$$

$$\left(-i\mu_0\right)\left(w_b + w_c + w_d + i\frac{\partial}{\partial t}\right)$$

$$\left\{ \overline{\overline{\overline{\sigma}}}^{NL(3)}\left(\vec{k}_b, w_b, \vec{k}_c, w_c, \vec{k}_d, w_d\right) \right.$$

$$\vec{E}_b(\vec{x}, t)\, \vec{E}_c(\vec{x}, t)\, \vec{E}_d(\vec{x}, t)$$

$$\left. + \cdots \cdots \right\}$$

$$+ \cdots \cdots$$

<div align="right">2.5-(7)</div>

Since each specific pair of modes appears twice in the double

summation in (6), the factor 1/2! does not occur in the sum over

specific pairs in (7). Similarly, since each specific triplet

of waves appears 3! times in the triple summation in (6), the factor 1/3! does not occur in the sum over specific triplets of waves in (7). The exponential factors immediately following the summation signs are slowly-varying functions of space and time since the wavevectors and frequencies in the exponent satisfy (3). As in (6), and also in 2.4-(8), 2.4-(10), the linear dispersion tensor $\vec{L}^{LIN}$ and the nonlinear conductivities $\vec{G}^{NL}(2)$, $\vec{G}^{NL}(3)$, ..., are contracted with the $\vec{E}$-vectors, and the wavevector-derivatives $\partial/\partial\vec{k}$ are contracted with the space-derivatives $\partial/\partial\vec{x}$. The expansions in orders of the wavevector-frequency derivatives and space-time derivatives can be written in closed form using exponentials of differential operators. This results in a form of equation more concise and exact than (7) but less physically transparent. This form has therefore been relegated to Appendix A2.

The set of equations (7) are the mode-coupling equations. They describe the space-time variation, in amplitude, phase, and polarization, of each wave-envelope $\vec{E}_a(\vec{x},t)$ due to nonlinear currents excited by other waves. They form a set of coupled nonlinear differential equations. The left-hand-side of each equation of the set relates the corresponding wave back to the linear system, defined by the linear conductivity together with Maxwell's equations. This enables one to classify known unstable interactions and predict new unstable interactions on the basis of pre-existing knowledge of the normal modes of the

linear medium. This is an aid to understanding the physical consequences of the growth of such instabilities and hence predicting mechanisms of saturation.

One does not attempt to retain all terms in each of the mode-coupling equations. Guided by the specific physical problem to be solved, one decides for each $\vec{E}_a$ where to truncate the corresponding equation (7). Note that in the simple coupling-of-modes theory the mode-coupling equations 2.4-(10) are truncated after the 2nd-order nonlinear conductivity and the linear and 2nd-order terms are then truncated after the 1st and zeroth order of space-time derivatives respectively. After truncation, the resulting set of coupled differential equations yields a simultaneous solution for all the $\vec{E}_a(\vec{x},t)$ as functions of space and time. This solution is used to estimate the magnitude of the discarded terms, thus providing an a posteriori check on the validity of the procedure. If necessary, the truncation may be revised to retain additional terms, and the procedure repeated. It is assumed that in this manner one arrives at a set of truncated mode-coupling equations which adequately describe the physical problem while retaining only one or two orders of the space-time derivatives of the $\vec{E}_a(\vec{x},t)$. This is what is meant physically by saying that the $\vec{E}_a(\vec{x},t)$ are assumed to be slowly-varying functions of space and time.

2.6 Linear Perturbations on an Undepleted, Unattenuated Pump-Wave

We now apply the generalized-coupling-of-modes theory of the previous section to the following physical problem. Consider a nonlinear medium supporting a pump-wave in self-consistent equilibrium with its own harmonics. Then small perturbations introduced, for example by thermal noise, will be coupled together if they are separated by multiples of the pump frequency and wavevector. Should this coupling be unstable, the perturbations will grow in a fashion which is approximately exponential, until such time as the cumulative effect of their growth in depleting the pump-wave becomes appreciable. In this section we derive equations which describe the phase of approximately exponential growth.

First consider the self-consistent equilibrium of the pump in the non-linear medium. In such a medium, the field at the fundamental pump wavevector and frequency, $\vec{E}_1(\vec{k}_1, w_1)$, cannot exist alone. Rather, the fundamental excites 2nd-order nonlinear currents at the 2nd harmonic $(2\vec{k}_1, 2w_1)$, 3rd-order nonlinear currents at the 3rd harmonic $(3\vec{k}_1, 3w_1)$, and so on. The nonlinear currents in turn excite electric fields and thus the pump field comprises a set of harmonics:

$$\vec{E}_{pump}(\vec{x}, t) = \sum_{n=1}^{\infty} \left[\vec{E}_n e^{in\vec{k}_1 \cdot \vec{x} - inw_1 t} + \begin{array}{c} \text{complex} \\ \text{conjugate} \end{array} \right]$$

2.6-(1)

The relations between these harmonics are investigated using the generalized-coupling-of-modes theory as follows. The set of interacting waves $\vec{E}_a(\vec{x},t)$ in 2.5-(4) is chosen to be just the set $\vec{E}_n$ of pump harmonics. Then each harmonic $\vec{E}_n$ is related to the others by the corresponding mode-coupling equation. This equation is obtained from the general mode-coupling equation 2.5-(7) by setting

$$\vec{E}_a(\vec{x},t) \equiv \vec{E}_n$$

$$(\vec{k}_a, w_a) \equiv (n\vec{k}_1, nw_1)$$

2.6-(2)

on the left-hand-side and

$$\vec{E}_b, \vec{E}_c, \vec{E}_d, \ldots \equiv \vec{E}_{n'}, \vec{E}_{n''}, \vec{E}_{n'''}, \ldots$$

$$(k_b, w_b), (k_c, w_c), (k_d, w_d), \ldots$$

$$\equiv (n'k_1, n'w_1), (n''k_1, n''w_1), (n'''k_1, n'''w_1), \ldots$$

2.6-(3)

on the right-hand-side. Here n', n'', n''', $\ldots$ are integers appropriately chosen so that the term in $\vec{E}_{n'}, \vec{E}_{n''}$ has $n = n' + n''$,

the term in $\vec{E}_n, \vec{E}_{n''} \vec{E}_{n'''}$ has $n = n' + n'' + n'''$, and so on. By hypothesis the pump wave including its harmonics is in equilibrium, so $\vec{E}_n$ is not a function of $(\vec{x}, t)$ and the space and time derivatives in 2.5-(7) can be set to zero. The complex conjugates in 2.6-(1) may be included by setting

$$\vec{E}_n^* = \vec{E}_{-n}$$

$$2.6-(4)$$

We assume that the pump-wave and its harmonics are undamped so that $(\vec{k}_1, w_1)$ is real. The mode-coupling equation relating $\vec{E}_n$ to the other harmonics then has the following form, illustrated for convenience by the examples $n = 1$, $n = 2$;

$$\vec{\overleftrightarrow{L}}^{LIN}(\vec{k}_1, w_1)\vec{E}_1 = -i\mu_o w_1 \Bigg\{$$

$$\Bigg[\overleftrightarrow{G}^{NL(2)}(2\vec{k}_1, 2w_1, -\vec{k}_1, -w_1)\vec{E}_2\vec{E}_1^*$$

$$+ \overleftrightarrow{G}^{NL(2)}(3\vec{k}_1, 3w_1, -2\vec{k}_1, -2w_1)\vec{E}_3\vec{E}_2^* + \cdots \Bigg]$$

$$+ \Bigg[\frac{1}{2}\overleftrightarrow{G}^{NL(3)}(\vec{k}_1, w_1, \vec{k}_1, w_1, -\vec{k}_1, -w_1)\vec{E}_1\vec{E}_1\vec{E}_1^*$$

$$+ \overleftrightarrow{G}^{NL(3)}(\vec{k}_1, w_1, 2\vec{k}_1, 2w_1, -2\vec{k}_1, -2w_1)\vec{E}_1\vec{E}_2\vec{E}_2^* + \cdots \Bigg] + \cdots \Bigg\}$$

$$2.6-(5)$$

$$\vec{L}^{LIN}(2\vec{k}_1, 2w_1)\vec{E}_2 = -2i\mu_o w_1$$

$$\left\{ \left[\frac{1}{2} \overset{\equiv}{G}^{NL(2)}(\vec{k}_1, w_1, \vec{k}_1, w_1)\vec{E}_1\vec{E}_1 \right. \right.$$

$$\left. + \overset{\equiv}{G}^{NL(2)}(3\vec{k}_1, 3w_1, -k_1, -w_1)\vec{E}_3\vec{E}_1^* + \cdots \right]$$

$$+ \left[\overset{\equiv}{G}^{NL(3)}(2\vec{k}_1, 2w_1, \vec{k}_1, w_1, -\vec{k}_1, -w_1)\vec{E}_2\vec{E}_1\vec{E}_1^* + \cdots \right]$$

$$\left. + \cdots \right\}$$

<div align="right">2.6-(6)</div>

Assuming that the self-consistent equilibrium of the pump-wave with all its harmonics has already been set up, let us try to follow the early-time history of a small disturbance by treating it as a linear perturbation.

Even in the linear approximation, because of the coupling due to the pump, one must consider a set of such small perturbations, at wavevectors and frequencies separated by multiples of the pump wavevector and frequency:

$$\vec{E}_{perturbation}(\vec{x}, t) =$$

$$\sum_{n=-\infty}^{\infty} \left[\vec{E}_{(n)}(\vec{x}, t) e^{i(\vec{k}_{(o)} + n\vec{k}_1)\cdot\vec{x} - i(w_{(o)} + nw_1)t} + c.c. \right]$$

<div align="right">2.6-(7)</div>

Here the indexing is arranged so that $w_{(o)}$ is the lowest of the
perturbation frequencies. For positive n, $\vec{E}_{(n)}$ is the wave-
envelope of the upper sideband of the corresponding pump-har-
monic. For negative n, $\vec{E}^*_{(n)}$ is the wave-envelope of the lower
sideband of the corresponding pump-harmonic.

To lowest order, the effect of the existence of these
perturbations on the previously existing pump-wave equilibrium
is given by equations such as (5) and (6) amended by the follow-
ing additional terms. On the left-hand-side are added terms in
the space-time variation of the pump-wave envelope, and on the
right-hand-side are added the nonlinear currents due to the
perturbing fields. Although these added terms are at least 2nd-
order in the small perturbations, they cannot be automatically
neglected in a linear theory of those perturbations. This is
because of their secular nature, which eventually causes their
effect on the pump to be appreciable. The amended versions of
(5) and (6) may be determined from the generalized-coupling-of-
modes theory as follows. Take the superposition of modes
2.5-(4) to include both the pump-harmonics (1) and the pump-
coupled perturbations (6). Then the generalized mode-coupling
equations 2.5-(7) comprise both equations describing the varia-
tions of the pump-harmonics $\vec{E}_n(\vec{x},t)$ and also equations describ-
ing the variations of the perturbation-fields $\vec{E}_{(n)}(\vec{x},t)$. The
former include the equations describing the variations of
$\vec{E}_1(\vec{x},t)$ and $\vec{E}_2(\vec{x},t)$, which are respectively

$$\vec{\overline{L}}^{LIN}(\vec{k}_1, w_1)\vec{E}_1 - i\frac{\partial \vec{\overline{L}}_1^{LIN}}{\partial \vec{k}}\frac{\partial \vec{E}_1}{\partial \vec{x}} + i\frac{\partial \vec{\overline{L}}_1^{LIN}}{\partial w}\frac{\partial \vec{E}_1}{\partial t} + \cdots =$$

$$-i\mu_0 w_1 \left\{ \left[\vec{\overline{\overline{G}}}^{NL(2)}(2\vec{k}_1, 2w_1, -\vec{k}_1, -w_1)\vec{E}_2\vec{E}_1^* \right.\right.$$

$$+ \vec{\overline{\overline{G}}}^{NL(2)}(3\vec{k}_1, 3w_1, -2\vec{k}_1, -2w_1)\vec{E}_3\vec{E}_2^*$$

$$+ \sum_{n=-\infty}^{\infty} \vec{\overline{\overline{G}}}^{NL(2)}\left(\vec{k}_{(0)}+(n+1)\vec{k}_1, w_{(0)}+(n+1)w_1,\right.$$

$$\left.\left. - \vec{k}_{(0)}-n\vec{k}_1, -w_{(0)}-nw_1\right)\vec{E}_{(n+1)}\vec{E}_{(n)}^* + \cdots \right]$$

$$+ \left[\frac{1}{2}\vec{\overline{\overline{G}}}^{NL(3)}(\vec{k}_1, w_1, \vec{k}_1, w_1, -\vec{k}_1, -w_1)\vec{E}_1\vec{E}_1\vec{E}_1^* \right.$$

$$+ \vec{\overline{\overline{G}}}^{NL(3)}(\vec{k}_1, w_1, 2\vec{k}_1, 2w_1, -2\vec{k}_1, -2w_1)\vec{E}_1\vec{E}_2\vec{E}_2^*$$

$$\left. + \cdots \right]$$

$$\left. + \cdots \cdots \right\}$$

$$2.6-(8)$$

$$\vec{\overline{L}}^{LIN}(2\vec{k}_1, 2w_1)\vec{E}_1 - i\frac{\partial \vec{\overline{L}}_2^{LIN}}{\partial \vec{k}}\frac{\partial \vec{E}_2}{\partial \vec{x}} + i\frac{\partial \vec{\overline{L}}_2^{LIN}}{\partial w}\frac{\partial \vec{E}_2}{\partial t}$$

$$+ \cdots \quad =$$

$$-i\mu_0 \omega_1 \left\{ \left[\frac{1}{2} \overset{\equiv}{G}{}^{NL(2)} \left(\vec{k}_1, \omega_1, \vec{k}_1, \omega_1\right) \vec{E}_1 \vec{E}_1 \right.\right.$$

$$+ \overset{\equiv}{G}{}^{NL(2)} \left(3\vec{k}_1, 3\omega_1, -\vec{k}_1, -\omega_1\right) \vec{E}_3 \vec{E}_1^*$$

$$+ \sum_{n=-\infty}^{\infty} \overset{\equiv}{G}{}^{NL(2)} \left(\vec{k}_{(0)} + (n+2)\vec{k}_1, \omega_{(0)} + (n+2)\omega_1,\right.$$
$$\left.\left. -\vec{k}_{(0)} - n\vec{k}_1, -\omega_{(0)} - n\omega_1\right) \vec{E}_{(n+2)} \vec{E}_{(n)}^* + \cdots \right]$$

$$+ \left[\overset{\equiv}{G}{}^{NL(3)} \left(2\vec{k}_1, 2\omega_1, \vec{k}_1, \omega_1, -\vec{k}_1, -\omega_1\right) \vec{E}_2 \vec{E}_1 \vec{E}_1^* + \cdots \right]$$

$$\left. + \cdots \cdots \right\}$$

<div align="right">2.6-(9)</div>

These describe the effect of the perturbing waves on the pump up to terms in 3$^{\text{rd}}$-order nonlinear conductivity. Higher-order space-time derivatives on the left-hand-side and all the space-time derivatives on the right-hand-side have been omitted.

As long as the accumulated effect of the perturbations on the pump, described by equations such as (8) and (9), remains small, the pump-harmonic wave-envelopes $\{\vec{E}_n\}$ can be taken to be constants. Now these pump-harmonic wave-envelopes also appear in the mode-coupling equation describing the variations of the

perturbation-field wave-envelopes $\{\vec{E}_{(n)}(\vec{x},t)\}$. In fact it is precisely these pump-harmonics which achieve the coupling between the various $\{\vec{E}_{(n)}\}$. Take the $\{\vec{E}_n\}$ to be constants, and retain the $\{\vec{E}_{(n)}\}$ only to first order. The linearized mode-coupling equations describing the variations of the perturbation-field wave-envelopes $\{\vec{E}_{(n)}\}$ become a set of linear space- and time-invariant differential equations. Thus they have solutions $\vec{E}_{(n)}(\vec{x},t)$ with exponential behavior. This exponential space-time dependence of the perturbation envelopes is just equivalent to a shift in wavevector and frequency of the perturbation carrier-waves. For instance, a solution with the $\{\vec{E}_{(n)}\}$ growing exponentially in time may be dealt with by redefining the frequencies of the perturbation carrier-waves to have a positive imaginary part. Without loss of generality, therefore, one may take the $\{\vec{E}_{(n)}\}$ to be constant and solve the mode-coupling equations for the basic perturbation wavevector and frequency $(\vec{k}_{(o)}, w_{(o)})$, now possibly complex, and the associated ratio between the $\{\vec{E}_{(n)}\}$. The mode-coupling equations to be solved now appear as follows:

$$\overleftrightarrow{L}^{LIN}\left(\vec{k}_{(0)}+n\vec{k}_1, \, w_{(0)}+nw_1\right)\vec{E}_{(n)} = -i\mu_0\left(w_{(0)}+nw_1\right)$$

$$\left\{\left[\overleftrightarrow{G}^{NL(2)}\left(\vec{k}_{(0)}+(n-1)\vec{k}_1, \, w_{(0)}+(n-1)w_1, \vec{k}_1, w_1\right)\vec{E}_{(n-1)}\vec{E}_1\right.\right.$$

$$+ \overleftrightarrow{G}^{NL(2)}\left(\vec{k}_{(0)}+(n+1)\vec{k}_1, \, w_{(0)}+(n+1)w_1, -\vec{k}_1, -w_1\right)\vec{E}_{(n+1)}\vec{E}_1^*$$

$$+ \overleftrightarrow{G}^{NL(2)}\left(\vec{k}_{(0)}+(n-2)\vec{k}_1, \, w_{(0)}+(n-2)w_1, 2\vec{k}_1, 2w_1\right)\vec{E}_{(n-2)}\vec{E}_2$$

$$+ \overleftrightarrow{G}^{NL(2)}\left(\vec{k}_{(0)}+(n+2)\vec{k}_1, \, w_{(0)}+(n+2)w_1, -2\vec{k}_1, -2w_1\right)\vec{E}_{(n+2)}\vec{E}_2^*$$

$$+ \dots \big]$$

$$+ \left[\frac{1}{2}\overleftrightarrow{G}^{NL(3)}\left(\vec{k}_{(0)}+(n-2)\vec{k}_1, \, w_{(0)}+(n-2)w_1, \vec{k}_1, w_1, \vec{k}_1, w_1\right)\right.$$
$$\vec{E}_{(n-2)}\,\vec{E}_1\,\vec{E}_1$$

$$+ \overleftrightarrow{G}^{NL(3)}\left(k_{(0)}+nk_1, w_{(0)}+nw_1, k_1, w_1, -k_1, -w_1\right)$$
$$\vec{E}_{(n)}\,\vec{E}_1\,\vec{E}_1^*$$

$$+ \frac{1}{2}\overleftrightarrow{G}^{NL(3)}\left(\vec{k}_{(0)}+(n+2)\vec{k}_1, \, w_{(0)}+(n+2)w_1, -k_1, -w_1,\right.$$
$$\left.-k_1, -w_1\right)\vec{E}_{(n+2)}\vec{E}_1^*\vec{E}_1^* + \dots\big]$$

$$+ \dots \Big\}$$

$$2.6 - (10)$$

Here each perturbation $\vec{E}_{(n)}$ couples to other perturbations
separated from it by integer multiples of the pump wavevector
and frequency $(\vec{k}_1, w_1)$. The coupling is via the appropriate
order of nonlinear conductivity and the appropriate harmonic(s)
of the pump.

Consider the set of variables which consists of the 3
spatial components of each $\vec{E}_{(n)}$. This infinite set of variables
satisfies the infinite set of scalar equations consisting of
the 3 spatial components of 2.6-(10), for each integer n. The
condition for this set of equations to be consistent is an
infinite determinantal equation, which does not involve the
$\{\vec{E}_{(n)}\}$ explicitly. This determinantal equation involves the
perturbation fields only through the quantities $(\vec{k}_{(o)}, w_{(o)})$.
This determinantal equation thus constitutes the dispersion
relation for small perturbations in the pump-permeated nonlinear
medium.

To sum up, as long as the perturbations cause the pump to
depart only slightly from its self-consistent equilibrium, a
linearized theory of the pump-coupled perturbations may be con-
structed. This theory yields a dispersion relation which may
predict growing pump-coupled perturbations. Such growth of
perturbations remains exponential in character only to the extent

that the resulting depletion or attenuation of the pump is negligible.

2.7 Use of Coupling Coefficients

The equations 2.6-(10) describe linear electric-field perturbations $\{\vec{E}_{(n)}\}$ coupled together by a pump-wave and its harmonics. The polarizations of the $\{\vec{E}_{(n)}\}$ are not known a priori. In fact, for any particular value of the perturbation-index (n), the vector equation 2.6-(10) comprises 3 scalar equations involving the 3 Cartesian components of $E_{(n)}$. The right-hand-sides of these scalar equations contain nonlinear currents which depend on the Cartesian components of $\vec{E}_{(n-1)}$, $\vec{E}_{(n+1)}$ and so on via the elements of the tensors $\overset{\leftrightarrow}{G}^{NL}(2)$, $\overset{\leftrightarrow}{G}^{NL}(3)$ and so on. However, from a physical viewpoint, the description of all interacting electric fields in terms of their polarization components along the same Cartesian axes may not be the most useful description. Rather, one wishes to express each electric-field perturbation $\vec{E}_{(n)}$ in terms of polarization components which have a definite physical meaning for electric fields at the carrier wavevector and frequency $(\vec{k}_{(o)} + n\vec{k}_1$, $w_{(o)} + nw_1)$ in the medium considered. The quantities which describe the coupling between such suitably chosen polarization components are known as coupling coefficients. These coupling coefficients are obtained from the nonlinear conductivities of the medium as will be shown here.

For each perturbation wavevector and frequency $(\vec{k}_{(o)} + n\vec{k}_1,$ $w_{(o)} + nw_1)$ occurring in the superposition of waves 2.6-(7), choose a physically appropriate basis triad of polarization vectors as follows. Express the field $\vec{E}_{(n)}$ in terms of a basis of eigenvectors $\{\vec{e}_{A(n)}\}$ of the linear dispersion tensor evaluated at the carrier wavevector and frequency. Here A indexes the members of the basis. Explicitly

$$\vec{E}_{(n)} \equiv \sum_A E_{A(n)}\vec{e}_{A(n)}$$

<div align="right">2.7-(1)</div>

where the $\{\vec{e}_{A(n)}\}$ are eigenvectors of $\vec{L}^{LIN}$ $(\vec{k}_{(o)} + n\vec{k}_1,$ $w_{(o)} + nw_1)$. Similarly one writes

$$\vec{E}_{(n\pm1)} \equiv \sum_A E_{A(n\pm1)}\vec{e}_{A(n\pm1)}$$

<div align="right">2.7-(2)</div>

where the $\{\vec{e}_{A(n\pm1)}\}$ are eigenvectors of $\vec{L}^{LIN}$ $(\vec{k}_{(o)} + (n \pm 1)\vec{k}_1,$ $w_{(o)} + (n \pm 1)w_1)$, and so on. In the same way, choose physically appropriate basis triads for expressing the polarization of the pump-wave and its harmonics. Set

$$\vec{E}_1 \equiv \sum_A E_{A1}\vec{e}_{A1}$$

<div align="right">2.7-(3)</div>

<div align="right">2.7-(3)</div>

where the $\{\vec{e}_{A1}\}$ are eigenvectors of $\vec{\vec{L}}^{LIN}(\vec{k}_1, w_1)$. Similarly, set

$$\vec{E}_2 \equiv \sum_A E_{A2}\vec{e}_{A2}$$

2.7-(4)

where the basis polarization vectors $\{\vec{e}_{A2}\}$ are eigenvectors of $\vec{\vec{L}}^{LIN}(2\vec{k}_1, 2w_1)$, and so on. Define a dimensionless form of the linear dispersion tensor for convenience:

$$\vec{\vec{D}}^{LIN}(\vec{k}, w) \equiv \frac{\vec{\vec{L}}^{LIN}(\vec{k}, w)}{\mu_0 \epsilon_0 w^2}$$

2.7-(5)

Now substitute (1) - (5) into 2.6-(10). The result is still a vector equation. To obtain 3 scalar equations from this vector equation, take its scalar product with each of 3 linearly independent vectors $\{\vec{f}_{A(n)}\}$ in turn. Choose this basis triad $\{\vec{f}_{A(n)}\}$ to be eigenvectors of the transposed linear dispersion tensor $\overset{\sim}{\vec{\vec{L}}}{}^{LIN}(\vec{k}_{(o)} + n\vec{k}_1, w_{(o)} + nw_1)$. Then each of the 3 scalar equations has the form

$$(D_{(n)})_{AA} \, E_{A(n)} = - \sum_B \sum_P \left(F^{NL(2)}_{(n-1),\,1}\right)_{ABP} E_{B(n-1)} E_{P1}$$

$$+ \left(F^{NL(2)}_{(n+1),\,-1}\right)_{ABP} E_{B(n+1)} E^*_{P1}$$

$$+ \left(F^{NL(2)}_{(n-2),\,2}\right)_{ABP} E_{B(n-2)} E_{P2}$$

$$+ \left(F^{NL(2)}_{(n+2),\,-2}\right)_{ABP} E_{B(n+2)} E^*_{P2}$$

$$+ \, \ldots$$

$$- \sum_B \sum_P \sum_Q \, \tfrac{1}{2} \left(F^{NL(3)}_{(n-2),\,1,\,1}\right)_{BPQ} E_{B(n-2)} E_{P1} E_{Q1}$$

$$+ \left(F^{NL(3)}_{(n),\,1,\,-1}\right)_{BPQ} E_{B(n)} E_{P1} E^*_{Q1}$$

$$+ \tfrac{1}{2} \left(F^{NL(3)}_{(n+2),\,-1,\,-1}\right)_{BPQ} E_{B(n)} E^*_{P1} E^*_{Q1}$$

$$+ \, \ldots$$

$$- \, \ldots \ldots$$

$$2.7\text{-}(6)$$

This equation relates one polarization component, specified by the index A, of the perturbation field $E_{(n)}$ to the various polarization components of the other perturbations to which it is coupled, and to the polarization components of the pump-wave and its harmonics. The quantities $\left(F^{NL(2)}\right)_{ABP}$, $\left(F^{NL(3)}\right)_{ABPQ}$,

and so on appearing in 2.7-(6) are the coupling coefficients.
The scalar dispersion function $(D_{(n)})_{AA}$ is defined in terms of
the linear dispersion tensor by

$$(D_{(n)})_{AA} \equiv \vec{f}_{A(n)} \cdot \overset{\leftrightarrow}{D}{}^{LIN}(\vec{k}_{(0)} + n\vec{k}_1, w_{(0)} + nw_1) \vec{e}_{A(n)}$$

$$2.7-(7)$$

The coupling coefficients are defined in terms of the nonlinear
conductivities of the medium by

$$\left(F^{NL(2)}_{b,\,c}\right)_{ABC} \equiv \frac{i\vec{f}_{Aa} \cdot \overset{\leftrightarrow}{G}{}^{NL(2)}(\vec{k}_b, w_b, \vec{k}_c, w_c) \, \vec{e}_{Bb}\vec{e}_{Cc}}{\epsilon_0 w_a}$$

$$2.7-(8)$$

provided $\vec{E}_a$, $\vec{E}_b$, $\vec{E}_c$ are any 3 interacting electric fields such
that the first of 2.5-(3) is satisfied, and by

$$\left(F^{NL(3)}_{b,\,c,\,d}\right)_{ABCD} \equiv \frac{i\vec{f}_{Aa} \cdot \overset{\leftrightarrow}{G}{}^{NL(3)}(\vec{k}_b, w_b, \vec{k}_c, w_c, \vec{k}_d, w_d) \, \vec{e}_{Bb}\vec{e}_{Cc}\vec{e}_{Dd}}{\epsilon_0 w_a}$$

$$2.7-(9)$$

provided $\vec{E}_a$, $\vec{E}_b$, $\vec{E}_c$, $\vec{E}_d$ satisfy the second of the wavevector
and frequency sum rules 2.5-(3), and so on. The coupling
coefficients are trivially invariant with respect to simultan-
eous identical permutations of b, c, d, $\cdots$ and B, C, D, $\cdots$.
In the form shown in (8) and (9), with the sum-frequency in the

denominator, they possess additional symmetries, as will be shown in section 2.8 for the case of the warm-fluid plasma model.

It is assumed in this section that the eigenvectors of the linear dispersion tensor $\overset{\leftrightarrow}{D}^{LIN}$ ($\vec{k}_{(o)} + n\vec{k}_1$, $w_{(o)} + nw_1$) are known in advance. This is despite the fact that the relation between $\vec{k}_{(o)}$ and $w_{(o)}$ is not known and in fact is the main object of the investigation. Such an assumption is justified for the case of laser-irradiated unmagnetized plasma treated in Chapter 3. Such a plasma is isotropic in the absence of the laser-pump. The eigenvectors of $\overset{\leftrightarrow}{D}^{LIN}$ ($\vec{k},w$) are then parallel and perpendicular to $\vec{k}$ for any $\vec{k}$. We know in advance that the consistency condition for the set of equations 2.6-(10) yields a dispersion relation connecting $\vec{k}_{(o)}$ and $w_{(o)}$. Thus we can choose $\vec{k}_{(o)}$ arbitrarily at the start, and the eigenvectors of $\overset{\leftrightarrow}{D}^{LIN}$ ($\vec{k}_{(o)} + n\vec{k}_1$, $w_{(o)} + nw_1$) will lie parallel and perpendicular to the perturbation wavevector $\vec{k}_{(o)} + n\vec{k}_1$. Knowing this, the basis triad for expressing the polarization of the perturbation field $\vec{E}_{(n)}$ can be appropriately chosen, and the set of equations 2.6-(10) converted to the form (6). The consistency condition for the set of equations 2.6-(10) is the same as that for the set (6), and the latter proves more convenient for computation. The consistency condition for the set of coupled equations (6) provides a restriction on $w_{(o)}$. Since the original choice of $\vec{k}_{(o)}$ was arbitrary, this restriction is just the dispersion relation connecting $\vec{k}_{(o)}$ and $w_{(o)}$ for the coupled

system.

In other cases it may not be possible to determine the exact eigenvectors of the linear dispersion tensor $\vec{\vec{D}}^{LIN}$ $(\vec{k}_{(o)} + n\vec{k}_1, w_{(o)} + nw_1)$ in advance of the determination of the exact values of $\vec{k}_{(o)}$ and $w_{(o)}$. Then the basis triad $\{\vec{e}_{A(n)}\}$ of polarization vectors and the triad $\{\vec{f}_{A(n)}\}$ employed in the definitions (7) - (9) must be chosen in some fashion, with no guarantee that the diagonal elements defined in (7) are the only scalar dispersion functions that appear. Rather, the left-hand-side of (6) must be replaced by the expression

$$\sum_{B} (D_{(n)})_{AB}\, E_{B(n)}$$

2.7-(10)

where off-diagonal elements of $(D_{(n)})$ are defined, as expected, by

$$(D_{(n)})_{AB} = \vec{f}_{A(n)} \cdot \vec{\vec{D}}^{LIN}(\vec{k}_{(0)} + n\vec{k}_1, w_{(0)} + nw_1)\,\vec{e}_{B(n)}$$

2.7-(11)

The form of the right-hand-side of (6) and the forms of the definitions (8) and (9) are unaltered.

2.8 Form of Nonlinear Conductivities in Warm-Fluid Plasma Model

Up to this point the conductivities $\vec{G}^{LIN}$, $\vec{G}^{NL}(2)$, $\vec{G}^{NL}(3)$ and so on have not been specified in any way. This section illustrates how, given the dynamic equations describing a nonlinear medium, one may proceed to derive the nonlinear conductivities of that medium in the wavevector-frequency domain. The nonlinear medium chosen as an illustrative example is a plasma, each of whose species is described by the warm-fluid model. The applications of this warm-fluid model are discussed.

The nonlinear conductivities will actually be displayed in the form of coupling coefficients. Recall the definitions 2.7-(8), 2.7-(9) of the coupling coefficients. It may be seen that, to within a factor, the Cartesian tensor components of nonlinear conductivity may be recovered from the coupling coefficients. This is done by choosing the triads of basis vectors which describe the polarization of the interacting waves so that they lie along the corresponding Cartesian axes. Thus it is enough to calculate the expressions for coupling coefficients.

The highest order of nonlinear conductivity for which the coupling coefficient will be calculated is the third. The calculations are set out in Appendix A4 in such a way that, it is hoped, the method of continuation to fourth-order nonlinear conductivity is clear. However, the fourth-order coupling-coefficient is expected to be extremely complex in form and is not essential for the interactions studied in the remainder of this thesis.

The warm-fluid model for a single species in the plasma is obtained from the Vlasov model for that species. We first outline the Vlasov model. We then obtain the fluid model by taking moments of the Vlasov equation in velocity space and truncating the resulting set of moment equations.

If one neglects many-body correlations in a 1-species plasma, such a plasma may be described by a smoothed-out distribution function $f(\vec{x}, \vec{v})$ in 6-dimensional phase-space. This function obeys the Boltzmann equation. in which the right-hand-side comprises all effects due to the actual particulate nature of the plasma:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}}$$

2.8-(1)

The Vlasov equation is obtained by neglecting the discreteness of the plasma particles, thus setting the right-hand-side of (1) equal to zero:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

2.8-(2)

Multiplying the Vlasov equation by successive powers of the velocity $\vec{v}$ and integrating over velocity-space, one forms a hierarchy of equations relating successive moments of the

velocity. This hierarchy may be truncated by making assumptions as to the form of some of these moments. In particular, make the zero-heat-flow assumption

$$\langle (\vec{v} - \langle \vec{v} \rangle) | \vec{v} - \langle \vec{v} \rangle |^2 \rangle = \vec{0}$$

2.8-(3)

where $\langle \rangle$ denotes the velocity-space average, and make the iso-tropic-pressure assumption

$$\langle (\vec{v} - \langle \vec{v} \rangle)(\vec{v} - \langle \vec{v} \rangle) \rangle = v_T^2 \vec{\vec{1}}$$

2.8-(4)

Then the truncated hierarchy of moment-equations governing $n(\vec{x},t)$ and $\langle \vec{v} \rangle$ $(\vec{x},t)$ is as follows:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \vec{x}} \cdot n \langle \vec{v} \rangle = 0$$

2.8-(5)

$$\frac{\partial \langle \vec{v} \rangle}{\partial t} + \langle \vec{v} \rangle \cdot \frac{\partial \langle \vec{v} \rangle}{\partial \vec{x}} + \frac{\gamma v_{To}^2}{n_0} \left(\frac{n}{n_0} \right)^{\gamma-2} \frac{\partial n}{\partial \vec{x}} = \frac{q}{m} (\vec{E} + \langle \vec{v} \rangle \times \vec{B})$$

2.8-(6)

The quantities γ, v_{To}, n_o in (6) arise as follows. The assumptions embodied in the equations (3) and (4) lead to the relation

$$\frac{n v_{T}^{2}}{n_0 v_{To}^{2}} = \left(\frac{n}{n_0}\right)^{\gamma}$$

<div align="right">2.8-(7)</div>

where γ has the value $\frac{5}{3}$. However, the pair of equations (5) and (6) may be used to define a model of the plasma, the so-called warm-fluid model, independent of any derivation from the pair of assumptions (3) and (4). In choosing the value of γ to be used in this model, one is guided by physical considerations and, wherever possible, by comparison with the results from the Vlasov model defined by (2).

When dealing solely with waves whose phase-velocity is greater than the electron thermal velocity, the value $\gamma = 3$ is used. When dealing solely with waves whose phase-velocity is much less than the electron thermal velocity, γ is taken to be 1 for the electrons and is ignored for the ions. Coupling-coefficients between high- and low-phase-velocity waves are also evaluated using $\gamma_e = 1$, $\gamma_i = 0$. These choices for the values of γ are justified by comparison with the results of Vlasov theory in Appendix A5.

Before actually calculating the coupling coefficients, recall that the conductivities define the constitutive relation between current and electric field. They describe the internal response of the system defined by (5) and (6) to an arbitrary electric field. The actual form of the electric field in the medium need only be considered at a later stage, when the calcu-

lated conductivities are substituted into the wave-equation
which governs the electric field:

$$\left\{ \frac{\partial}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}} \cdot - \frac{\partial}{\partial \bar{x}} \cdot \frac{\partial}{\partial \bar{x}} + \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right\} \vec{E} + \left\{ \mu_0 \frac{\partial}{\partial t} \right\} \vec{J} = \vec{0}$$

$$2.8-(8)$$

or equivalently into one of the mode-coupling equations
2.5-(7), 2.6-(8,9,10), or 2.7-(6). In all this work, the AC
magnetic field $\vec{B}$ is treated purely as a subsidiary quantity
defined in terms of the electric field by

$$\frac{\partial \vec{B}}{\partial t} = - \frac{\partial}{\partial \bar{x}} X \vec{E}$$

$$2.8-(9)$$

Thus the conductivities are found by imposing an arbitrary
electric field on the medium and calculating, using only (5),
(6), (9) and the definition

$$\vec{J}(\vec{x}, t) = \sum_{\substack{\text{species} \\ \pi}} q_\pi n_\pi(\vec{x}, t) \langle \vec{v}_\pi \rangle (\vec{x}, t)$$

$$2.8-(10)$$

of the resultant current response to this arbitrary field.

We proceed to calculate the linear and nonlinear conduc-
tivities in the wavevector-frequency domain. The linear con-
ductivity is needed not only for use in the linear dispersion

rO

tensor but also because quantities defined in the linear cal-
culation will be used in writing down the nonlinear conducti-
vities. Similarly, quantities arising in the second-order non-
linear calculation will be employed to facilitate the writing-
down of the third-order nonlinear conductivity. All variables
will have their zero-wavevector and -frequency components de-
noted by the subscript 0. The uniform dc electric field $\vec{E}_0$
will be taken to be zero. The uniform dc magnetic field $\vec{B}_0$
will be taken to be non-zero and to be included in the specifi-
cation of the medium, rather than being defined as a subsidiary
quantity by (9) as in the case of a finite-frequency magnetic
field. The particle species will be allowed to have drift
velocities $\langle v_\pi \rangle_0$. The space-average signs $\langle\rangle$ will be omitted
from the velocities. The species-index π will be omitted, on
the understanding that the final expressions for coupling-
coefficients are to be summed over species. This straight-
forward additivity of the coupling coefficients over species
is one of the advantages of the mode-coupling approach to coher-
ent wave-wave interaction.

First find the linear conductivity in the wavevector-fre-
quency domain. Impose an electric field containing a single
complex wave

$$\vec{E}(\vec{x},t) = \vec{E}_a\, e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \qquad 2.8\text{-}(11)$$

upon the medium defined by the continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \vec{x}} \cdot n\vec{v} = 0 \qquad\qquad 2.8\text{-}(12)$$

the momentum equation

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \frac{\partial \vec{v}}{\partial \vec{x}} + \frac{\gamma v_{To}^2}{n_0} \left(\frac{n}{n_0}\right)^{\gamma-2} \frac{\partial n}{\partial \vec{x}} = \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B})$$

$$2.8\text{-}(13)$$

and the current equation

$$\vec{J} = qn\vec{v} \qquad\qquad 2.8\text{-}(14)$$

The quantities n, $\vec{v}$, $\vec{J}$, $\vec{B}$ will then have parts which oscillate at the same wavevector and frequency $(\vec{k}_a, w_a)$. Denote these by the subscript a and use capitals for n and $\vec{v}$. Then equate all terms in (12), (13), (14) and (9) which oscillate at this wavevector and frequency to obtain

$$-iw_a N_a + i\vec{k}_a \cdot (n_0 \vec{V}_a + N_a \vec{v}_0) = 0$$

$$2.8\text{-}(15)$$

$$-iw_a\vec{V}_a + \vec{v}_0 \cdot i\vec{k}_a\vec{V}_a + \frac{\gamma v_{To}^2}{n_0} i\vec{k}_a N_a = \frac{q}{m}(\vec{E}_a + \vec{v}_0 \times \vec{B}_a + \vec{V}_a \times \vec{B}_0) \qquad 2.8\text{-}(16)$$

$$\vec{J}_a = q(n_0\vec{V}_a + N_a\vec{v}_0) \qquad\qquad 2.8\text{-}(17)$$

$$\vec{B}_a = \vec{k} \times \vec{E}_a / w_a \qquad\qquad 2.8\text{-}(18)$$

From (15), (17) and (18) the fluid velocity and density induced by the imposed electric field $\vec{E}_a$ are found to be given by

$$\vec{V}_a = \vec{\vec{M}}_a \frac{iq\vec{E}_a^{DR}}{mw_a^{DR}} \qquad N_a = n_0 \frac{\vec{k}_a \cdot \vec{V}_a}{mw_a^{DR}} \qquad\qquad 2.8\text{-}(19)$$

$$
\frac{
\begin{pmatrix}
\left(w_a^{DR}\right)^2 - \gamma\left(k_{ay}^2 + k_{az}^2\right)v_{To}^2 & \gamma k_{ax} k_{ay} v_{To}^2 - \dfrac{iw_{CYC}}{w_a^{DR}}\left(\left(w_a^{DR}\right)^2 - \gamma k_{az}^2 v_{To}^2\right) & \gamma k_{ax} k_{az} v_{To}^2 - \dfrac{iw_{CYC}}{w_a^{DR}} k_{ay} k_{az} v_{To}^2 \\[2em]
\gamma k_{ax} k_{ay} v_{To}^2 + \dfrac{iw_{CYC}}{w_a^{DR}}\left(\left(w_a^{DR}\right)^2 - \gamma k_{az}^2 v_{To}^2\right) & \left(w_a^{DR}\right)^2 - \gamma\left(k_{ax}^2 + k_{az}^2\right)v_{To}^2 & \gamma k_{ay} k_{az} v_{To}^2 + \dfrac{iw_{CYC}}{w_a^{DR}} k_{ax} k_{az} v_{To}^2 \\[2em]
\gamma k_{ax} k_{az} v_{To}^2 + \dfrac{iw_{CYC}}{w_a^{DR}} k_{ay} k_{az} v_{To}^2 & \gamma k_{ay} k_{az} v_{To}^2 - \dfrac{iw_{CYC}}{w_a^{DR}} k_{ax} k_{az} v_{To}^2 & \left(w_a^{DR}\right)^2 - w_{CYC}^2 - \gamma\left(k_{ax}^2 + k_{ay}^2\right)v_{To}^2
\end{pmatrix}
}{
\left(w_a^{DR}\right)^2 - |w_{CYC}|^2 - \gamma|k_a|^2 v_{To}^2 + \dfrac{|w_{CYC}|^2}{\left(w_a^{DR}\right)^2}\gamma k_{az}^2 v_{To}^2
}
$$

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2.8-(23)

Here $\vec{E}_a^{DR}$ and w_a^{DR} are respectively the electric vector and the frequency of the imposed field as seen by an observer moving with the drift velocity $\vec{v}_0$:

$$\vec{E}_a^{DR} \equiv \vec{E}_a + \vec{v}_0 \times \vec{B}_a \qquad\qquad 2.8\text{-}(20)$$

$$w_a^{DR} \equiv w_a - \vec{k}_a \cdot \vec{v}_0 \qquad\qquad 2.8\text{-}(21)$$

$\vec{M}_a$ is a mobility tensor, normalized such that in the absence of fluid pressure and steady magnetic field it has the value unity. We shall use mainly its definition in terms of vector operators:

$$\vec{M}_a \equiv \left(\vec{1} - \frac{\gamma v_{To}^2 \vec{k}_a \vec{k}_a}{\left(w_a^{DR}\right)^2} \cdot + \frac{iq\vec{B}_0}{mw_a^{DR}} X \right)^{-1}$$

$$2.8\text{-}(22)$$

Considered as a matrix operator on column vectors in Cartesian coordinates with the z-axis along $\vec{B}_0$, $\vec{M}_a$ has the explicit form shown in 2.8-(23). Here the "denominator" divides every element of the matrix, and w_{CYC} is the cyclotron frequency qB_0/m.

The expressions (19) for the velocity and density response of the medium are now substituted into (17) to obtain the current response. This current response may be written using the definition (21) as

$$\vec{J}_a = qn_0\left(1 + \frac{\vec{v}_0 \vec{k}_a}{w_a^{DR}} \cdot \right) \vec{V}_a \qquad\qquad 2.8\text{-}(24)$$

Using (19) and (21), the definition (19) may be written

$$\frac{\vec{E}_a^{DR}}{w_a^{DR}} = \left(1 + \frac{\vec{k}_a \vec{v}_0 \cdot}{w_a^{DR}}\right)\frac{\vec{E}_a}{w_a}$$

2.8-(25)

Thus the linear conductivity tensor in the wavevector-frequency domain has the appearance

$$\vec{G}^{LIN}(\vec{k}_a, w_a) = \frac{iq^2 n_0}{m w_a}\left(1 + \frac{\vec{v}_0 \vec{k}_a \cdot}{w_a^{DR}}\right)\vec{\vec{M}}_a\left(1 + \frac{\vec{k}_a \vec{v}_0 \cdot}{w_a^{DR}}\right)$$

2.8-(26)

and the normalized linear dispersion tensor defined by 2.4-(10), 2.7-(5) has the appearance

$$\vec{D}^{LIN}(\vec{k}_a, w_a) = \frac{(\vec{k}_a \vec{k}_a - k_a^2)}{\mu_0 \epsilon_0 w_a^2} + \vec{\vec{1}}$$

$$-\frac{w_p^2}{w_a^2}\left(1 + \frac{\vec{v}_0 \vec{k}_a \cdot}{w_a^{DR}}\right)\vec{\vec{M}}\left(1 + \frac{\vec{k}_a \vec{v}_0 \cdot}{w_a^{DR}}\right)$$

2.8-(27)

Here $w_p^2 = q^2 n_0 / m\epsilon_0$ is the square of the plasma frequency. The scalar dispersion functions defined by 2.7-(7) then are given by

$$(D_a)_{AB} = \frac{(\vec{f}_{Aa} \cdot \vec{k}_a)(\vec{k}_a \cdot \vec{e}_{Ba})}{\mu_0 \epsilon_0 w_a^2} + (\vec{f}_{Aa} \cdot \vec{e}_{Ba})\left(-\frac{k_a^2}{\mu_0 \epsilon_0 w_a^2} + 1\right)$$

$$-\frac{w_p^2}{w_a^2}\left(\vec{f}_{Aa}+\frac{(f_{Aa}\cdot\vec{v}_0)\,\vec{k}_a}{w_a^{DR}}\right)\cdot\vec{\vec{M}}_a\left(\vec{e}_{Ba}+\frac{(\vec{e}_{Ba}\cdot\vec{v}_0)\,\vec{k}_a}{w_a^{DR}}\right)$$

2.8-(28)

The term proportional to w_p^2 in (28) is due to linear conductivity. It is obtained from the linear conductivity in the same way that the coupling coefficients are obtained from the nonlinear conductivities. The coupling coefficients are sufficiently complex that they can be displayed conveniently only in terms of velocities rather than in terms of field polarizations. To show how this is done, we display the w_p^2 term of (28) in terms of velocities and densities rather than in terms of field polarizations as follows. We employ, not the polarization vectors $\vec{e}$ and $\vec{f}$, but the fluid velocities and densities which they would induce according to (19) were they present as unit fields within the medium. For this purpose $\vec{e}_{Ba}$ is of course regarded as having a wavevector and frequency $(\vec{k}_a, w_a)$. The vector $\vec{f}_{Aa}$ is the eigenvector (see section 2.7) of the transposed linear dispersion tensor; from (23) and (27) we find, for the lossless warm-fluid model considered, that

$$\vec{\vec{D}}^{\text{LIN}}_{\text{TRANSPOSE}}(\vec{k}_a, w_a) = \vec{\vec{D}}^{\text{LIN}}(\vec{k}_a, -w_a)$$

2.8-(29)

This provides a physical motivation for considering the unit field with electric vector $\vec{f}_{Aa} \equiv \vec{e}_{A\bar{a}}$ and frequency $(-\vec{k}_a, -w_a)$.

The scheme, again, is to express the contribution of the medium to (28), not in terms of the basis polarization vectors $\vec{e}_{Ba}$ and $\vec{f}_{Aa} \cong \vec{e}_{A\bar{a}}$, but rather in terms of the fluid velocity and density $\vec{v}_{Ba}$ and n_{Ba} induced by the normalized wave field $\vec{e}_{Ba}(\vec{k}_a, w_a)$, and the fluid velocity and density $\vec{v}_{A\bar{a}}$ and $n_{A\bar{a}}$ induced by the normalized wave field $\vec{f}_{Aa}(-\vec{k}_a, -w_a)$. Explicitly, the contribution of the medium to (28) will be rephrased in terms of

$$\vec{v}_{Ba} \equiv \overset{\Rightarrow}{M}_a \frac{iq\vec{e}_{Ba}^{DR}}{mw_a^{DR}}$$

2.8-(30)

$$n_{Ba} \equiv n_0 \frac{\vec{k}_a \cdot \vec{v}_{Ba}}{w_a^{DR}}$$

2.8-(31)

$$\vec{v}_{A\bar{a}} \cong \overset{\Rightarrow}{M}_{\bar{a}} \frac{iq\vec{e}_{Aa}^{DR}}{-mw_a^{DR}}$$

2.8-(32)

$$n_{A\bar{a}} \equiv n_0 \frac{\vec{k}_a \cdot \vec{v}_{A\bar{a}}}{w_a^{DR}}$$

2.8-(33)

Here the normalized quantities $\vec{v}_{Ba}$, n_{Ba} have been derived from $\vec{e}_{Ba}$ using (15), (19), (21) and (25) in just the same way that $\vec{V}_a$, N are derived from $\vec{E}_a$. Also the normalized quantities $\vec{v}_{A\bar{a}}$, $n_{A\bar{a}}$ are derived from $\vec{e}_{A\bar{a}} \cong \vec{f}_{Aa}$ in the same way but replacing $(\vec{k}_a, w_a)$ by $(-\vec{k}_a, -w_a)$ in the derivation. In terms of

these induced fluid velocities and densities, then, the w_p^2 term in (28) has the form

$$-\frac{mn_0}{\epsilon_0}\left\{\vec{v}_{A\bar{a}} \cdot \vec{v}_{Ba} + \gamma v_{To}^2 \frac{n_{A\bar{a}}n_{Ba}}{n_0^2} - \frac{iqB_0}{mw_a^{DR}} \cdot (\vec{v}_{A\bar{a}} \times \vec{v}_{Ba})\right\}$$

2.8-(34)

The form (34) is totally symmetric under simultaneous inter-change of the subscripts A and B and of the subscripts $\bar{a}$ and a. Note $w_{\bar{a}}^{DR} \equiv -w_a^{DR}$. The coupling coefficients due to nonlinear conductivity of the fluid medium will have forms more complex than, but similar in construction to, the expression (34).

Now find the second-order nonlinear conductivity in the frequency domain. Impose an electric field, comprising the superposition of 2 complex waves indexed by b and c respective-ly, upon the medium:

$$\vec{E}(\vec{x},t) = \vec{E}_b\, e^{i\vec{k}_b \cdot \vec{x} - iw_b t} + \vec{E}_c\, e^{i\vec{k}_c \cdot \vec{x} - iw_c t}$$

2.8-(35)

Now (9) and (12)-(14) are to be solved with (35) substituted in (9) and (13). The quantities n, $\vec{v}$, $\vec{J}$, and $\vec{B}$ occurring in (12)-(14) will then have parts which oscillate at the wavevector and frequency $(\vec{k}_b, w_b)$. Denote these by the subscript "b". They will also have parts oscillating at the wavevector and frequency $(\vec{k}_c, w_c)$ and denoted by the subscript "c". Since

the equations (12)-(14) are nonlinear, the quantities n, $\vec{v}$, $\vec{J}$, and $\vec{B}$ will further have parts oscillating at the sum-wavevector and frequency $(\vec{k}_b + \vec{k}_c, w_b + w_c)$. Denote these by the sub-script "b,c". The form of the relation between $\vec{J}_{b,c}$ and the imposed fields $\vec{E}_b$, $\vec{E}_c$ constitutes precisely the second-order nonlinear conductivity we seek. As in the case of the linear conductivity, the scalar components with respect to prescribed polarization basis vectors can be expressed in terms of velocities. This time, these scalar components are the coupling coefficients 2.7-(8). This time, the expression in terms of velocities is significantly more convenient than the expression in terms of polarizations.

Equate all terms in (12)-(14) which oscillate at the sum-wavevector and -frequency $(\vec{k}_b + \vec{k}_c, w_b + w_c)$ to obtain

$$-iw_{b+c}N_{b,c} + i\vec{k}_{b+c} \cdot (n_0 \vec{V}_{b,c} + N_b \vec{V}_c + N_c \vec{V}_b + N_{b,c}\vec{v}_0) = 0$$

<div align="right">2.8-(36)</div>

$$- iw_{b+c}\vec{V}_{b,c} + \vec{v}_0 \cdot i\vec{k}_{b+c}\vec{V}_{b+c} + \vec{V}_b \cdot i\vec{k}_c \vec{V}_c + \vec{V}_c \cdot i\vec{k}_b \vec{V}_b$$

$$+ \frac{\gamma v_{To}^2}{n_0}\left[i\vec{k}_{b+c}N_{b,c} + (\gamma-2)\frac{N_c}{n_0} i\vec{k}_b N_b + (\gamma-2)\frac{N_b}{n_0} i\vec{k}_c N_c \right]$$

$$= \frac{q}{m}\left(\vec{V}_{b,c} \times \vec{B}_0 + \vec{V}_b \times \vec{B}_c + \vec{V}_c \times \vec{B}_b \right)$$

<div align="right">2.8-(37)</div>

$$\vec{J}_{b,c} = q\left(n_0 \vec{V}_{b,c} + N_b \vec{V}_c + N_c \vec{V}_b + N_{b,c} \vec{v}_0\right)$$

<div align="right">2.8-(38)</div>

Here $(\vec{k}_{b+c}, w_{b+c})$ are abbreviations for $(\vec{k}_b + \vec{k}_c, w_b + w_c)$.
Also from (9)

$$\vec{B}_b = \vec{k}_b \times \vec{E}_b / w_b$$

$$\vec{B}_c = \vec{k}_c \times \vec{E}_c / w_c$$

<div align="right">2.8-(39)</div>

From the equations (36), (37) and (39) one may calculate the second-order nonlinear fluid velocity and density to be (see Appendix A3):

$$\vec{V}_{b,c} = \frac{\vec{\vec{M}}_{b+c}}{w_{b+c}^{DR}} \left[\vec{V}_b (\vec{k}_b \cdot \vec{V}_c) + \vec{V}_c (\vec{k}_c \cdot \vec{V}_b) \right.$$

$$+ \frac{\gamma v_{To}^2 \vec{k}_{b+c}}{w_{b+c}^{DR}} \left(\vec{k}_{b+c} \cdot \left(\vec{V}_b \frac{N_c}{n_0} + \vec{V}_c \frac{N_b}{n_0} \right) \right)$$

$$+ \gamma(\gamma - 2) v_{To}^2 \vec{k}_{b+c} \frac{N_b N_c}{n_0^2}$$

$$+ i\vec{V}_b \times \left(\vec{k}_c \cdot \frac{q\vec{E}_c^{DR}}{mw_c^{DR}}\right) + i\vec{V}_c \times \left(\vec{k}_b \cdot \frac{q\vec{E}_b^{DR}}{mw_b^{DR}}\right)\bigg]$$

<div align="right">2.8-(40)</div>

$$N_{b,c} = \frac{\vec{k}_{b+c}}{w_{b+c}^{DR}} \cdot \left(n_0 \vec{V}_{b,c} + N_b \vec{V}_c + N_c \vec{V}_b\right)$$

<div align="right">2.8-(41)</div>

Here w_{b+c}^{DR} and $\vec{M}_{b+c}$ are defined as in (21) and (22) with the wavevector and frequency $(\vec{k}_a, w_a)$ replaced by $(\vec{k}_{b+c}, w_{b+c}) \equiv (\vec{k}_b + \vec{k}_c, w_b + w_c)$. The quantities $\vec{V}_b$, N_b, $\vec{V}_c$, N_c constitute the linear responses of the warm fluid to the fields $\vec{E}_b (\vec{k}_b, w_b)$, $\vec{E}_c (\vec{k}_c, w_c)$ respectively as given by (19).

The expressions (40) and (41) for the second-order non-linear velocity and density response of the medium are now substituted into (38) to obtain the second-order nonlinear current response. This current response may be written

$$\vec{J}_{b,c} = qn_0 \left(1 + \frac{\vec{v}_0 \vec{k}_{b+c}}{w_{b+c}^{DR}}\right)\left(\vec{V}_{b,c} + \frac{N_b}{n_0} \vec{V}_c + \frac{N_c}{n_0} \vec{V}_b\right)$$

<div align="right">2.8-(42)</div>

Rather than work through the expressions for the nonlinear conductivity-tensor and for the coupling coefficients in terms of field polarizations, we shall immediately write the coupling coefficient in terms of normalized fluid velocities and densities

(see Appendix A3 for the calculation leading to this result).

$$\left(F_{b,c}^{NL(2)}\right)_{ABC} = -\frac{mn_0}{\epsilon_0}\left\{\frac{n_{A\bar{a}}}{n_0}\vec{v}_{Bb}\cdot\vec{v}_{Cc} + \frac{n_{Bb}}{n_0}\vec{v}_{Cc}\cdot\vec{v}_{A\bar{a}}\right.$$

$$+ \frac{n_{Cc}}{n_0}\vec{v}_{A\bar{a}}\cdot\vec{v}_{Bb} + \gamma(\gamma-2)\,v_{To}^2\,\frac{n_{A\bar{a}}n_{Bb}n_{Cc}}{n_0^3}$$

$$\left. + \frac{\vec{v}_{A\bar{a}}\cdot(\vec{v}_{Bb}\cdot\vec{v}_{Cc})}{w_{\bar{a}}^{DR}w_b^{DR}w_c^{DR}}\left(\vec{k}_b w_c^{DR} - \vec{k}_c w_b^{DR}\right)\cdot\frac{iqB_o}{m}\right\}$$

2.8-(43)

Here we have taken

$$(\vec{k}_a, w_a) = (\vec{k}_b, w_b) + (\vec{k}_c, w_c)$$

2.8-(44)

The normalized fluid velocities and densities appearing in (43) are the linear responses to the electric fields chosen as bases for describing the polarizations of (11) and (35). Explicitly (cf(30)-(33)):

$$\vec{v}_{A\bar{a}} \equiv \vec{\vec{M}}_{\bar{a}}\frac{iq\vec{e}_{A\bar{a}}^{DR}}{-mw_a^{DR}}$$

2.8-(45)

$$n_{A\bar{a}} \equiv n_0 \frac{\vec{k}_a \cdot \vec{v}_{A\bar{a}}}{w_a^{DR}}$$

2.8-(46)

$$\vec{v}_{Bb} \equiv \vec{\vec{M}}_b \frac{iq\vec{e}_{Bb}^{DR}}{mw_b^{DR}}$$

2.8-(47)

$$n_{Bb} \equiv \frac{\vec{k}_b \cdot \vec{v}_{Bb}}{w_b^{DR}}$$

2.8-(48)

$$\vec{v}_{Cc} \equiv \vec{\vec{M}}_c \frac{iq\vec{e}_{Cc}^{DR}}{mw_c^{DR}}$$

2.8-(49)

$$n_{Cc} \equiv n_0 \frac{\vec{k}_c \cdot \vec{v}_{Cc}}{w_c^{DR}}$$

2.8-(50)

The expression (43) for the coupling coefficient is totally symmetric under simultaneous identical permutations of the subscripts A, B, C and of the subscripts $\bar{a}$, b, c. (Compare (34).) For purely real wavevectors and frequencies this symmetry is in accord with the Manley-Rowe relations.

In similar fashion one finds the third-order nonlinear

conductivity in the wavevector-frequency domain, by imposing

upon the medium an electric field containing 3 complex waves;

$$\vec{E}(\vec{x}, t) = \vec{E}_b \, e^{i\vec{k}_b \cdot \vec{x} - iw_b t}$$

$$+ \vec{E}_c \, e^{i\vec{k}_c \cdot \vec{x} - iw_c t}$$

$$+ \vec{E}_d \, e^{i\vec{k}_d \cdot \vec{x} - iw_d t}$$

2.8-(51)

Now (9) and (12)-(14) are to be solved with (51) substituted in

(9) and (13). The medium specified by (12)-(14) then has a

third-order nonlinear response at the sum wavevector and fre-

quency $(\vec{k}_b + \vec{k}_c + \vec{k}_d, \; w_b + w_c + w_d)$. Equate all terms in (12)-

(14) which oscillate at this sum-wavevector and frequency.

This yields an expression for the third-order nonlinear current

in terms of linear and second-order nonlinear quantities. After

some algebra (see Appendix A3) one finds the third-order coup-

ling coefficient, in terms of the linear response of the fluid

to the electric fields chosen as polarization bases, and the

second-order responses to the same electric fields taken two at

a time. Explicitly

$$\left(F^{NL(3)}_{b,\,c,\,d}\right)_{ABCD}$$

$$= -\frac{m}{\epsilon_0}\Big[\, n_{A\bar{a}}\vec{v}_{Bb,\,Cc}\cdot\vec{v}_{Dd} + n_{A\bar{a}}\vec{v}_{Cc,\,Dd}\cdot\vec{v}_{Bb} + n_{A\bar{a}}\vec{v}_{Dd,\,Bb}\cdot\vec{v}_{Cc}$$

$$+ n_{Bb}\vec{v}_{Cc,\,Dd}\cdot\vec{v}_{A\bar{a}} + n_{Bb}\vec{v}_{Dd,\,A\bar{a}}\cdot\vec{v}_{Cc} + n_{Bb}\vec{v}_{A\bar{a},\,Cc}\cdot\vec{v}_{Dd}$$

$$+ n_{Cc}\vec{v}_{Dd,\,A\bar{a}}\cdot\vec{v}_{Bb} + n_{Cc}\vec{v}_{A\bar{a},\,Bb}\cdot\vec{v}_{Dd} + n_{Cc}\vec{v}_{Bb,\,Dd}\cdot\vec{v}_{A\bar{a}}$$

$$+ n_{Dd}\vec{v}_{A\bar{a},\,Bb}\cdot\vec{v}_{Cc} + n_{Dd}\vec{v}_{Bb,\,Cc}\cdot\vec{v}_{A\bar{a}} + n_{Dd}\vec{v}_{Cc,\,A\bar{a}}\cdot\vec{v}_{Bb}$$

$$+ n_0\vec{v}_{A\bar{a},\,Bb}\cdot\vec{v}_{Cc,\,Dd} + n_0\vec{v}_{A\bar{a},\,Cc}\cdot\vec{v}_{Bb,\,Dd} + n_0\vec{v}_{A\bar{a},\,Dd}\cdot\vec{v}_{Bb,\,Cc}$$

$$- \left(n_{A\bar{a},\,Bb}n_{Cc,\,Dd} + n_{A\bar{a},\,Cc}n_{Bb,\,Dd} + n_{A\bar{a},\,Dd}n_{Bb,\,Cc}\right)\gamma v^2_{To}/n_0$$

$$+ n_{A\bar{a}}n_{Bb}n_{Cc}n_{Dd}\,\gamma(\gamma-2)(\gamma-3)\,v^2_{To}/n^3_0$$

$$- \left(n_0\vec{v}_{A\bar{a},\,Bb} + n_{A\bar{a}}\vec{v}_{Bb} + n_{Bb}\vec{v}_{A\bar{a}}\right)\times\left(n_0\vec{v}_{Cc,\,Dd} + n_{Cc}\vec{v}_{Dd} + n_{Dd}\vec{v}_{Cc}\right)$$

$$\cdot\,\frac{iq\vec{B}_0}{mn_0 w^{DR}_{c+d}}$$

$$- \left(n_0\vec{v}_{A\bar{a},\,Cc} + n_{A\bar{a}}\vec{v}_{Cc} + n_{Cc}\vec{v}_{A\bar{a}}\right)\times\left(n_0\vec{v}_{Dd,\,Bb} + n_{Dd}\vec{v}_{Bb} + n_{Bb}\vec{v}_{Dd}\right)$$

$$\cdot\,\frac{iq\vec{B}_0}{mn_0 w^{DR}_{d+b}}$$

$$- \left(n_0\vec{v}_{A\bar{a},\,Dd} + n_{A\bar{a}}\vec{v}_{Dd} + n_{Dd}\vec{v}_{A\bar{a}}\right)\times\left(n_0\vec{v}_{Bb,\,Cc} + n_{Bb}\vec{v}_{Cc} + n_{Cc}\vec{v}_{Bb}\right)$$

$$\cdot\,\frac{iq\vec{B}_0}{mn_0 w^{DR}_{b+c}}$$

$$- n_0(\vec{v}_{A\bar{a}} \cdot (\vec{v}_{Cc} \cdot \vec{v}_{Dd})) \, \vec{v}_{Bb}$$

$$\cdot \left(\frac{\vec{k}_b^{DR}}{w_b} \left(\frac{\vec{k}_c^{DR}}{w_c} + \frac{\vec{k}_d^{DR}}{w_d} \right) + \frac{\vec{k}_{d+b}^{DR}}{w_{d+b}} \left(\frac{\vec{k}_d^{DR}}{w_d} - \frac{\vec{k}_b^{DR}}{w_b} \right) \right.$$

$$\left. + \frac{\vec{k}_{b+c}^{DR}}{w_{b+c}} \left(\frac{\vec{k}_b^{DR}}{w_b} - \frac{\vec{k}_c^{DR}}{w_c} \right) \right) \cdot \frac{iq\vec{B}_0^{DR}}{mw_a}$$

$$- n_0(\vec{v}_{A\bar{a}} \cdot (\vec{v}_{Dd} \cdot \vec{v}_{Bb})) \, \vec{v}_{Cc}$$

$$\cdot \left(\frac{\vec{k}_c^{DR}}{w_c} \left(\frac{\vec{k}_d^{DR}}{w_d} - \frac{\vec{k}_b^{DR}}{w_b} \right) + \frac{\vec{k}_{b+c}^{DR}}{w_{b+c}} \left(\frac{\vec{k}_b^{DR}}{w_b} - \frac{\vec{k}_c^{DR}}{w_c} \right) \right.$$

$$\left. + \frac{\vec{k}_{c+d}^{DR}}{w_{c+d}} \left(\frac{\vec{k}_c^{DR}}{w_c} - \frac{\vec{k}_d^{DR}}{w_d} \right) \right) \cdot \frac{iq\vec{B}_0^{DR}}{mw_a}$$

$$- n_0(\vec{v}_{A\bar{a}} \cdot (\vec{v}_{Bb} \cdot \vec{v}_{Cc})) \, \vec{v}_{Dd}$$

$$\cdot \left(\frac{\vec{k}_d^{DR}}{w_d} \left(\frac{\vec{k}_b^{DR}}{w_b} - \frac{\vec{k}_c^{DR}}{w_c} \right) + \frac{\vec{k}_{c+d}^{DR}}{w_{c+d}} \left(\frac{\vec{k}_c^{DR}}{w_c} - \frac{\vec{k}_d^{DR}}{w_d} \right) \right.$$

$$\left. + \frac{\vec{k}_{d+b}^{DR}}{w_{d+b}} \left(\frac{\vec{k}_d^{DR}}{w_d} - \frac{\vec{k}_b^{DR}}{w_b} \right) \right) \cdot \frac{iq\vec{B}_0^{DR}}{w_a} \Bigg]$$

Here we have taken

$$(\vec{k}_a, w_a) = (\vec{k}_b, w_b) + (\vec{k}_c, w_c) + (\vec{k}_d, w_d)$$

<div align="right">2.8-(53)</div>

The first-order normalized quantities appearing in (52) are defined as in (45)-(50). The second-order normalized quantities appearing in (52) are defined as being the second-order responses to the corresponding pair of unit basis fields according to (40) and (41). For instance

$$\vec{v}_{Bb,\,Cc} \equiv \frac{\vec{\overline{M}}_{b+c}}{w_{b+c}^{DR}} \left[\vec{v}_{Bb}(\vec{k}_b \cdot \vec{v}_{Cc}) + \vec{v}_{Cc}(\vec{k}_c \cdot \vec{v}_{Bb}) \right.$$

$$+ \frac{\gamma v_{To}^2 \vec{k}_{b+c}}{w_{b+c}^{DR}} \left(\vec{k}_{b+c} \cdot \left(\vec{v}_{Bb}\frac{n_{Cc}}{n_0} + \vec{v}_{Cc}\frac{n_{Bb}}{n_0} \right) \right)$$

$$+ \gamma(\gamma-2)v_{To}^2 \vec{k}_{b+c}\frac{n_{Bb}n_{Cc}}{n_0^2}$$

$$+ i\vec{v}_{Bb} \times \left(\vec{k}_c \times \frac{q\vec{e}_{Cc}^{DR}}{mw_c^{DR}} \right) + i\vec{v}_{Cc} \times \left(\vec{k}_b \times \frac{q\vec{e}_{Bb}^{DR}}{mw_b^{DR}} \right) \bigg]$$

<div align="right">2.8-(54)</div>

$$n_{Bb,\,Cc} \equiv \frac{\vec{k}_{b+c}}{w^{DR}_{b+c}} \cdot (n_0 \vec{v}_{Bb,\,Cc} + n_{Bb}\vec{v}_{Cc} + n_{Cc}\vec{v}_{Bb})$$

$$2.8\text{-}(55)$$

Consider the symmetry properties of the expression (52). The terms not explicitly involving $\vec{B}_o$ form a quantity which is trivially invariant under any simultaneous identical permutation of the subscripts A, B, C, D and of the subscripts $\bar{a}$, b, c, d. The first 3 out of the 6 terms involving $\vec{B}_o$ explicitly also form a quantity which is invariant in this same sense. This may be seen by using (53) and recalling that $w_{\bar{a}} \equiv -w_a$. The last 3 terms of (52) form a quantity which is certainly invariant under simultaneous identical permutations of B, C, D and of b, c, d. This lesser symmetry is obvious both by inspection and from the manner of derivation. The quantity formed by the last 3 terms if also invariant under permutation of all 4 subscripts A, B, C, D and $\bar{a}$, b, c, d, in the same sense as the rest of (52). This is proved in Appendix A4, making use of MACSYMA[7]. MACSYMA is a computer-based symbol-manipulation system, developed and maintained at Project MAC by Moses et al. Thus the whole expression (52) is of the form

$$\left(F_{b,\,c,\,d}^{NL(3)}\right)_{ABCD} \equiv X_{\bar{a},\;\;b,\;\;c,\;\;d}^{A,\;\;B,\;\;C,\;\;D}$$

$$2.8\text{-}(56)$$

where X is symmetric under simultaneous identical permutation of its upper and lower indices.

The most important uses of the conductivities $\vec{\vec{G}}^{\,LIN}$, $\vec{\vec{G}}^{\,NL(2)}$, $\vec{\vec{G}}^{\,NL(3)}$ and so on, lie in the generalized-coupling-of-modes equations 2.5-(7), where they appear along with their wavevector- and frequency- derivatives; in the equations 2.6-(4,5) describing pump-wave equilibrium; and in the equations 2.6-(10) (or equivalently 2.7-(6)) describing linearized perturbations about a pump-wave equilibrium.

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CHAPTER 3

THIRD-ORDER THEORY OF LASER-PLASMA-PELLET INTERACTIONS

3.1 Laser-Driven Instabilities

3.2 Geometry of Laser Pump

3.3 Self-Consistent Laser-Pump Equilibrium

3.4 Perturbations About Pump Equilibrium

    3.4.1 Description and Behavior of Coupled Perturbations

    3.4.2 Connection with Uncoupled Modes

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    3.4.4 Connection with Modified Instabilities

    3.4.5 Connection with Third-Order Instabilities

    3.4.6 Relation to Later Sections

3.5 Coupling Coefficients from General Formulas of Section 2.8

3.6 Coupling Coefficients from Differential Equation Approach

3.7 Coupled Equations Incorporating Physical Approximations

3.8 Isolation of Specific Instabilities from Determinantel Equation

3.9 Reduction to Theory of Kaw, White, et al.

CHAPTER 3

THIRD-ORDER THEORY OF
LASER-PLASMA-PELLET INTERACTIONS

3.1 Laser-Driven Instabilities

First we outline the course of this chapter and the relation of its vari-
ous sections to the results of the last chapter. In Chapter 2, the generalized-
coupling-of-modes theory was developed in section 2.5 and applied in sec-
tion 2.6 to the problem of perturbations about a pump-wave equilibrium.
In this chapter the pump-wave is specialized to be a laser-beam; section 3.2
relates the beam-geometry to the geometry of the pellet and surrounding
plasma. The results on pump-wave-equilibrium from section 2.6 are used
in 3.3 to deduce the self-consistent harmonic structure of the laser-pump,
in particular the amplitude and polarization of the 2nd harmonic. In Chap-
ter 2, those results from 2.6 which concerned the behavior of perturbations
about the pump-equilibrium were rephrased in 2.7 in terms of coupling coef-
ficients between particular wave polarizations. In section 3.4, the particu-
lar polarizations to be used in describing perturbations about the laser pump
are selected, and the physical grounds for their selection are explained. The
corresponding coupling coefficients are evaluated in two ways. In Chapter 2,
section 2.8 yielded general expressions for coupling coefficients. Section 3.5
uses these to evaluate coupling coefficients for the perturbations coupled by
the laser pump. Section 3.6 evaluates the same coefficients directly from
the differential equations describing the plasma dynamics in the presence
of the laser pump-wave. Both yield the same coupled equations describing
the behavior of the perturbations and incorporating physical approximations,

as displayed in 3. 7. The consistency condition for these coupled equations is a determinantal equation. Section 3.8 extracts subdeterminants from this, connecting specifically polarised components of the perturbations and hence describing specific laser-driven instabilities. Finally, 3.9 compares our results with those derived by certain other workers. These other workers have dealt with laser-driven instabilities by postulating some physical mechanisms for wave-wave interaction and making the corresponding approximations ab initio. The limitations of this approach will be discussed.

The aim of this chapter is to provide a systematic derivation of laser-driven instabilities in unmagnetized plasma and their corresponding 3-dimensional dispersion relations. The theoretical tools for this task were developed in Chapter 2. Chapter 4 will provide for each individual instability a simplified physical model and also an analysis of the propagation of initially localized disturbances.

3.2 Geometry of Laser Pump

The geometry of the laser-light irradiating the pellet is first idealized to be a converging wavefront perfectly centered on the pellet. The spherical geometry of that portion of the pellet plasma accessible to the laser light has curvature characterised by the inverse radius of the critical surface. This radius is expected to be about 600 microns in currently-proposed schemes. This figure is certainly much larger than the free-space wavelengths of lasers typically proposed for these schemes. Thus, inferring the wavefront geometry from the plasma geometry, one might conclude that a planar approximation would be reasonable. However, it is well to check the actual laser wavelength inside the plasma to see if this is still

small enough compared to the radius of the critical surface.

The local dispersion function gives the W. K. B. result

$$k^2 = \frac{w_1^2 - w_p^2}{c^2} \qquad \text{3.2-(1)}$$

where w_1 is the laser frequency. Of course, this breaks down near the critical surface, and one must have recourse to the exact Airy-function solutions as described by Ginzburg. These solutions are expressed in terms of the dimensionless parameter

$$\zeta = \ell \left(\frac{w^2}{c^2} \left| \frac{d\epsilon}{d\ell} \right|_{\ell=0} \right)^{1/3} \qquad \text{3.2-(2)}$$

where ℓ is measured outwards from the critical surface. These solutions are for a planar geometry, a linear dependence of dielectric function on position and full reflection of the laser light; nevertheless, they still serve as a guide. Taking the plasma dielectric function

$$\epsilon = 1 - \frac{(w_p(\ell))^2}{w^2} \qquad \text{3.2-(3)}$$

one has

$$\zeta = \left(\frac{8\pi^2}{L\lambda_{10}^2} \right)^{1/3} \ell$$

where λ_{10} is the free-space laser wavelength and L is the scale length of the plasma density gradient. The widest peak in the Airy-function solution stretches from about $\zeta = -1.5$ to about $\zeta = +2.5$. Thus, taking 1.09-micron

laser light and 600 micron scale-length for the plasma density gradient, the Airy-peak has a width of about 7.5 microns. This is much less than the radius of the critical surface so that a planar model for the pump geometry can give valid results.

A more serious issue is the effect of the swelling and broadening of the field pattern near the critical surface on the absorptive instabilities. The Airy function deviates appreciably from geometrical optics out to about $\zeta = 5.0$. Applying this to the above plasma-laser combination, the deviation is seen to extend about 9.4 microns out from the critical surface. This corresponds to the range of plasma densities

$$w_1 - \frac{w_1}{64} \lesssim w_p \lesssim w_1$$

Since ion-acoustic frequencies are less than the ion-plasma frequency which itself is roughly $w_p/40$, absorptive instabilities near the critical surface are subject to the broadening in effective k_ℓ brought about by the behavior of the pump-wave near its turning-point. However, these absorptive instabilities to lowest order depend only on k-matching in the direction perpendicular to the laser propagation, and so their growth is not affected.

The laser pump wave, then, is modeled as a planar wave having a local wavelength given by the local dispersion relation. The propriety of studying the behavior of the resulting laser-driven instabilities by taking the local plasma density and solving the corresponding homogeneous problem was discussed in subsection 1.2.3.

The polarisation of the planar wave is taken to be linear. The laser-pump is modeled as a wave propagating in the positive z-direction and

polarised linearly in the x-direction.   The cases of circular and elliptical polarizations are not dealt with in this work.

3.3 Self-Consistent Laser-Pump Equilibrium

In this section we look at the harmonic structure of the laser-pump. The oscillating electric field of the laser-fundamental

$$\vec{E}_{FUND}(\vec{x}, t) = \vec{E}_1 \, e^{i\vec{k}_1 \cdot \vec{x} - iw_1 t} + \text{complex conjugate} \qquad 3.3\text{-}(1)$$

induces a quivering motion of the electrons of the plasma:

$$\vec{v}_{FUND}(\vec{x}, t) = \vec{V}_1 \, e^{i\vec{k}_1 \cdot \vec{x} - iw_1 t} + \text{complex conjugate} \qquad 3.3\text{-}(2)$$

We consider a drift-free and unmagnetised plasma.   The transversely polarised fundamental causes no particle bunching in lowest order (see Figure F1).   Thus the linear response of the electrons is, from 2.8-(19) or simply from first principles, given by

$$\vec{V}_1 = \frac{i q_e \vec{E}_1}{m_e w_1} \qquad 3.3\text{-}(3)$$

This quivering takes place in the presence of the oscillating B-field of the fundamental.   The electrons feel a Lorentz force at twice the fundamental wavevector and frequency in the direction of the laser-beam propagation. The Lorentz force is directed toward the regions of instantaneous high field-magnitude.   However, the frequency of the Lorentz force lies well above the electrostatic resonant frequency of the electrons:

$$w_{\text{BOHM-GROSS}}(2k_1) \sim w_p \overset{\gtrsim}{} w_1 < 2w_1 \qquad\qquad 3.3\text{-}(4)$$

Also the phase-velocity of the Lorentz-force pattern is much greater than the velocity of any electron:

$$v_{Te} \ll c < v_{\text{phase}} = 2w_1/2k_1 \qquad\qquad 3.3\text{-}(5)$$

Thus the Lorentz force actually causes the electron density to be higher in the regions where the fundamental has a low electric field magnitude.

The physical mechanism we have just described constitutes coherent wave-wave coupling between the transversely-polarized fundamental taken twice and a longitudinally-polarized second harmonic. We can use the mode-coupling equations 2.6-(5,6), which govern the self-consistent harmonic structure of the pump, to look at the amplitude and polarization of the second harmonic. To lowest order 2.6-(5,6) become

$$\overset{\rightrightarrows}{L}^{\text{LIN}}(\vec{k}, w_1)\, \vec{E}_1 = \vec{0} \qquad\qquad 3.3\text{-}(6)$$

$$\overset{\rightrightarrows}{L}^{\text{LIN}}(2\vec{k}_1, 2w_1)\, \vec{E}_2 = -2i\mu_0 w_1 \left[\frac{1}{2} \overset{\rightrightarrows}{G}^{\text{NL}(2)}(\vec{k}_1, w_1, \vec{k}_1, w_1)\, \vec{E}_1 \vec{E}_1\right] \qquad 3.3\text{-}(7)$$

Using the notation of coupling coefficients introduced in 2.7, the vector equation (7) may be written as the triplet of scalar equations

$$(D_1)_{xx} E_{x2} = -\frac{1}{2}\left(F_{1,1}^{\text{NL}(2)}\right)_{xxx} E_{x1} E_{x1} \qquad\qquad 3.3\text{-}(8)$$

$$(D_1)_{yy} E_{y2} = -\frac{1}{2}\left(F_{1,1}^{\text{NL}(2)}\right)_{yxx} E_{x1} E_{x1} \qquad\qquad 3.3\text{-}(9)$$

$$(D_1)_{zz} E_{z2} = -\frac{1}{2}\left(F_{1,1}^{\text{NL}(2)}\right)_{zxx} E_{x1} E_{x1} \qquad\qquad 3.3\text{-}(10)$$

As mentioned in 3.2, the Cartesian axes are chosen such that the laser-

beam fundamental is polarized along x to lowest order. From 2.8-(43), in the absence of a steady magnetic field the coupling coefficient becomes the sum over species of

$$\left(F_{1,1}^{NL(2)}\right)_{xx} = -\frac{mn_0}{\epsilon_0}\left\{\frac{n_{A\bar{2}}}{n_0}\vec{v}_{x1}\cdot\vec{v}_{x1} + 2\frac{n_{x1}}{n_0}\vec{v}_{x1}\cdot\vec{v}_{A\bar{2}} + \gamma(\gamma-2)\,v_{T0}^2\,\frac{n_{A\bar{2}}n_{x1}^2}{n_0^3}\right\}$$

$$3.3-(11)$$

For a transversely polarized wave in unmagnetized plasma, the first-order response includes no particle bunching (see 2.8-(19)). Therefore

$$n_{x1} = n_{x\bar{2}} = n_{y\bar{2}} = 0 \qquad\qquad 3.3-(12)$$

and (11) becomes

$$\left(F_{1,1}^{NL(2)}\right)_{xxx} = \left(F_{1,1}^{NL(2)}\right)_{yxx} = 0 \qquad\qquad 3.3-(13)$$

$$\left(F_{1,1}^{NL(2)}\right)_{zxx} = -\frac{mn_0}{\epsilon_0}\frac{n_{z\bar{2}}}{n_0}\vec{v}_{x1}\cdot\vec{v}_{x1} \qquad\qquad 3.3-(14)$$

Now substitute (13), (14) into (8), (9), (10) and use the linear results 2.8-(19) and 2.8-(27). This yields equations specifying the polarization components of the 2nd harmonic of the laser:

$$E_{x2} = E_{y2} = 0 \qquad\qquad 3.3-(15)$$

$$\left(1 - \frac{w_p^2}{(2w_1)^2 - \gamma(2k_1)^2\,v_{Te}^2}\right)E_{z2} =$$

$$+ \frac{1}{2}\frac{m_e n_0}{\epsilon_0}\left(\frac{-2k_1}{-2w_1}\right)\frac{(-2w_1)^2}{\left((-2w_1)^2 - \gamma(-2k_1)^2\,v_{Te}^2\right)}\frac{iq_e}{m_e(-2w_1)}\,V_1^2 \qquad 3.3-(16)$$

Here V_1 is the electron quivering velocity induced by the laser fundamental, as given by (2) and (3). Ion contributions to (16) have been neglected. On the left-hand side of (16) the linear dispersion relation at twice the laser frequency can be evaluated accurately enough for our purposes without the ion term. The contributions of ions to the right-hand side of (16) are smaller than the electron contribution by the squared ratio of their quivering velocities, which is to say, by the squared inverse mass ratio. Use

$$q_e = -e \qquad\qquad 3.3-(17)$$

and the phase-velocity relations (5) to approximate (16) as follows

$$\left(1 - \frac{w_p^2}{4w_1^2}\right) E_{z2} \approx \frac{i}{4} \frac{m_e}{e} \frac{w_p^2}{w_1^2} k_1 V_1^2 \qquad\qquad 3.3-(18)$$

The phase relation between E_{z2} and E_{x1} implied by (18) is the same phase relation given by the Lorentz-force argument quoted previously.

The second harmonic of the laser pump has here been found (see Figure 3.3F2) in terms of the uncorrected fundamental $\vec{E}_1$. This has been done by using the mode-coupling equations typified by 2.6-(5,6) in their simplest form, namely (6,7). Of course, one can now go back and find the correction to the fundamental due to the presence of the 2nd harmonic, and the 3rd-order self-correction to the fundamental. One can also extend the series of mode-coupling equations 2.6-(5,6) to find the amplitude and polarization of the 3rd harmonic. We briefly sketch these further corrections.

Extend (6) to include the effect of the 2nd harmonic and of the self-

correction to the fundamental:

$$\vec{\vec{L}}^{LIN}(\vec{k}_1, w_1)\, E_1 = -i\mu_0 w_1 \left\{ \vec{\vec{\vec{G}}}^{NL(2)}(2\vec{k}_1, 2w_1, -\vec{k}_1, -w_1)\, \vec{E}_2\vec{E}_1^* \right.$$

$$\left. + \frac{1}{2}\, \vec{\vec{\vec{G}}}^{NL(3)}(\vec{k}_1, w_1, \vec{k}_1, w_1, -\vec{k}_1, -w_1)\, \vec{E}_1\vec{E}_1\vec{E}_1^* \right\} \qquad 3.3\text{-}(19)$$

On the right-hand side, $\vec{E}_1$ and $\vec{E}_2$ will, of course, be taken to have their uncorrected values (1) and (16). Using the notation of coupling coefficients introduced in 2.7, the vector equation (19) may be written as the triplet of scalar equations

$$(D_1)_{AA} E_{A1} = -\left(F_{2,\bar{1}}^{NL(2)}\right)_{Azx} E_{z2} E_{x1}^*$$

$$- \frac{1}{2}\left(F_{1,1,\bar{1}}^{NL(3)}\right)_{Axxx} E_{x1} E_{x1} E_{x1}^* \qquad A = x, y, z \qquad 3.3\text{-}(20)$$

The form 2.8-(43) for the second-order coupling coefficient, the form 2.8-(52) for the third-order coupling coefficient, and the form 2.8-(40) for the second-order velocities are used to evaluate (20). The right-hand side of (20) vanishes for A = y, z, and the A = x component yields

$$(D_1)_{xx} E_{x1} = \frac{w_p^4 k_1^2 |V_1^2|}{8w_1^6}\left(1 - \frac{w_p^2}{4w_1^2}\right)^{-1} E_{x1} + \frac{w_p^2 k_1^2 |V_1^2|}{2w_1^4}\, E_{x1} \qquad 3.3\text{-}(21)$$

The two correction terms may be combined on the right-hand side and the scalar dispersion function for electromagnetic waves inserted on the left-hand side. Then the wavenumber of the laser, shifted due to the nonlinearity of the plasma medium, satisfies

$$\left(-\frac{k_1^2 c^2}{w_1^2} + 1 - \frac{w_p^2}{w_1^2} \right) = \frac{w_p^2}{w_1^2} \frac{k_1^2 |V_1^2|}{\left(4w_1^2 - w_p^2 \right)} \qquad 3.3-(23)$$

For a given input laser frequency w_1, the laser wavenumber at a fixed

plasma density given by w_p is decreased from the wavenumber derived

from the linear electromagnetic dispersion relation. The decrease is

given by the ratio

$$\frac{k_1^{CORRECTED}}{k_1^{LINEAR\ PLASMA}} = \frac{1}{\sqrt{1 + \frac{|V_1^2|}{c^2} \frac{w_p^2}{\left(4w_1^2 - w_p^2 \right)}}} \qquad 3.3-(24)$$

This wavenumber correction is the only effect of medium nonlinearity on

the laser fundamental, at least to the order specified by (19). The polari-

zation of the fundamental is unaltered. (See Figure F3.) Extend the series

of mode-coupling equations (6, 7) to find the amplitude and polarization of

the 3rd harmonic:

$$\vec{\vec{L}}^{LIN}(3\vec{k}_1, 3w_1) \vec{E}_3 = -3i\mu_0 w_1 \left[\vec{\vec{G}}^{NL(2)}(\vec{k}_1, w_1, 2\vec{k}_1, 2w_1) \vec{E}_1 \vec{E}_2 \right.$$

$$\left. + \frac{1}{3!} \vec{\vec{G}}^{NL(3)}(\vec{k}_1, w_1, \vec{k}_1, w_1, \vec{k}_1, w_1) \vec{E}_1 \vec{E}_1 \vec{E}_1 \right] \quad 3.3-(25)$$

The vector equation (25) may be written as the triplet of scalar equations

E_{x1}, polarization of laser fundamental

V_{x1}, quivering motion of electrons to lowest order

k_{1z}, propagation of laser beam

B_{y1}
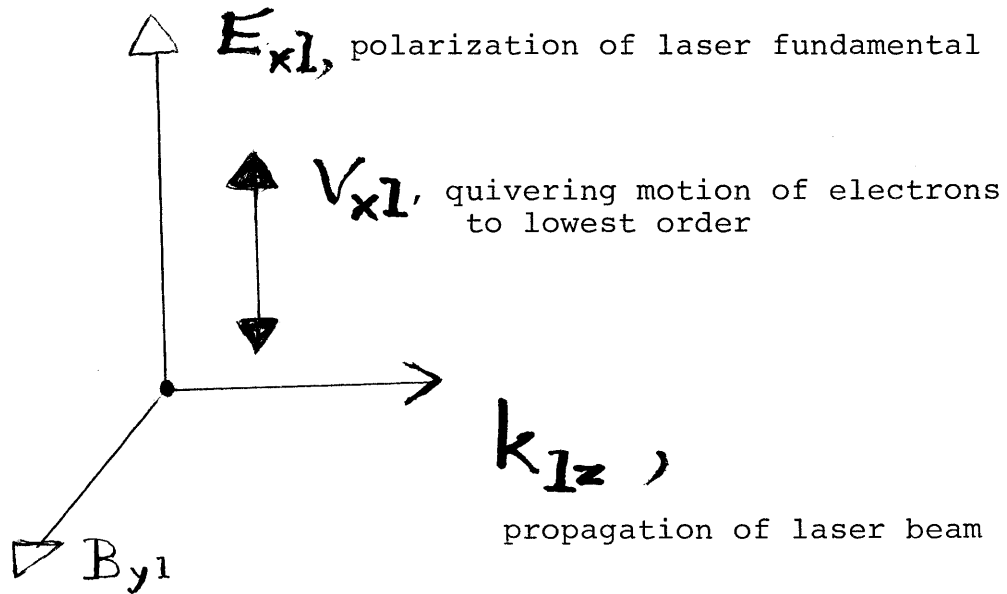
Figure 3.3 F1 LASER-PUMP EQUILIBRIUM TO FIRST ORDER

E_{x1}'

k_{1z}

E_{z2}
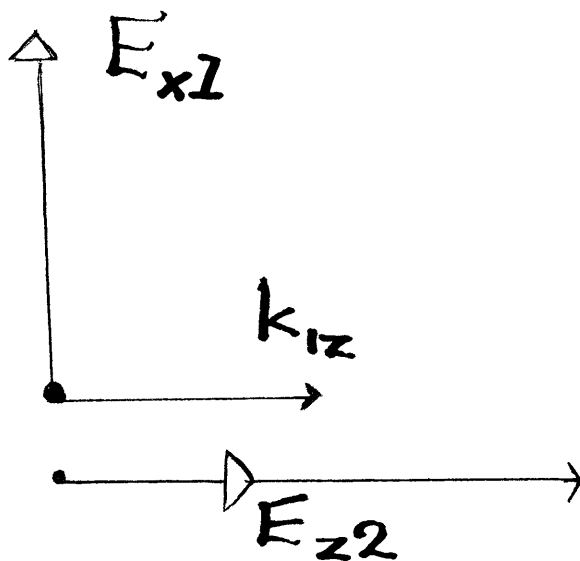
Figure 3.3 F2 LASER-PUMP EQUILIBRIUM TO SECOND ORDER
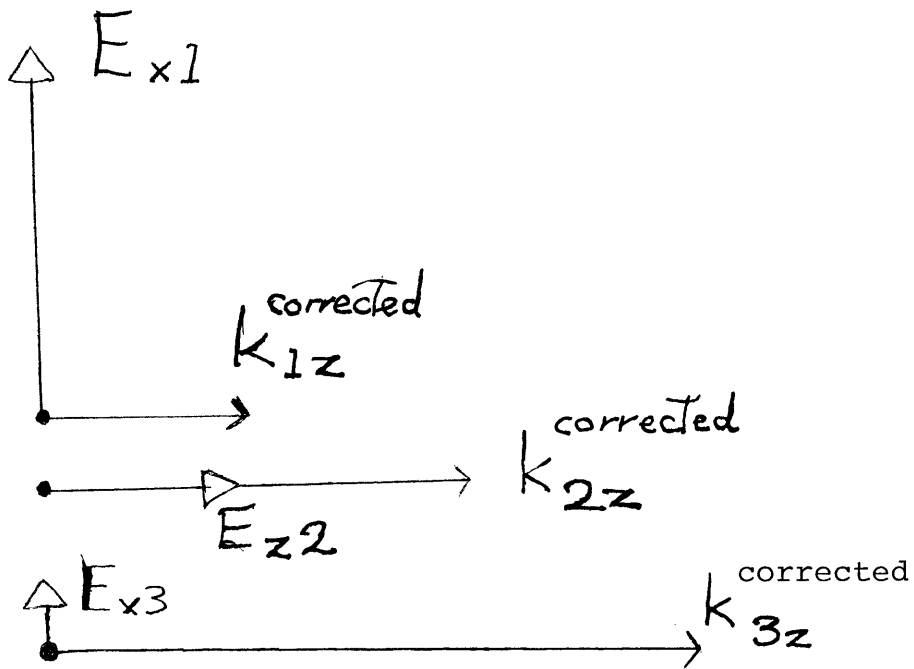
Figure 3.3 F3   LASER-PUMP EQUILIBRIUM TO THIRD ORDER

$$(D_3)_{AA} E_{A3} = -\left(F_{1,2}^{NL(2)}\right)_{Axz} E_{1x} E_{2z}$$

$$- \frac{1}{6}\left(F_{1,1,1}^{NL(3)}\right)_{Axxx} E_{1x} E_{1x} E_{1x} \qquad A = x, y, z \qquad 3.3\text{-}(26)$$

The right-hand side of (26) vanishes for $A = y, z$ and the $A = x$ component yields

$$(D_3)_{xx} E_{x3} = \frac{i}{24}\frac{m_e}{e}\frac{w_p^4 k_1^2 V_1^3}{w_1^5}\left(1 - \frac{w_p^2}{4w_1^2}\right)^{-1} + \frac{i}{6}\frac{m_e}{e}\frac{w_p^2 k_1^2 V_1^3}{w_1^3} \qquad 3.3\text{-}(27)$$

The two terms may be combined on the right-hand side and the scalar dispersion function for electromagnetic waves inserted on the left-hand side. Here again V_1 is the quivering velocity induced by the laser fundamental, as given by (2) and (3). The ions may be neglected in the dispersion function on the left since that is evaluated at three times the laser frequency, and in the nonlinear conductivity term on the right since ion contributions are smaller by at least the squared inverse mass ratio. The amplitude and polarization of the 3rd harmonic electric field of the laser pump are thus given by (see Figure F3)

$$\left(-\frac{9k_1^2 c^2}{9w_1^2} + 1 - \frac{w_p^2}{9w_1^2}\right) E_{x3} = \frac{i}{6}\frac{m_e}{e}\frac{w_p^2 k_1^2 V_1^3}{w_1^3}\left(1 - \frac{w_p^2}{4w_1^2}\right)^{-1} \qquad 3.3\text{-}(28)$$

Let us now consider the application to succeeding work on laser-driven instabilities of the detailed harmonic structure of the laser pump. In particular, let us consider the influence on laser-driven instabilities of the

self-correction to the fundamental, of the existence of a 2nd-harmonic field, and of the existence of a 3rd-harmonic field.

In the work which is to follow, the symbol k_1 will be understood to refer to the actual value of the laser wavenumber. Thus the wavenumber self-correction (24) will not appear explicitly in the dispersion relations which will be derived for specific unstable interactions. Further, the form of these unstable dispersion relations will not be sensitive to whether or not the laser wavenumber and frequency lie on the exact linear electromagnetic dispersion curve. Therefore the self-correction (23, 24) will not be referred to again.

In the work which is to follow, perturbations about the equilibrium described in this section will be considered. No more than 3 such perturbations will be considered in dealing with any one specific instability. These perturbations will comprise, say, $\vec{E}_{(1)}, \vec{E}_{(0)}, \vec{E}_{(-1)}$ in the notation of 2.6-(7). The 2nd-harmonic content of the laser pump, $\vec{E}_1$, is capable of coupling together the two extreme members of the set of 3 perturbations, via a 2nd-order nonlinear current. However, this function of coupling together $\vec{E}_{(1)}$ and $\vec{E}_{(-1)}$ can also be performed by the laser fundamental, $\vec{E}_1$, entering twice into a 3rd-order nonlinear current. This may be seen explicitly by referring to 2.6-(10) which describes pump-coupled perturbations, and picking out terms on the right-hand side which contain $\vec{E}_{(n\pm2)}$. It will be demonstrated in some detail in section 3.5 that the laser fundamental $\vec{E}_1$ acting twice achieves a stronger coupling than the laser 2nd harmonic $\vec{E}_2$ acting once. This demonstration depends only on the fact that the phase velocities of the perturbation fields and the electron thermal velocity

satisfy the following for all laser-driven instabilities of interest:

$$\left|\frac{w_{(0)}}{k_{(0)}}\right| \ll c \qquad\qquad 3.3\text{-}(29)$$

$$v_T \ll c \qquad\qquad 3.3\text{-}(30)$$

$$v_T \ll \left|\frac{w_{(\pm 1)}}{k_{(\pm 1)}}\right| \qquad\qquad 3.3\text{-}(31)$$

$$\left|\frac{w_{(0)}}{k_{(0)}}\right| \lesssim \left|\frac{w_{(\pm 1)}}{k_{(\pm 1)}}\right| \qquad\qquad 3.3\text{-}(32)$$

Note that these restrictions are looser than those imposed by Kaw, White, et al.[1] In particular, (32) is replaced in their work by

$$\left|\frac{w_{(0)}}{k_{(0)}}\right| \ll \left|\frac{w_{(\pm 1)}}{k_{(\pm 1)}}\right| \qquad\qquad 3.3\text{-}(33)$$

This point is elaborated in section 3.9. To repeat, the laser 2nd-harmonic field, $\vec{E}_2$, existing as part of the self-considente laser-pump equilibrium in the nonlinear plasma medium, will be shown not to be important for the parametric instabilities driven by that laser pump.

A fortiori, the laser 3rd-harmonic field, $\vec{E}_3$, may be neglected in work on laser-driven plasma instabilities.

3.4 Perturbations about Pump Equilibrium

3.4.1 DESCRIPTION AND BEHAVIOR OF COUPLED PERTURBATIONS

In section 2.7 we discussed how to choose for each perturbation a separate basis triad of polarization vectors, in terms of which to express the

electric field of the perturbation. When each such basis triad is chosen to consist of eigenvectors of the corresponding linear dispersion tensor, evaluated at the perturbation wavevector and frequency, the coupling of the polarization components is described by 2.7-(6). In this section we make an explicit choice of basis triads for describing the polarization of perturbations which are coupled together by the laser pump described in 3.3.

Consider the linear dispersion tensor for unmagnetized drift-free plasma, evaluated at the wavevector and frequency $(\vec{k}_{(n)}, w_{(n)})$ of a perturbation field $\vec{E}_{(n)}$. The warm-fluid model gives (cf. 2.8-(27))

$$\vec{D}^{LIN}(\vec{k}_{(n)}, w_{(n)}) = \frac{\vec{k}_{(n)}\vec{k}_{(n)} - k_{(n)}^2}{\mu_0 \epsilon_0 w_{(n)}^2} + 1$$

$$- \sum_{\text{species } \pi} \frac{w_{p\pi}^2}{w_{(n)}^2} \left(1 + \frac{\gamma_\pi v_{T\pi}^2 \vec{k}_{(n)}\vec{k}_{(n)}}{w_{(n)}^2 - \gamma_\pi k_{(n)}^2 v_{T\pi}^2} \right) \qquad \text{3.4-(1)}$$

This tensor operator has as one eigenvector $\vec{k}_{(n)}$, with eigenvalue

$$(D_{(n)})_{SS} \equiv 1 - \sum_{\text{species } \pi} \frac{w_{p\pi}^2}{w_{(n)}^2 - \gamma_\pi k_{(n)}^2 v_{T\pi}^2} \qquad \text{3.4-(2)}$$

Any vector perpendicular to $\vec{k}_{(n)}$ is also an eigenvector, with eigenvalue

$$(D_{(n)})_{NN} = (D_{(n)})_{MM} = -\frac{k_{(n)}^2}{\mu_0 \epsilon_0 w_{(n)}^2} + 1 - \sum_{\text{species } \pi} \frac{w_{p\pi}^2}{w_{(n)}^2} \qquad \text{3.4-(3)}$$

The prescription of section 2.7 is to choose the basis triad $\{\vec{e}_{A(n)}\}$ to be

eigenvectors of $\vec{\vec{D}}^{LIN}(\vec{k}_{(n)}, w_{(n)})$. This fixes one member of the triad to be the electrostatically polarized unit field, i.e., the unit field in the direction of the wavevector;

$$\vec{e}_{S(n)} = \vec{k}_{(n)}/|\vec{k}_{(n)}| \qquad\qquad 3.4\text{-}(4)$$

The other two members of the triad can be any two electromagnetically polarized unit fields, i.e., any two unit vectors perpendicular to (4). It will later prove convenient to choose these two as follows. Choose one to be perpendicular to the polarization of the laser fundamental. This can always be done, as follows:

$$\vec{e}_{N(n)} = (\vec{k}_{(n)} \times \vec{e}_{1x})/|\vec{k}_{(n)} \times \vec{e}_{1x}| \qquad\qquad 3.4\text{-}(5)$$

Choose the other electromagnetic polarization, $\vec{e}_{M(n)}$, perpendicular to both $\vec{e}_{S(n)}$ and $\vec{e}_{N(n)}$. This ensures that $\vec{e}_{M(n)}$ has the greatest possible projection on $\vec{e}_{x1}$. In other words, the direction of $\vec{e}_{M(n)}$ is aligned as closely with the direction of polarization of the laser as possible consistent with retaining its electromagnetic character.

$$\vec{e}_{M(n)} = ((\vec{k}_{(n)} \times \vec{e}_{1x}) \times \vec{k}_{(n)})/|(\vec{k}_{(n)} \times \vec{e}_{1x}) \times \vec{k}_{(n)}| \qquad\qquad 3.4\text{-}(6)$$

The prescription of section 2.7 is to choose the triad $\{\vec{f}_{A(n)}\}$ to be eigen-vectors of $\vec{\vec{D}}^{LIN \ TRANSPOSE}(\vec{k}_{(n)}, w_{(n)})$. From (1) the tensor $\vec{\vec{D}}^{LIN}(\vec{k}_{(n)}, w_{(n)})$ is symmetric and so one can choose

$$\left. \begin{array}{l} \vec{f}_{M(n)} = \vec{e}_{M(n)} \\[1em] \vec{f}_{N(n)} = \vec{e}_{N(n)} \\[1em] \vec{f}_{S(n)} = \vec{e}_{S(n)} \end{array} \right\} \qquad\qquad 3.4\text{-}(7)$$

For the laser-driven instabilities considered in this chapter it will be enough to consider 3 perturbation fields, $\vec{E}_{(1)}$, $\vec{E}_{(0)}$, and $\vec{E}_{(-1)}$. One uses 2 perturbation fields, $\vec{E}_{(0)}$ and $\vec{E}_{(-1)}$, in describing the Brillouin, Raman, 2-plasmon and decay instabilities together with their modified forms. One uses 3 perturbation fields $\vec{E}_{(1)}$, $\vec{E}_{(0)}$, and $\vec{E}_{(-1)}$ in describing the filamentation-modulation and oscillating-two-stream instabilities. In preparation for later work, therefore, one considers the coupling via the laser pump of the 3 perturbations $\vec{E}_{(1)}$, $\vec{E}_{(0)}$, and $\vec{E}_{(-1)}$. One wishes to do this in terms of the 3 polarization components of each of the 3 perturbations. Use (4), (5), and (6) to write each perturbation field in terms of the basis triad appropriate for its wavevector and frequency:

$$\vec{E}_{(1)} = E_{M(1)}\,\vec{e}_{M(1)} + E_{N(1)}\,\vec{e}_{N(1)} + E_{S(1)}\,\vec{e}_{S(1)}$$

$$\vec{E}_{(0)} = E_{M(0)}\,\vec{e}_{M(0)} + E_{N(0)}\,\vec{e}_{N(0)} + E_{S(0)}\,\vec{e}_{S(0)} \qquad \text{3.4-(8)}$$

$$\vec{E}_{(-1)} = E_{M(-1)}\,\vec{e}_{M(-1)} + E_{N(-1)}\,\vec{e}_{N(-1)} + E_{S(-1)}\,\vec{e}_{S(-1)}$$

The equations relating the amplitudes of the polarization components are obtained by substituting in 2.7-(6). There are 9 such equations, one for each $E_{A(n)}$, $A = M, N, S$, $n = 1, 0, -1$. These equations will be written down to first order in the laser 2nd harmonic and to the 2nd order in the laser fundamental, as follows:

$$(D_{(1)})_{AA} E_{A(1)} + E_{x1} E_{x1}^{*} \sum_{B} \left(F_{(1),1,-1}^{NL(3)} \right)_{ABxx} E_{B(1)} + E_{x1} \sum_{B} \left(F_{(0),1}^{NL(2)} \right)_{ABx} E_{B(0)}$$

$$+ \frac{1}{2} E_{x1}^{2} \sum_{B} \left(F_{(-1),1,1}^{NL(3)} \right)_{ABxx} E_{B(-1)} + E_{z2} \sum_{B} \left(F_{(-1),2}^{NL(2)} \right)_{ABz} E_{B(-1)} = 0$$

separately for each of $A = M, N, S$

3.4-(9)

$$E_{x1}^* \sum_B \left(F_{(1),-1}^{NL(2)} \right)_{ABx} E_{B(1)} + (D_{(0)})_{AA} E_{A(0)}$$

$$+ E_{x1} E_{x1}^* \sum_B \left(F_{(0),1,-1}^{NL(3)} \right)_{ABxx} E_{B(0)}$$

$$+ E_{x1} \sum_B \left(F_{(-1),1}^{NL(2)} \right)_{Bx} E_{B(-1)} = 0$$

separately for each of A = M, N, S   3.4-(10)

$$\frac{1}{2} E_{x1}^{*2} \sum_B \left(F_{(1),-1,-1}^{NL(3)} \right)_{ABxx} E_{B(1)}$$

$$+ E_{z2}^* \sum_B \left(F_{(1),-2}^{NL(2)} \right)_{ABz} E_{B(1)} + E_{x1}^* \sum_B \left(F_{(0),-1}^{NL(2)} \right)_{ABx} E_{B(0)}$$

$$+ (D_{(-1)})_{AA} E_{A(-1)} + E_{x1} E_{x1}^* \sum_B \left(F_{(-1),1,-1}^{NL(3)} \right)_{ABxx} E_{B(-1)} = 0$$

separately for each of A = M, N, S

3.4-(11)

These 9 equations are linear in the 9 quantities $\{\vec{E}_{A(n)}\}$ and can therefore be written in matrix form. Using partitioned matrix notation they can be written as in (12). Care must be exercised in using this form of the coupled equations. The quantities $\{\vec{F}\}$ do not transform like tensors since their subscripts range over different basis triads. Further the quantities indicated by $\{\vec{E}_{(n)}\}$ in (12) are column matrices of the components of the perturbation fields each referred to a different basis as shown in (8). Equation (12) essentially describes the pump-induced coupling between perturbation fields whose polarizations are described in terms of polarizations of of normal modes of the plasma. This is so even though the frequencies of

$$
\begin{pmatrix}
\vec{D}_{(1)} + |E_1|^2 \vec{\vec{F}}^{NL(3)}_{(1),1,-1} & E_1 \vec{\vec{F}}^{NL(2)}_{(0),1} & \dfrac{1}{2}E_1^2 \vec{\vec{F}}^{NL(3)}_{(-1),1,1} + E_2 \vec{\vec{F}}^{NL(2)}_{(-1),2} \\[4ex]
E_1^* \vec{\vec{F}}^{NL(2)}_{(1),-1} & \vec{D}_{(0)} + |E_1|^2 \vec{\vec{F}}^{NL(3)}_{(0),1,-1} & E_1 \vec{\vec{F}}^{NL(2)}_{(-1),1} \\[4ex]
\dfrac{1}{2}E_1^{*2} \vec{\vec{F}}^{NL(3)}_{(1),-1,-1} + E_1 \vec{\vec{F}}^{NL(2)}_{(1),-2} & E_1^* \vec{\vec{F}}^{NL(2)}_{(0),-1} & \vec{D}_{(-1)} + |E_1|^2 \vec{\vec{F}}^{NL(3)}_{(-1),1,1}
\end{pmatrix}
\begin{pmatrix}
\vec{E}_{(1)} \\[4ex] \vec{E}_{(0)} \\[4ex] \vec{E}_{(-1)}
\end{pmatrix}
=
\begin{pmatrix}
\vec{0} \\[4ex] \vec{0} \\[4ex] \vec{0}
\end{pmatrix}
$$

3.4-(12)

the coupled perturbations may be displaced from their normal-mode values.

Upon evaluating the submatrices $\overset{\rightrightarrows}{F}$ of equation (12) with the physical parameters appropriate to laser-driven plasma instabilities (see section 3.5 and Appendix A5), the couplings

$$E_2^{(*)}\overset{\rightrightarrows}{F}{}^{NL(2)}_{(\ ),\pm2}$$

due to the second-harmonic electric-field component of the pump are seen to be negligible. The couplings

$$E_1^{(*)}E_1^{(*)}\overset{\rightrightarrows}{F}{}^{NL(3)}_{(\ ),\pm1,\pm1}$$

due to the fundamental of the pump acting twice are seen to be sums of parts

$$E_1^{(*)}E_1^{(*)}\overset{\triangle}{\overset{\rightrightarrows}{F}}{}^{NL(3)}_{(\ ),\pm1,\pm1}$$

say which are physically appreciable and other parts

$$E_1^{(*)}E_1^{(*)}\overset{\blacktriangle}{\overset{\rightrightarrows}{F}}{}^{NL(3)}_{(\ ),\pm1,\pm1}$$

say which are physically negligible for the case of unmagnetized plasma. These quantities are all evaluated and discussed in section 3.5 and Appendix A5. The low-frequency self-correction

$$E_1 E_1^* \overset{\rightrightarrows}{F}{}^{NL(3)}_{(0),1,-1}$$

plays in any case no physical role and can be omitted. With these simplifications (12) acquires the form (13). The physical significance of the various submatrices of (13) and their connection with various classes of instabilities will be discussed in subsections 3.4.2-3.4.6 below.

$$
\begin{pmatrix}
\vec{\vec{D}}_{(1)} + |E_1|^2 \vec{\vec{\vec{F}}}^{NL(3)}_{(1),1,-1} & E_1 \vec{\vec{F}}^{NL(2)}_{(0),1} & \frac{1}{2}E_1^2 \vec{\vec{\vec{F}}}^{NL(3)}_{(-1),1,1} \\[2ex]
E_1^* \vec{\vec{F}}^{NL(2)}_{(1),-1} & \vec{\vec{D}}_{(0)} & E_1 \vec{\vec{F}}^{NL(2)}_{(-1),1} \\[2ex]
\frac{1}{2}E_1^{*2} \vec{\vec{\vec{F}}}^{NL(3)}_{(1),-1,-1} & E_1^* \vec{\vec{F}}^{NL(2)}_{(0),-1} & \vec{\vec{D}}_{(-1)} + |E_1|^2 \vec{\vec{\vec{F}}}^{NL(3)}_{(-1),1,-1}
\end{pmatrix}
\begin{pmatrix}
\vec{E}_{(1)} \\[2ex]
\vec{E}_{(0)} \\[2ex]
\vec{E}_{(-1)}
\end{pmatrix}
=
\begin{pmatrix}
\vec{0} \\[2ex]
\vec{0} \\[2ex]
\vec{0}
\end{pmatrix}
$$

3.4-(13)
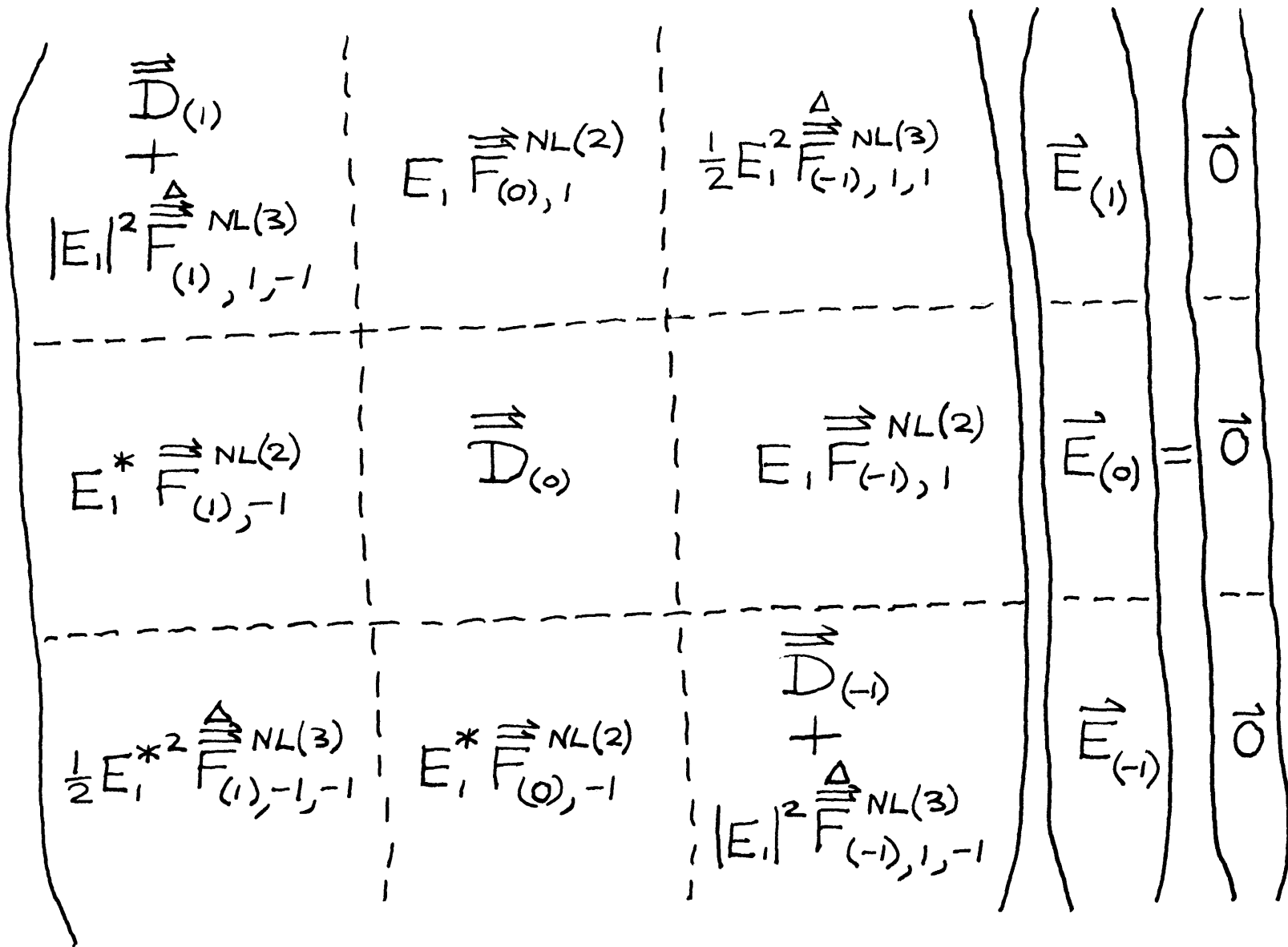
The consistency condition for the set of coupled equation (13) is that the determinant of the be zero. Indeed, the condition that this determinant be zero constitutes precisely the dispersion relation for the coupled system. However, the solution of the full 9×9 determinantal equation will not be attempted. Rather the approach will be to use physical arguments to break (13) in several different ways.

Study the form of (13) further in the light of its intended use for describing laser-driven instabilities. Consider the physical meaning of the various submatrices into which the large matrix on the left-hand side is partitioned. It is profitable to do this even though these submatrices have not yet been explicitly evaluated.

3.4.2 CONNECTION WITH UNCOUPLED MODES

The submatrices $\{D_{(n)}\}$, $n = 1, 0, -1$, are, we recall, just the linear dispersion tensors evaluated at the wavevectors and frequencies $(k_{(n)}, w_{(n)})$. Indeed, were the laser pump to be switched off, the coupled equation (13) would revert to a set of 3 separate uncoupled equations of the form

$$\vec{\vec{D}}_{(n)}^{\text{LIN}} \vec{E}_{(n)} = 0 \qquad n = 1, 0, -1 \qquad\qquad 3.4\text{-}(14)$$

This is to be expected physically since in the absence of the laser pump there is no coupling between sufficiently small (linear) perturbations and therefore no reason for their dispersion relations to differ from the linear dispersion relation of the plasma. Also, because of the choice of bases (4)-(6), the submatrices $\{\vec{\vec{D}}_{(n)}\}$ not only lie on the diagonal of (13) but are themselves diagonal. Thus the vectors $\{\vec{E}_{(n)}\}$, $n = 1, 0, -1$, appearing in (13), display the electric field of each pump-coupled perturbation as a

linear superposition of the corresponding pump-free normal-mode polarizations. For the unmagnetized plasma, as mentioned in section 2.7 and made explicit by (4)-(6), this can be done just from knowledge of the $\{\vec{k}_{(n)}\}$, n = 1, 0, -1, and it does not matter that the $\{w_{(n)}\}$ of the actual pump-coupled perturbations may not lie on the corresponding pump-free normal modes.

3.4.3 CONNECTION WITH UNMODIFIED INSTABILITIES

Consider the submatrices whose existence depends on the 2nd-order nonlinear conductivity of the plasma, and which involve only the fundamental of the laser. They are of the forms

$$E_1 \vec{\vec{F}}^{NL(2)}_{(n),1} \quad , \quad E_1^* \vec{\vec{F}}^{NL(2)}_{(n),-1} \qquad\qquad 3.4\text{-}(15)$$

They lie just off the leading diagonal of (12), and describe the simplest non-trivial coupling brought about by the laser pump. Indeed, since they involve only 2nd-order currents and involve no space-time derivatives, they fall within the simple coupling-of-modes theory of Section 2.4 as well as within the generalized-coupling-of-modes theory of Section 2.5. The submatrices of type (15) together with the submatrices $\{\vec{\vec{D}}_{(n)}\}$ suffice for the description of the so-called "unmodified" laser-driven instabilities. These unmodified laser-driven instabilities arise as follows. First assume that the laser pump has been switched off completely. Then (13) decouples into the 3 vector equations (14). Now there exist values of $(\vec{k}_{(0)}, w_{(0)})$ such that two adjacent members of the set of 3 vector equations (13) have non-trivial solutions simultaneously. That is to say, there exist values of

$(\vec{k}_{(0)}, w_{(0)})$ such that for some choice of the polarizations A and B

Points labelled I are initial loci of
unmodified instability
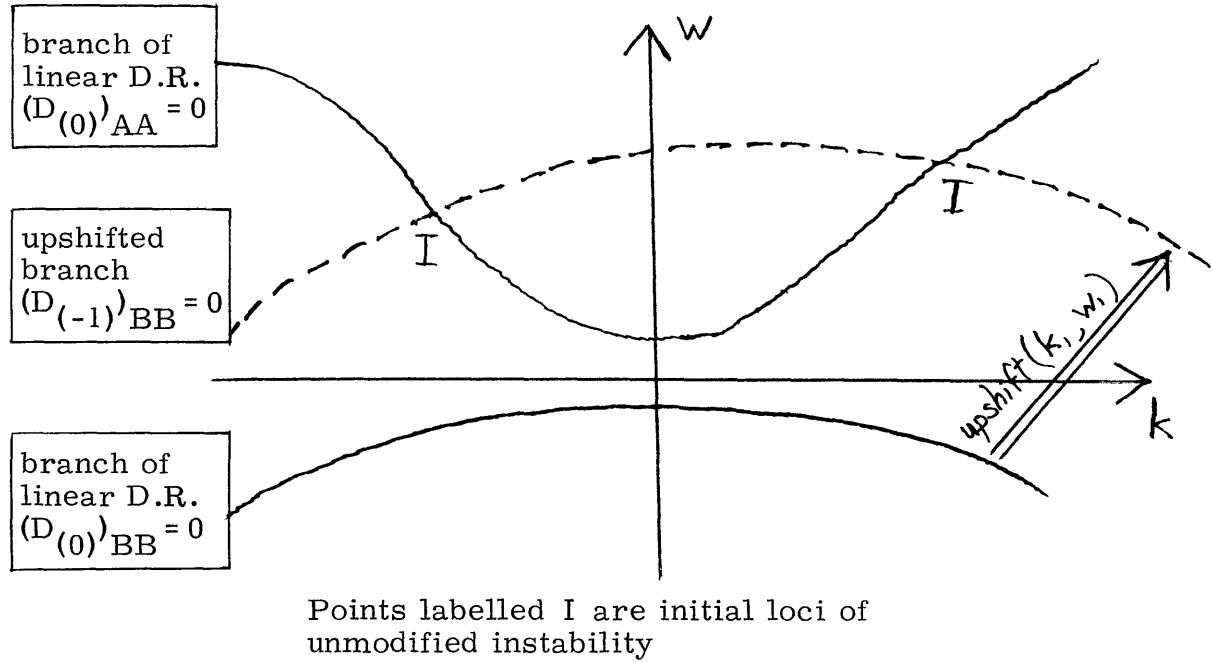
Figure 3.4.F1

$$(D_{(0)})_{AA} E_{A(0)} = 0 \qquad E_{A(0)} \neq 0 \qquad\qquad 3.4\text{-}(16)$$

$$(D_{(-1)})_{BB} E_{B(-1)} = 0 \qquad E_{B(-1)} \neq 0 \qquad\qquad 3.4\text{-}(17)$$

These values are found at points in the wavevector-frequency domain where
a branch of the linear dispersion relation intersects a branch of the same
linear dispersion relation upshifted by the pump wavevector and frequency,
as shown in Figure F1. Note that it makes no sense to use the pump wave-
vector and frequency in locating these intersections, even though the pump
is supposed to be switched off. This is because these intersections become
the initial loci of instability as the laser-pump field is brought up from
zero to some finite level. This is not to say that the problem of temporally

or spatially inhomogeneous pump is being attacked here. Rather the discussion is one of the behavior of solutions of the homogeneous problem, i.e., the roots of (13), as a function of the constant laser amplitude E_1. Returning to (16) and (17), consider what happens to the double root in $(\vec{k}_{(0)}, w_{(0)})$ as E_1, the laser field, is increased from zero to some very small value. The double root splits into 2 closely adjacent roots. There is no need to find these roots from the full 9×9 equation (13); they can be found approximately from the 2×2 equation

$$\begin{pmatrix} (D_{(0)})_{AA} & E_{1x}(F_{(-1),1})_{ABx} \\ E_{1x}^{*}(F_{(0),-1})_{BAx} & D_{(-1)BB} \end{pmatrix} \begin{pmatrix} E_{A(0)} \\ E_{B(-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3.4-(18)

Equating the 2×2 determinant of the matrix to zero, one obtains the dispersion relation for the corresponding "unmodified" laser-driven instability. This dispersion relation is valid for sufficiently low values of the laser-pump field E_{x1}, and correspondingly low growth rates of the coupled perturbations. Higher values of the laser-pump field E_{x1} lead to higher growth rates of the coupled perturbations. When $(D_{(0)})_{AA}$ describes a low-frequency ion-acoustic mode, the growth rate can be comparable with the normal-mode frequency. Then the approximate equation (8) no longer describes the coupled perturbations accurately, and the unmodified laser-driven instability must be replaced by the "modified" laser-driven instability. The modified laser-driven instabilities incorporate the effects of of 3rd-order nonlinear conductivity, to which we now turn.

3.4.4 CONNECTION WITH MODIFIED INSTABILITIES

Consider the submatrices of (13) whose existence depends on the 3rd-order nonlinear conductivity of the plasma, and which further lie on the leading diagonal of (12). These are of the form

$$E_1 E_1^* \overset{=}{F}{}^{NL(3)}_{(n),1,-1} \qquad \qquad 3.4\text{-}(19)$$

These constitute pump-induced self-corrections to the linear dispersion tensor at the perturbation wavevector and frequency. Their importance for laser-driven instabilities arises as follows. Take a low-frequency perturbation described by the ion-acoustic scalar dispersion function $(D_{(0)})_{SS}$, coupled to a high-frequency perturbation described by $(D_{(-1)})_{BB}$;

$$\left| w_{(0)} \right| \ll \left| w_{(-1)} \right| \qquad \qquad 3.4\text{-}(20)$$

Let the pump field be sufficiently strong so that the growth rate of the coupled perturbations is comparable to the ion-acoustic frequency, but still much less than the frequency of the normal mode from which the high-frequency perturbation stems. Then $(D_{(0)})_{SS}$ is large, and the pump-induced self-correction to $(D_{(-1)})_{BB}$ must be included in (18);

$$
\begin{pmatrix}
(D_{(0)})_{SS} & E_{1x}\left(F^{NL(2)}_{(-1),1}\right)_{SBx} \\
E_{1x}^*\left(F^{NL(2)}_{(0),-1}\right)_{BSx} & (D_{(-1)})_{BB} + E_{1x}E_{1x}^*\left(F^{NL(3)}_{(-1),1,-1}\right)_{BBxx}
\end{pmatrix}
\begin{pmatrix}
E_{A(0)} \\
E_{B(-1)}
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

Equating the 2 × 2 determinant of the matrix to zero, one obtains the dispersion relation for the corresponding "modified" laser-driven instability.

3.4.5 CONNECTION WITH THIRD-ORDER INSTABILITIES

Finally, consider the submatrices which occupy the upper-right and lower-left entries in the partitioned matrix (12). These contain terms whose existence depends on the 2nd-order nonlinear conductivity of the plasma, and which involve the 2nd-harmonic component of the laser pump. They also contain terms whose existence depends on the 3rd-order nonlinear conductivity of the plasma, and which involve the laser fundamental. The importance of the submatrices for laser-driven instabilities arises as follows. Start as before from the uncoupled equations (13) which govern the perturbations in the absence of the pump. There exist values of $(\vec{k}_{(0)}, w_{(0)})$ such that the two extreme members of the set of 3 vector equations (13) have nontrivial solutions simultaneously. That is to say, there exist values of $(\vec{k}_{(0)}, w_{(0)})$ such that for some choice of the polarizations A and B,

$$(D_{(1)})_{AA} E_{A(1)} = 0 \qquad E_{A(1)} \neq 0 \qquad\qquad 3.4\text{-}(22)$$

$$(D_{(-1)})_{BB} E_{B(-1)} = 0 \qquad E_{B(-1)} \neq 0 \qquad\qquad 3.4\text{-}(23)$$

These values are found at points in the wavevector-frequency domain where a branch of the linear dispersion relation downshifted by the pump wavevector and frequency intersects a branch of the linear dispersion relation upshifted by the pump wavevector and frequency, as shown in Figure F2. Again these intersections become the initial loci of instability as the laser-pump field is brought up from zero to some finite level. Again the discussion implies no temporal or spatial inhomogeneity, but merely concerns the behavior of solutions of the homogeneous problem (12) as a function of the laser field E_1. Returning to (22) and (23), consider what happens to

$$
\begin{pmatrix}
(D_{(1)})_{AA} + |E'_{1x}|^2 (\overset{\Delta}{F}{}^{NL(3)}_{(1),1,-1})_{AAxx} & E_{1x}(F^{NL(2)}_{(0),1})_{ASx} & E^2_{1x}(\overset{\Delta}{F}{}^{NL(3)}_{(-1),1,1})_{ABxx} \\[2em]
E^*_{1x}(F^{NL(2)}_{(1),-1})_{SAx} & (D_{(0)})_{SS} & E_{1x}(F^{NL(2)}_{(-1),1})_{SBx} \\[2em]
E^{*2}_{1x}(F^{NL(3)}_{(1),-1,-1})_{BAxx} & E^*_{1x}(F^{NL(2)}_{(0),-1})_{BSx} & (D_{(-1)})_{BB} + |E_{1x}|^2 (\overset{\Delta}{F}{}^{NL(3)}_{(-1),1,-1})_{BBxx}
\end{pmatrix}
\begin{pmatrix}
E_{A(1)} \\[2em]
E_{S(0)} \\[2em]
E_{B(-1)}
\end{pmatrix}
=
\begin{pmatrix}
0 \\[2em]
0 \\[2em]
0
\end{pmatrix}
$$

3.4-(24)

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Branch of linear $disp^n rel^n$:
$(D_{(0)})_{AA} = 0$

Branch downshifted by (k_1, w_1) :
$(D_{(1)})_{AA} = 0$

Upshifted Branch :
$(D_{(-1)})_{BB} = 0$

Branch of linear $disp^n rel^n$:
$(D_{(0)})_{BB} = 0$

w

k

downshift $(-k_1, -w_1)$

upshift (k_1, w_1)

I

I
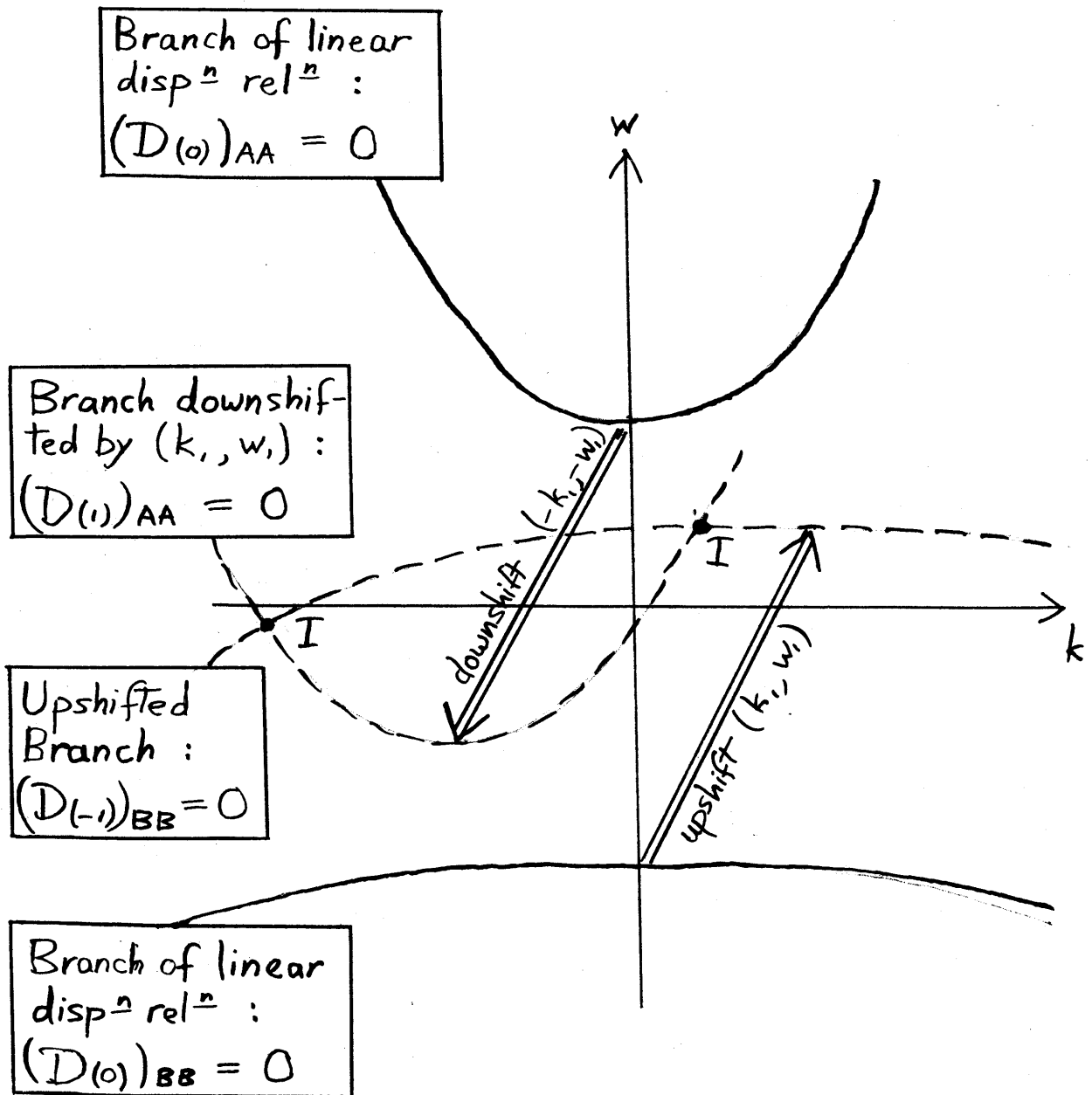
Points labelled I are initial loci of 3rd-order instability

Figure 3.4 F2

the double root in $(\vec{k}_{(0)}, w_{(0)})$ as E_1, the laser field, is increased from zero to some very small value. At first one might think that the roots could be found approximately from the 2 X 2 equation

$$\begin{pmatrix} {}^{(D}{}_{(1)}{}^){}_{AA} & E_{1x}^2 \left({}^{\triangle}_{F}{}^{NL(3)}_{(-1),1,1} \right)_{ABxx} \\ E_{1x}^{*2} \left({}^{\triangle}_{F}{}^{NL(3)}_{(1),-1,-1} \right)_{BAxx} & {}^{(D}{}_{(-1)}{}^)_{BB} \end{pmatrix} \begin{pmatrix} E_{A(1)} \\ E_{B(-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by analogy with (17). However, since this direct coupling is via matrix elements of order E_1^2, it is of the same order as indirect coupling via the intermediate low-frequency wave. Further it is of the same order as the self-corrections to ${}^{(D}{}_{(1)}{}^)_{AA}, {}^{(D}{}_{(-1)}{}^)_{BB}$. Thus even for very small laser-pump field intensities, the course of the double root $(\vec{k}_{(0)}, w_{(0)})$ of (22) and (23) can only be followed using the equation (24). Equating the 3 X 3 determinant of the matrix in (24) to zero, one obtains the dispersion relation for the corresponding laser-driven instability, which is "intrinsically 3rd-order" in that 2nd-order nonlinear conductivity will not suffice to describe it even at very low laser-pump field amplitudes. Note that the low-frequency wave need not lie on or near the ion-acoustic dispersion relation.

3.4.6 RELATION TO LATER SECTIONS

The preceding discussion provides the physical motivation for evaluating the submatrices of (13) in detail. In section 3.5, the general forms for warm-fluid coupling coefficients are used to evaluate the submatrices. Appropriate physical approximations are used to eliminate terms not important for laser-driven instabilities in unmagnetized plasma, and thus

justify the replacement of (12) by (13). The details are relegated to Appendix A5. The remaining physically important parts of the submatrices are evaluated again in section 3.6 by a quicker, but less general, alternative method. The self-consistent pump-wave equilibrium is modeled by a simplified oscillating equilibrium with harmonic content removed. The fluid equations are linearized about this oscillating equilibrium and these linearized fluid equations are used directly to find the behavior of small perturbations in the presence of the pump wave.

The preceding discussion has provided a rather general division of laser-drived instabilities into unmodified instabilities, modified instabilities, and intrinsically 3rd-order instabilities. This general division will be used as a guide in section 3.8, where specific instabilities will be characterized by the polarization, wavevector and frequency of the interacting waves. The corresponding reduced matrix equations, of types (18), (21), or (24) as the case may be, will furnish the determinantal equations which constitute the dispersion relations for the specific instabilities.

3.5 Coupling Coefficients from General Formulas of Section 2.8

The submatrices of the partitioned matrix equation 3.4-(13) will here be evaluated by using the general formula 2.8-(27) for the linear dispersion tensor, the general formula 2.8-(43) for the 2nd-order nonlinear coupling coefficients, and the general formula 2.8-(52) for the 3rd-order nonlinear coupling coefficients. Any intermediate quantities appearing in these formulae will be calculated from other equations of section 2.8 as required. The formulae 2.8-(27,43,52) are used omitting the steady magnetic field B_0 and omitting the drift velocity $\vec{v}_0$. Only the electron contribution to

the nonlinear coupling coefficients is included, although both electron and ion contributions to each linear dispersion tensor are included, at least initially.

Consider the entries in the partitioned matrix 3.4-(13) in increasing order of nonlinearity. Because of the choice of polarization bases 3.4-(4,5,6), the linear dispersion tensors appearing in 3.4-(13) acquire a diagonal representation, with entries given by 3.4-(2,3);

$$
\vec{\vec{D}}_{(n)}^{LIN} =
\begin{pmatrix}
-\dfrac{k_{(n)}^2}{\mu_0 \epsilon_0 w_{(n)}^2} + 1 - \displaystyle\sum_\pi \dfrac{w_p^2}{w_{(n)}^2} & 0 & 0 \\[4ex]
0 & -\dfrac{k_{(n)}^2}{\mu_0 \epsilon_0 w_{(n)}^2} + 1 - \displaystyle\sum_\pi \dfrac{w_p^2}{w_{(n)}^2} & 0 \\[4ex]
0 & 0 & 1 - \displaystyle\sum_\pi \dfrac{w_{p\pi}^2}{w_{(n)}^2 - \gamma_\pi k_{(n)}^2 v_{T\pi}^2}
\end{pmatrix}
$$

$$3.5\text{-}(1)$$

The 2nd-order nonlinear entries in 3.4-(13) are evaluated from 2.8-(43) with the simplifications that $\vec{B}_0 = \vec{0}$, $\vec{v}_0 = \vec{0}$, and that $n_a/n_0 = 0$ for any wave $\vec{e}_a$ polarized transversely to its own direction of propagation. Taking only the electron contributions, one obtains

$$E_{x1}\left(F^{NL(2)}_{(n),\,1}\right)_{ABx} = -\frac{mn_0}{\epsilon_0}\left\{\sum_n^{\overline{A(n+1)}}\frac{\overline{A(n+1)}}{n_0}\vec{v}_{B(n)}\cdot\vec{V}_{x1}\right.$$

$$\left. +\frac{n_{B(n)}}{n_0}\vec{v}_{\overline{A(n+1)}}\cdot\vec{V}_{x1}\right\} \qquad 3.5\text{-}(2)$$

$$E^*_{x1}\left(F^{NL(2)}_{(n),\,-1}\right)_{ABx} = -\frac{mn_0}{\epsilon_0}\left\{\sum_n^{\overline{A(n-1)}}\frac{\overline{A(n-1)}}{n_0}\vec{v}_{B(n)}\cdot\vec{V}^*_{x1}\right.$$

$$\left. +\frac{n_{B(n)}}{n_0}\vec{v}_{\overline{A(n-1)}}\cdot\vec{V}^*_{x1}\right\} \qquad 3.5\text{-}(3)$$

To calculate the normalized velocities $\{\vec{v}\}$ and densities $\{n\}$, refer to equations (45)-(50) of section 2.8. In the absence of $\vec{B}_0$ and $\vec{v}_0$, the dimensionless mobility tensor $\overset{\leftrightarrow}{M}$ as well as the dimensionless dispersion tensor $\overset{\leftrightarrow}{D}^{LIN}$ is diagonal with our choice of basis polarizations. Thus for any unit basis polarization vector $\vec{e}_{Cc}$, say, the linear response $\vec{v}_{Cc}$ lies in the direction of $\vec{e}_{Cc}$ and satisfies

$$\vec{v}_{Cc} = \frac{iq}{mw_c}\vec{e}_{Cc} \qquad , \qquad \vec{e}_{Cc}\perp\vec{k}_c \qquad 3.5\text{-}(6)$$

$$= \frac{iqw_c}{m\left(w_c^2 - \gamma k_c^2 v_T^2\right)} \qquad \vec{e}_{Cc} \quad , \quad \vec{e}_{Cc}\parallel\vec{k}_c \qquad 3.5\text{-}(7)$$

$$n_{CC} = 0 \qquad , \qquad \vec{e}_{Cc}\perp\vec{k}_c \qquad 3.5\text{-}(8)$$

$$= \frac{iqk_c}{m\left(w_c^2 - \gamma k_c^2 v_T^2\right)} \qquad , \qquad \vec{e}_{Cc}\parallel\vec{k}_c \qquad 3.5\text{-}(9)$$

$$E_1 \overset{\Rightarrow}{F}_{(n),1}^{NL(2)} = -w_P^2 V_1$$

$$
\begin{pmatrix}
0 & 0 & \left[k_{(n)}\left(\vec{e}_{M(n+1)} \cdot \vec{e}_{x1}\right)\right] \Big/ \left[\left(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2\right) w_{(n+1)}\right] \\
0 & 0 & 0 \\
\dfrac{\left[k_{(n+1)}\left(\vec{e}_{M(n)} \cdot \vec{e}_{x1}\right)\right]}{\left[\left(w_{(n+1)}^2 - \gamma k_{(n+1)}^2 V_T^2\right) w_{(n)}\right]} & 0 & \dfrac{\left[k_{(n+1)} w_{(n)}\left(\vec{e}_{S(n)} \cdot \vec{e}_{x1}\right) + k_{(n)} w_{(n+1)}\left(\vec{e}_{S(n+1)} \cdot \vec{e}_{x1}\right)\right]}{\left[\left(w_{(n+1)}^2 - \gamma k_{(n+1)}^2 V_T^2\right)\left(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2\right)\right]}
\end{pmatrix}
$$

$$\vec{E}_1^* \overrightarrow{F}_{(n),-1}^{NL(2)} = -w_P^2 \vec{V}_1^*$$

$$
\begin{pmatrix}
0 & 0 & \dfrac{[k_{(n)}(\vec{e}_{M(n-1)} \cdot \vec{e}_{x1})]}{[(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2) w_{(n-1)}]} \\[4mm]
0 & 0 & 0 \\[4mm]
\dfrac{[k_{(n-1)}(\vec{e}_{M(n)} \cdot \vec{e}_{x1})]}{[(w_{(n-1)}^2 - \gamma k_{(n-1)}^2 V_T^2) w_{(n)}]} & 0 & \dfrac{[k_{(n-1)} w_{(n)}(\vec{e}_{S(n)} \cdot \vec{e}_{x1}) + k_{(n)} w_{(n-1)}(\vec{e}_{S(n-1)} \cdot \vec{e}_{x1})]}{[(w_{(n-1)}^2 - \gamma k_{(n-1)}^2 V_T^2)(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2)]}
\end{pmatrix}
$$

3.5-(i)

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$$|E_1|^2 \,\overset{\triangle}{\underset{\Rightarrow}{F}}{}^{NL(3)}_{(\pm 1),1,-1} = -w_p^2 |V_1|^2$$

$$
\begin{pmatrix}
\dfrac{[k_{(0)}^2 (\vec{e}_{M(\pm 1)} \cdot \vec{e}_{x1})^2]}{[w_{(\pm 1)}^2 (w_{(0)}^2 - \gamma k_{(0)}^2 V_T^2)]} & 0 & \dfrac{[k_{(0)}^2 w_{(\pm 1)}^2 (\vec{e}_{M(\pm 1)} \cdot \vec{e}_{x1})(\vec{e}_{S(\pm 1)} \cdot \vec{e}_{x1}) + k_{(\pm 1)} k_{(0)} w_{(\pm 1)} w_{(0)} (\vec{e}_{M(\pm 1)} \cdot \vec{e}_{x1})(\vec{e}_{S(0)} \cdot \vec{e}_{x1})]}{[w_{(\pm 1)}^2 (w_{(\pm 1)}^2 - \gamma k_{(\pm 1)}^2 V_T^2)(w_{(0)}^2 - \gamma k_{(0)}^2 V_T^2)]} \\[2ex]
0 & 0 & 0 \\[2ex]
\dfrac{[k_{(0)}^2 w_{(\pm 1)}^2 (\vec{e}_{M(\pm 1)} \cdot \vec{e}_{x1})(\vec{e}_{S(\pm 1)} \cdot \vec{e}_{x1}) + k_{(0)} k_{(\pm 1)} w_{(0)} w_{(\pm 1)} (\vec{e}_{M(\pm 1)} \cdot \vec{e}_{x1})(\vec{e}_{S(0)} \cdot e_{x1})]}{[w_{(\pm 1)}^2 (w_{(\pm 1)}^2 - \gamma k_{(\pm 1)}^2 V_T^2)(w_{(0)}^2 - \gamma k_{(0)}^2 V_T^2)]} & 0 & \dfrac{[k_{(0)}^2 w_{(\pm 1)}^2 (\vec{e}_{S(\pm 1)} \cdot \vec{e}_{x1})^2 + 2 k_{(0)} k_{(\pm 1)} w_{(0)} w_{(\pm 1)} (\vec{e}_{S(\pm 1)} \cdot e_{x1})(\vec{e}_{S(0)} \cdot \vec{e}_{x1}) + k_{(\pm 1)}^2 \gamma k_{(0)}^2 V_T^2 (\vec{e}_{S(0)} \cdot \vec{e}_{x1})^2]}{[w_{(\pm 1)}^2 (w_{(\pm 1)}^2 - \gamma k_{(\pm 1)}^2 V_T^2)(w_{(0)}^2 - \gamma k_{(0)}^2 V_T^2)]}
\end{pmatrix}
$$

3.5-(12)

Therefore the dot products of velocities occurring in (2) and (3), which embody the effects of the 3-dimensional geometry of the problem, translate immediately into dot products of polarization vectors. Therefore those submatrices in 3.4-(13) which describe second-order coupling can be written down at once. The second-order coupling which involves only the fundamental of the laser pump is, from (2) and (3), described by the submatrices (10) and (11).

The 3rd-order nonlinear entries in 3.4-(13) are evaluated from 2.8-(52) taking advantage of the facts that $\vec{B}_0 = 0$, and that $n_a/n_0 = 0$ for any wave $\vec{e}_a$ polarized transversely to its own direction of propagation. Also no nonlinear response exists with both zero wavevector and zero frequency. The details of the calculation have been relegated to Appendix A5.

The results for the submatrices appearing on the leading diagonal of 3.4-(13) are shown in (12). This expression is a good approximation to the coupling quantity

$$\left|E_1\right|^2 \vec{\vec{F}}^{NL(3)}_{(\pm 1), 1, -1}$$

provided that the phase velocities of the perturbations and the thermal velocity of the electron together satisfy the inequalities 3.3-(29)-(32). These inequalities are in fact satisfied for all laser-driven instabilities of interest. Note that these inequalities do not necessitate a phase-velocity disparity between $\vec{E}_{(0)}$ and $\vec{E}_{(-1)}$. Use is made of 3.3-(29)-(32) to show that there does, however, exist a phase-velocity disparity

$$\frac{w_{(0)}}{k_{(0)}} \ll \left|\frac{w_{(\pm 2)}}{k_{(\pm 2)}}\right| \qquad\qquad 3.5\text{-}(13)$$

and an inequality

$$v_T \ll \left| \frac{w_{(\pm 2)}}{k_{(\pm 2)}} \right|$$ 3.5-(14)

Then processes which involve the high frequency of the plasma at $(k_{(\pm 2)}$, $w_{(\pm 2)})$ can be neglected in calculating (see Appendix A5) the pump-induced self-corrections to the linear dispersion tensors. The result is the expression (12). These submatrices on the diagonal of 3.4-(13) comprise the pump-induced self-corrections to the linear dispersion tensors at the perturbation wavevectors and frequencies. Their importance for laser-driven instabilities was discussed in subsections 3.4.4 and 3.4.5.

The submatrices in the top-right and bottom-left corners of 3.4-(13) comprise pump-induced coupling between perturbations separated by twice the pump wavevector and frequency. Their role in laser-driven instabilities was discussed in subsection 3.4.5. These submatrices are calculated in Appendix A5 and the results are shown in (15) and (16). Again it can be shown (see Appendix A5) that the expressions displayed in (15),(16) are good approximations to the coupling quantities

$$E_1^2 \overset{\rightleftharpoons}{F}_{(-1),1,1}^{NL(3)}$$

$$E_1^{*2} \overset{\rightleftharpoons}{F}_{(1),-1,-1}^{NL(3)}$$

Again one uses only the inequalities 3.3-(29)-(32) which govern laser-driven instabilities. The processes which involve the high-frequency response of the plasma at $(\vec{k}_{\pm 2}, w_{\pm 2})$ are neglected in calculating the pump-induced coupling between $\vec{E}_{(-1)}$ and $\vec{E}_{(1)}$. This results in the expressions

$$\frac{1}{2}\vec{E}_{x1}^2 \stackrel{\triangle}{\underset{=}{F}}_{NL(3)}^{(-1),1,1} = -w_P^2 V_1^2$$

$$
\begin{pmatrix}
\dfrac{[k_{(o)}^2(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})]}{[(w_{(o)}^2-\gamma k_{(o)}^2 V_T^2)w_{(1)}w_{(-1)}]} & 0 & \dfrac{\left[k_{(o)}^2 w_{(-1)}(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})+\ k_{(o)}k_{(-1)}w_{(o)}(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(o)}\cdot\vec{e}_{x1})\right]}{[w_{(1)}(w_{(o)}^2-\gamma k_{(o)}^2 V_T^2)(w_{(-1)}^2-\gamma k_{(-1)}^2 V_T^2)]} \\[4pt]
0 & 0 & 0 \\[4pt]
\dfrac{\left[k_{(o)}^2 w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})+\ k_{(o)}k_{(1)}w_{(o)}(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})(\vec{e}_{S(o)}\cdot\vec{e}_{x1})\right]}{[w_{(-1)}(w_{(o)}^2-\gamma k_{(o)}^2 V_T^2)(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2)]} & 0 & \dfrac{\left[k_{(o)}^2 w_{(1)}w_{(-1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})+\ k_{(o)}k_{(-1)}w_{(o)}w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(o)}\cdot\vec{e}_{x1})+\ k_{(o)}k_{(1)}w_{(o)}w_{(-1)}(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})(\vec{e}_{S(o)}\cdot\vec{e}_{x1})+\ \gamma k_{(o)}^2 V_T^2 k_{(1)}k_{(-1)}(\vec{e}_{S(o)}\cdot\vec{e}_{x1})^2\right]}{\left[(w_{(o)}^2-\gamma k_{(o)}^2 V_T^2)(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2)(w_{(-1)}^2-\gamma k_{(-1)}^2 V_T^2)\right]}
\end{pmatrix}
$$

$$\frac{1}{2}E_{x1}^{*2} \stackrel{\triangle}{=\!=\!=} F^{NL(3)}_{(1),-1,-1} = -w_P^2 V_1^{*2}$$

$$
\begin{pmatrix}
\dfrac{[k_{(0)}^2(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})]}{[(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2)w_{(1)}w_{(-1)}]} & 0 & \dfrac{[k_{(0)}^2 w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})+k_{(0)}k_{(1)}w_{(0)}(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})]}{[w_{(-1)}(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2)(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2)]} \\[4ex]
0 & 0 & 0 \\[2ex]
\dfrac{[k_{(0)}^2 w_{(-1)}(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})+k_{(0)}k_{(-1)}w_{(0)}(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})]}{[w_{(1)}(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2)(w_{(-1)}^2-\gamma k_{(-1)}^2 V_T^2)]} & 0 & \dfrac{\begin{array}{l}[k_{(0)}^2 w_{(1)}w_{(-1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})+\\ k_{(0)}k_{(-1)}w_{(0)}w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})+\\ k_{(0)}k_{(1)}w_{(0)}w_{(-1)}(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})+\\ \gamma k_{(0)}^2 V_T^2 k_{(1)}k_{(-1)}(\vec{e}_{S(0)}\cdot\vec{e}_{x1})^2]\end{array}}{\begin{array}{l}[(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2)(w_{(1)}^2-\gamma k_{(1)}V_T^2)\\ (w_{(-1)}^2-\gamma k_{(-1)}V_T^2)]\end{array}}
\end{pmatrix}
$$

(15) and (16).

The physically useful part, then, of the partitioned matrix equation 3.4-(12) is the partitioned matrix equation 3.4-(13). The equation 3.4-(13) describes perturbations coupled together by the laser fundamental via processes involving the plasma response at $(\vec{k}_{(0)}, w_{(0)})$ and at $(\vec{k}_{(\pm1)}, w_{(\pm1)})$. Present in the original equation 3.4-(12), but omitted on physical grounds, are coupling via the laser 2nd harmonic, and coupling via processes involving the plasma response at $(\vec{k}_{\pm2}, w_{\pm2})$ and $(\vec{k}_{(\pm2)}, w_{(\pm2)})$. The physical grounds in question are the approximations 3.3-(29)-(32) which we repeat for emphasis:

$$\left| \frac{w_{(0)}}{k_{(0)}} \right| \ll c$$

$$v_T \ll c$$

$$v_T \ll \left| \frac{w_{(\pm1)}}{k_{(\pm1)}} \right|$$

$$\left| \frac{w_{(0)}}{k_{(0)}} \right| \lesssim \left| \frac{w_{(\pm1)}}{k_{(\pm1)}} \right|$$

The last of these does not necessitate a phase-velocity disparity between perturbations $\vec{E}_{(0)}, \vec{E}_{(-1)}$, unlike the case of the theory of Kaw, White, et al. Also present in the original equation 3.4-(12) but omitted from (30) is the self-correction to the low-frequency perturbation. This low-frequency self-correction does not contribute to the structure of the laser-driven instabilities, as was discussed in the latter part of section 3.4.

Equation 3.4-(13), describing the physically important part of the

coupling between perturbations brought about by the laser pump, can also be derived from a simpler approach. This approach is set out in section 3.6 and involves expanding the fluid equations directly about a simplified version of the oscillating equilibrium set up by the pump. This approach is worth setting out for the sake of comparison with the present section and with other treatments of laser-driven instabilities appearing in the literature.

Whether equation 3.4-(13) be arrived at from the general forms for nonlinear coupling coefficients, as in this section, or by expanding about an oscillating equilibrium, as in section 3.6, it has one very convenient feature. Those submatrices within the partitioned matrix which describe nonlinear coupling each have zero in the row and column describing coupling to the polarizations $\vec{e}_{N(n)}$. The polarization components $E_{N(1)}$, $E_{N(0)}$, $E_{N(-1)}$ are therefore decoupled. Equation (30), which is a 9×9 matrix equation, factors into the 3 uncoupled equations

$$(D_{(n)})_{NN} E_{N(n)} = 0 \qquad n = 1, 0, -1 \qquad\qquad 3.5\text{-}(31)$$

which describe electromagnetic perturbations polarized perpendicularly to the laser pump, and a 6×6 matrix equation. The 6×6 matrix equation describes laser-induced coupling between electrostatic perturbation and electromagnetic perturbations polarized as nearly as possible along the laser polarization direction. This reduction to a 6×6 equation was the motive for choosing the polarization vectors as in 3.4-(4,5,6).

3.6 Coupling Coefficients from a Differential Equation Approach

In section 3.5 certain coupling coefficients, important for laser-driven instabilities, were evaluated from their general forms found in section 2.8. In this section the same coupling coefficients are found by a somewhat more ad hoc approach, akin to certain other treatments of laser-driven instabilities appearing in the literature. This more direct approach yields the same values for the coupling coefficients to within the physical approximations 3.3-(29)-(32).

The "differential equation approach" to deriving coupling coefficients proceeds as follows. The steady magnetic field and the drift velocities are set to zero ab initio. The warm-fluid equations 2.8-(12,13,14)

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \vec{x}} \cdot n\vec{v} = 0 \qquad\qquad 3.6-(1)$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \frac{\partial \vec{v}}{\partial \vec{x}} + \frac{v_{T0}^2}{n_0} \left(\frac{n}{n_0}\right)^{\gamma-2} \frac{\partial n}{\partial \vec{x}} = \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B}) \qquad\qquad 3.6-(2)$$

$$J = qn\vec{v} \qquad\qquad 3.6-(3)$$

are expanded directly about a simplified version of the oscillating equilibrium set up by the pump. A small perturbing electric field at one wavevector and frequency $(\vec{k}_{(n)}, w_{(n)})$ then excites nonlinear currents at $(\vec{k}_{(n)},$ $w_{(n)})$, $(\vec{k}_{(n\pm1)}, w_{(n\pm1)})$, $(\vec{k}_{(n\pm2)}, w_{(n\pm2)})$, and so on. These nonlinear currents are found directly and yield the coupling coefficients of 3.5. More strictly, they yield the coupling coefficients

$$E_1 \vec{\vec{F}}^{NL(2)}_{(n),1}, \qquad E_1^* \vec{\vec{F}}^{NL(2)}_{(n),-1}, \qquad |E_1|^2 \vec{\vec{F}}^{NL(3)}_{(n),1,-1} \qquad\qquad 3.6-(4)$$

exactly; they yield for the coupling coefficients

$$\frac{1}{2} E_1^2 \overset{\rightrightarrows}{F}_{(-1),\,1,\,1}^{NL(3)} + E_2 \vec{F}_{(-1),\,2}^{NL(2)}, \quad \frac{1}{2} E_1^{*2} \overset{\rightrightarrows}{F}_{(1),\,-1,\,1}^{NL(3)} + E_2^* \vec{F}_{(1),\,-2}^{NL(2)}$$

$$3.6\text{-}(5)$$

only their physically important parts, namely, in the notation of section 3.5,

$$\frac{1}{2} E_1^2 \overset{\triangle}{\underset{}{\overset{\rightrightarrows}{F}}}_{(-1),\,1,\,1}^{NL(3)}, \quad \frac{1}{2} E_1^{*2} \overset{\triangle}{\underset{}{\overset{\rightrightarrows}{F}}}_{(1),\,-1,\,1}^{NL(3)}$$

$$3.6\text{-}(6)$$

These results are satisfactory within the physical approximations 3.3-(29)-(32). They suffice for the description of laser-driven instabilities embodied in equation 3.4-(13), which is in fact the description to be used in later sections.

The oscillating equilibrium set up by the pump is approximated as follows. The steady magnetic field and the drift velocity for each species are taken to be zero. The electric field of the laser is approximated by its fundamental component (cf. 3.3-(1))

$$\vec{E}_{FUND}(\vec{x}, t) = E_1 \, e^{i\vec{k}_1 \cdot \vec{x} - i w_1 t} + \text{complex conjugate}$$

$$3.6\text{-}(7)$$

The resultant quivering motion of the electrons of the plasma is approximated by its fundamental component (cf. 3.3-(2))

$$\vec{v}_{FUND}(\vec{x}, t) = \vec{V}_1 \, e^{i\vec{k}_1 \cdot \vec{x} - i w_1 t} + \text{complex conjugate}$$

$$3.6\text{-}(8)$$

The fundamental component of the quivering motion is itself approximated by the linear response to the fundamental component of the field (cf. 3.3-(3))

$$\vec{V}_1 = \frac{iq_e \vec{E}_1}{m_e w_1} \qquad\qquad 3.6\text{-}(9)$$

The warm-fluid equations (1),(2),(3) are linearized about the approximate oscillating equilibrium (7),(8),(9) just as if it were the exact oscillating equilibrium. The resulting small-signal equations are, for each species in the plasma,

$$\frac{\partial n_{SMALL}}{\partial t} + \frac{\partial}{\partial \vec{x}} \cdot (n_{SMALL}\vec{v}_{FUND} + n_0 \vec{v}_{SMALL}) = 0 \qquad\qquad 3.6\text{-}(10)$$

$$\frac{\partial \vec{v}_{SMALL}}{\partial t} + \vec{v}_{SMALL} \cdot \frac{\partial \vec{v}_{FUND}}{\partial \vec{x}} + \vec{v}_{FUND} \cdot \frac{\partial \vec{v}_{SMALL}}{\partial \vec{x}} + \frac{\gamma v_{T0}^2}{n_0} \frac{\partial n_{SMALL}}{\partial \vec{x}}$$

$$= \frac{q}{m} (\vec{E}_{SMALL} + \vec{v}_{SMALL} \times \vec{B}_{FUND} + \vec{v}_{FUND} \times \vec{B}_{SMALL}) \qquad 3.6\text{-}(11)$$

$$\vec{J} = q(n_{SMALL}\vec{v}_{FUND} + n_0\vec{v}_{SMALL}) \qquad\qquad 3.6\text{-}(12)$$

Here the non-steady magnetic fields are subsidiary quantities defined in terms of electric fields by

$$-\dot{\vec{B}}_{FUND} = \nabla \times \vec{E}_{FUND} \qquad\qquad 3.6\text{-}(13)$$

$$-\dot{\vec{B}}_{SMALL} = \nabla \times \vec{E}_{SMALL} \qquad\qquad 3.6\text{-}(14)$$

The differential equations (10),(11),(12) describe the response of the pump-modulated oscillating medium to a small perturbing field. In particular, a small perturbing field $\vec{E}_{(n)}$ at one wavevector and frequency $(\vec{k}_{(n)}, w_{(n)})$ excites a response in the form of currents at $(\vec{k}_{(n)}, w_{(n)})$, $(\vec{k}_{(n\pm1)}, w_{(n\pm1)})$, $(\vec{k}_{(n\pm2)}, w_{(n\pm2)})$, and so on. These currents are calculated in Appendix A6,

and the results are as follows. The perturbation field $\vec{E}_{(n)}$ gives rise to a first-order current at $(\vec{k}_{(n)}, w_{(n)})$:

$$\frac{i\vec{J}_{(n)}}{\epsilon_0 w_{(n)}} = -\frac{w_p^2}{w_{(n)}^2}\left(1 + \frac{\gamma v_T^2 \vec{k}_{(n)}\vec{k}_{(n)}}{w_{(n)}^2 - \gamma k_{(n)}^2 v_T^2}\right) \cdot \vec{E}_{(n)} \qquad 3.6-(15)$$

The perturbation field $E_{(n)}(\vec{k}_{(n)}, w_{(n)})$, together with the positive frequency component of the pump, gives rise to a 2nd-order current at $(\vec{k}_{(n+1)}, w_{(n+1)})$:

$$\frac{iJ_{(n),1}}{\epsilon_0 w_{(n+1)}} = -\frac{w_p^2}{w_{(n)} w_{(n+1)}}\left(1 + \frac{\gamma v_T^2 \vec{k}_{(n+1)}\vec{k}_{(n+1)}}{w_{(n+1)}^2 - \gamma k_{(n+1)}^2 v_T^2}\right)$$

$$\cdot \left(\frac{\vec{V}_1 \vec{k}_{(n)}}{w_{(n)}} + \frac{\vec{k}_{(n+1)}\vec{V}_1}{w_{(n+1)}}\right) \cdot \left(1 + \frac{\gamma v_T^2 \vec{k}_{(n)}\vec{k}_{(n)}}{w_{(n)}^2 - \gamma k_{(n)}^2 v_T^2}\right) \cdot \vec{E}_{(n)}$$

$$3.6-(16)$$

The perturbation field $\vec{E}_{(n)}(k_{(n)}, w_{(n)})$, together with the negative frequency component of the pump, gives rise to a 2nd-order current at $(\vec{k}_{(n-1)}, w_{(n-1)})$:

$$\frac{iJ_{(n),-1}}{\epsilon_0 w_{(n-1)}} = -\frac{w_p^2}{w_{(n)} w_{(n-1)}}\left(1 + \frac{\gamma v_T^2 \vec{k}_{(n-1)}\vec{k}_{(n-1)}}{w_{(n-1)}^2 - k_{(n-1)}^2 v_T^2}\right)$$

$$\cdot \left(\frac{\vec{V}_1^* \vec{k}_{(n)}}{w_{(n)}} + \frac{\vec{k}_{(n-1)}\vec{V}_1^*}{w_{(n+1)}}\right) \cdot \left(1 + \frac{\gamma v_T^2 \vec{k}_{(n)}\vec{k}_{(n)}}{w_{(n)}^2 - \gamma k_{(n)}^2 v_T^2}\right) \cdot \vec{E}_{(n)}$$

$$3.6-(17)$$

$\vec{E}_{(n)}(\vec{k}_{(n)}, w_{(n)})$ and the positive-frequency pump component acting twice

give rise to a third-order current at $(\vec{k}_{(n+2)}, w_{(n+2)})$:

$$\frac{iJ_{(n),1,1}}{\epsilon_0 w_{(n+2)}} = -\frac{w_p^2}{w_{(n)}w_{(n+2)}}\left(1 + \frac{\gamma v_T^2 \vec{k}_{(n+2)}\vec{k}_{(n+2)}}{w_{(n+2)}^2 - \gamma k_{(n+2)}^2 v_T^2}\right) \cdot \left[\frac{1}{w_{(n+1)}^2 - \gamma k_{(n+1)}^2 v_T^2}\right.$$

$$\left(\vec{V}_1 k_{(n+1)}^2 \vec{V}_1 + \frac{\vec{k}_{(n+2)}}{w_{(n+2)}}\gamma v_T^2(\vec{k}_{(n+1)}\cdot\vec{V}_1)^2 \frac{\vec{k}_{(n)}}{w_{(n)}}\right.$$

$$+ V_1 w_{(n+1)}(\vec{k}_{(n+1)}\cdot\vec{V}_1)\frac{\vec{k}_{(n)}}{w_{(n)}}$$

$$\left.\left.+ \frac{\vec{k}_{(n+2)}}{w_{(n+2)}}w_{(n+1)}(\vec{k}_{(n+1)}\cdot\vec{V}_1)\vec{V}_1\right)\right] \cdot \left(1 + \frac{\gamma v_T^2 \vec{k}_{(n)}\vec{k}_{(n)}}{w_{(n)}^2 - \gamma k_{(n)}^2 v_T^2}\right)\cdot\vec{E}_{(n)}$$

$$3.6-(18)$$

$\vec{E}_{(n)}(\vec{k}_{(n)}, w_{(n)})$ and the negative-frequency pump-component acting twice give rise to a third-order current at $(\vec{k}_{(n-2)}, w_{(n-2)})$:

$$\frac{iJ_{(n),-1,-1}}{\epsilon_0 w_{(n-2)}} = -\frac{w_p^2}{w_{(n)}w_{(n-2)}}\left(1 + \frac{\gamma v_T^2 \vec{k}_{(n-2)}\vec{k}_{(n-2)}}{w_{(n-2)}^2 - \gamma k_{(n-2)}^2 v_T^2}\right) \cdot \left[\frac{1}{w_{(n-1)}^2 - \gamma k_{(n-1)}^2 v_T^2}\right.$$

$$\left\{\vec{V}_1^* k_{(n-1)}^2 \vec{V}_1^* + \frac{\vec{k}_{(n-2)}}{w_{(n-2)}}\gamma v_T^2(\vec{k}_{(n-1)}\cdot\vec{V}_1^*)^2 \frac{\vec{k}_{(n)}}{w_{(n)}}\right.$$

$$3.6-(19)$$

$$+ \vec{V}_1^* w_{(n-1)}(\vec{k}_{(n-1)}\cdot\vec{V}_1^*)\frac{\vec{k}_{(n)}}{w_{(n)}}$$

$$\left.\left.+ \frac{\vec{k}_{(n-2)}}{w_{(n-2)}}w_{(n-1)}(\vec{k}_{(n-1)}\cdot\vec{V}_1^*)\vec{V}_1^*\right\}\right] \cdot \left(1 + \frac{\gamma v_T^2 \vec{k}_{(n)}\vec{k}_{(n)}}{w_{(n)}^2 - \gamma k_{(n)}^2 v_T^2}\right)\cdot\vec{E}_{(n)}$$

$\vec{E}_{(n)}(\vec{k}_{(n)}, w_{(n)})$ and the positive- and negative-frequency pump components acting in succession give rise to a third-order current at $(\vec{k}_{(n)}, w_{(n)})$ which forms a self-correction to the perturbation:

$$
\frac{iJ_{(n),1,-1}}{\epsilon_0 w_{(n)}} = -\frac{w_p^2}{w_{(n)}^2}\left(1 + \frac{\gamma v_T^2 \vec{k}_{(n)}\vec{k}_{(n)}}{w_{(n)}^2 - \gamma k_{(n)}^2 v_T^2}\right) \cdot \left[\frac{1}{w_{(n-1)}^2 - \gamma k_{(n-1)}^2 v_T^2}\right.
$$

$$
\left\{\vec{V}_1 k_{(n-1)}^2 \vec{V}_1^* + \frac{\vec{k}_{(n)}}{w_{(n)}}\gamma v_T^2 (\vec{k}_{(n-1)}\cdot\vec{V}_1)\left(\vec{k}_{(n-1)}\cdot\vec{V}_1^*\right)\frac{\vec{k}_{(n)}}{w_{(n)}}\right.
$$

$$
\left. + \vec{V}_1 w_{(n-1)}\left(\vec{k}_{(n-1)}\cdot\vec{V}_1^*\right)\frac{\vec{k}_{(n)}}{w_{(n)}}\right\} + \frac{1}{w_{(n+1)}^2 - \gamma k_{(n+1)}^2 v_T^2}
$$

$$
\left\{\vec{V}_1^* k_{(n+1)}^2 \vec{V}_1 + \frac{\vec{k}_{(n)}}{w_{(n)}}\gamma v_T^2\left(\vec{k}_{(n+1)}\cdot\vec{V}_1^*\right)(\vec{k}_{(n+1)}\cdot\vec{V}_1)\frac{\vec{k}_{(n)}}{w_{(n)}}\right.
$$

$$
\left.\left. + \vec{V}_1^* w_{(n+1)}(\vec{k}_{(n+1)}\cdot\vec{V}_1)\frac{\vec{k}_{(n)}}{w_{(n)}} + \frac{\vec{k}_{(n)}}{w_{(n)}}w_{(n+1)}\left(\vec{k}_{(n+1)}\cdot\vec{V}_1^*\right)\vec{V}_1\right\}\right]
$$

$$
\cdot\left(1 + \frac{\gamma v_T^2 \vec{k}_{(n)}\vec{k}_{(n)}}{w_{(n)}^2 - \gamma k_{(n)}^2 v_T^2}\right)\cdot\vec{E}_{(n)} \qquad\qquad 3.6\text{-}(20)
$$

In section 2.7, the coupling coefficients corresponding to second- and third-order nonlinear currents were defined by equations 2.7-(8) and 2.7-(9), respectively. Substituting (16) into 2.7-(8), one arrives at an expression for the matrix of coupling coefficients

$$E_1 \overset{\leftrightarrow}{F}^{NL(2)}_{(n),\,1}$$

3.6-(21)

which is identical to that of 3.5-(10). Similarly, substituting (17) into 2.7-(8), one arrives at an expression for

$$E_1^* \overset{\leftrightarrow}{F}_{(n),\,-1}$$

3.6-(22)

which is identical to that of 3.5-(11). Moving on to the third-order non-linear currents, one may substitute (18) into 2.7-(9), and set $n = -1$. The result is identical to the expression for

$$\frac{1}{2} E_1^2 \overset{\triangle}{\overset{\leftrightarrow}{F}}{}^{NL(3)}_{(-1),\,1,\,1}$$

3.6-(23)

set out in 3.5-(26). The expression 3.5-(26) was obtained from the general coupling coefficient by omitting the effects of the 2nd-harmonic electric field and the 2nd-harmonic quivering velocity set up by the laser pump. The agreement between 3.6-(23) and 3.5-(26) is thus to be expected, since in this section the oscillating equilibrium set up by the laser was simplified by omitting the 2nd harmonic of the electric field and the 2nd harmonic of the quivering velocity. Similarly, one may substitute (19) into 2.7-(9), and set $n = 1$. The result is identical to the expression for

$$\frac{1}{2} E_1^{*2} \overset{\triangle}{\overset{\leftrightarrow}{F}}{}^{NL(3)}_{(1),\,-1,\,1}$$

3.6-(24)

set out in 3.5-(27). Again the identity occurs because both the expression 3.5-(27) and the differential-equation approach of this section ignore the self-consistent 2nd-harmonic electric field and 2nd-harmonic quivering velocity present in the pump wave. Finally one may substitute (20) into 2.7-(9). The result is identical to the expression for

$$|E_1|^2 \vec{\vec{F}}_{(n),1,-1} \qquad\qquad\qquad 3.6\text{-}(25)$$

set out in 3.5-(15).

In this section we have determined the extent to which an oscillating-equilibrium treatment can recover the results of generalized-coupling-of-modes theory. An oscillating-equilibrium in which the 2nd-harmonic electric field and quivering velocity are neglected can turnish 3rd-order coupling coefficients only if the neglect is justified on physical grounds. This is so for the case of unmagnetized plasma and electromagnetic pump, because in this case all pump-driven instabilities satisfy the kinematic inequalities 3.3-(29)-(32). This in turn is because the normal-mode structure in unmagnetized plasma is relatively simple. In magnetized plasma the normal-mode structure in the absence of the pump is much richer than for the unmagnetized case. Thus the introduction of a pump wave gives rise to a much greater variety of wave-wave interactions than in the unmagnetized case. Some of these wave- ave interactions will constitute pump-driven instabilities, and some of these pump-driven instabilities will involve 3rd-order nonlinear conductivity. Because of the rich structure in $(\vec{k}, w)$ space of the interacting waves, there will occur pump-driven instabilities involving 3rd-order nonlinear conductivity for which no kinematic inequalities that would justify neglecting the 2nd-harmonic content of the pump can be found. For such instabilities in magnetized plasma, then, the treatment which starts from a simplified oscillating equilibrium may not be valid. In particular, any treatments of magnetized-plasma analogues of the oscillating-two-stream, self-focusing, modified-decay or modified-Brillouin instabilities which start from an oscillating

equilibrium must either include 2nd-harmonic components in that pump equilibrium or explain on kinematic grounds why they can be neglected.

Returning to the unmagnetized-plasma case, the conclusion of this section is that the physically important terms, describing the behavior of small perturbations coupled by a laser pump in unmagnetized plasma, can be derived for illustrative purposes from an oscillating equilibrium treatment as well as from the general results of section 2.8. In the next section, the results of this and the previous section will be assembled for easy reference.

3.7 Coupled Equations Incorporating Physical Approximations

In section 2.8 the warm-fluid model of a drifting, magnetized plasma was used to derive the coupling coefficients which characterize coherent wave-wave interaction in general. In section 3.5 these general results were specialized to laser-driven instabilities in drift-free, unmagnetized plasma. This specialization allowed the kinematic inequalities (3.3-(29)-(32))

$$\left| \frac{w_{(0)}}{k_{(0)}} \right| \ll c \qquad \qquad 3.7\text{-}(1)$$

$$v_T \ll c \qquad \qquad 3.7\text{-}(2)$$

$$v_T \ll \left| \frac{w_{(\pm 1)}}{k_{(\pm 1)}} \right| \qquad \qquad 3.7\text{-}(3)$$

$$\left| \frac{w_{(0)}}{k_{(0)}} \right| < \left| \frac{w_{(\pm 1)}}{k_{(\pm 1)}} \right| \qquad \qquad 3.7\text{-}(4)$$

to be used. The behavior of the pump-coupled perturbation fields was then

$$
\begin{pmatrix}
\begin{array}{c} (D_{(1)})_{MM}+|E_1^2| \\ \left(\overset{\Delta}{F}^{NL(3)}_{(1),1,-1}\right)_{MMxx} \end{array} &
\begin{array}{c} |E_1^2| \\ \left(\overset{\Delta}{F}^{NL(3)}_{(1),1,-1}\right)_{MSxx} \end{array} &
\begin{array}{c} E_1 \\ \left(F^{NL(2)}_{(0),1}\right)_{MSx} \end{array} &
\begin{array}{c} \tfrac{1}{2}E_1^2 \\ \left(\overset{\Delta}{F}^{NL(3)}_{(-1),1,1}\right)_{MMxx} \end{array} &
\begin{array}{c} \tfrac{1}{2}E_1^2 \\ \left(\overset{\Delta}{F}^{NL(3)}_{(-1),1,1}\right)_{MSxx} \end{array} \\[1em]
\begin{array}{c} |E_1^2| \\ \left(\overset{\Delta}{F}^{NL(3)}_{(1),1,-1}\right)_{SMxx} \end{array} &
\begin{array}{c} (D_{(1)})_{SS}+|E_1^2| \\ \left(\overset{\Delta}{F}^{NL(3)}_{(1),1,-1}\right)_{SSxx} \end{array} &
\begin{array}{c} E_1 \\ \left(F^{NL(2)}_{(0),1}\right)_{SSx} \end{array} &
\begin{array}{c} \tfrac{1}{2}E_1^2 \\ \left(\overset{\Delta}{F}^{NL(3)}_{(-1),1,1}\right)_{SMxx} \end{array} &
\begin{array}{c} \tfrac{1}{2}E_1^2 \\ \left(\overset{\Delta}{F}^{NL(3)}_{(-1),1,1}\right)_{SSxx} \end{array} \\[1em]
\begin{array}{c} E_1^* \\ \left(F^{NL(2)}_{(1),-1}\right)_{SMx} \end{array} &
\begin{array}{c} E_1^* \\ \left(F^{NL(2)}_{(1),-1}\right)_{SSx} \end{array} &
(D_{(0)})_{SS} &
\begin{array}{c} E_1 \\ \left(F^{NL(2)}_{(-1),1}\right)_{SMx} \end{array} &
\begin{array}{c} E_1 \\ \left(F^{NL(2)}_{(-1),1}\right)_{SSx} \end{array} \\[1em]
\begin{array}{c} \tfrac{1}{2}E_1^{*2} \\ \left(\overset{\Delta}{F}^{NL(3)}_{(1),-1,-1}\right)_{MMxx} \end{array} &
\begin{array}{c} \tfrac{1}{2}E_1^{*2} \\ \left(\overset{\Delta}{F}^{NL(3)}_{(1),-1,-1}\right)_{MSxx} \end{array} &
\begin{array}{c} E_1^* \\ \left(F^{NL(2)}_{(0),-1}\right)_{MSx} \end{array} &
\begin{array}{c} (D_{(-1)})_{MM}+|E_1^2| \\ \left(\overset{\Delta}{F}^{NL(3)}_{(-1),1,-1}\right)_{MMxx} \end{array} &
\begin{array}{c} |E_1^2| \\ \left(\overset{\Delta}{F}^{NL(3)}_{(-1),1,-1}\right)_{MSxx} \end{array} \\[1em]
\begin{array}{c} \tfrac{1}{2}E_1^{*2} \\ \left(\overset{\Delta}{F}^{NL(3)}_{(1),-1,-1}\right)_{SMxx} \end{array} &
\begin{array}{c} \tfrac{1}{2}E_1^{*2} \\ \left(\overset{\Delta}{F}^{NL()}_{(1),-1,-1}\right)_{SSxx} \end{array} &
\begin{array}{c} E_1^* \\ \left(F^{NL(2)}_{(0),-1}\right)_{SSx} \end{array} &
\begin{array}{c} |E_1^2| \\ \left(\overset{\Delta}{F}^{NL(3)}_{(-1),1,-1}\right)_{SMxx} \end{array} &
\begin{array}{c} (D_{(-1)})_{SS}+|E_1^2| \\ \left(\overset{\Delta}{F}^{NL(3)}_{(-1),1,-1}\right)_{SSxx} \end{array}
\end{pmatrix}
\begin{pmatrix} E_{M(1)} \\[1em] E_{S(1)} \\[1em] E_{S(0)} \\[1em] E_{M(-1)} \\[1em] E_{S(-1)} \end{pmatrix}
=
\begin{pmatrix} 0 \\[1em] 0 \\[1em] 0 \\[1em] 0 \\[1em] 0 \end{pmatrix}
$$

3.7-(5)

The matrix equation 3.7-(6) appears in sectioned form on the succeeding 3 pages. The complete equation is reconstituted by placing the three sections side by side.

$$\left(\mathbb{D}_{(1)}\right)_{mm}$$
$$-|V_1^2|\,w_P^2\left[k_{(0)}^2\left(\vec{e}_{m(1)}\cdot\vec{e}_{x1}\right)^2\right]/$$
$$\left[w_{(1)}^2\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)\right]$$

$$-|V_1^2|\,w_P^2\left[k_{(0)}^2 w_{(1)}^2\left(\vec{e}_{m(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(1)}\cdot\vec{e}_{x1}\right)+\right.$$
$$\left. k_{(1)}k_{(0)}w_{(1)}w_{(0)}\left(\vec{e}_{m(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(0)}\cdot\vec{e}_{x1}\right)\right]/$$
$$\left[w_{(1)}^2\left(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2\right)\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)\right]$$

$$-|V_1^2|\,w_P^2\left[k_{(0)}^2 w_{(1)}^2\left(\vec{e}_{m(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(1)}\cdot\vec{e}_{x1}\right)\right.$$
$$\left.+k_{(0)}k_{(1)}w_{(0)}w_{(1)}\left(\vec{e}_{m(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(0)}\cdot\vec{e}_{x1}\right)\right]/$$
$$\left[w_{(1)}^2\left(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2\right)\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)\right]$$

$$\left(\mathbb{D}_{(1)}\right)_{ss}-|V_1^2|\,w_P^2\left[k_{(0)}^2 w_{(1)}^2\left(\vec{e}_{s(1)}\cdot\vec{e}_{x1}\right)^2+\right.$$
$$\left. 2k_{(0)}k_{(1)}w_{(0)}w_{(1)}\left(\vec{e}_{s(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(0)}\cdot\vec{e}_{x1}\right)\right]/$$
$$\left[w_{(1)}^2\left(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2\right)\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)\right]$$

$$-V_1^*\,w_P^2\left[k_{(0)}\left(\vec{e}_{m(1)}\cdot\vec{e}_{x1}\right)\right]/$$
$$\left[\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)w_{(1)}\right]$$

$$-V_1^*\,w_P^2\left[k_{(0)}w_{(1)}\left(\vec{e}_{s(1)}\cdot\vec{e}_{x1}\right)+k_{(1)}w_{(0)}\left(\vec{e}_{s(0)}\cdot\vec{e}_{x1}\right)\right]/$$
$$\left[\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)\left(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2\right)\right]$$

$$-V_1^{*2}w_P^2\left[k_{(0)}^2\left(\vec{e}_{m(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{m(-1)}\cdot\vec{e}_{x1}\right)\right]/$$
$$\left[\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)w_{(1)}w_{(-1)}\right]$$

$$-V_1^{*2}w_P^2\left[k_{(0)}^2 w_{(1)}\left(\vec{e}_{s(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{m(-1)}\cdot\vec{e}_{x1}\right)+\right.$$
$$\left. k_{(0)}k_{(1)}w_{(0)}\left(\vec{e}_{m(-1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(0)}\cdot\vec{e}_{x1}\right)\right]/$$
$$\left[\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)\left(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2\right)w_{(-1)}\right]$$

$$-V_1^{*2}w_P^2\left[k_{(0)}^2 w_{(-1)}^2\left(\vec{e}_{m(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(-1)}\cdot\vec{e}_{x1}\right)\right.$$
$$\left.+k_{(0)}k_{(-1)}w_{(0)}\left(\vec{e}_{m(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(0)}\cdot\vec{e}_{x1}\right)\right]/$$
$$\left[\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)\left(w_{(-1)}^2-\gamma k_{(-1)}^2 V_T^2\right)w_{(1)}\right]$$

$$-V_1^{*2}w_P^2\left[k_{(0)}^2 w_{(1)}w_{(-1)}\left(\vec{e}_{s(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(-1)}\cdot\vec{e}_{x1}\right)+\right.$$
$$k_{(0)}k_{(-1)}w_{(0)}w_{(1)}\left(\vec{e}_{s(1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(0)}\cdot\vec{e}_{x1}\right)+$$
$$\left. k_{(0)}k_{(1)}w_{(0)}w_{(-1)}\left(\vec{e}_{s(-1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(0)}\cdot\vec{e}_{x1}\right)\right]/$$
$$\left[\left(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2\right)\left(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2\right)\left(w_{(-1)}^2-\gamma k_{(-1)}^2 V_T^2\right)\right]$$

$-V_1 \omega_P^2 [k_{(0)} (\vec{e}_{m(1)} \cdot \vec{e}_{x1})]/$ $[(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2) \omega_{(1)}]$	$-V_1^2 \omega_P^2 [k_{(0)}^2 (\vec{e}_{m(1)} \cdot \vec{e}_{x1})(\vec{e}_{m(-1)} \cdot \vec{e}_{x1})]/$ $[(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2) \omega_{(1)} \omega_{(-1)}]$
$-V_1 \omega_P^2 [k_{(1)} \omega_{(0)} (\vec{e}_{s(0)} \cdot \vec{e}_{x1})$ $+ k_{(0)} \omega_{(1)} (\vec{e}_{s(1)} \cdot \vec{e}_{x1})]/$ $[(\omega_{(1)}^2 - \gamma k_{(1)}^2 v_T^2)(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2)]$	$-V_1^2 \omega_P^2 [k_{(0)}^2 \omega_{(1)} (\vec{e}_{s(1)} \cdot \vec{e}_{x1})(\vec{e}_{m(-1)} \cdot \vec{e}_{x1}) +$ $k_{(0)} k_{(1)} \omega_{(0)} (\vec{e}_{m(-1)} \cdot \vec{e}_{x1})(\vec{e}_{s(0)} \cdot \vec{e}_{x1})]/$ $[(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2)(\omega_{(1)}^2 - \gamma k_{(1)}^2 v_T^2) \omega_{(-1)}]$
$(D_{(0)})_{ss}$	$-V_1 \omega_P^2 [k_{(0)} (\vec{e}_{m(-1)} \cdot \vec{e}_{x1})]/$ $[(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2) \omega_{(-1)}]$
$-V_1^* \omega_P^2 [k_{(0)} (\vec{e}_{m(-1)} \cdot \vec{e}_{x1})]/$ $[(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2) \omega_{(-1)}]$	$(D_{(-1)})_{mm} - |V_1^2| \omega_P^2 [k_{(0)}^2 (\vec{e}_{m(-1)} \cdot \vec{e}_{x1})^2]/$ $[\omega_{(-1)}^2 (\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2)]$
$-V_1^* \omega_P^2 [k_{(-1)} \omega_{(0)} (\vec{e}_{s(0)} \cdot \vec{e}_{x1})$ $+ k_{(0)} \omega_{(-1)} (\vec{e}_{s(-1)} \cdot \vec{e}_{x1})]/$ $[(\omega_{(-1)}^2 - \gamma k_{(-1)}^2 v_T^2)(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2)]$	$-|V_1^2| \omega_P^2 [k_{(0)}^2 \omega_{(-1)}^2 (\vec{e}_{m(-1)} \cdot \vec{e}_{x1})(\vec{e}_{s(-1)} \cdot \vec{e}_{x1})$ $+ k_{(0)} k_{(-1)} \omega_{(0)} \omega_{(-1)} (\vec{e}_{m(-1)} \cdot \vec{e}_{x1})(\vec{e}_{s(0)} \cdot \vec{e}_{x1})]/$ $[\omega_{(-1)}^2 (\omega_{(-1)}^2 - \gamma k_{(-1)}^2 v_T^2)(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2)]$

$$
\begin{pmatrix}
\dfrac{-V_1^2 \omega_p^2 \left[k_{(0)}^2 \omega_{(-1)} (\vec{e}_{m(1)} \cdot \vec{e}_{x1})(\vec{e}_{s(-1)} \cdot \vec{e}_{x1}) + k_{(0)} k_{(-1)} \omega_{(0)} (\vec{e}_{m(1)} \cdot \vec{e}_{x1})(\vec{e}_{s(0)} \cdot \vec{e}_{x1}) \right]}{\left[(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2)(\omega_{(-1)}^2 - \gamma k_{(-1)}^2 v_T^2) \omega_{(1)}^2 \right]} \\[3mm]
\dfrac{-V_1^2 \omega_p^2 \left[k_{(0)}^2 \omega_{(1)} \omega_{(-1)} (\vec{e}_{s(1)} \cdot \vec{e}_{x1})(\vec{e}_{s(-1)} \cdot \vec{e}_{x1}) + k_{(0)} k_{(-1)} \omega_{(0)} \omega_{(1)} (\vec{e}_{s(1)} \cdot \vec{e}_{x1})(\vec{e}_{s(0)} \cdot \vec{e}_{x1}) + k_{(0)} k_{(1)} \omega_{(0)} \omega_{(-1)} (\vec{e}_{s(-1)} \cdot \vec{e}_{x1})(\vec{e}_{s(0)} \cdot \vec{e}_{x1}) \right]}{\left[(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2)(\omega_{(1)}^2 - \gamma k_{(1)}^2 v_T^2)(\omega_{(-1)}^2 - \gamma k_{(-1)}^2 v_T^2) \right]} \\[3mm]
\dfrac{-V_1 \omega_p^2 \left[k_{(0)} \omega_{(-1)} (\vec{e}_{s(-1)} \cdot \vec{e}_{x1}) + k_{(-1)} \omega_{(0)} (\vec{e}_{s(0)} \cdot \vec{e}_{x1}) \right]}{\left[(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2)(\omega_{(-1)}^2 - \gamma k_{(-1)}^2 v_T^2) \right]} \\[3mm]
\dfrac{-|V_1|^2 \omega_p^2 \left[k_{(0)}^2 \omega_{(-1)}^2 (\vec{e}_{m(-1)} \cdot \vec{e}_{x1})(\vec{e}_{s(-1)} \cdot \vec{e}_{x1}) + k_{(-1)} k_{(0)} \omega_{(-1)} \omega_{(0)} (\vec{e}_{m(-1)} \cdot \vec{e}_{x1})(\vec{e}_{s(0)} \cdot \vec{e}_{x1}) \right]}{\left[\omega_{(-1)}^2 (\omega_{(-1)}^2 - \gamma k_{(-1)}^2 v_T^2)(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2) \right]} \\[3mm]
(D_{(-1)})_{ss} - \dfrac{|V_1|^2 \omega_p^2 \left[k_{(0)}^2 \omega_{(1)}^2 (\vec{e}_{s(-1)} \cdot \vec{e}_{x1})^2 + 2 k_{(0)} k_{(-1)} \omega_{(0)} \omega_{(-1)} (\vec{e}_{s(-1)} \cdot \vec{e}_{x1})(\vec{e}_{s(0)} \cdot \vec{e}_{x1}) \right]}{\left[\omega_{(-1)}^2 (\omega_{(-1)}^2 - \gamma k_{(-1)}^2 v_T^2)(\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2) \right]}
\end{pmatrix}
\begin{pmatrix}
E_{m(1)} \\[3mm]
E_{s(1)} \\[3mm]
E_{s(0)} \\[3mm]
E_{m(-1)} \\[3mm]
E_{s(-1)}
\end{pmatrix}
=
\begin{pmatrix}
0 \\[3mm]
0 \\[3mm]
0 \\[3mm]
0 \\[3mm]
0
\end{pmatrix}
$$

3.7-(6)

given by 3.4-(13). The form of 3.4-(13) was rederived in 3.6 by directly expanding the warm-fluid equations about a simplified version of the oscillating equilibrium set up by the laser pump. In this section we set out 3.4-(13) explicitly, taking advantage of the fact that it is effectively not a 9×9 but a 6×6 equation, as discussed at the end of section 3.5. Further, we neglect the low-frequency electromagnetic response of the plasma at $(\vec{k}_{(0)}, w_{(0)})$. The form that 3.4-(13) assumes is then the 5×5 matrix equation written down formally in (5) and more explicitly in (6). Of course, no attempt will be made to solve this matrix equation directly for the relation between $\vec{k}_{(0)}$ and $w_{(0)}$. Rather, the discussion in the latter part of section 3.4 will be used as a guide in extracting submatrices from (5) which will be identified with specific instabilities. This extraction of specific instabilities will be performed in section 3.8.

3.8 Isolation of Specific Instabilities from the Full Determinantal Equation

Chapter 2 provided a general framework for talking about coherent-wave-wave interactions in nonlinear media. A process of specialization led up to a discussion in the latter part of section 3.4 on finding and classifying laser-driven instabilities in drift-free unmagnetized plasma. That discussion will be amplified in this section. Details of the polarizations, wavevectors, and frequencies of the interacting coherent waves will subdivide the broad categories of instabilities which were defined by 3.4-(18), 3.4-(21), and 3.4-(24), respectively. Those 3 broad categories — simple parametric instabilities, modified parametric instabilities, and intrinsically 3rd-order instabilities — will in this manner furnish between them all the

well-known laser-driven instabilities.

In section 3.4 the behavior of small perturbations coupled together by the pump was shown to be described by a set of coupled equations, written down in matrix form as 3.4-(13). The consistency condition for this set of coupled equations was that the determinant of the matrix in 3.4-(13) should be zero. Indeed, the condition that this determinant be zero constituted the dispersion relation for the coupled system. However, the solution of the full 9×9 determinantal equation was not attempted. Rather, the behavior of the roots of the full determinantal equation as a function of laser-pump intensity was considered. It was shown that the behavior of each root could be found approximately from an appropriately reduced set of coupled equations. A root which could be found from a reduced matrix equation of type 3.4-(18) was termed a simple parametric instability. Similarly, roots found from reduced equations of type 3.4-(21) and 3.4-(24) were termed modified parametric instabilities and intrinsically third-order instabilities, respectively.

In this section we proceed in a similar manner, but specify in more detail the polarizations and frequencies of the interacting waves. The behavior of small perturbation coupled together by the pump is here represented not by the 9×9 matrix equation 3.4-(12) but the the 5×5 matrix equation 3.7-(5). This preliminary reduction is justified physically in section 3.5. Again no attempt is made to solve the full 5×5 determinantal equation. Rather, the behavior of the roots of 3.7-(5) as a function of laser intensity is considered. First set the laser-pump electric-field intensity to zero in 3.7-(5). Locate the double roots of the determinantal equation. As the laser-pump intensity is brought up from zero, each

double root splits, and the behavior of each root pair as a function of pump intensity may be followed approximately. This is done using for each root pair the appropriate reduced matrix equation, just as described in section 3.4. The details follow.

The 9×9 matrix equation 3.4-(12) described the behavior of 3 electric-field perturbations with wavevectors and frequencies separated by the pump wavevector and frequency. Each of the 3 perturbations $\vec{E}_{(n)}$ had an electrostatically polarized component, of amplitude $E_{s(n)}$, an electromagnetically polarized component perpendicular to the pump field, of amplitude $E_{N(n)}$, and an electromagnetically polarized component at the smallest angle to the pump field, of amplitude $E_{M(n)}$. The equation 3.4-(12) described how these 9 electric-field components were coupled together by the laser pump wave. To reduce 3.4-(12) to 3.4-(13), physical approximations were introduced, based on the normal-mode structure of unmagnetized plasma and the consequent restrictions on the phase velocities of interacting waves. Within these approximations, the electromagnetically polarized components perpendicular to the laser field, of amplitude $E_{N(n)}$, were decoupled. In 3.7 the other electromagnetic component of the low-frequency perturbation, $E_{M(0)}$, was dropped and the resulting 5×5 equation 3.7-(5) forms the basis of the present section. Interesting solutions of 3.7-(5) have not all of the coupled perturbation components $E_{M(1)}, E_{S(1)}, E_{S(0)}, E_{M(-1)}, E_{S(-1)}$ equal to zero. For interesting solutions, therefore, the determinant of the 5×5 matrix must be zero:

$$|M| = 0 \qquad\qquad 3.8\text{-}(1)$$

where

$$M = M(E_1) \qquad\qquad 3.8\text{-}(2)$$

is the 5×5 matrix in 3.7-(5), with the emphasis on its functional dependence on the laser-pump electric-field intensity E_1. For zero pump-field E_1, (1) reduces to the product

$$(D_{(1)})_{MM}(D_{(1)})_{SS}(D_{(0)})_{SS}(D_{(-1)})_{MM}(D_{(-1)})_{SS} = 0 \qquad\qquad 3.8\text{-}(3)$$

The double roots of (3) will be located in turn and the behavior of the corresponding roots of (1) followed as a function of E_1.

The equation (3) has a double root at any value of $(\vec{k}_{(0)}, w_{(0)})$ such that

$$(D_{(0)})_{SS} = 0 \qquad\qquad 3.8\text{-}(4)$$

and

$$(D_{(-1)})_{MM} = 0 \qquad\qquad 3.8\text{-}(5)$$

simultaneously. This has an immediate physical meaning in terms of the linear dispersion diagram describing the normal modes of the unmagnetized plasma. Equation (4) states that the wavevector and frequency

$$(\vec{k}_{(0)}, w_{(0)})$$

satisfy either the ion-acoustic-wave dispersion relation or the electron-plasma-wave dispersion relation. Equation (5) states that the wavevector and frequency

$$(\vec{k}_{(-1)}, w_{(-1)}) \equiv (\vec{k}_{(0)}, w_{(0)}) - (\vec{k}_1, w_1)$$

satisfy the electromagnetic-branch dispersion relation.

How is one to locate the values of $(\vec{k}_{(0)}, w_{(0)})$ which satisfy (4) and (5)

simultaneously, bearing in mind that $\vec{k}_{(0)}$ is a 3-dimensional vector and that $\vec{k}_1$, the pump wavevector, is nonzero? The values of $(\vec{k}_{(0)}, w_{(0)})$ satisfying (4) lie on a 3-dimensional hypersurface, $H_{(0)SS}$, say, in the 4-dimensional $(\vec{k}, w)$ domain. This hypersurface $H_{(0)SS}$ has 4 sheets, 2 sheets comprising the ion-acoustic branch and 2 sheets comprising the electron-plasma branch. Each branch has 2 sheets because positive and negative frequencies are included. The values of $(\vec{k}_{(0)}, w_{(0)})$ satisfying (5) lie on another 3-dimensional hypersurface, $H_{(-1)MM}$, say, in the 4-dimensional $(\vec{k}, w)$ domain. This hypersurface $H_{(-1)MM}$ has 2 sheets, comprising the electromagnetic branch upshifted by the pump wavevector and frequency $(\vec{k}_1, w_1)$. The values of $(\vec{k}_{(0)}, w_{(0)})$ which satisfy (4) and (5) simultaneously lie in a 2-dimensional manifold which is the intersection of the 3-dimensional hypersurfaces $H_{(0)SS}$ and $H_{(-1)MM}$. These geometrical objects are difficult to visualize. The 2-dimensional and 3-dimensional plane cross sections of the 4-dimensional $(\vec{k}, w)$ domain are useful in this regard. Such cross sections will be displayed in Chapter 4 as an aid to understanding the 3-(space)-dimensional dispersion relations for specific instabilities.

Now follow the behavior of any double root, satisfying (4) and (5) for $E_1 = 0$, as the magnitude of E_1 increases from zero in (1) and (2). From the discussion in section 3.4, we know that the pair of roots into which the double root splits can be found approximately from a reduced determinantal equation. Substituting B = M in 3.4-(18) we deduce

---

UNMODIFIED BRILLOUIN AND UNMODIFIED RAMAN INSTABILITIES

$$
\begin{vmatrix}
(D_{(0)})_{SS} & E_{x1}\left(F^{NL(2)}_{(-1),\,1}\right)_{SMx} \\[2ex]
E^{*}_{x1}\left(F^{NL(2)}_{(0),\,-1}\right)_{MSx} & (D_{(-1)})_{MM}
\end{vmatrix} = 0 \qquad\qquad 3.8\text{-}(6)
$$

Suppose the double root (4),(5) was chosen so that $(\vec{k}_{(0)}, w_{(0)})$ lay on the ion-acoustic branch and $(\vec{k}_{(-1)}, w_{(-1)})$ on the electromagnetic branch for $E_1 = 0$. Then the pair of roots of (6), into which the double root splits for small nonzero E_1, describe the unmodified Brillouin instability. Suppose, on the other hand, that the double root (4),(5) was chosen so that $(\vec{k}_{(0)}, w_{(0)})$ lay on the electron-plasma branch and $(\vec{k}_{(-1)}, w_{(-1)})$ on the electromagnetic branch for $E_1 = 0$. Then the pair of roots of (6), into which the double root splits, describe the unmodified Raman instability.

Now follow the behavior of the root pair as the magnitude of the laser pump field E_1 is increased further. Suppose the growth rate of the instability becomes of the order of $w_{(0)}$, but still much smaller than $w_{(-1)}$. This can happen for the Brillouin instability, and in highly underdense plasma for the Raman instability. Then, from the discussion in section 3.4, we know that the root pair can be found approximately from the reduced determinantal equation (cf. 3.4-(21) with B = M).

MODIFIED BRILLOUIN AND MODIFIED RAMAN INSTABILITY

$$\begin{vmatrix} (D_{(0)})_{SS} & E_{x1}\left(F^{NL(2)}_{(-1),\,1}\right)_{SMx} \\ E^{*}_{x1}\left(F^{NL(2)}_{(0),\,-1}\right)_{MSx} & (D_{(-1)})_{MM} + |E_{x1}|^{2}\left(F^{NL(3)}_{(-1),1,-1}\right)_{MMxx} \end{vmatrix} = 0$$

3.8-(7)

This describes the modified Brillouin instability or modified Raman insta-

bility, depending on the region of wavevector-frequency space chosen for

$(\vec{k}_{(0)}, w_{(0)})$.

The equation (3) has a double root at any value of $(\vec{k}_{(0)}, w_{(0)})$ such that

$$(D_{(0)})_{SS} = 0 \qquad\qquad\qquad 3,8\text{-}(8)$$

and

$$(D_{(-1)})_{SS} = 0 \qquad\qquad\qquad 3.8\text{-}(9)$$

Equation (8) states that the wavevector and frequency

$$(\vec{k}_{(0)}, w_{(0)})$$

satisfy either the ion-acoustic-wave dispersion relation or the electron-

plasma-wave dispersion relation. Equation (9) will be taken to mean that

$$(\vec{k}_{(-1)}, w_{(-1)}) \equiv (\vec{k}_{(0)}, w_{(0)}) - (\vec{k}_{1}, w_{1})$$

satisfy the electron-plasma-wave dispersion relation.

Locate the values of $(\vec{k}_{(0)}, w_{(0)})$ satisfying (8),(9) as follows. Define

the hypersurface $H_{(0)SS}$ to comprise the ion-acoustic and electron-plasma

branches of the linear dispersion hypersurface, as before. Define the hypersurface $H_{(-1)SS}$ to comprise the electron-plasma branch, upshifted by the pump wavevector and frequency. Then the intersection of $H_{(0)SS}$, $H_{(-1)SS}$ contains the simultaneous roots of (8) and (9).

Now follow the behavior of a double root for nonzero laser pump field E_1. Again refer to the discussion of section 3.4 and substitute $B = S$ in 3.4-(1). The double root splits into the pair of roots satisfying

UNMODIFIED PLASMON-PHONON AND 2-PLASMON INSTABILITIES

$$\begin{vmatrix} {}^{(}D_{(0)})_{SS} & E_{x1}\left(F^{NL(2)}_{(-1),1}\right)_{SSx} \\ E_{x1}^{*}\left(F^{NL(2)}_{(0),-1}\right)_{SSx} & {}^{(}D_{(-1)})_{SS} \end{vmatrix} = 0 \qquad 3.8\text{-}(10)$$

Suppose the double root (8),(9) was chosen so that $(\vec{k}_{(0)}, w_{(0)})$ lay on the ion-acoustic branch and $(\vec{k}_{(-1)}, w_{(-1)})$ on the electron-plasma branch. Then (10) describes the unmodified plasmon-phonon or so-called "decay" instability. Suppose, on the other hand, that the double root from which the roots of (10) originated was chosen with $(\vec{k}_{(0)}, w_{(0)})$ and $(\vec{k}_{(-1)}, w_{(-1)})$ both lying on the electron-plasma branch. Then (10) describes the unmodified 2-plasmon instability.

Now suppose the laser field intensity E_1 increased so that the growth rate of the instability becomes comparable to $w_{(0)}$ but still much smaller than $w_{(-1)}$. This can happen for the plasmon-phonon instability but not for the 2-plasmon instability. Then, substituting $B = S$ in 3.4-(21), one finds

$$\begin{vmatrix} (D_{(0)})_{SS} & E_{x1}\left(F^{NL(2)}_{(-1),1}\right)_{SSx} \\[2mm] E^{*}_{x1}\left(F^{NL(2)}_{(0),-1}\right)_{SSx} & (D_{(-1)})_{SS} + |E^{2}_{x1}|\left(F^{NL(3)}_{(-1),1,-1}\right)_{SSxx} \end{vmatrix} = 0$$

This would describe the modified plasmon-phonon instability, were it not for the following fact. The values of $\vec{k}_{(0)}$ which yield plasmon-phonon instability lie near values of $\vec{k}_{(0)}$ which yield the oscillating-two-stream instability, to be discussed later in this section. Raise the laser-pump field intensity E_1 such that the growth rate predicted by (10) becomes comparable to the ion-acoustic frequency. Then the plasmon-phonon instability becomes strongly affected by the oscillating-two-stream instability, and can not be considered in isolation from the OTS.

The equation (3) is satisfied at the origin,

$$(\vec{k}_{(0)}, w_{(0)}) = (\vec{0}, 0)$$

where the hypersurfaces

$$(D_{(1)})_{MM} = 0 \qquad\qquad\qquad 3.8\text{-}(11)$$

$$(D_{(-1)})_{MM} = 0 \qquad\qquad\qquad 3.8\text{-}(12)$$

are tangent to each other. The structure here is more complicated than a double root, nevertheless the methods of section 3.4 still apply. Equations (11),(12) state that the wavevectors and frequencies

$$(\vec{k}_{(\pm1)}, w_{(\pm1)}) \equiv (\vec{k}_{(0)}, w_{(0)}) \pm (\vec{k}_1, w_1) = (\pm\vec{k}_1, \pm w_1)$$

satisfy the electromagnetic dispersion relation.

The hypersurface (11) comprises the electromagnetic branch down-

shifted by the laser wavevector and frequency. The hypersurface (12) comprises the electromagnetic branch upshifted by the laser wavevector and frequency. The laser wavevector itself incorporates a self-correction 3.3-(24) even in the absence of any perturbations, but this self-correction is via processes which depend on the plasma response at the laser 2nd harmonic. Such processes are systematically neglected in 3.7-(5), as discussed in sections 3.5 and 3.6. Thus (11) and (12) can be used as a guide to the location of the unstable roots for nonzero E_1.

For nonzero E_1, look for unstable roots in the neighborhood of the origin. Substitute A = B = M in 3.4-(24) and set the determinant of the matrix equal to zero. The result is, to within the physical approximations introduced in 3.5,

FILAMENTATION AND MODULATION INSTABILITIES

$$
\begin{vmatrix}
\begin{matrix} (D_{(1)})_{MM}\, + \\ |E_{x1}^2|\left(F_{(1),1,-1}^{NL(3)}\right)_{MMxx} \end{matrix} & E_{x1}\left(F_{(0),1}^{NL(2)}\right)_{MSx} & E_{x1}^2\left(F_{(-1),1,1}^{NL(3)}\right)_{MMxx} \\[2em]
E_{x1}^*\left(F_{(1),-1}^{NL(2)}\right)_{SMx} & (D_{(0)})_{SS} & E_{x1}\left(F_{(-1),1}^{NL(2)}\right)_{SMx} \\[2em]
E_{x1}^{*2}\left(F_{(1),-1,-1}^{NL(3)}\right)_{MMxx} & E_{x1}^*\left(F_{(0),-1}^{NL(2)}\right)_{MSx} & \begin{matrix} (D_{(-1)})_{MM}\, + \\ |E_{x1}^2|\left(F_{(-1),1,-1}\right)_{MMxx} \end{matrix}
\end{vmatrix} = 0
$$

3.8-(13)

This determinantal equation has pairs of roots in the neighborhood of the origin. These roots describe the filamentation or self-modulation instabilities, depending on whether $\vec{k}_{(0)}$ lies perpendicular or not to the

pump wavevector $\vec{k}$. Omitting the first row and column one recovers equation (7). Thus the modified Brillouin instability is also contained in (13). However, the $\vec{k}$-vectors characteristic of filamentation and self-modulation differ greatly from the $\vec{k}$-vectors characteristic of Brillouin instability, so that (7) and (13) may be used separately.

The equation (3) has a double root at any value of $(\vec{k}_{(0)}, w_{(0)})$ such that

$$(D_{(1)})_{SS} = 0 \qquad\qquad 3.8\text{-}(14)$$

$$(D_{(-1)})_{SS} = 0 \qquad\qquad 3.8\text{-}(15)$$

Physically this means that the wavevectors and frequencies

$$(\vec{k}_{(1)}, w_{(n)}) \equiv (\vec{k}_{(0)}, w_{(0)}) + (\vec{k}_1, w_1)$$

$$(\vec{k}_{(-1)}, w_{(-1)}) \equiv (\vec{k}_{(0)}, w_{(0)}) - (\vec{k}_1, w_1)$$

lie respectively on the positive and negative frequency sheets of the electron-plasma-wave dispersion relation.

In the 4-dimensional $(\vec{k}, w)$ domain, the values of $(\vec{k}_{(0)}, w_{(0)})$ satisfying (14) and (15) may be located as follows. Define the hypersurface $H_{(1)SS}$ to consist of the electron-plasma branch, downshifted by the pump wavevector and frequency. Define the hypersurface $H_{(-1)SS}$ to consist of the electron-plasma branch, upshifted by the pump wavevector. Then the intersection of $H_{(1)SS}, H_{(-1)SS}$ contains the simultaneous roots of (14) and (15).

For nonzero E_1, follow the behavior of the double root as it splits into a root pair describing unstable interaction. Substitute A = B = S in 3.4-(24) and set the determinant of the resulting matrix equal to zero. The

result is, to within the physical approximations introduced in 3.5,

---

OSCILLATING-TWO-STREAM AND MODIFIED PLASMON-PHONON
INSTABILITIES

$$
\begin{vmatrix}
\begin{array}{l} (D_{(1)})_{SS} + \\ |E_{x1}^2|\left(F_{(1),1,-1}^{NL(3)}\right)_{SSxx} \end{array} & E_{x1}\left(F_{(0),1}^{NL(2)}\right)_{SSx} & E_{x1}^2\left(F_{(-1),1,1}^{NL(3)}\right)_{SSxx} \\[3ex]
E_{x1}^*\left(F_{(1),-1}^{NL(2)}\right)_{SSx} & (D_{(0)})_{SS} & E_{x1}\left(F_{(-1),1}^{NL(2)}\right)_{SSx} \\[3ex]
E_{x1}^{*2}\left(F_{(1),-1,-1}^{NL(3)}\right)_{SSxx} & E_{x1}^*\left(F_{(0),-1}^{NL(2)}\right)_{SSx} & \begin{array}{l} (D_{(-1)})_{SS} + \\ |E_{x1}^2|\left(F_{(-1),1,-1}^{NL(3)}\right)_{SSxx} \end{array}
\end{vmatrix} = 0
$$

3.8-(16)

This determinantal equation has pairs of roots which for low enough values
of E_1 lie near the double roots of (14),(15). These roots describe the so-
called "non-oscillatory" or "oscillating-two-stream" instability. On
omitting the first row and column one recovers also the determinantal
equation pertaining to the modified plasmon-phonon instability. As ex-
plained previously, the OTS affects the modified plasmon-phonon insta-
bility so that the unstable roots describing the latter must also be found
from the full equation (16).

There exists a certain plasma density, or equivalently, a certain
value of w_p/w_1, such that equation (3) has the multiple root

$$(D_{(0)})_{SS} = 0 \qquad\qquad\qquad 3.8\text{-}(20)$$

$$(D_{(-1)})_{MM} = 0 \qquad\qquad\qquad 3.8\text{-}(21)$$

$$(D_{(-1)})_{SS} = 0 \qquad\qquad 3.8\text{-}(22)$$

At this plasma density, the wavevector and frequency $(\vec{k}_{(0)}, w_{(0)})$ lie on the electron-plasma dispersion hypersurface when the wavevector and frequency

$$(\vec{k}_{(-1)}, w_{(-1)}) \equiv (\vec{k}_{(0)}, w_{(0)}) - (\vec{k}_1, w_1)$$

$$\equiv (\vec{0}, -w_p)$$

lie on the tangent point of the electron-plasma and electromagnetic dispersion hypersurfaces. The behavior of the unstable roots corresponding to (20),(21),(22) for finite pump amplitude is given by a slight extension of 3.4-(18). Include two polarizations at the frequency $(\vec{k}_{(-1)}, w_{(-1)})$ in the determinantal equation, then the unstable roots are given by

COALESCED RAMAN-2-PLASMON INSTABILITY

$$\begin{vmatrix} (D_{(0)})_{SS} & E_{x1}\left(F^{NL(2)}_{(-1),1}\right)_{SMx} & E_{x1}\left(F^{NL(2)}_{(-1),1}\right)_{SSx} \\ E^*_{x1}\left(F^{NL(2)}_{(0),-1}\right)_{MSx} & (D_{(-1)})_{MM} & 0 \\ E^*_{x1}\left(F^{NL(2)}_{(0),-1}\right)_{SSx} & 0 & (D_{(-1)})_{SS} \end{vmatrix} = 0$$

$$3.8\text{-}(23)$$

These roots describe the coalesced Raman-2-plasmon instability. For this description to be useful, the plasma density must be near that for which (20),(21),(22) hold exactly.

3.9 Reduction to Theory of Kaw, White et al.

Certain physical approximations were used to reduce the equation 3.4-(12), describing perturbations coupled together by a laser pump wave, to the simpler form 3.4-(13). These physical approximations are valid for laser-driven instabilities which come about by coherent wave-wave interaction in unmagnetized plasma. These physical approximations 3.7-(1, 2, 3, 4) do not include the assumption of a phase velocity disparity between the perturbation waves. On supplementing the inequalities 3.7-(1, 2, 3, 4) with such a phase-velocity disparity, one arrives at the theory of Kaw, White et al. This theory does not cover, for instance, the 2-plasmon instability. The theory of Kaw, White et al. reduces all laser-driven instabilities to the action of 2 physical mechanisms, to be described below.

For an electromagnetic pump in unmagnetized plasma, it is certainly true that the pump-wave phase velocity satisfies the inequality

$$c < \left| \frac{w_1}{k_1} \right| \qquad\qquad 3.9\text{-}(1)$$

Since electrons are material particles,

$$v_T \ll c$$

The kinematics of the electromagnetic branch of the unmagnetized plasma dispersion relation forbid the decay of the pump wave into electromagnetic waves alone. In each laser-driven instability at least one decay product is an electrostatic wave, with characteristic phase velocity much less than the speed of light;

$$\left| \frac{w_{(0)}}{k_{(0)}} \right| \ll c$$

Now in each specific laser-driven instability, the decay products, which is to say, the coupled perturbations, are separated by the pump wavevector and frequency. The pump frequency is greater than the plasma frequency and so at most one decay product can lie in the low-frequency ion-acoustic region of $(\vec{k}, w)$ space. The other decay products must be electron-plasma or electromagnetic waves, with characteristic phase velocities much greater than the electron thermal velocity;

$$v_T \ll \left| \frac{w_{(\pm 1)}}{k_{(\pm 1)}} \right| \qquad \qquad 3.9\text{-}(4)$$

However, not every laser-driven instability has a low-frequency decay product and thus the most that can be said about the relative phase velocities of the coupled perturbations is that

$$\left| \frac{w_{(0)}}{k_{(0)}} \right| \lesssim \left| \frac{w_{(\pm 1)}}{k_{(\pm 1)}} \right| \qquad \qquad 3.9\text{-}(5)$$

Approximate equality holds for instance in the case of the 2-plasmon instability. The inequalities (2)-(5) were used in section 3.4 to derive equation 3.4-(13) which describes the behavior of small perturbations coupled together by the laser pump wave. The equation 3.4-(13) was written out more explicitly in 3.7-(5) and 3.7-(6), and specific instabilities were extracted from 3.7-(5) by the work of section 3.8. In this section we investigate the consequences of tightening the condition (5) to a requirement of phase-velocity disparity;

$$\left|\frac{W_{(0)}}{k_{(0)}}\right| \ll \left|\frac{W_{(\pm 1)}}{k_{(\pm 1)}}\right| \qquad\qquad 3.9\text{-}(6)$$

The immediate effect is to simplify the equation 3.7-(6) to the form (7),

where

$$\mu_{M(1)} \equiv \vec{e}_{M(1)} \cdot \vec{e}_{x1} \qquad\qquad 3.9\text{-}(8)$$

and so on. The factors of type (8) incorporate all the 3-dimensional geo-

metrical effects in the form of direction cosines. These pertain to the

angles between the polarizations of the high-phase-velocity perturbations

and the polarization of the laser-pump fundamental. On inspecting the ma-

trix in (7), simple relations between the entries become evident. The sim-

plified coupling coefficients appearing in the 1st, 2nd, 4th, and 5th rows

are related to quantities appearing in the 3rd row as follows (cf. 3.7-(5)).

$$\left|E_1^2\right|\left(F_{(1),1,-1}^{\triangle NL(3)}\right)_{ABxx} = \frac{\vec{e}_{A(1)}}{\epsilon_0 W_{(1)}} \cdot \vec{e}_{x1} q_e \vec{V}_1 \left[\frac{\vec{k}_{(0)} E_1^*\left(F_{(1),-1}^{NL(2)}\right)_{SB}}{q_e/\epsilon_0}\right] \qquad 3.9\text{-}(9)$$

$$E_1\left(F_{(0),1}^{NL(2)}\right)_{ASx} = \frac{\vec{e}_{A(1)}}{\epsilon_0 W_{(1)}} \cdot \vec{e}_{x1} q_e \vec{V}_1 \left[\frac{\vec{k}_{(0)} D_e}{q_e/\epsilon_0}\right] \qquad 3.9\text{-}(10)$$

$$\frac{1}{2} E_1^2\left(F_{(-1),1,1}^{\triangle NL(3)}\right)_{ABxx} = \frac{\vec{e}_{A(1)}}{\epsilon_0 W_{(1)}} \cdot \vec{e}_{x1} q_e \vec{V}_1 \left[\frac{\vec{k}_{(0)} E_1\left(F_{(-1),1}^{NL(2)}\right)_{SB}}{q_e/\epsilon_0}\right] \qquad 3.9\text{-}(11)$$

$$\frac{1}{2} E_1^{*2}\left(F_{(1),-1,-1}^{\triangle NL(3)}\right)_{ABxx} = \frac{\vec{e}_{A(-1)}}{\epsilon_0 W_{(-1)}} \cdot \vec{e}_{x1} q_e \vec{V}_1^* \left[\frac{k_{(0)} E_1^*\left(F_{(1),-1}^{NL(2)}\right)_{SB}}{q_e/\epsilon_0}\right] \qquad 3.9\text{-}(12)$$

$$
\begin{pmatrix}
(D_{(1)})_{mm} - \dfrac{\mu_{m(1)}^2 k_{(0)}^2 |V_1|^2 w_P^2}{w_{(1)}^2\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{m(1)}\mu_{s(1)} k_{(0)}^2 |V_1|^2 w_P^2}{w_{(1)}^2\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{m(1)} k_{(0)} V_L w_P^2}{w_{(1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{m(1)}\mu_{m(-1)} k_{(0)}^2 V_1^2 w_P^2}{w_{(1)} w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{m(1)}\mu_{s(-1)} k_{(0)}^2 V_1^2 w_P^2}{w_{(1)} w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} \\[3ex]
-\dfrac{\mu_{m(1)}\mu_{s(1)} k_{(0)}^2 |V_1|^2 w_P^2}{w_{(1)}^2\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & (D_{(1)})_{ss} - \dfrac{\mu_{s(1)}^2 k_{(0)}^2 |V_1|^2 w_P^2}{w_{(1)}^2\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{s(1)} k_{(0)} V_1 w_P^2}{w_{(1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{s(1)}\mu_{m(-1)} k_{(0)}^2 V_1^2 w_P^2}{w_{(1)} w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{s(1)}\mu_{s(-1)} k_{(0)}^2 V_1^2 w_P^2}{w_{(1)} w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} \\[3ex]
-\dfrac{\mu_{m(1)} k_{(0)} V_1^* w_P^2}{w_{(1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{s(1)} k_{(0)} V_1^* w_P^2}{w_{(1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & (D_{(0)})_{ss} & -\dfrac{\mu_{m(-1)} k_{(0)} V_1 w_P^2}{w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{s(-1)} k_{(0)} V_1 w_P^2}{w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} \\[3ex]
-\dfrac{\mu_{m(-1)}\mu_{m(1)} k_{(0)}^2 V_1^{*2} w_P^2}{w_{(1)} w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{m(-1)}\mu_{s(1)} k_{(0)}^2 V_1^{*2} w_P^2}{w_{(1)} w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{s(-1)} k_{(0)} V_1^* w_P^2}{w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & (D_{(-1)})_{mm} - \dfrac{\mu_{m(-1)}^2 k_{(0)}^2 |V_1|^2 w_P^2}{w_{(-1)}^2\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{m(-1)}\mu_{s(-1)} k_{(0)}^2 |V_1|^2 w_P^2}{w_{(-1)}^2\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} \\[3ex]
-\dfrac{\mu_{s(-1)}\mu_{m(1)} k_{(0)}^2 V_1^{*2} w_P^2}{w_{(1)} w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{s(-1)}\mu_{s(1)} k_{(0)}^2 V_1^{*2} w_P^2}{w_{(1)} w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{s(-1)} k_{(0)} V_1^* w_P^2}{w_{(-1)}\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & -\dfrac{\mu_{s(-1)}\mu_{m(-1)} k_{(0)}^2 |V_1|^2 w_P^2}{w_{(-1)}^2\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)} & (D_{(-1)})_{ss} - \dfrac{\mu_{s(-1)}^2 k_{(0)}^2 |V_1|^2 w_P^2}{w_{(-1)}^2\left(w_{(0)}^2 - \gamma k_{(0)}^2 v_T^2\right)}
\end{pmatrix}
\begin{pmatrix}
E_{m(1)} \\[2ex] E_{s(1)} \\[2ex] E_{s(0)} \\[2ex] E_{m(-1)} \\[2ex] E_{s(-1)}
\end{pmatrix}
=
\begin{pmatrix}
0 \\[2ex] 0 \\[2ex] 0 \\[2ex] 0 \\[2ex] 0
\end{pmatrix}
$$

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3.9-(7)

$$E_1^* \left(F_{(0),-1}^{NL(2)} \right)_{ASx} = \frac{\vec{e}_{A(-1)}}{\epsilon_0 w_{(-1)}} \cdot \vec{e}_{x1} q_e V_1^* \left[\frac{\vec{k}_{(0)} D_e}{q_e/\epsilon_0} \right]$$
$$\text{3.9-(13)}$$

$$|E_1|^2 \left(F_{(-1),1,-1}^{NL(3)} \right)_{ABxx} = \frac{\vec{e}_{A(-1)}}{\epsilon_0 w_{(-1)}} \cdot \vec{e}_{x1} q_e V_1^* \left[\frac{k_{(0)} E_1 \left(F_{(-1),1}^{NL(2)} \right)_{SB}}{q_e/\epsilon_0} \right]$$

$$\text{3.9-(14)}$$

Here

$$\left(D_{(0)} \right)_{SS} \equiv 1 + D_e + D_i$$
$$\text{3.9-(15)}$$

$$D_e = - \frac{\omega_p^2}{\omega_{(0)}^2 - \gamma k_{(0)}^2 v_T^2}$$
$$\text{3.9-(16)}$$

The relations (9)-(11) admit a simple physical explanation. The quantities in square brackets are just the linear and nonlinear contributions to the low-frequency electron density at $(\vec{k}_{(0)}, w_{(0)})$ from unit perturbation fields. The quantities in square brackets, multiplied by $\vec{e}_{x1} q_e V_1$, are the high-frequency nonlinear currents at $(\vec{k}_{(1)}, w_{(1)})$ caused by the low-frequency electron density-bunches quivering in the laser field. Similarly, the relations (12)-(14) may be physically explained. The quantities in square brackets are again the normalized contribution to the low-frequency electron bunching at $(\vec{k}_{(0)}, w_{(0)})$. The quantities in square brackets, multiplied by $\vec{e}_{x1} q_e V_1^*$, are the high-frequency nonlinear currents at $(\vec{k}_{(-1)}, w_{(-1)})$ which are generated when the electron bunches quiver in the laser field. Thus the relations (9)-(14) state that the high-frequency perturbations are excited by currents consisting of low-frequency electron-bunches quivering in the

laser field. To complete the physical explanation of how laser-coupled perturbations can be driven unstable, one needs to explain physically how high-frequency perturbations can lead to low-frequency electron-bunching. To do this we return to the basic warm-fluid equations.

Consider the warm-fluid momentum equation for a single species in the form

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + \frac{1}{n} \nabla p = \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B}) \qquad \text{3.9-(17)}$$

This contains 2 nonlinearities, namely, a convective and a Lorentz nonlinearity, through which a low phase-velocity perturbation is excited by a high-phase-velocity perturbation acting with the laser field. Take the linear terms of (17) to describe the low phase-velocity perturbation and the nonlinear terms to be functions of the high phase-velocity perturbation field and the laser field. For both the latter

$$\dot{\vec{v}} = q\vec{E}/m \qquad \text{3.9-(18)}$$

Use this and the relation

$$\dot{\vec{B}} = -\nabla \times \vec{E} \qquad \text{3.9-(19)}$$

to combine the convective and Lorentz nonlinearities:

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{n} \nabla p + \nabla \left(\frac{v^2}{2} \right) = \frac{q\vec{E}}{m} \qquad \text{3.9-(20)}$$

The combined nonlinearity has the form of the gradient of an effective pressure of coherent oscillations, analogous to the preexisting thermal pressure of random motions. This has a component at $(\vec{k}_{(0)}, w_{(0)})$ due to

beating between the quivering motion due to the high-frequency perturbation and the quivering motion due to the laser. The low-frequency response to the resulting low-frequency pressure variations for unit perturbation field gives the form of the nonlinear coupling coefficients appearing in the 3rd row of (7). These therefore satisfy the relations

$$E_1\left(F^{NL(2)}_{(-1),1}\right)_{SBx} = D_e k_{(0)} V_1 \vec{e}_{x1} \cdot \frac{\vec{e}_{B(-1)}}{w_{(-1)}} \qquad 3.9\text{-}(21)$$

$$E_1^*\left(F^{NL(2)}_{(1),-1}\right)_{SBx} = D_e k_{(0)} V_1^* \vec{e}_{x1} \cdot \frac{\vec{e}_{B(1)}}{w_{(1)}} \qquad 3.9\text{-}(22)$$

In Chapter 4 the physical mechanisms driving specific instabilities will be gone through in detail and the mechanisms described in this section will be incorporated where appropriate.

Chapter 4

SPECIFIC INSTABILITIES IN LASER-PLASMA-PELLET INTERACTION

4.1   Outline of Chapter

4.2   Brillouin Instability

4.3   Modified Brillouin Instability

4.4   Filamentational and Modulational Instabilities

4.5   Plasmon-Phonon (Decay) Instability

4.6   Modified Plasmon-Phonon Instability, treated together
       with Oscillating Two-Stream Instability

4.7   Raman Instability

4.8   Two-Plasmon Instability

4.9   Coalesced Raman-Two-Plasmon Instability

4.1 Outline of Chapter

The general formalism developed in Chapter 2 for describing coherent wave-wave interactions was applied in Chapter 3 to laser-driven instabilities. In particular, Section 3.8 extracted the dispersion relations for specific instabilities. Those specific dispersion relations form the basis of this chapter.

In this chapter, each laser-driven instability is dealt with in detail. The interacting coherent waves are specified. The expected consequences of the instability for the laser-fusion concept are briefly outlined. Each instability is physically characterised either as leading mainly to absorption of laser energy by the plasma, or as leading mainly to reflection of laser energy by the plasma. Further, the range of plasma densities, and hence the spatial region of pellet-blowoff-plasma, in which the instability can exist, is delineated.

For each laser-driven instability, the physical mechanisms leading to growth of the perturbations are described. In some cases, the description of Section 3.9 will prove useful for this purpose. The full three-dimensional dispersion relation is written down, using the physical approximations appropriate to those waves which are interacting. The variation of the growthrate as the directions of the propagation vectors are varied in three dimensions is traced back to the physics of the

interacting waves.

The stability analysis of Briggs and Bers is used to obtain, from the dispersion relation, the corresponding time-asymptotic Green's function. This constitutes the response of the unstable system to an initial spatially localized pulse-excitation. The stability analysis is carried out in one dimension in the direction of maximum growth. For three-dimensional stability analysis of the unmodified instabilities, the interested reader is referred to the work of Bers and Chambers[1]. Computations of three-dimensional pulse responses for modified and 3rd order laser-driven instabilities are relegated to future work.

The physical parameters for the stability analysis are those of a 1 KeV plasma irradiated by a neodymium-glass laser with vacuum wavelength 1.06μ. The laser E-field is taken to be that of a beam of intensity 10^{15} watts/cm^2. For some coalesced and/or 3rd-order instabilities, the Green's function describing the pulse response may have a complicated structure. If such is the case for a particular laser-driven instability, the corresponding stability analysis is repeated using a lower laser-pump intensity. The physical discussion of sections 3.4 and 3.8 can then be used to clarify the structure of the Green's function.

4.2 The Brillouin Instability

The Brillouin instability is a coherent wave-wave inter-
action in which an electromagnetic perturbation and a low-
frequency electrostatic perturbation are coupled together by
the laser pump. The Brillouin instability occurs at low enough
pump-wave intensities so that the perturbations closely resemble
a linear electromagnetic wave and an ion-acoustic wave respec-
tively. In quantum language, the laser photon decays into
another photon and a phonon. As discussed in sections 3.4 and
3.8, the locus of the instability in the $(\vec{k},w)$ 4-space is found
from the intersection of the ion-acoustic dispersion hypersur-
face and the upshifted electromagnetic dispersion hypersurface.
However, the physical mechanisms driving the Brillouin insta-
bility, and the most important consequences of the instability's
existence, can be studied adequately using only the case of
collinear propagation. Thus, the locus of the instability will
be illustrated on a $(\vec{k}_z,w)$ diagram first. Then, an attempt will
be made to sketch the kinematics using two spatial dimensions,
with the vectors $\vec{k}$ lying in a plane containing the propagation
vector of the laser. The locus of the instability will thus be
illustrated on a $(k_z,\vec{k}_\perp,w)$ diagram. The kinematics are invar-
iant with respect to rotation about the direction of laser
propagation.

The Brillouin instability leads to scattering of laser
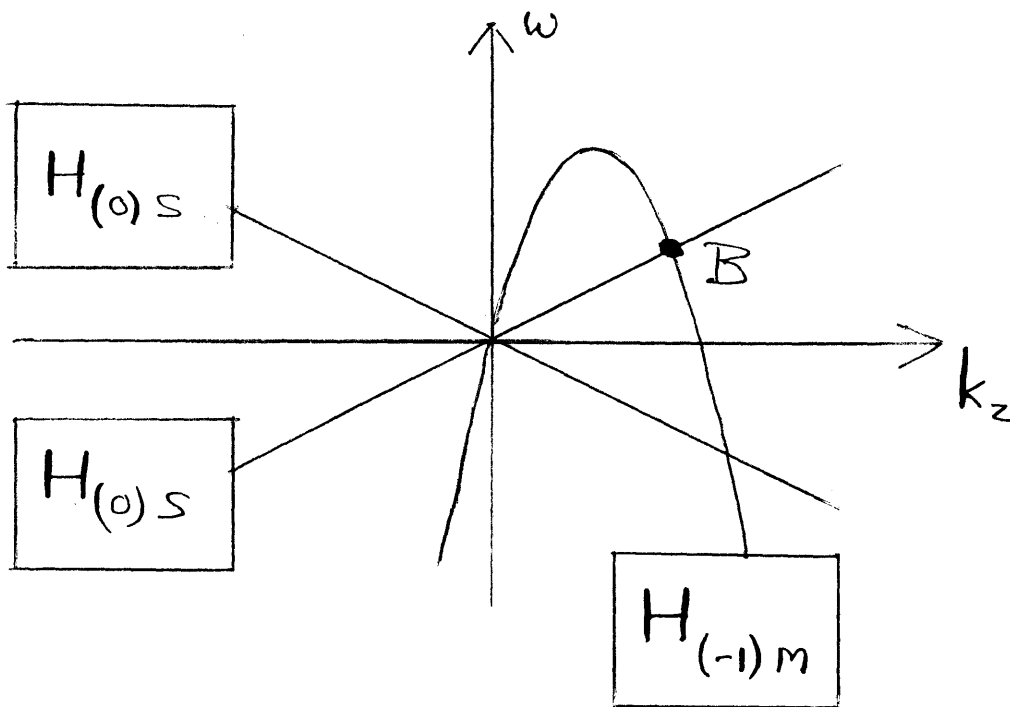energy by the plasma. Since the frequency of the scattered

Fig 4.2.1.

PROPAGATION COLLINEAR WITH LASER
PROPAGATION.

Here $H_{(0)S}$ is the ion-acoustic dispersion curve.

$H_{(-1)M}$ is the electromagnetic dispersion curve, upshifted by the laser wavevector and frequency.

B is the locus of Brillouin instability, near which the wavenumber and frequency ($k_{(0)z}$, $w_{(0)}$) of the low-frequency perturbation lie for small pump-wave intensities.

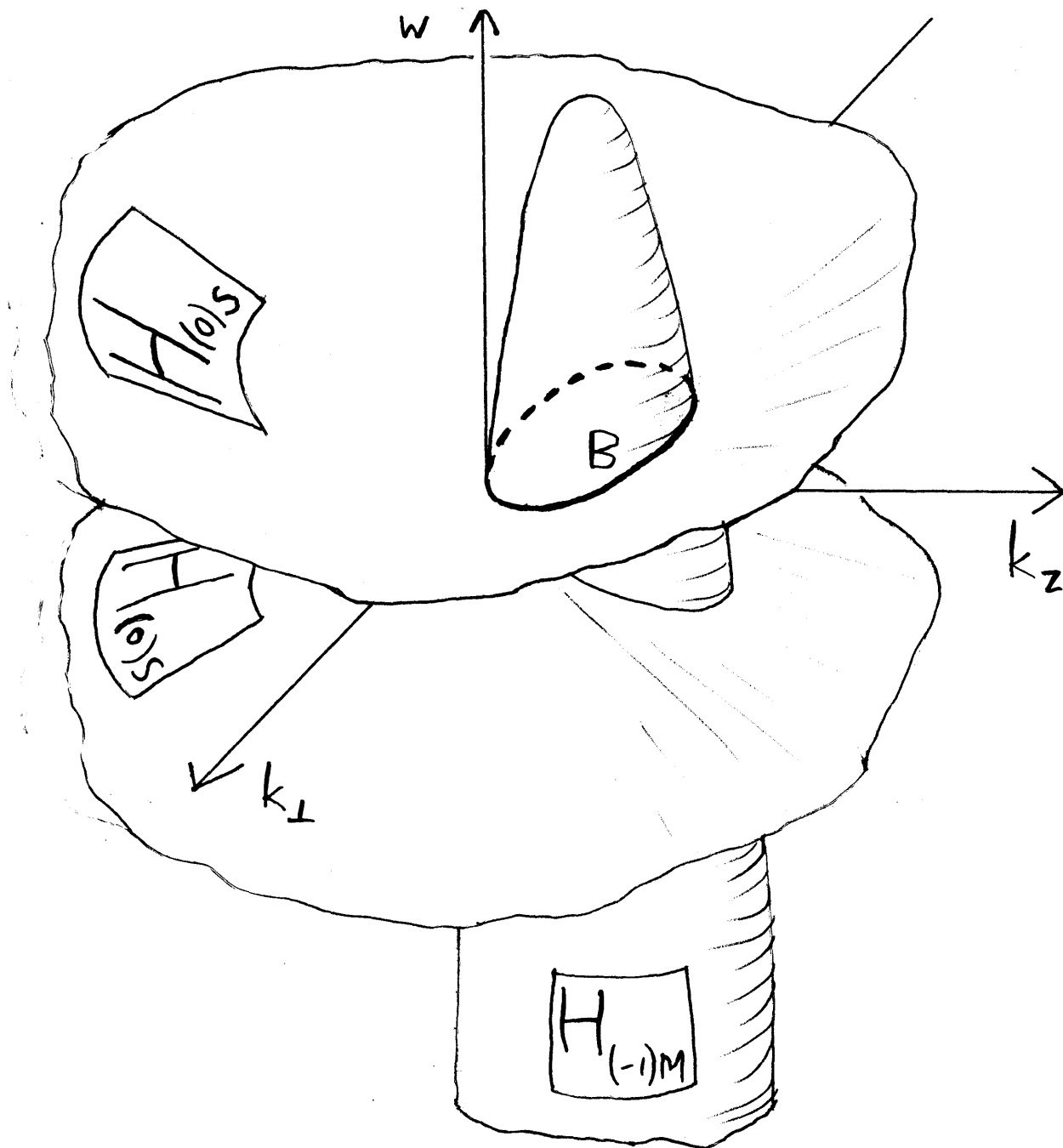The origin is also a point of intersection of the dispersion curves but does not yield instabilities.

Fig. 4.2.2 : PROPAGATION IN PLANE CONTAINING DIRECTION OF LASER PROPAGATION

Here $H_{(0)s}$ is the ion-acoustic dispersion surface.

$H_{(-1)m}$ is the upshifted electromagnetic dispersion surface.

B is the locus of Brillouin instability. The values of $(\vec{k}_{(0)}, w_{(0)})$ lie near this locus.

electromagnetic wave is so much greater than that of the ion-acoustic wave, the Manley-Rowe relations predict very little absorption of laser energy by the plasma. The instability can occur at any plasma density into which the laser pump-wave can propagate, that is, any plasma density less than the critical density

$$w_p < w_1 \qquad . \qquad\qquad 4.2-(1)$$

In terms of the spherical pellet geometry, this means that the Brillouin instability can occur in the region outside the (spherical) critical surface.

Having indicated the kinematics of the interacting waves, let us now move on to the dynamics underlying the instability. In section 3.9, it was shown that two high-phase velocity waves beating together can cause, by "radiation pressure", a low-phase velocity modulation of the electron density. More strictly, a term appears in the low-frequency fluid momentum equation which has the form of an effective pressure due to coherent oscillations. In this section, one of the two high-phase velocity waves is the laser pump and the other is the electromagnetic perturbation.

In section 3.9, it was further shown that the low-phase-velocity electron density modulation, or "bunching", oscillating or "quivering" in the laser field constitutes a current which can regenerate the other high-frequency perturbation. Thus, depending on the relative phases and three-dimensional

alignment of the interacting waves, a regenerative or positive feedback loop can be set up, causing the high- and low-phase-velocity perturbations to grow together. This constitutes the laser-driven instability.

We complete the description of the physical driving mechanisms by considering the simplest case -- that of collinear propagation, the point B in Fig. 1. Recall that the laser pump-wave propagates along z and is polarized along x, with wavenumber and frequency (k_1, w_1). The point B describes an ion-acoustic wave with wavenumber and frequency $(k_{(0)}, w_{(0)})$. also propagating in the positive z-direction. This ion-acoustic perturbation is coupled to an electromagnetic perturbation, with wavenumber and frequency

$$(k_{(-1)}, \; w_{(-1)}) \; \equiv \; (k_{(0)}, w_{(0)}) \; - \; (k_1, w_1) \quad . \qquad 4.2\text{-}(2)$$

In terms of the more usual convention that all frequencies should be positive, the wavenumber and frequency of the electromagnetic perturbation can be taken to be

$$(-k_{(-1)}, \; -w_{(-1)}) \; \equiv \; (k_1, w_1) \; - \; (k_{(0)}, \; w_{(0)}) \quad . \quad 4.2\text{-}(3)$$

The electromagnetic perturbation is thus a backscattered wave. The propagation vectors look like this:

$\vec{k}_1$ (laser pump)

$\vec{k}_{(0)}$ (ion-acoustic wave)   z

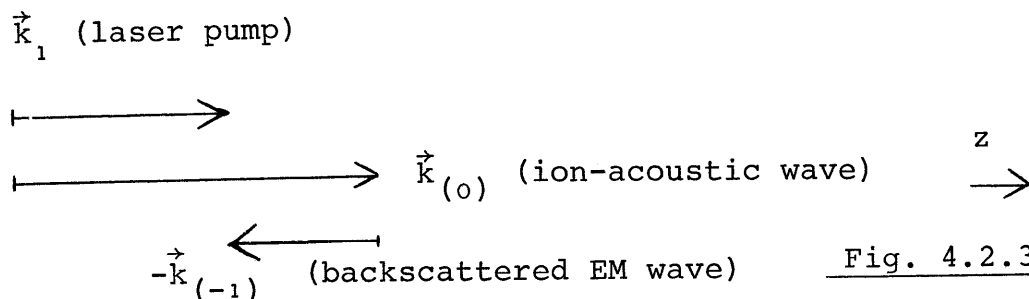$-\vec{k}_{(-1)}$ (backscattered EM wave)   Fig. 4.2.3

The physical picture of the mechanisms driving the instability appears as follows. The laser and the backscattered wave form beats leading to a pattern of effective pressure of oscillations. This pattern has the wavenumber $k_{(o)}$, which is about twice the wavenumber of either the laser pump or the backscattered electromagnetic wave, and moves forward in the z-direction at the sound speed. If the pattern were stationary in the plasma, the electrons would move into the regions of minimum high-frequency field, dragging the ions with them, so that striations of higher plasma density would form in the troughs of the "radiation-pressure" patterns. However, the pattern is moving forward at the sound speed, and so the striations of higher plasma density form on the forward slopes of the radiation pressure pattern (see figures next page).

The phase relationship between the radiation pressure pattern and the plasma density striations is determined a posteriori from the coupling coefficient matrix (5) governing the instability. Thus, the description presented here of physical mechanisms is presented as an aid to insight and does not constitute an independent derivation of the unstable dispersion relation.

It may easily be checked that this phase relation is such as to feed energy into the acoustic oscillations. Recall that in a sound wave, the velocity in the direction of propagation is in phase with the density. The radiation
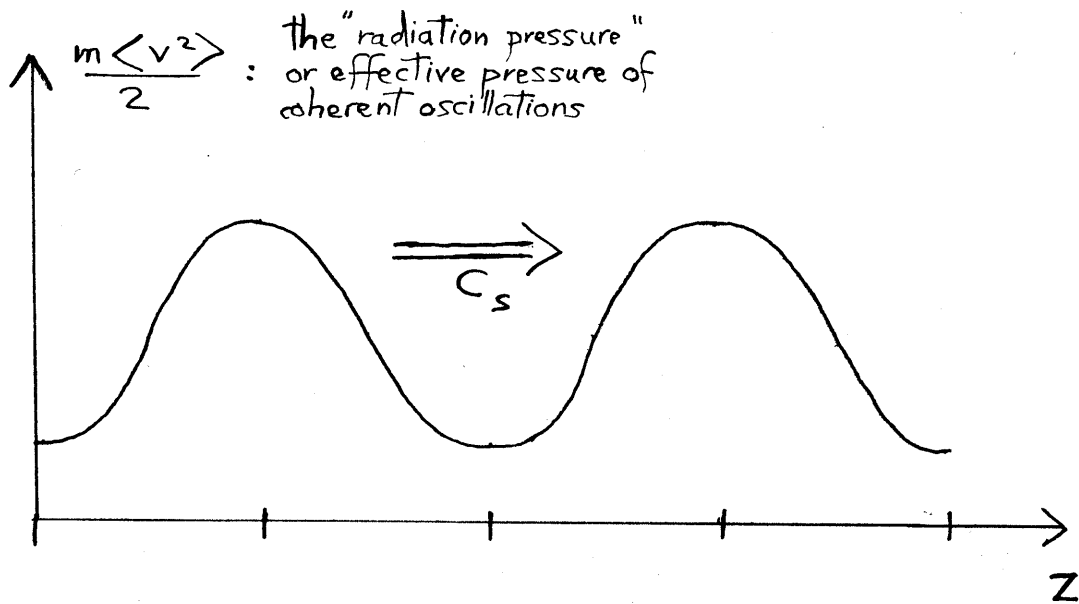
$\dfrac{m\langle v^2\rangle}{2}$: the "radiation pressure" or effective pressure of coherent oscillations

$\overrightarrow{} \atop C_s$

Z

Fig 4.2.4

regions of higher plasma density (striations)
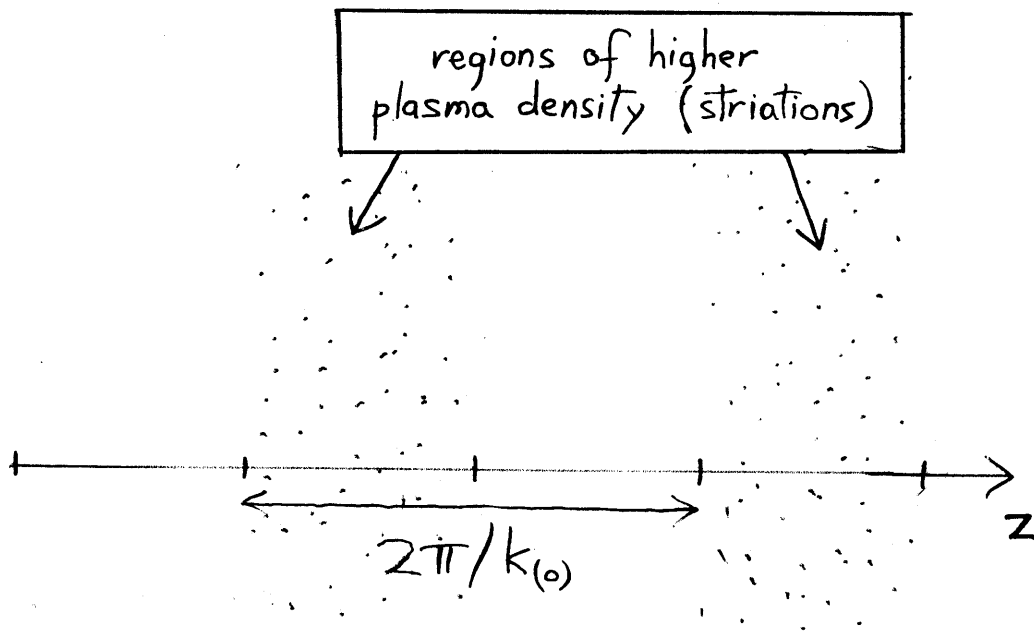
$2\pi/k_{(0)}$

Z

Fig. 4.2.5

pressure pattern therefore does net work on the plasma density striation, since it exerts a forward force on forward-moving plasma, and a backward force on backward-moving plasma. The feedback loop may be closed, as before, by noting that the high-frequency current formed by electrons oscillating in the x-directed laser field now has a variation in the z-direction. One may think of the striated plasma with its x-oscillating electrons as constituting an endfire-antenna-array which regenerates the backward electromagnetic perturbation in the z-direction. Alternatively, one may think of the density striations as forming a lattice which backscatters the incident laser.

The full three-dimensional dispersion relation for the Brillouin instability may be written down from 3.8-(6). Using physical approximations appropriate to the ion-acoustic and electromagnetic waves, 3.8-(6) becomes

$$
\begin{vmatrix}
1 + \dfrac{w_{pe}^2}{k_{(o)}^2 V_{Te}^2} - \dfrac{w_{pi}^2}{w_{(o)}^2} & -\dfrac{V_1 w_p^2 [\vec{e}_{x1} \cdot \vec{e}_{M(-1)}]}{\left| w_{(-1)} \right| k_{(o)} V_{Te}^2} \\[4ex]
\dfrac{V_1^* w_p^2 [\vec{e}_{x1} \cdot \vec{e}_{M(-1)}]}{\left| w_{(-1)} \right| k_{(o)} V_{Te}^2} & -\dfrac{k_{(-1)}^2 c^2}{w_{(-1)}^2} + 1 - \dfrac{w_p^2}{w_{(-1)}^2}
\end{vmatrix} = 0
$$

4.2-(4)

Since the unmodified Brillouin instability is defined only for growthrates small compared to the ion-acoustic frequency at the point of onset (e.g., point B in Fig. 1), the ion-

acoustic dispersion function occurring in (4) and, a fortiori, the electromagnetic dispersion function, may be linearized about that point B. For convenience, first factor out some powers of $k_{(o)}\lambda_{De}$. Then Taylor-expand the dispersion function to first order, about some wavevector and frequency $(\vec{k}_{(o)}, w_{(o)})$ on the intersection of the ion-acoustic dispersion hypersurface and the pump-upshifted electromagnetic dispersion hypersurface. Then (4) becomes

$$
0 = \left|
\begin{array}{cc}
\left[\dfrac{2(\Delta w - \Delta\vec{k}\cdot\vec{e}_{s(o)}C_s)}{w_{IA}}\right]\dfrac{1}{k_{(o)}^2\lambda_{De}^2} & -\left[\dfrac{V_1}{V_{Te}}\ \dfrac{w_p[\vec{e}_{x_1}\cdot\vec{e}_{M(-1)}]}{w_{EM}}\right]\dfrac{1}{k_{(o)}\lambda_{De}} \\[4ex]
-\left[\dfrac{V_1^*}{V_{Te}}\ \dfrac{w_p}{w_{EM}}[\vec{e}_{x_1}\cdot\vec{e}_{M(-1)}]\right]\dfrac{1}{k_{(o)}\lambda_{De}} & \dfrac{2(\Delta w + \Delta\vec{k}\cdot\vec{e}_{s(-1)}\left|V_g^{EM}\right|)}{-w_{EM}}
\end{array}
\right|
$$

<div align="right">4.2-(5)</div>

Here, $w_{IA} \equiv w_{(o)} \equiv k_{(o)}c_s$ is the ion-acoustic frequency

$w_{EM} \equiv -w_{(-1)}$ is the frequency of the backscattered electromagnetic wave

$\vec{e}_{s(o)}c_s \equiv \vec{V}_g^{IA}$ is the ion-acoustic group velocity

$-\vec{e}_{s(-1)}\left|V_g^{EM}\right| \equiv \vec{V}_g^{EM}$ is the group velocity of the backscattered electromagnetic wave.

Introduce $V_{Laser} \equiv |\vec{V}_{Fund}(\vec{x},t)|_{Peak} \equiv 2|V_1|$ 4.2-(6)

[See 3.3 - (2)]. Expand (5); the result is the dispersion rela-
tion for the three-dimensional Unmodified Brillouin instability:

$$(\Delta w - \Delta\vec{k}\cdot\vec{e}_{s(o)}C_s)(\Delta w + \Delta\vec{k}\cdot\vec{e}_{s(-1)}|V_g^{EM}|)$$

$$= -\frac{V_{Laser}^2}{16V_{Te}^2} w_p^2 \frac{w_{IA}}{w_{EM}} (\vec{e}_{x_1}\cdot\vec{e}_{M(-1)})^2 \qquad . \qquad 4.2-(7)$$

The maximum growthrate is obtained from this dispersion relation
by setting $\Delta\vec{k} = \vec{0}$, which yields

$$\Delta w = i \frac{V_{Laser}}{4 V_{Te}} w_p \sqrt{\frac{w_{IA}}{w_{EM}}} |\vec{e}_{x_1}\cdot\vec{e}_{M(-1)}| \qquad 4.2-(8)$$

This growthrate depends on the three-dimensional nature of the
kinematics via the square-root factor and on the three-dimens-
ional nature of the polarizations via the last factor. This
last factor describes the effectivenesss of electrons oscil-
lating in the laser field along $\vec{e}_{x_1}$ as a means of regenerating
the scattered electromagnetic wave polarized along $e_{M(-1)}$. For
electromagnetic radiation scattered at an angle θ to the laser
polarization, i.e., such that

$$\vec{e}_{x_1}\cdot\vec{k}_{(-1)}/|k_{(-1)}| \equiv \vec{e}_{x_1}\cdot\vec{e}_{s(-1)} = \cos\theta \qquad 4.2-(9)$$

the effectiveness of regeneration by electrons oscillating in the
laser field is (see Fig. 6):

$$|\vec{e}_{x_1} \cdot \vec{e}_{M(-1)}| = \sin \theta \qquad\qquad 4.2\text{-}(10)$$

The unit polarizations 3.4-(4,5,6) were indeed chosen so that, for a fixed $\vec{k}_{(n)}$, the polarization $\vec{e}_{M(n)}$ would be the electromagnetic polarization closest to the polarization of the laser. Thus, the purely geometrical dependence of (8) on the scattering angle may be dexcribed as follows. Recall that the Cartesian axes were chosen such that the laser propagates along z and is polarized along x. The last factor in (8) tells us that the Brillouin interaction is most effective in scattering electromagnetic radiation into the y-z plane. The Brillouin interaction is not effective at all in scattering electromagnetic radiation into the x-direction.

The three-dimensional dispersion relation (7) can be used to perform a three-dimensional stability analysis in the manner described by Bers and Briggs. The time-asymptotic Green's function is evaluated by a saddle-point method. This yields the space-time history of a disturbance excited by an initially localized pulse. Time-asymptotically, this history associates with every observer moving away from the initial excitation point a corresponding exponential growthrate of the disturbance as seen by that observer. The time-asymptotic development of the disturbance can thus be indicated by a graph of observed growthrate as a function of observer velocity. In two and three dimensions, this graph becomes a contour map.

The dispersion relation (7) employs linearized dispersion relations for the two perturbations which enter the coupling. The kinematics of the interaction are the same for all planes
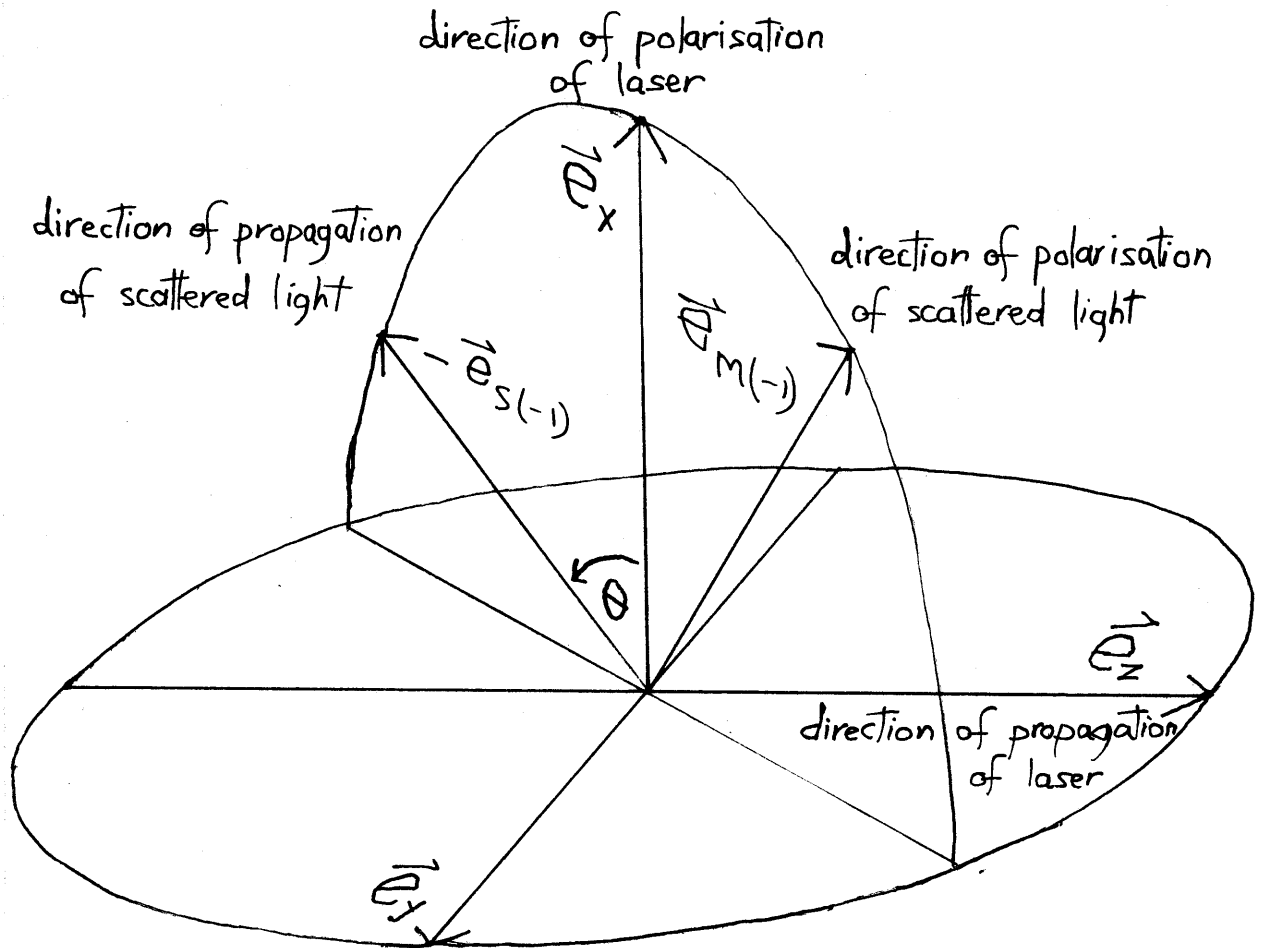
direction of polarisation of laser

$\vec{e}_x$

direction of propagation of scattered light

direction of polarisation of scattered light

$-\vec{e}_{S(-1)}$

$\vec{e}_{M(-1)}$

θ

$\vec{e}_z$

direction of propagation of laser

$\vec{e}_y$

Fig. 4.2.6.   PROPAGATION AND POLARISATION OF SCATTERED LIGHT ILLUSTRATED BY UNIT VECTORS.

$$k_{(0)} \vec{e}_{s(0)} \equiv k_{(0)}$$

Ion-Acoustic Wavevector

$$-k_{(-1)} \vec{e}_{s(-1)}$$
$$\equiv -\vec{k}_{(-1)}$$

Scattered Light
Wavevector

$$k_1 \vec{e}_z \equiv \vec{k}_1$$

Laser Wavevector
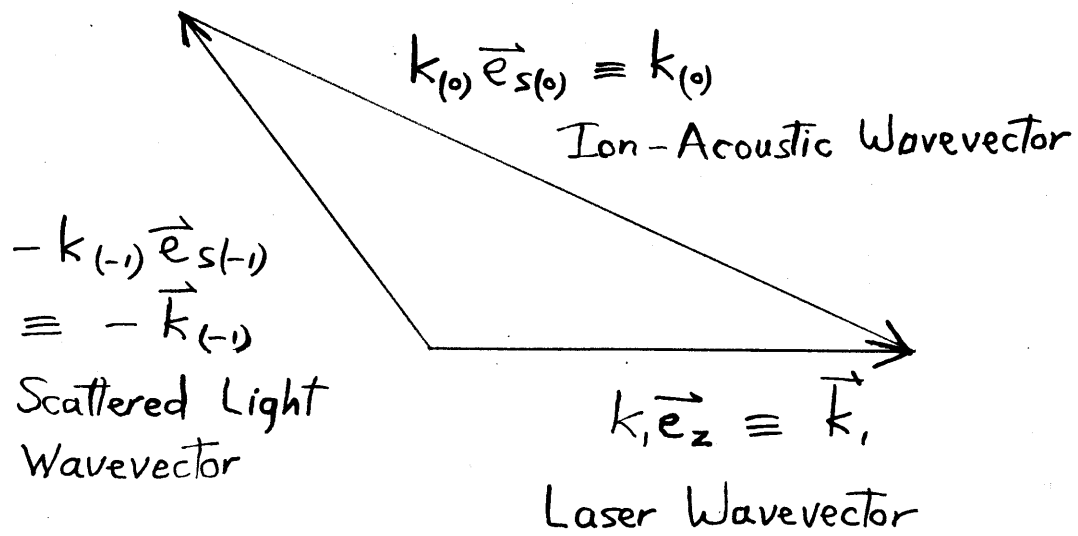
Fig. 4.2.7. PROPAGATION VECTORS OF LASER AND OF DECAY PRODUCTS.

containing the direction of propagation of the laser. The dynamics of the interaction are the same for all planes containing the direction of propagation of the laser except for the last factor in (8), whose effect on the coupling was described above. These features of the dispersion relation (7) allow the three-dimensional stability analysis of the unmodified Brillouin instability to be reduced to a two-dimensional problem, as shown by Bers and Chambers. The contour map of observed growthrate versus observer velocity is plotted in the plane of maximum interaction, namely the plane of propagation perpendicular to the laser polarization. The contour map in other planes is obtained by multiplying the growthrates by the appropriate values of the geometrical factor (10). Thus, the full three-dimensional pulse response may be built up.

For purposes of comparison with the modified case to be described in the next section, we sketch here that part of the pulse response generated by propagation collinear with the direction of propagation of the laser pump. This is the one-dimensional cross section along the z-axis of the full three-dimensional response. Its leading edge advances, in the direction of laser propagation, at the group velocity C_s of the ion-acoustic wave. Its rear edge propagates backwards, along the laser beam, at the group velocity V_g^{EM} of the backscattered electromagnetic wave. The exponential growthrate is positive for intermediate velocities and is greatest at that velocity which is the algebraic mean of the two edge velocities. This

maximum growthrate is the same as (8), the maximum growth-rate for real $\Delta \vec{k}$, deduced directly from the dispersion relation (7). The pulse response in the direction of laser propagation, then, looks like this:

Fig. 4.2.8
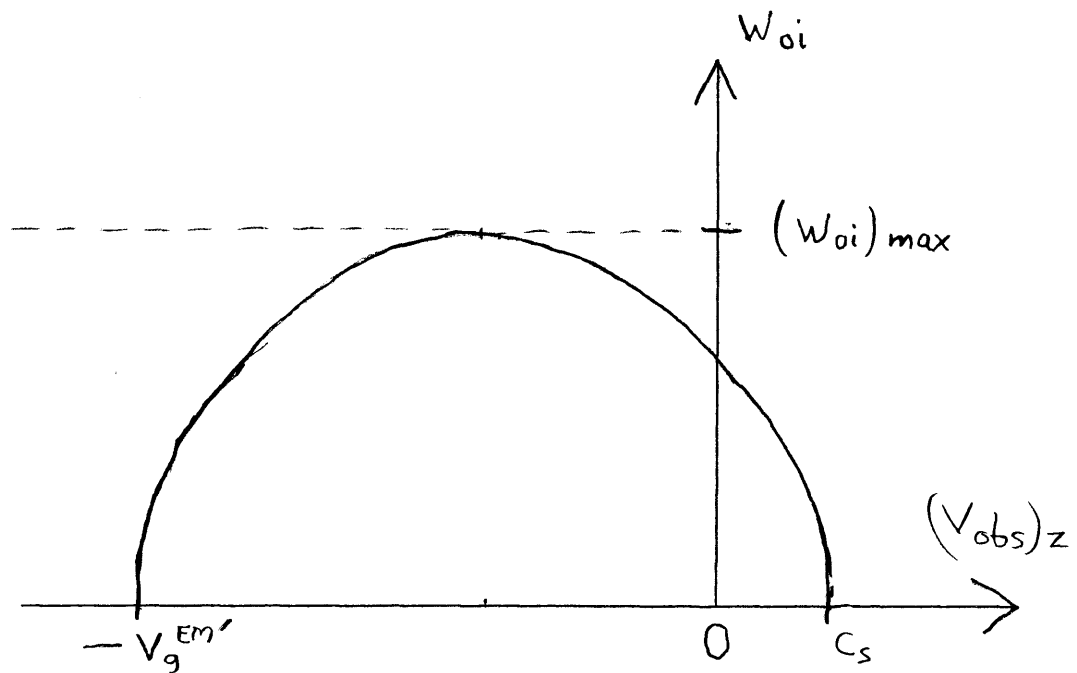
Observed growthrate of disturbance vs. observer velocity, for observer moving along laser beam.

$$(w_{oi})_{Max} = i \frac{V_{Laser}}{4\, V_{Te}}\, w_p \sqrt{\frac{w_{IA}}{w_{EM}}}$$

4.3 The Modified Brillouin Instability

The modified Brillouin instability, like the unmodified,
is a coherent wave-wave interaction in which the laser pump
couples together an electromagnetic perturbation and a low-
frequency electrostatic perturbation. However, the modified
Brillouin instability occurs at pump-wave intensities which
are high enough so that the growthrate is comparable to the
frequency of the low-frequency perturbation. Thus, the latter
no longer closely resembles an ion-acoustic normal mode of the
plasma. Its wavevector and frequency $(\vec{k}_{(o)}, w_{(o)})$ are no long-
er restricted to be near the intersections of the uncoupled
normal mode dispersion surfaces in $(\vec{k}, w)$ space illustrated in
Figs. 4.2.1 and 4.2.2.

The physical driving mechanisms are the same as for the
unmodified Brillouin instability; only the exact phase rela-
tions are changed. Again, the laser beam and the scattered
electromagnetic wave beat and induce density striations in
the plasma. The electron density striations oscillating in
the laser field form an antenna array which retransmits and
regenerates the electromagnetic perturbation.

The physical import of the modified Brillouin instability
for the laser-pellet fusion process is that the modified
growthrates increase with laser intensity more slowly than the
unmodified calculations would indicate. The qualitative
effect of the Brillouin instability on the laser-fusion pro-
cess is not changed by the modification. The modified

Brillouin instability is still a mechanism for scattering
the incident laser energy, and to that extent preventing the
laser energy from being absorbed by the plasma.

The dispersion relation 4.2-(4) for the unmodified Bril-
louin instability contains, in the top-left corner of the
determinant, the ion acoustic dispersion function evaluated at
the wavevector and frequency $(\vec{k}_{(0)}, w_{(0)})$ of the electrostatic
decay product. For the unmodifed Brillouin instability that
electrostatic decay product closely resembles an ion-acoustic
wave and the ion-acoustic dispersion function is small. For
modified Brillouin instability that resemblance fades and the
ion-acoustic dispersion function is not small. Therefore, its
cofactor in the determinant must be evaluated to higher order
in $|v_1|$. The full three-dimensional relation for the modi-
fied Brillouin instability may be written down from 3.8-(7).
In this equation, the ion-acoustic dispersion function has a
cofactor, which comprises the electromagnetic dispersion func-
tion supplemented by the self-correction brought about by the
pump. Using the physical approximations appropriate to the
ion-acoustic and electromagnetic waves, 3.8-(7) becomes

$$\left|\begin{array}{cc} 1 + \dfrac{w_{pe}^2}{k_{(o)}^2 V_{Te}^2} - \dfrac{w_{pi}^2}{w_{(o)}^2} & -\dfrac{V_1 w_p [\vec{e}_{x1} \cdot \vec{e}_{m(-)}]}{V_{Te}[-w_{(-1)}]} \dfrac{w_p}{k_{(o)} V_{Te}} \\[3em] -\dfrac{V_1^* w_p [\vec{e}_{x1} \cdot \vec{e}_{m(-1)}]}{V_{Te}[-w_{(-1)}]} \dfrac{w_p}{k_{(o)} V_{Te}} & -\dfrac{k_{(-1)}^2 c^2}{w_{(-1)}^2} + 1 - \dfrac{w_p^2}{w_{(-1)}^2} - \dfrac{|V_1^2| w_p^2 [\vec{e}_{x1} \cdot \vec{e}_{M(-1)}]^2}{V_{Te}^2 \; w_{(-1)}^2} \end{array}\right| = 0$$

<div align="right">4.3-(1)</div>

Expanding, one obtains -- using 4.2-(6), the dispersion relation

<div style="border:1px solid black; padding:1em;">

Three-dimensional Modified Brillouin Instability

$$\underbrace{\left(1 - \dfrac{k_{(o)}^2 c_s^2}{w_{(o)}^2}\right)}_{\substack{\text{ion-acoustic}\\ \text{dispersion}\\ \text{function}}} \underbrace{\left(1 - \dfrac{w_p^2 + k_{(-1)}^2 c^2}{w_{(-1)}^2}\right)}_{\substack{\text{electromagnetic}\\ \text{dispersion}\\ \text{function}}} = - \underbrace{\dfrac{V_{Laser}^2}{4 \; V_{Te}^2} \dfrac{w_p^2}{w_{(-1)}^2}}_{\substack{\text{coupling}\\ \text{constant}}} \underbrace{[\vec{e}_{x1} \cdot \vec{e}_{M(-1)}]}_{\substack{\text{geometri-}\\ \text{cal}\\ \text{factor}}} \underbrace{\dfrac{k_{(o)}^2 \; c_s^2}{w_{(o)}^2}}_{\substack{\text{effect of}\\ \text{modifi-}\\ \text{cation}}}$$

<div align="right">4.3-(2)</div>

</div>

The electromagnetic dispersion function may still be linearized about a wavevector and frequency lying on the locus of the original unmodified Brillouin instability, as illustrated in Figs. 4.1.1, 4.1.2. Then Eq. (2) acquires the form
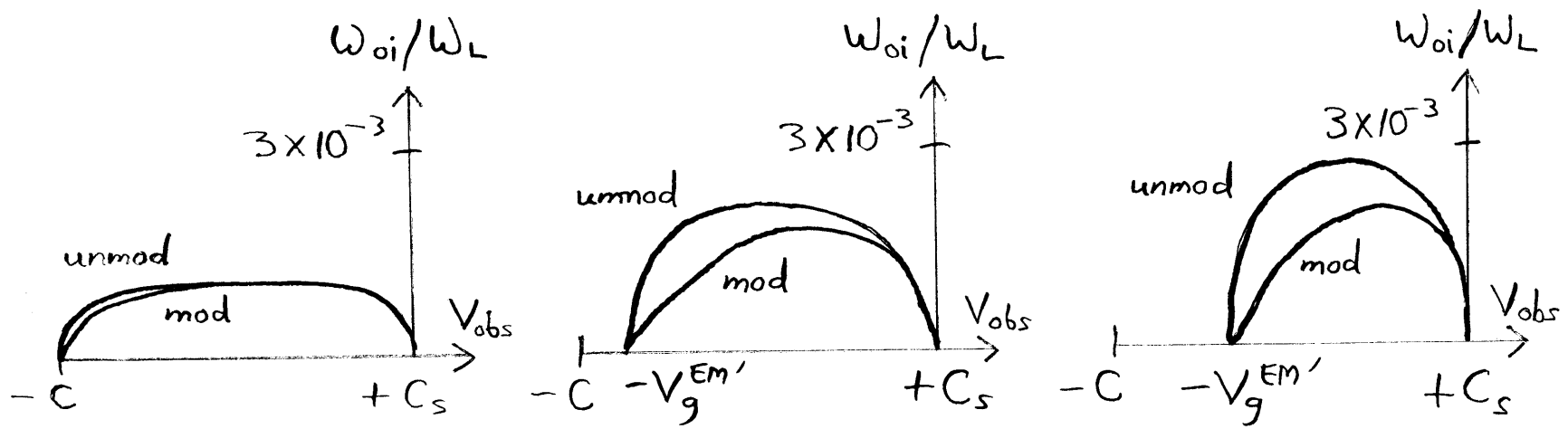
$$\left(1 - \frac{k^2_{(0)} c^2_s}{w^2_{(0)}}\right)\left(\frac{\Delta w + \Delta \vec{k} \cdot \vec{e}_{s\,(-1)} \, |V^{EM}_g|}{w_{EM}}\right) = -\frac{V^2_{Laser}}{8V_{Te}} \frac{w^2_p}{w^2_{EM}} [\vec{e}_{x_1} \cdot \vec{e}_{M\,(-1)}] \frac{k^2_{(0)} c^2_s}{w^2_{(0)}}$$

$$4.3-(3)$$

This is the form used for stability analysis of the modified Brillouin instability. The modification is brought about by high growthrates. The highest growthrates for the unmodified Brillouin stability analysis occur for propagation collinear with the direction of laser propagation. The effect of the modification is therefore expected to be most pronounced in that direction. A good idea of the type and extent of the changes in pulse propagation characteristics brought about by the modification can therefore be obtained by inspecting a one-dimensional cross section of the modified Brillouin pulse-response taken along the z- axis, and comparing it with the unmodified Brillouin pulse-response. The unmodified Brillouin pulse-response was illustrated in Fig. 4.2.6. The modified Brillouin pulse-response is illustrated below, with the unmodified result included for comparison. The changes in pulse propagation brought about by modification are as follows.

The velocities of the forward and backward edges are un-altered. The maximum gorwthrate is reduced and occurs for a smaller backward velocity. The extent of these effects is about 30% for the laser power level considered - namely, 10^{15} watts/cm^2. Since the Brillouin instability can occur for any plasma density below the critical density, a range of values of

$$\frac{\Delta(W_{oi}^{max})}{(W_{oi}^{max})_{unmod}}$$

$$\quad\quad -0.05 \quad\quad -0.15 \quad\quad -0.26$$

$$\frac{\Delta(V_{obs}(W_{oi}^{max}))}{(V_{obs}(W_{oi}^{max}))_{unmod}}$$

$$\quad\quad -0.09 \quad\quad -0.18 \quad\quad -0.29$$

$$n/n_{critical}$$

$$\quad\quad 0.06 \quad\quad 0.25 \quad\quad 0.56$$

Fig. 4.3.1.

w_p/w_1 between zero and unity is included in the figure.

Consider any one plasma density and its associated pulse-propagation diagram in Fig. 1. The highest growthrate for any observer velocity is also the highest for any real $\vec{k}$, obtained from the dispersion relation (3). For the unmodified Brillouin dispersion relation 4.2-(7), the highest growthrate for any real $\vec{k}$ is obtained by setting $\Delta\vec{k} = 0$, thus taking $\vec{k}$ to have the same value as for a vanishingly small pump intensity. For the modified Brillouin dispersion relation (3), this is no longer necessarily true; the value of real $\vec{k}_{(o)}$ for which the imaginary part of $w_{(o)}$ is greatest may shift around. Nevertheless, a lower bound for the maximum growthrate may be obtained by estimating the imaginary part of $w_{(o)}$ from the dispersion relation (3) with $\vec{k}_{(o)}$ set to a fixed value such that

$$\vec{k}_{(o)} C_s = w_{IA} \quad , \qquad\qquad 4.3-(4)$$

say,

has the same value as for a vanishingly small pump intensity. Then the dispersion relation (3) becomes approximately

$$\left(w_{(o)}^2 - w_{IA}^2 \right)\left(\frac{w_{(o)} - w_{IA}}{w_{EM'}} \right) = - \frac{V_{Laser}^2}{8V_{Te}^2} \frac{w_p^2}{w_{EM}} w_{IA}^2 [\vec{e}_{x_1} \cdot \vec{e}_{m(-1)}]^2 .$$

$$4.3-(5)$$

In the limit that the growthrate $w_{(o)i}$ greatly exceeds the original ion-acoustic frequency w_{IA}, this cubic equation for $w_{(o)}$ simplifies to

$$w_{(0)}^3 = -\frac{V_{Laser}^2}{8V_{Te}^2} \frac{w_p^2 w_{IA}^2}{w_{EM}} [\vec{e}_{x_1} \cdot \vec{e}_{m(-1)}]^2 . \qquad 4.3\text{-}(6)$$

One root of this equation has an imaginary part

$$w_{(0)i} = \frac{\sqrt{3}}{4} \left(\frac{V_{Laser}}{V_{Te}}\right)^{2/3} \left(\frac{w_p^2 w_{IA}^2}{w_{EM}}\right)^{1/3} [\vec{e}_{x_1} \cdot \vec{e}_{m(-1)}]^{2/3} . \qquad 4.3\text{-}(7)$$

This result is the basis of the oft-quoted statement that for modified instability, the growthrate varies as the 2/3 power of the laser field intensity. As was discussed earlier, the maximum growthrate observed in an unstable pulse is the same as the maximum imaginary part of the frequency root of the dispersion relation as the wavevector ranges over all real values. No simple approximation is known for this.

4.4   The Filamentational and Modulational Instabilities

The filamentational and modulational instabilities are coherent wave-wave interactions in which two electromagnetic perturbations and a low-frequency electrostatic perturbation are coupled together by the laser pump. The two electromagnetic perturbations have wavevectors and frequencies close to the wavevector and frequency of the laser. Indeed, the superposition of the two electromagnetic perturbations and the original laser pump field is equivalent to a modulated laser beam. The wavevector and frequency of the modulation are very small, and coincide with the wavevector and frequency of the electrostatic perturbation. When the modulation wavevector is exactly perpendicular to the laser wavevector, so that the light intensity varies across the laser beam but not along it, the term "filamentation instability" is used. When the modulation wavevector is not quite perpendicular to the laser wavevector, so that there is a slow modulation of the light intensity along any one ray of the beam, the term "modulation instability" is used. The electrostatic perturbation is not an ion-acoustic wave, although the ion-acoustic dispersion function is used to describe it. In quantum language, two laser photons exchange a virtual phonon.

The locus of the instabilities in $(\vec{k},w)$ space, as discussed in sections 3.4 and 3.8, is the region surrounding the point of tangency of two hypersurfaces. These two hypersurfaces

are the negative branch of the electromagnetic dispersion surface, upshifted by the pump wavevector and frequency, and the positive branch of the electromagnetic dispersion surface, downshifted by the pump wavevector and frequency. The laser pump itself satisfies the electromagnetic dispersion relation, to within the physical approximations introduced in 3.3-(29-32). Thus, the point of tangency, near which the values of $[\vec{k}_{(o)}, w_{(o)}]$ describing the electrostatic perturbation must lie, is just the origin of $(\vec{k}, w)$ space. These values of the modulation wavevector $\vec{k}_{(o)}$ cannot lie exactly at the origin, since that would be physically meaningless. The locus of the instability will be illustrated on four diagrams.

First, there will be a (k_z, w) diagram for modulation wavevectors parallel to the laser wavevector, which should be compared with the (k_z, w) diagram 4.2.1., which describes Brillouin instability. Secondly, there will be a $(k_z, k_\perp, w)$ diagram for modulation wavevectors in any one plane through the laser wavevector, which should be compared with the $(k_z, k_\perp, w)$ diagram 4.2.2. Thirdly, there will be a $(k_\perp, w)$ diagram describing the kinematics of the "filamentation instability". This $(k_\perp, w)$ diagram may be obtained by taking the $k_z = 0$ cross section of the $(k_z, k_\perp, k_w)$ diagram; taking another cross section of this same diagram at a slight angle to the $k_\perp$-axis, one obtains a fourth diagram describing the kinematics of "modulational instability". The diagrams follow.

Fig. 4.4.1 : KINEMATICS FOR MODULATION VECTOR
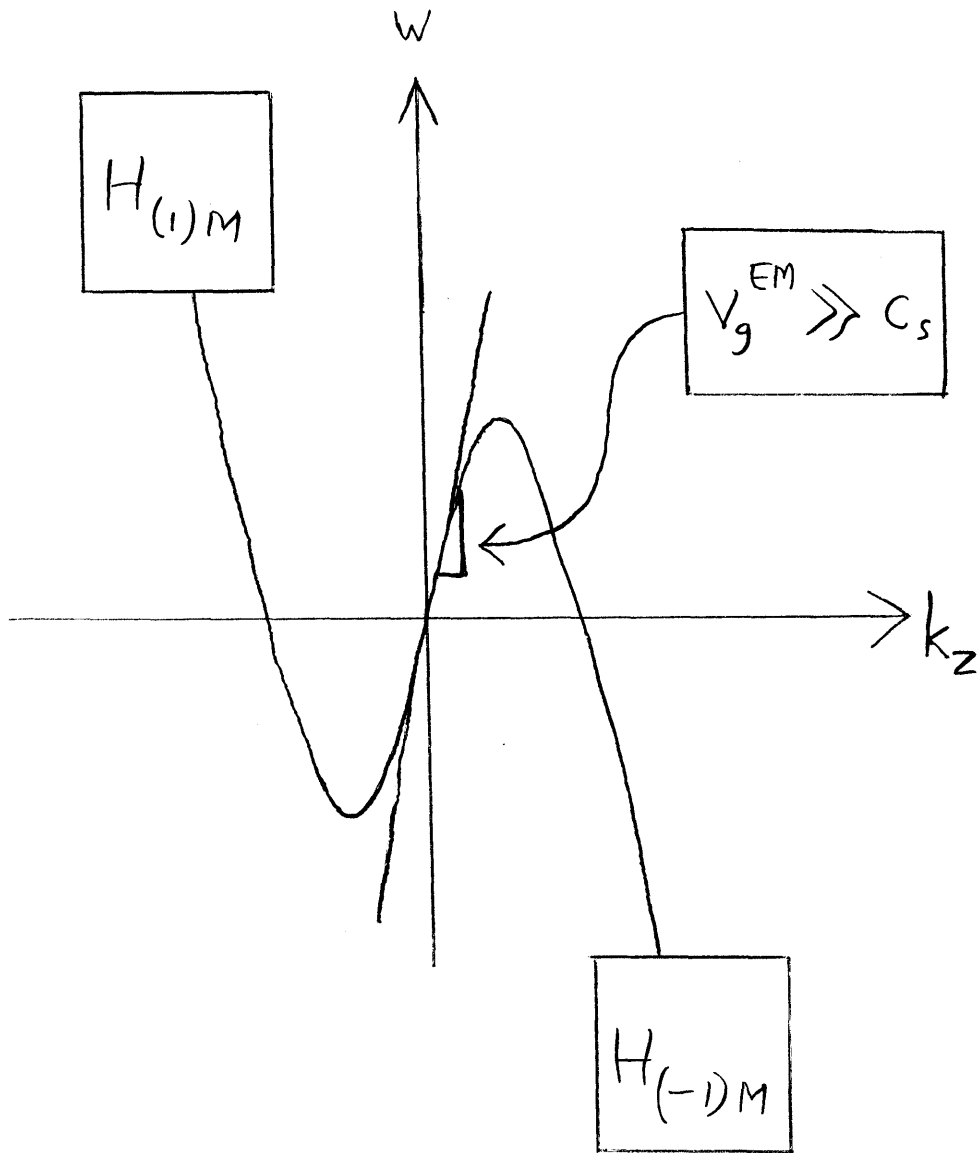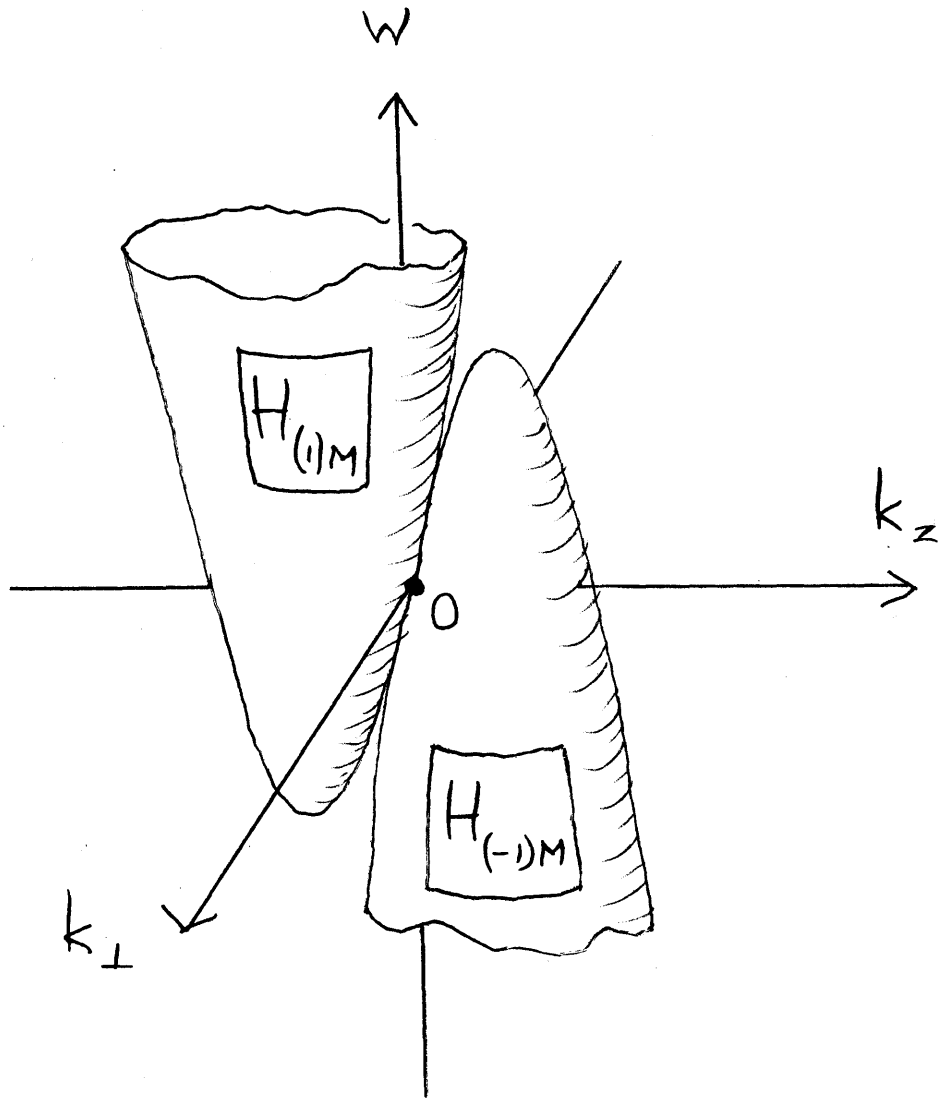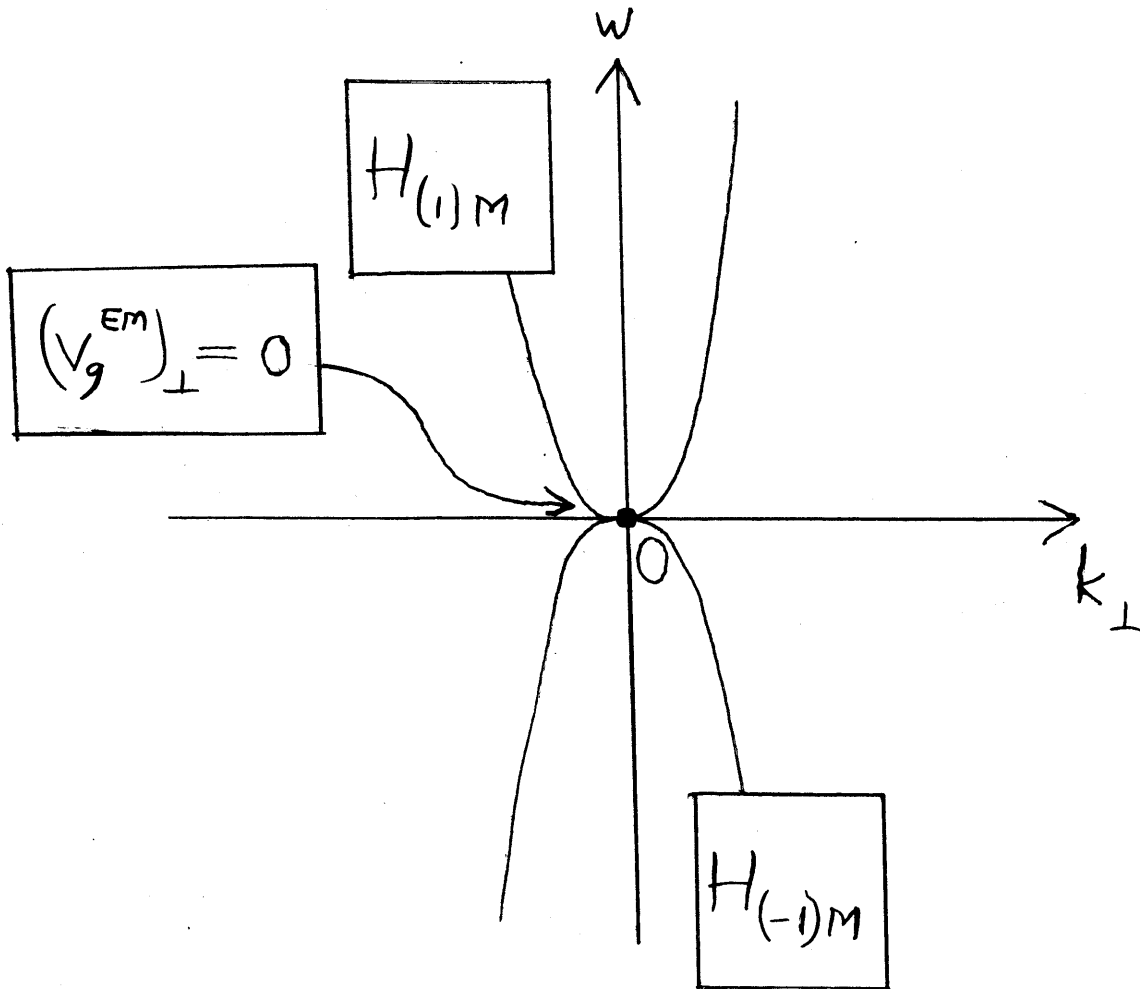$\vec{k}_{(o)}$ PARALLEL TO LASER WAVEVECTOR
$\vec{k}_1$. NO INSTABILITY.

Fig. 4.4.2. KINEMATICS FOR MODULATION WAVECTOR $\vec{k}_{(0)}$
IN PLANE CONTAINING LASER WAVEVECTOR $\vec{k}_1$

$(\vec{k}_{(0)}, w_{(0)})$ lies near 0.

Fig. 4.4.3 : KINEMATICS FOR FILAMENTATION
INSTABILITY : MODULATION WAVEVECTOR $\vec{k}_{(0)}$
PERPENDICULAR TO LASER WAVEVECTOR $\vec{k}_1$
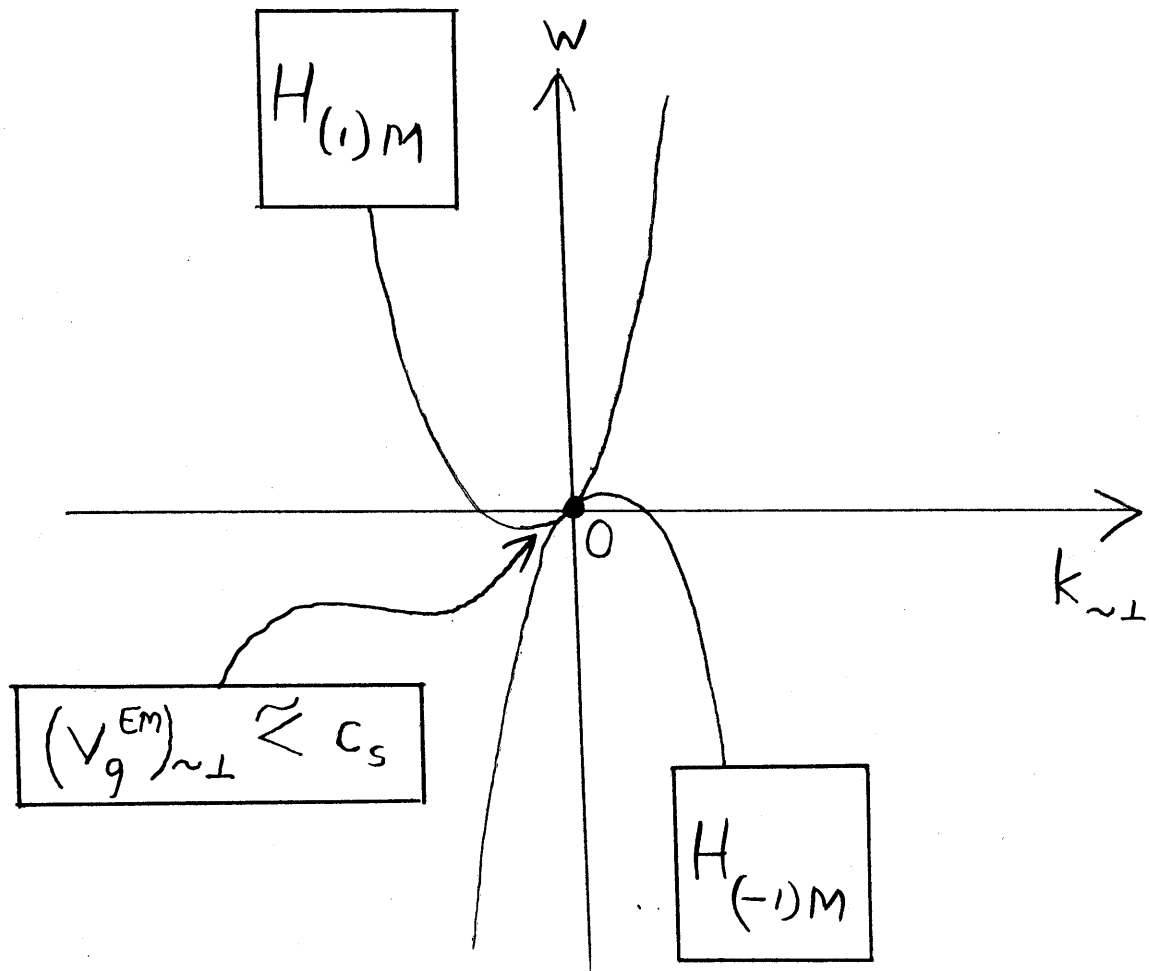

$(k_{(0)}, w_{(0)})$ lies near 0.

Fig 4.4.4 : KINEMATICS FOR MODULATIONAL INSTABILITY:
MODULATION WAVEVECTOR $\vec{k}_{(0)}$ LIES ALMOST PERPENDICULAR
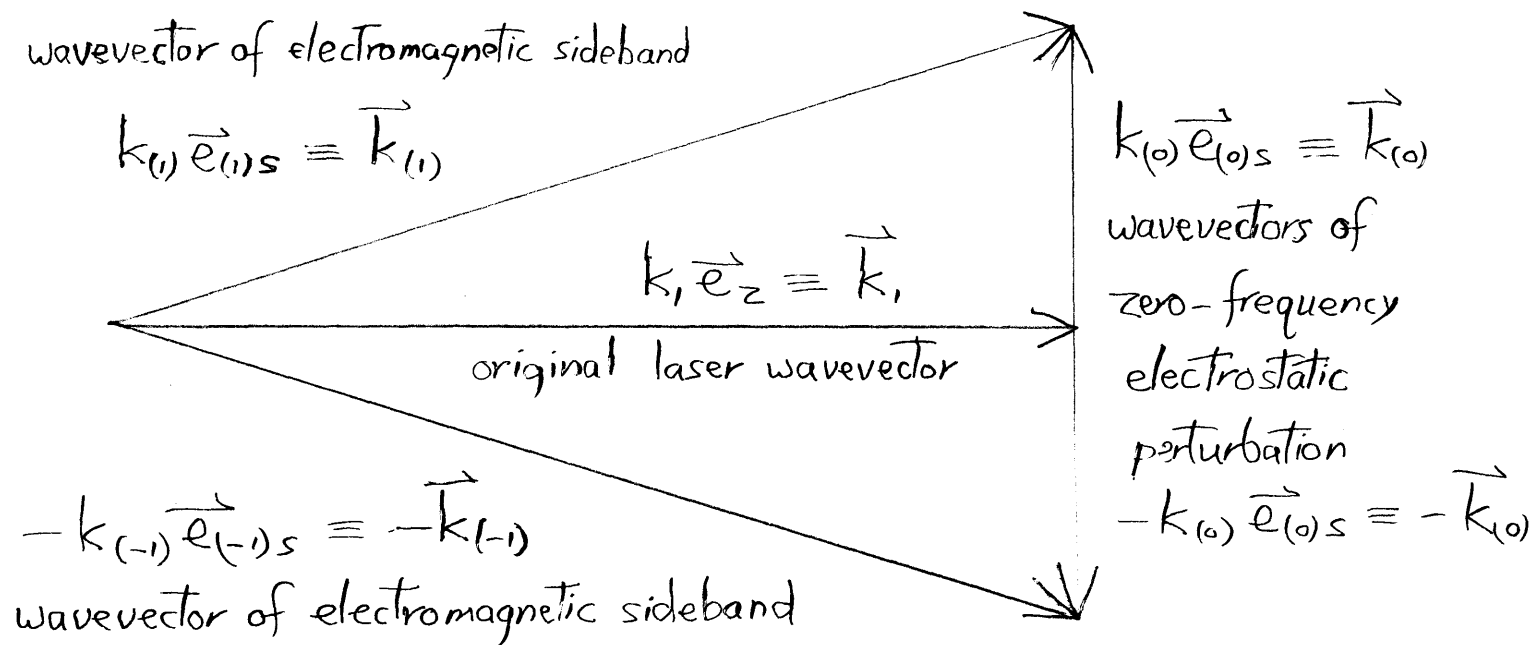TO LASER WAVEVECTOR $\vec{k}_1$.

$(k_{(0)}, w_{(0)})$ lies near 0.

Since the modulation wavevector $\vec{k}_{(0)}$ is so small, both electromagnetic perturbations propagate in a direction very close to the original laser beam direction. Thus, the filamentational and modulational instabilities do not lead to scattering of laser energy by the plasma. Rather, laser energy is fed into the low-frequency electrostatic perturbation which must eventually heat the plasma nonlinearly. As with the Brillouin instability, the filamentational and modulational instabilities can occur for any plasma density less than the critical density

$$w_p < w_1 \quad .$$

The physical mechanisms underlying these instabilities are describable in macroscopic terms. Suppose that the laser pump-wave acquires a slight modulation, or equivalently, that two small electromagnetic perturbations are superposed on the steady laser pump. Further suppose that the phase velocity of this modulation pattern is less than the ion-sound-speed c_s. Then the radiation pressure pattern due to the modulation of the laser intensity will lead to density striations in the plasma. These density striations constitute variations in the dielectric constant in the plasma. Since the dielectric constant of a plasma decreases with density, these variations in dielectric constant focus the laser light into the less dense plasma regions. This increases the radiation pressure in those regions, expelling the plasma further and increasing

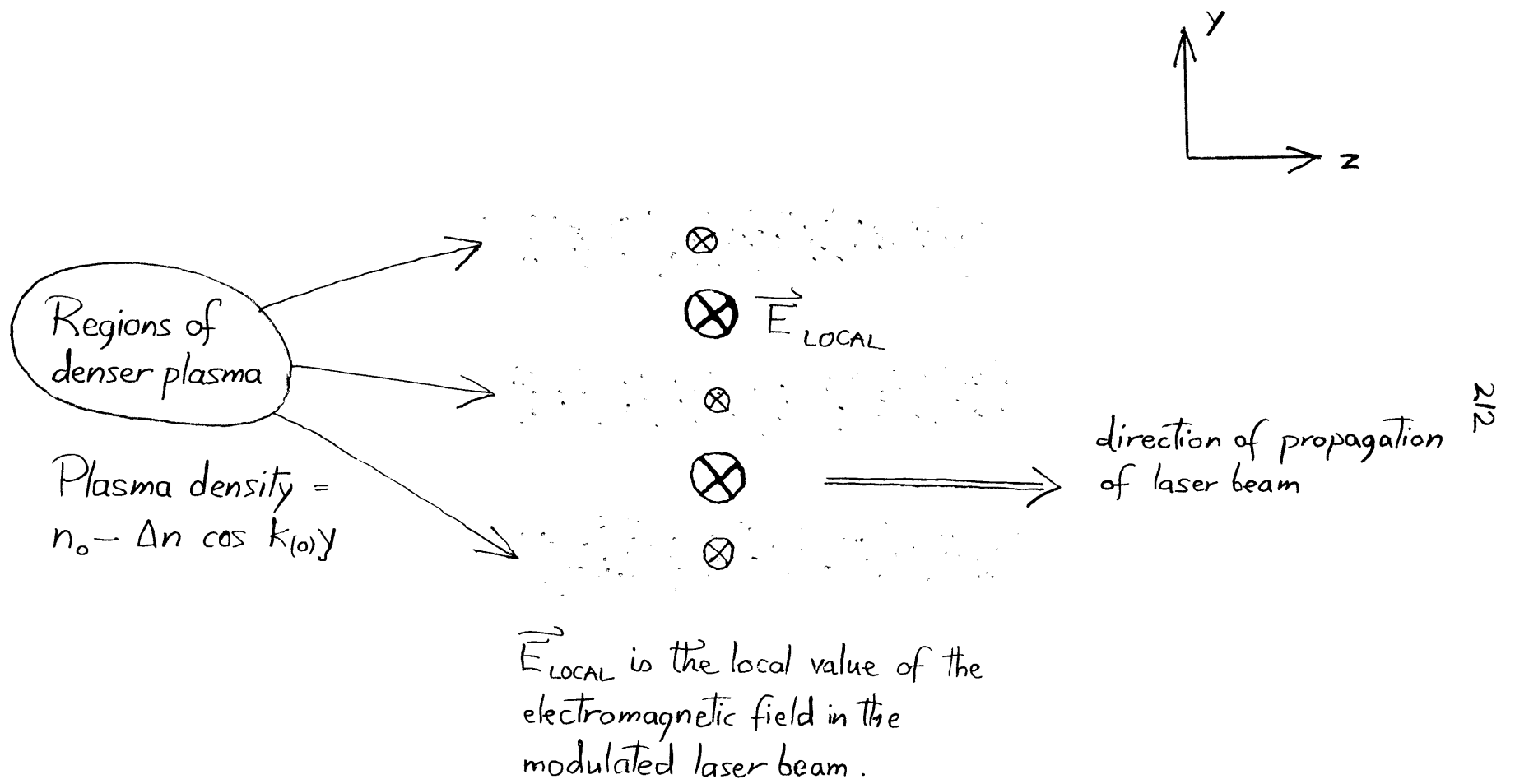Fig. 4.4.5.   WAVE VECTOR MATCHING FOR FILAMENTATION
           INSTABILITY

wavevector of electromagnetic sideband

$$k_{(1)}\,\vec{e}_{(1)s} \equiv \vec{k}_{(1)}$$

$$k_{(0)}\,\vec{e}_{(0)s} \equiv \vec{k}_{(0)}$$

wavevectors of

zero-frequency

$$k_1\,\vec{e}_z \equiv \vec{k}_1$$

original laser wavevector

electrostatic

perturbation

$$-k_{(-1)}\,\vec{e}_{(-1)s} \equiv -\vec{k}_{(-1)}$$

$$-k_{(0)}\,\vec{e}_{(0)s} \equiv -\vec{k}_{(0)}$$

wavevector of electromagnetic sideband

y

z

Regions of denser plasma

Plasma density $= n_0 - \Delta n \cos k_{(0)} y$

$\vec{E}_{LOCAL}$

direction of propagation of laser beam

$\vec{E}_{LOCAL}$ is the local value of the electromagnetic field in the modulated laser beam.

Fig. 4.4.6 : SIDEVIEW OF FILAMENTATION INSTABILITY , $\vec{k}_{(0)} \perp \vec{E}.$

the depth of the density striations. These mechanisms are most easily visualized for the case when the modulation is directly across the laser beam and its phase velocity is zero. This is the "filamentational instability".

In the filamentational instability, the laser beam is modulated in intensity across the direction of propagation. The plasma acquires density striations in the form of parallel slabs in the direction of laser propagation. The performance of the resulting parallel-slab dielectric waveguide as a means of confining the laser radiation to the regions between the slab can be calculated. This performance factor for the slab-waveguide, together with radiation pressure calculations of the rate at which the plasma slab structure is increasing its strength, yield the growthrate for the filamentational instability, as will now be shown.

The growthrate of the filamentational instability will later be shown to depend very little on the orientation of the density striations with respect to the polarization of the laser, provided these striations have a wavelength much longer than that of the laser. For illustrative purposes, we take the plasma density to have sinusoidal variations in a direction perpendicular to the laser polarization, so that the "slabs" of denser plasma lie parallel to the laser polarization (see next page, Fig. 4.4.6).

Treat the plasma density striations as a given static structure, namely a sinusoidal-index-profile dielectric-slab waveguide. The performance of this waveguide may be found by

standard methods. The result is, to first order in the index-variations,

$$E_{Local} = \left\{ 1 + \frac{w_1^2}{k_{(o)}^2 c^2} \frac{\Delta\varepsilon}{\varepsilon_o} \cos k_{(o)} y \right\} E_1 \qquad . \qquad 4.4-(1)$$

Now the refractive index is a decreasing function of plasma density

$$\frac{\varepsilon}{\varepsilon_o} = 1 - \frac{w_p^2}{w_1^2} = 1 - \frac{1}{w_1^2} \frac{nq^2}{m\varepsilon_o} \qquad . \qquad 4.4-(2)$$

Use (1) and (2) to find the variation in the radiation pressure or effective pressure of coherent oscillations, thus:

$$n_o \Delta \langle V_{Local}^2 \rangle = \frac{2}{k_{(o)}^2 c^2} \left(-\Delta n \, w_{po}^2 \right) \langle V_{Laser}^2 \rangle \qquad 4.4-(3)$$

$$\Delta p_{coherent \atop oscillations} = n_o \Delta \langle V_{Local}^2 \rangle / 2 \qquad 4.4-(4)$$

Also

$$\Delta p_{Thermal} = V_T^2 \Delta n \qquad . \qquad 4.4-(5)$$

Thus, the radiation force overcomes the electron thermal pressure gradient, provided

$$\frac{w_{po}^2}{k_{(o)}^2 c^2} \frac{V_{Laser}^2}{2} > V_T^2 \qquad 4.4-(6)$$

$$\frac{V_{Laser}^2}{4c^2} = \frac{|V_1^2|}{c^2} > \frac{k_{(o)}^2 V_{Te}^2}{2w_p^2} = \frac{k_{(o)}^2 c_s^2}{2w_{pi}^2} \qquad . \qquad 4.4-(7)$$

For a fixed laser intensity, (7) delimits the range of unstable k values; this range lies in the neighborhood of the origin (cf. Fig. 3). When the inequality (7) is satisfied, the striations can grow in time with growthrate γ and all quantities associated with them have temporal variation $e^{\gamma t}$. The electrons are forced into the higher plasma density regions, dragging the ions with them. To the extent that the neutralization is not complete, there will exist a small electrostatic field with wavevector and frequency

$$[\vec{k}_{(o)}, w_{(o)}] \;\; = \;\; [\vec{k}_{(o)}\vec{e}_y, \; i\gamma] \qquad . \qquad\qquad 4.4\text{-}(8)$$

This is the low-frequency electrostatic perturbation mentioned at the beginning of this section. For the ions, the linearized quasistatic fluid equations are

$$\frac{\partial V_i}{\partial t} \;\; = \;\; \frac{q_i}{m_i} E_{static} \qquad\qquad\qquad 4.4\text{-}(9)$$

$$\frac{\partial n_i}{\partial t} \;\; = \;\; - n_{oi} \nabla \cdot V_i \qquad\qquad\qquad 4.4\text{-}(10)$$

These quasistatic quantities have temporal variation $e^{\gamma t}$, therefore

$$\frac{\Delta n_i}{n_{oi}} = \frac{k_{(o)}q_i}{\gamma^2 m_i} \; E_{static}^{peak} \qquad . \qquad\qquad 4.4\text{-}(11)$$

For the electrons, the linearized quasistatic fluid equations are

$$\frac{\partial V_e}{\partial t} = - \frac{\nabla_p}{n_o} - \frac{\nabla <V^2>}{2} + \frac{q_e}{m_e} E_{static} \qquad 4.4-(12)$$

$$\frac{\partial n_e}{\partial t} = - n_{oe} \nabla \cdot V_e^{static} \qquad . \qquad 4.4-(13)$$

These quasistatic quantities have temporal variation $e^{\gamma t}$, therefore

$$\frac{\Delta n_e}{n_{oe}} = \left(\frac{k_{(o)} \, q_e}{\left(\gamma^2 + k_{(o)}^2 V_{Te}^2 - \frac{V_{Laser}^2}{2c^2} w_{pe}^2 \right) m_e} \right) E_{static}^{peak} \qquad . \quad 4.4-(14)$$

Substituting in Poisson's equation and assuming quasineutrality, one obtains the following:

$$\gamma^2 = \frac{V_{Laser}^2}{2c^2} \, w_{pi}^2 - k_{(o)}^2 c_s^2 \qquad . \qquad 4.4-(15)$$

The more exact result from the generalized coupling modes method used later in this section is that the growthrate satisfies the biquadratic

$$\gamma^2 = \frac{V_{Laser}^2}{2c^2} \, w_{pi}^2 \left[\frac{k_{(o)}^4 c^4}{k_{(o)}^4 c^4 + 4\gamma^2 w_1^2} \right] - k_{(o)}^2 c_s^2 \quad . \quad 4.4-(16)$$

The bracketed quantity is the correction factor to the wave-guide performance, due to the finite temporal growthrate of its refractive index profile. For striations with wavevector $\vec{k}_{(o)}$ not perpendicular to the laser polarization $\vec{e}_{x_1}$, there is a further geometrical correction factor. This is more conveniently introduced later when treating the filamentation
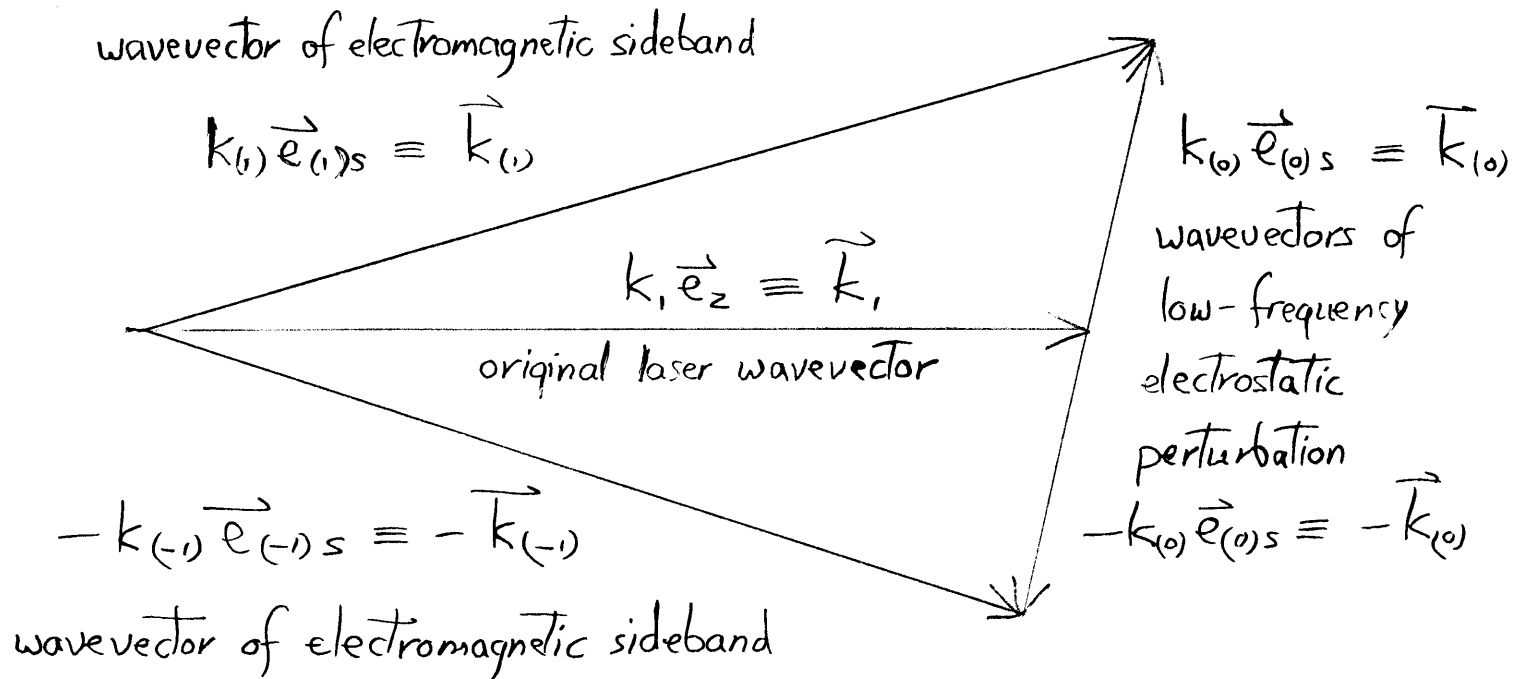
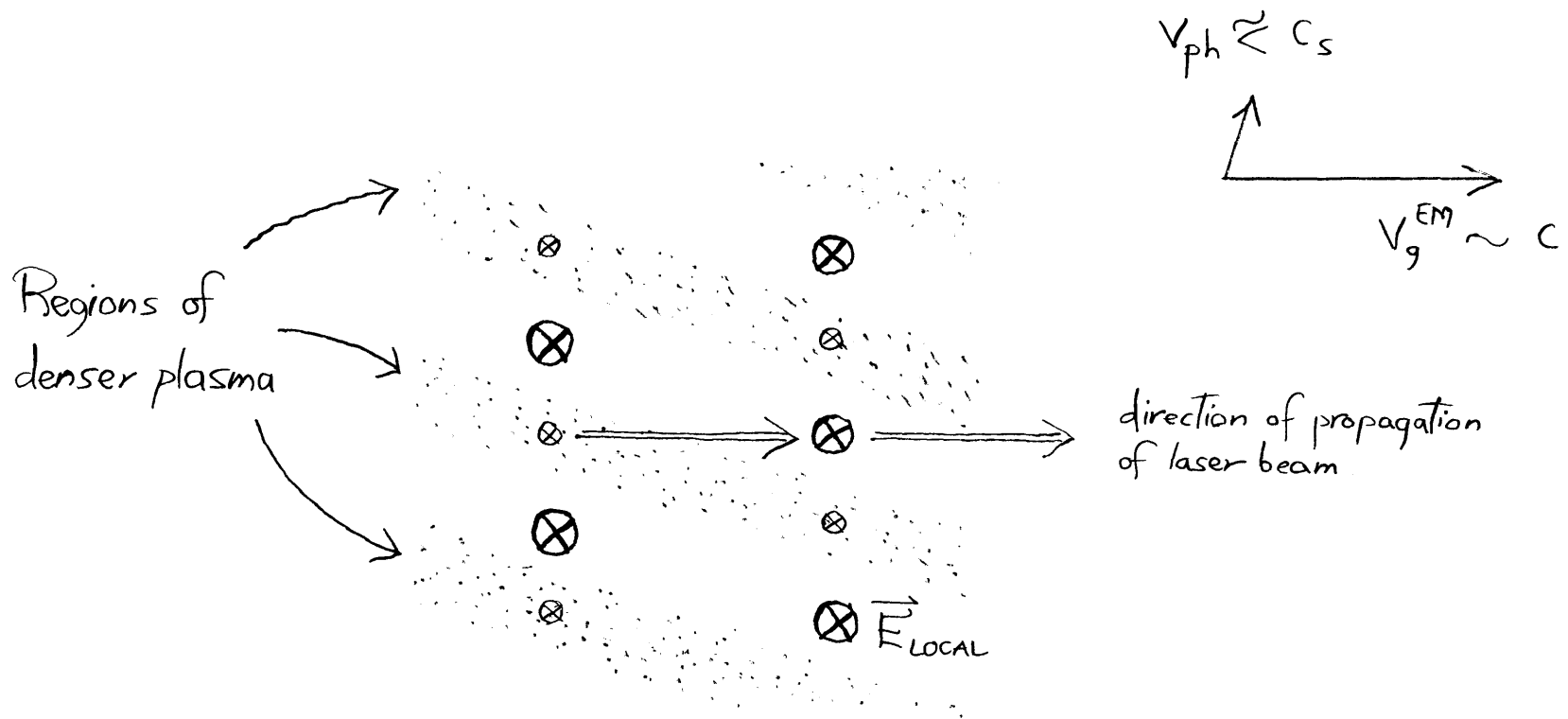instability using the general scheme worked out in Chapter 3.

For the filamentation instability, the density striations and the associated electrostatic perturbation have a wavevector $\vec{k}_{(o)}$ perpendicular to the laser wavevector $\vec{k}_1$. For the modulational instability, the striation wavevector is still almost perpendicular to the laser wavevector. This allows the laser modulation pattern to move with the laser beam group velocity, while the phase velocity of the striations is still less than or comparable to the ion-sound speed. (See Figs. 4.4.6 and 4.4.7).

On taking a one-dimensional cross section of the system parallel to the laser propagation direction, one sees the modulation envelope of the laser radiation proceeding with the group velocity of the laser radiation and the modulation increasing in depth as the instability grows, hence the name, "modulation instability". However, this 1-D picture is misleading in that the density striations, since they involve both electrons and ions, can move only at the sound speed. Thus, the modulational instability is intrinsically at least two-dimensional. The dispersion relation for the modulation instability will now be derived along with that for the filamentation instability, using the general scheme for laser-driven instabilities set out in section 3.8.

The full three-dimensional dispersion for the filamentation and modulation instabilities may be written down from

Fig. 4.4.7. WAVEVECTOR MATCHING FOR MODULATION INSTABILITY.

wavevector of electromagnetic sideband

$$k_{(1)} \vec{e}_{(1)s} \equiv \vec{k}_{(1)}$$

$$k_{(0)} \vec{e}_{(0)s} \equiv \vec{k}_{(0)}$$

wavevectors of low-frequency electrostatic perturbation

$$k, \vec{e}_z \equiv \vec{k},$$

original laser wavevector

$$-k_{(-1)} \vec{e}_{(-1)s} \equiv -\vec{k}_{(-1)}$$

$$-k_{(0)} \vec{e}_{(0)s} \equiv -\vec{k}_{(0)}$$

wavevector of electromagnetic sideband

$V_{ph} \lesssim C_s$

$V_g^{EM} \sim C$

Regions of denser plasma

direction of propagation of laser beam

$\vec{E}_{LOCAL}$

$\vec{E}_{LOCAL}$ is the local value of the electromagnetic field in the modulated laser beam.

Fig 4.4.6   SIDEVIEW OF MODULATION INSTABILITY.

3.8-(13). Using the appropriate approximations, 3.8-(13)

becomes

$$
\begin{vmatrix}
1 - \dfrac{w_p^2 + k_{(1)}^2 c^2}{w_{(1)}^2} + |X^2| & \dfrac{X}{k_{(o)}\lambda_D} & XY \\[4ex]
\dfrac{X^*}{k_{(o)}\lambda_D} & \dfrac{1}{k_{(o)}^2 \lambda_D^2}\left[1 - \dfrac{k_{(o)}^2 c_s^2}{w_{(o)}^2}\right] & \dfrac{Y}{k_{(o)}\lambda_D} \\[4ex]
X^*Y^* & \dfrac{Y^*}{k_{(o)}\lambda_D} & 1 - \dfrac{w_p^2\, k_{(-1)}^2 c^2}{w_{(-1)}^2} + |Y^2|
\end{vmatrix} = 0
$$

$$4.4\text{-}(17)$$

where $X = \dfrac{V_1}{V_{Te}}\dfrac{w_p}{w_{(1)}}(\vec{e}_{x1}\cdot\vec{e}_{M(1)})$, $\qquad Y = \dfrac{V_1}{V_{Te}}\dfrac{w_p}{w_{(-1)}}(\vec{e}_{x1}\cdot\vec{e}_{M(-1)})$

Expanding (17), one obtains the dispersion relation

$$
1 - \frac{k_{(o)}^2 c_s^2}{w_{(o)}^2} = \frac{V_{Laser}^2}{4V_{Te}^2}\frac{k_{(o)}^2 c_s^2}{w_{(o)}^2}\left\{ \frac{w_p^2[\vec{e}_{x1}\cdot\vec{e}_{M(1)}]^2}{w_{(1)}^2 - w_p^2 - k_{(1)}^2 c^2} \right.
$$

$$
\left. + \frac{w_p^2\,[\vec{e}_{x1}\cdot\vec{e}_{M(-1)}]^2}{w_{(-1)}^2 - w_p^2 - k_{(-1)}^2 c^2} \right\} \qquad . \qquad 4.4\text{-}(18)
$$

Expand the electromagnetic dispersion functions about the

point $\left(\vec{k}_{(o)}, w_{(o)}\right) = (\vec{0}, 0)$, keeping all orders in $\vec{k}_{(o)}$ and

first order in $w_{(o)}$. The result is the dispersion relation

for

Three-dimensional filamentation and modulation instability

$$1 - \frac{k^2_{(0)}c^2_s}{w^2_{(0)}} = \frac{V^2_{Laser}}{8V^2_{Te}} \frac{k^2_{(0)}c^2_s}{w^2_{(0)}} \frac{w_p}{w_1} \left\{ \frac{w_p[\vec{e}_{x1}\cdot\vec{e}_{M(1)}]^2}{w_{(0)} - \vec{k}_{(0)}\cdot\vec{V}^{Laser}_g - (k^2_{(0)}c^2/2w_1)} \right.$$

$$\left. - \frac{w_p(\vec{e}_{x1}\cdot\vec{e}_{M(-1)})^2}{w_{(0)} - \vec{k}_{(0)}\cdot\vec{V}^{Laser}_g + (k^2_{(0)}c^2/2w_1)} \right\}$$

4.4-(19)

Equation (19) was obtained by modeling the instability as a four-wave interaction, involving the unmodulated laser-pump $\vec{E}_{x1}$, two closely adjacent sidebands $\vec{E}_{M(1)}$ and $\vec{E}_{M(-1)}$ and the low-frequency electrostatic perturbation $\vec{E}_{s(0)}$ whose wave-vector and frequency is that separating the electromagnetic sidebands from the laser fundamental. For the filamentation instability, the angle between the sideband polarization and the polarization of the laser fundamental is zero when the striation wavevector is perpendicular to the polarization of the laser fundamental. This makes the geometrical factors in (19) equal to their maximum value of unity and thus yields the maximum filamentation growthrate for a given laser power. This is the case illustrated in Fig. 5. In fact, on setting

$$\vec{k}_{(0)}\cdot\vec{V}^{Laser}_g = 0$$
$$\vec{e}_{x1}\cdot\vec{e}_{M(1)} = 1 \qquad\qquad 4.4-(20)$$
$$\vec{e}_{x1}\cdot\vec{e}_{M(-1)} = 1$$

in (19) one recovers the growthrate formula (16).

4.5 Plasmon-Phonon (Decay) Instability

In the plasmon-phonon instability, two electrostatic perturbations, one high-frequency and one low-frequency, are coupled together by the laser pump. The unmodified instability occurs at low enough pump-wave intensities so that the perturbations closely resemble an electron-plasma wave and an ion-acoustic wave, respectively. In quantum language, the laser photon decays into a plasmon and a phonon. This plasmon-phonon instability was historically one of the first coupled-mode instabilities to be considered in plasmas, and so became known as "the" parametric instability, or "the" decay instability.

Since the perturbations are electrostatic, the instability is excited most strongly when these perturbations propagate parallel to the electric field of the laser. This parallelism cannot be exact, since the wavevector diagram is two-dimensional and the laser wavevector is finite:

(ion-acoustic $\vec{k}_{(0)}$ $-\vec{k}_{(-1)}$ (electron-plasma wave)
wave)

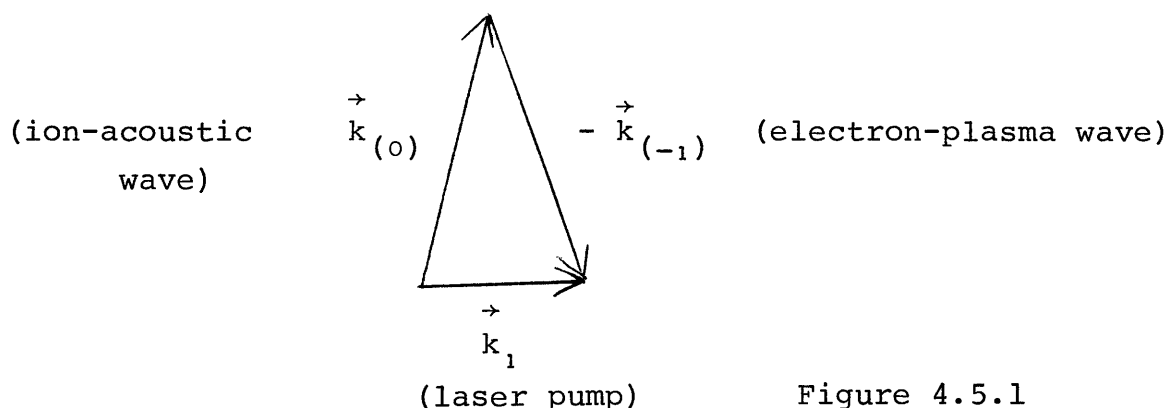$\vec{k}_1$

(laser pump)   Figure 4.5.1

However, the parallelism can be attained approximately;
in the plasmon-phonon instability, as distinct from the
Brillouin, filamentation and modulation instabilities,
the perturbation wavelengths are characteristically much
shorter than the laser wavelength. The locus of the insta-
bility in $(\vec{k}, w)$ space is shown first on a $(k_z, k_\perp, w)$ diagram
and secondly on a $(k_\perp, w)$ diagram. The latter describes the
kinematics of the instability with the ion-acoustic pertur-
bation propagating exactly in the direction of the laser
polarization and the electron-plasma perturbation propagat-
ing almost exactly in that direction. This case is conven-
ient for illustrating the results of stability analysis and
comparing them with the results of stability analysis of the
modified plasmon-phonon instability described in section 4.6.

Since both growing perturbations are electrostatic, the
plasmon-phonon instability is an absorptive instability; it
occurs only for plasma densities near the critical surface

$$w_p \sim w_l \qquad . \qquad\qquad 4.5 - (1)$$

The physical mechanisms driving the instability are as
follows. Consider a small perturbation in the form of an
electron plasma wave propagating approximately parallel to
the electric field of the laser, with a wavelength much shorter
than that of the laser, but a frequency about the same as that
of the laser. On the spatial scale of this plasma wave, the

Fig. 4.5.2 : KINEMATICS OF PLASMON-PHONON INSTABILITY;
PROPAGATION IN PLANE CONTAINING DIRECTION OF LASER
PROPAGATION.
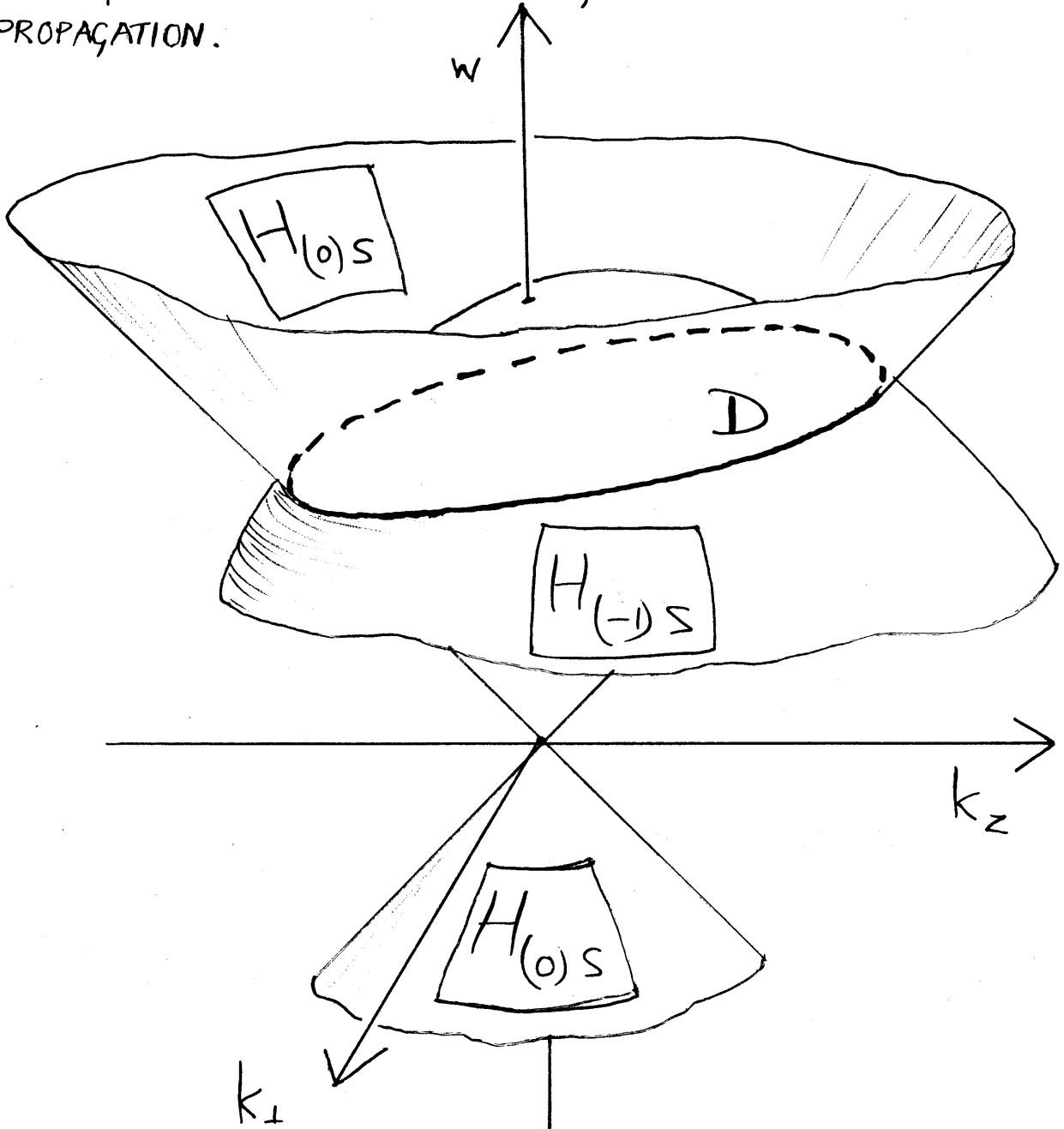
Here $H_{(o)s}$ is the ion-acoustic dispersion surface, $H_{(-1)s}$ the electron-plasma dispersion surface. D is the locus of the "decay" instability. The values of $(\vec{k}_{(o)}, w_{(o)})$ lie near this locus.

Fig. 4.5.3.   KINEMATICS WITH PHONON PROPAGATING
PARALLEL TO ELECTRIC FIELD OF LASER
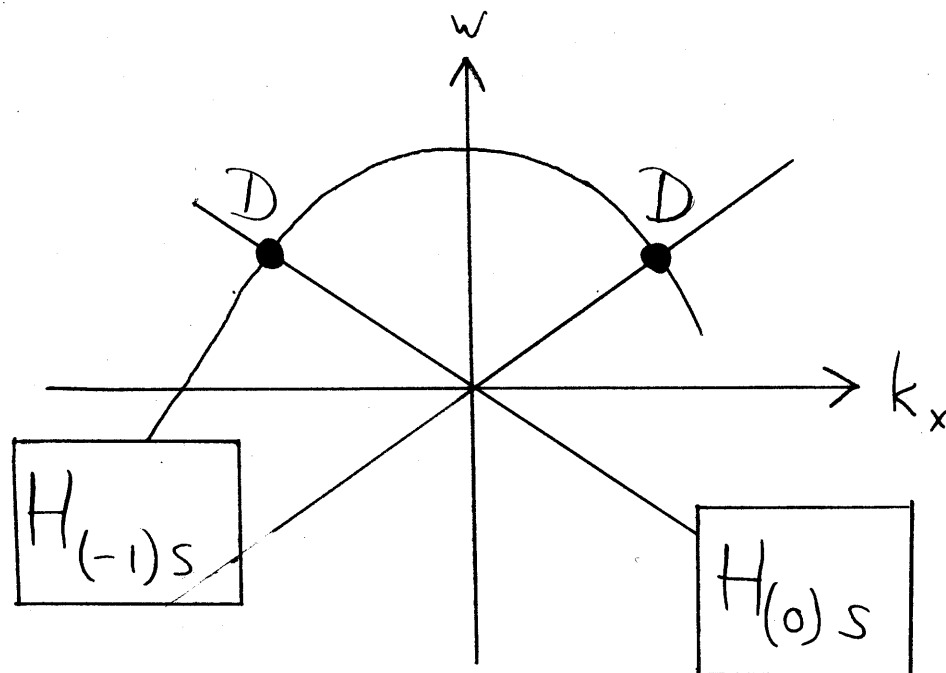
The values of $(k_{(o)x}, w_{(o)})$ lie near one of the points
labelled "D".

electric field of the laser can be taken to be spatially uni-
form. Thus, the radiation pressure pattern has about the

same wavelength as the plasma wave and moves slowly parallel

to the laser polarization. If the radiation pressure pattern

were stationary, the plasma as a whole would be pushed into its

"troughs". When the radiation pressure pattern moves with the

ion-sound speed c_s, the plasma tends to "ride" on the forward

slopes of the pressure peaks. This is made more explicit in

the following diagram, which consists of three successive

snapshots of the same line parallel to the x-axis, taken at

time intervals of a quarter of the laser period. The phase

relations involved are derived by finding the eigenvectors of

the matrix (2) and are used here only to illustrate the physics
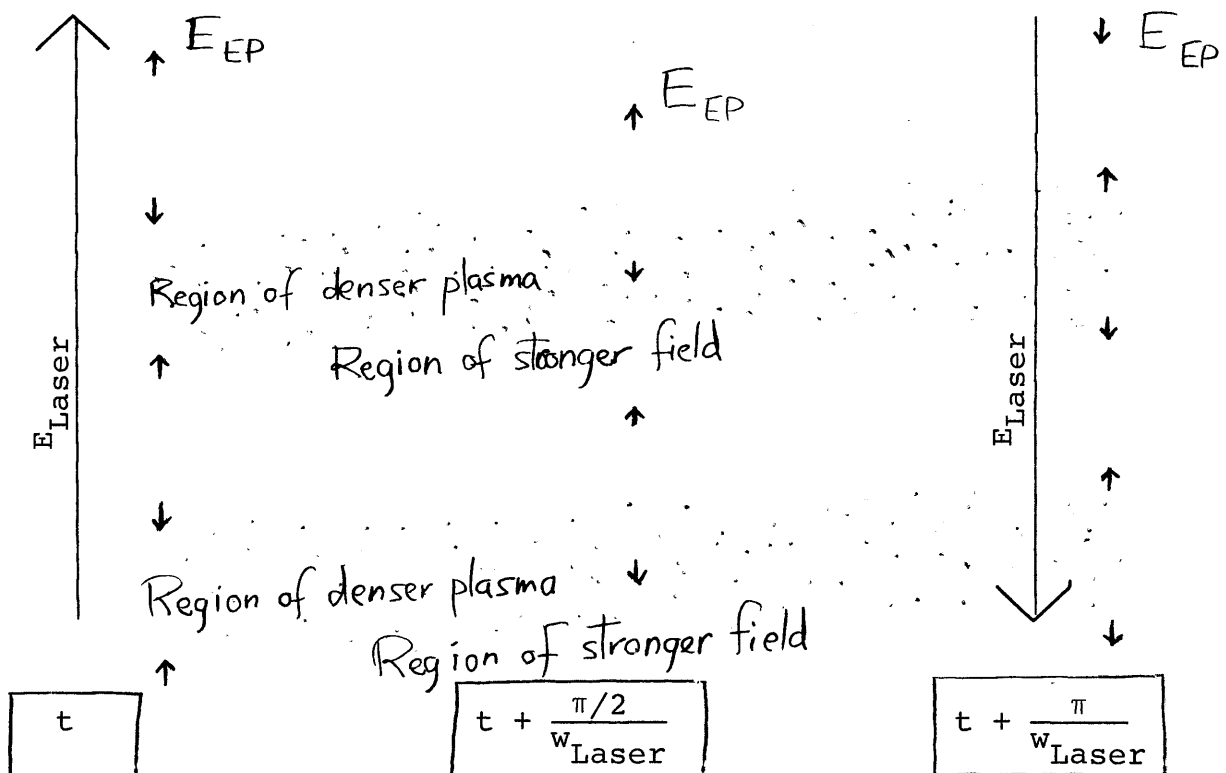
of the interaction.

Fig. 4.5.4 DYNAMICS OF PLASMON-PHONON INSTABILITY

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In Fig. 3, the electron-plasma wave is propagating downward.
Since its frequency is slightly less than that of the laser,
it moves through slightly less than its own wavelength in the
time π/w_{Laser} that it takes for the laser field to reverse it-
self. Thus, the radiation pressure pattern moves slowly up-
wards.

The regions of high R.M.S. electric field sweep the higher
density striations ahead of them. This explains how a pertur-
bation in the form of an electron-plasma wave can give rise to
an ion-sound wave. To complete the feedback loop, we now explain
how the density striations in the presence of the laser beam
can drive the electron-plasma wave. To do this, merely note
that the quivering motion of the electrons in the higher den-
sity striations, as these electrons are acted on by the laser
field,constitutes a current. This current is a maximum at
$t = (\pi/2w_{Laser})$, and opposes the electron-plasma wave field.
The resulting negative dissipation causes the electron plasma
wave to grow.

Note that, if the plasma wave frequency were slightly
higher than that of the laser, the radiation pressure pattern
would move downward . The higher density plasma striations
would then appear on the lower side of the radiation pressure
maxima, and the plasma wave would be suppressed; this is to be
expected, since the Manley-Rowe relations prevent unstable up-
conversion in a single three-wave interaction. Note further
that, if the plasma wave frequency were the same as the laser

frequency the radiation pressure pattern would be stationary. The higher density plasma striations would then occur in the regions of minimum radiation pressure. This last situation occurs in the oscillating two-stream instability, to be dealt with in section 4.6

The full three-dimensional dispersion relation for the plasmon-phonon instability may be written down from 3.8-(10). Using physical approximations appropriate to the ion-acoustic and electron-plasma waves, 3.8-(10) becomes

$$
\begin{vmatrix}
\dfrac{1}{k^2_{(0)} \lambda^2_D} \left[1 - \dfrac{k^2_{(0)} c^2_s}{w^2_{(0)}} \right] & -\dfrac{1}{k_{(0)} \lambda_D} \left[\dfrac{V_1}{V_{Te}} \dfrac{w_p}{|w_{(-1)}|} [\vec{e}_{x_1} \cdot \vec{e}_{s\,(-1)}] \right] \\[3ex]
-\dfrac{1}{k_{(0)} \lambda_D} \left[\dfrac{V^*_1}{V_{Te}} \dfrac{w_p}{|w_{(-1)}|} [\vec{e}_{x_1} \cdot \vec{e}_{s\,(-1)}] \right] & 1 - \dfrac{w^2_p}{w^2_{(-1)} - 3k^2_{(-1)} V^2_{Te}}
\end{vmatrix} = 0
$$

4.5 -(2)

As for the unmodified Brillouin instability, expand the determinant and linearize both the ion-acoustic and electron-plasma dispersion function about the wavevector and frequency of some point on the instability locus D of Fig. 2. The result is the dispersion relation for

---

Three-dimensional Plasmon-phonon Instability

$$\left(\Delta w - \Delta \vec{k} \cdot \vec{e}_{s(0)} c_s \right) \left(\Delta w + \Delta \vec{k} \cdot \vec{e}_{s(-1)} \left| v_g^{EP} \right| \right)$$

$$= - \frac{V_{Laser}^2}{16 V_{Te}^2} \; w_p^2 \; \frac{w_{IA}}{w_{EP}} \; [\vec{e}_{x_1} \cdot \vec{e}_{s(-1)}]^2 \qquad . \qquad 4.5\text{-}(3)$$

The maximum growthrate for real $\Delta \vec{k}$ is obtained from this dispersion relation by setting $\Delta \vec{k} = \vec{0}$, which yields

$$\Delta w = i \, \frac{V_{Laser}}{4 V_{Te}} \, w_p \sqrt{\frac{w_{IA}}{w_{EP}}} \, \left| \vec{e}_{x_1} \cdot \vec{e}_{s(-1)} \right| \qquad . \qquad 4.5\text{-}(4)$$

This growthrate depends on the three-dimensional nature of the kinematics via the square-root factor, and on the three-dimensional nature of the polarization via the last factor. The kinematics are the same in any plane containing the laser propagation vector. The dynamics depend on the effectiveness of the quivering of the electrons in the electric field of the laser as a means of driving the plasma-wave perturbation. This effectiveness is given by the last factor in (4).

The three-dimensional dispersion relation (4) can be used to perform a stability analysis in the manner prescribed by Bers and Briggs. As in the case of the unmodified Brillouin instability, the dispersion relation (3) employs linearized dispersion functions for the two perturbations which enter the coupling. The kinematics of the coupling are the same for all

planes containing the direction of propagation of the laser.
The dynamics of the interaction are the same for all planes
containing the direction of propagation of the laser except
for the factor

$$\left| \vec{e}_{x_1} \cdot \vec{e}_{s(-1)} \right| \qquad . \qquad \qquad 4.5\text{-}(5)$$

These features of the dispersion relation (3) allow the three-
dimensional stability analysis to be reduced to a two-dimen-
sional problem, as shown by Bers and Chambers. The contour
map of observed growthrate vs observer velocity is plotted in
the plane containing the maximum interaction, namely the x-z
plane which contains the x-directed electric field of the
laser. For any other plane through the z-axis, making an
angle α, say, with the x-z plane, the pulse response diagram
can be obtained from the pulse response diagram in the x-z
plane. This is done simply by multiplying all growthrates by
cos α.

For low laser-pump intensities, the contributions of
third order conductivity to the dispersion relation (2) may be
neglected. Further, one may linearize the dispersion functions
for the perturbations which enter the coupling, thus arriving
at the dispersion relation (3) for the unmodified plasmon-
phonon instability. For high enough laser pump intensities,
the growthrate for the plasmon-phonon instability becomes com-
parable to the phonon frequency. Then, the exact ion-acoustic

dispersion function must be used for the low-frequency elec-
trostatic perturbation. Also, the self-correction due to
third-order conductivity must be added in to the electron-
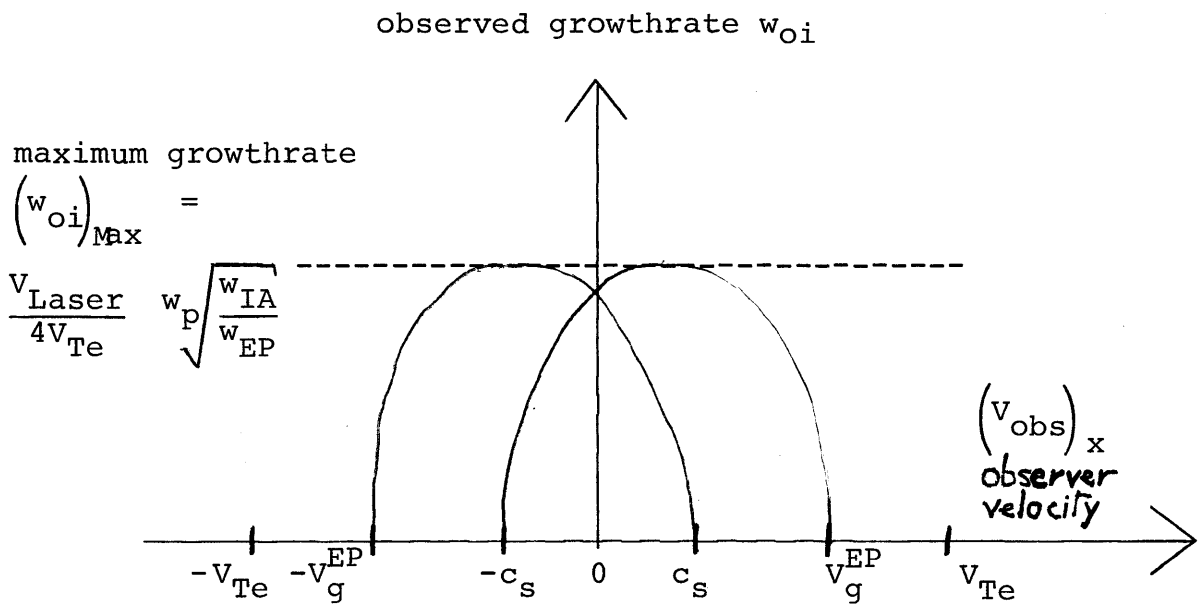plasma dispersion function.

These steps will yield in the next section the dispersion
relation for the modified plasmon-phonon instability. To
examine the effect of this modification on the time-asymp-
totic pulse response characteristics of the unstable system,
it will be useful to compare selected one-dimensional cross
sections of the modified and unmodified pulse-response plots.
Since the modification is an effect due to large growthrate,
the cross section chosen for the comparison should include
the observer velocities for which the unmodifed growthrates
are greatest. From the behavior of the geometrical factor
(5), one may infer that the unmodifed growthrates are greatest
when the plasmon has a group velocity in the x-direction. The
comparison of unstable pulse cross sections is actually car-
ried out using the physical approximation that the laser
wavelength is much shorter than that of the perturbation:

$$|k_1| \ll |k_{(0)}| , |k_{(-1)}| \qquad . \qquad\qquad 4.5\text{-}(6)$$

This approximation does not affect the physics of the
plasmon-phonon interaction, and was in fact used in the des-
cription of the driving physical mechanisms (cf. Fig. 4). To
within this approximation, the plasmon and phonon group velo-
cities together with the corresponding observer velocities can

all be taken to lie along the x-axis. This approximation is convenient for the stability analysis of the modified instability and is made here to facilitate comparison. The unmodified time-asymptotic pulse response along the x-axis has the following form:

Fig. 4.5.5   TIME-ASYMPTOTIC PULSE RESPONSE;

COLLINEAR APPROXIMATION

observed growthrate w_{oi}

maximum growthrate

$$\left(w_{oi}\right)_{Max} =$$

$$\frac{V_{Laser}}{4V_{Te}} \quad w_p\sqrt{\frac{w_{IA}}{w_{EP}}}$$

$\left(V_{obs}\right)_x$

observer velocity

$-V_{Te}$ $-V_g^{EP}$ $-c_s$ 0 c_s V_g^{EP} V_{Te}

4.6 Modified Plasmon-phonon Instability, Treated Together with Oscillating Two-stream Instability

In the oscillating two-stream instability, two high-frequency electrostatic perturbations and one low-frequency electrostatic perturbation are coupled together by the laser pump. The high-frequency perturbations are electron-plasma waves driven below their resonant frequencies; the low-frequency perturbation is a pattern of density striations in the plasma which moves with phase velocity less than the ion-sound speed. In quantum language, two laser photons become two plasmons by exchanging a virtual phonon. The dispersion relation to be derived for the oscillating two-stream instability will possess additional roots which pertain to the modified plasmon-phonon instability.

Since the perturbations are all electrostatic, the oscillating two-stream, like the plasmon-phonon, is excited most strongly when these perturbations propagate parallel to the electric field of the laser. Again, since the laser wavevector is finite, this parallelism cannot be exact. However, for the purpose of explaining the physical mechanisms which drive the instability and for the purpose of finding the pulse response of the instability, the perturbations may be taken to propagate parallel to the laser polarization.

The locus of the non-oscillatory instability in $(\vec{k},w)$ space is shown first on a $(k_z, k_\perp, w)$ diagram and secondly on a $(k_\perp, w)$

Fig. 4.6.1 KINEMATICS OF OSCILLATING-TWO-STREAM INSTABILITY : PROPAGATION IN PLANE CONTAINING LASER PROPAGATION VECTOR
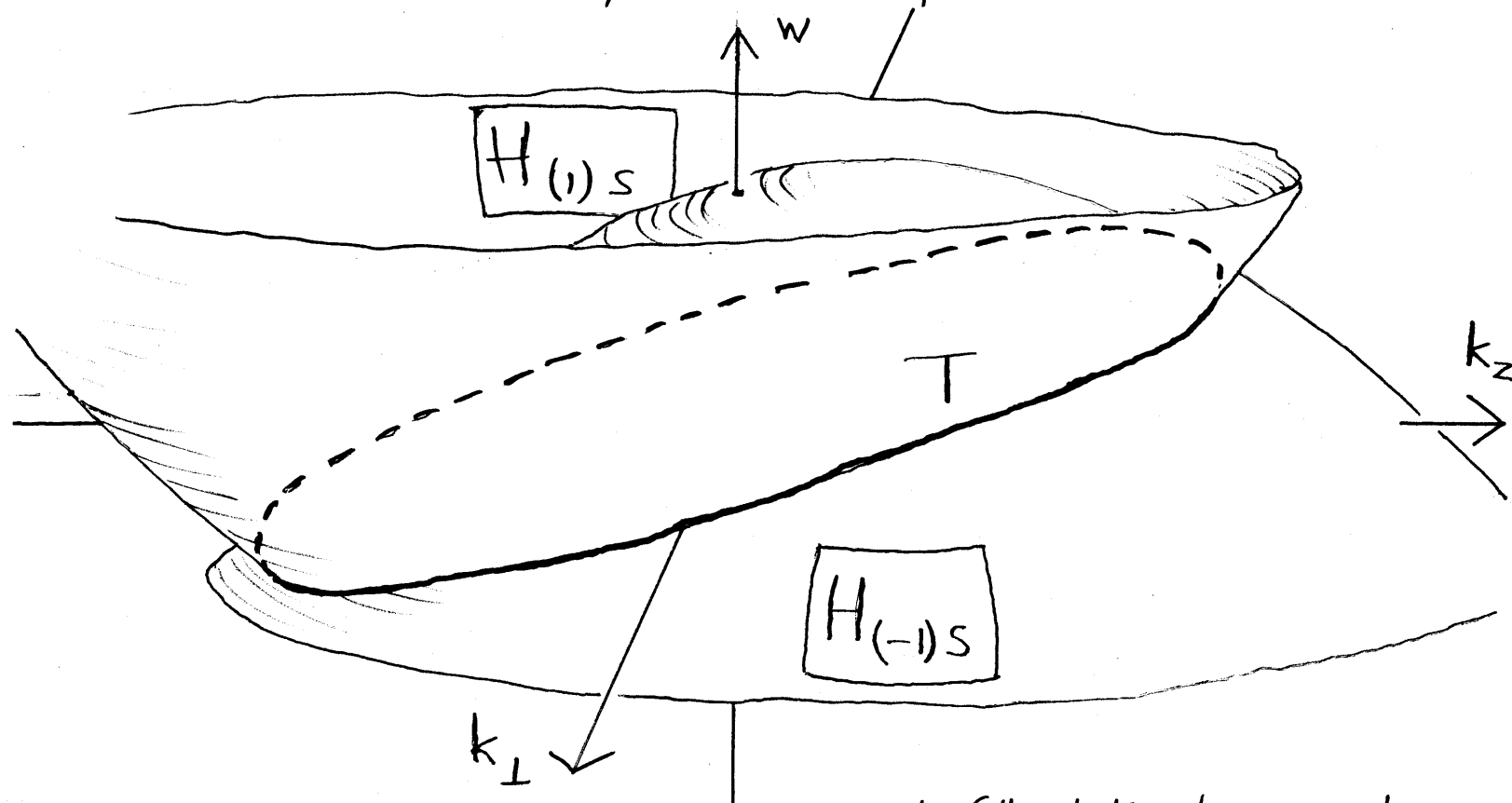
Here $H_{(-1)s}$ is the pump-upshifted negative-frequency sheet of the electron-plasma-wave dispersion surface. $H_{(1)s}$ is the pump-downshifted positive-frequency sheet of the same surface.
T is the locus of the oscillating-two-stream. The values of $(\vec{k}_{(0)}, w_{(0)})$ lie near this locus T for small enough values of the laser-pump intensity.

Fig. 4.6.2. KINEMATICS OF OSCILLATING-TWO-STREAM INSTABILITY; PROPAGATION IN DIRECTION OF ELECTRIC FIELD OF LASER.
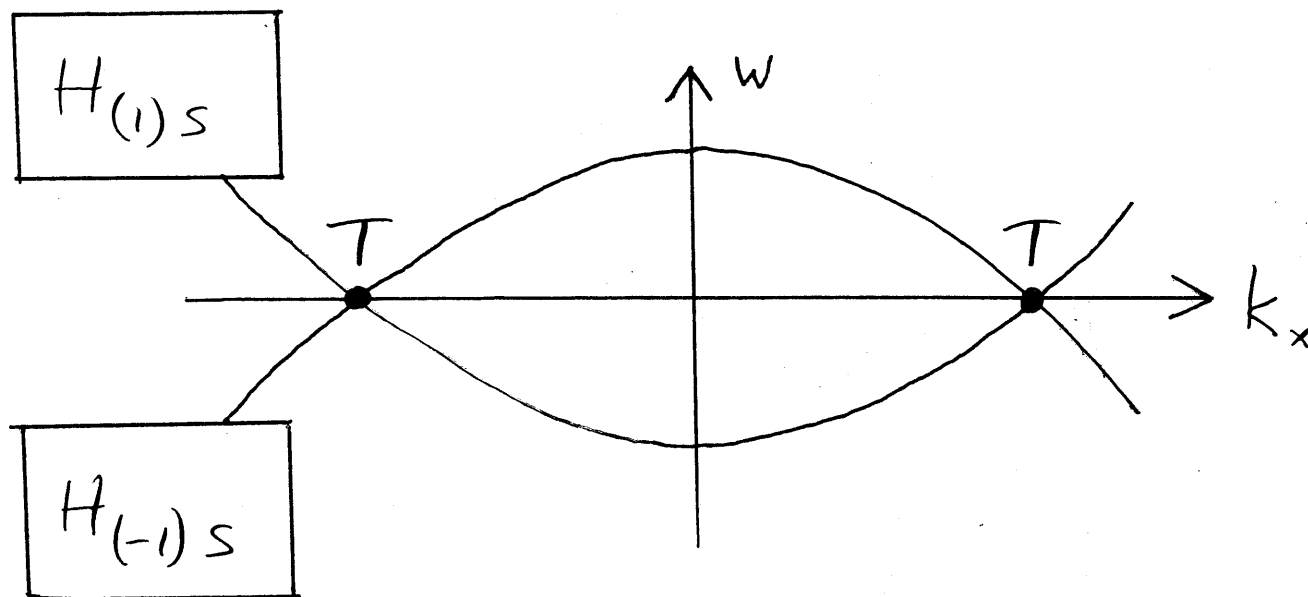
The points T are the loci of the oscillating-two-stream instability. The values of $(k_{(0)}, w_{(0)})$ lie near the points T for small enough values of the laser pump intensity.

diagram, where $k_\perp$ is a wavevector component in any selected direction perpendicular to the laser wavevector.
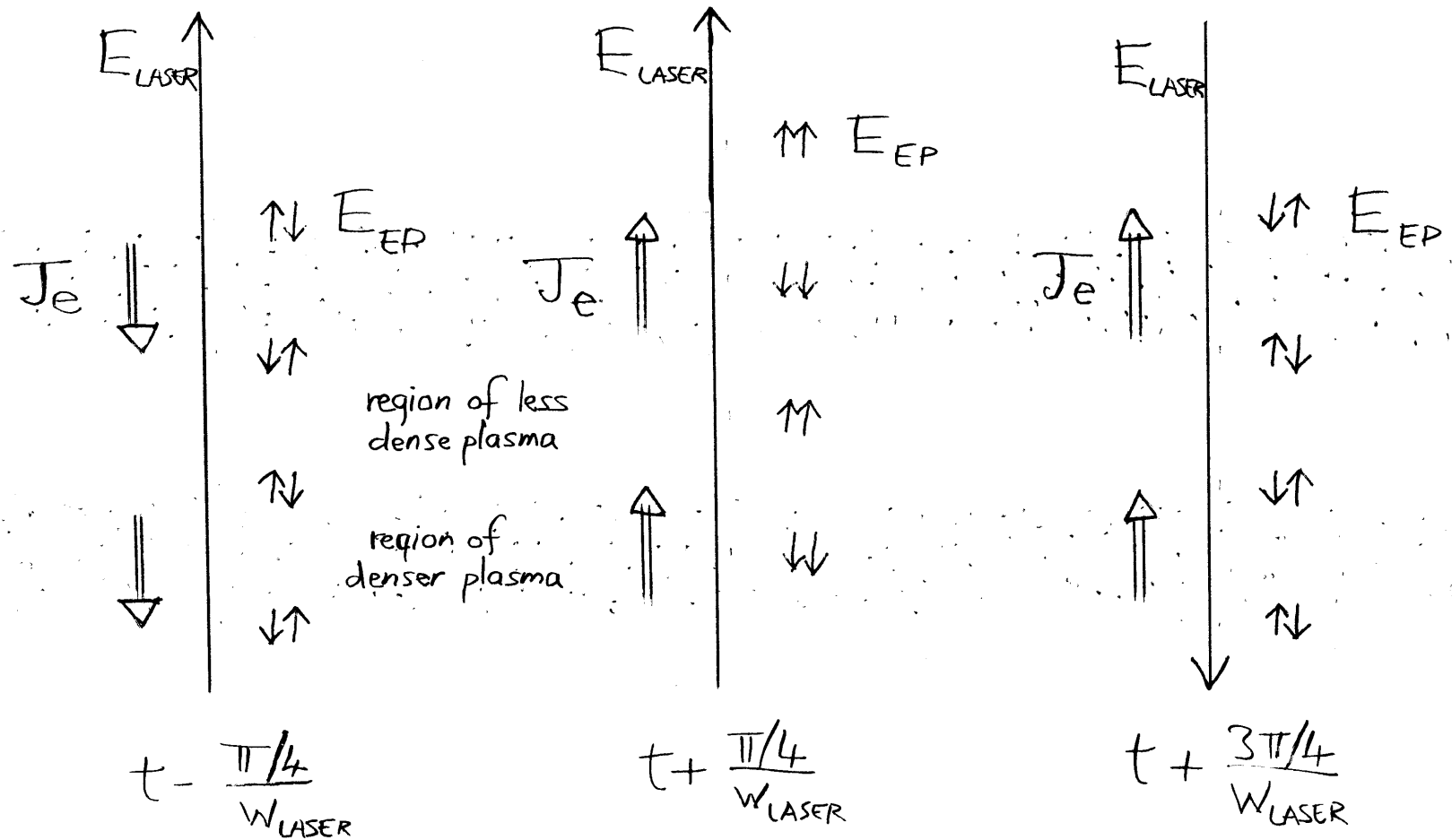
Since all three growing perturbations are electrostatic, the oscillating two-stream instability is an absorptive one. It occurs only for plasma densities near the critical surface

$$w_p \sim w_1 \qquad . \qquad \qquad 4.6\text{-}(1)$$

The physical mechanisms driving the instability are as follows. As for the plasmon-phonon instability, consider perturbations having wavelength small compared to that of the laser, and propagating parallel to the electric field of the laser. In particular, consider two electron plasma waves of equal amplitude, with frequency equal to the pump frequency, propagating in opposite directions. Their electric fields form a standing wave pattern. Let this standing wave have a temporal maximum one eighth of a cycle after the electric field of the laser has its maximum ----- the phase relations which actually hold at the maximum growthrate. These are derived by substituting the maximum growthrate and associated optimum real wavevector into (2) and solving for the eigenvector.

Then, the total high-frequency field amplitude has spatial minima into which the plasma will be driven to form density striations. The process is illustrated in Fig. 4.6.3 by three successive snapshots of the same line parallel to the x-axis,

Fig. 4.6.3. DYNAMICS OF OSCILLATING-TWO-STREAM INSTABILITY

E_{LASER} $\uparrow\downarrow$ E_{EP} J_e

$\downarrow\uparrow$

region of less dense plasma

$\uparrow\downarrow$

region of denser plasma

$\downarrow\uparrow$

$$t - \frac{\pi/4}{w_{LASER}}$$

E_{LASER} $\uparrow\uparrow$ E_{EP} J_e

$\downarrow\downarrow$

$\uparrow\uparrow$

$\downarrow\downarrow$

$$t + \frac{\pi/4}{w_{LASER}}$$

E_{LASER} $\downarrow\uparrow$ E_{EP} J_e

$\uparrow\downarrow$

$\downarrow\uparrow$

$\uparrow\downarrow$

$$t + \frac{3\pi/4}{w_{LASER}}$$

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taken at time intervals of one quarter of the laser period, but now starting one eighth of a laser period before the maximum of the laser electric field.

To complete the regenerative feedback, merely note that the quivering motion of the striated electrons in the laser field constitutes a current. This striated current has a maximum at $t + (\pi/2)w_{Laser}$, and the standing wave has a maximum in the opposite sense at $t + (\pi/4)/w_{Laser}$. The dissipation $E \cdot J$ is negative and therefore regenerates the two electron plasma waves.

The full three-dimensional dispersion relation for the

oscillating two-stream instability may be written down from 3.8-(16). Using physical approximations appropriate to the two electron-plasma waves and to the ion-acoustic regime, respectively, Eq. 3.8-(16) becomes [cf. 4.4-(17)]:

$$
\begin{vmatrix}
1 - \dfrac{w_p^2}{w_{(1)}^2 - 3k_{(1)}^2 V_{Te}^2} + |X|^2 & \dfrac{X}{k_{(0)}\lambda_D} & XY \\[3em]
\dfrac{X^*}{k_{(0)}\lambda_D} & \dfrac{1}{k_{(0)}^2\lambda_D^2}\; 1 - \dfrac{k_{(0)}^2 c_s^2}{w_{(0)}^2} & \dfrac{Y}{k_{(0)}\lambda_D} \\[3em]
X^*Y^* & \dfrac{Y^*}{k_{(0)}\lambda_D} & 1 - \dfrac{w_p}{w_{(-1)}^2 - 3k_{(-1)}^2 V_{Te}^2} + |Y|^2
\end{vmatrix} = 0
$$

$$4.6-(2)$$

where $X = \dfrac{V_1}{V_{Te}}\; \dfrac{w_p}{w_{(1)}}\; [\vec{e}_{x_1} \cdot \vec{e}_{s(1)}]$, $Y = \dfrac{V_1}{V_{Te}}\; \dfrac{w_p}{w_{(-1)}}\; [\vec{e}_{x_1} \cdot \vec{e}_{s(-1)}]$.

Expanding Eq. (2), one obtains the dispersion relation [cf. 4.4-18)]:

$$
1 - \frac{k_{(0)}^2 c_s^2}{w_{(0)}^2} = \frac{V_{Laser}^2}{4V_{Te}^2}\; \frac{k_{(0)}^2 c_s^2}{w_{(0)}^2}\; \frac{w_p^2[w_{(1)}^2 - 3k_{(1)}^2 V_{Te}^2][\vec{e}_{x_1} \cdot \vec{e}_{s(1)}]^2}{w_{(1)}^2 [w_{(1)}^2 - w_p^2 - 3k_{(1)}^2 V_{Te}^2]}
$$

$$
+ \frac{w_p^2[w_{(-1)}^2 - 3k_{(-1)}^2 V_{Te}^2]\,[\vec{e}_{x_1} \cdot \vec{e}_{s(-1)}]^2}{w_{(-1)}^2 [w_{(-1)}^2 - w_p^2 - 3k_{(-1)}^2 V_{Te}^2]} \qquad 4.6-(3)
$$

Now expand the electron-plasma dispersion functions as Taylor series in wavevector and frequency about some point on the instability locus, T. Retain first order in wavevector and frequency. The result is the dispersion relation for

---

Three-dimensional Oscillating Two-stream Instability

and Three-dimensional Modified Plasmon-phonon Instability

$$1 - \frac{k^2_{(o)} c^2_s}{w^2_{(o)}} = \frac{V^2_{Laser}}{8V^2_{Te}} \frac{k^2_{(o)} c^2_s}{w^2_{(o)}} \left\{ \frac{w^4_p [\vec{e}_{x_1} \cdot \vec{e}_{s(_1)}]^2}{w^3_{EP'}[\Delta w - \Delta \vec{k} \cdot \vec{e}_{s(_1)} |V^{EP'}_g|]} \right.$$

$$\left. - \frac{w^4_p [\vec{e}_{x_1} \cdot \vec{e}_{s(-_1)}]^2}{w^3_{EP}[\Delta w + \Delta \vec{k} \cdot \vec{e}_{s(-_1)} |V^{EP}_g|]} \right\}$$

4.6-(4)

Here,

$$(\Delta \vec{k}, \Delta w) = [\vec{k}_{(o)} - \vec{k}_T, \, w_{(o)} - w_T] \qquad 4.5-(5)$$

where
$(\vec{k}_T, w_T)$ is some point on the instability locus T in Fig. 1.

Also,

$$w_{EP'} \equiv w_T + w_1 \quad , \qquad \vec{V}^{EP'}_g$$

$$w_{EP} \equiv |w_T - w_1| \quad , \qquad \vec{V}^{EP}_g$$

are the corresponding plasmon frequencies and group velocities, determined from the displaced dispersion surface passing through T.

The maximum growthrate for real $\vec{k}$ can be obtained approximately by treating the oscillating two-stream as a one-dimensional instability with variation only along the x-axis. The

oscillating two-stream kinematics are then described com-
pletely by Fig. 3. The dispersion relation (4) reduces to

$$1 - \frac{k^2_{(o)} c^2_s}{w^2_{(o)}} = \frac{V^2_{Laser}}{8V^2_{Te}} \frac{k^2_{(o)} c^2_s}{w^2_{(o)}} \frac{w^2_p}{w^4_{EP}} \left\{ \frac{w_{EP}}{\Delta w - \Delta k \; V^{EP}_g} \right.$$

$$\left. - \frac{w_{EP}}{\Delta w + \Delta k \; V^{EP}_g} \right\} \qquad . \qquad\qquad 4.6-(6)$$

For low enough laser-pump intensities, the ion-acoustic dis-
persion function may be replaced by its value at the point
T in Fig. 2. Then Eq. (6) becomes

$$- 1 = \frac{V^2_{Laser}}{8V^2_{Te}} \frac{w^4_p}{w^4_{EP}} \frac{2w_{EP} \; \Delta k \; V^{EP}_g}{\Delta w^2 - (\Delta k \; V^{EP}_g)^2} \qquad , \qquad\qquad 4.6-(7)$$

which may be termed the dispersion relation for the "unmodi-
fied OTS". Equation (7) may be written

$$\Delta w^2 = - \Gamma^2_0 + \left[(\Delta k V^{EP}_g) - \Gamma_0 \right]^2 \qquad\qquad 4.6-(8)$$

where

$$\Gamma_0 \equiv \frac{V^2_{Laser}}{8V^2_{Te}} \frac{w^4_p}{w^4_{EP}} \; w_{EP} \qquad\qquad 4.6-(9)$$

is the maximum growthrate for real wavevector, attained at

$$\Delta k = \Gamma_0 / V^{EP}_g \qquad . \qquad\qquad 4.6-(10)$$

One may rearrange Eq. (7) slightly differently to obtain the Nishikawa result. Note that each decay plasmon has a frequency which is lower than that of a free plasmon with the same wavevector, by the frequency

$$\delta \equiv \Delta k \ v_g^{EP} \qquad . \qquad \qquad 4.6-(11)$$

Note also that the unstable roots of (8) are pure imaginary

$$\Delta w = i\gamma \qquad \qquad 4.6-(12)$$

Substituting (1) and (12) into (7) and rearranging, one obtains

$$\frac{V_{Laser}^2}{8 \ V_{Te}^2} \ \frac{w_p^4}{w_{EP}^4} = \frac{\gamma^2 + \delta^2}{2\delta w_{EP}} \quad , \qquad \qquad 4.6-(13)$$

the well known result of Nishikawa. However, the results of Silin and of Friedberg and Marder, concerning cold plasma driven by an electric field oscillating below the plasma frequency, cannot be recovered from our theory. This is because the strength of the coupling between modes in our theory is effectively expressed in terms of the expansion parameter V_{Laser}/V_{Te} , which goes to infinity as V_{Te} goes to zero.

A stability analysis conducted on the low-pump intensity, one-dimensional dispersion relation (8) yields the following results. An initially localized excitation develops time-asymptotitically into an unstable pulse, with pulse edges propagating at the equal and opposite group

velocities of the two interacting plasmons. The maximum

growthrate is seen by a stationary observer; this maximum

growthrate is that given by Eq. (9). The time-asymptotic

behavior of the pulse may be described by a plot of observed

growthrate vs. observer velocity (compare Fig. 4.5.5):

Fig. 4.6.4 TIME-ASYMPTOTIC PULSE RESPONSE: COLLINEAR

APPROXIMATION AND LOW PUMP-WAVE INTENSITY

observed growthrate w_{oi}

maximum growthrate

$(w_{oi})_{Max} = \Gamma_o$

observer velocity $(V_{obs})_x$

$-V_{Te}$ $-V_g^{EP}$ $-c_s$ 0 c_s V_g^{EP} V_{Te}
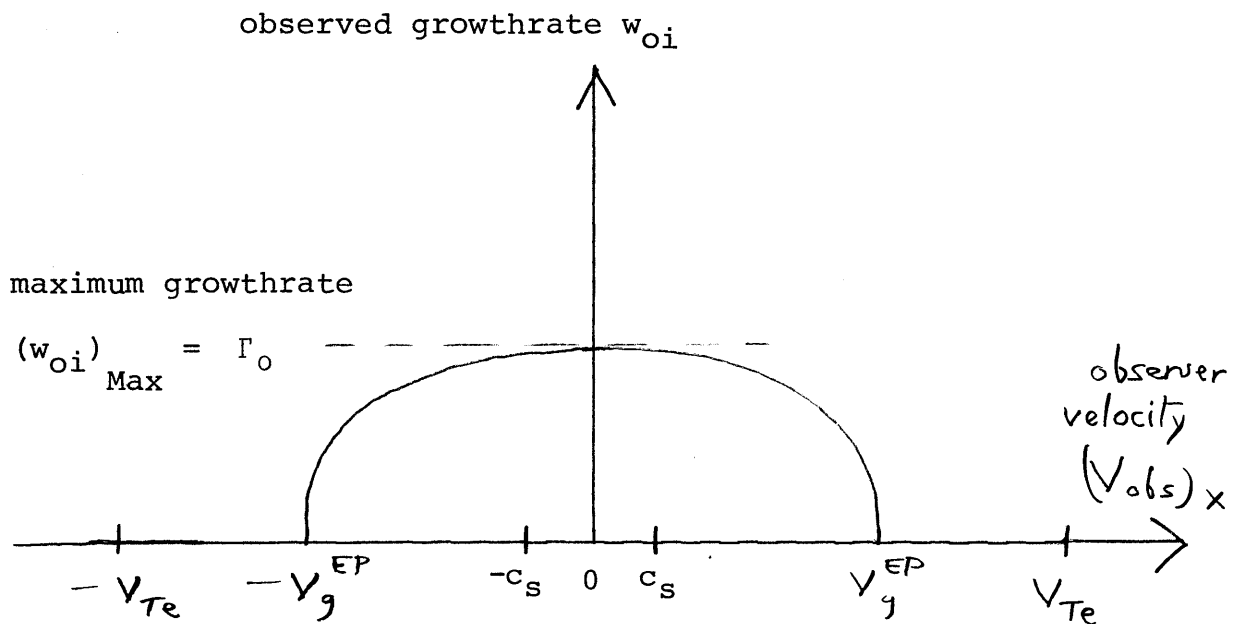
For higher intensities of the laser pump, (8) is insuffic-

ient. One may then use the exact ion-acoustic dispersion

function while retaining the one-dimensional approximation,

thus returning to (6). A stability analysis conducted on

the dispersion relation (6) yields the combined time-asymp-

totic pulse response for the oscillating two-stream insta-

bility together with the modified plasmon-phonon instability.

The dispersion relation for the latter is contained in (6).
Indeed, the upper-left 2×2 submatrix and lower-right 2×2
submatrix of (2) describe the modified plasmon-phonon insta-
bility, although as discussed in section 3.8, these descrip-
tion are not satisfactory in isolation from the oscillating
two-stream. The results of the stability analysis are shown
below for various laser power levels. For sufficiently low
pump intensities, the results resemble a superposition of
the pulse responses for the unmodified plasmon-phonon (see
Fig. 4.5.5), and the low power oscillating two-stream (Fig.
4.6.4). For higher pump intensities, the plasmon-phonon and
OTS instabilities affect each other strongly, then coalesce;
finally, the OTS becomes dominant.

Fig. 4.6.5. Combined Pulse-Response for DECAY & OTS : both have almost their unmodified forms.
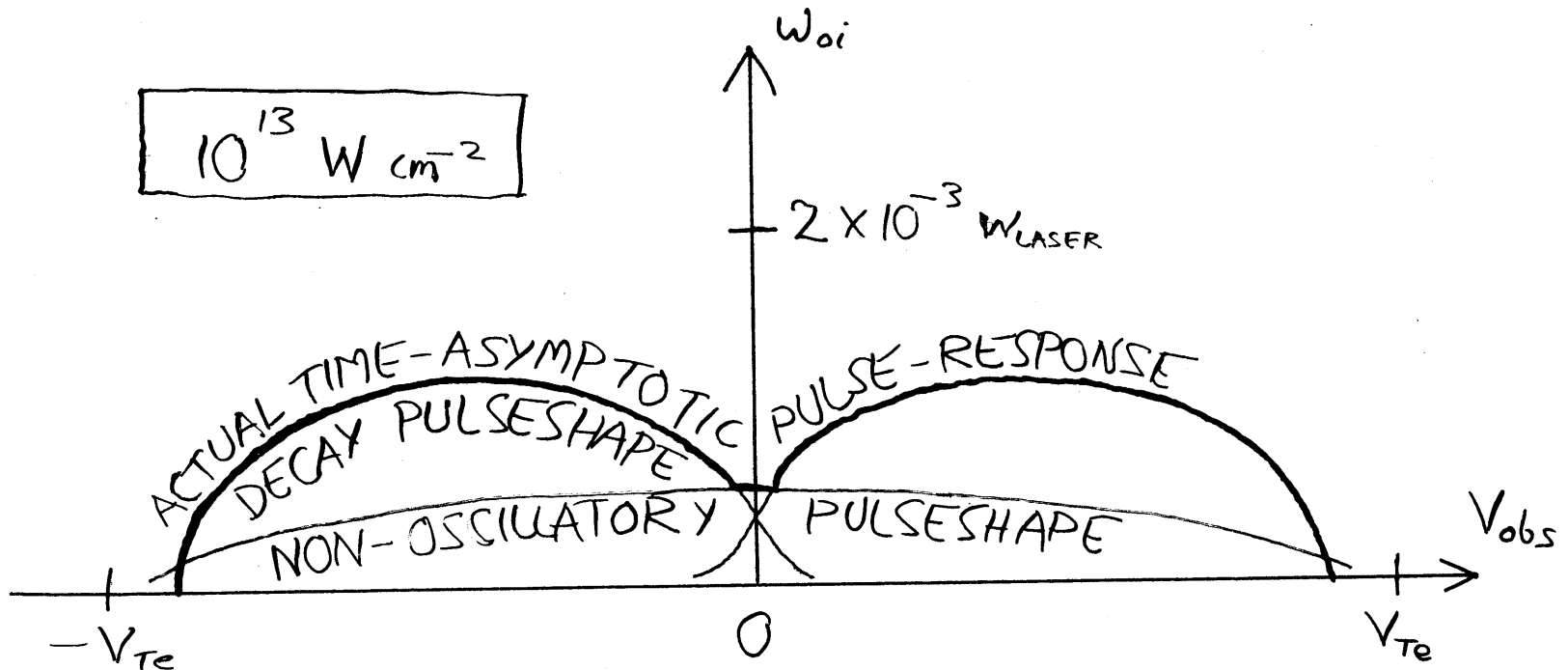
Fig 4.6.6. Combined Pulse Response for DECAY and OTS ; Strong Mutual Effect, both Modified and Unmodified Pulse-Responses shown for comparison.
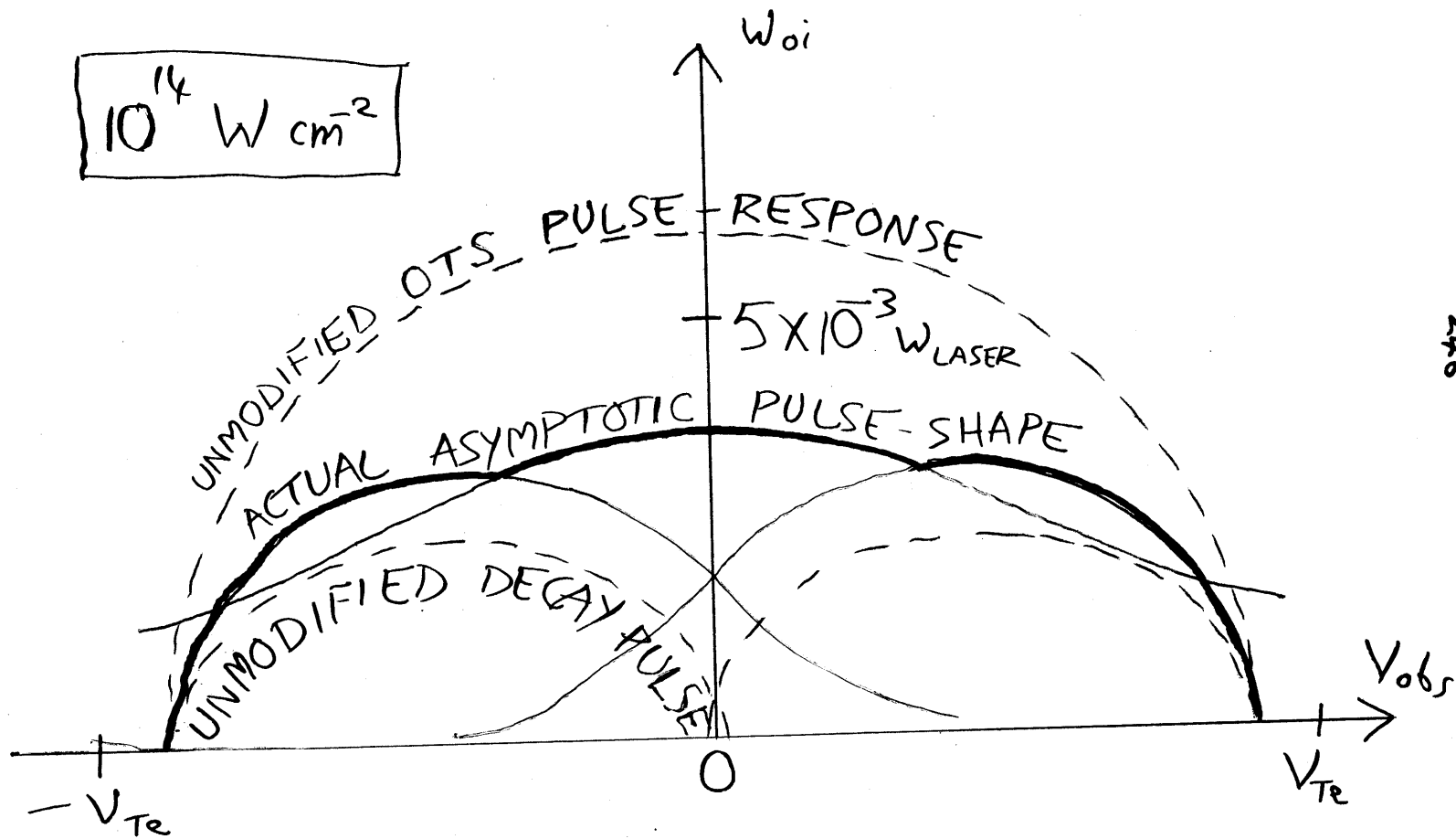
Fig. 4.6.7.   Coalesced Pulse-Response of DECAY & OTS : Unmodified
Pulse-Responses for Decay & OTS shown for comparison.
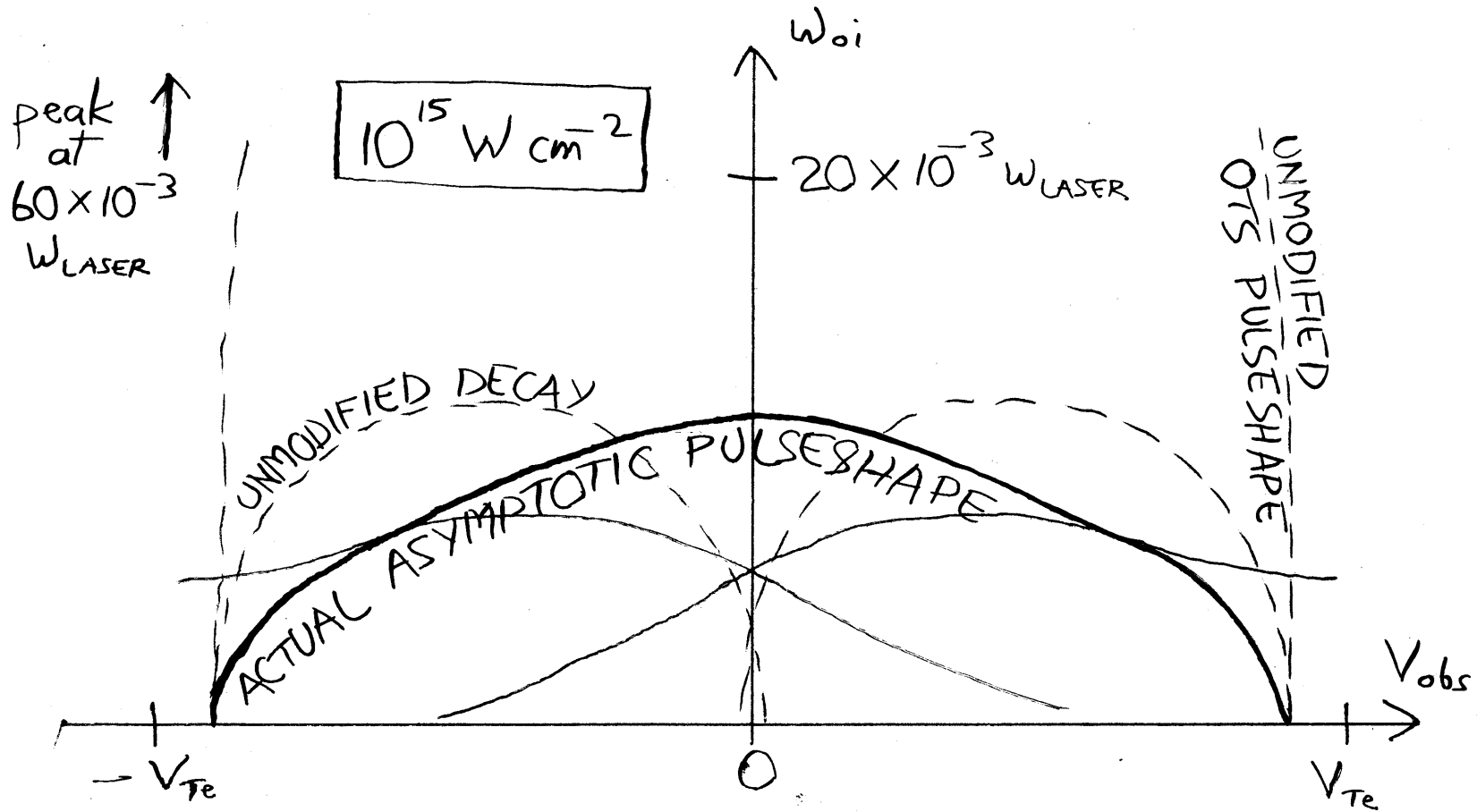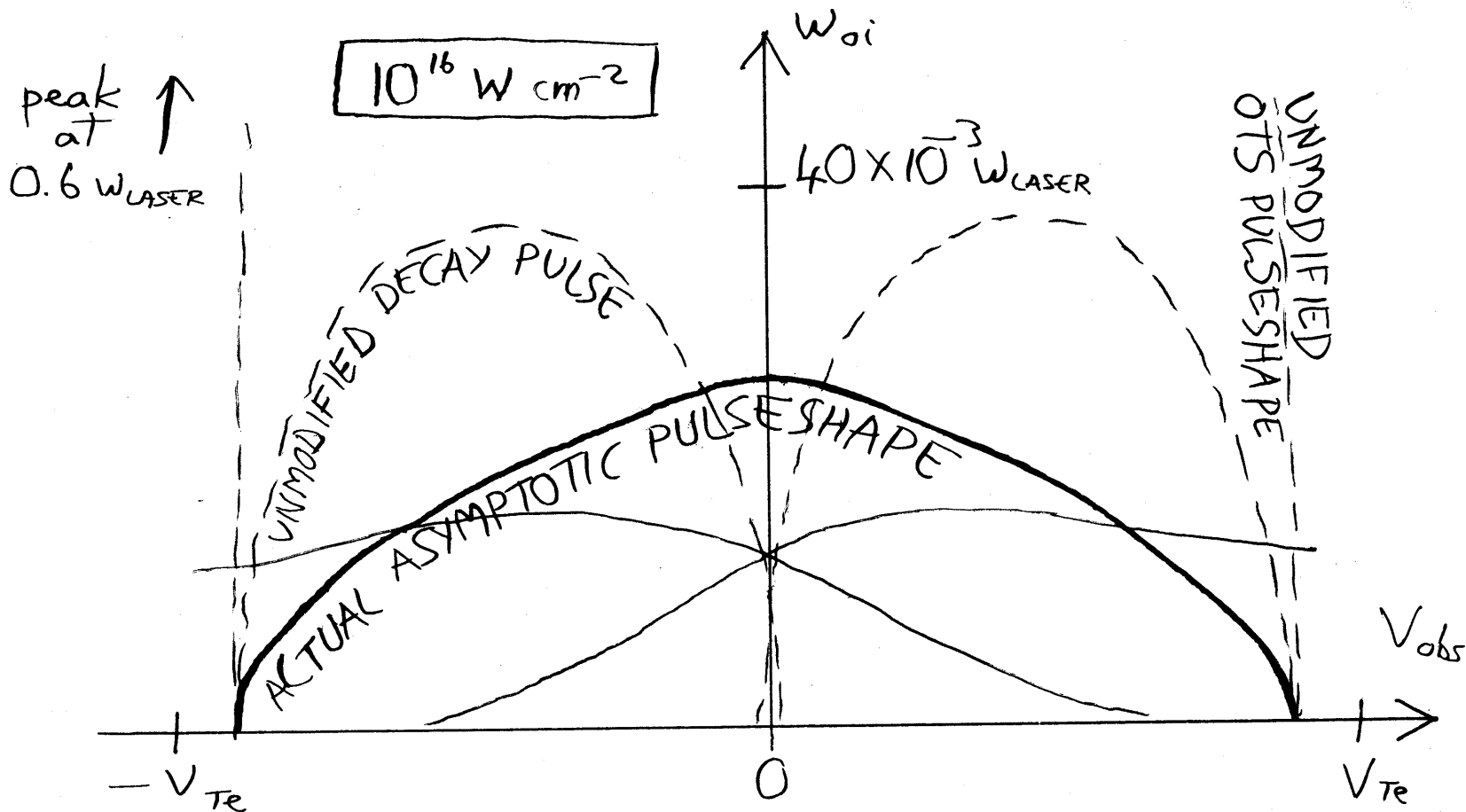

peak
at
60×10^{-3}
W_{LASER}

W_{oi}

$\boxed{10^{15} \, W \, cm^{-2}}$

$-20 \times 10^{-3} \, W_{LASER}$

UNMODIFIED
OTS PULSESHAPE

UNMODIFIED DECAY

ACTUAL ASYMPTOTIC PULSESHAPE

V_{obs}

$-V_{Te}$

O

V_{Te}

Fig. 4.6.8. Pulse-Reponse of DECAY dominated by that of OTS. Unmodified Pulse-Responses shown for comparison.

peak at 0.6 W_{LASER}

10^{16} W cm^{-2}

W_{oi}

$40 \times 10^{-3} W_{LASER}$

UNMODIFIED DECAY PULSE

UNMODIFIED OTS PULSESHAPE

ACTUAL ASYMPTOTIC PULSESHAPE

V_{obs}

$-V_{Te}$

0

V_{Te}

248

4.7 The Raman Instability

In the Raman instability, an electromagnetic perturbation and an electrostatic perturbation are coupled together by the laser pump. The perturbations closely resemble a freely-propagating electromagnetic wave and an electron-plasma wave, respectively. In quantum language, the laser photon decays into another photon and a plasmon. As discussed in Sections 3.4 and 3.8, the locus of the instability in the $(\vec{k},w)$ four-space is found from the intersection of the electron-plasma dispersion hypersurface and the upshifted electromagnetic dispersion hypersurface. This locus is illustrated on a (k_z,w) diagram and a $(k_z,k_\perp,w)$ diagram.

The Raman instability leads to scattering of laser energy by the plasma. The instability can occur at any plasma density less than one-quarter of the critical density:

$$w_p < w_1/2 \qquad . \qquad \qquad 4.7\text{-}(1)$$

The dynamics of the instability bear some resemblance to those of the Brillouin instability described in section 4.2. As for the Brillouin instability, its dynamics can best be described in the case where both electromagnetic and electrostatic perturbations propagate in the same direction as the laser. The kinematics for this case are shown in Fig. 1. Consider a small backscattered electromagnetic perturbation. This beats with the incoming laser beam to produce a pattern of radiation pressure minima, into which the electrons are

Fig. 4.7.1. PROPAGATION COLLINEAR WITH LASER
PROPAGATION
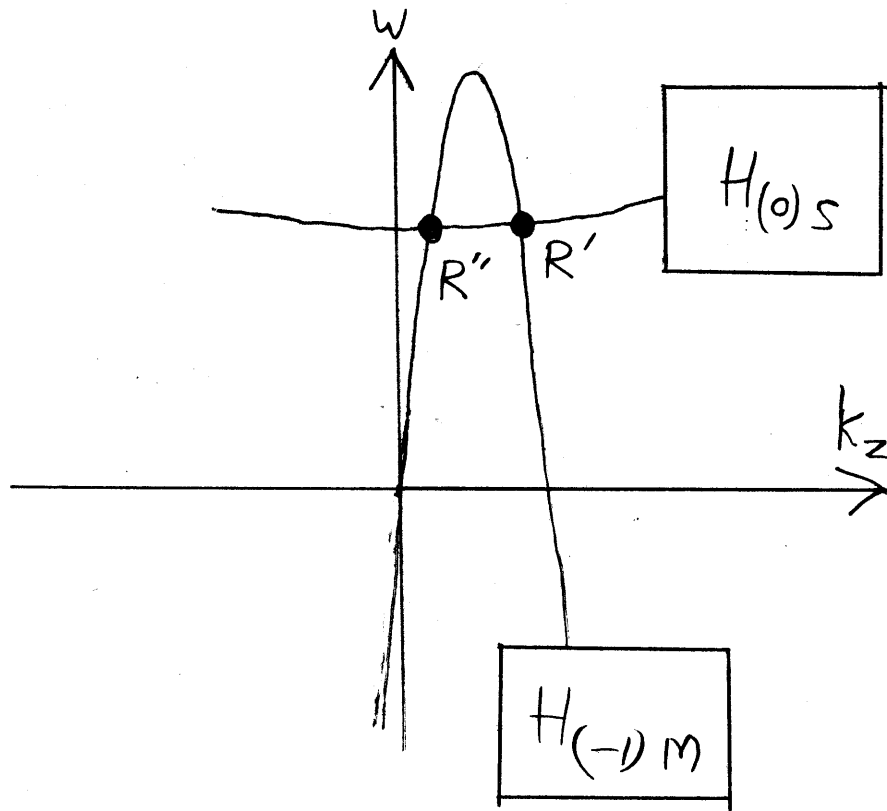
Here $H_{(0)s}$ is the electron-plasma dispersion curve.
$H_{(-1)m}$ is the electromagnetic dispersion curve, upshifted by
the laser wavevector and frequency.
The points labelled R', R'' are the loci of Raman
instability, near which the wavenumber and frequency of
$(k_{(0)z}, \omega_{(0)})$ of the electrostatic perturbation lie for small
pump-wave intensities.

Fig. 4.7.2. PROPAGATION IN PLANE CONTAINING DIRECTION OF LASER PROPAGATION
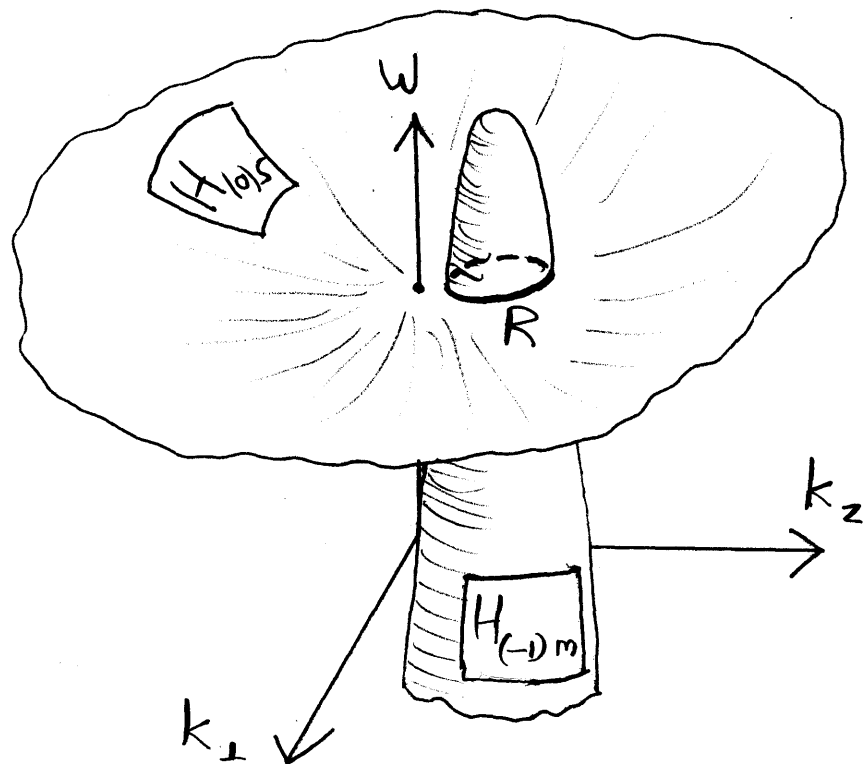
$H_{(0)s}$ is the electron-plasma dispersion surface

$H_{(-1)m}$ is the upshifted electromagnetic dispersion surface

R is the locus of Raman instability. The values of $\left(\vec{k}_{(0)}, w_{(0)}\right)$ lie near this locus.

forced to form a pattern of electron density striations.
When this pattern has a wavevector and frequency near the
electron-plasma wave dispersion surface, the formation of
striations is enhanced. This explains how an electromagnetic
perturbation can generate a perturbation having the form of
an electron-plasma wave. The regenerative or positive feed-
back loop is closed by showing that the electron striations
can regenerate the electromagnetic perturbation. This happens
as in the Brillouin instability. The striated electrons os-
cillating in the laser field constitute an endfire-antenna-
array which reradiates the backscattered electromagnetic wave.

The full three-dimensional dispersion relation for the
Raman instability may be written down from 3.8-( ). Using
the physical approximations appropriate to the electron-plasma
and electromagnetic waves, 3.8-( ) becomes

$$
\begin{vmatrix}
1 - \dfrac{w_p^2}{w_{(0)}^2 - 3k_{(0)}^2 V_{Te}^2} & \dfrac{k_{(0)} V_1}{\left| w_{(-1)} \right|} \dfrac{w_p^2 [\vec{e}_{x_1} \cdot \vec{e}_{M(-1)}]}{[w_{(0)}^2 - 3k_{(0)}^2 V_{Te}^2]} \\[3em]
\dfrac{k_{(0)} V_1^*}{\left| w_{(-1)} \right|} \dfrac{w_p^2}{[w_{(0)}^2 - 3k_{(0)}^2 V_{Te}^2]} & 1 - \dfrac{w_p^2 + k_{(-1)}^2 c^2}{w_{(-1)}^2}
\end{vmatrix} = 0
$$

4.7-(2)

The leading diagonal of (2) contains the electron-plasma
wave and electromagnetic-wave dispersion functions. Linearize
these about some wavevector and frequency belonging to the
instability locus R of Fig. 2, for which

$$\left(\vec{k}_{(o)}, w_{(o)}\right) \equiv (\vec{k}_{EP}, w_{EP}) \qquad \qquad 4.7\text{-}(3)$$

$$[\vec{k}_{(-1)}, w_{(-1)}] \equiv (-\vec{k}_{EM}, -w_{EM}) \equiv [\vec{k}_{(o)} - \vec{k}_1, w_{(o)} - w_1] \ . \qquad 4.7\text{-}(4)$$

Then, (2) becomes

$$\begin{vmatrix} \dfrac{2[\Delta w - \Delta\vec{k}\cdot\vec{e}_{s(o)} \, |v_g^{EP}|]}{w_{EP}} & \dfrac{k_{EP}V_1}{w_{EM}} \ \dfrac{w_p^2}{w_{EP}^2} \ [\vec{e}_{x1} \cdot \vec{e}_{M(-1)}] \\[4mm] \dfrac{k_{EP}V_1^*}{w_{EM}} \ \dfrac{w_p^2}{w_{EP}^2} [\vec{e}_{x1} \cdot \vec{e}_{M(-1)}] & \dfrac{2[\Delta w + \Delta\vec{k}\cdot\vec{e}_{s(-1)} \, |v_g^{EM}|]}{-w_{EM}} \end{vmatrix} = 0$$

$$4.7\text{-}(5)$$

Expand (5) and use 4.2-(6). The result is the dispersion relation for the

---

Three-dimensional Raman Instability

$$[\Delta w - \Delta\vec{k}\cdot\vec{e}_{s(o)} \, |v_g^{EP}|][\Delta w + \Delta\vec{k}\cdot\vec{e}_{s(-1)} \, |v_g^{EM}|]$$

$$= -k_{EP}^2 V_{Laser}^2 \cdot \frac{w_p^4}{w_{EM}w_{EP}^3} \ [\vec{e}_{x1} \cdot \vec{e}_{M(-1)}]^2 \qquad 4.7\text{-}(6)$$

The maximum growthrate for real $\vec{k}$ is obtained from this dispersion relation by setting $\Delta\vec{k} = \vec{0}$, which yields

$$\Delta w = i \, k_{EP}V_{Laser} \, \frac{w_p^2}{\sqrt{w_{EP}^3 w_{EM}}} \, \left| \vec{e}_{x1} \cdot \vec{e}_{M(-1)} \right| \qquad 4.7\text{-}(7)$$

This growthrate depends on the three-dimensional nature of the electrical polarizations via the last factor. This last factor describes the effectiveness of electrons, quivering in the laser field along $\vec{e}_{x_1}$, as a means of regenerating the scattered electromagnetic wave, polarized along $\vec{e}_{M(-_1)}$. This effectiveness factor is unity for electromagnetic radiation propagating in the y-z plane, i.e., scattering perpendicular to the laser polarization. The effectiveness factor is zero for electromagnetic propagation along the laser polarization, so there is no scattering in the x-direction.

This geometrical dependence on scattering angle is the same as that in the Brillouin instability described in section 4.2.

The dispersion relation (6) has been used by Bers and Chambers to derive two-dimensional cross sections of the three-dimensional time-asymptotic pulse response. The cross section in the y-z plane is found first. The cross section of the three-dimensional pulse in other planes through the z-axis is then found by multiplying the observed growthrate at each observer velocity by the appropriate value of the effectiveness factor. Thus, the full three-dimensional pulse response is built up.

For comparison with the Brillouin instability, we sketch here that part of the pulse response generated by propagation along the z-axis, that is, propagation collinear with the

direction of propagation of the laser pump. This is the cross
section along the z-axis of the full three-dimensional pulse
response. This pulse response has two lobes corresponding to
the two points R',R" of the instability locus in Fig. 1. The
edges of the lobe corresponding to R' propagate at the corres-
ponding plasmon and photon group velocities which are the slopes
of the dispersion curves through R'. Similarly, the edges of
the lobe corresponding to R" propagate at plasmon and photon
group velocities given by the slopes of the dispersion curves
through R". The maximum growthrate in each lobe is the same
as that for real $\Delta \vec{k}$, given in (7), calculated from the disper-
sion relation (6) linearized about R' or R" as appropriate.
The pulse response in the direction of propagation, then,
looks like this:
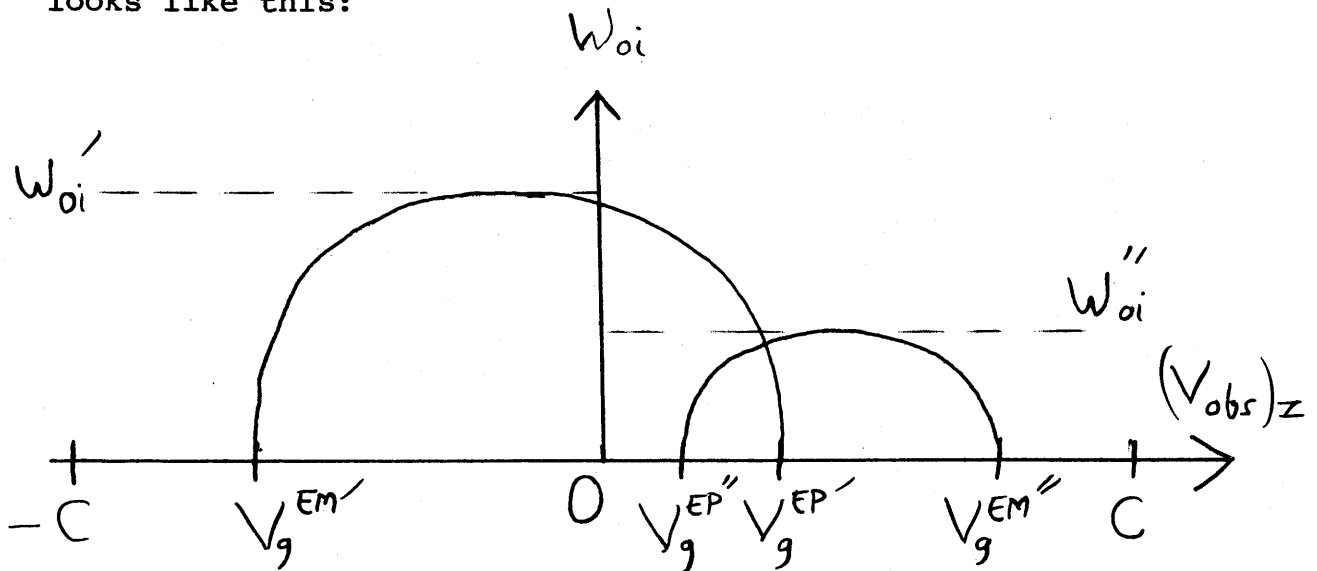
Observed growthrate of disturbance vs. observer velocity, for
observer moving along laser beam, $w'_{oi} = k_{EP'} V_{Laser} w_p^2 / \sqrt{w_{EP'}^3 w_{EM'}}$,

$w''_{oi} = k_{EP''} V_{Laser} w_p^2 / \sqrt{w_{EP''}^3 w_{EM''}}$.

4.8 The Two-Plasmon Instability

In the two-plasmon instability, two electrostatic per-
turbations, both closely resembling electron-plasma waves,
are coupled together by the laser pump. In quantum lang-
uage, the laser photon decays into two plasmons. As dis-
cussed in sections 3.4 and 3.8, the locus of the instability
in $(\vec{k},w)$ four-space is found from the intersection of the
positive-frequency electron-plasma dispersion hypersurface
with the upshifted negative-frequency electron-plasma dis-
persion hypersurface.

This locus is illustrated on a (k_z,w) diagram and a
$(k_z,k_\perp,w)$ diagram. As discussed later in this section, the
two-plasmon instability has zero growthrate for collinear
propagation and another zero for propagation perpendicular
to the laser wavevector, so that maximum growthrates occur
for plasmons propagating around 45° to the direction of
laser propagation. The kinematics of this case are sketched
on a $(|k|,w)$ diagram.

The two-plasmon instability leads to absorption of
laser energy by the plasma. The instability can only occur
at plasma densities lying just outside the quarter-critical
surface:

$$w_p \sim w_1/2 \quad - \qquad\qquad 4.8-(1)$$

The dynamics of the instability are more complex than for
the other absorptive instabilities, namely, the plasmon-

Fig. 4.8.1   KINEMATICS OF COLLINEAR PROPAGATION — NO INSTABILITY.
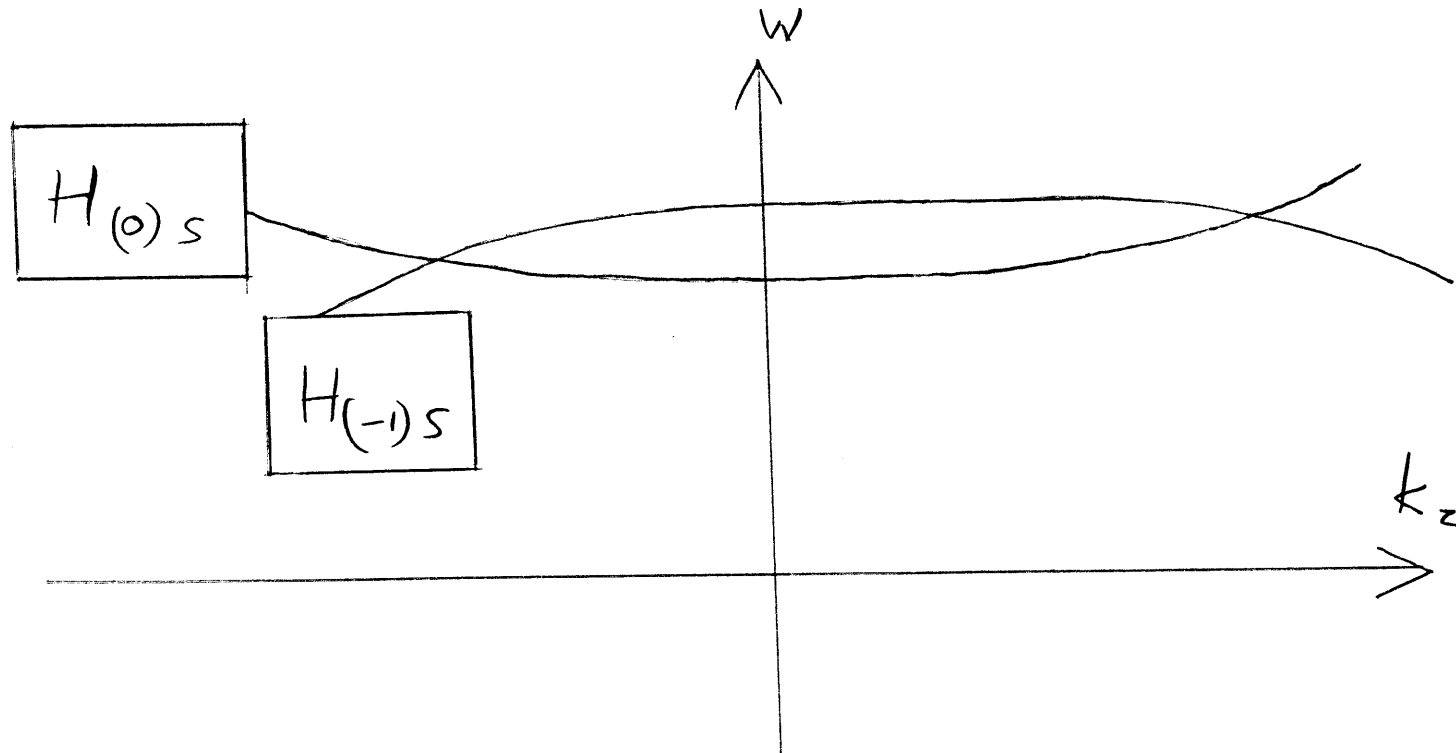
Here $H_{(0)s}$ is the electron-plasma dispersion curve
$H_{(-1)s}$ is the negative-frequency branch of the electron-plasma dispersion curve, upshifted by the laser wavenumber and frequency.

Fig. 4.8.2. KINEMATICS IN PLANE CONTAINING DIRECTION OF LASER PROPAGATION
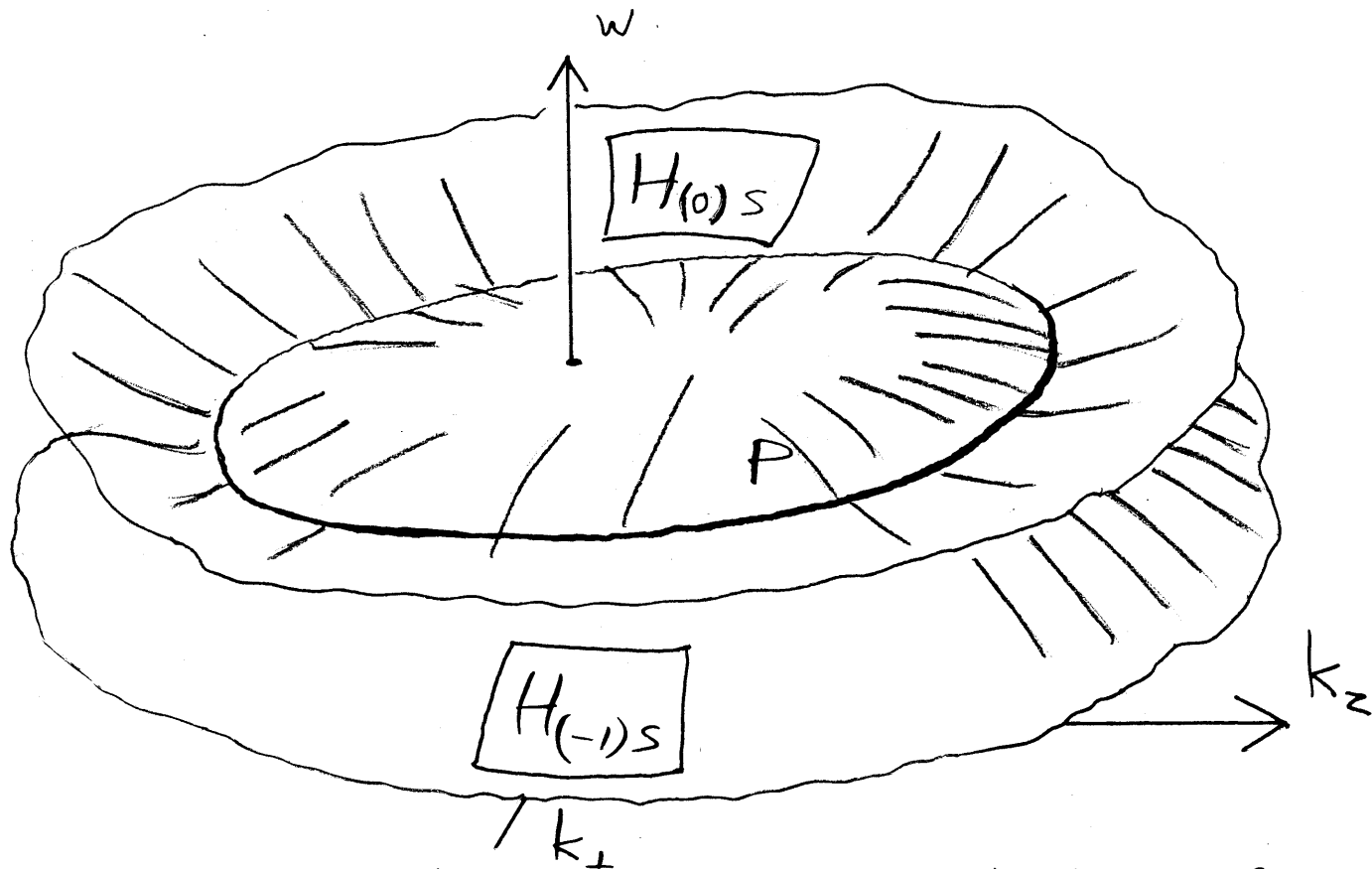
Here $H_{(0)S}$ is the electron-plasma dispersion surface, $H_{(-1)S}$ is the upshifted negative-frequency electron plasma dispersion surface, P is the locus of the 2-plasmon instability. The values of $(\vec{k}_{(0)}, w_{(0)})$ lie near this locus.

Fig. 4.8.3. KINEMATICS OF PROPAGATION AT $45°$ TO LASER PROPAGATION.
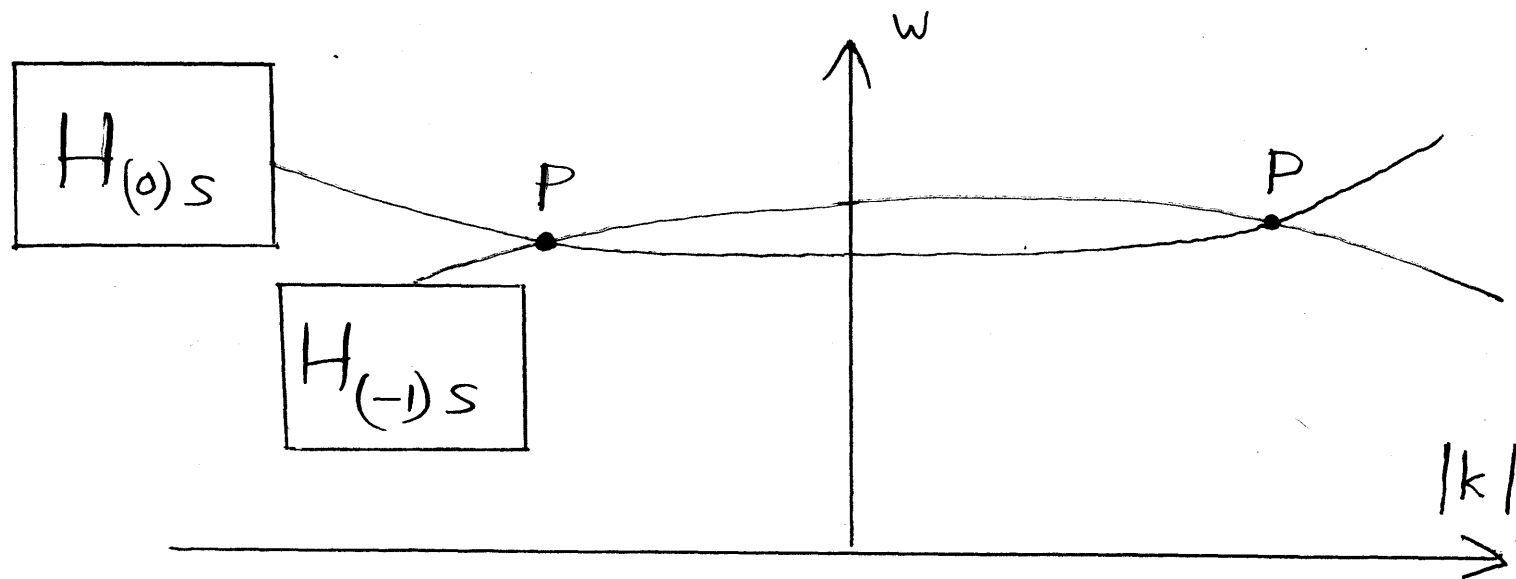
Here $H_{(0)S}$ and $H_{(-1)S}$ are curves obtained by taking the appropriate cross-sections of the surfaces $H_{(0)S}$, $H_{(-1)S}$ in Fig. 2. The points P are obtained by taking the appropriate cross-section of the ring P in Fig. 2.

phonon and non-oscillatory instabilities. These other insta-
bilities have decay products which fall naturally into high-
and low-phase-velocity groups. The high-phase-velocity decay
products or perturbations combine with the laser field to
create a radiation pressure pattern which has a low phase
velocity. This pattern reinforces the density striations
of the low-phase-velocity perturbation. The electron density
is modulated along with the plasma density, so that when
the electrons quiver in the laser field, the resulting cur-
rent has a component which regenerates the high-phase-velo-
city perturbation.

In the case of the two-plasmon instability, the two
decay products have roughly equal phase velocities. The
above physical mechanism must be invoked twice; the first
time one considers the radiation pressure from one partic-
ular plasma wave and the modulated electron current due to
the other; the second time, the rôles of the plasma waves
are reversed. The coupling coefficient is thus the sum of
two terms. These terms cancel for propagation perpendicular
to the laser propagation. These terms are initially zero
for propagation parallel to the laser polarization.
Thus, the maximum growthrates are attained when the decay
plasmons propagate at roughly $45°$ to both the propagation
and polarization vectors of the laser.

The full three-dimensional dispersion relation for
the two-plasmon instability may be written from 3.8-( ).

Using the physical approximations valid for the two electron-plasma waves, 3.8-( ) becomes

$$
\begin{vmatrix}
1 - \dfrac{w_p^2}{w_{(0)}^2 - 3k_{(0)}^2 V_{Te}^2} & V_1 \dfrac{w_{(0)}|W_{(-1)}|}{w_p^2} \left\{ \dfrac{[\vec{e}_{x_1} \cdot \vec{e}_{s(-1)}]}{w_{(0)}/k_{(0)}} - \dfrac{[\vec{e}_{x_1} \cdot \vec{e}_{s(0)}]}{|w_{(-1)}|/k_{(-1)}} \right\} \\[4mm]
V_1^* \dfrac{w_{(0)}|w_{(-1)}|}{w_p^2} \left(\dfrac{[\vec{e}_{x_1} \cdot \vec{e}_{s(-1)}]}{w_{(0)}/k_{(0)}} - \dfrac{[\vec{e}_{x_1} \cdot \vec{e}_{s(0)}]}{|w_{(-1)}|/k_{(-1)}} \right) & 1 - \dfrac{w_p^2}{w_{(-1)}^2 - 3k_{(-1)}^2 V_{Te}^2}
\end{vmatrix} = 0
$$

4.8-(2)

The leading diagonal of (2) contains the two electron-plasma wave dispersion functions. Linearize these about some wave-vector and frequency belonging to the instability locus P of Fig. 2, for which

$$[\vec{k}_{(0)}, \ w_{(0)}] \equiv [\vec{k}_{EP'}, \ w_{EP'}] \qquad 4.8\text{-}(3)$$

$$[\vec{k}_{(-1)}, w_{(-1)}] \equiv [\vec{k}_{(0)} - \vec{k}_1, w_{(0)} - w_1] \equiv (-\vec{k}_{EP'}, - w_{EP})$$

$$\equiv [k_{EP} \ \vec{e}_{s(-1)}, \ - w_{EP}] \qquad . \qquad 4.8\text{-}(4)$$

Then, (2) becomes

$$
\begin{vmatrix}
\dfrac{2(\Delta w - \Delta\vec{k} \cdot \vec{e}_{s(0)} |v_g^{EP'}|)}{w_{EP'}} & V_1 \dfrac{w_{EP'}w_{EP}}{w_p^2} \left\{ \dfrac{[\vec{e}_{x_1} \cdot \vec{e}_{s(-1)}]}{v_{ph}^{EP'}} - \dfrac{[\vec{e}_{x_1} \cdot \vec{e}_{s(0)}]}{v_{ph}^{EP}} \right\} \\[5mm]
V_1^* \dfrac{w_{EP'}w_{EP}}{w_p^2} \left\{ \dfrac{[\vec{e}_{x_1} \cdot \vec{e}_{s(-1)}]}{v_{ph}^{EP'}} - \dfrac{[\vec{e}_{x_1} \cdot \vec{e}_{s(0)}]}{v_{ph}^{EP}} \right\} & \dfrac{2(\Delta w + \Delta\vec{k} \cdot \vec{e}_{s(-1)} |v_g^{EP}|)}{- w_{EP}}
\end{vmatrix}
$$

$$= 0 \ . \qquad 4.8\text{-}(5)$$

Expand Eq. (5) and use 4.2-(6); the result is the dispersion relation for the

---

Three-dimensional Two-plasmon Instability

$$(\Delta w - \Delta \vec{k} \cdot \vec{e}_{s(o)} \, | V_g^{EP'} |)(\Delta w + \Delta \vec{k} \cdot \vec{e}_{s(-1)} \, | V_g^{EP} |) \;=$$

$$-\frac{V_{Laser}^2}{16} \frac{w_{EP}^2 w_{EP'}^2}{w_p^4} \left[\frac{[\vec{e}_{x_1} \cdot \vec{e}_{s(-1)}]}{V_{ph}^{EP'}} - \frac{[\vec{e}_{x_1} \cdot \vec{e}_{s(o)}]}{V_{ph}^{EP}} \right]^2 \qquad 4.8\text{-}(6)$$

The terms in the square bracket are both identically zero when the plasma waves propagate parallel to the laser propagation vector;

$$\vec{k}_{(o)} \; || \; \vec{k}_{(-1)} \; || \; \vec{k}_1 \;=\; k_1 \vec{e}_z \quad . \qquad\qquad 4.8\text{-}(7)$$

They cancel exactly when the plasma-wave propagation vectors form an isosceles triangle with the laser propagation vector:

$$k_{EP'} \equiv |\vec{k}_{(o)}| = |\vec{k}_{(-1)}| \equiv k_{EP} \quad . \qquad\qquad 4.8\text{-}(8)$$

This happens when both plasmon wavevectors are almost perpendicular to the laser wavevector, since the characteristic electron-plasma wavenumbers are much greater than the characteristic laser wavenumbers. In order for the square bracket to be a maximum, one requires plasmon wavevectors, hence plasmon group velocities, at roughly $45°$ to both the laser wavevector and the laser polarization. The stability analysis

conducted on the dispersion relation (6) thus yields a pulse response whose cross section in the x-z plane has four lobes at $45°$ to the x- and z-axes. Cross sections in other planes through the laser wavevector are obtained by multiplying all growthrates by $|\cos \alpha|$ where α is the angle betwen the desired plane and the x-z plane.

4.9   The Coalesced Raman-Two-plasmon Instability

In this instability, two perturbations are coupled together by the laser pump. One is electrostatic; the other has a polarization which is neither electrostatic nor electromagnetic, but is to be determined from the coupling equations.

The electrostatic perturbation closely resembles an electron-plasma wave. The other perturbation closely resembles a linear superposition of an electron plasma wave and an electromagnetic wave. Indeed, its wavevector and frequency lie close to the electron-plasma dispersion hypersurface and also close to the electromagnetic dispersion hypersurface, simultaneously.   In quantum language, the laser photon decays into two plasmons or into a plasmon and a photon, with a fixed probability for the choice. The locus of the instability in the $(\vec{k},w)$ four-space is the region where the locus of Raman instability, shown in Fig. 4.7.2, closely approaches that of two-plasmon instability, shown in Fig. 4.8.2. The close approach occurs only very near the quarter-critical surface:

$$w_p \sim w_l/2 \qquad .$$

4.9-(1)

The close approach is illustrated on a (k_z,w) and on a $(k_z,k_\perp,w)$ diagram, which are obtained, respectively, by superimposing Figs. 4.7.1 and 4.8.1, and superimposing Figs. 4.7.2 and 4.8.2. The plasma density is chosen so that the close approach becomes a coincidence on the (k_z,w) diagram and a tangency on the

Fig 4.9.1. PROPAGATION COLLINEAR WITH LASER PROPAGATION.
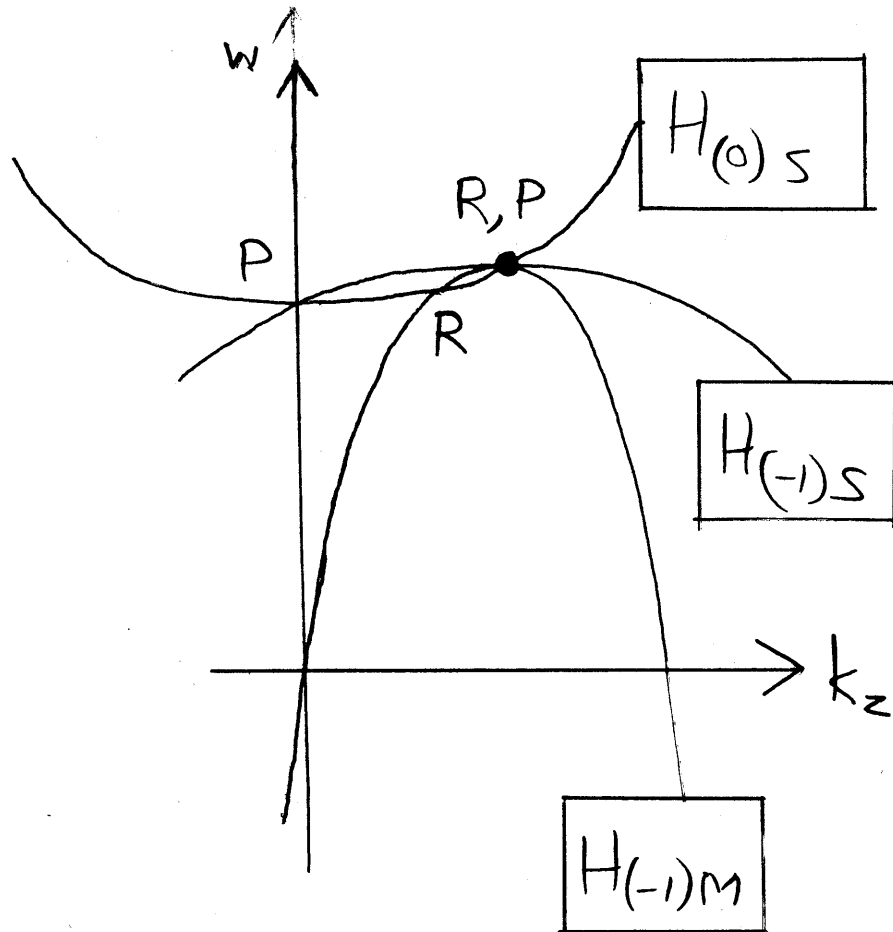
Here $H_{(0)S}$ is the electron-plasma dispersion curve ; $H_{(-1)S}$ and $H_{(-1)M}$ are the electron-plasma and electromagnetic dispersion curves, upshifted by the laser wavevector and frequency. The point labelled R, P is the locus of the coalesced Raman-2-Plasmon instability, near which the wavenumber and frequency $(k_{(0)}, w_{(0)})$ of the purely electrostatic perturbation lie for small pump-wave intensities.

Fig. 4.9.2   PROPAGATION IN PLANE CONTAINING DIRECTION
OF LASER PROPAGATION
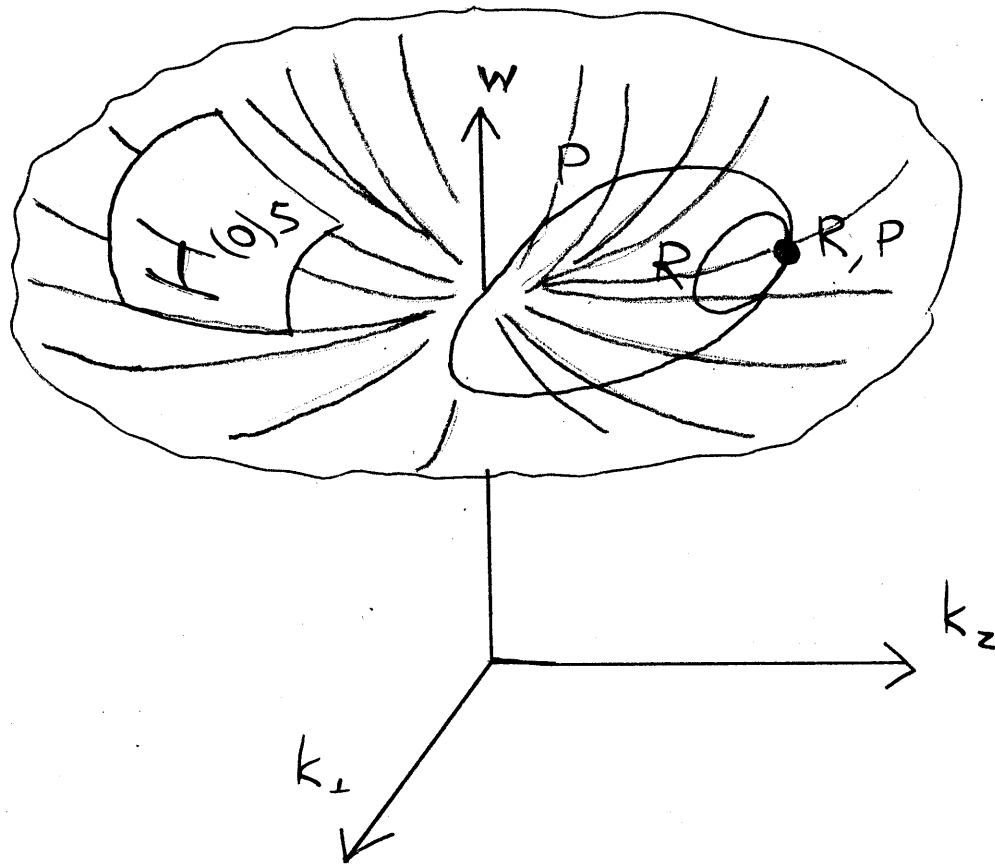
Here $H_{(0)s}$ is the electron-plasma dispersion surface. $H_{(-1)s}$ and $H_{(-1)m}$ have been omitted. The point labelled R, P is the locus of the instability. The values of $(\vec{k}_{(0)}, w_{(0)})$ lie near this locus.

$(k_z, k_\perp, w)$ diagram.

The absorptive or reflective nature of the instability depends on the ratio of electrostatic polarization to electromagnetic polarization in the mixed decay product, i.e., the ratio

$$\left| E_{(-1)s} \right| \;:\; \left| E_{(-1)M} \right| \quad . \qquad\qquad 4.9\text{-}(1)$$

The full three-dimensional dispersion relation for the coalesced instability, together with the ratio of the electrostatic and electromagnetic field components in the mixed perturbation, may be obtained from the coupled equations [cf. 4.7-(2) and 4.8-(2)] valid for low pump power; the result is shown in Eq. 4.9-(2).

Linearize the dispersion functions lying on the leading diagonal about the wavevectors and frequencies

$$\left[\vec{k}_{(o)}, \; w_{(o)} \right] \equiv \left(\vec{k}_{EP}, \; w_{EP} \right) \qquad\qquad 4.9\text{-}(3)$$

$$\left[\vec{k}_{(-1)}, \; w_{(-1)} \right] \equiv \left(\vec{0}, \; - w_{EP'} \right) \qquad\qquad 4.9\text{-}(4)$$

$$\equiv \left(\vec{0}, \; - w_{EM'} \right) \qquad\qquad 4.9\text{-}(5)$$

respectively. Note that the group velocities at (4) and (5) are zero. Further approximate by setting

$$\left| w_{(-1)} \right| \;\overset{\sim}{=}\; w_{(o)} \;\overset{\sim}{=}\; w_p \quad ; \qquad\qquad 4.9\text{-}(6)$$

then 4.9-(2) simplifies to 4.9-(7).

The dispersion relation is obtained by taking the determinant of the matrix and is

$$
\begin{pmatrix}
1 - \dfrac{w_p^2}{w_{(0)}^2 - 3k_{(0)}^2 V_{Te}^2} & \dfrac{k_{(0)} V_1}{|w_{(-1)}|}\dfrac{w_p^2[\vec{e}_{x1}\cdot\vec{e}_{M(-1)}]}{[w_{(0)}^2 - 3k_{(0)}^2 V_{Te}^2]} & \dfrac{k_{(0)} V_1 |w_{(-1)}|}{w_p^2}[\vec{e}_{x1}\cdot\vec{e}_{s(-1)}] \\[4ex]
\dfrac{k_{(0)} V_1^*}{|w_{(-1)}|}\dfrac{w_p^2[\vec{e}_{x1}\cdot\vec{e}_{M(-1)}]}{[w_{(0)}^2 - 3k_{(0)}^2 V_{Te}^2]} & 1 - \dfrac{w_p^2 + k_{(-1)}^2 c^2}{w_{(-1)}^2} & 0 \\[4ex]
\dfrac{k_{(0)} V_1^* |w_{(-1)}|}{w_p^2}[\vec{e}_{x1}\cdot\vec{e}_{s(-1)}] & 0 & 1 - \dfrac{w_p^2}{w_{(-1)}^2 - 3k_{(-1)}^2 V_{Te}^2}
\end{pmatrix}
\begin{pmatrix}
E_{(0)s} \\[4ex] E_{(-1)M} \\[4ex] E_{(-1)s}
\end{pmatrix}
=
\begin{pmatrix}
0 \\[4ex] 0 \\[4ex] 0
\end{pmatrix}
$$

4.9-(2)

$$
\begin{pmatrix}
\dfrac{2\left[\Delta w - \Delta \vec{k}\cdot \vec{e}_{z(0)}\, V_g^{EP}\right]}{w_p} & \dfrac{k_{EP}V_1}{w_p}\left[\vec{e}_{x1}\cdot \vec{e}_{M(-1)}\right] & \dfrac{k_{EP}V_1}{w_p}\left[\vec{e}_{x1}\cdot \vec{e}_{s(-1)}\right] \\[3em]
\dfrac{k_{EP}V_1^{*}}{w_p}\left[\vec{e}_{x1}\cdot \vec{e}_{M(-1)}\right] & \dfrac{2\,\Delta w}{-w_p} & 0 \\[3em]
\dfrac{k_{EP}V_1^{*}}{w_p}\left[\vec{e}_{x1}\cdot \vec{e}_{s(-1)}\right] & 0 & \dfrac{2\Delta w}{-w_p}
\end{pmatrix}
\begin{pmatrix}
E_{(0)s} \\[3em]
E_{(-1)M} \\[3em]
E_{(-1)s}
\end{pmatrix}
=
\begin{pmatrix}
0 \\[3em]
0 \\[3em]
0
\end{pmatrix}
$$

4.9-(7)

> ### Three-dimensional Coalesced Raman Two-Plasmon Instability
>
> $$\Delta w \left(\Delta w - \Delta k_z V_g^{EP} \right) = - \frac{k_{EP}^2 V_{Laser}^2}{16} \qquad .$$
>
> <div align="right">4.9-(8)</div>

This assumes a very simple form for two reasons; the plasma density was chosen so that an exact coalescence between the Raman and two-plasmon instabilities occurred even at vanishingly small pump powers (see Figs. 1 and 2); and terms in Δk^2 were omitted from the expansion of the on-diagonal dispersion functions in (7). The maximum growthrate for real $\Delta \vec{k}$ is obtained from (8) by setting

$$\Delta k_z = 0 \qquad\qquad 4.9-(9)$$

whereupon the growthrate becomes

$$\Delta w_i = k_{EP} V_{Laser}/4 \qquad . \qquad 4.9-(10)$$

The time-asymptotic pulse response calculated from (8) is essentially one-dimensional and has the form

Fig. 4.9.3   TIME-ASYMPTOTIC PULSE RESPONSE
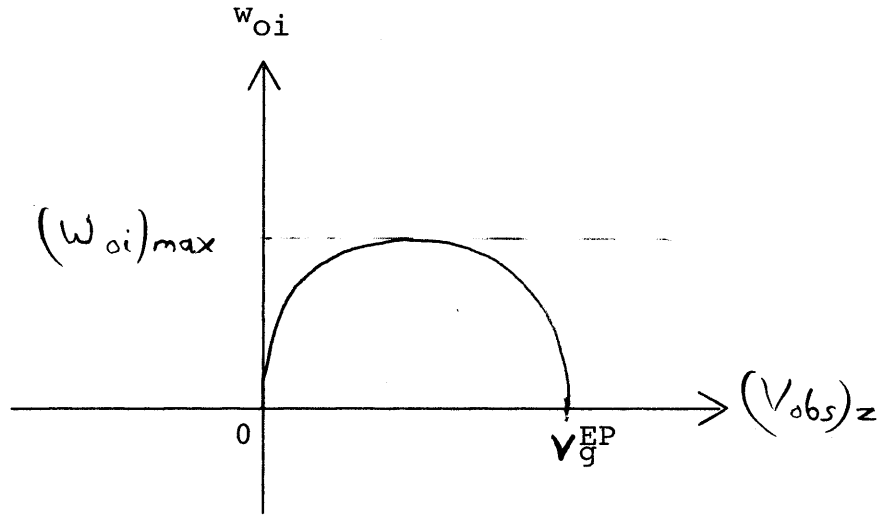
Here, w_{oi} is the observed growthrate, $\vec{V}_{obs}$ is the observer velocity. The maximum observed growthrate $(w_{oi})_{Max}$ for any observer velocity $\vec{V}_{obs}$ is the same as the maximum growthrate $(\Delta w_i)_{Max}$ derived from the dispersion relation (8) for any real $\Delta \vec{k}$, and so

$$(w_{oi})_{Max} = (\Delta w_i)_{Max} = k_{EP} V_{Laser}/4 \quad . \qquad 4.9\text{-}(11)$$

The ratio of electrostatic field component to electromagnetic field component may be recovered for a given $\Delta \vec{k}$ by substitution in the coupling equation (7).

The growing instability has the character of Raman sidescatter for $\Delta \vec{k}$ in the y-direction, two-plasmon absorption for $\Delta \vec{k}$ in the x-direction and a mixed instability for $\Delta \vec{k}$ lying in an arbitrary direction in the x-y plane. For moderately

high pump powers the terms in Δk^2 should be retained at least in the electromagnetic dispersion function, whereupon the dispersion relation for the coalesced instability becomes

$$(\Delta w - \Delta k_z v_g^{EP}) = -\frac{k_{EP}^2 V_{Laser}^2}{16}\left\{\frac{[\vec{e}_{x1} \cdot \vec{e}_{M(-1)}]^2}{\Delta w + (\Delta k^2 c^2/2w_p)} + \frac{[\vec{e}_{x1} \cdot \vec{e}_{s(-1)}]^2}{\Delta w}\right\}$$

4.9-(11)

One may also investigate the coalesced Raman-two-plasmon instability for plasma densities such that the tangency of Raman and two-plasmon loci illustrated in Fig. 2 becomes merely a close approach. Then, for the appropriate dispersion relation, one must return to (2). At sufficiently low pump intensities, one finds separate Raman and two-plasmon instabilities, and the coalescence in this case occurs at some finite value of the intensity of the laser pump.

CHAPTER 5

RESULTS AND CONCLUSIONS

5.1. Specific Accomplishments

In the main body of this thesis, namely Chapter 2, 3,
and 4, we proceeded from the general to the specific. A
generalized-coupling-of-modes theory was first developed to
deal with coherent wave-wave interaction in a nonlinear me-
dium. Then the case of a pump-wave with vanishingly small
depletion and attenuation was considered. The medium was
specialized to a warm-fluid magnetized plasma with drifts.
Finally the pump-wave was specialized to a laser-beam, and
the steady magnetic field and the drift-velocities were set
to zero. Thus the well-known laser-driven instabilities were
recovered, with the advantages over certain other treatments
that their 3-dimensional behavior was automatically included
and that successively higher corrections to their dispersion
relations and Green functions could be ordered and calculated
systematically. We proceed to describe these accomplishments
in more detail.

In section 2.4 the usual coupling-of-modes equations were
first recovered, starting from a general homogeneous medium
as defined by its conductivity. This constitutive relation
between electric current and electric field was allowed to be
non-local, thus giving rise to dispersion, and weakly nonlinear,
thus giving rise to mode-mode coupling and slow variation in
mode amplitudes. The procedure was repeated in section 2.5 to
obtain the generalized-coupling-of-modes equations. The non-

linear conductivity was no longer restricted to be 2nd-order
in the electric field, and the slow variation was no longer
restricted to being described by its first-order spacetime
derivatives. This generalized-coupling-of-modes theory forms
the basis for the remainder of the thesis.

The behavior of small-amplitude waves in the presence of
a much stronger pump-wave was then formulated in terms of the
theory. The equations 2.6-(8,9) describe the spatial attenu-
ation and temporal depletion of the first and second harmonic
of a pump-wave due to growing perturbations. The manner in
which these two equations were derived by substitution in
the more general equation 2.5-(7) is straightforward, and
allows one to extend the equation 2.6-(8,9) to higher-order
terms than those written down, and to extend the series of
two equations to describe attenuation and depletion of third
and higher pump-wave harmonics if necessary. Of course any
changes in the polarization of the pump brought about by the
interaction are also comprised in 2.6-(8,9). In cases where
the effect of the growing perturbations on the pump is negli-
gible, as for example in the early stages of their growth out
of thermal noise, those generalized-coupling-of-modes equations

which describe the behavior of the perturbations themselves

acquire a simple form. In these cases the complexity of the

problem is confined to the calculation of the coupling coeffi-

cients.

The actual forms of the coupling coefficients which

describe coherent wave-wave interactions were computed in

Section 2.8 for the warm-fluid plasma model. These forms are

symmetric in all the interacting waves. This model symmetry

did not follow immediately from the method of derivation, and

indeed a certain amount of algebraic manipulation, involving repeated use of vector identities and wavevector-frequency sum-rules, was necessary to demonstrate the total symmetry.

The computation of coupling coefficients in unmagnetized Vlasov plasma, and of coupling coefficients for electrostatic waves propagating parallel to the magnetic field in magnetized Vlasov plasma, present no difficulty. The plasma is regarded as a continuous superposition of cold beams, and the expressions 2.8-(43,52) are summed over these beams. The sums take the form of integrals over velocity space, which are evaluated according to the Landau prescription.

Starting from the generalized-coupling-of modes theory of Section 2.5, Chapter 2 thus furnishes a computational framework sufficient to deal with linear perturbations about an undepleted, unattenuated pump-wave in a plasma described by the warm-fluid model.

This computational framework is capable of generating dispersion relations for instabilities in a magnetized warm-fluid plasma, containing any number of beams and permeated by any number of undepleted unattenuated pump-waves. This framework is considered to have been proved out by being used, in Chapter 3 and 4, to derive dispersion relations and stability analyses for laser-driven instabilities in unmagnetized drift-free plasma. The usefulness of the framework is considered proved by the fact that the three-dimensional nature of the polarization and propagation of the interacting waves is auto-

matically included in the computations, and also by the fact

that fresh instabilities and modified or coalescent versions

of well-known instabilities can be systematically investigated.

In Chapter 3 the self-consistent harmonic structure of

the pump-wave equilibrium was investigated, and then the

behavior of linear perturbations about this equilibrium was

formulated in terms of coupling brought about by the pump-

wave. This was done for the special case of a pump-wave con-

sisting of a uniform laser beam, and a nonlinear medium consis-

ting of an unmagnetized plasma, each species in the plasma

characterized by a single temperature and a zero drift velocity.

The nonlinear coupling between perturbing waves was computed

from the nonlinear conductivity of the electrons only. The

self-consistent harmonics of the pump-wave were determined to

an order consistent with the order of the couplings which

they bring about. It was shown that for the laser-driven

instabilities in unmagnetized plasma, the effects of the medi-

um nonlinearity on the pump-wave itself can be neglected when

computing the couplings leading to those instabilities.

In Chapter 4 the well-known laser-driven instabilities were

recovered. The physical mechanisms driving them were discussed

and their 3-dimensional dispersion relations were derived.

These 3-dimensional dispersion relations were not derived

from the discussions of physical mechanisms, but from the gen-

eral work on laser-coupled perturbations carried out in Chapter

4. The discussions of physical mechanisms were limited to 1 or

2 dimensions; their purpose is not to derive dispersion rela-
tions but rather to illustrate the consequences of the growth
of the instabilities and to suggest saturation mechanisms.

Stability analyses were performed in order to obtain
time-asymptotic Green's functions. The three-dimensional
time-asymptotic pulse-response to an initially localized ex-
citation of a specific instability is determined once the
three-dimensional dispersion relation for that instability is
given. For unmodified instabilities involving only the laser
pump and two decay products, with the dispersion functions
for the decay products approximated by first-order Taylor
expansions in wavevector and frequency, the pulse-response
problem was solved by Bers and Chambers.[1] For modified in-
stabilities and instabilities involving three decay products,
the calculation of three-dimensional pulse-responses becomes
considerably more complex. However, good physical insight can
be gained by calculating one-dimensional pulse-responses,
which are cross-sections of the three-dimensional pulse-
shapes, along those directions which can be seen by inspection
of the dispersion relations to be directions of greatest
growth.

For those instabilities having as one of their decay pro-
ducts a low-frequency disturbance in the ion-acoustic regime,
the following statement holds: the curve of increasing in-
stability-growthrate versus increasing laser-intensity tends
to flatten out at growthrates of a few times the "ion-acoustic

frequency". This statement is deliberately left qualitative
and contains quotation marks, for the following reason. The
highest growthrate of the instability occurs at a value of
the wavevector which itself is a function of the laser inten-
sity. The "ion-acoustic frequency", which appears in formulae
purporting to describe the asymptotic dependence of growth-
rate on laser intensity, must therefore itself depend on
that intensity, in a way which can not be known exactly with-
out solving the dispersion relation.

5.2 Relevance to Laser-Pellet Interactions

The growthrate calculations of Chapter 4 lead to the con-
clusions that at realistic power intensities using realistic
laser wavelengths - 10^{15} watts/cm^2 at around 1 micron free-
space wavelength, say - all laser-driven instabilities have
growthtimes of the same order, namely between $\frac{1}{2}$ and 5 pico-
seconds. Thus all these laser-driven instabilities will grow
contemporaneously in the laser-irradiated plasma. The state
of the plasma when nonlinear saturation sets in will be de-
termined by the competitive evolution of all the instabilities
up to that time. Since these instabilities have their maxi-
mum growthrates in different spatial directions, an appropriate
theory of the instabilities must not only derive them system-
atically but furnish their 3-dimensional behavior. The theory
of laser-driven instabilities described in this thesis does
both these things.

The above convergence of growthrates is brought about be-
cause at high pump power-intensities the laser-driven instabil-
ities assume their modified forms. These modified forms re-
quire 3rd-order plasma conductivity for their correct descrip-
tion. The effect of the modification is to reduce growthrates,
and the effect is stronger the higher the unmodified growth-
rates. Thus the above convergence is effected.

One aspect of the instabilities which was not described
in Chapter 4 is the relation between the damping rates of
uncoupled modes and the thresholds of the instabilities that
occur when the same modes are coupled together by the pump.
Damping is easily incorporated into the instability dispersion
relations. The dispersion functions for uncoupled modes,
which appear in the dispersion relation describing a specific
instability, are adjusted slightly. Instead of having zeros
at real wavevectors and real frequencies separated by the
pump wavevector and frequency, the dispersion functions are
chosen so that each has a zero at the real wavevector and at
the complex frequency with negative imaginary part equal to
the damping rate of the corresponding uncoupled mode. This
can be done in two main ways. One may insert a more exact
form of the uncoupled-mode dispersion function; for example,
one may replace the warm-fluid dispersion-functions used in
Chapter 4 by the more exact Vlasov dispersion-functions which
incorporate Landau damping. Alternatively one may insert a

phenomenological damping rate Γ simply by replacing each un-coupled dispersion function D ($\vec{k}$,w) by an amended dispersion function:

$$D^{damped}(\vec{k}, w) \equiv \vec{D}(\vec{k}, w + i\Gamma)$$

5.2-(1)

One finds that at the high laser power intensities previously mentioned the effect of damping on the maximum growthrates of laser-driven instabilities is relatively slight. The finite threshold intensities due to damping are orders of magnitude below the laser intensities actually used. However the presence of damping has important effects on the spatial evolution of instabilities. The wavenumbers, and hence Landau damping rates, of perturbations satisfying the sum rules are dependent on the plasma density and on the directions of the wavevectors considered. Thus the presence of damping can confine a specific instability to a narrower range of plasma densities and propagation directions, while leaving its maximum growthrate almost unaffected. This effect is discussed in detail by Bers and Chambers.[2]

As an illustration of the use of phenomenological damping take the case of the unmodified Brillouin instability considered in section 4.2. The 3-dimensional dispersion relation for this instability is set out in equation 4.2-(7). The left-hand-

side of this equation is the product of the two dispersion

functions for the uncoupled perturbations, namely the electro-

magnetic dispersion function and the ion-acoustic dispersion

function. Inserting phenomenological damping rates Γ_{EM} and

Γ_{IA} respectively, one obtains the dispersion relation for

unmodified Brillouin instability caused by laser-induced

coupling between lightly-damped modes:

$$\left(\Delta\omega + i\Gamma_{IA} - \left| V_g^{IA} \right| \vec{e}_{s(o)} \cdot \overrightarrow{\Delta k} \right)$$

$$\left(\Delta\omega + i\Gamma_{EM} + \left| V_g^{EM} \right| \vec{e}_{s(-1)} \cdot \overrightarrow{\Delta k} \right)$$

$$= -\frac{V_{LASER}^2}{16\, V_{Te}^2} \frac{\omega_p^2 \omega_{IA}^2}{\omega_{EM}} \left(\vec{e}_{x1} \cdot \vec{e}_{s(-1)} \right)^2$$

5.2-(2)

This has the form of a three-dimensional two-coupled-mode

instability for which the stability analysis has been per-

formed analytically[1] and numerically[2] by Bers and

Chambers.[1] The pulse-response cross-section in the direction

of maximum growth was shown in Fig. 4.2-(4) for laser-induced

coupling between undamped modes. For coupling between damped

modes, the pulse-response curve is lowered as follows.[2] For

an observer travelling at the group velocity V_g^{IA} of the ion-

acoustic wave, the observed growthrate is reduced by the ion-

acoustic damping rate Γ_{IA}. For an observer travelling at the

group velocity $-V_g^{EM}$ of the back-scattered electromagnetic

wave, the observed growthrate of the instability is reduced

TIME-ASYMPTOTIC PULSE-RESPONSE FOR UNMODIFIED

BRILLOUIN INSTABILITY WITH DAMPED DECAY PRODUCTS
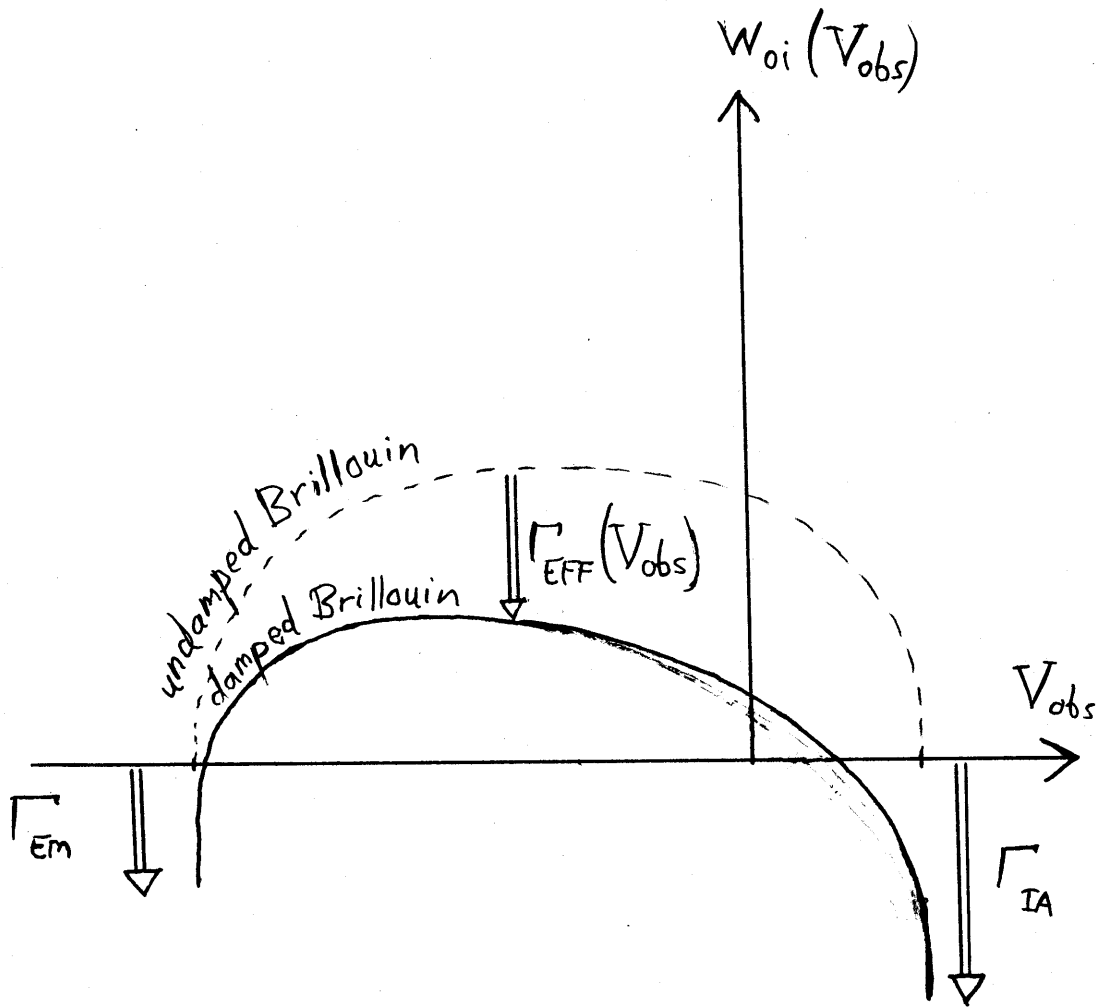
Fig. 5.2.1

by the electromagnetic damping rate Γ_{EM}. For an observer travelling at an intermediate velocity V_{obs}, the observed growthrate of the instability is reduced by an effective damping rate Γ_{EFF} which is a linear interpolation between Γ_{IA} and Γ_{EM}:

$$\Gamma_{EFF}\left(V_{obs}\right) = \frac{\left(V_g^{IA} - V_{obs}\right)\Gamma_{EM} + \left(V_{obs} + V_g^{EM}\right)\Gamma_{IA}}{V_g^{IA} + V_g^{EM}}$$

5.2-(3)

Note that the edges of the undamped pulse thus acquire negative growthrates. The range of observer velocities for which the observed growthrate is positive is narrowed down and thus damping helps to localize the instability spatially as mentioned earlier in this section.

For modified and 3rd-order instabilities, the effect of damping on the time-asymptotic pulse-response can not be described by such a simple prescription. However the thresholds for all these laser-driven instabilities can be easily calculated from phenomenological damping rates and are as follows: Brillouin Threshold:

$$\frac{V_{LASER}^2}{V_{Te}^2} = 16 \frac{\Gamma_{EM}\Gamma_{IA}}{W_{EM}W_{IA}} \frac{W_{EM}^2}{W_{PE}^2}$$

5.2-(4)

Raman Threshold:

$$\frac{V_{LASER}^2}{c^2} = 16 \frac{\Gamma \, \Gamma_{EM}}{W_{PE}^1 \, W_{EM}} \frac{k_{EM}^2}{k_{EP}^2}$$

5.2-(5)

Plasmon-Phonon Threshold:

$$\frac{V_{LASER}^2}{V_{Te}^2} = 16 \frac{\Gamma_{EP} \, \Gamma_{IA}}{W_{PE} \, W_{IA}} \frac{W_{LASER}^2}{W_{PE}^2}$$

5.2-(6)

2-Plasmon Threshold:

$$\frac{V_{LASER}^2}{V_{Te}^2} = 8 \frac{\Gamma_{EP}^2}{k_{EP}^2 \, V_{Te}^2}$$

5.2-(7)

Filamentation Threshold:

$$\frac{V_{LASER}^2}{V_{Te}^2} = 8 \frac{\Gamma_{EM}}{W_{LASER}} \frac{W_{LASER}^2}{W_{PE}^2}$$

5.2-(8)

Oscillating-Two-Stream Threshold:

$$\frac{V_{LASER}^2}{V_{Te}^2} \;=\; \gamma \; \frac{\Gamma_{Ep}}{W_{pe}} \; \frac{W_{LASER}^2}{W_{pe}^2}$$

5.2-(9)

No separate thresholds are quoted for modified instabilities
since modification only occurs at high growthrates.

The calculations of Chapter 4 are confined to the linear
behavior of perturbations about the laser pump-equilibrium.
However the physical mechanisms of each instability serve as a
guide to the probable method of nonlinear saturation. An
instability having an electron plasma wave as one decay product
can saturate by the wave growing to an amplitude comparable with
the pump-wave amplitude, or by that wave growing to an ampli-
tude sufficient to trap electrons lying within the main body of
the distribution function. An instability having a plasma
density modulation as one decay product can saturate by having
that modulation grow to a gross macroscopic alteration in plas-
ma density, which destroys the frequency relations necessary
for the instability to grow. Which of these saturation mech-
anisms takes effect first depends on the intensity of the laser
pump and on the fixed ratio between the various exponentially
growing electric field perturbations in the linear phase of
the instability. The latter can be determined in a straight-

forward manner by back-substitution in the coupling matrix
from which the dispersion relation is derived.

5.3 Extension to other problems

The calculations that were actually carried through to
completion in Chapter 3 and 4 concerned a very restricted
case of a pump-permeated plasma. However the theoretical
framework of Chapter 2 can accomodate a much wider variety
of cases. These may include magnetic fields, many pump-
waves and/or particle-beams, attenuated and depleted pumps,
species described by the Vlasov model, and, with only slight
modification, weakly inhomogeneous plasma. This variety is
required for investigating realistic heating schemes both
in laser-pellet and in magnetic-confinement experiments.

We recall that in the course of Chapter 2 and the first
three sections of Chapter 3, the generalized-coupling-of-modes
theory suffered successive specializations. In a sense, each
specialization furnished a test case for the more general
formulation preceding it. In this section, we reverse the
process and go back from the specific to the general. At each
stage, we ask the question: if we remove a particular res-
triction, how do we replicate the calculations which followed
after it, in such a way as to still obtain specific results,
and what is accomplished thereby? This question will turn out

to be a fruitful one. The theoretical framework set out in
Chapter 2 is broad enough that definite prescriptions can be
given for finding the behavior of, say, parametric instabili-
ties in strongly-magnetized plasma driven by an attenuated
electrostatic pump-wave. This is so even though there is
neither time nor space to perform the relevant calculations
in this thesis. The significance of the theoretical frame-
work will thus be brought out physically by exploring and
assessing its adequacy for progressively wider classes of
physical problems. This will be taken as far as the case of
weakly inhomogeneous plasma; the framework of Chapter 2 was
set out in terms of a homogeneous medium, but the modification
of this framework to accomodate inhomogeneity is easy, at least
so long as the coupling terms can be described by a local
approximation. Finally, a possible strategy for investigating
arbitrarily-high-order 3-dimensional couplings in magnetized
plasma, based on a conservation theorem approach, will be
outlined.

The computations in the fourth and later sections of
Chapter 3 and in Chapter 4 were performed for an unmagnetized
drift-free plasma permeated by a single electromagnetic pump-
wave. Without relaxing the restrictions of Sections 2-6,7,8
to undepleted pumps and warm-fluid plasma, the computations in
Chapter 3 could have been performed in an exactly analogous
fashion for the case of magnetized plasma, with each particle

species containing several beams or several temperature components. Several pump-waves could have been employed some possibly electrostatic, some possibly with large self-consistent harmonic components. In a sense the computations actually performed in Chapter 4 were a test run, to confirm the validity of the theoretical formulation and to gain experience in translating that formulation into actual dispersion relations. The success of this test run implies that we have at our disposal a computational framework for any problem involving fluid modes and undepleted pumps. This framework will be used for problems in self-magnetized laser-irradiated pellets, beam-heated target plasmas, and RF heating in Tokamaks.

As a next step, one considers how that computational framework - essentially comprising the coupled equations 2.7-(6) for linear perturbations about an undepleted pump, together with the expressions 2.8-(43,52) for warm-fluid coupling coefficients - can be broadened. Chapter 2 certainly provides the prescription for computing the depletion and attenuation of pump-waves due to growing instabilities. Chapter 2 certainly also provides the prescription for computing the effect of the resulting temporal and spatial evolution of the pump on the coupling coefficients themselves. One may further consider weakly inhomogeneous plasmas by relatively slight modifications of the basic arguments in Sections 2.4

and 2.5, and finally one may investigate the form taken by coupling coefficients in Vlasov plasma.

The spatial attenuation and temporal depletion of a pump-wave and its attendant harmonics due to the unstable growth of pump-coupled perturbations are described by equations of type 2.6-(8,9). Having equations describing the space-time behavior of the pump amplitude, one may then solve the system of equations consisting of these conjoined with equations of type 2.6-(10) which fix the local instantaneous growthrate of the unstable perturbations. This solves the "attenuated and depleted pump" problem. However, to do this is to neglect the effects of the finite rate of spatial and temporal variation in the pump amplitude on the coupling between the perturbations. The effects of this temporal evolution and spatial gradient of the pump amplitude can be found by adding to 2.6-(10) the corresponding terms in the pump-amplitude spacetime derivatives. These extra terms are prescribed by the form of the generalized-coupling-of-modes equation 2.5-(7). This solves the "depleted and evolving pump" problem. These approaches will be used in considering the following: time-tailored laser pulses impinging on target pellets; strongly-absorbed laser-beam pumps in pellet plasmas; RF pumps in Tokamaks which have a resonance-cone structure; and time-tailored RF heating in Tokamaks.

The simple-coupling-of-modes equations were set up in Section 2.5 starting from a medium described by a non-local conductivity, and then taking the medium to be homogeneous so that the conductivity depended only on the space-time vector joining the site of the electric field to the site of the resulting electric current. The assumption of homogeneity can be relaxed to that of weak inhomogeneity (see Appendix A7) whereupon the coupling-of-modes equation 2.4-(11) is replaced by the following equation:

$$
\vec{\overline{L}}\left(\vec{k}_a, w_a, \vec{x}, t\right)\vec{E}_a(\vec{x}, t) - i\,\frac{\partial\vec{\overline{L}}}{\partial\vec{k}}\,\frac{\partial\vec{E}_a}{\partial\vec{x}} + i\,\frac{\partial\vec{\overline{L}}}{\partial w}\,\frac{\partial\vec{E}_a}{\partial t}
$$

$$
- \frac{i}{2}\left(\left(\frac{\partial}{\partial\vec{k}}\cdot\frac{\partial}{\partial\vec{x}} + \frac{\partial}{\partial w}\frac{\partial}{\partial t}\right)\vec{\overline{L}}\right)\vec{E}_a
$$

$$
+ \frac{1}{2!}\left(\left(\frac{\partial\vec{k}_a}{\partial\vec{x}}\cdot\frac{\partial^2}{\partial\vec{k}\partial\vec{k}} + 2\,\frac{\partial\vec{k}_a}{\partial t}\cdot\frac{\partial^2}{\partial\vec{k}\partial w} + \frac{\partial w_a}{\partial t}\frac{\partial^2}{\partial w^2}\right)\vec{\overline{L}}\right)\vec{E}_a =
$$

$$
\sum_{\substack{\text{those pairs }(b,c)\\ \text{which drive }\vec{E}_a}} exp\left[i\int_{(\vec{0},0)}^{(\vec{x},t)}\left(\left(-\vec{k}_a+\vec{k}_b+\vec{k}_c\right)\cdot d\vec{x} - \left(-w_a+w_b+w_c\right)dt\right)\right]
$$

$$
\left(-i\mu_0\right)\left(w_b+w_c\right)\vec{\overline{G}}^{NL(2)}\left(\vec{k}_b, w_b, \vec{k}_c, w_c, \vec{x}, t\right)\vec{E}_b(\vec{x}, t)\vec{E}_c(\vec{x}, t)
$$

Here the local wavevector and frequency

$$\vec{k}_a(\vec{x}, t) \quad , \quad w_a(\vec{x}, t)$$

<div align="right">5.4-(2)</div>

satisfy the phase-uniqueness conditions

$$\nabla \times \vec{k}_a = \vec{0}$$

$$\nabla w_a = \dot{\vec{k}}_a$$

<div align="right">5.4-(3)</div>

In order for the equation (1) to be applicable, the exponential phase-factor on the right-hand-side must be a slowly-varying function of space and time; thus (1) can only be used within some region where

$$\vec{k}_a(\vec{x}, t) \cong \vec{k}_b(\vec{x}, t) + \vec{k}_c(\vec{x}, t)$$

$$w_a(\vec{x}, t) \cong w_b(\vec{x}, t) + w_c(\vec{x}, t)$$

<div align="right">5.4-(4)</div>

for the modes considered. The inhomogeneity of the medium appears in (1) via terms on the left-hand-side which are of the order of the inverse scalelength and scaletime of the inhomogeneity, and via the exponential phase-factor which multiplies the nonlinear term on the right-hand-side. For the usual laser-pellet parameters, the nonlinear terms describing laser-driven instabilities are strong enough so that

the effect of the exponential phase-factor is more important
than the effect of the linear inhomogeneity terms on the left
of (1), as shown by Chambers and Bers.[2]   The coupled equations
of type (1), with the linear inhomogeneity terms omitted, have
been used by Rosenbluth[3] and several others to study blowoff
plasmas. We hope to extend this work to pellets with non-
uniform illumination resulting in non-uniform surface tempera-
ture, and also to RF heating in magnetic confinement devices,
taking into account the effect of plasma inhomogeneity on the
parametric coupling.

Finally, the symmetries of the expressions for warm-
fluid coupling-coefficients computed in Section 2.8 suggest -
though they do not guarantee the success of - an alternative
method for computing such coupling-coefficients which should
be especially valuable for the case of Vlasov plasma. This
method will now be described in more detail.

We hope to rederive the coupling coefficients in a
manner which is a priori symmetric in all the interacting waves,
and which is algebraically less involved. We hope to do this
by finding a quantity, analogous to energy density, which
satisfies a conservation theorem, and expanding this quantity
in terms of physical variables pertaining to the interacting
waves. The second-order part of this quantity will regenerate
the linear dispersion tensor. The third-order part of the
conserved quantity will furnish 3-wave coupling-coefficients,
the fourth-order part 4-wave coupling-coefficients, and so on.

If the above scheme proves successful, it will be tried on the magnetized Vlasov plasma. Again expressions for the coupling coefficients have been derived by expanding the equations of motion. These coupling coefficients have been limited to 3-wave interaction, and further limited to certain directions of propagation.[4] The proof of their symmetry is indirect. Again we would hope to generate coupling coefficients as coefficients in the expansion of some conserved quantity, thus obtaining them in a straightforward manner and in an a priori symmetric form. Such a simplified, systematic derivation of three-dimensional coupling-coefficients in magnetized Vlasov plasma is needed before a certain practical possibility can be investigated. This is the possibility of injecting microwave energy into Tokamaks and heating the plasma by coupling to ion-cyclotron modes, rather than to the warm-fluid modes previously considered by Bers and Karney.[5]

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APPENDICES

A.1   Derivation of the Simple-Coupling-of-Modes Equation

The conductivity of the medium is given by 2.4-(5);

$$\vec{J}(\vec{x}, t) = \int\int d\vec{\xi} \ d\tau \ \vec{G}^{LIN} (-\vec{\xi}, -\tau) \ \vec{E}(\vec{x} + \vec{\xi}, \ t + \tau)$$

$$+ \frac{1}{2!} \int\int\int\int d\vec{\xi}' \ d\tau' \ d\vec{\xi}'' \ d\tau'' \ \vec{G}^{NL(2)} (-\vec{\xi}', -\tau', -\vec{\xi}'', -\tau'')$$

$$\vec{E}(\vec{x} + \vec{\xi}', \ t + \tau') \ \vec{E}(\vec{x} + \vec{\xi}'', \ t + \tau'')$$

$$\text{A1-(1)}$$

As in 2.4-(6), take

$$\vec{E}(\vec{x}, t) = \sum_a \vec{E}_a(\vec{x}, t) \ e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \qquad\qquad \text{A1-(2)}$$

Treat the nonlinear conductivity as a small quantity.   The space-time derivatives of the $\{\vec{E}_a\}$ will be also treated as small quantities. Substitute (2) into (1).   Expand the $\{\vec{E}_a\}$ appearing in the linear current as 1st-order Taylor series about $(\vec{x}, t)$.   Approximate the $\{\vec{E}_a\}$ appearing in the nonlinear current by $\{\vec{E}_a(\vec{x}, t)\}$.   The result is

$$\vec{J}(\vec{x}, t) = \sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \boxed{\int\int d\vec{\xi} \ d\tau \ \vec{G}^{LIN} (-\vec{\xi}, -\tau) \ e^{i\vec{k}_a \cdot \vec{\xi} - iw_a \tau}}$$

$$\left(\vec{E}_a(\vec{x}, t) + \vec{\xi} \cdot \frac{\partial \vec{E}_a}{\partial \vec{x}} + \tau \frac{\partial \vec{E}_a}{\partial t} \right)$$

$$+ \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c) \cdot \vec{\xi} - i(w_b + w_c)t} \boxed{\int \int \int \int d\vec{\xi}' \, d\tau' \, d\vec{\xi}'' \, d\tau''}$$

$$\boxed{\vec{G}^{NL(2)} (-\vec{\xi}', -\tau', -\vec{\xi}'', -\tau'') e^{i\vec{k}_b \cdot \vec{\xi}' - iw_b\tau'} e^{i\vec{k}_c \cdot \vec{\xi}'' - iw_c\tau''}}$$

$$\vec{E}_b(\vec{x}, t) \, \vec{E}_c(\vec{x}, t)$$

A1-(3)

The subexpressions outlined by boxes are just the Fourier-transformed linear and nonlinear conductivities respectively. In terms of these,

$$\vec{J}(\vec{x}, t) = \sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \left(\vec{G}^{LIN} (\vec{k}_a, w_a) \vec{E}_a(\vec{x}, t) \right.$$

$$- \frac{i \partial \vec{G}^{LIN}}{\partial \vec{k}_a} \frac{\partial \vec{E}_a}{\partial \vec{x}} + \frac{i \partial \vec{G}^{LIN}}{\partial w_a} \cdot \frac{\partial \vec{E}_a}{\partial t}$$

$$+ \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c) \cdot \vec{x} - i(w_b + w_c)t} \vec{G}^{NL(2)} (\vec{k}_b, w_b, \vec{k}_c, w_c)$$

$$\vec{E}_b(\vec{x}, t) \, \vec{E}_c(\vec{x}, t)$$

A1-(4)

Differentiate (4) with respect to time, keeping 1st-order space-time derivatives of the $\{\vec{E}_a\}$ in the linear-current term and no space-time derivatives of the $\{\vec{E}_a\}$ in the nonlinear-current term. The result can be written

$$\mu_o \, \dot{\vec{J}}(\vec{x},t) = \sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \left[-i\mu_o w_a \, \vec{G}^{LIN}(\vec{k}_a, w_a) \, \vec{E}_a(\vec{x},t) \right.$$

$$- \frac{i\partial}{\partial \vec{k}_a} \left(-i\mu_o w_a \, \vec{G}^{LIN}(\vec{k}_a, w_a) \right) \frac{\partial \vec{E}_a}{\partial \vec{x}}$$

$$\left. + \frac{i\partial}{\partial w_a} \left(-i\mu_o w_a \, \vec{G}^{LIN}(\vec{k}_a, w_a) \right) \frac{\partial \vec{E}_a}{\partial t} \right]$$

$$+ \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c) \cdot \vec{x} - i(w_b + w_c)t} (-i\mu_o)(w_b + w_c)$$

$$\vec{G}^{NL(2)}(\vec{k}_b, w_b, \vec{k}_c, w_c) \, \vec{E}_b(\vec{x},t) \, \vec{E}_c(\vec{x},t)$$

$$\text{A1-(5)}$$

The remaining terms appearing in Maxwell's wave equation 2.4-(8) are

$$-\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) - \mu_o \epsilon_o \ddot{\vec{E}} \equiv \sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t}$$

$$\left[\left(\vec{k}_a - \frac{i\partial}{\partial \vec{x}} \right) \times \left(\left(\vec{k}_a - \frac{i\partial}{\partial \vec{x}} \right) \times \right) + \mu_o \epsilon_o \left(w_a + \frac{i\partial}{\partial t} \right)^2 \right] \vec{E}_a(\vec{x},t)$$

$$\text{A1-(6)}$$

To first order in space-time derivatives of the $\{\vec{E}_a\}$, these terms can be written

$$-\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) - \mu_o \epsilon_o \ddot{\vec{E}} = \sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t}$$

$$\left[\vec{k}_a \times (\vec{k}_a \times) + \mu_o \epsilon_o w_a^2 \right] \vec{E}_a(\vec{x},t)$$

$$+ \frac{i\partial}{\partial \vec{k}_a} \left[\vec{k}_a \times (\vec{k}_a \times) + \mu_o \epsilon_o w_a^2 \right] \frac{\partial \vec{E}_a}{\partial \vec{x}}$$

$$+ \frac{i\partial}{\partial w_a} \left[\vec{k}_a \times (\vec{k}_a \times) + \mu_o \epsilon_o w_a^2 \right] \frac{\partial \vec{E}_a}{\partial t}$$

A1-(7)

Combining (5) and (7), we find that Maxwell's wave equation 2.4-(8) has the appearance

$$
\sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \left\{ \overset{\leftrightarrow}{L}{}^{LIN} (\vec{k}_a, w_a) \vec{E}_a (\vec{x}, t) \right.
$$

$$
\left. - \frac{i\partial \overset{\leftrightarrow}{L}{}^{LIN}}{\partial \vec{k}_a} \frac{\partial \vec{E}_a}{\partial \vec{x}} + \frac{i\partial \overset{\leftrightarrow}{L}{}^{LIN}}{\partial w_a} \frac{\partial \vec{E}_a}{\partial t} \right\}
$$

$$
= \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c) \cdot \vec{x} - i(w_b + w_c)t} (-i\mu_o)(w_b + w_c)
$$

$$
\overset{\leftrightarrow}{G}{}^{NL(2)} (\vec{k}_b, w_b, \vec{k}_c, w_c) \vec{E}_b (\vec{x}, t) \vec{E}_c (\vec{x}, t)
$$

A1-(8)

where the operator $\overset{\leftrightarrow}{L}{}^{LIN}$ is the linear dispersion tensor defined by

$$
\overset{\leftrightarrow}{L}{}^{LIN} (\vec{k}, w) \equiv \vec{k} \times (\vec{k} \times) + \mu_o \epsilon_o w^2 + i\mu_o w \overset{\leftrightarrow}{G}{}^{LIN} (\vec{k}, w)
$$

A1-(9)

In (8), $\overset{\leftrightarrow}{L}{}^{LIN}$ is contracted with $\vec{E}_a$ and $\partial/\partial \vec{k}_a$ is contracted with $\partial/\partial \vec{x}$. Equations (8) and (9) yield 2.4-(9), 2.4-(10) respectively.

A.2 Derivation of the Generalized-Coupling-of-Modes Equation

The conductivity of the medium is now given by 2.5-(2);

$$\vec{J}(\vec{x}, t) = \int \int d\vec{\xi} \; d\tau \; \vec{\vec{G}}^{LIN} (-\vec{\xi}, -\tau) \; \vec{E}(\vec{x} + \vec{\xi}, \; t + \tau)$$

$$+ \frac{1}{2!} \int \int \int \int d\vec{\xi}' \; d\tau' \; d\vec{\xi}'' \; d\tau'' \; \vec{\vec{G}}^{NL(2)} (-\vec{\xi}', -\tau', -\vec{\xi}'', -\tau'')$$

$$\vec{E}(\vec{x} + \vec{\xi}', \; t + \tau') \; \vec{E}(\vec{x} + \vec{\xi}'', \; t + \tau'')$$

$$+ \frac{1}{3!} \int \int \int \int \int \int d\vec{\xi}' \; d\tau' \; d\vec{\xi}'' \; d\tau'' \; d\vec{\xi}''' \; d\tau'''$$

$$\vec{\vec{\vec{G}}}^{NL(3)} (-\vec{\xi}, -\tau', -\vec{\xi}'', -\tau'', -\vec{\xi}''', -\tau''')$$

$$\vec{E}(\vec{x} + \vec{\xi}', \; t + \tau') \; \vec{E}(\vec{x} + \vec{\xi}'', \; t + \tau) \; \vec{E}(\vec{x} + \vec{\xi}''', \; t + \tau''')$$

$$+ \; \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{A2-(1)}$$

Again take as in 2.5-(4)

$$\vec{E}(\vec{x}, t) = \sum_a \vec{E}_a (\vec{x}, t) \; e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \qquad\qquad \text{A2-(2)}$$

Again the nonlinear conductivities and the space-time derivatives of the $\{E_a\}$ will be treated as small quantities in some sense. These small quantities will be used as expansion parameters. Unlike Appendix A1, this appendix retains all orders in these small quantities, at least form- ally. The user of the final generalized-coupling-of-modes equation should retain such orders of nonlinear conductivity and such orders of

space-time derivatives of the $\{\vec{E}_a\}$ as he judges to be significant for his physical problem. Substitute (2) into (1). Expand the $\{\vec{E}_a\}$ appearing in the integrands as Taylor series about $(\vec{x}, t)$. The result is

$$\vec{J}(\vec{x}, t) = \sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \boxed{\int\!\!\int d\vec{\xi}\, d\tau\ \vec{G}^{LIN}(-\vec{\xi}, -\tau)}$$

$$\boxed{e^{i\vec{k}_a \cdot \vec{\xi} - iw_a \tau}} \exp\left(\vec{\xi} \cdot \frac{\partial}{\partial \vec{x}} + \tau \frac{\partial}{\partial t}\right) \vec{E}_a(\vec{x}, t)$$

$$+ \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c)\cdot\vec{x} - i(w_b + w_c)t} \boxed{\int\!\!\int d\vec{\xi}'\, d\tau'\, d\vec{\xi}''\, d\tau''}$$

$$\boxed{\vec{G}^{NL(2)}(-\vec{\xi}', -\tau', -\vec{\xi}'', -\tau'')\, e^{i\vec{k}_b \cdot \vec{\xi}' - iw_b \tau'}\, e^{i\vec{k}_c \cdot \vec{\xi}'' - iw_c \tau''}}$$

$$\exp\left(\vec{\xi}' \cdot \frac{\partial}{\partial \vec{x}_b} + \tau' \frac{\partial}{\partial t_b} + \vec{\xi}'' \cdot \frac{\partial}{\partial \vec{x}_c} + \tau'' \frac{\partial}{\partial t_c}\right) \vec{E}_b(\vec{x}, t)\, \vec{E}_c(\vec{x}, t)$$

$$+ \frac{1}{3!} \sum_b \sum_c \sum_d e^{i(\vec{k}_b + \vec{k}_c + \vec{k}_d)\cdot\vec{x} - i(w_b + w_c + w_d)t}$$

$$\boxed{\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int d\vec{\xi}'\, d\tau'\, d\vec{\xi}''\, d\tau''\, d\vec{\xi}'''\, d\tau'''\ \vec{G}^{NL(3)}(-\vec{\xi}', -\tau', -\vec{\xi}'', -\tau'', -\vec{\xi}''', -\tau''')}$$

$$\boxed{e^{i\vec{k}_b \cdot \vec{\xi}' - iw_b \tau'}\, e^{i\vec{k}_c \cdot \vec{\xi}'' - iw_c \tau''}\, e^{i\vec{k}_d \cdot \vec{\xi}''' - iw_d \tau'''}}$$

$$\exp\left(\vec{\xi}' \cdot \frac{\partial}{\partial \vec{x}_b} + \tau' \frac{\partial}{\partial t_b} + \vec{\xi}'' \cdot \frac{\partial}{\partial \vec{x}_c} + \tau'' \frac{\partial}{\partial t_c} + \vec{\xi}''' \cdot \frac{\partial}{\partial \vec{x}_d} + \tau''' \frac{\partial}{\partial t_d}\right)$$

$$\vec{E}_b(\vec{x}, t)\, \vec{E}_c(\vec{x}, t)\, \vec{E}_d(\vec{x}, t)$$

$$+\ \cdots\cdots \tag{A2-(3)}$$

Here $\partial/\partial\vec{x}_b$, $\partial/\partial t_b$ are understood to act only on $\vec{E}_b(\vec{x}, t)$, and so on.

The subexpressions outlined by boxes are just the Fourier transforms of the linear, 2nd-order, 3rd-order, $\cdots$ conductivities. In terms of these,

$$\vec{J}(\vec{x}, t) = \sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \exp\left(-\frac{i\partial}{\partial \vec{k}_a} \cdot \frac{\partial}{\partial \vec{x}} + \frac{i\partial}{\partial w_a} \frac{\partial}{\partial t}\right) \vec{G}^{LIN}(\vec{k}_a, w_a) \vec{E}_a(\vec{x}, t)$$

$$+ \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c) \cdot \vec{x} - i(w_b + w_c)t}$$

$$\exp\left(-\frac{i\partial}{\partial \vec{k}_b} \cdot \frac{\partial}{\partial \vec{x}_b} + \frac{i\partial}{\partial w_b} \frac{\partial}{\partial t_b} - \frac{i\partial}{\partial \vec{k}_c} \cdot \frac{\partial}{\partial \vec{x}_c} + \frac{i\partial}{\partial w_c} \frac{\partial}{\partial t_c}\right)$$

$$\vec{G}^{NL(2)}(\vec{k}_b, w_b, \vec{k}_c, w_c) \vec{E}_b(\vec{x}, t) \vec{E}_c(\vec{x}, t)$$

$$+ \frac{1}{3!} \sum_b \sum_c \sum_d e^{i(\vec{k}_b + \vec{k}_c + \vec{k}_d) \cdot \vec{x} - i(w_b + w_c + w_d)t}$$

$$\exp\left(-\frac{i\partial}{\partial \vec{k}_b} \cdot \frac{\partial}{\partial \vec{x}_b} + \frac{i\partial}{\partial w_b} \frac{\partial}{\partial t_b} - \frac{i\partial}{\partial \vec{k}_c} \cdot \frac{\partial}{\partial \vec{x}_c} + \frac{i\partial}{\partial w_c} \frac{\partial}{\partial t_c} - \frac{i\partial}{\partial \vec{k}_d} \cdot \frac{\partial}{\partial \vec{x}_d}\right.$$

$$\left. + \frac{i\partial}{\partial w_d} \frac{\partial}{\partial t_d}\right)$$

$$\vec{G}^{NL(3)}(\vec{k}_b, w_b, \vec{k}_c, w_c, \vec{k}_d, w_d) \vec{E}_b(\vec{x}, t) \vec{E}_c(\vec{x}, t) \vec{E}_d(\vec{x}, t)$$

$$+ \cdots \cdots \qquad\qquad\qquad \text{A2-(4)}$$

Here $\partial/\partial \vec{x}_b$, $\partial/\partial t_b$ are understood to act only on $\vec{E}_b(\vec{x}, t)$, and so for c, d, $\cdots$.

Compare (4) with the approximate form for the current used in the simple-coupling-of-modes theory, A1-(4). The expression A1-(4)

comprises the above linear current to first order in the space-time derivatives of the $\{\vec{E}_a\}$ and the above 2nd-order nonlinear current to zeroth order in those derivatives.

As an illustration, one may expand A2-(4), displaying higher orders of space-time derivative and higher orders of nonlinear conductivity than are present in A1-(4). When the first few terms of A2-(4) are thus explicitly displayed, the result is 2.5-(5).

From (4), an expression for $\mu_o \dot{\vec{J}}$ may easily be derived. The remaining terms in Maxwell's wave-equation, 2.4-(8), will be written in the form

$$-\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) - \mu_o \epsilon_o \ddot{\vec{E}} = \sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t}$$

$$\exp\left(-\frac{i\partial}{\partial \vec{k}_a} \cdot \frac{\partial}{\partial \vec{x}} + \frac{i\partial}{\partial w_a} \frac{\partial}{\partial t}\right) \left[\vec{k}_a \times (\vec{k}_a \times) + \mu_o \epsilon_o w_a^2\right] \vec{E}_a (\vec{x}, t)$$

A2-(5)

The only non-zero derivatives that the operator in square brackets possesses are the first and second with respect to wavevector and the first and second with respect to frequency. The exponential-operator notation is used merely for convenience in combining (5) with the expression for $\mu_o \dot{\vec{J}}$. The result is Maxwell's wave equation 2.4-(8) in the form

$$\sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \exp\left(-\frac{i\partial}{\partial \vec{k}_a} \cdot \frac{\partial}{\partial \vec{x}} + \frac{i\partial}{\partial w_a} \frac{\partial}{\partial t}\right) \vec{L}^{LIN} (\vec{k}_a, w_a) \vec{E}_a(\vec{x}, t)$$

$$= \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c) \cdot \vec{x} - i(w_b + w_c)t} \exp\left(-\frac{i\partial}{\partial \vec{k}_b} \cdot \frac{\partial}{\partial \vec{x}_b} + \frac{i\partial}{\partial w_b} \frac{\partial}{\partial t_b}\right.$$

$$\left. -\frac{i\partial}{\partial \vec{k}_c} \cdot \frac{\partial}{\partial \vec{x}_c} + \frac{i\partial}{\partial w_c} \frac{\partial}{\partial t_c}\right)(-i\mu_o)(w_b + w_c) \vec{\vec{G}}^{NL(2)}(\vec{k}_b, w_b, \vec{k}_c, w_c)$$

$$\vec{E}_b(\vec{x}, t) \, \vec{E}_c(\vec{x}, t)$$

$$+ \frac{1}{3!} \sum_b \sum_c \sum_d e^{i(\vec{k}_b + \vec{k}_c + \vec{k}_d) \cdot \vec{x} - i(w_b + w_c + w_d)t}$$

$$\exp\left(-\frac{i\partial}{\partial \vec{k}_b} \cdot \frac{\partial}{\partial \vec{x}_b} + \frac{i\partial}{\partial w_b} \frac{\partial}{\partial t_b} - \frac{i\partial}{\partial \vec{k}_c} \cdot \frac{\partial}{\partial \vec{x}_c} + \frac{i\partial}{\partial w_c} \frac{\partial}{\partial t_c}\right.$$

$$\left. - \frac{i\partial}{\partial \vec{k}_d} \cdot \frac{\partial}{\partial \vec{x}_d} + \frac{i\partial}{\partial w_d} \frac{\partial}{\partial t_d}\right)(-i\mu_o)(w_b + w_c + w_d)$$

$$\vec{\vec{\vec{G}}}^{NL(3)}(\vec{k}_b, w_b, \vec{k}_c, w_c, \vec{k}_d, w_d) \vec{E}_b(\vec{x}, t) \vec{E}_c(\vec{x}, t) \vec{E}_d(\vec{x}, t)$$

$$+ \quad \cdots \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{A2-(6)}$$

Here the operator $\vec{\vec{L}}^{LIN}$ is the linear dispersion tensor defined by

$$\vec{\vec{L}}^{LIN}(\vec{k}, w) \equiv \vec{k} X(\vec{k}X) + \mu_o \epsilon_o w^2 + i\mu_o w \vec{\vec{G}}^{LIN}(\vec{k}, w) \qquad \text{A2-(7)}$$

Compare (6) with the simple-coupling-of-modes equation A1-(8).
The equation A1-(8) comprises the linear term of the above to first order

in the space-time derivatives of the $\{\vec{E}_a\}$, equated to the 2nd-order non-linear current taken to zeroth order in those derivatives.

As an illustration one may expand A2-(6), displaying higher orders of space-time derivatives and higher orders of nonlinear conductivity than are present in A1-(8). When the first few terms of A2-(6) are explicitly displayed in this manner, the result is 2.5-(6).

One may also write A2-(6) formally as

$$\sum_a e^{i\vec{k}_a \cdot \vec{x} - iw_a t} \vec{L}^{\text{LIN}} \left(\vec{k}_a + \frac{i\partial}{\partial \vec{x}}, \ w_a + \frac{i\partial}{\partial t} \right) \vec{E}_a$$

$$= \frac{1}{2!} \sum_b \sum_c e^{i(\vec{k}_b + \vec{k}_c) \cdot \vec{x} - i(w_b + w_c)t} (-i\mu_o)(w_b + w_c + \frac{i\partial}{\partial t})$$

$$\vec{\vec{G}}^{\text{NL}(2)} \left(\vec{k}_b - \frac{i\partial}{\partial \vec{x}_b}, \ w_b + \frac{i\partial}{\partial t_b}, \ \vec{k}_c - \frac{i\partial}{\partial \vec{x}_c}, \ w_c + \frac{i\partial}{\partial t_c} \right) \vec{E}_b \vec{E}_c$$

$$+ \frac{1}{3!} \sum_b \sum_c \sum_d e^{i(\vec{k}_b + \vec{k}_c + \vec{k}_d) \cdot \vec{x} - i(w_b + w_c + w_d)t} (-i\mu_o)(w_b + w_c + w_d + \frac{i\partial}{\partial t})$$

$$\vec{\vec{\vec{G}}}^{\text{NL}(3)} \left(\vec{k}_b - \frac{i\partial}{\partial \vec{x}_b}, \ w_b + \frac{i\partial}{\partial t_b}, \ \vec{k}_c - \frac{i\partial}{\partial \vec{x}_c}, \ w_c + \frac{i\partial}{\partial t_c}, \right.$$

$$\left. \vec{k}_d - \frac{i\partial}{\partial \vec{x}_d}, \ w_d + \frac{i\partial}{\partial t_d} \right) \vec{E}_b \vec{E}_c \vec{E}_d$$

$$+ \ \cdots \cdots$$

A2-(8)

A.3   Calculation of Coupling Coefficients for
Warm-Fluid Model of Plasma

Coupling coefficients are additive over species.   For each species, the warm-fluid model is defined by (see 2.8-(12,13,14))

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \frac{\partial \vec{v}}{\partial \vec{x}} + \frac{\gamma v_{To}^2}{n_o} \left(\frac{n}{n_o}\right)^{\gamma-2} \frac{\partial n}{\partial \vec{x}} = \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B}) \qquad \text{A3-(1)}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \vec{x}} \cdot n\vec{v} = 0 \qquad \text{A3-(2)}$$

The steady magnetic field $\vec{B}_o$ is given.   The rest of the magnetic field is a subsidiary quantity defined by

$$\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial \vec{x}} \times \vec{E} \qquad \text{A3-(3)}$$

The electric current is

$$\vec{J} = qn\vec{v} \qquad \text{A3-(4)}$$

Consider 2nd-order conductivity.   Impose fields $\vec{E}_b$, $\vec{E}_c$ upon the medium. (1) and (2) each have a component at the sum-wavevector and frequency $(\vec{k}_{b+c}, w_{b+c})$, namely

$$-i w_{b+c} \vec{V}_{b,c} + \vec{v}_o \cdot i\vec{k}_{b+c} \vec{V}_{b,c} + \vec{V}_a \cdot i\vec{k}_b \vec{V}_b + \frac{\gamma v_{To}^2}{n_o} \left[i\vec{k}_{b+c} N_{b,c} + \right.$$

$$\left. + (\gamma - 2) \frac{N_b}{n_o} i\vec{k}_c N_c \right] = \frac{q}{m} (\vec{V}_{b,c} \times \vec{B}_o + \vec{V}_b \times \vec{B}_c)$$

$$\text{A3-(5)}$$

$$-iw_{b+c} N_{b,c} + i\vec{k}_{b+c} \cdot (n_o \vec{V}_{b,c} + N_b \vec{V}_c + N_{b,c}\vec{v}_o) = 0 \qquad \text{A3-(6)}$$

Here terms other than those in $\vec{V}_{b,c}$ and $N_{b,c}$ are summed over the 2 possible permutations of the subscripts b, c. Rewrite (4) using the definition 2.8-(21)

$$\frac{N_{b,c}}{n_o} = \frac{\vec{k}_{b+c}}{w_{b+c}} \cdot \left(\vec{V}_{b+c} + \vec{V}_b \frac{N_c}{n_o}\right) \qquad \text{A3-(7)}$$

Substitute (7) into (5)

$$\left(-iw_{b+c}^{DR} + \frac{\gamma v_{To}^2\, i\vec{k}_{b+c}\vec{k}_{b+c}}{w_{b+c}^{DR}} \cdot + \frac{q\vec{B}_o}{m} \times\right)\vec{V}_{b,c}$$

$$= -i\vec{k}_b \cdot \vec{V}_c \vec{V}_b - \frac{\gamma v_{To}^2\, i\vec{k}_{b+c}\vec{k}_{b+c}}{w_{b+c}} \cdot \vec{V}_b N_c$$

$$- i\gamma(\gamma-2)v_{To}^2 \vec{k}_c \frac{N_b N_c}{n_o^2} + \vec{V}_b \times \left(\vec{k}_c \times \frac{q\vec{E}_c}{mw_c}\right)$$

$$\text{A3-(8)}$$

Use the linear results 2.8-(19, 25) and the definition (22) to get

$$\vec{V}_{b,c} = \vec{M}_{b+c}^{-1} \left[\left\{ \frac{\vec{k}_{b+c}}{w_{b+c}^{DR}} \vec{V}_b \cdot \vec{V}_c \right\} + \frac{\gamma v_{To}^2 \vec{k}_{b+c}\vec{k}_{b+c}}{(w_{b+c}^{DR})^2} \cdot \vec{V}_b \frac{N_c}{n_o} \right.$$

$$+ \left. \left\{ \gamma(\gamma-2)v_{To}^2 \frac{\vec{k}_{b+c}}{w_{b+c}^{DR}} \frac{N_b N_c}{n_o^2} \right\} + \vec{V}_b \times \left(\frac{\vec{k}_c}{w_c^{DR}} \times \left(\frac{iq\vec{B}_o}{m} \times \vec{V}_c \right) \right) \right]$$

$$\text{A3-(9)}$$

where terms on the right-hand side other than those in curly brackets are

summed over the 2 permutations of b, c. The electric current (4) has a component at the sum-wavevector and frequency $(\vec{k}_{b+c}, w_{b+c})$, namely

$$\vec{J}_{b,c} = n_o q \left(\vec{V}_{b,c} + \vec{V}_b \frac{N_c}{n_o} + \vec{v}_o N_{b,c} \right) \qquad \text{A3-(10)}$$

(10) and (6) yield (see 2.8-(42))

$$\vec{J}_{b,c} = q n_o \left(1 + \frac{\vec{v}_o \vec{k}_{b+c}}{w_{b+c}^{DR}} \cdot \right) \left(\vec{V}_{b,c} + \vec{V}_b \frac{N_c}{n_o} \right) \qquad \text{A3-(11)}$$

Now go directly to coupling coefficient defined by 2.7-(8). Take

$$(\vec{k}_a, w_a) = (\vec{k}_b, w_b) + (\vec{k}_c, w_c) \qquad \text{A3-(12)}$$

and write

$$(-\vec{k}_a, -w_a) = (\vec{k}_{\bar{a}}, w_{\bar{a}}) \qquad \text{A3-(13)}$$

$$\vec{f}_{Aa} = \vec{e}_{A\bar{a}} \qquad \text{A3-(14)}$$

Then

$$(F_{b,c}^{NL(2)})_{ABC} \equiv \frac{i \, \vec{e}_{A\bar{a}} \cdot \vec{j}_{Bb,Cc}}{\epsilon_o \, w_a} \qquad \text{A3-(15)}$$

$$= - \frac{i q n_o}{\epsilon_o} \frac{\vec{e}_{A\bar{a}}}{w_{\bar{a}}} \cdot \left(1 + \frac{\vec{v}_o \vec{k}_{\bar{a}}}{w_{\bar{a}}^{DR}} \cdot \right) \left(\vec{v}_{Bb,Cc} + \vec{v}_{Bb} \frac{n_{Cc}}{n_o} \right) \qquad \text{A3-(16)}$$

(from (11))

(Here terms on the right-hand-side other than those in $\vec{j}_{Bb, Cc}$ and $\vec{v}_{Bb, Cc}$ and $n_{Bb, Cc}$ are summed over simultaneous interchange of B, C and b, c.)

$$\left(F_{b, c}^{NL(2)}\right)_{ABC} = -\frac{iqn_o}{\epsilon_o} \frac{\vec{e}_{A\bar{a}}^{DR}}{w_{\bar{a}}^{DR}} \cdot \left(\vec{v}_{Bb, Cc} + \vec{v}_{Bb} \frac{n_{Cc}}{n_o}\right) \qquad \text{A3-(17)}$$

(from 2.8-(25))

$$= -\frac{mn_o}{\epsilon_o} \vec{v}_{A\bar{a}} \cdot \left(\vec{\vec{M}}_{\bar{a}}^{-1}\right)^{TRANSPOSE} \left(\vec{v}_{Bb, Cc} + \vec{v}_{Bb} \frac{n_{Cc}}{n_o}\right) \qquad \text{A3-(18)}$$

(from 2.8-(32))

$$= -\frac{mn_o}{\epsilon_o} \vec{v}_{A\bar{a}} \cdot \vec{\vec{M}}_{\bar{a}}^{-1} \left(\vec{v}_{Bb, Cc} + \vec{v}_{Bb} \frac{n_{Cc}}{n_o}\right) \qquad \text{A3-(19)}$$

(from 2.8-(22 or 23))

From (11) and the definition 2.8-(22) of $\vec{\vec{M}}_{a}$

$$\left(F_{b, c}^{NL(2)}\right)_{ABC} = -\frac{mn_o}{\epsilon_o} \vec{v}_{A\bar{a}} \cdot \left[\left\{ \frac{\vec{k}_a}{w_a^{DR}} \vec{v}_{Bb} \cdot \vec{v}_{Cc} \right\} \right.$$

$$+ \frac{\gamma v_{To}^2 \vec{k}_a \vec{k}_a}{(w_a^{DR})^2} \cdot \vec{v}_{Bb} \frac{n_{Cc}}{n_o} + \left\{ \gamma(\gamma-2)v_{To}^2 \frac{\vec{k}_a}{w_a^{DR}} \frac{n_{Bb} n_{Cc}}{n_o^2} \right\}$$

$$+ \vec{v}_{Bb} \times \left(\frac{\vec{k}_c}{w_c^{DR}} \times \left(\frac{iq\vec{B}_o}{m} \times \vec{v}_{Cc} \right) \right)$$

$$+ \left(1 - \frac{\gamma v_{To}^2 \vec{k}_a \vec{k}_a}{(w_a^{DR})^2} \cdot + \frac{iq\vec{B}_o}{m} \times \right) \vec{v}_{Bb} \frac{n_{Cc}}{n_o} \left. \right]$$

$$\text{A3-(20)}$$

Here all terms other than those in the curly braces are summed over interchange of Bb, Cc. Now use 2.8-(33), rearrange, and write out the sum of permuted terms explicitly:

$$\left(F_{b,c}^{NL(2)}\right)_{ABC} \equiv \frac{i\,\vec{e}_{A\bar{a}} \cdot \vec{j}_{Bb,Cc}}{\epsilon_o w_a}$$

$$= -\frac{m}{\epsilon_o}\left[n_{A\bar{a}}\,\vec{v}_{Bb}\cdot\vec{v}_{Cc} + n_{Bb}\,\vec{v}_{Cc}\cdot\vec{v}_{A\bar{a}} + n_{Cc}\,\vec{v}_{A\bar{a}}\cdot\vec{v}_{Bb}\right.$$

$$+ \gamma(\gamma-2)v_{To}^2\,\frac{n_{A\bar{a}}\,n_{Bb}n_{Cc}}{n_o^2}$$

$$\left. + n_o\left(\frac{\vec{v}_{A\bar{a}}\cdot\vec{v}_{Bb} \times \vec{v}_{Cc}}{w_{\bar{a}}^{DR}\,w_b^{DR}\,w_c^{DR}}\right)\left(\vec{k}_b w_c^{DR} - \vec{k}_c w_b^{DR}\right)\cdot\frac{iq\vec{B}_o}{m}\right]$$

$$\text{A3-(21)}$$

Consider 3rd-order conductivity. Impose 3 fields $\vec{E}_b$, $\vec{E}_c$, $\vec{E}_d$ upon the medium. (1) and (2) each have a component at the sum-wavevector and frequency ($\vec{k}_{b+c+d}$, w_{b+c+d}), namely

$$-iw_{b+c+d}\,\vec{V}_{b,c,d} + \vec{v}_o\cdot i\vec{k}_{b+c+d}\,\vec{V}_{b,c,d} + \vec{V}_b\cdot i\vec{k}_{c+d}\,\vec{V}_{c,d}$$

$$+ \vec{V}_{c,d}\cdot i\vec{k}_b\vec{V}_b + \frac{\gamma v_{To}^2}{n_o}\left(i\vec{k}_{b+c+d}N_{b,c,d} + (\gamma-2)\frac{N_b}{n_o}i\vec{k}_{c+d}N_{c,d}\right.$$

$$\left. + (\gamma-2)\frac{N_{c,d}}{n_o}i\vec{k}_b N_b + (\gamma-2)(\gamma-3)\frac{N_c N_d}{n_o^2}i\vec{k}_b N_b\right)$$

$$= \frac{q}{m}\left(\vec{V}_{b,c,d} \times \vec{B}_o + \vec{V}_{c,d} \times \vec{B}_b\right)$$

$$\text{A3-(22)}$$

$$-iw_{b+c+d} N_{b,\,c,\,d} + i\vec{k}_{b+c+d} \cdot (n_o \vec{V}_{b,\,c,\,d} + N_b \vec{V}_{c,\,d}$$

$$+ N_{c,\,d} \vec{V}_b + N_{b,\,c,\,d} \vec{V}_o) \;=\; 0$$

<div align="right">A3-(23)</div>

Here all terms other than those in $\vec{V}_{b,\,c,\,d}$ and $N_{b,\,c,\,d}$ are summed over t the 3 cyclic permutations of b, c, d. Rewrite (23) using the definition 2.8-(21)

$$\frac{N_{b,\,c,\,d}}{n_o} \;=\; \frac{\vec{k}_{b+c+d}}{w^{DR}_{b+c+d}} \cdot \left(\vec{V}_{b,\,c,\,d} + \vec{V}_{c,\,d}\frac{N_b}{n_o} + \vec{V}_b \frac{N_{c,\,d}}{n_o}\right)$$

<div align="right">A3-(24)</div>

Substitute (24) in (22)

$$\left(-iw^{DR}_{b+c+d} + \gamma v^2_{To} \frac{i\vec{k}_{b+c+d}\vec{k}_{b+c+d}}{w^{DR}_{b+c+d}} \cdot \; + \frac{q\vec{B}_o}{m} X\right)\vec{V}_{b,\,c,\,d}$$

$$= \; -i\vec{k}_b \cdot \vec{V}_{c,\,d}\vec{V}_b - i\vec{k}_{c+d} \cdot \vec{V}_b \vec{V}_{c,\,d} + \vec{V}_{c,\,d} X \left(\vec{k}_b X \frac{q\vec{E}_b}{mw_b}\right)$$

$$- \gamma v^2_{To} \frac{i\vec{k}_{b+c+d}\vec{k}_{b+c+d}}{w^{DR}_{b+c+d}} \cdot \left(\vec{V}_{c,\,d}\frac{N_b}{n_o} + \vec{V}_b \frac{N_{c,\,d}}{n_o}\right)$$

$$- \gamma(\gamma-2) v^2_{To} \, i\vec{k}_{b+c+d} \frac{N_b N_{c,\,d}}{n^2_o}$$

$$- \left\{\gamma(\gamma-2)(\gamma-3) v^2_{To} \, i\vec{k}_{b+c+d} \frac{N_b N_c N_d}{n^3_o}\right\}$$

<div align="right">A3-(25)</div>

Here all terms on the right-hand-side other than those in curly braces

are summed over cyclic permutations of b, c, d.   Now use the linear

results 2.8-(19, 25) and the definition (22)

$$
\vec{V}_{b,c,d} = \vec{M}_{b+c+d} \left[\left\{ \frac{\vec{k}_{b+c+d} \vec{V}_b \vec{V}_{c,d}}{w^{DR}_{b+c+d}} - \frac{\vec{V}_b \times (\vec{k}_{c+d} \times \vec{V}_{c,d})}{w^{DR}_{b+c+d}} \right. \right.
$$

$$
+ \frac{\vec{V}_{c,d} \times (\vec{k}_b \times (iq\vec{B}_o/mw_b \times \vec{V}_b))}{w^{DR}_{b+c+d}}
$$

$$
+ \gamma v^2_{To} \frac{\vec{k}_{b+c+d} \vec{k}_{b+c+d}}{(w^{DR}_{b+c+d})^2} \cdot \left(\vec{V}_{c,d} \frac{N_b}{n_o} + \vec{V}_b \frac{N_{c,d}}{n_o} \right)
$$

$$
+ \gamma(\gamma-2) v^2_{To} \frac{\vec{k}_{b+c+d} N_b N_{c,d}}{w^{DR}_{b+c+d} \, n^2_o}
$$

$$
+ \left\{ \gamma(\gamma-2)(\gamma-3) v^2_{To} \frac{\vec{k}_{b+c+d}}{w^{DR}_{b+c+d}} \frac{N_b N_c N_d}{n^3_o} \right\} \right]
$$

$$
\text{A3-(26)}
$$

The electric current (4) has a component at the sum-wavevector and

frequency $(\vec{k}_{b+c+d}, w_{b+c+d})$, namely

$$
\vec{J}_{b+c+d} = n_o q \left(\vec{V}_{b,c,d} + \vec{V}_{c,d} \frac{N_b}{n_o} + \vec{V}_b \frac{N_{c,d}}{n_o} + \vec{v}_c N_{b,c,d} \right) \quad \text{A3-(27)}
$$

(27) and (23) yield

$$
\vec{J}_{b,c,d} = q n_o \left(1 + \frac{\vec{v}_o \vec{k}_{b+c+d}}{w^{DR}_{b+c+d}} \cdot \right) \left(\vec{V}_{b,c,d} + \vec{V}_{c,d} \frac{N_b}{n_o} + \vec{V}_b \frac{N_{c,d}}{n_o} \right) \quad \text{A3-(28)}
$$

Now go directly to coupling coefficient defined by 2.7-(9). Take

$$(\vec{k}_a, w_a) = (\vec{k}_b, w_b) + (\vec{k}_c, w_c) + (\vec{k}_d, w_d) \qquad \text{A3-(29)}$$

and write

$$(-\vec{k}_a, -w_a) = (\vec{k}_{\bar{a}}, w_{\bar{a}}) \qquad \text{A3-(30)}$$

$$\vec{f}_{Aa} = \vec{e}_{A\bar{a}} \qquad \text{A3-(31)}$$

Then

$$\left(F^{NL(3)}_{b,c,d}\right)_{ABCD} \equiv \frac{i\,\vec{e}_{A\bar{a}} \cdot \vec{j}_{Bb,\,Cc,\,Dd}}{\epsilon_o\,w_a} \qquad \text{A3-(32)}$$

$$= -\frac{mn_o}{\epsilon_o}\,\vec{v}_{A\bar{a}} \cdot \vec{M}_a^{-1}\left(\vec{v}_{Bb,\,Cc,\,Dd} + \vec{v}_{Cc,\,Dd}\frac{n_{Bb}}{n_o} + \vec{v}_{Bb}\frac{n_{Cc,\,Dd}}{n_o}\right)$$

(cf (19)   A3-(33)

From (28) and the definiton 2.8-(22) of $\vec{M}_a$

$$\left(F^{NL(3)}_{b,c,d}\right)_{ABCD} = -\frac{mn_o}{\epsilon_o}\left[\vec{v}_{A\bar{a}} \cdot \left[\frac{\vec{k}_a}{w_a^{DR}}\vec{v}_{Bb} \cdot \vec{v}_{Cc,\,Dd}\right.\right.$$

$$-\frac{\vec{v}_{Bb} \times (\vec{k}_{c+d} \times \vec{v}_{Cc,\,Dd})}{w_a^{DR}} + \frac{\vec{v}_{Cc,\,Dd} \times (\vec{k}_b \times (iqB_o/mw_b^{DR} \times \vec{v}_b))}{w_a^{DR}}$$

$$+ \gamma v_{To}^2 \frac{\vec{k}_a\vec{k}_a}{(w_a^{DR})^2} \cdot \left(\vec{v}_{Cc,\,Dd}\frac{n_{Bb}}{n_o} + \vec{v}_{Bb}\frac{n_{Cc,\,Dd}}{n_o}\right)$$

$$+ \gamma(\gamma-2)v_{To}^2\frac{\vec{k}_a}{w_a^{DR}}\frac{n_{Bb}n_{Cc,\,Dd}}{n_o^2} + \left\{\gamma(\gamma-2)(\gamma-3)v_{To}^2\frac{\vec{k}_a}{w_a^{DR}}\frac{n_{Bb}n_{Cc}n_{Dd}}{n_o^3}\right\}$$

$$+ \vec{v}_{Cc,Dd}\frac{n_{Bb}}{n_o} + \vec{v}_{Bb}\frac{n_{Cc,Dd}}{n_o}\Big]$$

$$+ \left(-\vec{v}_{A\bar{a}}\cdot\gamma v_{To}^2\frac{\vec{k}_a\vec{k}_a}{(w_a^{DR})^2} + \vec{v}_{A\bar{a}}\times\frac{iq\vec{B}_o}{mw_a^{DR}}\right)\cdot\left(\vec{v}_{Cc,Dd}\frac{n_{Bb}}{n_o} + \vec{v}_{Bb}\frac{n_{Cc,Dd}}{n_o}\right)\Big]$$

<div align="right">A3-(34)</div>

Here all terms on the right-hand-side other than those in curly braces are summed over simultaneous identical cyclic permutations of B, C, D and b, c, d. Rewrite (34) using the 2nd-order result (7) to substitute for $n_{Cc,Dd}$:

$$\left(F^{NL(3)}_{b,c,d}\right)_{ABCD} = -\frac{mn_o}{\epsilon_o}\left[\frac{n_{A\bar{a}}}{n_o}\vec{v}_{Bb}\cdot\vec{v}_{Cc,Dd} + \frac{n_{Bb}}{n_o}\vec{v}_{A\bar{a}}\cdot\vec{v}_{Cc,Dd}\right.$$

$$-\frac{\vec{v}_{A\bar{a}}\times\vec{v}_{Bb}\cdot\vec{k}_{c+d}\times\vec{v}_{Cc,Dd}}{w_a^{DR}} - \frac{\vec{v}_{A\bar{a}}\times(\vec{k}_b\times(iq\vec{B}_o/mw_b^{DR}\times\vec{v}_{Bb}))}{w_a^{DR}}\cdot\vec{v}_{Cc,Dd}$$

$$+\frac{\vec{v}_{A\bar{a}}\times(\vec{k}_b\times(iq\vec{B}_o/mw_a^{DR}\times\vec{v}_{Bb}))}{w_{c+d}^{DR}}\cdot\vec{v}_{Cc,Dd}$$

$$+\left(\gamma(\gamma-2)v_{To}^2\frac{n_{A\bar{a}}n_{Bb}}{n_o^2} + \vec{v}_{A\bar{a}}\cdot\vec{v}_{Bb} + \vec{v}_{A\bar{a}}\times\frac{iq\vec{B}_o}{mw_a^{DR}}\cdot\vec{v}_{Bb}\right)\frac{\vec{k}_{c+d}}{w_{c+d}^{DR}}$$

$$\cdot\left(\vec{v}_{Cc,Dd} + \vec{v}_{Cc}\frac{n_{Dd}}{n_o} + \vec{v}_{Dd}\frac{n_{Cc}}{n_o}\right) + \frac{n_{Bb}}{n_o}\vec{v}_{A\bar{a}}\times\frac{iq\vec{B}_o}{mw_a^{DR}}\cdot\vec{v}_{Cc,Dd}$$

$$+ \vec{v}_{A\bar{a}}\times\frac{iq\vec{B}_o}{mw_a^{DR}}\cdot\vec{v}_{Bb}\frac{n_{Cc,Dd}}{n_o} + \boxed{\left\{\gamma(\gamma-2)(\gamma-3)\frac{n_{A\bar{a}}n_{Bb}n_{Cc}n_{Dd}}{n_o^4}\right\}}\;\Big]$$

<div align="right">A3-(35)</div>

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The boxed term forms a quantity invariant under simultaneous identical permutations of all 4 subscripts A, B, C, D and all 4 subscripts a, b, c, d. Rewrite the remainder, $\left(\hat{F}^{NL(3)}_{b,\,c,\,d}\right)_{ABCD}$ say, using the following pair of 2nd-order results derived from (7) and (9) respectively.

$$\frac{n_{A\bar{a},\,Bb}}{n_o} = \frac{\vec{k}_{c+d}}{w^{DR}_{c+d}} \cdot \left(\vec{v}_{A\bar{a},\,Bb} + \vec{v}_{A\bar{a}}\frac{n_{Bb}}{n_o} + \vec{v}_{Bb}\frac{n_{A\bar{a}}}{n_o}\right)$$

A3-(36)

$$\vec{v}_{A\bar{a},\,Bb} - \gamma v^2_{To}\frac{\vec{k}_{c+d}\vec{k}_{c+d}}{(w^{DR}_{c+d})^2}\cdot\vec{v}_{A\bar{a},\,Bb} - \frac{iq\vec{B}_o}{mw^{DR}_{c+d}}\times\vec{v}_{A\bar{a},\,Bb}$$

$$= \frac{\vec{k}_{c+d}}{w^{DR}_{c+d}}\left(\vec{v}_{A\bar{a}}\cdot\vec{v}_{Bb} + \gamma(\gamma-2)v^2_{To}\frac{n_{A\bar{a}}\,n_{Bb}}{n^2_o}\right)$$

$$- \frac{\vec{v}_{A\bar{a}}\times(\vec{k}_b\times(iq\vec{B}_o/mw^{DR}_b\times\vec{v}_{Bb}))}{w^{DR}_{c+d}}$$

$$- \frac{\vec{v}_{Bb}\times(\vec{k}_a\times(iq\vec{B}_o/mw^{DR}_a\times\vec{v}_{A\bar{a}}))}{w^{DR}_{c+d}}$$

$$+ \gamma v^2_{To}\frac{\vec{k}_{c+d}\vec{k}_{c+d}}{(w^{DR}_{c+d})^2}\cdot\left(\vec{v}_{A\bar{a}}\frac{n_{Bb}}{n_o} + \vec{v}_{Bb}\frac{n_{A\bar{a}}}{n_o}\right)$$

A3-(37)

The result for the remainder of the 3rd-order coupling coefficient is

$$\left(\hat{F}^{NL(3)}_{b,\,c,\,d}\right)_{ABCD} = -\frac{mn_o}{\epsilon_o}\left[\frac{n_{A\bar{a}}}{n_o}\vec{v}_{Bb}\cdot\vec{v}_{Cc,\,Dd} + \frac{n_{Bb}}{n_o}\vec{v}_{Aa}\cdot\vec{v}_{Cc,\,Dd}\right.$$

$$-\frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{k}_{c+d} \times \vec{v}_{Cc,Dd}}{w_a^{DR}} + \frac{\vec{v}_{Aa} \times \left(\vec{k}_b \times \left(iq\vec{B}_o/mw_a^{DR} \times \vec{v}_{Bb}\right)\right)}{w_{c+d}^{DR}} \cdot \vec{v}_{Cc,Dd}$$

$$+ \boxed{\;\vec{v}_{A\bar{a},Bb} \cdot \vec{v}_{Cc,Dd}\;} + \boxed{\;\vec{v}_{A\bar{a},Bb} \cdot \left(\vec{v}_{Cc}\frac{n_{Dd}}{n_o} + \vec{v}_{Dd}\frac{n_{Cc}}{n_o}\right)\;}$$

$$- \boxed{\;\gamma v_{To}^2 \frac{n_{A\bar{a},Bb}\, n_{Cc,Dd}}{n_o^2}\;} + \boxed{\;\vec{v}_{A\bar{a},Bb} \times \frac{iq\vec{B}_o}{mw_{c+d}^{DR}} \cdot \vec{v}_{Cc,Dd}\;}$$

$$+ \vec{v}_{A\bar{a},Bb} \times \frac{iq\vec{B}_o}{mw_{c+d}^{DR}} \cdot \left(\vec{v}_{Cc}\frac{n_{Dd}}{n_o} + \vec{v}_{Dd}\frac{n_{Cc}}{n_o}\right)$$

$$+ \frac{\vec{v}_{Bb} \times \left(\vec{k}_a \times \left(iq\vec{B}_o/mw_a^{DR} \times \vec{v}_{A\bar{a}}\right)\right)}{w_{c+d}^{DR}} \cdot \left(\vec{v}_{Cc,Dd} + \vec{v}_{Cc}\frac{n_{Dd}}{n_o} + \vec{v}_{Dd}\frac{n_{Cc}}{n_o}\right)$$

$$+ \frac{\vec{v}_{A\bar{a}} \times \left(\vec{k}_b \times \left(iq\vec{B}_o/mw_b^{DR} \times \vec{v}_{Bb}\right)\right)}{w_{c+d}^{DR}} \cdot \left(\vec{v}_{Cc}\frac{n_{Dd}}{n_o} + \vec{v}_{Dd}\frac{n_{Cc}}{n_o}\right)$$

$$+ \frac{n_{Bb}}{n_o}\, \vec{v}_{Aa} \times \frac{iq\vec{B}_o}{mw_a^{DR}} \cdot \vec{v}_{Cc,Dd}$$

$$+ \vec{v}_{A\bar{a}} \times \frac{iq\vec{B}_o}{mw_a^{DR}} \cdot \vec{v}_{Bb} \frac{\vec{k}_{c+d}}{w_{c+d}^{DR}} \cdot \left(\vec{v}_{Cc,Dd} + \vec{v}_{Cc}\frac{n_{Dd}}{n_o} + \vec{v}_{Dd}\frac{n_{Cc}}{n_o}\right)\Bigg]$$

A3-(38)

The right-hand-side is again summed over cyclic permutations of Bb, Cc, Dd. The sum of the boxed terms is a quantity invariant under all permutations of Aā, Bb, Cc, Dd. The remainder is a quantity,

$\left(\hat{\hat{F}}^{NL(3)}_{b,\,c,\,d}\right)_{ABCD}$ say, which may be written

$$\left(\hat{\hat{F}}^{NL(3)}_{b,\,c,\,d}\right)_{ABCD} = -\frac{mn_o}{\epsilon_o}\left[-\frac{\vec{v}_{Aa} \times \vec{v}_{Bb} \cdot \vec{k}_{c+d} \times \vec{v}_{Cc,\,Dd}}{w_a^{DR}}\right.$$

$$+ \boxed{\frac{n_{A\bar{a}}}{n_o}\vec{v}_{Bb} \times \frac{iq\vec{B}_o}{mw_{c+d}^{DR}} \cdot \vec{v}_{Cc,\,Dd}} + \frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc,\,Dd}}{w_{c+d}^{DR}}\frac{\vec{k}_a}{w_a^{DR}} \cdot \frac{iq\vec{B}_o}{m}$$

$$+ \frac{\vec{v}_{Aa} \times (\vec{k}_b \times (iq\vec{B}_o/mw_a^{DR} \times \vec{v}_{Bb}))}{w_{c+d}^{DR}} \cdot \vec{v}_{Cc,\,Dd}$$

$$- \boxed{\frac{iq\vec{B}_o}{mw_{c+d}^{DR}} \times \vec{v}_{A\bar{a},\,Bb} \cdot \overset{\wwave}{\frac{n_{Cc}}{n_o}\vec{v}_{Dd}}}$$

$$+ \frac{\vec{v}_{Bb} \times (\vec{k}_a \times (iq\vec{B}_o/mw_a^{DR} \times \vec{v}_{A\bar{a}}))}{w_{c+d}^{DR}} \cdot \frac{n_{Cc}}{n_o}\vec{v}_{Dd}$$

$$+ \frac{\vec{v}_{A\bar{a}} \times (\vec{k}_b \times (iq\vec{B}_o/mw_b^{DR} \times \vec{v}_{Bb}))}{w_{c+d}^{DR}} \cdot \frac{n_{Cc}}{n_o}\vec{v}_{Dd}$$

$$+ \boxed{\frac{n_{Bb}}{n_o}\vec{v}_{A\bar{a}} \times \frac{iq\vec{B}_o}{mw_{c+d}^{DR}} \cdot \vec{v}_{Cc,\,Dd}} - \frac{w_b^{DR}}{w_a^{DR}}\frac{n_{Bb}}{n_o}\vec{v}_{A\bar{a}} \times \frac{iq\vec{B}_o}{mw_{c+d}^{DR}} \cdot \vec{v}_{Cc,\,Dd}$$

$$+ \vec{v}_{A\bar{a}} \times \frac{iq\vec{B}_o}{mw_a^{DR}} \cdot \vec{v}_{Bb}\frac{\vec{k}_{c+d}}{w_{c+d}^{DR}} \cdot \left(\vec{v}_{Cc,\,Dd} + \overset{\wwave}{\frac{n_{Cc}}{n_o}\vec{v}_{Dd}}\right)$$

A3-(39)

Here wavy overscores denote summation over interchange of Cc, Dd, carried out before the general summation over cyclic permutation of Bb, Cc, Dd common to the whole right-hand-side. The boxed terms sum to a quantity invariant under all permutations of Aa, Bb, Cc, Dd.

The remainder, $\left(\hat{\hat{F}}^{NL(3)}_{b,\,c,\,d}\right)_{ABCD}$ say, may be rewritten using the 2nd-order result derived from (9)

$$\vec{k}_{c+d} \times \vec{v}_{Cc,\,Dd} = -\frac{\vec{k}_{c+d}}{w^{DR}_{c+d}} \times \left(\frac{iq\vec{B}_o}{m} \times \vec{v}_{Cc,\,Dd}\right)$$

$$+ \frac{\vec{k}_{c+d}}{w^{DR}_{c+d}} \times \left(\vec{v}_{Cc} \times \left(\vec{k}_d \times (iq\vec{B}_o/mw^{DR}_d \times \vec{v}_{Dd})\right)\right)$$

<div align="right">A3-(40)</div>

The result is

$$\left(\hat{\hat{F}}^{NL(3)}_{b,\,c,\,d}\right)_{ABCD} = -\frac{mn_o}{\epsilon_o}\left[\frac{-\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{k}_{c+d} \times (iq\vec{B}_o/mw^{DR}_{c+d} \times \vec{v}_{Cc,\,Dd})}{w^{DR}_a}\right.$$

$$- \vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \frac{\vec{k}_{c+d} \times \left(\vec{v}_{Cc} \times \left(\vec{k}_d \times (iq\vec{B}_o/mw^{DR}_d \times \vec{v}_{Dd})\right)\right)}{w^{DR}_a w^{DR}_{c+d}}$$

$$+ \vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc,\,Dd} \frac{\vec{k}_a}{w^{DR}_a} \cdot \frac{iq\vec{B}_o}{mw^{DR}_{c+d}}$$

$$+ \frac{\vec{v}_{A\bar{a}} \times \left(\vec{k}_b \times (iq\vec{B}_o/mw^{DR}_a \times \vec{v}_{Bb})\right)}{w^{DR}} \cdot \vec{v}_{Cc,\,Dd}$$

$$+ \frac{\vec{v}_{Bb} \times \left(\vec{k}_a \times (iq\vec{B}_o/mw^{DR}_a \times \vec{v}_{A\bar{a}})\right)}{w^{DR}_{c+d}} \cdot \frac{n_{Cc}}{n_o}\vec{v}_{Dd}$$

$$+ \frac{\vec{v}_{A\bar{a}} \times \left(\vec{k}_b \times (iq\vec{B}_o/mw^{DR}_b \times \vec{v}_{Bb})\right)}{w^{DR}_{c+d}} \cdot \frac{n_{Cc}}{n_o}\vec{v}_{Dd}$$

$$- \frac{w_b^{DR}}{w_a^{DR}} \frac{n_{Bb}}{n_o} \vec{v}_{A\bar{a}} \times \frac{iq\vec{B}_o}{mw_{c+d}^{DR}} \cdot \vec{v}_{Cc,Dd}$$

$$+ \vec{v}_{A\bar{a}} \times \frac{iq\vec{B}_o}{mw_a^{DR}} \cdot \vec{v}_{Bb} \frac{\vec{k}_{c+d}}{w_{c+d}^{DR}} \cdot \left(\vec{v}_{Cc,Dd} + \frac{n_{Cc}}{n_o} \vec{v}_{Dd} \right)]$$

$$A3\text{-}(41)$$

The terms in $\vec{v}_{Cc,Dd}$ cancel. The remaining terms may be written

$$\left(F_{b,c,d}^{NL(3)} \right)_{ABCD} = - \frac{mn_o}{\epsilon_o}$$

$$\left[- \frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc}}{w_a^{DR}} \frac{\vec{k}_{c+d}}{w_{c+d}^{DR}} \cdot \vec{k}_d \times (iq\vec{B}_o/mw_d^{DR} \times \vec{v}_{Dd}) \right.$$

$$+ \frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb}}{w_a^{DR}} \cdot \vec{k}_d \times (iq\vec{B}_o/mw_d^{DR} \times \vec{v}_{Dd}) \frac{\vec{k}_{c+d}}{w_{c+d}^{DR}} \cdot \vec{v}_{Cc}$$

$$\boxed{ \begin{aligned} &+ \frac{n_{A\bar{a}}}{n_o} \vec{v}_{Bb} \times \frac{iq\vec{B}_o}{mw_{c+d}^{DR}} \cdot \frac{n_{Cc}}{n_o} \vec{v}_{Dd} \\ &+ \frac{n_{Bb}}{n_o} \vec{v}_{A\bar{a}} \times \frac{iq\vec{B}_o}{mw_{c+d}^{DR}} \cdot \frac{n_{Cc}}{n_o} \vec{v}_{Dd} \end{aligned} }$$

$$+ \frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb}}{w_{c+d}^{DR}} \cdot \left(\frac{\vec{k}_a}{w_a^{DR}} \cdot \frac{iq\vec{B}_o}{m} \right) \frac{n_{Cc}}{n_o} \vec{v}_{Dd}$$

$$- \frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb}}{w_{c+d}^{DR}} \cdot \left(\frac{\vec{k}_b}{w_b^{DR}} \cdot \frac{iq\vec{B}_o}{m} \right) \frac{n_{Cc}}{n_o} \vec{v}_{Dd}$$

$$\left. - \vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \frac{iq\vec{B}_o}{mw_a^{DR}} \frac{\vec{k}_{c+d}}{w_{c+d}^{DR}} \cdot \frac{n_{Cc}}{n_o} \vec{v}_{Dd} \right]$$

$$A3\text{-}(42)$$

The boxed terms sum to a quantity totally symmetric in the subscripts

$A\bar{a}$, Bb, Cc, Dd.   The remainder may be written

$$\left(\hat{\hat{\hat{F}}}^{NL(3)}_{b,c,d}\right)_{ABCD} = -\frac{mn_o}{\epsilon_o}\left[-\frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb}}{w^{DR}_a} \cdot \frac{{}^n Dd}{n_o}\vec{v}_{Cc}\left(\frac{iq\vec{B}_o}{m} \cdot \frac{\vec{k}_{c+d}}{w^{DR}_{c+d}}\right)\right.$$

$$+ \frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc}}{w^{DR}_a}\left(\frac{\vec{k}_d}{w^{DR}_d} \cdot \frac{iq\vec{B}_o}{m}\right)\vec{v}_{Dd} \cdot \frac{\vec{k}_{c+d}}{w^{DR}_{c+d}}$$

$$+ \frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb}}{w^{DR}_{c+d}} \cdot \frac{{}^n Dd}{n_o}\vec{v}_{Cc}\left(\frac{iq\vec{B}_o}{m} \cdot \frac{\vec{k}_a}{w^{DR}_a}\right) - \vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc}\left(\frac{\vec{k}_c}{w_c} \cdot \frac{iq\vec{B}_o}{m}\right)\vec{v}_{Dd} \cdot \frac{\vec{k}_{c+d}}{w^{DR}_{c+d}}$$

$$\left. - \frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb}}{w^{DR}_{c+d}} \cdot \frac{{}^n Dd}{n_o}\vec{v}_{Cc}\left(\frac{iq\vec{B}_o}{m} \cdot \frac{\vec{k}_b}{w^{DR}_b}\right)\right]$$

<div align="right">A3-(43)</div>

$$\left(\hat{\hat{\hat{F}}}^{NL(3)}_{b,c,d}\right)_{ABCD} = -\frac{mn_o}{\epsilon_o}$$

$$\left[\frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb}}{w^{DR}_{c+d}} \cdot \frac{{}^n Dd}{n_o}\vec{v}_{Cc}\frac{iq\vec{B}_o}{m} \cdot \left(-\frac{\vec{k}_{c+d}}{w^{DR}_a} + \frac{\vec{k}_a}{w^{DR}_a} - \frac{\vec{k}_b}{w^{DR}_b}\right)\right.$$

$$\left. + \frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc}}{w^{DR}_a}\vec{v}_{Dd} \cdot \frac{\vec{k}_{c+d}}{w^{DR}_{c+d}}\frac{iq\vec{B}_o}{m} \cdot \left(\frac{\vec{k}_d}{w^{DR}_d} - \frac{\vec{k}_c}{w^{DR}_c}\right)\right]$$

<div align="right">A3-(44)</div>

The right-hand-side of (44) is first summed over interchange of Cc, Dd

and then summed over cyclic permutations of Bb, Cc, Dd.   It is the

same as the following, summed over cyclic permutations of Bb, Cc, Dd:

$$\left(\hat{\hat{\hat{F}}}^{NL(3)}_{b,\,c,\,d}\right)_{ABCD} =$$

$$-\frac{mn_o}{\epsilon_o}\left[\frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc}}{\frac{DR}{w_{c+d}}} \vec{v}_{Dd} \cdot \frac{\vec{k}_d}{\frac{DR}{w_d}} \frac{iq\vec{B}_o}{m} \cdot \left(-\frac{\vec{k}_{c+d}}{\frac{DR}{w_a}} + \frac{\vec{k}_a}{\frac{DR}{w_a}} - \frac{\vec{k}_b}{\frac{DR}{w_b}}\right)\right.$$

$$-\frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc}}{\frac{DR}{w_{b+d}}} \vec{v}_{Dd} \cdot \frac{\vec{k}_d}{\frac{DR}{w_d}} \frac{iq\vec{B}_o}{m} \cdot \left(-\frac{\vec{k}_{b+d}}{\frac{DR}{w_a}} + \frac{\vec{k}_a}{\frac{DR}{w_a}} - \frac{\vec{k}_c}{\frac{DR}{w_c}}\right)$$

$$+\frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc}}{\frac{DR}{w_{c+d}}} \vec{v}_{Dd} \cdot \frac{\vec{k}_{c+d}}{\frac{DR}{w_a}} \frac{iq\vec{B}_o}{m} \cdot \left(\frac{\vec{k}_d}{\frac{DR}{w_d}} - \frac{\vec{k}_c}{\frac{DR}{w_c}}\right)$$

$$\left. -\frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc}}{\frac{DR}{w_{b+d}}} \vec{v}_{Dd} \cdot \frac{\vec{k}_{b+d}}{\frac{DR}{w_a}} \frac{iq\vec{B}_o}{m} \cdot \left(\frac{\vec{k}_d}{\frac{DR}{w_d}} - \frac{\vec{k}_b}{\frac{DR}{w_b}}\right)\right]$$

$$\text{A3-(45)}$$

$$A\left(_a\hat{\hat{\hat{F}}}^{NL(3)}_{b,\,c,\,d}\right)_{BCD} = -\frac{mn_o}{\epsilon_o}\frac{\vec{v}_{A\bar{a}} \times \vec{v}_{Bb} \cdot \vec{v}_{Cc}}{\frac{DR}{w_a}} \vec{v}_{Dd} \cdot \left[\frac{\vec{k}_d}{\frac{DR}{w_d}}\left(\frac{\vec{k}_c}{\frac{DR}{w_c}} - \frac{\vec{k}_c}{\frac{DR}{w_b}}\right)\right.$$

$$\left. +\frac{\vec{k}_{c+d}}{\frac{DR}{w_{c+d}}}\left(\frac{\vec{k}_d}{\frac{DR}{w_d}} - \frac{\vec{k}_c}{\frac{DR}{w_c}}\right) + \frac{\vec{k}_{c+d}}{\frac{DR}{w_{b+d}}}\left(\frac{\vec{k}_b}{\frac{DR}{w_b}} - \frac{\vec{k}_d}{\frac{DR}{w_d}}\right)\right] \cdot \frac{iq\vec{B}_o}{m}$$

$$\text{A3-(46)}$$

This quantity summed over cyclic permutations of Bb, Cc, Dd can be shown (see Appendix A6) to be completely symmetric in $A\bar{a}$, Bb, Cc, Dd. Thus the whole third-order coupling coefficient is completely symmetric in $A\bar{a}$, Bb, Cc, Dd. Reassemble the whole coefficient by bringing back the boxed terms which were successively set aside in (35), (38), (39), and (42). The result is

$$\left(F_{b,c,d}^{NL(3)}\right)_{ABCD} =$$

$$-\frac{m}{\epsilon_o}\left[(n_{A\bar{a}}\vec{v}_{Bb} + n_{Bb}\vec{v}_{A\bar{a}})\cdot\vec{v}_{Cc,Dd} + \vec{v}_{A\bar{a},Bb}\cdot(n_{Cc}\vec{v}_{Dd} + n_{Dd}\vec{v}_{Cc})\right.$$

$$+\ n_o\,\vec{v}_{A\bar{a},Bb}\cdot\vec{v}_{Cc,Dd} - \frac{\gamma v_{To}^2}{n_o}\,n_{A\bar{a},Bb}\,n_{Cc,Dd}$$

$$+\ \left\{\gamma(\gamma-2)(\gamma-3)\frac{v_{To}^2}{n_o^3}\,n_{A\bar{a}}\,n_{Bb}\,n_{Cc}\,n_{Dd}\right\}$$

$$-\left(n_o\vec{v}_{A\bar{a},Bb} + n_{A\bar{a}}\vec{v}_{Bb} + n_{Bb}\vec{v}_{A\bar{a}}\right) X \left(n_o\vec{v}_{Cc,Dd} + n_{Cc}\vec{v}_{Dd} + n_{Dd}\vec{v}_{Cc}\right)\cdot\frac{iq\vec{B}_o}{mn_o w_{c+d}^{DR}}$$

$$+\ n_o\,(\vec{v}_{A\bar{a}} X \vec{v}_{Bb}\cdot\vec{v}_{Cc})\,\vec{v}_{Dd}\cdot\left(\frac{\vec{k}_d}{w_d^{DR}}\left(\frac{\vec{k}_c}{w_c^{DR}} - \frac{\vec{k}_b}{w_b^{DR}}\right) + \frac{\vec{k}_{c+d}}{w_{c+d}^{DR}}\left(\frac{\vec{k}_d}{w_d^{DR}} - \frac{\vec{k}_c}{w_c^{DR}}\right)\right.$$

$$\left.\left.+\ \frac{\vec{k}_{b+d}}{w_{b+d}^{DR}}\left(\frac{\vec{k}_b}{w_b^{DR}} - \frac{\vec{k}_d}{w_d^{DR}}\right)\right)\right)\cdot\frac{iq\vec{B}_o}{mw_{\bar{a}}^{DR}}\ \right]$$

$$\text{A3-(47)}$$

In (47) the terms on the right-hand-side other than those in curly braces are summed over cyclic permutations of Bb, Cc, Dd. Writing out the summation explicitly,

$$\left(F_{b,c,d}^{NL(3)}\right)_{ABCD} =$$

$$-\frac{m}{\epsilon_o}\left[(n_{A\bar{a}}\vec{v}_{Bb} + n_{Bb}\vec{v}_{A\bar{a}})\cdot\vec{v}_{Cc,Dd} + (n_{Cc}\vec{v}_{Dd} + n_{Dd}\vec{v}_{Cc})\cdot\vec{v}_{A\bar{a},Bb}\right.$$

$$+ \ (n_{A\bar{a}}\vec{v}_{Cc} + n_{Cc}\vec{v}_{A\bar{a}}) \cdot \vec{v}_{Dd, Bb} + (n_{Dd}\vec{v}_{Bb} + n_{Bb}\vec{v}_{Dd}) \cdot \vec{v}_{A\bar{a}, Cc}$$

$$+ \ (n_{A\bar{a}}\vec{v}_{Dd} + n_{Dd}\vec{v}_{A\bar{a}}) \cdot \vec{v}_{Bb, Cc} + (n_{Bb}\vec{v}_{Cc} + n_{Cc}\vec{v}_{Bb}) \cdot \vec{v}_{A\bar{a}, Dd}$$

$$+ \ n_o \vec{v}_{A\bar{a}, Bb} \cdot \vec{v}_{Cc, Dd} + n_o \vec{v}_{A\bar{a}, Cc} \cdot \vec{v}_{Dd, Bb} + n_o \vec{v}_{A\bar{a}, Dd} \cdot \vec{v}_{Bb, Cc}$$

$$- \ (n_{A\bar{a}, Bb} \, n_{Cc, Dd} + n_{A\bar{a}, Cc} \, n_{Dd, Bb} + n_{A\bar{a}, Dd} \, n_{Bb, Cc}) \, \gamma v_{To}^2 / n_o$$

$$+ \ n_{A\bar{a}} \, n_{Bb} \, n_{Cc} \, n_{Dd} \, \gamma(\gamma-2)(\gamma-3) \, v_{To}^2 / n_o^3$$

$$- \ (n_o \vec{v}_{A\bar{a}, Bb} + n_{A\bar{a}}\vec{v}_{Bb} + n_{Bb}\vec{v}_{A\bar{a}}) \ X \ (n_o \vec{v}_{Cc, Dd} + n_{Cc}\vec{v}_{Dd} + n_{Dd}\vec{v}_{Cc}) \cdot \frac{iq\vec{B}_o}{mn_o w_{c+d}^{DR}}$$

$$- \ (n_o \vec{v}_{A\bar{a}, Cc} + n_{A\bar{a}}\vec{v}_{Cc} + n_{Cc}\vec{v}_{A\bar{a}}) \ X \ (n_o \vec{v}_{Dd, Bb} + n_{Dd}\vec{v}_{Bb} + n_{Bb}\vec{v}_{Dd}) \cdot \frac{iq\vec{B}_o}{mn_o w_{d+b}^{DR}}$$

$$- \ (n_o \vec{v}_{A\bar{a}, Dd} + n_{A\bar{a}}\vec{v}_{Dd} + n_{Dd}\vec{v}_{A\bar{a}}) \ X \ (n_o \vec{v}_{Bb, Cc} + n_{Bb}\vec{v}_{Cc} + n_{Cc}\vec{v}_{Bb}) \cdot \frac{iq\vec{B}_o}{mn_o w_{b+c}^{DR}}$$

$$+ \ n_o (\vec{v}_{A\bar{a}} \ X \ \vec{v}_{Bb} \cdot \vec{v}_{Cc}) \vec{v}_{Dd} \cdot \left(\frac{\vec{k}_d}{w_d^{DR}} \left(\frac{\vec{k}_c}{w_c^{DR}} - \frac{\vec{k}_b}{w_b^{DR}} \right) + \frac{\vec{k}_{c+d}}{w_{c+d}^{DR}} \left(\frac{\vec{k}_d}{w_d^{DR}} - \frac{\vec{k}_c}{w_c^{DR}} \right) \right.$$

$$\left. + \ \frac{\vec{k}_{b+d}}{w_{b+d}^{DR}} \left(\frac{\vec{k}_b}{w_b^{DR}} - \frac{\vec{k}_d}{w_d^{DR}} \right) \right) \cdot \frac{iq\vec{B}_o}{mw_{\bar{a}}^{DR}} + n_o (\vec{v}_{A\bar{a}} \ X \ \vec{v}_{Cc} \cdot \vec{v}_{Dd}) \vec{v}_{Bb} \ \cdot$$

$$\left(\frac{\vec{k}_b}{w_b^{DR}} \left(\frac{\vec{k}_d}{w_d^{DR}} - \frac{\vec{k}_c}{w_c^{DR}} \right) + \frac{\vec{k}_{d+b}}{w_{d+b}^{DR}} \left(\frac{\vec{k}_b}{w_b^{DR}} - \frac{\vec{k}_d}{w_d^{DR}} \right) + \frac{\vec{k}_{c+b}}{w_{c+b}^{DR}} \left(\frac{\vec{k}_c}{w_c^{DR}} - \frac{\vec{k}_b}{w_b^{DR}} \right) \right) \cdot \frac{iq\vec{B}_o}{mw_{\bar{a}}^{DR}}$$

$$+ \; n_o(\vec{v}_{A\bar{a}} \times \vec{v}_{Dd} \cdot \vec{v}_{Bb}) \vec{v}_{Cc} \cdot$$

$$\left(\frac{\vec{k}_c}{w_c}DR\left(\frac{\vec{k}_b}{w_b}DR - \frac{\vec{k}_d}{w_d}DR\right) + \frac{\vec{k}_{b+c}}{w_{b+c}}DR\left(\frac{\vec{k}_c}{w_c}DR - \frac{\vec{k}_b}{w_b}DR\right) + \frac{\vec{k}_{d+c}}{w_{d+c}}DR\left(\frac{\vec{k}_d}{w_d}DR - \frac{\vec{k}_c}{w_c}DR\right)\right) \cdot \frac{iq\vec{B}_o}{mw_{\bar{a}}}DR \; \Big]$$

A3-(48)

The prescription for evaluating (48) is as follows. The normalized mobility tensors $\vec{M}$ are found from 2.8-(22 or 23). The first-order velocities and densities are then found from 2.8-(19). The second-order velocities are found from (9) and the second-order densities from (7). Both the 2nd-order coupling coefficient (21) and the 3rd-order coupling coefficient (48) are to be summed over particle species. They are also to be summed over different drift-velocity and temperature-components for a single species, if such exist in the plasma.

A. 4   Verification of Symmetry of 3rd-Order Coupling
Coefficient Using MACSYMA

The 3rd-order coupling coefficient $\left(F_{b,\,c,\,d}^{NL(3)}\right)_{ABCD}$ for a warm-fluid

plasma is displayed in A3-(48) or equivalently in 2.8-(52).   The displayed

form is constructed from first- and second-order velocities and densities

which characterize the interacting waves.   The method of construction

is totally symmetric in the 4 interacting waves, except for the last 3

terms.   The sum of the last 3 terms forms a quantity, $\left(\widehat{\widehat{F}}_{b,\,c,\,d}\right)_{ABCD}$

say, which is certainly symmetric in the 3 waves characterized by the

subscripts Bb, Cc, Dd.   We shall show that despite its appearance it is

also symmetric in all 4 subscripts Aā, Bb, Cc, Dd.   This will be done

using MACSYMA - Project MAC's SYmbolic MAnipulation system.   This

is a highly interactive computer system, implemented and maintained at

M.I.T.'s Project MAC by Joel Moses and his group.   No attempt will be

made to describe the workings of MACSYMA.   Rather, the console

session will be reproduced.   Only those few features of MACSYMA used

in the calculation will appear, and their function will be obvious.   Of

course, the range and power of MACSYMA's facilities are by no means

limited to the relatively simple operations appearing on the printout in

this appendix.

From 2.8-(52)

$$\left(\widehat{\widehat{\widehat{\widehat{F}}}}_{b,\,c,\,d}\right)_{ABCD} = + mn_o/\epsilon_o$$

$$\left[\left(\vec{v}_{A\bar{a}}\cdot\vec{v}_{Cc} \times \vec{v}_{Dd}\right)\vec{v}_{Bb}\cdot\right.$$

$$\left(\frac{\vec{k}_b}{w_b^{DR}}\left(\frac{\vec{k}_c}{w_c^{DR}}-\frac{\vec{k}_d}{w_d^{DR}}\right)+\frac{\vec{k}_{d+b}}{w_{d+b}^{DR}}\left(\frac{\vec{k}_d}{w_d^{DR}}-\frac{\vec{k}_b}{w_b^{DR}}\right)+\frac{\vec{k}_{b+c}}{w_{b+c}^{DR}}\left(\frac{\vec{k}_b}{w_b^{DR}}-\frac{\vec{k}_c}{w_c^{DR}}\right)\right))\cdot\frac{iq\vec{B}_o}{mw_{\bar{a}}^{DR}}$$

$$+ \ (\vec{v}_{A\bar{a}} \cdot \vec{v}_{Dd} \times \vec{v}_{Bb}) \vec{v}_{Cc} \ \cdot$$

$$\left(\frac{\vec{k}_c}{w_c^{DR}} \left(\frac{\vec{k}_d}{w_d^{DR}} - \frac{\vec{k}_b}{w_b^{DR}} \right) + \frac{\vec{k}_{b+c}}{w_{b+c}^{DR}} \left(\frac{\vec{k}_b}{w_b^{DR}} - \frac{\vec{k}_c}{w_c^{DR}} \right) + \frac{\vec{k}_{c+d}}{w_{c+d}^{DR}} \left(\frac{\vec{k}_c}{w_c^{DR}} - \frac{\vec{k}_d}{w_d^{DR}} \right) \right) \cdot \frac{iq\vec{B}_o}{mw_{\bar{a}}^{DR}}$$

$$+ \ (\vec{v}_{A\bar{a}} \cdot \vec{v}_{Bb} \times \vec{v}_{Cc}) \vec{v}_{Dd} \ \cdot$$

$$\left(\frac{\vec{k}_d}{w_d^{DR}} \left(\frac{\vec{k}_b}{w_b^{DR}} - \frac{\vec{k}_c}{w_c^{DR}} \right) + \frac{\vec{k}_{c+d}}{w_{c+d}^{DR}} \left(\frac{\vec{k}_c}{w_c^{DR}} - \frac{\vec{k}_d}{w_d^{DR}} \right) + \frac{\vec{k}_{d+b}}{w_{d+b}^{DR}} \left(\frac{\vec{k}_d}{w_d^{DR}} - \frac{\vec{k}_b}{w_b^{DR}} \right) \right) \cdot \frac{iq\vec{B}_o}{mw_{\bar{a}}^{DR}}$$

<div align="right">A4-(1)</div>

Consider the vector

$$\vec{u}_{A\bar{a}} \ \equiv \ \vec{v}_{A\bar{a}} (\vec{v}_{Bb} \cdot \vec{v}_{Cc} \times \vec{v}_{Dd})$$

and the 3 other distinct vectors formed from it by even permutations of the 4 subscripts of the $\{\vec{v}\}$

$$\vec{u}_{Bb} \ \equiv \ \vec{v}_{Bb} (\vec{v}_{A\bar{a}} \cdot \vec{v}_{Dd} \times \vec{v}_{Cc})$$

$$\vec{u}_{Cc} \ \equiv \ \vec{v}_{Cc} (\vec{v}_{A\bar{a}} \cdot \vec{v}_{Bb} \times \vec{v}_{Dd})$$

$$\vec{u}_{Dd} \ \equiv \ \vec{v}_{Dd} (\vec{v}_{A\bar{a}} \cdot \vec{v}_{Cc} \times \vec{v}_{Bb})$$

<div align="right">A4-(2)</div>

An even permutation of the subscripts of the $\{\vec{v}\}$ induces the same even permutation of the subscripts of the $\{\vec{u}\}$. An odd permutation of the subscripts of the $\{\vec{v}\}$ induces the same odd permutation of the subscripts of the $\{\vec{u}\}$, together with an overall sign change. Further, using vector identities,

$$\vec{u}_{A\bar{a}} + \vec{u}_{Bb} + \vec{u}_{Cc} + \vec{u}_{Dd} \equiv 0 \qquad \text{A4-(3)}$$

Define

$$\vec{k}_{\bar{a}} \cdot \frac{iqn_o\vec{B}_o}{\epsilon_o} \equiv \ell_{\bar{a}} \qquad \text{A4-(4)}$$

and so for the subscripts b, c, d. Then

$$\ell_{\bar{a}} + \ell_b + \ell_c + \ell_d = 0 \qquad \text{A4-(5)}$$

In terms of these variables

$$\left(\hat{\hat{\hat{\hat{F}}}}_{b,\,c,\,d}\right)_{ABCD} =$$

$$-\vec{u}_{Bb} \cdot \left(\frac{\vec{k}_b}{{}_w DR_b}\left(\frac{\ell_c}{{}_w DR_c} - \frac{\ell_d}{{}_w DR_d}\right) + \frac{\vec{k}_{d+b}}{{}_w DR_{d+b}}\left(\frac{\ell_d}{{}_w DR_d} - \frac{\ell_b}{{}_w DR_b}\right) + \frac{\vec{k}_{b+c}}{{}_w DR_{b+c}}\left(\frac{\ell_b}{{}_w DR_b} - \frac{\ell_c}{{}_w DR_c}\right) \right) / {}_w DR_{\bar{a}}$$

$$-\vec{u}_{Cc} \cdot \left(\frac{\vec{k}_c}{{}_w DR_c}\left(\frac{\ell_d}{{}_w DR_d} - \frac{\ell_b}{{}_w DR_b}\right) + \frac{\vec{k}_{b+c}}{{}_w DR_{b+c}}\left(\frac{\ell_b}{{}_w DR_b} - \frac{\ell_c}{{}_w DR_c}\right) + \frac{\vec{k}_{c+d}}{{}_w DR_{c+d}}\left(\frac{\ell_c}{{}_w DR_c} - \frac{\ell_d}{{}_w DR_d}\right) \right) / {}_w DR_{\bar{a}}$$

$$-\vec{u}_{Dd} \cdot \left(\frac{\vec{k}_d}{{}_w DR_d}\left(\frac{\ell_b}{{}_w DR_b} - \frac{\ell_c}{{}_w DR_c}\right) + \frac{\vec{k}_{c+d}}{{}_w DR_{c+d}}\left(\frac{\ell_c}{{}_w DR_c} - \frac{\ell_d}{{}_w DR_d}\right) + \frac{\vec{k}_{d+b}}{{}_w DR_{d+b}}\left(\frac{\ell_d}{{}_w DR_d} - \frac{\ell_b}{{}_w DR_b}\right) \right) / {}_w DR_{\bar{a}}$$

$$\text{A4-(6)}$$

Interchanging the 2 subscripts A$\bar{a}$, Bb on the right-hand-side of (1) transforms it into the quantity (note we have used (10) and the immediately preceding discussion)

$$Q = \left(-\vec{u}_{Bb} - \vec{u}_{Cc} - \vec{u}_{Dd}\right) \cdot$$

$$\left(\frac{\vec{k}_{\bar{a}}}{w_{\bar{a}}^{DR}}\left(\frac{\ell_c}{w_c^{DR}}-\frac{\ell_d}{w_d^{DR}}\right)+\frac{\vec{k}_{d+\bar{a}}}{w_{d+\bar{a}}^{DR}}\left(\frac{\ell_d}{w_d^{DR}}-\frac{\ell_{\bar{a}}}{w_{\bar{a}}^{DR}}\right)+\frac{\vec{k}_{\bar{a}+c}}{w_{\bar{a}+c}^{DR}}\left(\frac{\ell_{\bar{a}}}{w_{\bar{a}}^{DR}}-\frac{\ell_c}{w_c^{DR}}\right)\right)/w_b^{DR}$$

$$+\vec{u}_{Cc}\cdot\left(\frac{\vec{k}_c}{w_c^{DR}}\left(\frac{\ell_d}{w_d^{DR}}-\frac{\ell_{\bar{a}}}{w_{\bar{a}}^{DR}}\right)+\frac{\vec{k}_{\bar{a}+c}}{w_{\bar{a}+c}^{DR}}\left(\frac{\ell_{\bar{a}}}{w_{\bar{a}}^{DR}}-\frac{\ell_c}{w_c^{DR}}\right)+\frac{\vec{k}_{c+d}}{w_{c+d}^{DR}}\left(\frac{\ell_c}{w_c^{DR}}-\frac{\ell_d}{w_d^{DR}}\right)\right)/w_b^{DR}$$

$$+\vec{u}_{Dd}\cdot\left(\frac{\vec{k}_d}{w_d^{DR}}\left(\frac{\ell_{\bar{a}}}{w_{\bar{a}}^{DR}}-\frac{\ell_c}{w_c^{DR}}\right)+\frac{\vec{k}_{c+d}}{w_{c+d}^{DR}}\left(\frac{\ell_c}{w_c^{DR}}-\frac{\ell_d}{w_d^{DR}}\right)+\frac{\vec{k}_{d+\bar{a}}}{w_{d+\bar{a}}^{DR}}\left(\frac{\ell_d}{w_d^{DR}}-\frac{\ell_{\bar{a}}}{w_{\bar{a}}^{DR}}\right)\right)/w_b^{DR}$$

$$\text{A4-(7)}$$

(6) and (7) may be written as say

$$\left(\hat{\hat{\hat{\hat{F}}}}_{b,c,d}\right)_{ABCD} \equiv -\vec{u}_{Bb}\cdot\overrightarrow{VECB}$$

$$-\vec{u}_{Cc}\cdot\overrightarrow{VECC}$$

$$-\vec{u}_{Dd}\cdot\overrightarrow{VECD} \qquad \text{A4-(8)}$$

$$Q \equiv (-\vec{u}_{Bb}-\vec{u}_{Cc}-\vec{u}_{Dd})\cdot\overrightarrow{TVECB}$$

$$+\vec{u}_{Cc}\cdot\overrightarrow{TVECC}$$

$$+\vec{u}_{Dd}\cdot\overrightarrow{TVECD} \qquad \text{A4-(9)}$$

Also

$$\vec{k}_{\bar{a}}+\vec{k}_b+\vec{k}_c+\vec{k}_d = \vec{0} \qquad \text{A4-(10)}$$

$$w_{\bar{a}}^{DR}+w_b^{DR}+w_c^{DR}+w_d^{DR} = 0 \qquad \text{A4-(11)}$$

The MACSYMA system is given the form of $\overrightarrow{VECB}$, $\overrightarrow{VECC}$, $\overrightarrow{VECD}$,

$\overrightarrow{\text{TVECB}}$, $\overrightarrow{\text{TVECC}}$, $\overrightarrow{\text{TVECD}}$, appearing in (6) and (7). It is then given the relations (3), (10) and (11). Using evaluation followed by rational simplification we find that MACSYMA returns the results (see printout)

$$\overrightarrow{\text{VECB}} \quad - \quad \overrightarrow{\text{TVECB}} \quad = \quad \vec{0}$$

$$\overrightarrow{\text{VECC}} \quad + \quad \overrightarrow{\text{TVECC}} \quad - \quad \overrightarrow{\text{TVECB}} \quad = \quad \vec{0}$$

$$\overrightarrow{\text{VECD}} \quad + \quad \overrightarrow{\text{TVECD}} \quad - \quad \overrightarrow{\text{TVECB}} \quad = \quad \vec{0}$$

A4-(12)

Therefore

$$Q \equiv \left(\overset{\wedge\wedge\wedge\wedge}{F}_{b,\,c,\,d} \right)_{ABCD}$$

A4-(13)

Therefore the right-hand-side of (1) is, despite its appearance, invariant under interchange of the subscripts $A\bar{a}$, Bb. It is clearly invariant under any permutation of the 3 subscripts Bb, Cc, Dd. It is therefore totally invariant under any permutation of all 4 subscripts $A\bar{a}$, Bb, Cc, Dd. Therefore the whole 3rd-order coupling coefficient for the warm-fluid plasma model is symmetric in the 4 interacting waves. The actual console session is reproduced on the following 10 pages.

macsym ∧K

THIS IS MACSYMA 250

FIX 250 DSK MACSYM BEING LOADED
LOADING DONE

(C1)   vecb1 : k[b]/w[b];

(D1)
$$\frac{K_B}{W_B}$$

(C2)   vecb1: vecb1 *(ℓ[c]/w[c] - ℓ[d]/w[d]);

(D2)
$$\frac{K_B(\frac{L_C}{W_C} - \frac{L_D}{W_D})}{W_B}$$

.

(C3) vecb2 : (k[d]+k[b]) /(w[d]+w[b]) ;

(D3)
$$\frac{K_D + K_B}{W_D + W_B}$$

(C4) vecb2 : vecb2*(ℓ[d]/w[d] - ℓ[b]/w[b]);

(D4)
$$\frac{(K_D + K_B) \left(\frac{L_D}{W_D} - \frac{L_B}{W_B}\right)}{W_D + W_B}$$

(C5) vecb3 : (k[b]+k[c]) /(w[b]+w[c]);

(D5)
$$\frac{K_C + K_B}{W_C + W_B}$$

(C6) vecb3 : vecb3*(ℓ[b]/w[b] - ℓ[c]/w[c]) ;

(D6)
$$\frac{(K_C + K_B) \left(\frac{L_B}{W_B} - \frac{L_C}{W_C}\right)}{W_C + W_B}$$

(C7) ℓineℓ : 70 $

(C8) vecb : (vecb1 + vecb2 + vecb3) /w[abar];

(D8)
$$\frac{(K_D + K_B)\,(\frac{L_D}{W_D} - \frac{L_B}{W_B})}{W_D + W_B} + \frac{K_B(\frac{L_C}{W_C} - \frac{L_D}{W_D})}{W_B} + \frac{(K_C + K_B)\,(\frac{L_B}{W_B} - \frac{L_C}{W_C})}{W_C + W_B}}{W_{ABAR}}$$

(C9)   genvec : subst([b = bb, c = cc, d = dd], vecb) ;

$$
(D9) \quad \left(\frac{(K_{DD} + K_{BB}) \, (\frac{L_{DD}}{W_{DD}} - \frac{L_{BB}}{W_{BB}})}{W_{DD} + W_{BB}} + \frac{K_{BB} \, (\frac{L_{CC}}{W_{CC}} - \frac{L_{DD}}{W_{DD}})}{W_{BB}} \right.
$$

$$
\left. + \frac{(K_{CC} + K_{BB}) \, (\frac{L_{BB}}{W_{BB}} - \frac{L_{CC}}{W_{CC}})}{W_{CC} + W_{BB}} \right) / W_{ABAR}
$$

(C10) vecc : subst([bb = c, cc = d, dd = b], genvec) ;

(D10)
$$\dfrac{\dfrac{(K_D + K_C)\,(\dfrac{L_C}{W_C} - \dfrac{L_D}{W_D})}{W_D + W_C} + \dfrac{K_C\,(\dfrac{L_D}{W_D} - \dfrac{L_B}{W_B})}{W_C} + \dfrac{(K_C + K_B)\,(\dfrac{L_B}{W_B} - \dfrac{L_C}{W_C})}{W_C + W_B}}{W_{ABAR}}$$

(C11) vecd : subst([bb = d, dd = c, cc = b], genvec) ;

(D11)
$$\dfrac{\dfrac{(K_D + K_C)\,(\dfrac{L_C}{W_C} - \dfrac{L_D}{W_D})}{W_D + W_C} + \dfrac{(K_D + K_B)\,(\dfrac{L_D}{W_D} - \dfrac{L_B}{W_B})}{W_D + W_B} + \dfrac{(\dfrac{L_B}{W_B} - \dfrac{L_C}{W_C})\,K_D}{W_D}}{W_{ABAR}}$$

(C12)   tvecb : subst([abar = bb, b = aabar], vecb)$

(C13)   tvecb : subst([bb = b, aabar = abar], tvecb) ;

(D13) $\left(\dfrac{(K_D + K_{ABAR})\, (\frac{L_D}{W_D} - \frac{L_{ABAR}}{W_{ABAR}})}{W_D + W_{ABAR}} + \dfrac{K_{ABAR}\, (\frac{L_C}{W_C} - \frac{L_D}{W_D})}{W_{ABAR}} \right.$

$\left. + \dfrac{(K_C + K_{ABAR})\, (\frac{L_{ABAR}}{W_{ABAR}} - \frac{L_C}{W_C})}{W_C + W_{ABAR}} \right) / W_B$

(C14)   tvecc : subst([abar = bb, b = aabar], vecc)$

(C15)   tvecc : subst([bb = b, aabar = abar], tvecc);

(D15) $(\dfrac{(K_D + K_C)\,(\frac{L_C}{W_C} - \frac{L_D}{W_D})}{W_D + W_C} + \dfrac{K_C\,(\frac{L_D}{W_D} - \frac{L_{ABAR}}{W_{ABAR}})}{W_C}$

$+ \;\dfrac{(K_C + K_{ABAR})\,(\frac{L_{ABAR}}{W_{ABAR}} - \frac{L_C}{W_C})}{W_C + W_{ABAR}})\, /W_B$

(C16)   tvecd : subst([abar = bb, b = aabar], vecd)\$

(C17)   tvecd : subst([bb = b, aabar = abar], tvecd);

$$
\text{(D17)} \quad \frac{(K_D + K_C)\,(\frac{L_C}{W_C} - \frac{L_D}{W_D})}{W_D + W_C} + \frac{(K_D + K_{ABAR})\,(\frac{L_D}{W_D} - \frac{L_{ABAR}}{W_{ABAR}})}{W_D + W_{ABAR}}
$$

$$
+ \frac{(\frac{L_{ABAR}}{W_{ABAR}} - \frac{L_C}{W_C})\,K_D}{W_D}) / W_B
$$

(C18) $\ell[\text{abar}]: -\ell[b] -\ell[c] -\ell[d]$;

(D18) $-L_D - L_C - L_B$

(C19) $k[\text{abar}]: -k[b] -k[c] -k[d]$;

(D19) $-K_D - K_C - K_B$

(C20) $w[\text{abar}]: -w[b] -w[c] -w[d]$;

(D20) $-W_D - W_C - W_B$

(C21)   ratsimp(ev(vecb - tvecb));

(D21)   0

(C22)   ratsimp(ev(vecc + tvecc - tvecb));

(D22)   0

(C23)   ratsimp(ev(vecd + tvecd - tvecb));

(D23)   0

A5. Coupling Coefficients From General Formulas of Section 2.8

The submatrices of the partitioned matrix equation 3.4-(13) will here be evaluated by using the general formula 2.8-(27) for the linear dispersion tensor, the general formula 2.8-(43) for the second-order nonlinear coupling coefficients, and the general formula 2.8-(52) for the third-order nonlinear coupling coefficients. Any intermediate quantities appearing in these formulae will be calculated from other equations of Section 2.8 as required. The formulae 2.8-(27, 43, 52) are used omitting the steady magnetic field $\vec{B}_0$ and omitting the drift velocity $\vec{v}_0$. Second-order nonlinear entries in (12) are evaluated from 2.8-(43) with the simplifications that $\vec{B}_0 = \vec{0}$, $\vec{v}_0 = \vec{0}$, and that $n_a/n_0 = 0$ for any wave $\vec{E}_a$ polarized tranversely to its own direction of propagation. Taking only the electron contributions, one obtains

$$E_{x1}\left(F^{NL(2)}_{(n),1}\right)_{ABX} = -\frac{mn_0}{\varepsilon_0}\left\{\frac{n_{A\overline{(n+1)}}}{n_0}\vec{v}_{B(n)}\cdot\vec{v}_{x1}\right.$$

$$\left. + \frac{n_{B(n)}}{n_0}\vec{v}_{A\overline{(n+1)}}\cdot\vec{v}_{x1}\right\} \quad \text{A5-(1)}$$

$$E_{x1}^{*}\left(F^{NL(2)}_{(n),-1}\right)_{ABX} = -\frac{mn_0}{\varepsilon_0}\left\{\frac{n_{A\overline{(n-1)}}}{n_0}\vec{v}_{B(n)}\cdot\vec{v}_{x1}^{*}\right.$$

$$\left. + \frac{n_{B(n)}}{n_0}\vec{v}_{A\overline{(n-1)}}\cdot\vec{v}_{x1}^{*}\right\} \quad \text{A5-(2)}$$

$$E_{z2}\left(F^{NL(2)}_{(-1),2}\right)_{ABZ} = -\frac{mn_0}{\varepsilon_0}\left\{\frac{n_{A(1)}}{n_0}\vec{v}_{B(-1)}\cdot\vec{v}_{z2} + \frac{n_{B(-1)}}{n_0}\vec{v}_{A(1)}\right.$$

$$\cdot\vec{v}_{z2} + \frac{N_{z2}}{n_0}\vec{v}_{A(1)}\cdot\vec{v}_{B(-1)}$$

$$\left. + \frac{\gamma(\gamma-2)v_T^2\, n_{A(1)}n_{B(-1)}N_{z2}}{n_0^3}\right\}$$

$$\text{A5-(3)}$$

$$E_{z2}^*\left(F^{NL(2)}_{(1),-2}\right)_{ABZ} = -\frac{mn_0}{\varepsilon_0}\left\{\frac{n_{A(-1)}}{n_0}\vec{v}_{B(1)}\cdot\vec{v}_{z2}^* + \frac{n_{B(1)}}{n_0}\vec{v}_{A(-1)}\right.$$

$$\cdot\vec{v}_{z2}^* + \frac{N_{z2}^*}{n_0}\vec{v}_{A(-1)}\cdot\vec{v}_{B(1)}$$

$$\left. + \frac{\gamma(\gamma-2)v_T^2\, n_{A(-1)}n_{B(1)}N_{z2}^*}{n_0^3}\right\}$$

$$\text{A5-(4)}$$

To calculate the normalized velocities $\{\vec{v}\}$ and densities $\{n\}$, refer to equations (45)-(50) of Section 2.8. In the absence of $\vec{B}_0$ and $\vec{v}_0$, the dimensionless mobility tensor $\overset{\Rightarrow}{M}$ as well as the dimensionless dispersion tensor $\overset{\Rightarrow}{D}^{LINEAR}$ is diagonal with our choice of basis polarizations. Thus for any unit basis polarization vector $\vec{e}_{Cc}$ say, the linear response $\vec{v}_{Cc}$ lies in the direction of $\vec{e}_{Cc}$ and satisfies

$$\vec{v}_{Cc} = \frac{iq}{mw_c}\vec{e}_{Cc} \qquad\qquad \vec{e}_{Cc}\perp\vec{k}_c \qquad \text{A5-(5)}$$

$$\vec{v}_{Cc} = \frac{iqw_c}{m(w_c^2 - k_c^2 v_T^2)} e_{Cc} \qquad \vec{e}_{Cc} \parallel \vec{k}_c \qquad \text{A5-(6)}$$

$$n_{Cc} = 0 \qquad \vec{e}_{Cc} \perp \vec{k}_c \qquad \text{A5-(7)}$$

$$n_{Cc} = \frac{iqk_c}{m(w_c^2 - k_c^2 v_T^2)} \qquad \vec{e}_{Cc} \parallel \vec{k}_c \qquad \text{A5-(8)}$$

Therefore the dot-products of velocities occurring in (1)-(4), which embody the effects of the three-dimensional geometry of the problem, translate immediately into dot-products of polarization vectors. Therefore those submatrices in 3.4-(12) which describe second-order nonlinear coupling can be written down at once. The second-order coupling which involves the second harmonic of the laser-pump is, from (3) and (4), described by the submatrices (9) and (10).

The third-order nonlinear entries in 3.4-(12) are evaluated from 2.8-(52) taking advantage of the facts that $\vec{B}_0 = \vec{0}$, $\vec{v}_0 = \vec{0}$ and that $n_a/n_0 = 0$ for any wave E_a polarized transversely to its own direction of propagation. Also no nonlinear response exists with both zero wavevector and zero frequency.

The third-order nonlinear entries on the leading diagonal of 3.4-(12) are given by the terms surviving after substitution into 2.8-(52);

$$|E_{x1}|^2 \left(F_{(n),1,-1}^{NL(3)} \right)_{ABxx} = -\frac{mn_0}{\varepsilon_0} |E_{x1}|^2 \left[\frac{n_{A(n)}}{n_0} \vec{v}_{x1} \cdot \vec{v}_{B(n),x1} \right.$$

$$+ \vec{v}_{A(\overline{n}),x\bar{1}} \cdot \frac{n_{B(n)}}{n_0} \vec{v}_{x1} + \vec{v}_{A(\overline{n}),x\bar{1}} \cdot \vec{v}_{B(n),x1} + \frac{n_{A(\overline{n})}}{n_0} \vec{v}_{x1} \cdot \vec{v}_{B(\overline{n}),x\bar{1}}$$

$$+ \vec{v}_{A(\overline{n}),x1} \cdot \frac{n_{B(n)}}{n_0} \vec{v}_{x\bar{1}} + \vec{v}_{A(\overline{n}),x1} \cdot \vec{v}_{B(n),x\bar{1}} - \frac{v_T^2}{n_0^2}\left(n_{A(\overline{n}),x\bar{1}} \; n_{B(n),x1} \right.$$

$$\left. + n_{A(\overline{n}),x1} \; n_{B(n),x\bar{1}} \right) \Bigg] \qquad\qquad \text{A5-(11)}$$

The second-order $\left\{n\right\}$ are expressed in terms of the second-order $\left\{\vec{v}\right\}$ by 2.8-(41). The second-order $\left\{\vec{v}\right\}$ are expressed in terms of first-order quantities most conveniently by A3-(9). The first-order quantities satisfy (5)-(8) because of the choice of basis polarizations. Use 2.8-(41) to express the second-order $\left\{n\right\}$ in terms of the second-order $\left\{\vec{v}\right\}$;

$$\left| E_{x1} \right|^2 \left(F^{NL(3)}_{(n),1,-1} \right)_{ABxx} = -\frac{mn_0}{\varepsilon_0} \left| E_{x1} \right|^2 \Bigg[\left(\vec{v}_{A(n),x\bar{1}} + \frac{n_{A(\overline{n})}}{n_0} \vec{v}_{x\bar{1}} \right)$$

$$\cdot \left(1 - \frac{\gamma v_T^2 \vec{k}_{(n+1)} \vec{k}_{(n+1)}}{w^2_{(n-1)}} \right) \cdot \left(\vec{v}_{B(n),x1} + \frac{n_{B(n)}}{n_0} \vec{v}_{x1} \right)$$

$$+ \left(\vec{v}_{A(n),x1} + \frac{n_{A(\overline{n})}}{n_0} \vec{v}_{x1} \right) \cdot \left(1 - \frac{\gamma v_T^2 \vec{k}_{(n-1)} \vec{k}_{(n+1)}}{w^2_{(n-1)}} \right)$$

$$\cdot \left(\vec{v}_{B(n),x\bar{1}} + \frac{n_{B(n)}}{n_0} \vec{v}_{x\bar{1}} \right) - 2 \frac{n_{A(\overline{n})}}{n_0} \frac{n_{B(n)}}{n_0} \vec{v}_{x\bar{1}} \cdot \vec{v}_{x1} \Bigg] \qquad \text{A5-(12)}$$

$$E_{z2}\vec{\vec{F}}^{NL(2)}_{(-1),2} = -\omega_p^2 \vec{V}_{z2}$$

$$
\begin{pmatrix}
\dfrac{[k_2(\vec{e}_{M(1)}\cdot\vec{e}_{M(-1)})]}{[w_2 w_{(1)} w_{(-1)}]} &
\dfrac{[k_2(\vec{e}_{M(1)}\cdot e_{N(-1)})]}{[w_2 w_{(1)} w_{(-1)}]} &
\dfrac{[w_2 k_{(-1)}(\vec{e}_{M(1)}\cdot\vec{e}_{z2}) + w_{(-1)} k_2(\vec{e}_{M(1)}\cdot\vec{e}_{S(-1)})]}{[w_2 w_{(1)}(w_{(-1)}^2 - \gamma k_{(-1)}^2 V_T^2)]} \\[4ex]
\dfrac{[k_2(\vec{e}_{N(1)}\cdot\vec{e}_{M(-1)})]}{[w_2 w_{(1)} w_{(-1)}]} &
\dfrac{[k_2(e_{N(1)}\cdot\vec{e}_{N(-1)})]}{[w_2 w_{(1)} w_{(-1)}]} &
\dfrac{[w_2 k_{(-1)}(\vec{e}_{N(-1)}\cdot\vec{e}_{z2}) + k_2 w_{(-1)}(\vec{e}_{N(1)}\cdot\vec{e}_{S(-1)})]}{[w_2 w_{(-1)}(w_{(-1)}^2 - \gamma k_{(-1)}^2 V_T^2)]} \\[4ex]
\dfrac{[w_2 k_{(1)}(\vec{e}_{M(-1)}\cdot\vec{e}_{z2}) + k_2 w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{M(-1)})]}{[w_2 w_{(-1)}(w_{(1)}^2 - \gamma k_{(1)}^2 V_T^2)]} &
\dfrac{[w_2 k_{(1)}(\vec{e}_{N(-1)}\cdot\vec{e}_{z2}) + k_2 w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{N(-1)})]}{[w_2 w_{(-1)}(w_{(1)}^2 - \gamma k_{(1)}^2 V_T^2)]} &
\dfrac{\begin{array}{l}[w_2 k_{(1)} w_{(-1)}(\vec{e}_{S(-1)}\cdot\vec{e}_{z2}) \\ + w_2 k_{(-1)} w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{z2}) \\ + k_2 w_{(1)} w_{(-1)}(\vec{e}_{S(1)}\cdot\vec{e}_{S(-1)}) \\ + \gamma(\gamma-2) V_T^2 k_{(1)} k_{(-1)} k_2]\end{array}}{[w_2(w_{(-1)}^2 - \gamma k_{(-1)}^2 V_T^2)(w_{(1)}^2 - \gamma k_{(1)}^2 V_T^2)]}
\end{pmatrix}
$$

$$\overset{*}{E}_{zz}\,\overset{\rightleftharpoons}{\vec{F}}{}^{NL(2)}_{(1),-2} = -\omega_p^2\,\overset{*}{V}_{z2}$$

$$
\begin{pmatrix}
\dfrac{[k_2(\vec{e}_{M(-1)}\cdot\vec{e}_{M(1)})]}{[w_2 W_{(1)}W_{(-1)}]} &
\dfrac{[k_2(\vec{e}_{N(1)}\cdot e_{M(-1)})]}{[w_2 W_{(1)}W_{(-1)}]} &
\dfrac{[w_2 k_{(1)}(\vec{e}_{M(-1)}\cdot\vec{e}_{z2}) + w_{(1)}k_2(\vec{e}_{S(1)}\cdot\vec{e}_{M(-1)})]}{[w_2 w_{(-1)}(w_{(1)}^2 - \gamma k_{(1)}^2 V_T^2)]} \\[4ex]
\dfrac{[k_2(\vec{e}_{M(1)}\cdot\vec{e}_{N(-1)})]}{[w_2 W_{(1)}W_{(-1)}]} &
\dfrac{[k_2(\vec{e}_{N(1)}\cdot\vec{e}_{N(-1)})]}{[w_2 W_{(1)}W_{(-1)}]} &
\dfrac{[w_2 k_{(1)}(\vec{e}_{N(-1)}\cdot\vec{e}_{z2}) + k_2 w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{N(-1)})]}{[w_2 w_{(-1)}(w_{(1)}^2 - \gamma k_{(1)}^2 V_T^2)]} \\[4ex]
\dfrac{[w_2 k_{(-1)}(\vec{e}_{M(1)}\cdot\vec{e}_{z2}) + k_2 w_{(-1)}(\vec{e}_{S(-1)}\cdot\vec{e}_{M(1)})]}{[w_2 w_{(1)}(w_{(-1)}^2 - \gamma k_{(-1)}^2 V_T^2)]} &
\dfrac{[w_2 k_{(-1)}(\vec{e}_{N(1)}\cdot\vec{e}_{z2}) + k_2 w_{(-1)}(\vec{e}_{N(1)}\cdot\vec{e}_{S(-1)})]}{[w_2 w_{(1)}(w_{(-1)}^2 - \gamma k_{(-1)}^2 V_T^2)]} &
\dfrac{[w_2 k_{(1)}w_{(-1)}(\vec{e}_{S(-1)}\cdot\vec{e}_{z2}) + w_2 k_{(-1)}w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{z2}) + k_2 w_{(1)}w_{(-1)}(\vec{e}_{S(1)}\cdot\vec{e}_{S(-1)}) + \gamma(\gamma-2)V_T^2 k_{(1)}k_{(-1)}k_2]}{[w_2(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2)(w_{(-1)}^2 - \gamma k_{(-1)}^2 V_T^2)]}
\end{pmatrix}
$$

Now express the second-order $\left\{\vec{v}\right\}$ in terms of first-order quantities using A3-(9);

$$|E_{xl}|^2\left(F^{NL(3)}_{(n),1,-1}\right)_{ABxx} = -\frac{mn_0}{\varepsilon_0}|E_{xl}|^2$$

$$\left[\left(\frac{\vec{k}_{(n+1)}}{w_{(n+1)}}\vec{v}_{A\overline{(n)}}\cdot\vec{v}_{x\overline{1}} + \frac{n_{A\overline{(n)}}}{n_0}\vec{v}_{x\overline{1}}\right)\cdot\left(1 - \frac{\gamma v_T^2\vec{k}_{(n+1)}\vec{k}_{(n+1)}}{w^2_{(n+1)}}\right)^{-1}\right.$$

$$\cdot\left(\frac{\vec{k}_{(n+1)}}{w_{(n+1)}}\vec{v}_{B(n)}\cdot\vec{v}_{x1} + \frac{n_{B(n)}}{n_0}\vec{v}_{x1}\right) + \left(\frac{\vec{k}_{(n-1)}}{w_{(n-1)}}\vec{v}_{A\overline{(n)}}\cdot\vec{v}_{x1} + \frac{n_{A\overline{(n)}}}{n_0}\vec{v}_{x1}\right)$$

$$\cdot\left(1 - \frac{\gamma v_T^2\vec{k}_{(n-1)}\vec{k}_{(n-1)}}{w^2_{(n-1)}}\right)^{-1}\cdot\left(\frac{\vec{k}_{(n-1)}}{w_{(n-1)}}\vec{v}_{B(n)}\cdot\vec{v}_{x\overline{1}} + \frac{n_{B(n)}}{n_0}\vec{v}_{x\overline{1}}\right)$$

$$\left. - \frac{n_{A\overline{(n)}}}{n_0}\vec{v}_{x\overline{1}}\cdot\frac{n_{B(n)}}{n_0}\vec{v}_{x1} - \frac{n_{A\overline{(n)}}}{n_0}\vec{v}_{x1}\cdot\frac{n_{B(n)}}{n_0}\vec{v}_{x\overline{1}}\right] \qquad \text{A5-(13)}$$

Use the operator identities which are valid for our choice of basis (see 3.4)

$$\left(1 - \frac{\gamma v_T^2\vec{k}_{(n+1)}\vec{k}_{(n+1)}}{w^2_{(n+1)}}\right)^{-1} \equiv \vec{e}_{M(n+1)}\vec{e}_{M(n+1)} + \vec{e}_{N(n+1)}\vec{e}_{N(n+1)}$$

$$+ \vec{e}_{s(n+1)}\vec{e}_{s(n+1)}\frac{w^2_{(n+1)}}{w^2_{(n+1)}-\gamma k^2_{(n+1)}v_T^2}$$

$$\equiv 1 + \frac{\gamma k^2_{(n+1)}v_T^2}{w^2_{(n+1)}-\gamma k^2_{(n+1)}v_T^2}\vec{e}_{s(n+1)}\vec{e}_{s(n+1)}$$

$$\text{A5-(14)}$$

Express the first-order $\left\{n\right\}$ in terms of the first-order $\left\{\vec{v}\right\}$ using 2.8-(19). Then (13) becomes

$$|E_{x1}|^2\left(F^{NL(3)}_{(n),1,-1}\right)_{ABxx} = -\frac{mn_0}{\varepsilon_0}$$

$$\left[\frac{w^2_{(n+1)}}{w^2_{(n+1)}-\gamma k^2_{(n+1)}v^2_T}\left\{\frac{k^2_{(n+1)}}{w^2_{(n+1)}}\left(\vec{v}_{A(n)}\cdot\vec{v}_{x\overline{1}}\right)\left(\vec{v}_{B(n)}\cdot\vec{v}_{x1}\right)+\left(\vec{v}_{A(n)}\cdot\vec{v}_{x\overline{1}}\right)\right.\right.$$

$$\left(\frac{\vec{k}_{(n)}}{w_{(n)}}\vec{v}_{B(n)}\right)\left(\frac{\vec{k}_{(n+1)}}{w_{(n+1)}}\vec{v}_{x1}\right)+\left(\vec{v}_{B(n)}\vec{v}_{x1}\right)\left(\frac{\vec{k}_{(n)}}{w_{(n)}}\cdot\vec{v}_{A(n)}\right)\left(\frac{\vec{k}_{(n+1)}}{w_{(n+1)}}\cdot\vec{v}_{x\overline{1}}\right)$$

$$+\frac{\gamma k^2_{(n+1)}v^2_T}{w^2_{(n+1)}-\gamma k^2_{(n+1)}v^2_T}\left\{\left(\frac{\vec{k}_{(n)}}{w_{(n)}}\cdot\vec{v}_{A(n)}\right)\left(\frac{\vec{k}_{(n)}}{w_{(n)}}\cdot\vec{v}_{B(n)}\right)\left(\vec{v}_{x\overline{1}}\cdot\vec{e}_{s(n+1)}\right)\right.$$

$$\left.\left(\vec{v}_{x1}\cdot\vec{e}_{s(n+1)}\right)\right\}+\frac{w^2_{(n-1)}}{w^2_{(n-1)}-\gamma k^2_{(n-1)}v^2_T}\left\{\frac{k^2_{(n-1)}}{w^2_{(n-1)}}\left(\vec{v}_{A(n)}\cdot\vec{v}_{x1}\right)\left(\vec{v}_{B(n)}\cdot\vec{v}_{x\overline{1}}\right)\right.$$

$$+\left(\vec{v}_{A(n)}\cdot\vec{v}_{x1}\right)\left(\frac{\vec{k}_{(n)}}{w_{(n)}}\cdot\vec{v}_{B(n)}\right)\left(\frac{\vec{k}_{(n-1)}}{w_{(n-1)}}\cdot\vec{v}_{x\overline{1}}\right)+\left(\vec{v}_{B(n)}\cdot\vec{v}_{x\overline{1}}\right)\left(\frac{\vec{k}_{(n)}}{w_{(n)}}\cdot\vec{v}_{A(n)}\right)$$

$$\left(\frac{\vec{k}_{(n-1)}}{w_{(n-1)}}\cdot\vec{v}_{x1}\right)\right\}+\frac{\gamma k^2_{(n-1)}v^2_T}{w^2_{(n-1)}-\gamma k^2_{(n-1)}v^2_T}\left\{\left(\frac{\vec{k}_{(n)}}{w_{(n)}}\cdot\vec{v}_{A(n)}\right)\left(\frac{\vec{k}_{(n)}}{w_{(n)}}\cdot\vec{v}_{B(n)}\right)\right.$$

$$\left.\left.\left(\vec{v}_{x1}\cdot\vec{e}_{s(n+1)}\right)\left(\vec{v}_{x1}\cdot\vec{e}_{s(n+1)}\right)\right\}\right] \qquad \text{A5-(15)}$$

Let A,B range independently over the polarizations M, N, S.

Use the facts that

$$\vec{v}_{x1} = v_{x1}\vec{e}_x$$
$$\vec{v}_{x\overline{1}}^* = v^*_{x1}\vec{e}_x \qquad \text{A5-(16)}$$

$$\ldots]^2 \; \vec{F}_{(n),1,-1}^{NL(3)} = -|V_1^2|\,\omega_P^2$$

Left upper block:

$$\left. k_{(n+1)}^2 \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)^2\right] \big/$$
$$w_{(n)}^2 \left(w_{(n+1)}^2 - \gamma k_{(n+1)}^2 V_T^2\right)\Big]$$
$$\left[k_{(n-1)}^2 \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)^2\right] \big/$$
$$\left[w_{(n)}^2 \left(w_{(n-1)}^2 - \gamma k_{(n-1)}^2 V_T^2\right)\right]$$

Center column: 0

Right upper block:

$$\left[k_{(n+1)}^2 w_{(n)}^2 \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n)}\cdot\vec{e}_{x1}\right) + \right.$$
$$\left. k_{(n)} k_{(n+1)} w_{(n)} w_{(n+1)} \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n+1)}\cdot\vec{e}_{x1}\right)\right] \big/$$
$$\left[w_{(n)}^2 \left(w_{(n+1)}^2 - \gamma k_{(n+1)}^2 V_T^2\right)\left(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2\right)\right]$$
$$+ \left[k_{(n-1)}^2 w_{(n)}^2 \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n)}\cdot\vec{e}_{x1}\right) + \right.$$
$$\left. k_{(n)} k_{(n-1)} w_{(n)} w_{(n-1)} \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n-1)}\cdot\vec{e}_{x1}\right)\right] \big/$$
$$\left[w_{(n)}^2 \left(w_{(n-1)}^2 - \gamma k_{(n-1)}^2 V_T^2\right)\left(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2\right)\right]$$

Middle row: $0 \qquad 0 \qquad 0$

Left lower block:

$$k_{(n+1)}^2 w_{(n)}^2 \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n)}\cdot\vec{e}_{x1}\right) + $$
$$k_{(n)} k_{(n+1)} w_{(n)} w_{(n+1)} \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n+1)}\cdot\vec{e}_{x1}\right)\right] \big/$$
$$\left. w_{(n)}^2 \left(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2\right)\left(w_{(n+1)}^2 - \gamma k_{(n+1)}^2 V_T^2\right)\right]$$
$$k_{(n-1)}^2 w_{(n)}^2 \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n)}\cdot\vec{e}_{x1}\right) + $$
$$k_{(n)} k_{(n-1)} w_{(n)} w_{(n-1)} \left(\vec{e}_{m(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n-1)}\cdot\vec{e}_{x1}\right)\right] \big/$$
$$w_{(n)}^2 \left(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2\right)\left(w_{(n-1)}^2 - \gamma k_{(n-1)}^2 V_T^2\right)\right]$$

Center column lower: 0

Right lower block:

$$\left[k_{(n+1)}^2 w_{(n)}^2 \left(\vec{e}_{s(n)}\cdot\vec{e}_{x1}\right)^2 + \right.$$
$$2 k_{(n)} k_{(n+1)} w_{(n)} w_{(n+1)} \left(\vec{e}_{s(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n+1)}\cdot\vec{e}_{x1}\right)$$
$$\left. + k_{(n)}^2 \gamma k_{(n+1)}^2 V_T^2 \left(\vec{e}_{s(n+1)}\cdot\vec{e}_{x1}\right)^2\right] \big/$$
$$\left[w_{(n)}^2 \left(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2\right)^2 \left(w_{(n+1)}^2 - \gamma k_{(n+1)}^2 V_T^2\right)\right]$$
$$+ \left[k_{(n-1)}^2 w_{(n)}^2 \left(\vec{e}_{s(n)}\cdot\vec{e}_{x1}\right)^2 + \right.$$
$$2 k_{(n)} k_{(n-1)} w_{(n)} w_{(n-1)} \left(\vec{e}_{s(n)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(n-1)}\cdot\vec{e}_{x1}\right)$$
$$\left. + k_{(n)}^2 \gamma k_{(n-1)}^2 V_T^2 \left(\vec{e}_{s(n-1)}\cdot\vec{e}_{x1}\right)^2\right] \big/$$
$$\left[w_{(n)}^2 \left(w_{(n)}^2 - \gamma k_{(n)}^2 V_T^2\right)^2 \left(w_{(n-1)}^2 - \gamma k_{(n-1)}^2 V_T^2\right)\right]$$

A5

and the forms (5,6) for the first-order velocities. One generates from (16) precisely the elements of (17).

These submatrices on the diagonal of 3.4-(12) comprise the pump-induced self-corrections to the linear dispersion tensors at the perturbation wavevectors and frequencies. Their importance for laser-driven instabilities was discussed in the latter part of Section 3.4. The self-correction to the low-frequency dispersion tensor, namely the submatrix

$$\left|E_1\right|^2 \; \overset{\leftrightarrow}{F}{}^{NL(3)}_{(0),1,-1} \qquad\qquad A5\text{-}(18)$$

plays no important role. The self-corrections to the high-frequency dispersion tensors, namely the submatrices

$$\left|E_1\right|^2 \; \vec{F}{}^{NL(3)}_{(+1),1,-1} \qquad\qquad A5\text{-}(19)$$

are important for laser-driven instabilities. However, not all the terms in the explicit form (15) need be used. Rather, each of (17) may be expressed as the sum of two submatrices, say

$$\left|E_1\right|^2 \; \overset{\leftrightarrow}{F}{}^{NL(3)}_{(\underline{+}1),1,-1} = \left|E_1\right|^2 \; \overset{\leftrightarrow}{F}{}^{\Delta\,NL(3)}_{(\underline{+}1),1,-1} + \left|E_1\right|^2 \; \overset{\leftrightarrow}{F}{}^{\blacktriangle\,NL(3)}_{(\underline{+}1),1,-1}$$

$$A5\text{-}(20)$$

Here the first object on the right-hand side describes processes which involve the low-frequency response of the plasma at $(k_{(o)}, w_{(o)})$ and is displayed in (21). The second object on the right-hand side describes processes which involve the

$$|E_1|^2 F^{NL(3)}_{(\pm 1),\,1,\,-1} = -w_p^2 |V_1|^2$$

$$
\begin{pmatrix}
\dfrac{\left[k^2_{(\pm 2)}\left(\vec{e}_{m(\pm 2)}\cdot\vec{e}_{x1}\right)^2\right]}{\left[w^2_{(\pm 1)}\left(w^2_{(\pm 2)}-\gamma k^2_{(\pm 2)}V_T^2\right)\right]} & 0 & \dfrac{\left[k^2_{(\pm 2)}w^2_{(\pm 1)}\left(\vec{e}_{m(\pm 1)}\cdot\vec{e}_{x1}\right)\left(e_{s(\pm 1)}\cdot\vec{e}_{x1}\right)+k_{(\pm 1)}k_{(\pm 2)}w_{(\pm 1)}w_{(\pm 2)}\left(\vec{e}_{m(\pm 1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s(\pm 2)}\cdot\vec{e}_{x1}\right)\right]}{\left[w^2_{(\pm 1)}\left(w^2_{(\pm 1)}-\gamma k^2_{(\pm 1)}V_T^2\right)\left(w^2_{(\pm 2)}-\gamma k^2_{(\pm 2)}V_T^2\right)\right]} \\[3em]
0 & 0 & 0 \\[2em]
\dfrac{\left[k^2_{(\pm 2)}w^2_{(\pm 1)}\left(\vec{e}_{m(\pm 1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s'(\pm 1)}\cdot\vec{e}_{x1}\right)+k_{(\pm 2)}k_{(\pm 1)}w_{(\pm 2)}w_{(\pm 1)}\left(\vec{e}_{m(\pm 1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s'(\pm 2)}\cdot\vec{e}_{x1}\right)\right]}{\left[w^2_{(\pm 1)}\left(w^2_{(\pm 1)}-\gamma k^2_{(\pm 1)}V_T^2\right)\left(w^2_{(\pm 2)}-\gamma k^2_{(\pm 2)}V_T^2\right)\right]} & 0 & \dfrac{\left[k^2_{(\pm 2)}w^2_{(\pm 1)}\left(\vec{e}_{s'(\pm 1)}\cdot\vec{e}_{x1}\right)^2+2k_{(\pm 2)}k_{(\pm 1)}w_{(\pm 2)}w_{(\pm 1)}\left(\vec{e}_{s'(\pm 1)}\cdot\vec{e}_{x1}\right)\left(\vec{e}_{s'(\pm 2)}\cdot\vec{e}_{x1}\right)+k^2_{(\pm 1)}\gamma k^2_{(\pm 2)}V_T^2\left(\vec{e}_{s'(\pm 2)}\cdot e_{x1}\right)^2\right]}{\left[w^2_{(\pm 1)}\left(w^2_{(\pm 1)}-\gamma k^2_{(\pm 1)}V_T^2\right)\left(w^2_{(\pm 2)}-\gamma k^2_{(\pm 2)}V_T^2\right)\right]}
\end{pmatrix}
$$

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high-frequency response of the plasma at $(k_{(\underline{+}2)}, w_{(\underline{+}2)})$, and is displayed in (22).

It is also required to calculate the submatrices appearing in the top-right and bottom-left corners of 3.4-(12). For zero magnetic field, zero drift-velocity, and transversely polarized pump these submatrices have elements given by the sum of (9,10) and (23,24) below. The third-order nonlinear entries in the top-right and bottom-left corners of 3.4-(12) are given by the terms surviving after substitution into 2.8-(52);

$$\frac{1}{2} E_{x1}^2 \left(F^{NL(3)}_{(-1),1,1} \right)_{ABxx} = -\frac{mn_0}{\varepsilon_0} E_{x1}^2$$

$$\left[\frac{n_{A\overline{(1)}}}{n_0} \vec{v}_{x1} \cdot \vec{v}_{B(-1),x1} + \vec{v}_{A\overline{(1)},x1} \cdot \frac{n_{B(-1)}}{n_0} \vec{v}_{x1} \right.$$

$$\left. + \vec{v}_{A\overline{(1)},x1} \cdot \vec{v}_{B(-1),x1} - \frac{\gamma v_T^2}{n_0} n_{A\overline{(1)},x1} \, n_{B(-1),x1} \right] - \frac{1}{2} \frac{mn_0}{\varepsilon_0} E_{x1}^2$$

$$\left[\frac{n_{A\overline{(1)}}}{n_0} \vec{v}_{B(-1)} \cdot \vec{v}_{x1,x1} + \frac{n_{B(-1)}}{n_0} \vec{v}_{A\overline{(1)}} \cdot \vec{v}_{x1,x1} + \vec{v}_{A\overline{(1)},B(-1)} \cdot \vec{v}_{x1,x1} \right.$$

$$\left. - \frac{\gamma v_T^2}{n_0} n_{A\overline{(1)},B(-1)} n_{x1,x1} \right] \qquad\qquad \text{A5-(23)}$$

$$\frac{1}{2} E_{x1}^{*2} \left(F_{(1),-1,-1}^{NL(3)} \right)_{ABxx} = -\frac{mn_0}{\epsilon_0} E_{x1}^{*2}$$

$$\left[\frac{n_{A\overline{(-1)}}}{n_0} \vec{v}_{x\overline{I}} \cdot \vec{v}_{B(1),x\overline{I}} + \vec{v}_{A\overline{(-1)},x\overline{I}} \cdot \frac{n_{B(1)}}{n_0} \vec{v}_{x\overline{I}} + \vec{v}_{A\overline{(-1)},x\overline{I}} \cdot \vec{v}_{B(1),x\overline{I}} \right.$$

$$\left. - \frac{\gamma v_T^2}{n_0} n_{A\overline{(-1)},x\overline{I}} \; n_{B(1),x\overline{I}} \right] - \frac{1}{2} \frac{mn_0}{\epsilon_0} E_{x1}^{*2}$$

$$\left[\frac{n_{A\overline{(-1)}}}{n_0} \vec{v}_{B(1)} \cdot \vec{v}_{x\overline{I},x\overline{I}} + \frac{n_{B(1)}}{n_0} \vec{v}_{A\overline{(-1)}} \cdot \vec{v}_{x\overline{I},x\overline{I}} + \vec{v}_{A\overline{(-1)},B(1)} \vec{v}_{x\overline{I},x\overline{I}} \right.$$

$$\left. - \frac{\gamma v_T^2}{n_0} n_{A\overline{(-1)},B(1)} n_{x\overline{I},x\overline{I}} \right] \qquad\qquad\qquad\qquad A5\text{-}(24)$$

Again the second-order $\left\{ n \right\}$ are expressed in terms of the second-order $\left\{ \vec{v} \right\}$ by 2.8-(41). The second-order $\left\{ \vec{v} \right\}$ are expressed in terms of first-order quantities most conveniently by A3-(9). The first-order quantities satisfy (5)-(8) since the basis polarizations are eigenvectors of their respective mobility tensors M . Split (23,24) into $\overset{\triangle}{F}$ terms not involving the quantities $(\vec{v}_{x1,x1}, n_{x1,x1}, \vec{v}_{x\overline{I},x\overline{I}}, n_{x\overline{I},x\overline{I}})$ and $\overset{\blacktriangle}{F}$ terms which do involve these quantities. Use 2.8-(41) to express all second-order $\left\{ n \right\}$ in terms of second-order $\left\{ \vec{v} \right\}$;

$$\frac{1}{2} E_{x1}^2 \left(\overset{\Delta}{F}{}^{NL(3)}_{(-1),1,1} \right)_{ABxx} = -\frac{mn_0}{\epsilon_0} E_{x1}^2$$

$$\left[\left(\vec{v}_{\overline{A(1)},x1} + \frac{n_{\overline{A(1)}}}{n_0} \vec{v}_{x1} \right) \cdot \left(1 - \frac{\gamma v_T^2 \vec{k}_{(o)} \vec{k}_{(o)}}{w_{(o)}^2} \right) \cdot \left(\vec{v}_{B(-1),x1} + \frac{n_{B(-1)}}{n_0} \vec{v}_{x1} \right) \right.$$

$$\left. - \frac{n_{\overline{A(1)}}}{n_0} \vec{v}_{x1} \cdot \frac{n_{B(1)}}{n_0} \vec{v}_{x1} \right] \qquad\qquad \text{A5-(25)}$$

$$\frac{1}{2} E_{x1}^{*2} \left(\overset{\Delta}{F}{}^{NL(3)}_{(1),-1,-1} \right)_{ABxx} = -\frac{mn_0}{\epsilon_0} E_{x1}^{*2}$$

$$\left[\left(\vec{v}_{\overline{A(-1)},x\overline{1}} + \frac{n_{\overline{A(-1)}}}{n_0} \vec{v}_{x\overline{1}} \right) \cdot \left(1 - \frac{\gamma v_T^2 \vec{k}_{(o)} \vec{k}_{(o)}}{w_{(o)}^2} \right) \cdot \left(\vec{v}_{B(1),x\overline{1}} + \frac{n_{B(1)}}{n_0} \vec{v}_{x\overline{1}} \right) \right.$$

$$\left. - \frac{n_{\overline{A(-1)}}}{n_0} \vec{v}_{x\overline{1}} \cdot \frac{n_{B(1)}}{n_0} \vec{v}_{x\overline{1}} \right] \qquad\qquad \text{A5-(26)}$$

$$\frac{1}{2} E_{x1}^2 \left(\overset{\Delta}{F}{}^{NL(3)}_{(-1),1,1} \right)_{ABxx} = -\frac{1}{2} \frac{mn_0}{\epsilon_0} E_{x1}^2$$

$$\left[\left(\vec{v}_{\overline{A(1)},B(-1)} + \frac{n_{\overline{A(1)}}}{n_0} \vec{v}_{B(-1)} + \frac{n_{B(-1)}}{n_0} \vec{v}_{\overline{A(1)}} \right) \cdot \left(1 - \frac{\gamma v_T^2 \vec{k}_2 \vec{k}_2}{w_2^2} \right) \cdot \vec{v}_{x1,x1} \right]$$

$$\text{A5-(27)}$$

$$\frac{1}{2} E_{x1}^{*2} \left(\overset{\Delta}{F}{}^{NL(3)}_{(1),-1,-1} \right)_{ABxx} = -\frac{1}{2} \frac{mn_0}{\epsilon_0} E_{x1}^{*2}$$

$$\left[\left(\vec{v}_{\overline{A(-1)},B(1)} + \frac{n_{\overline{A(-1)}}}{n_0} \vec{v}_{B(1)} + \frac{n_{B(1)}}{n_0} \vec{v}_{A(-1)} \right) \cdot \left(1 - \frac{\gamma v_T^2 \vec{k}_2 \vec{k}_2}{w_2^2} \right) \cdot \vec{v}_{x\overline{1},x\overline{1}} \right]$$

$$\text{A5-(28)}$$

Now express the second-order $\{\vec{v}\}$ in terms of first-order

quantities using A3-(9);

$$\frac{1}{2} E_{x1}^2 \left(\underset{F(-1),1,1}{\Delta NL(3)}\right)_{ABxx} = \frac{mn_0}{\epsilon_0} E_{x1}^2$$

$$\left[\left(\frac{\vec{k}_{(o)}}{w_{(o)}}\left(\vec{v}_{A(1)}\cdot\vec{v}_{x1}\right) + \frac{n_{A(1)}}{n_0}\vec{v}_{x1}\right)\cdot\left(1 - \frac{\gamma v_T^2 \vec{k}_{(o)}\vec{k}_{(o)}}{w_{(o)}^2}\right)^{-1}\right.$$

$$\left.\cdot\left(\frac{\vec{k}_{(o)}}{w_{(o)}}\left(\vec{v}_{B(-1)}\cdot\vec{v}_{x1}\right) + \frac{n_{B(-1)}}{n_0}\vec{v}_{x1}\right) - \frac{n_{A(1)}}{n_0}\vec{v}_{x1}\cdot\frac{n_{B(1)}}{n_0}\vec{v}_{x1}\right] \quad \text{A5-(29)}$$

$$\frac{1}{2} E_{x1}^{*2} \left(\underset{F(1),-1,-1}{\Delta NL(3)}\right)_{ABxx} = \frac{mn_0}{\epsilon_0} E_{x1}^{*2}$$

$$\left[\left(\frac{\vec{k}_{(o)}}{w_{(o)}}\vec{v}_{A(-1)}\vec{v}_{x\bar{1}} + \frac{n_{A(-1)}}{n_0}\vec{v}_{x\bar{1}}\right)\cdot\left(1 - \frac{\gamma v_T^2 \vec{k}_{(o)}\vec{k}_{(o)}}{w_{(o)}^2}\right)^{-1}\right.$$

$$\left.\left(\frac{\vec{k}_{(o)}}{w_{(o)}}\left(\vec{v}_{B(1)}\cdot\vec{v}_{x\bar{1}}\right) + \frac{n_{B(1)}}{n_0}\vec{v}_{x\bar{1}}\right) - \frac{n_{A(-1)}}{n_0}\vec{v}_{x\bar{1}}\cdot\frac{n_{B(1)}}{n_0}\vec{v}_{x\bar{1}}\right] \quad \text{A5-(30)}$$

$$\frac{1}{2} E_{x1}^2 \left(\underset{F(-1),1,1}{\Delta NL(3)}\right)_{ABxx} = -\frac{1}{2}\frac{mn_0}{\epsilon_0} E_{x1}^2$$

$$\left[\left(\frac{\vec{k}_2}{w_2}\left(\vec{v}_{A(1)}\cdot\vec{v}_{B(-1)}\right) + \frac{n_{A(1)}}{n_0}\vec{v}_{B(-1)} + \frac{n_{B(-1)}}{n_0}\vec{v}_{A(1)}\right)\right.$$

$$\left.\left(1 - \frac{\gamma v_T^2 \vec{k}_2 \vec{k}_2}{w_2^2}\right)^{-1}\cdot\frac{\vec{k}_2}{w_2}v_{x1}^2\right] \quad \text{A5-(31)}$$

$$\frac{1}{2} E_{x1}^{*2} \left(F_{(1),-1,-1}^{NL(3)} \right)_{ABxx} = -\frac{1}{2} \frac{mn_0}{\varepsilon_0} E_{x1}^{*2}$$

$$\left[\left(\frac{\vec{k}_2}{w_2} \left(\vec{v}_{A(-1)} \cdot \vec{v}_{B(1)} \right) + \frac{n_{A(-1)}}{n_0} \vec{v}_{B(1)} + \frac{n_{B(1)}}{n_0} \vec{v}_{A(-1)} \right) \right.$$

$$\left. \left(1 - \frac{\gamma v_T^2 \vec{k}_2 \vec{k}_2}{w_2^2} \right)^{-1} \cdot \frac{\vec{k}_2}{w_2} v_{xI}^2 \right] \qquad \qquad A5\text{-}(32)$$

Again use the operator identities (14) valid for the choice of basis made in 3.4. Also express the first-order $\{n\}$ in terms of the first-order $\{\vec{v}\}$ using 2.8-(19). Then (29),(30) become

$$\frac{1}{2} E_{x1}^2 \left(F_{(-1),1,1}^{NL(3)} \right)_{ABxx} = -\frac{mn_0}{\varepsilon_0}$$

$$\left[\frac{w_{(o)}^2}{w_{(o)}^2 - \gamma k_{(o)}^2 v_T^2} \left\{ \frac{k_{(o)}^2}{w_{(o)}^2} \left(\vec{v}_{A(1)} \cdot \vec{v}_{x1} \right) \left(\vec{v}_{B(-1)} \cdot \vec{v}_{x1} \right) + \left(\vec{v}_{A(1)} \cdot \vec{v}_{x1} \right) \left(\vec{v}_{B(-1)} \right. \right. \right.$$

$$\left. \cdot \frac{\vec{k}_{(-1)}}{w_{(-1)}} \right) \left(\frac{\vec{k}_{(o)}}{w_{(o)}} \cdot \vec{v}_{x1} \right) + \left(\vec{v}_{B(-1)} \cdot \vec{v}_{x1} \right) \left(\frac{\vec{k}_{(1)}}{w_{(1)}} \cdot \vec{v}_{A(1)} \right) \left(\frac{\vec{k}_{(o)}}{w_{(o)}} \cdot \vec{v}_{x1} \right) \right\}$$

$$\left. + \frac{\gamma k_{(o)}^2 v_T^2}{w_{(o)}^2 - k_{(o)}^2 v_T^2} \left\{ \left(\frac{\vec{k}_{(1)}}{w_{(1)}} \cdot \vec{v}_{A(1)} \right) \left(\frac{\vec{k}_{(-1)}}{w_{(-1)}} \cdot \vec{v}_{B(-1)} \right) \left(\vec{v}_{x1} \cdot \vec{e}_{s(o)} \right) \left(\vec{v}_{x1} \cdot \vec{e}_{s(o)} \right) \right\} \right]$$

$$A5\text{-}(33)$$

$$\frac{1}{2} E^{*2}_{x1}\left(F^{NL(3)}_{(1),-1,-1}\right)_{ABxx} = -\frac{mn_0}{\varepsilon_0}$$

$$\left[\frac{w^2_{(o)}}{w^2_{(o)}-\gamma k^2_{(o)}v^2_T}\left\{\frac{k^2_{(o)}}{w^2_{(o)}}\left(\vec{v}_{A(-1)}\cdot\vec{v}_{x1}\right)\left(\vec{v}_{B(1)}\cdot\vec{v}_{x1}\right) + \left(\vec{v}_{A(-1)}\cdot\vec{v}_{x1}\right)\right.\right.$$

$$\left(\frac{\vec{k}_{(1)}}{w_{(1)}}\cdot\vec{v}_{B(1)}\right)\left(\frac{\vec{k}_{(o)}}{w_{(o)}}\cdot\vec{v}_{x1}\right) + \left(\vec{v}_{B(1)}\cdot\vec{v}_{x1}\right)\left(\frac{\vec{k}_{(-1)}}{w_{(-1)}}\cdot\vec{v}_{A(-1)}\right)\left(\frac{\vec{k}_{(o)}}{w_{(o)}}\cdot\vec{v}_{x1}\right)\right\}$$

$$\left. + \frac{\gamma k^2_{(o)}v^2_T}{w^2_{(o)}-\gamma k^2_{(o)}v^2_T}\left\{\left(\frac{\vec{k}_{(-1)}}{w_{(-1)}}\cdot\vec{v}_{A(-1)}\right)\left(\frac{\vec{k}_{(1)}}{w_{(1)}}\cdot\vec{v}_{B(1)}\right)\left(\vec{v}_{x1}\cdot\vec{e}_{s(o)}\right)\left(\vec{v}_{x1}\cdot\vec{e}_{s(o)}\right)\right\}\right]$$

<div align="right">A5-(34)</div>

Let A,B range independently over the polarizations M, N, S, and use (16) together with the forms (5), (6) for the first-order velocities. One generates from (33) and (34) precisely the elements of (35) and (36), respectively.

It is desired to add (3) and (4) to (31) and (32), respectively to obtain matrix elements describing the total coupling due to the plasma response at the laser second harmonic. Use 3.3-(16) to express the second harmonic laser field in terms of the field of the fundamental. One may obtain the relations

$$V_{z2} = \frac{1}{2}\frac{k_2}{w_2}V^2_{x1}\frac{w^2_2 w^2_p}{\left(w^2_2-\gamma k^2_2 v^2_T\right)\left(w^2_2-\gamma k^2_2 v^2_T-w^2_p\right)}$$

<div align="right">A5-(37)</div>

$$\frac{1}{2} E_{x1}^2 \overset{\triangle}{\overset{=}{F}}{}^{NL(3)}_{(-1),1,1} = -w_p^2 \nabla_1^2$$

$$
\begin{pmatrix}
\dfrac{\left[k_{(0)}^2(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})\right]}{\left[(w_{(0)}^2-\gamma k_{(0)}^2 v_T^2)w_{(1)}w_{(-1)}\right]} & 0 & \dfrac{\left[k_{(0)}^2 w_{(-1)}(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})+ k_{(0)}k_{(-1)}w_{(0)}(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})\right]}{\left[w_{(1)}(w_{(0)}^2-\gamma k_{(0)}^2 v_T^2)(w_{(-1)}^2-\gamma k_{(-1)}^2 v_T^2)\right]} \\[2ex]
0 & 0 & 0 \\[2ex]
\dfrac{\left[k_{(0)}^2 w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})+ k_{(0)}k_{(1)}w_{(0)}(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})\right]}{\left[w_{(-1)}(w_{(0)}^2-\gamma k_{(0)}^2 v_T^2)(w_{(1)}^2-\gamma k_{(1)}^2 v_T^2)\right]} & 0 & \dfrac{\left[\begin{array}{l}k_{(0)}^2 w_{(1)}w_{(-1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})+ \\ k_{(0)}k_{(-1)}w_{(0)}w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})+ \\ k_{(0)}k_{(1)}w_{(0)}w_{(-1)}(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})+ \\ \gamma k_{(0)}^2 v_T^2 k_{(1)}k_{(-1)}(\vec{e}_{S(0)}\cdot\vec{e}_{x1})^2\end{array}\right]}{\left[(w_{(0)}^2-\gamma k_{(0)}^2 v_T^2)(w_{(1)}^2-\gamma k_{(1)}^2 v_T^2)(w_{(-1)}^2-\gamma k_{(-1)}^2 v_T^2)\right]}
\end{pmatrix}
$$

A5-(35)

$$\frac{1}{2}E_{x1}^{*2}\overset{\Delta}{=}F_{(1),-1,-1}^{NL(3)} = -w_P^2 V_1^{*2}$$

$$\begin{pmatrix} \dfrac{[k_{(0)}^2(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})]}{[(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2)w_{(1)}w_{(-1)}]} & 0 & \dfrac{[k_{(0)}^2 w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})+k_{(0)}k_{(1)}w_{(0)}(\vec{e}_{M(-1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})]}{[w_{(-1)}(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2)(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2)]} \\[4ex] 0 & 0 & 0 \\[4ex] \dfrac{[k_{(0)}^2 w_{(-1)}(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})+k_{(0)}k_{(-1)}w_{(0)}(\vec{e}_{M(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})]}{[w_{(1)}(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2)(w_{(-1)}^2-\gamma k_{(-1)}^2 V_T^2)]} & 0 & \dfrac{\begin{array}{l}[k_{(0)}^2 w_{(1)}w_{(-1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})+\\ k_{(0)}k_{(-1)}w_{(0)}w_{(1)}(\vec{e}_{S(1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})+\\ k_{(0)}k_{(1)}w_{(0)}w_{(-1)}(\vec{e}_{S(-1)}\cdot\vec{e}_{x1})(\vec{e}_{S(0)}\cdot\vec{e}_{x1})+\\ \gamma k_{(0)}^2 V_T^2 k_{(1)}k_{(-1)}(\vec{e}_{S(0)}\cdot\vec{e}_{x1})^2]\end{array}}{\begin{array}{l}[(w_{(0)}^2-\gamma k_{(0)}^2 V_T^2)(w_{(1)}^2-\gamma k_{(1)}^2 V_T^2)\\ \qquad (w_{(-1)}^2-\gamma k_{(-1)}^2 V_T^2)]\end{array}} \end{pmatrix}$$

$$V_{z2} + \frac{1}{2}\frac{k_2}{w_2}V_{x1}^2\frac{w_2^2}{\left(w_2^2-\gamma k_2^2 v_T^2\right)} = \frac{1}{2}\frac{k_2}{w_2}V_{x1}^2\frac{w_2^2}{\left(w_2^2-\gamma k_2^2 v_T^2-w_p^2\right)} \quad \text{A5-(38)}$$

Using these one finds

$$E_{z2}\left(F_{(-1),2}^{NL(2)}\right)_{ABz} + \frac{1}{2}E_{x1}^2\left(F_{(-1),1,1}^{NL(3)}\right)_{ABxx} = -\frac{1}{2}\frac{mn_0}{\varepsilon_0}V_{x1}^2$$

$$\left[\frac{w_2^2}{w_2^2-\gamma k_2^2 v_T^2-w_p^2}\left\{\frac{k_2^2}{w_2^2}\left(\vec{v}_{A(1)}\cdot\vec{v}_{B(-1)}\right) + \left(\frac{\vec{k}_{(1)}\cdot\vec{v}_{A(1)}}{w_{(1)}}\right)\left(\frac{\vec{k}_2}{w_2}\cdot\vec{v}_{B(-1)}\right)\right.\right.$$

$$\left. + \left(\frac{\vec{k}_{(-1)}\cdot\vec{v}_{B(-1)}}{w_{(-1)}}\right)\left(\frac{\vec{k}_2}{w_2}\cdot\vec{v}_{A(1)}\right) + \frac{w_2^2 w_p^2}{\left(w_2^2-\gamma k_2^2 v_T^2\right)\left(w_2^2-\gamma k_2^2 v_T^2-w_p^2\right)}\right.$$

$$\left.\left\{\frac{k_2^2}{w_2^2}\gamma(\gamma-2)v_T^2\left(\frac{\vec{k}_{(1)}\cdot\vec{v}_{A(1)}}{w_{(1)}}\right)\left(\frac{\vec{k}_{(1)}\cdot\vec{v}_{A(1)}}{w_{(1)}}\right)\left(\frac{\vec{k}_{(-1)}\cdot\vec{v}_{B(-1)}}{w_{(-1)}}\right)\right\}\right] \quad \text{A5-(39)}$$

$$E_{z2}^*\left(F_{(1),-2}^{NL(2)}\right)_{ABz} + \frac{1}{2}E_{x1}^{*2}\left(F_{(1),-1,-1}^{NL(3)}\right)_{ABxx} = -\frac{1}{2}\frac{mn_0}{\varepsilon_0}V_{x1}^{*2}$$

$$\left[\frac{w_2^2}{w_2^2-\gamma k_2^2 v_T^2-w_p^2}\left\{\frac{k_2^2}{w_2^2}\left(\vec{v}_{A(-1)}\cdot\vec{v}_{B(1)}\right) + \left(\frac{\vec{k}_{(-1)}\cdot\vec{v}_{A(-1)}}{w_{(-1)}}\right)\left(\frac{\vec{k}_2}{w_2}\cdot\vec{v}_{B(1)}\right)\right.\right.$$

$$\left. + \left(\frac{\vec{k}_{(1)}\cdot\vec{v}_{B(1)}}{w_{(1)}}\right)\left(\frac{\vec{k}_2}{w_2}\cdot\vec{v}_{A(-1)}\right) + \frac{w_2^2 w_p^2}{\left(w_2^2-\gamma k_2^2 v_T^2\right)\left(w_2^2-\gamma k_2^2 v_T^2-w_p^2\right)}\right.$$

$$\left.\left\{\frac{k_2^2}{w_2^2}\gamma(\gamma-2)v_T^2\left(\frac{\vec{k}_{(-1)}\cdot\vec{v}_{A(-1)}}{w_{(1)}}\right)\left(\frac{\vec{k}_{(1)}\vec{v}_{B(1)}}{w_{(1)}}\right)\right\}\right] \quad \text{A5-(40)}$$

$$E_{z2}\vec{F}^{NL(2)}_{(-1),2} + \frac{1}{2}E_{x1}^2\vec{\vec{F}}^{NL(3)}_{(-1),1,1} = -\frac{1}{2}w_p^2\vec{V}_{x1}^2$$

$$\left(\begin{array}{ccc}
\dfrac{\left[k_2^2(\vec{e}_{m(1)}\cdot\vec{e}_{m(-1)})\right]}{\left[w_{(1)}w_{(-1)}(w_2^2-\gamma k_2^2 v_T^2-w_p^2)\right]} & \dfrac{\left[k_2^2(\vec{e}_{m(1)}\cdot\vec{e}_{N(-1)})\right]}{\left[w_{(1)}w_{(-1)}(w_2^2-\gamma k_2^2 v_T^2-w_p^2)\right]} & \dfrac{\left[k_2^2 w_{(1)}w_{(-1)}(\vec{e}_{m(1)}\cdot\vec{e}_{s(-1)})+k_2 k_{(1)}w_2 w_{(-1)}(\vec{e}_{s(-1)}\cdot\vec{e}_{z2})\right]}{\left[w_{(1)}^2(w_{(-1)}^2-\gamma k_{(-1)}^2 v_T^2)(w_2^2-\gamma k_2^2 v_T^2-w_p^2)\right]} \\[25pt]
\dfrac{\left[k_2^2(\vec{e}_{N(1)}\cdot\vec{e}_{m(-1)})\right]}{\left[w_{(1)}w_{(-1)}(w_2^2-\gamma k_2^2 v_T^2-w_p^2)\right]} & \dfrac{\left[k_2^2(\vec{e}_{N(1)}\cdot\vec{e}_{N(-1)})\right]}{\left[w_{(1)}w_{(-1)}(w_2^2-\gamma k_2^2 v_T^2-w_p^2)\right]} & \dfrac{\left[k_2^2 w_{(1)}w_{(-1)}(\vec{e}_{N(1)}\cdot\vec{e}_{s(-1)})+k_2 k_{(1)}w_2 w_{(-1)}(\vec{e}_{s(-1)}\cdot\vec{e}_{z2})\right]}{\left[w_{(1)}^2(w_{(-1)}^2-\gamma k_{(-1)}^2 v_T^2)(w_2^2-\gamma k_2^2 v_T^2-w_p^2)\right]} \\[25pt]
\dfrac{\left[k_2^2 w_{(1)}w_{(-1)}(\vec{e}_{s(1)}\cdot\vec{e}_{m(-1)})+k_2 k_{(-1)}w_2 w_{(1)}(\vec{e}_{s(1)}\cdot\vec{e}_{z2})\right]}{\left[w_{(-1)}^2(w_{(1)}^2-\gamma k_{(1)}^2 v_T^2)(w_2^2-\gamma k_2^2 v_T^2-w_p^2)\right]} & \dfrac{\left[k_2^2 w_{(1)}w_{(-1)}(\vec{e}_{s(1)}\cdot\vec{e}_{N(-1)})+k_2 k_{(-1)}w_2 w_{(1)}(\vec{e}_{s(1)}\cdot e_{z2})\right]}{\left[w_{(-1)}^2(w_{(1)}^2-\gamma k_{(1)}^2 v_T^2)(w_2^2-\gamma k_2^2 v_T^2-w_p^2)\right]} & \dfrac{\left[k_2^2 w_{(1)}w_{(-1)}(w_2^2-\gamma k_2^2 v_T^2)(\vec{e}_{s(1)}\cdot\vec{e}_{s(-1)})+k_2 k_{(1)}w_2 w_{(-1)}(w_2^2-\gamma k_2^2 v_T^2)(\vec{e}_{s(1)}\cdot\vec{e}_{z2})+k_2 k_{(-1)}w_2 w_{(1)}(w_2^2-\gamma k_2^2 v_T^2)(e_{s(-1)}\cdot\vec{e}_{z2})+\gamma(\gamma-2)k_2^2 v_T^2 k_{(1)}k_{(-1)}w_p^2\right]}{\left[(w_{(1)}^2-\gamma k_{(1)}^2 v_T^2)(w_{(-1)}^2-\gamma k_{(-1)}^2 v_T^2)(w_2^2-\gamma k_2^2 v_T^2)(w_2^2-\gamma k_2^2 v_T^2-w_p^2)\right]}
\end{array}\right)$$

A5-(41)

$$E_{zz}^{*}\, \vec{\vec{F}}^{NL(2)}_{(1),-2} + \frac{1}{2} E_{x1}^{*2}\, \vec{\vec{\vec{F}}}^{NL(3)}_{(1),-1,-1} \;=\; -\frac{1}{2}\,\omega_P^2\, V_{x1}^{*2}$$

$$
\begin{pmatrix}
\dfrac{[k_z^2(\vec{e}_{m(1)}\cdot\vec{e}_{m(-1)})]}{[\omega_{(1)}\omega_{(-1)}(\omega_2^2-\gamma k_z^2 v_T^2-\omega_P^2)]}
&
\dfrac{[k_z^2(\vec{e}_{N(1)}\cdot\vec{e}_{m(-1)})]}{[\omega_{(1)}\omega_{(-1)}(\omega_2^2-\gamma k_z^2 v_T^2-\omega_P^2)]}
&
\dfrac{[k_z^2\omega_{(1)}\omega_{(-1)}(\vec{e}_{s(1)}\cdot\vec{e}_{m(-1)})+k_z k_{(-1)}\omega_z\omega_{(1)}(\vec{e}_{s(1)}\cdot\vec{e}_{z2})]}{[\omega_{(-1)}^2(\omega_{(1)}^2-\gamma k_{(1)}^2 v_T^2)(\omega_2^2-\gamma k_z^2 v_T^2-\omega_P^2)]}
\\[28pt]
\dfrac{[k_z^2(\vec{e}_{m(1)}\cdot\vec{e}_{N(-1)})]}{[\omega_{(1)}\omega_{(-1)}(\omega_2^2-\gamma k_z^2 v_T^2-\omega_P^2)]}
&
\dfrac{[k_z^2(\vec{e}_{N(1)}\cdot\vec{e}_{N(-1)})]}{[\omega_{(1)}\omega_{(-1)}(\omega_2^2-\gamma k_z^2 v_T^2-\omega_P^2)]}
&
\dfrac{[k_z^2\omega_{(1)}\omega_{(-1)}(\vec{e}_{s(1)}\cdot\vec{e}_{N(-1)})+k_z k_{(1)}\omega_z\omega_{(-1)}(\vec{e}_{s(1)}\cdot\vec{e}_{z2})]}{[\omega_{(-1)}^2(\omega_{(1)}^2-\gamma k_{(1)}^2 v_T^2)(\omega_2^2-\gamma k_z^2 v_T^2-\omega_P^2)]}
\\[28pt]
\dfrac{\begin{array}{c}[k_z^2\omega_{(1)}\omega_{(-1)}(\vec{e}_{m(1)}\cdot\vec{e}_{s(-1)})\\+k_z k_{(1)}\omega_z\omega_{(-1)}(\vec{e}_{s(-1)}\cdot\vec{e}_{z2})]\end{array}}{\begin{array}{c}[\omega_{(1)}^2(\omega_{(-1)}^2-\gamma k_{(-1)}^2 v_T^2)\\(\omega_2^2-\gamma k_z^2 v_T^2-\omega_P^2)]\end{array}}
&
\dfrac{\begin{array}{c}[k_z^2\omega_{(1)}\omega_{(-1)}(\vec{e}_{N(1)}\cdot\vec{e}_{s(-1)})+\\k_z k_{(1)}\omega_z\omega_{(-1)}(\vec{e}_{s(-1)}\cdot e_{z2})]\end{array}}{\begin{array}{c}[\omega_{(1)}^2(\omega_{(-1)}^2-\gamma k_{(-1)}^2 v_T^2)\\(\omega_2^2-\gamma k_z^2 v_T^2-\omega_P^2)]\end{array}}
&
\dfrac{\begin{array}{c}[k_z^2\omega_{(1)}\omega_{(-1)}(\omega_2^2-\gamma k_z^2 v_T^2)(\vec{e}_{s(1)}\cdot\vec{e}_{s(-1)})\\+k_z k_{(-1)}\omega_z\omega_{(1)}(\omega_2^2-\gamma k_z^2 v_T^2)(\vec{e}_{s(1)}\cdot\vec{e}_{z2})\\+k_z k_{(1)}\omega_z\omega_{(-1)}(\omega_2^2-\gamma k_z^2 v_T^2)(\vec{e}_{s(-1)}\cdot\vec{e}_{z2})\\+\gamma(\gamma-2)k_z^2 v_T^2\,k_{(1)}k_{(-1)}\omega_P^2]\end{array}}{\begin{array}{c}[(\omega_{(1)}^2-\gamma k_{(1)}^2 v_T^2)(\omega_{(-1)}^2-\gamma k_{(-1)}^2 v_T^2)\\(\omega_2^2-\gamma k_z^2 v_T^2)(\omega_2^2-\gamma k_z^2 v_T^2-\omega_P^2)]\end{array}}
\end{pmatrix}
$$

A5-(42)

Let A,B range independently over the polarizations M, N, S and use (16) together with the forms (6,7) for the first-order velocities. One generates from (39) and (40) precisely the elements of (41) and (42), respectively.

The submatrices in the top-right and bottom-left corners of 3.4-(12) comprise direct pump-induced coupling between perturbations separated by twice the pump wavevector and frequency. Their role in laser-driven instabilities was discussed in the latter part of Section 3.4.

The results for these submatrices are shown in the following fashion. The terms of (23), (34) not containing the quantities $(\vec{v}_{x1,x1}, \vec{v}_{x\bar{1},x\bar{1}}, n_{x1,x1}, n_{x\bar{1},x\bar{1}})$ are fashioned into the submatrices

$$\frac{1}{2} E_{x1}^2 \overset{\triangle}{\underset{F}{\rightleftharpoons}} NL(3)_{(-1),1,1} \quad , \quad \frac{1}{2} E_{x1}^{*2} \overset{\triangle}{\underset{F}{\rightleftharpoons}} NL(3)_{(1),-1,-1} \quad \text{say.}$$

These submatrices describe processes which involve the low-frequency response of the plasma at $(k_{(o)}, w_{(o)})$ and are displayed in (35) and (36). The terms of (23) and (24) which do contain the nonlinear responses $(\vec{v}_{x1,x1}, \vec{v}_{x\bar{1},x\bar{1}}, n_{x1,x1}, n_{x\bar{1},x\bar{1}})$ at the laser second harmonic due to the electric field of the laser fundamental are fashioned into the submatrices

$$\frac{1}{2} E_{x1}^2 \overset{\blacktriangle}{\underset{F}{\rightleftharpoons}} NL(3)_{(-1),1,1} \quad , \quad \frac{1}{2} E_{x1}^{*2} \overset{\blacktriangle}{\underset{F}{\rightleftharpoons}} NL(3)_{(1),-1,-1} \quad \text{say.}$$

These are added to (12) and (13) which describe coupling due to
the pre-existing electric field of the laser second harmonic,
and the sums are displayed in (41) and (42). Equations (41)
and (42) contain the coupling processes which involve the high-
frequency response of the plasma at $(2k_1, 2w_1)$.

A6. <u>Motivation for Choice of Values for γ in the Fluid Model</u>

In this appendix certain quantities are evaluated using the warm-fluid model and the Vlasov model in turn. The results of the latter are used as a guide in choosing the most appropriate values of γ for use in the former. The quantities are evaluated for collinearly propagating electrostatic waves in an unmagnetized drift-free plasma. The quantities in question are the contribution of a single species to the linear dispersion function and the contribution of a single species to the second-order nonlinear coupling coefficient. The Vlasov model will yield separate results, and hence indicate separate choices of γ, for phase-velocities much greater than the spec ies thermal-velocity and for phase-velocities much less than the species thermal-velocity.

First look at the contribution of a single species to the linear dispersion function. Using the warm-fluid model this contribution is, from 2.8-(28)

$$\left((D_a)_{AA} \right) \text{ single species } = -\frac{w_p^2}{w_a^2} \frac{w_a^2}{w_a^2 - \gamma k_a^2 v_{T_0}^2} \qquad \text{A6-(1)}$$

Now evaluate this quantity using the Vlasov model. One gets the usual Landau result

$$\left((D_a)_{AA} \right) \text{ single species } = -\frac{w_p^2}{k_a^2} \int_U dv \, \frac{2f_0(v)/2v}{v - (w_a/k_a)} \qquad \text{A6-(2)}$$

Here the variable of integration v and the velocity distribution function $f_0(v)$ are one-dimensional in the direction of k_a, and the mark ∪ indicates that the contour of integration goes below the singularity in the complex plane, as prescribed by Landau. On taking the distribution function to be Maxwellian and employing the plasma dispersion function described by Fried and Conte, one finds the contribution

$$\left((D_a)_{AA}\right) \text{ single species} = -\frac{w_p^2}{2k_a^2 v_{T_0}^2} Z'\left(\frac{w_a}{\sqrt{2}k_a v_{T_0}}\right) \qquad \text{A6-(3)}$$

For phase-velocities much greater than the thermal velocity, the asymptotic expansion of the Z'-function gives

$$\left((D_a)_{AA}\right) \text{ single species} = -\frac{w_p^2}{w_a^2}\left(1 + \frac{3k_a^2 v_{T_0}^2}{w_a^2} + \ldots\right) \qquad \text{A6-(4)}$$

This clearly corresponds to taking $\gamma=3$ in the fluid model. For phase-velocities much less than the thermal velocity, the power-series expansion of the Z'-function gives

$$\left((D_a)_{AA}\right) \text{ single species} = -\frac{w_p^2}{w_a^2}\left(-\frac{w_a^2}{k_a^2 v_{T_0}^2} - i\sqrt{\frac{\pi}{2}}\frac{w_a^3}{k_a^3 v_{T_0}^2}\right.$$

$$\left. + \frac{w_a^4}{k_a^4 v_{T_0}^4} + \ldots\right) \qquad \text{A6-(5)}$$

The real part corresponds to taking $\gamma=1$ in the fluid model. The imaginary part is the Landau damping term which cannot be recovered directly from the fluid model. Secondly, look at the contribution of a single species to the coupling coefficient describing the effect of second-order nonlinear conductivity. Using the warm-fluid model, the contribution is

$$(F_{b,c})_{AAA} = -\frac{mn_0}{\varepsilon_0} \left(\frac{k_{\bar{a}}}{w_{\bar{a}}} + \frac{k_b}{w_b} + \frac{k_c}{w_c} + \gamma(\gamma-2)v_{T_0}^2 \frac{k_{\bar{a}}k_b k_c}{w_{\bar{a}}w_b w_c} \right)$$

$$\frac{w_{\bar{a}}w_b w_c}{\left(w_{\bar{a}}^2 - \gamma k_{\bar{a}}^2 v_{T_0}^2\right)\left(w_b^2 - \gamma k_b^2 v_{T_0}^2\right)\left(w_c^2 - \gamma k_c^2 v_{T_0}^2\right)} \left(\frac{-iq^3}{m^3}\right)$$

$$A6-(6)$$

where

$$(k_{\bar{a}}, w_{\bar{a}}) \equiv -(k_a, w_a) \equiv -(k_b, w_b) - (k_c, w_c)$$

The corresponding expression in the Vlasov model may be deduced from the fact that coupling coefficients are additive over particle populations. Since the particle-velocity distribution-function in the Vlasov model may be thought of as describing a continuous superposition of cold beamlets, the coupling-coefficient in the Vlasov model has the form of the coupling-coefficient for a cold, drifting particle population, weighted by the distribution function and integrated over all velocities:

$$(F_{b,c})_{AAA} = -\frac{mn_0}{\varepsilon_0} \int dv\, f_0(v) \frac{\left(\frac{k_{\bar{a}}}{w_{\bar{a}}^v} + \frac{k_b}{w_b^v} + \frac{k_c}{w_c^v}\right)}{w_{\bar{a}}^v w_b^v w_c^v} \frac{-iq^3}{m^3} \quad A6-(7)$$

This result can also be obtained directly from the Vlasov equation.

Here

$$w_a^v \equiv w_a - k_a v \Bigg\}$$

$$w_b^v \equiv w_b - k_b v \Bigg\} \qquad\qquad\qquad A6\text{-}(8)$$

$$w_c^v \equiv w_c - k_c v \Bigg\}$$

are the wave frequencies as observed in a frame moving with the beamlet which has velocity v.

Again take the distribution-function to be Maxwellian, and consider the behavior of the integral (A6-(7)) for various limiting cases of the phase-velocities. When all three phase-velocities are much greater than the thermal velocity of the species, asymptotic expansion of the integral (A6-(7)) gives

$$(F_{b,c})_{AAA} \equiv -\frac{mn_0}{\varepsilon_0}\Bigg\{\left(\frac{k_a}{w_a} + \frac{k_b}{w_b} + \frac{k_c}{w_c}\right)\left(1 + 3v_{T_0}^2\left(\frac{k_a^2}{w_a^2} + \frac{k_b^2}{w_b^2} + \frac{k_c^2}{w_c^2}\right)\right)$$

$$+ 3v_{T_0}^2\frac{k_a k_b k_c}{w_a w_b w_c}\Bigg\}\frac{1}{w_a w_b w_c}\left(\frac{-iq^3}{m^3}\right) \qquad A6\text{-}(9)$$

This clearly corresponds to taking $\gamma=3$ in the fluid model. When one phase-velocity, say w_b/k_b, is much smaller than the thermal-velocity, but the other two are much greater, an appropriate combined expansion of the integral (A6-(7)) gives

$$(F_{b,c})_{AAA} = -\frac{mn_0}{\varepsilon_0}\left\{1 + v_{T_0}^2\left(\frac{k_a^2}{w_a^2} + \frac{k_a k_c}{w_a w_c} + \frac{k_c^2}{w_a^2}\right)\right\}\frac{k_b}{w_b}$$

$$\frac{w_b}{w_a(-k_b^2 v_{T_0}^2)w_c}\left(\frac{-iq^3}{m^3}\right) \qquad\qquad \text{A6-(10)}$$

This corresponds to taking $\gamma = 1$ in the fluid model, with the proviso that $\gamma(\gamma-2)$ should be replaced by $+1$ whenever it appears. This unsatisfactory feature is of no consequence if (A6-(10)) is approximated by its leading term as is done for use in laser-driven instabilities.

BIOGRAPHICAL NOTE

Duncan Charles Watson was born in Adlington, Cheshire, England on October 3, 1943. After graduating from Manchester Grammar (High) School he was admitted to Oxford University in September 1962. From that institution he received in June 1965 the degree of Bachelor of Arts in Pure and Applied Mathematics, and in June 1966 the Diploma in Advanced Mathematics for proficiency in group theory and quantum mechanics.

From 1966 until 1970 Mr. Watson divided his time between university and industry. He worked under Professor Paul Roman of Boston University in the field of elementary particles, receiving the degree of Master of Science in Physics in May 1970. He also worked at the Air Force Cambridge Research Laboratories, Bedford, on molecular spectrocopy, and at Arthur D. Little, Inc., Cambridge, on atmospheric physics.

In September 1970 Mr. Watson was admitted to Massachusetts Institute of Technology as a graduate student in Electrical Engineering to work under Professor Abraham Bers within the Research Laboratory of Electronics. In February 1972 he received the degree of Master of Science in Electrical Engineering. His thesis was entitled "Plasma Oscillations in an Applied Electric Field." During the summer of 1972 he worked on symbolic computation by computers at Project MAC. Since that time he has worked in the field of nonlinear wave-wave interactions and plasma instabilities, culminating in this

thesis. He is an associate member of Sigma Xi, a student member of the Institute of Electronic and Electrical Engineers and a student member of the American Association for the Advancement of Science.