PLANE STRESS AND BENDING OF PLATES BY METHOD OF ARTICULATED FRAMEWORK

By

Alexander Paul Hrennikoff

Graduate, Institute of Communication Engineering
Moscow, Russia, 1920

B.A.Sc., University of British Columbia
Vancouver, Canada, 1930

M.A.Sc., University of British Columbia
Vancouver, Canada, 1933

Submitted in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF SCIENCE

from the

Massachusetts Institute of Technology

1940

Signature of Author ....

Department of Civil Engineering, May 9, 1940

Signature of Professor
in Charge of Research ....

Signature of Chairman of Department
Committee on Graduate Students ..

Cambridge, Mass.
May 9, 1940.

Professor George W. Swett,
Secretary of the Faculty,
Massachusetts Institute of Technology,
Cambridge, Mass.

Dear Sir:

In partial fulfillment of the requirements for the degree of Doctor of Science in Civil Engineering, I present this thesis entitled, "Plane Stress and Bending of Plates by Method of Articulated Framework".

Respectfully submitted,

Alexander P. Hrennikoff

ACKNOWLEDGEMENT.

The author wishes to express his thanks to Dr. J. B. Wilbur, who acted as his supervisor, for valuable suggestions and criticism.

TABLE OF CONTENTS

Table of Contents (continued)

SYNOPSIS

This treatise is an exposition of the framework method of stress analysis and of its application to two fields of elasticity, that of two-dimensional stress discussed in Chapter I, and the field of bending of plates treated in Chapter II. Main justification for introduction of this new method lies in the fact that in both fields mentioned, solutions of the theory of elasticity are available only in a limited number of special cases.

The framework analysis is essentially an arithmetical procedure applicable to any problem within the above classification, dealing with a rectangular plate. The results are only approximate but believed to be fairly accurate in most cases.

Both the idea and the development of the method are original, except perhaps for the general background of the method of moment distribution of Professor Hardy Cross. By the time the main features of the method had been established, the author's attention was called to papers by W. Riedel[1)] and K. Wieghardt[2)] dealing with the same subject of plane stress by framework, but in a somewhat different and more restricted manner. Figure 17, Chapter I and supporting discussion have been somewhat influenced by the second of these papers. At the same time the author became aware of application of string network to deflection of a thin membrane in the paper by Christopherson and

1), 2)... Numerals refer to items in Bibliography.

Professor Southwell[3] in connection with torsion of a triangular shaft and in the book of H. Marcus[4] in connection with bending of plates.

The presentation of the subject in the two following chapters proceeds along similar lines. First, it is demonstrated that the state of two-dimensional stress and strain on the one hand, and the flexural state in a bent plate, on the other hand, are faithfully preserved if the solid medium is replaced by a framework of articulated elastic bars of infinitesimal size and of certain definite patterns extending over the whole plate, while the acting loads and the restraints are left unchanged.

Theoretically, the size of the unit of pattern must be infinitesimal in order to imitate truly the mechanical behaviour of the continuous material, but such framework would be impossible of solving, and, for practical reasons, the size of the unit is taken not only finite, but fairly large, so as to make the solution less laborious.

An extensive investigation is conducted into various types of pattern, satisfying the conditions for equivalence of the framework and the plate, and the framework constants, i.e., the cross-sectional areas of the members in two-dimensional stress and the moments of inertia in the flexural state, are determined. Triangular, rectangular and square types of pattern are thus examined in Chapter I and two square patterns in Chapter II. Peculiarly enough, most of these patterns are capable only of imitating a solid material with Poisson's ratio one-third, however, a square unit applicable to any value of ν has also been found.

A method of successive movements, a procedure reminding moment distribution of Professor Hardy Cross, and consisting in moving joints and groups of joints, one after the other, toward equilibrium, has been adopted in solution of two-dimensional framework. A special form of keeping the record of stresses has been devised and the quantities known as "distribution factors" have been derived for use in the process of balancing the framework.

A more elaborate procedure of bringing to equilibrium a flexural framework has been made necessary by the greater number of degrees of freedom of the joints in such framework. Quantities known as "reaction factors" and "influence factors" have been devised in addition to the distribution factors, and with their assistance the distribution has been made possible.

The method of successive movements has proved to be a workable equivalent of a great number of simultaneous equations, which otherwise would be required in order to solve the problem of framework.

The labour of distribution may be considerably lowered by the use of the principle of symmetry and antisymmetry, when dealing with symmetrical plates. Thus, should a plate be symmetrical about two axes, and should it be loaded with an unsymmetrical load, it is possible to find the required solution by solving one quarter of the plate four times and then combining the results.

The last step in the framework method is the interpretation of the framework or conversion of the bar stresses into the plate stresses. Simple rules are given for this operation both in plane stress and in plate bending.

The test of two-dimensional framework method comes when it is applied to solution of a deep rectangular beam carrying a uniform load - one of the few cases of rectangular bodies whose exact solution by elasticity is known. Square frameworks with $\nu = \frac{1}{3}$ and $\nu = 0$ are used in solution and later the number of units in the former framework is doubled in order to determine the speed of convergence of the approximate solution to the true one. The results have been found gratifying, especially with $\nu = \frac{1}{3}$.

A somewhat generalized gusset plate at the top chord of a roof truss has been selected for the second problem handled by square framework with $\nu = \frac{1}{3}$. After the first solution the number of units has been doubled and the problem resolved. A close agreement of the two sets of stresses has been found.

The flexural framework with $\nu = \frac{1}{3}$ is tested on an example of a square plate with clamped edges loaded with a concentrated load at the centre. The results are compared with those derived by an approximate method of difference equations by H. Marcus[4)] and with exact formulae of Professor Timoshenko[7)]. On strength of comparison with exact solution the framework results are excellent in relation to the deflections, good for the bending moments, but unsatisfactory with regard to torsional moments and some shears. The discrepancy is attributed to imperfect stress interpretation. Most of the framework results are better than the ones by the method of Marcus.

In conclusion, some unsolved problems susceptible to handling by the framework method and some new fields, into which the method may be projected, are discussed briefly, and the advantages and disadvantages of the method are enumerated.

I.

Framework Method Applied to Two-Dimensional Stress.

1. Plane Stress and Plane Strain.

Two-dimensional problems of the theory of elasticity are characterized by independence of stresses from one of the coordinates, taken here as z, which means that the stresses are dependent only on x and y.

There are two kinds of two-dimensional stress problems, plane stress and plane strain. In the former type $\sigma_z = 0$ and so are τ_{xz} and τ_{yz}, the stress system being thus limited to σ_x, σ_y and τ_{xy} only. This state of stress represents fairly well what happens in thin plates when they are acted upon by forces lying in their planes, and distributed uniformly throughout the thickness, with no forces acting normally to the plates.

The state of plane strain or plane deformation occurs in long prismatical or cylindrical bodies under the action of forces constant along the length, and having no components in the longitudinal direction. The deformations produced are such that $\epsilon_z = 0$, so that the points do not move in the direction of length. σ_z is not zero here but it is independent of z, and is expressible through σ_x and σ_y. At the same time τ_{xz} and τ_{yz} are both zeros.

In both these types of two-dimensional problems a thin slice cut out from the body by sections perpendicular to z axis may be considered as representative of the state of stress in the whole body.

Theoretically, the three independent stress components σ_x, σ_y and τ_{xy}, characterizing a two-dimensional state of stress, can be found by solving three differential equations pertaining to this condition in such a way that the stresses or displacements found satisfy the stress or displacement conditions at the boundary. These three differential equations for plane stress are as follows: [5]

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0 \end{aligned}\right\} \quad \text{(a)}$$

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2\,(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x\,\partial y} \quad \text{(b)}$$

For plane strain the first two equations are the same, but the third one is somewhat different, and may be obtained from (b) by replacing ν with $\frac{\nu}{1-\nu}$.

Equations (a) are derived exclusively from statics by considering an infinitesimal element of the body, and are independent of any physical characteristics of the material. The third equation is an equation of continuity and elasticity. When it is first written in the form

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} \quad \text{(c)}$$

it merely represents the statement of geometric continuity of the material. Strains are replaced by stresses by means of elasticity conditions, which in the absence of σ_z are:

$$\left.\begin{aligned} \epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y); \\ \epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x); \\ \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy}; \end{aligned}\right\} \quad \text{(d)}$$

[5] Pages 21-23.

and that transforms (c) into (b). Equation (b) thus implies that the material retains its geometrical continuity after the deformation and at the same time behaves according to the laws of elasticity, as characterized by the constants E and ν.

Equation (c) holds also for the plane strain, but relations (d) do not hold in view of $\sigma_z \neq 0$, and as a result, the following equivalent relations are present instead of them:

$$\left.\begin{aligned} \epsilon_x &= \frac{1}{E}\left[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y\right] = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right) \\ \epsilon_y &= \frac{1}{E}\left[(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x\right] = \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right) \\ \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} = \frac{2\left(1+\frac{\nu}{1-\nu}\right)}{E/1-\nu^2}\tau_{xy} \end{aligned}\right\} \quad \text{(e)}$$

Their substitution into (c) results in the third equation of plane strain, equivalent to (b) in plane stress.

A comparison of equations (e) with equations (d) shows that deformability of a material with elastic constants E and ν under conditions of plane strain is the same as the deformability of a material under plane stress with elastic constants E' and ν' such that

$$\left.\begin{aligned} E' &= \frac{E}{1-\nu^2} \\ \nu' &= \frac{\nu}{1-\nu} \end{aligned}\right\} \quad \text{(f)}$$

Since the first two differential equations are identical for the two kinds of two-dimensional stress it is evident, that the stresses σ_x, σ_y and τ_{xy} under plane strain conditions are the same as the corresponding stresses under plane stress in a material with elastic properties modified according to (f), acted upon by the same forces in the presence of similar boundary conditions.

The differential equations (a) and (b) taken by themselves allow an infinite number of mathematical solutions, but among all these there is only one which represents the true physical solution. This is the one that satisfies the boundary conditions. Determination of the proper type of function, capable of satisfying both the differential equations and the boundary conditions, presents in most cases a formidable mathematical problem. Apparent inadequacy of known mathematical functions and methods accounts for the paucity of the solved cases, and at the same time provides an inducement for discovery of methods other than those of pure mathematical analysis such as the one described in this treatise.

2. Framework Analogy

Imagine a plane framework of bars of infinitesimal length and cross section joined by means of pins perpendicular to its plane and forming the same ever-repeating pattern in all the parts of it. The framework so formed is given the same external outline and dimensions as the thin plate subjected to plane stress, which is under investigation. Furthermore, the same forces and boundary conditions are imposed on the framework as on the actual plate, which it proposes to replace, all the forces being applied at the framework joints.

The action of external forces, imposed on the framework, causes direct stresses in the bar members and produces elastic distortion of the structure. A conception of "stress in the framework", as distinct from stresses in individual bars, will be introduced now and will be made analogous to the ordinary idea of stress in continuous material. Referring to Figure 1, the stresses on plane AA are defined as,

$$\sigma_y = \frac{\Sigma(P_y)}{dx} \qquad \text{and} \qquad \tau_{yx} = \frac{\Sigma(P_x)}{dx} \qquad \text{(a)}$$

in other words, normal stress on any plane in the framework is simply the sum of the normal components of bar stresses on the length dx of this plane, divided by dx, with a similar definition for the tangential stress.

This definition implies that the size of the pattern <u>a</u> is considered an infinitesimal of a higher order than dx, otherwise the value of stress would be indefinite, depending on inclusion or exclusion of the last bars that happen to be on either side at the ends of the length dx. However, once <u>a</u> is of the second order of smallness compared to dx the presence or absence of these bars in the sum is immaterial.

In special cases the relation of plane of stress to the framework pattern may be such, that the manner of cutting of the bars by the plane repeats itself continually along the plane. This is the case for planes BB and CC in Figure 1, the repeating length being designated by λ. It is clear that should the plane of stress satisfy this requirement the length λ may be taken in place of dx in equations (a) and the sums of stress components will naturally refer to this repeating length.

A question arises as to whether it is possible to devise the form of the infinitesimal pattern of bar arrangement and to determine the geometrical properties of it, such as cross-sectional areas of bar members and the angles between them, in such a manner that the stresses in the framework, as well as the deformations and displacements will be identical with those of the plate prototype, whose action the framework purports to imitate. This question can be answered by reference to the differential equations of the two-dimensional stress. The two equations of statics are applicable to the framework just the same as to a continuous material of plate in view of the similarity of the definition of stress in both cases. The third equation, expressed in terms of strains, (c), Art.1, is equally applicable, but its other form (b), Art.1, written in terms of stresses, holds only if the framework satisfies the deformability conditions (d), Art.1 or (e), Art.1. Should this be the case, the framework stresses are subject to the same differential equations as the stresses in the plate prototype, and in the presence of similar acting forces and boundary conditions they are bound to assume the same values as the corresponding stresses in the original plate. Furthermore, equality of stresses leads to equality in deformations and displacements, so that the framework becomes a true full-scale model of the plate. The necessary and sufficient condition for a complete mechanical equivalence of a framework and plate is therefore the deformability of the framework in accordance with the laws of elasticity described by the constants E and ν. The form of the pattern, satisfying these conditions, is suitable for any two-dimensional problem, involving a material of the same E and ν, irrespective of the nature of the applied forces and of the boundary conditions.

The conclusion just reached may be also arrived at from the general consideration of the mechanical behaviour of the plate and of its frame analogue. Forces acting on the body are transmitted through it, on the way to reactions, creating a state of stress and deformation, whose character is determined by two factors only, statics and elastic deformability. In order to make the influence of these two factors felt in the framework in the same manner as in the plate, it is necessary to prescribe infinitesimal size for the mesh in order to satisfy the first factor and an equal with plate deformability in order to satisfy the second.

In connection with the latter requirement, it may be stated that the framework should have no preferred directions as to its deformability.

3. <u>General Remarks on Pattern.</u>

Determination of pattern consists in assumption of a certain form for it and in testing its suitability, deriving at the same time the values for the necessary geometrical characteristics. While it is not inconceivable that some irregular pattern may be applicable theoretically, the practical use of such pattern would be inconvenient. For this reason, only forms possessing two axes of symmetry have been tried and such axes have been consistently used for the coordinate axes.

In the following discussion, conditions of plane stress are assumed everywhere, except when it is specifically stated that plane strain is being considered.

The law of elastic deformability of the framework with no preferred direction may be conveniently stated in terms of the following three conditions, explained with reference to Figures 2(a), 2(b), and 2(c), although other equivalent formulations are possible:

1. The framework is loaded uniformly with load p per unit length on X plane and νp on Y plane, Figure 2a. The resultant deformations should be the same as in the plate prototype. Therefore,

$$\epsilon_x = \frac{p(1-\nu^2)}{aE_x t} \qquad \text{(a)}$$

$$\epsilon_y = 0 \qquad \text{(b)}$$

where t is the thickness of the plate. The deformations of the framework ϵ_x and ϵ_y are expressed in terms of the geometry of the framework and of the cross-sectional areas of the bars, assumed to be made of the material with the same E as the plate.

Thus, the first condition results in two equations (a) and (b) serving to determine the framework characteristics.

2. Reversing the planes X and Y on which the loads p and νp are applied, two similar equations are produced, Figure 2b.

$$\epsilon_x = 0 \qquad \text{(c)}$$

$$\epsilon_y = \frac{p(1-\nu^2)}{aE_y t} \quad . \qquad \text{(d)}$$

It will be noted that the equation (c) is not independent and follows from (b) by Betti's reciprocal theorem. The second condition gives thus only one new equation (d) for determining the framework characteristics.

3. Uniform tangential load p per unit length is applied on X and Y planes, Figure 2c, and the resultant shear deformation is

$$\gamma_{xy} = \frac{2(1+\nu)p}{tE} . \qquad \text{(e)}$$

The four independent equations (a), (b), (d) and (e) allow to determine four characteristics of the assumed framework pattern. Should the number of characteristics be larger than four, the ones in excess of this figure may be assigned at will and should the number be less than four, the pattern type is either totally unfit or else is suitable only with some additional qualifications, as will be explained later.

In this discussion of the number of equations available for determination of the framework constants it has been assumed that the geometrical properties of the framework are different in the directions of the two axes of symmetry X and Y. If, however, the framework has similar properties in X and Y directions, condition 2 does not provide any new information, not contained in condition 1. In that case, only three independent equations, (a), (b) and (e) are available for finding the pattern constants.

A little thought will show that a framework type possessing requisite deformability under the three above mentioned load conditions will deform properly in any state of uniform stress, since any such state may be considered as a superposition of the three cases discussed.

Kinds of Framework.

4. Triangular Pattern.

Framework of Figure 3 is characterized by three constants, α, A and A_1, where A is the cross-sectional area of each vertical bar, and A_1 of each inclined one. Since the pattern is different in X and Y directions four equations are available for finding three constants, and consequently the type may prove suitable only under some special conditions, which become apparent after completion of the analysis. The size of the mesh a may be considered as an arbitrary infinitesimal length, and the problem consists in finding the constants in terms of a, E, ν and t, the latter being the thickness of the plate prototype.

Condition 1. Referring to Figure 3, let P be the horizontal load per joint, i.e., per vertical length 2b, and let S and S_1 be the stresses in the vertical and in the inclined bars respectively. Then, in view of the fact that the plate prototype has no vertical extension, the vertical loads per joint are alternately S and $(\nu P \cot\alpha - S)$, since $\frac{a}{b} = \cot\alpha$.

Before writing the equations (a) and (b) of the previous paragraph it is necessary to find the stresses S and S_1.

From a horizontal equilibrium of an outside joint:

$$2 S_1 \cos\alpha = P$$

or

$$S_1 = \frac{P}{2\cos\alpha} \qquad \text{(a)}$$

From the vertical equilibrium of one of the upper outside joints

$$S + 2\,S_1 \sin\alpha = \nu P \cot\alpha - S \;; \quad \text{or}$$

$$S = \frac{\nu P}{2}\cot\alpha - S_1 \sin\alpha = \frac{\nu P}{2}\cot\alpha - \frac{P}{2}\tan\alpha = \frac{P}{2}(\nu\cot\alpha - \tan\alpha);\quad \text{(b)}$$

Having found the bar stresses it is not hard to find the strains of the framework ϵ_x and ϵ_y ;

$$\epsilon_y = \frac{S}{AE} = \frac{P}{2AE}(\nu\cot\alpha - \tan\alpha) \qquad \text{(c)}$$

Unit strain of the inclined bar is

$$\epsilon_l = \frac{S_1}{A_1 E} = \frac{P}{2A_1 E\cos\alpha}. \qquad \text{(d)}$$

The three lengths which before the deformation are a, b and l form a right triangle both before and after the deformation. Therefore,

$$a^2 = l^2 - b^2$$

Differentiating,

$$a\,da = l\,dl - b\,db \qquad \text{(e)}$$

These differentials may be considered as the corresponding elastic changes

$$da = \epsilon_x a;$$

$$dl = \epsilon_l l = \frac{Pa}{2A_1E\cos^2\alpha}\;;$$

$$db = \epsilon_y b = \frac{Pa}{2AE}(\nu\cot\alpha - \tan\alpha)\tan\alpha = \frac{Pa}{2AE}(\nu - \tan^2\alpha).$$

Substitution of these values into (e) gives,

$$\epsilon_x a^2 = \frac{Pa^2}{2A_1E\cos^3\alpha} - \frac{Pa^2}{2AE}\tan\alpha\,(\nu - \tan^2\alpha);$$

or

$$\epsilon_x = \frac{P}{2E}\left[\frac{1}{A_1\cos^3\alpha} - \frac{\tan\alpha(\nu - \tan^2\alpha)}{A}\right]. \qquad \text{(f)}$$

Elastic extension of the plate prototype in X direction is $\frac{P(1-\nu^2)}{2a\tan\alpha\ Et}$, and Y direction zero.

The two equations of the framework, therefore, are:

$$\frac{P}{2E}\left[\frac{1}{A_1\cos^3\alpha} - \frac{\tan\alpha(\nu-\tan^2\alpha)}{A}\right] = \frac{P(1-\nu^2)}{2a\tan\alpha\ Et} \qquad (g)$$

and $\frac{P}{2AE}(\nu\cot\alpha - \tan\alpha) = 0;$ (h)

After simplification:

$$\left[\frac{1}{A_1\cos^3\alpha} - \frac{\tan\alpha(\nu-\tan^2\alpha)}{A}\right] = \frac{(1-\nu^2)}{at\tan\alpha} \qquad (g_1)$$

$$\tan^2\alpha = \nu \qquad (h_1)$$

Condition 2. In Figure 4 the framework is loaded with horizontal forces $P\nu\tan\alpha$ per joint and vertical forces S and (P–S) at alternate joints. The same symbols S and S_1 designate the stresses in the vertical and inclined members, although these values are, of course, different from the first case. As before:

$S_1 = \nu P\,\frac{\tan\alpha}{2\cos\alpha}$ (k) and

$S + 2S_1\sin\alpha = P-S$ or

$S = \frac{P}{2} - S_1\sin\alpha = \frac{P}{2}(1 - \nu\tan^2\alpha)$ (1)

$\epsilon_y = \frac{S}{AE} = \frac{P}{2AE}(1 - \nu\tan^2\alpha)$ (m)

while the determination of ϵ_x is not needed.

Equating ϵ_y to $\frac{P(1-\nu^2)}{2atE}$ gives the third equation:

$$\frac{P}{2AE}(1-\nu\tan^2\alpha) = \frac{P(1-\nu^2)}{2atE}\ ;\ \text{or}\ A = \frac{at}{1-\nu^2}(1-\nu\tan^2\alpha). \qquad (n)$$

Condition 3. The framework is loaded with horizontal forces P per joint and vertical forces $P\tan\alpha$ per joint as shown in Figure 5.

Vertical members are evidently unstressed, while members inclined in one direction have stress + S_1, and those inclined in the other direction - S_1.

From equilibrium of an outside joint:

$$2\, S_1 \operatorname{Cos}\alpha = P \; ; \qquad \text{or} \qquad S_1 = \frac{P}{2 \operatorname{Cos}\alpha} \; . \tag{p}$$

Deformation of a diagonal member

$$dl = \frac{S_1 l}{A_1 E} = \frac{Pa}{2 A_1 E \operatorname{Cos}^2\alpha} \; . \tag{r}$$

Considering the distortion of one of the triangles ABC (Figure 6) and assuming that the vertical member keeps its alignment in space, the displacement BD of the vertex B will be vertical in view of the equal and opposite in sign deformations of the members AB and BC. Shear strain of the framework

$$\gamma_{xy} = \angle DOB = \frac{DB}{a} = \frac{dl}{a \operatorname{Sin}\alpha} = \frac{P}{2 A_1 E \operatorname{Sin}\alpha \operatorname{Cos}^2\alpha} \; .$$

Equating this value to the shear strain in the plate under corresponding shear loads the fourth and last equation is obtained:

$$\frac{P}{2 A_1 E \operatorname{Sin}\alpha \operatorname{Cos}^2\alpha} = \frac{P}{2atG} = \frac{P\, 2(1+\nu)}{2atE} \; ;$$

$$A_1 = \frac{at}{2(1+\nu) \operatorname{Sin}\alpha \operatorname{Cos}^2\alpha} \; . \tag{s}$$

Equations (g_1), (h_1), (n) and (s) should be solved for A, A_1 and α .

$$\left.\begin{aligned} &\text{From } (h_1) \quad \tan\alpha = \sqrt{\nu} \; ; \\ &\text{Then from (n)} \quad A = at \; ; \\ &\text{From (s)} \quad A_1 = \frac{at}{2}\sqrt{\frac{1+\nu}{\nu}} \; ; \end{aligned}\right\} \tag{t}$$

Substitution of these values into the remaining equation (g_1), which must also be satisfied, leads to $\nu = \frac{1}{3}$. With this value for ν expressions (t) become:

$$\left.\begin{aligned} \alpha &= 30^\circ \\ A = A_1 &= at \end{aligned}\right\} \qquad \text{(u)}$$

Thus, a triangular framework may be used in plane stress problems as an analogue of a material with only one value of $\nu = \frac{1}{3}$; the triangles must be equilateral, and the cross-sectional area of each bar must be $A = \frac{\sqrt{3}}{2} a_1 t$, where a_1 is the side of the triangle and t the thickness of the plate prototype.

5. Simple Square Pattern with Intersecting Diagonals.

The type shown in Figure 7 consists of squares with two intersecting diagonals in each. All horizontal and vertical members have the same area A, while all diagonal members an area A_1. It is immaterial whether to consider the two diagonals in each unit as totally disconnected or pinned at their intersection at the centre. The pattern is characterized by two constants A and A_1. The number of equations available for determination of these is three, since the fourth equation drops out on account of identity of the framework in x and y directions. It may be expected, therefore, that the pattern is suitable only for some particular values of Poisson's ratio.

Condition 1. The framework is loaded with horizontal forces P per joint and vertical forces νP per joint (Figure 7). Determine stresses in the members. Since the framework has no vertical extension, $S_2 = 0$. From vertical equilibrium of an outside joint $S_1 = \frac{\nu P}{\sqrt{2}}$, and from horizontal equilibrium of an outside joint $S + \sqrt{2}\, S_1 = P$ or $S = (1-\nu)\, P$. The horizontal strain in the framework is $\epsilon_x = \frac{S}{AE} = \frac{(1-\nu)P}{AE}$ (a), while the vertical strain is found on the basis of deformations in the horizontal and diagonal members in the following manner, similar to the previous article, see Figure 8.

$$b^2 = l^2 - a^2 \quad \text{and} \quad bdb = ldl - ada \qquad \text{(b)}$$

$$db = b\epsilon_y = a\epsilon_y \ ,$$

$$da = a\epsilon_x = \frac{(1-\nu)Pa}{AE} \ ,$$

$$dl = \frac{S_1 l}{A_1 E} = \frac{\nu P a\sqrt{2}}{\sqrt{2}\,A_1 E} = \frac{\nu P a}{A_1 E} \ .$$

Substituting these expressions into (b)

$$\epsilon_y = \frac{\nu P\sqrt{2}}{A_1 E} - \frac{(1-\nu)P}{AE} = \frac{P}{E}\left(\frac{\nu\sqrt{2}}{A_1} - \frac{1-\nu}{A}\right) \ . \qquad \text{(c)}$$

Equating ϵ_x to $\frac{P(1-\nu^2)}{atE}$ and ϵ_y to zero

$$\frac{(1-\nu)P}{AE} = \frac{(1-\nu^2)P}{atE} \quad \text{or} \quad A = \frac{at}{1+\nu} \qquad \text{(d)}$$

$$\text{and} \quad \frac{P}{E}\left(\frac{\nu\sqrt{2}}{A_1} - \frac{1-\nu}{A}\right) = 0 \quad \text{or} \quad A_1 = \frac{\nu\sqrt{2}A}{1-\nu} = \frac{\nu\sqrt{2}}{1-\nu^2}at \quad \text{(e).}$$

Condition ~~3~~ 2. When tangential loads P are applied at outside joints, as shown in Figure 9, the horizontal and vertical bars are unstressed, while the diagonals are stressed with equal tensions and compressions $S_1 = \frac{P}{\sqrt{2}}$, and consequently, are changed in length by the amount $\delta = \frac{S_1 l}{A_1 E} = \frac{Pa}{A_1 E}$, (f), which transforms each square into a *rhombus*, see Figure 10. The shear strain of the framework γ_{xy} is found from

$$\tan\left(\frac{\pi}{4} - \frac{\gamma_{xy}}{2}\right) = \frac{\frac{a}{\sqrt{2}} - \frac{\delta}{2}}{\frac{a}{\sqrt{2}} + \frac{\delta}{2}} = \frac{1 - \frac{\delta}{a\sqrt{2}}}{1 + \frac{\delta}{a\sqrt{2}}} \ . \qquad \text{(f)}$$

Since $\tan\left(\frac{\pi}{4} - \frac{\gamma_{xy}}{2}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\gamma_{xy}}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\gamma_{xy}}{2}} = \frac{1 - \frac{1}{2}\gamma_{xy}}{1 + \frac{1}{2}\gamma_{xy}}$ (g)

the equations (g) and (f) give

$$\gamma_{xy} = \frac{\delta\sqrt{2}}{a} = \frac{P\sqrt{2}}{A_1 E} \qquad \text{(h)}$$

Equating this expression to the shear strain of the plate gives the third framework equation

$$\frac{P\sqrt{2}}{A_1 E} = \frac{P}{atG} = \frac{P\,2(1+\nu)}{atE} \qquad \text{from which}$$

$$A_1 = \frac{at}{\sqrt{2}\,(1+\nu)} \quad . \tag{k}$$

A_1 can satisfy expressions (k) and (e) only if

$$\frac{at}{\sqrt{2}(1+\nu)} = \frac{\nu\sqrt{2}}{1-\nu^2}\,at; \quad \text{which gives:}$$

$$\nu = \frac{1}{3}\ ; \quad A = \frac{3}{4}at, \quad \text{and } A_1 = \frac{3}{4\sqrt{2}}\,at \quad . \tag{l}$$

Thus, the type of the pattern considered here is good only for the same particular value of Poisson's ratio $\nu = \frac{1}{3}$.

6. <u>Rectangle With Two Diagonals</u> .

It has been expected that transformation of the square form of the previous pattern into an oblong would make the framework suitable for imitating a plate with any arbitrary value of the Poisson's ratio in view of the fact that four equations would be available to determine four characteristics, i.e., three areas and one ratio of the sides of the rectangle. For this reason an analysis of an oblong framework has been undertaken.

Condition 1. The framework is loaded with vertical forces P and horizontal forces $\frac{\nu P}{k}$ as shown in Figure 11, where k is the ratio between the horizontal and vertical sides of the rectangle. The stresses produced in horizontal, vertical and diagonal members are designated respectively by S, S_1 and S_2 while their respective areas are A, A_1 and A_2. Since there is no deformation of the framework in horizontal direction $S_1 = 0$. From statics:

$$2\ S_2\ \text{Cos}\alpha = \frac{\gamma P}{k} \qquad \text{or} \qquad S_2 = \frac{\gamma P}{2k\,\text{Cos}\alpha} \tag{a}$$

$$S + 2\ S_2\ \text{Sin}\alpha = P \qquad \text{or} \qquad S = P - \frac{\gamma P}{k^2} = P(1-\frac{\gamma}{k^2}) \tag{b}$$

since $k = \text{Cot}\alpha$.

Vertical strain $\epsilon_y = \dfrac{P(1-\frac{\gamma}{k^2})}{AE}$. (c)

Strain in the diagonal $\epsilon_l = \dfrac{\gamma P}{2k\ \text{Cos}\alpha\ A_2 E}$. (d)

From Figure 12

$$bdb = ldl - ada \qquad \text{or} \qquad \epsilon_x k^2 a^2 = \epsilon_l l^2 - \epsilon_y a^2$$

from which $\epsilon_x = \dfrac{1}{k^2}\left[\dfrac{\gamma P}{2k\,\text{Cos}\alpha\,A_2 E\,\text{Sin}^2\alpha} - \dfrac{P(1-\frac{\gamma}{k^2})}{AE}\right]$ (e)

The first two equations of framework are:

$$\frac{P(1-\frac{\gamma}{k^2})}{AE} = \frac{P(1-\gamma^2)}{katE} \qquad \text{or} \qquad A = \frac{(k-\frac{\gamma}{k})}{(1-\gamma^2)}\,at \tag{f}$$

and $\dfrac{\gamma P}{2k\ \text{Cos}\alpha\ \text{Sin}^2\alpha A_2 E} - \dfrac{P(1-\frac{\gamma}{k^2})}{AE} = 0$ leading to

$$A_2 = \frac{\gamma at}{2(1-\gamma^2)\ \text{Cos}\alpha\ \text{Sin}^2\alpha} \ . \tag{g}$$

Condition 2. With loads as in Figure 13,

$S = 0$, then by statics

$$S_2 = \frac{\gamma k P}{2\,\text{Sin}\alpha}; \qquad \text{and } S_1 = P - 2\ S_2\ \text{Cos}\alpha = P\ (1-\gamma k^2);$$

Therefore, $\epsilon_x = \dfrac{P}{A_1 E}(1-\gamma k^2)$ (h)

and the third equation is,

$$\frac{P}{A_1 E}(1-\gamma k^2) = \frac{P(1-\gamma^2)}{atE} \qquad \text{or} \qquad A_1 = \frac{1-\gamma k^2}{1-\gamma^2}\,at \ . \tag{k}$$

Condition 3. Under shear load, as shown in Figure 14, only diagonal members are stressed with compressions and tensions

$S_2 = \dfrac{P}{2\cos\alpha}$, and the change in length of a diagonal is

$\delta = \dfrac{Pa}{2\cos\alpha\,\sin\alpha\;A_2 E}$ which after substitution for A_2 of its expression (g) becomes $\delta = \dfrac{P\sin\alpha\,(1-\nu^2)}{\nu t E}$ (l). The shear strain of the framework (Figure 15) then is:

$$\gamma_{xy} = \frac{\delta\sec\alpha}{a} = \frac{P(1-\nu^2)}{k\nu t a E} \qquad \text{(m)}$$

and the fourth equation of the framework is

$$\frac{P(1-\nu^2)}{k\nu t a E} = \frac{P\,2(1+\nu)}{k a t E},$$

which after simplification gives $\nu = \frac{1}{3}$ while the ratio of the sides of the oblong k cancels out. Substitution of this value of into (f), (g) and (k) gives,

$$A = \frac{3}{8}\,\frac{3k^2-1}{k}\,ta\;; \qquad (1)$$

$$A_1 = \frac{3}{8}(3-k^2)\;ta\;; \qquad (2)$$

$$A_2 = \frac{3}{16}\,\frac{(1+k^2)^{3/2}}{k}\,ta\;; \qquad (3)$$

Thus, a somewhat unexpected result of this analysis is that the oblong framework, like two other previously discussed types, is only capable of representing a material with $\nu = \frac{1}{3}$, while the ratio of the sides of the oblong k may be arbitrary. The repeated recurrence of the figure $\frac{1}{3}$, which by the way is not far from the true value for metals, presents a curious fact.

It may be mentioned in brackets that outside of the range $\sqrt{\frac{1}{3}} \leqslant k \leqslant \sqrt{3}$ the areas A or A_1 in expressions (n) become negative, which makes the framework outside of these limits a physical impossibility. That, however, does not make the framework any less suitable for use in calculations in connection with problems of two-dimensional stress.

7. Square Pattern for an Arbitrary Value of ν.

It has been seen that unsuitability of the triangular and square patterns of framework for any arbitrary Poisson's ratio is traceable to deficiency in the number of the pattern characteristics by one unit. Therefore, it has been expected that introduction of an additional characteristic should remedy the situation.

Figure 16 represents the framework of square pattern modified by introduction of auxiliary horizontal and vertical members pinned to the diagonals. The cross-sectional area A_2 of these auxiliary members is the third characteristic of the pattern in addition to A, the area of the main horizontal and vertical members, and A_1, the area of the diagonals, both inside the square centre portion of each unit and outside of it. The centre portion, or heart, is made a square one-half of the size a of the unit.

Deformability of this type of framework will be computed on the basis of displacements of the main joints, i.e., the joints at the connections of the main horizontal and vertical members, in other words, strains in the framework will be measured by changes in position of the main joints. The external loads may be applied at the main joints only.

It follows from the conditions of equilibrium of the secondary joints that under any loading the four auxiliary members of each unit carry equal stresses. Parts of each diagonal inside and outside of the heart are stressed differently, but the two outside parts of every diagonal are stressed equally under any loading, which is evident from equilibrium of the two secondary joints on the same diagonal.

Condition 1. As shown in Figure 16, the framework is loaded with vertical forces P and horizontal forces γP at the exterior main joints. Since $\epsilon_x = 0$, the stress in the main horizontal members $S_3 = 0$.

From horizontal equilibrium of an outside main joint $S_1 = \frac{\gamma P}{\sqrt{2}}$ (a). From vertical equilibrium of an outside joint $S = P - \sqrt{2}\, S_1 = (1-\gamma)\, P$ (b). From symmetry and from the fact that all four auxiliary members have equal stresses it follows that the heart remains square in shape after the deformation. This consideration permits to determine the stresses S_2 and S_{10}, since from elastic considerations $\frac{S_2}{A_2} = \frac{S_{10}}{A_1}$, and from statics of a secondary joint $S_{10} + \sqrt{2}\, S_2 = S_1 = \frac{\gamma P}{\sqrt{2}}$, where from

$$S_{10} = \frac{\gamma P}{\sqrt{2} + 2\frac{A_2}{A_1}} \qquad \text{(c)}$$

and $$S_2 = \frac{\gamma P}{2 + \sqrt{2}\frac{A_1}{A_2}} . \qquad \text{(d)}$$

Strains between the main joints are:

vertical $$\epsilon_y = \frac{S}{AE} = \frac{(1-\gamma)P}{AE} \qquad \text{(e)}$$

and along the diagonal $$\epsilon_\ell = \frac{S_1 + S_{10}}{2A_1E} = \frac{\gamma P\left(1 + \frac{1}{\sqrt{2}}\frac{A_2}{A_1}\right)}{\sqrt{2}EA_1\left(1+\sqrt{2}\frac{A_2}{A_1}\right)} . \qquad \text{(f)}$$

As was shown in article 5

$$\epsilon_x = \frac{1}{a^2}(ldl - bdb) = 2\epsilon_\ell - \epsilon_y = \frac{\sqrt{2}\gamma P\left(1+\frac{1}{\sqrt{2}}\frac{A_2}{A_1}\right)}{EA_1\left(1+\sqrt{2}\frac{A_2}{A_1}\right)} - \frac{(1-\gamma)P}{AE} . \qquad \text{(g)}$$

The two equations of framework are:

$$\frac{(1-\gamma)P}{AE} = \frac{(1-\gamma^2)P}{atE} \quad \text{or} \quad A = \frac{at}{1+\gamma} \qquad \text{(h)}$$

and $$\frac{\sqrt{2}\gamma P}{EA_1}\,\frac{1+\frac{1}{\sqrt{2}}\frac{A_2}{A_1}}{1+\sqrt{2}\frac{A_2}{A_1}} - \frac{(1-\gamma)P}{AE} = 0 \quad \text{or} \quad \frac{\sqrt{2}A_1 + A_2}{A_1 + \sqrt{2}A_2} = \frac{(1-\gamma^2)A_1}{\gamma a t} . \qquad \text{(j)}$$

Condition 3. The shear loads P, applied to the framework in the same manner as in Figure 9 cause no stresses in the auxiliary members, which follows from considerations of symmetry. Therefore, all that has been said in connection with this condition in the article 5, dealing with simple square pattern, is equally applicable here, and leads to the expression (k) of that article:

$$A_1 = \frac{at}{\sqrt{2}\,(1+\nu)} \quad .$$

This value substituted into (j) gives an expression for A_2, which completes the set of values for different areas:

$$A = \frac{at}{1+\nu} \quad , \qquad (4)$$

$$A_1 = \frac{at}{\sqrt{2}\,(1+\nu)} \quad , \qquad (5)$$

$$A_2 = \frac{3\nu-1}{2(1+\nu)(1-2\nu)}\,at \quad . \qquad (6)$$

It may be seen that the previously discussed simple square pattern is a special case of the form now considered, since for $\nu = \frac{1}{3}$, $A_2 = 0$, and the auxiliary members thus disappear. A and A_1 are always positive, while A_2 may be either positive or negative. As ν varies between its physically possible limits $\frac{1}{2}$ and 0, A_2 changes all way from plus infinity through zero to $-\frac{1}{2}at$. The negative sign of A_2 makes impossible construction of a physical model of the framework for values of ν below $\frac{1}{3}$, but it does not affect its applicability in computations.

Attention is called to the fact that the type of the pattern here considered may become structurally unstable under compressive stress, in view of the absence of elastic restraint to rotation of the heart about an axis perpendicular to the plane of the framework. This instability, however, does not affect the bar stresses, which continue to be unique, as long as the distortions are small. Mathematical suitability of the square pattern with auxiliary members is thus not impared by instability.

The type discussed in this article completes the list of patterns that have been investigated. Other types are, of course, conceivable.

8. Applicability of Discussed Patterns to Different ν's and to Plane Strain.

The square pattern for $\nu = \frac{1}{3}$, by virtue of its simplicity, has important advantages over the square type for any arbitrary ν; for this reason, it is profitable to handle problems involving materials with any ν by means of simple square framework for $\nu = \frac{1}{3}$. Such substitution is perfectly correct if the boundary conditions are represented by some known forces or stresses applied at the boundary, with no external forces acting within the boundary. Stresses under these circumstances do not depend on ν. This deduction follows from independence from ν of the Airy's stress function in the absence of the body forces.[5]

If, on the other hand, external forces are applied within the boundary, or the boundary conditions involve some restraints to displacements, then the state of stress depends on ν, and substitution of a framework with a different ν is not correct.

---

5) Page 25.

Thus, the applicability of any framework with $\nu = \frac{1}{3}$ is really much wider than what might be expected from its title. The present discussion explains the reason for prominence given the framework with $\nu = \frac{1}{3}$ in this treatise.

Formulae developed so far for the values of pattern characteristics have presupposed the presence of plane stress. If it is necessary to apply the framework method to plane strain, these formulae should be modified by replacing E and ν with $\frac{E}{1-\nu^2}$ and $\frac{\nu}{1-\nu}$ respectively, as has been explained in article 1. It follows from this, that the first three types of pattern will be suitable in plane strain problems only for a material with ν determined from:

$$\frac{\nu}{1-\nu} = \frac{1}{3} \qquad \text{or} \qquad \nu = \frac{1}{4} \quad .$$

The necessary areas of bars in the square pattern with auxiliary members will be

$$A = (1-\nu)\ at \quad , \tag{4a}$$

$$A_1 = \frac{1-\nu}{\sqrt{2}}\ at \quad , \tag{5a}$$

$$A_2 = \frac{(4\nu-1)(1-\nu)}{2(1-3\nu)}\ at \quad . \tag{6a}$$

In calculating deflections, the bars must be considered as endowed with modulus of elasticity $\frac{E}{1-\nu^2}$. In all these expressions E and ν are the true modulus and the true Poisson's ratio of the material subjected to plane strain.

9. Framework With Finite Size of Unit.

It will be recalled that a framework of a suitable pattern is strictly equivalent to a plate only if the size of the unit is infinitesimal. However, solution of such infinitesimal framework would not be in any way different from solution of the plate prototype since it would apparently require handling of the same differential equations and boundary conditions.

In order to make the framework method suitable for practical use the size of the unit must be taken finite, which transforms the method from an exact into an approximate one. As a justification for this procedure a hypothesis is propounded that the use of finite units results in a fairly close approximation of stresses and deformations to their exact values, even if the size of the unit is fairly large.

This statement seems plausible if it is realized that the bar stresses in a finite framework satisfy exactly the equations of statics for a portion of the plate separated along any section, and at the same time they satisfy, to some measure, the requirements of continuity by virtue of continuity of the framework at the joints. Therefore, the plate stresses, obtained by proper interpretation of these bar stresses, may be expected not to diverge far from the true ones.

This consideration, however, does not constitute a proof of the above statement. It is felt, that the truth of the hypothesis can be demonstrated only by comparison of a number of framework solutions with the solutions by the theory of elasticity. One such comparison will be given later. Since its results are good, it carries with it a strong argument for the soundness of the hypothesis, although, of course,a single example cannot be taken in all fairness as a conclusive evidence.

Theoretically, the framework method may be applied to any plane stress problem irrespective of the boundary conditions and of the character of forces. Figure 17 shows a plate of a curvilinear outline indicated by dotted line, acted upon by forces applied both at the boundary and at interior points. A framework of proper pattern and size of unit is inscribed into the boundary. The acting forces are all transferred to the nearest main framework joints by statically determined pairs of bars, such as ab, aN or fh, fg etc. This procedure implies that the deformation of the marginal area of the plate between the dotted line and the boundary of the framework proper, i.e., the lines NA, AB, BC etc., is disregarded.

Use in the same problem of units of different sizes or even of different patterns is theoretically permissible. Half-size units near the border may bring framework closer to the actual boundary.

Marginal bars of the framework, i.e., bars on lines NA, AB, BC, CD etc. must have areas equal to one half of the areas of regular bars. This requirement should be evident, if one considers deformability of framework near the edge in conditions of uniform stress, referred to in previous articles as conditions 1 and 2. For the same reason, a bar belonging to units of different kinds, such as ABC in Figure 18, should have an area which is average for the two kinds.

Framework of any rectangular pattern is a structure highly statically indeterminate. To illustrate this point a two by three simple square framework, Figure 19, will be considered, although the same reasoning applies to the oblong framework. Assuming no restraints of the joints, there will be found one statically unknown quantity for each square of the first horizontal row and for the first square of the second row, with two unknowns for each of the subsequent squares of the second row. This makes the total number of unknowns eight. With joint restraints the number of unknowns will be even higher.

Squares with auxiliary members have the same number of unknowns as the simple squares, since from comparison of these two kinds of units it follows that the more complicated type of unit possesses four more joints and eight more members than the simpler type, so that the number of additional equations of statics, eight, is just sufficient to determine eight extra stresses.

In triangular framework the statical unknowns are less numerous, so that a two by three structure of Figure 20 has only two of them.

Large number of unknowns shows the formidable character of the problem of framework stress analysis (at least with rectangular units) and presents a good reason for keeping down the number of units by increasing their size. At the same time this puts limitation to applicability of the method to plates of curved or irregular outline and to the use, in the same problem, of units of different types, which would tend to multiply the number of unknowns.

These considerations restrict the use of triangular pattern to plates bounded by straight lines at 60° to each other, and make the square and rectangular patterns primarily useful only for plates of simple rectangular outline. Only rectangular patterns will be dealt with in the following discussion.

10. Outline of Methods of Framework Stress Analysis.

A. Method of Least Work.

Application of this method to framework is in no way different from its use for stress analysis of statically indeterminate structures in general. The large number of simultaneous equations for determination of statical unknowns makes the method very laborious and virtually unusable when many units are involved.

B. Solution by Displacements.

Referring to Figure 19, each joint has two degrees of freedom and two components of displacement caused by the elastic distortion of the structure. Without interference with the structural action of the framework one of its joints may be considered fixed and the freedom of another restricted to one degree. Taking components of displacements of different joints as unknowns, expressing in terms of them the stresses in all the members of framework, and setting up two equations of statics for the stresses meeting at each joint, a system of simultaneous equations is obtained, involving joint displacements as unknowns. Three of these equations are not independent from others since the external forces acting on the framework, which naturally is in equilibrium, must satisfy three equations of statics. In example of Figure 19 this procedure results in a system of $12(2)-3 = 21$ simultaneous equations against eight by Least Work.

The relative disadvantage of this method in comparison with Least Work is decreased as the number of units is increased, and as restraints are added to the framework, since such restraints add to the number of statical unknowns but cut out some of the displacement components.

The displacement method has been discussed here not for its value in itself, but rather because it leads to a method of successive movements, used throughout in the present investigation, a method which is in reality a practicable adaptation of the displacement method.

C. Method of Successive Movements

If the elastic displacements of the joints are found and the joints are brought into their true displaced positions, the stresses and external forces applied at each joint are mutually balanced. Instead of finding displacements from equations, one can make guesses at them, one after the other, on the basis of forces applied at different joints, displace the joints one by one by amounts guessed, calculate after each displacement the stresses in the members, brought about by these displacements, and from the bar stresses determine the remaining unbalanced forces at the joints and correct them again and again by similar procedures until a close balance is established at all joints. This constitutes the essence of the method, which, as may be easily seen, resembles somewhat the method of moment distribution of Professor Hardy Cross.

A circumstance highly favourable to use of this method in framework analysis is the identity of the pattern in all parts of the framework. For this reason, if a joint gets a unit horizontal displacement while the adjacent joints remain fixed, stresses brought about in members radiating from it are the same as the stresses in corresponding members caused by unit horizontal displacement of some other joint. Such stresses or rather values proportional to them will be referred to as distribution factors, and their determination will be undertaken now.

11. Determination of Distribution Factors.

A. Simple Square Pattern.

Let joint 0 (Figure 21) move upward a distance Δ , while all adjacent joints are held against any movement. Question is, what are the stresses brought about by this action.

According to (1), Art.5, the areas of horizontal and vertical members are $A = \frac{3}{4}$ at, and of diagonals $A_1 = \frac{3}{4\sqrt{2}}$ at. Stresses in horizontal members are evidently zero. Stress in vertical members is $S = \frac{AE\Delta}{a} = \frac{3}{4}Et\Delta$. (a). Since change in length of a diagonal is $\frac{\Delta}{\sqrt{2}}$, its stress is $S_1 = \frac{A_1 E \frac{\Delta}{\sqrt{2}}}{a\sqrt{2}} = \frac{3\sqrt{2}}{16} Et\Delta$ (b). H or V component of $S_1 = \frac{3}{16} Et\Delta$ (c).

It is not the stresses in the diagonals but their horizontal and vertical components that are used in balancing the joints. It will be noticed that comp. $S_1 : S = \frac{1}{4} : 1$. The figures $\frac{1}{4}$ and 1 are the distribution factors of the simple square framework. They are simply ~~ratios between~~ stress components corresponding to certain joint movement parallel to one of the axes, the value of the movement not being stated as immaterial. Distribution factors possess signs. On the side toward which the movement is made they are negative for compression, and on the opposite side, positive for tension. Their values are shown in circles in Figure 21. It is emphasized here that the values for diagonals represent not the stresses, but their vertical or horizontal components.

The distribution procedure will be described in full detail later; however, in order to give now some preliminary idea of the use of these factors, reference is made to Figure 22, representing a part of the framework, whose central joint is acted upon by a vertical force 100 and by a horizontal force 50, indicated by arrows. It is required to move this joint toward balance.

The joint is first moved upward 32 units, which causes stresses 32 and 8 in verticals and diagonals respectively, as recorded on the corresponding members. By the way, 8 is, of course, not the stress but a component of it. As a result of this movement, unbalanced forces 8 appear at the joints A, C, E and G and unbalanced forces 32 at the joints B and F, as shown by the arrows, while the force 100 at the joint O is reduced to $100 - \left[(4)\ 8 + (2)\ 32\right] = 4$.

All this means that an unbalanced force 96 has disappeared from joint O, while two forces each equal to $\frac{1}{3}$ of it have appeared at the joints B and F, and forces equal to $\frac{1}{12}$ of it have come to joints A, C, E and G. Horizontal unbalanced forces have also come to the latter four joints.

A similar procedure disposes of a larger part of the horizontal unbalanced force 50. The additional stresses caused by the new movement and the resultant unbalanced forces are all recorded on the diagram.

It will be recalled that a marginal member has an area twice smaller than that of an interior member parallel to it. For this reason its distribution factor is also twice smaller; this is indicated in Figure 23, giving factors for a horizontal movement of joint O to the right.

Factors corresponding to movement of a marginal joint in the direction normal to the margin are in no way different from the regular values of these factors.

B. Square Pattern With Auxiliary Members.

In view of similarity of the distributed factors in all four units adjacent to the joint undergoing a displacement, it is quite sufficient to consider a single unit, Figure 24, whose joint O moves upward a distance Δ, while adjacent main joints A, B, C do not move. The four inner joints move as much as necessary for equilibrium.

The areas of the members have been found in (4), (5), (6) of Art. 7 : $A = \frac{at}{1+\nu}$, $A_1 = \frac{at}{\sqrt{2}(1+\nu)}$ and $A_2 = \frac{3\nu-1}{2(1+\nu)(1-2\nu)}\,at$. Stress in the horizontal member AO is evidently zero, while stress in the vertical member,

$$S = \frac{AE\Delta}{a} = \frac{tE\Delta}{1+\nu} \quad . \tag{d}$$

It has been pointed out that the four auxiliary members carry equal stresses S_2, while the two outer portions of each diagonal are also stressed equally. Symbols S_a and S_p are selected to designate these latter stresses. The subscripts a and p stand for the words active and passive, which suggest whether the ends of the diagonal in question move (active) or do not move (passive).

Of five unknown stresses S_a, S_p, S_{ao}, S_{po} and S_2 only the first two are needed for conversion into distribution factors, the remaining three have no importance in themselves.

The following five equations are available, two of which are of statics and three of deformation.

1. From equilibrium of an interior joint of the passive diagonal

$$S_p = S_2 \sqrt{2} + S_{po} \tag{e}$$

2. Same for the active diagonal

$$S_a = S_2 \sqrt{2} + S_{ao} \tag{f}$$

3. The total length of the passive diagonal remains unchanged

$$S_p + S_{po} = 0 \tag{g}$$

4. The total length of the active diagonal changes by $\frac{\Delta}{\sqrt{2}}$

$$\frac{S_a + S_{ao}}{2} \; \frac{a\sqrt{2}}{A_1\ E} = \frac{\Delta}{\sqrt{2}} \tag{h}$$

5. The deformation of the heart of the unit is such that elongation of ED = $\frac{1}{\sqrt{2}}$ (elongation of EH + elongation of HD), see Figure 24 and 25. This follows from the fact that the diagonals remain orthogonal after the deformation, since the auxiliary members stay equal in length.

$$\frac{S_2 \frac{a}{2}}{A_2\ E} = \frac{(S_{po} + S_{ao})}{\sqrt{2}} \; \frac{a}{2\sqrt{2}} \; \frac{1}{EA_1} \tag{k}$$

The following expressions are obtained from the solution of these equations.

$$S_a = \frac{1}{4\sqrt{2}\,(1-\nu)}\, Et\Delta \quad , \qquad (l)$$

$$S_p = \frac{3\nu-1}{4\sqrt{2}\,(1-\nu^2)}\, Et\Delta \quad , \qquad (m)$$

$$S_{ao} = \frac{3-5\nu}{4\sqrt{2}\,(1-\nu^2)}\, Et\Delta \quad , \qquad (n)$$

$$S_{po} = \frac{1-3\nu}{4\sqrt{2}\,(1-\nu^2)}\, Et\Delta \quad , \qquad (p)$$

$$S_2 = \frac{3\nu-1}{4\,(1-\nu^2)}\, Et\Delta \quad . \qquad (q)$$

Thus:

$$S = \frac{Et\Delta}{(1+\nu)} \quad , \qquad (d)$$

$$\text{V or H Comp. of } S_a = \frac{Et\Delta}{8(1-\nu)} \quad , \qquad (r)$$

$$\text{V or H Comp. of } S_p = \frac{3\nu-1}{8(1-\nu^2)}\, Et\Delta \quad . \qquad (s)$$

The distribution factors designated by f with proper subscripts are taken proportional to these values as follows:

$$\left.\begin{aligned} f &= 1 \; ; \\ f_a &= \frac{1+\nu}{8(1-\nu)} \; ; \\ f_p &= \frac{3\nu-1}{8(1-\nu)} \; . \end{aligned}\right\} \qquad (t)$$

It is needless to say that for the members on the compression side of the displaced joint the distribution factors have the same expressions only with minus signs. The value of the factor for a marginal member adjacent to a joint, which is moved parallel to the margin, is again $\frac{1}{2}$.

It will be noticed that if the heart of a unit is cut by a section, such as XX in Figure 24, the resultant of the stresses in the three bars cut is equal to the stress in the outside part of the diagonal S_p or S_a, as the case may be, and the statical effect in any part of the framework is not altered if the units are imagined as having simple diagonals carrying proper stresses S_a and S_p without any auxiliary bars. For this reason, the distribution procedure in this type of the framework is identical with the simple type discussed in the previous article, except for different values of distribution factors and for the fact that on displacement of a joint both diagonals of a unit adjacent to it get stressed, while in case of framework for $\nu = \frac{1}{3}$ only active diagonals are affected.

Equations (t) may be rewritten in terms of the reciprocal of Poisson's ratio, $m = \frac{1}{\nu}$.

$$\left.\begin{aligned} f_s &= 1 \quad , \\ f_a &= \frac{m+1}{8(m-1)} \quad , \\ f_p &= \frac{3-m}{8(m-1)} \quad . \end{aligned}\right\} \qquad (7)$$

Numerical values of these factors for different values of m are given in Table 1, Plate 24.

Attention may be called to simple round figures of the factors for a number of different values of m, such as 2, 3, 5, 7, 11 and ∞. This simplicity has much to do with the ease of carrying out the distribution, and with the practicability of the method in general. The framework with m = 3 is, however, the easiest of all to handle, in view of the absence of stress in the passive diagonals.

C. Oblong Pattern.

Horizontal Displacement. The areas of the members according to equations (1), (2) and (3), Art. 6, are:

$$A = \frac{3}{8}\,\frac{3\kappa^2-1}{\kappa}\,ta, \qquad A_1 = \frac{3}{8}(3-\kappa^2)\,ta \quad \text{and} \quad A_2 = \frac{3}{16}\,\frac{(1+\kappa^2)^{3/2}}{\kappa}\,ta.$$

Elongation of horizontal member is Δ, while that of the diagonal is $\Delta \cos\alpha$ (see Figure 26).

$$S_1 = \frac{A_1 E\Delta}{\kappa a} = \frac{3}{8}\,\frac{3-\kappa^2}{\kappa}\,Et\Delta \quad , \qquad (v)$$

$$\text{H Comp. of } S_2 = \frac{\kappa}{\sqrt{1+\kappa^2}}\,\frac{3}{16}\,\frac{(1+\kappa^2)^{3/2}}{\kappa}\quad ta\ \frac{E\Delta\frac{\kappa}{\sqrt{1+\kappa^2}}}{a\sqrt{1+\kappa^2}} = \frac{3}{16}\kappa Et\Delta ; (w)$$

The distribution factors are obtained by changing these stresses in proportion, so that S_1 becomes unity.

$$\left.\begin{aligned} f_{s_1} &= 1 \quad , \\ f_{s_2 H} &= \frac{\kappa^2}{2(3-\kappa^2)} \end{aligned}\right\} \qquad (8)$$

The letter H in the subscript signifies that the corresponding factor refers to horizontal component of the stress in the diagonal, and that the movement of the joint, causing it, is horizontal.

Vertical Displacement. While elongation of the vertical member is Δ , that of the diagonal is $\Delta \operatorname{Sin} \alpha$. (see Figure 27) .

$$S = \frac{AE\Delta}{a} = \frac{3}{8}\,\frac{3\kappa^2-1}{\kappa}\,Et\Delta \quad , \qquad (y)$$

$$\text{V. Comp of } S_2 = \frac{1}{\sqrt{1+\kappa^2}}\,\frac{3}{16}\,\frac{(1+\kappa^2)^{3/2}}{\kappa}\,ta\ \frac{E\Delta\frac{1}{\sqrt{1+\kappa^2}}}{a\sqrt{1+\kappa^2}} = \frac{3}{16}\,\frac{Et\Delta}{\kappa} ; (z)$$

Distribution factors are:

$$\left.\begin{aligned} f_s &= 1 \quad , \\ f_{s_2 v} &= \frac{1}{2(3\kappa^2-1)} \quad . \end{aligned}\right\} \qquad (9)$$

Distribution procedure in oblong framework involves a complication, which has been absent in square framework, and which arises from inequality of the vertical and horizontal components of stresses in diagonals. It will be recalled that each joint displacement is accompanied by recording on the corresponding members the stress components caused by that displacement, so that the remaining unbalanced forces at each joint may be easily found by adding proper components. While in the square framework one figure written on the diagonal member suffices to show either of the two components of stress in it, since they both are equal, in oblong framework two figures might apparently be needed on each diagonal. Such double recording would complicate considerably the distribution form, and would make the method less workable.

Fortunately, however, there is a convenient way out, obviating the necessity of keeping double sets of figures on the diagonals. It consists in dividing, before the distribution is started, all horizontal acting forces, i.e., forces parallel to long dimension of the framework, by κ, while the vertical acting forces are left unchanged. After that, the distribution is proceeded with in ordinary manner, using the values of the factors just derived in (8) and (9) and writing single values on the diagonals. As this procedure is followed, figures written down on the members after each horizontal movement have the following meaning: on the horizontal members they are κ times less than the true stresses, and on the diagonal members, κ times less than the horizontal components of their stresses, or else, equal to the true values of their vertical components. Therefore, it is correct to balance the latter directly against the results of vertical distributions, representing true values of vertical components and stresses.

The distribution procedure with oblong framework may then be summarized as follows:

1. Divide all horizontal acting forces by κ.

2. Distribute all the vertical and the modified horizontal forces in an ordinary manner, using the factors (8) and (9).

3. Interpret the resultant figures on the members in the following manner: figures on the vertical (short) members are the true stresses in them; figures on horizontal members should be multiplied by κ in order to obtain the true stresses; figures on the diagonals represent the true vertical components of their stresses.

Numerical values of the distribution factors for different values of κ are given in Table 2, Plate 24.

It may be observed from equation (2), Art. 6, that for $\kappa = \sqrt{3}$, $A_1 = 0$, i.e., the horizontal members are absent. For $\kappa > \sqrt{3}$, A_1 becomes negative, which, however, does not affect the solution. A and A_2 are always positive.

12. Distribution Procedure in Simple Square Framework.

A. General.

Single joint movement parallel to one of the axes, explained in previous article, forms the basis for distribution procedure, but, although it has its proper place, its exclusive use would be highly cumbersome. In order to shorten the distribution and to make it more practicable, block movements are resorted to. Any such movement is quite legitimate as long as the stresses brought about by it may be clearly visualized. It does not pay to undertake too complicated a movement, whose stresses could not be easily calculated mentally, for fear of error. As long as the stresses are properly calculated and recorded the continuity of the framework is preserved, although the statics is not satisfied. The purpose of movements is to work closer and closer toward the state when all joints are in equilibrium. An error in recorded stress is tantamount to break in continuity of the framework.

A number of typical movements is given below. Although the movements are general to any kind of the framework, the values of stresses given all refer to simple square pattern for $\nu = \frac{1}{3}$. It is reminded, that what is referred to and recorded as stress in the diagonal is actually its V or H component.

B. Types of Block Movements, $\nu = \frac{1}{3}$.

a. Movement in a row, Figure 28. All joints, but F, G and H remain fixed. Joint F moves a units to the right which causes stress a in EF and stresses $\pm \frac{a}{4}$ in the four diagonals radiating from F. At the same time, joint G moves b units away from F, i.e., (a + b) units altogether, and joint H, c units away from G, i.e., (a + b + c) altogether. It will be noticed that while the horizontal members are stressed b and c, the diagonals meeting at G carry stresses $\pm \frac{a+b}{4}$ and the diagonals CH and MH have stresses $\frac{a+b+c}{4}$. All the stresses thus produced can be easily calculated mentally, and there is no difficulty visualizing the signs of stresses in the diagonals.

b. Movement in a row in a marginal panel, Figure 29, is analogous to the one just discussed. The only difference is due to twice smaller cross-sectional area of the marginal members compared to the interior ones.

c. Combined movement of a corner joint, Figure 30. Joint A at the corner of the framework moves a units down and as much to the right. Stress in the diagonal by superposition is evidently a.

d. Shear distortion, Figure 31. The lower row of joints or the whole lower part of the framework, is moved horizontally 4 κ units. All diagonals of the panel get stressed with $\pm\ \kappa$.

e. Direct stress, Figure 32. All lower part of the framework is moved down bodily a units. All verticals of the panel, except the marginal ones, are stressed a, and all diagonals are stressed $\frac{a}{4}$.

f. Interior block displacement, Figure 33. Block BDHF is moved to the right a units. Members AB and EF are stressed a, while all diagonals affected carry stresses $\pm\ \frac{a}{4}$, whose signs may be visualized.

g. Rotation about an exterior joint, Figure 34. The right part of the framework is rotated about O, so that joint E moves horizontally a units to the right. Therefore, joints F and G move 2a and 3a units respectively, which explains the stresses produced.

A modification of this movement is illustrated in Figure 35, in which there are two equal rotations in opposite directions about O_1 and O_2. Joints B and E move away from each other $\frac{a}{2}$ units each, joints C and F, as well as D and G do likewise, only move proportionately farther.

Rotations, such as explained in the last two examples are advantageous in problems involving antisymmetry, such as bending of a simple beam. This point will be discussed fully later.

h. Rotation about an interior joint is illustrated in Figure 36, in which the block OABC is rotated about point O.

k. Shear combined with tension, Figure 37. In this case lines AB, EF and GH are rotated separately through equal angles in the same direction about the points A, E and F. Stresses in the left vertical bay are the same as in case of rotation of Figure 34, while in the portion of the framework to the right of AD all diagonals are stressed $\pm \frac{a}{4}$.

All these movements and some of their combinations, as well as the single joint displacements are used in the distribution. Just which of these movements should be selected at any time, and how big should be the distortion is rather hard to state. General principle is, of course, to work towards balance of the framework and to move those joints first that are most unbalanced, but the ability to apply this principle to the best advantage depends mostly on experience. On the other hand, improper movements do not invalidate the work, but only retard the progress.

C. <u>Technique of Distribution and Manner of Recording</u>.

The technique and the form used in carrying out the distribution are explained on an imaginary example of Figure 38. It is felt that the use of any of the actual somewhat complicated problems for the purpose of explanation may lead to confusion.

Figure 38 represents a 2 x 3 framework acted upon by forces applied at most of the joints. A colored pencil is suitable for stating these forces and their arrows on the diagram. The four top joints are assumed restrained in vertical direction, while one of them, A, is prevented to move at all. The problem may be considered to represent one-half of a symmetrical plate loaded symmetrically, the line of restraint AD, corresponding to the axis of symmetry.

It was pointed out in the previous article that the motions of the joints are accompanied by shift of the unbalanced forces from joint to joint, while the algebraic sum of the unbalanced components at all the joints remains constant at all times. The purpose of distribution in the present problem is to shift vertical unbalanced forces to the line AD where they may be resisted by the restraining reactions, and to move those horizontal components, that will not get mutually balanced, to the joint A, the only joint restrained against a horizontal motion.

No reasons will be given for the motions which bring the framework into a state of near-equilibrium. A bona fide solution would require more movements, and that would only obscure the technique.

Each movement is accompanied by recording the stresses on the members affected, using a plus sign for tension and a minus for compression. When this is done, all the joints at both extremities of the affected members are gone over, and the unbalanced forces acting on these joints are modified by taking into consideration the newly added stresses. The resultant state of equilibrium of each joint is recorded by means of vertical and horizontal arrows, with appropriate numbers, in the direction of unbalanced forces. The original figure is simply rubbed out and replaced by a new one. After that, a new movement is carried out and the procedure repeated.

The following movements have been used:

1. Shear movement of the joints on the line KN to the right, causing stresses of $\pm$ 10 in all diagonals of the lower horizontal bay.

2. Vertical compression movement causing stresses -4 in all interior verticals, -2 in marginal verticals and -1 in all diagonals.

3. Rotation of the block EHNK about the joint E. As a result of it the verticals BF, CG and DH shorten 8, 16 and 24 units respectively, the latter vertical being a marginal one getting a stress -12. The diagonals get stresses: FA and FC, -2; GB and GD, -4; and HC, -6.

4. Horizontal movement on the middle line, in which joint G moves 8 units to the left and joint H approaches G by 4 units. Diagonals radiating from G get stressed $\pm$ 2, and diagonals HC and HM, -3.

5. Similar movement on the bottom line with joint L moving away from M, 8 units, and K away from L, 20 units.

6. Same on top line, with the members AB, BC and CD stressed in tension respectively, 6, 10 and 12.

The joint forces remaining after all these movements may be seen in Figure 38 near each joint in form of arrows. While most of these arrows represent unbalanced forces, those at point A and the vertical arrows at the other top joints are balanced by the reactions. However, in spite of this, they must be recorded for the purpose of check, as will be explained presently. In order to distinguish these forces from the unbalanced ones and from the acting forces, it is advantageous to use for them a pencil of a different colour.

In making the movements it is not practical to attempt to achieve a close balance of any of the joints at once. The best procedure, facilitated by simplicity of the ratio of the distribution factors exactly one to four, has been found to be as follows. If the unbalanced forces are expressed by thousands, first use distortions in round figures of thousands and hundreds of units. After some time, when the remaining unbalanced forces get reduced to hundreds, use distortions expressed in tens of units and so on. It is never wise to use small or fractional distortions when there are still fairly large unbalanced forces.

In any given problem it takes a certain amount of time to lower the order of unbalanced forces by one unit, this time being roughly independent, whether the order is lowered from thousands to hundreds or from hundreds to tens, etc.

D. Current Checks of Distribution.

In spite of the great simplicity of calculations associated with the distribution and consisting mainly in addition, subtraction and division by four and two of round figures, after hundreds and thousands of these actions, mostly made mentally, errors are bound to crop in. In a solution requiring sometimes several days to accomplish it would be unwise to proceed to any great length without some effective checks at regular intervals, and a method of distribution would be useless without such checks being available.

Two types of possible errors may conceivably occur. First - recording wrong stress on the member, the error consisting in wrong number, wrong sign or in an omission to record a stress in the member at all. Such an error is equivalent to failure to preserve continuity in the framework.

The measure adopted to remedy this sort of error is to go over the distribution after every 4 - 8 hours of work, dotting with a coloured pencil every stress that has been found correct. Experience has shown that, although the original sequence of movements is likely to be forgotten by the time of checking, the kind of every movement can always be reconstructed and the check can always be accomplished.

Except for one or two times early in the game, this check has revealed no errors and for this reason it has been discontinued in the last few problems on distribution. After experience has been gained, this check is scarcely needed, if the work is carried on carefully and without hurry.

Second kind of error is incorrect joint force resulting from erroneous addition. The effect of this error is that a joint, which is apparently in equilibrium, is actually out of balance. The experience has been that no reasonable amount of care insures freedom from this error. Fortunately, however, an easy check is available, consisting in adding up the joint forces and comparing the sums with the original ones.

In the example of Figure 38, the sums of the active forces at the beginning were 65 ↑ and 15 ←, and the sums of the joint forces at the end, including the coloured arrows on the top line of joints, are found to be equal to the same figures, and this constitutes the check.

If there is a discrepancy, all the joints are gone over, and the joint forces are recomputed and compared with their previously recorded values. As a preliminary step for this operation the bar stresses must be found by adding up several figures resulting from separate movements. This is shown in Figure 38. Recomputation of the joint forces is facilitated by the use of the form shown in Figure 39, reproducing this calculation for the problem of Figure 38. The form is self-explanatory.

No error in joint forces will be left undetected after this check. However, it has been found unwise to rely too much on it and to compute the joint forces at the time of distribution hurriedly or carelessly, since this check consumes considerable time.

It has been found most satisfactory to use this check two or three times in the course of a complicated distribution, applying it every time when the order of unbalanced forces has been lowered by approximately two units. The last application should follow completion of distribution.

E. Final Check.

Although proper current checks, as described in the previous sub-article, make an error in the distribution improbable, it is desirable to have some method of testing the correctness of the final result, especially if some of the checks have been waived. This can be easily done in a small fraction, perhaps one tenth, of the time required for original solution of the framework.

First step in this final check consists in distorting the structure in such a manner that all horizontal and vertical members are stressed to the values found in solution. This usually can be done in a number of ways, and anyone of them is permissible, if it is consistent with the restraints of the structure. Stresses present in the diagonals after this distortion are generally not their true stresses.

Second step consists in giving the structure shear distortions, so that the stresses in some diagonals get changed to their true values. All remaining diagonals must then also get their true values, which constitutes the check.

Figure 40 represents such a check of the solution of Figure 38 by means of the following operations:

a. Distortion of the left vertical row, keeping joint A fixed, so that the stresses in AE and EK are -2. Then stresses in the diagonals are: in EB, -1; in EL, +1; in KF, -2.

b. Similar operations, in succession, on the other vertical rows, keeping the top joints fixed in order to satisfy the conditions of restraint.

c. Distortion of the top horizontal row, leaving joint A stationary.

d. Distortion of the row EH. In the absence of prescribed restraining conditions, joint H has been assumed stationary, although any other joint might have been taken as such.

e. Distortion of the bottom row, with joint K assumed stationary.

Stresses in the diagonals are added up and found different from the true stresses, for which reason two additional operations are necessary:

f. Horizontal shear distortion of the lower bay of such magnitude that the stress in any of its diagonals assumes its true value. By comparing stresses in members EF in Figure 40 and 38 the required shear distortion must be such as to produce stress + 1 in diagonals parallel to EF and - 1 in the opposite ones. After applying this distortion the bottom bay diagonals are all found correct.

g. Shear distortion of the top bay causing stress - 3 in diagonals parallel to AF and + 3 in the ones parallel to EB. After this movement the values of all top diagonals check.

It is evident that vertical shear distortions are not necessary since they would be inconsistent with the conditions of restraint at the top line of joints, however, had the framework been free at that line it would be required to superimpose some additional vertical shear distortions.

13. Distribution in a Square Framework with $\gamma = 0$.

The whole procedure of distribution and recording described in the preceding article in connection with square framework for $\gamma = \frac{1}{3}$ holds good for a square pattern involving any other value of γ, although the numerical values of distribution factors are different, and the pattern of arbitrary ν possesses a peculiarity of causing stress in the passive diagonals.

In view of some work, referred to later in this treatise, involving square pattern of $\nu = 0$, the distribution factors and the stresses for some simple block movements in this type of pattern are given here.

As is evident from Figure 41, a simple displacement of one of the joints results in equal and opposite in sign stresses in the diagonals of the unit. Since any distortion of a unit is merely a combination of displacements of its four corners, stresses in the two diagonals of the same unit must always be equal and opposite in sign. This peculiarity of the square pattern for $\nu = 0$ accounts for some additional curious properties noted below.

Figures 42 and 43 give stresses due to row movements in interior and marginal rows. They are all computed by superposition of the values corresponding to the movements of separate joints. Figure 44 refers to the displacement of the framework corner along the diagonal.

Figure 45 shows stresses caused by a horizontal shear movement; unlike the previous movements, the stresses of this case are the same as for $\nu = \frac{1}{3}$.

Figure 46 corresponds to a direct stress distortion. A peculiar feature of it is that the diagonals are unstressed.

In the internal block displacement of Figure 47 the diagonals above BD and below FH carry the same stresses as in the framework with $\nu = \frac{1}{3}$.

In rotation about point O of the part of the framework to the right of O G, represented in Figure 48, the diagonals in all three units are stressed equally.

According to Figure 49, if the parts of the framework to the left and to the right of the bay shown are rotated about O_1 and O_2 away from each other through equal angles, the diagonals stay unstressed.

14. <u>Reduction of Distribution to a Different Value of Poisson's Ratio</u>.

Square patterns for ν^s other than $\frac{1}{3}$ are more complicated to work with than for $\nu = \frac{1}{3}$ for two reasons. First, there is participation of passive diagonals and the computer has to remember two values of distribution factors instead of one, i.e., one value for active and the other for passive diagonals. This condition, of course, causes the stresses corresponding to block movements to come out more complicated, except, perhaps, for $\nu = 0$.

Apart from this general condition, applicable to all ν^s, there is a second difficulty affecting those ν^s, whose factors are not expressed in round figures. Thus, for $\nu = 0.3$, $f_a = \frac{13}{56}$ and $f_p = -\frac{1}{56}$. Evidently, with such values of factors, mental computations during distributions are impossible, while the scheme of working in round figures, first, in hundreds of units, then in tens, etc. also falls through.

This second difficulty is particularly serious and it makes a direct solution for a Poisson's ratio like the above mentioned $\nu = 0.3$ impracticable.

There is, however, a method by means of which the difficulty may sometimes be overcome. This method, described below, allows to reduce a solution made for one value of Poisson's ratio to a different value of it; in other words, the problem may be solved for a convenient value of ν, and then a second solution is made for the difference in ν^s .

Let f_a and f_p (Figure 50) be the distribution factors in the active and passive diagonals for a certain value of ν, in other words, they are the stresses in the diagonals or rather their components, when the joint A has been displaced vertically an amount necessary to cause stress unity in the bar AB.

Consider a unit of Poisson's ratio ν in a framework that has been solved for certain load conditions. Let the stresses,(i.e., components of stresses) in the diagonals of this unit be $X_a + X_p$ in one of them and $Y_a + Y_p$ in the other (Figure 51). The part of the stress X_a is caused by the active participation of the diagonal EG, i.e., by movements of the joints E and G, while the passive part X_p is caused by displacements of the joints H and F. Similarly, Y_a is the result of movements of H and F, whole Y_p is due to motions of E and G. Stresses in the horizontal and vertical bars of the unit are S, S_1, S_2 and S_3.

Suppose now that the framework, while holding the load, has changed its value of Poisson's ratio from ν to ν' and m has changed to m'. The question is, how would this change affect the values of bar stresses.

Let the values of distribution factors for γ' be f_a' and f_p'. From (7), Art. 11,

$$f_a = \frac{m+1}{8(m-1)} \quad \text{and } f_p = \frac{3-m}{8(m-1)} \quad .$$

Therefore, $f_a - f_p = \frac{1}{4} = f_a' - f_p'$ and

$$f_a' - f_a = f_p' - f_p \qquad \text{(a)}$$

Call the ratio $r = \dfrac{f_a}{f_p} = \dfrac{m+1}{3-m}$. (b)

Now, imagine that all the movements by means of which the framework of γ has been balanced are repeated on the framework γ' and in doing that the joints are given each time the same number of units of displacement as was the case originally in the framework γ. As a result of this, the horizontal and vertical bars are carrying now in framework γ' the same stresses as they were carrying before in the framework γ, even though their strains are different by virtue of different areas.

The stresses in the diagonals are, however, not the same as before. Thus, the diagonal EG carries more stress by the amount:

$$\Delta X = \frac{f_a' - f_a}{f_a} X_a + \frac{f_p' - f_p}{f_p} X_p$$

which by using (a) and (b) transforms into

$$\Delta X = \frac{f_a' - f_a}{f_a} (X_a + r X_p).$$

Evidently, $r X_p = Y_a$, therefore,

$$\Delta X = \Delta Y = \frac{f_a' - f_a}{f_a} (X_a + Y_a) \qquad (10)$$

because the same expression must hold by analogy also for the other diagonal of the unit.

The original stresses were in static equilibrium at all joints of the framework; the stresses which are present now in the framework ν' are different from the original ones by the extra stresses in the diagonals ΔX and ΔY, and, consequently, they are out of equilibrium to the extent of these extra stresses. Therefore, it is necessary to make a distribution in the framework ν' for these extra stresses in the diagonals, which condition is shown in Figure 52. A superposition of the result of this solution on the original distribution in the framework ν gives true stresses in the framework ν'.

The unbalanced forces caused by ΔX will be small if ν' is close to ν, consequently, the distribution for them will be brief, which is the justification of the method. It is possible to make for convenience this second distribution using framework ν instead of ν' and to apply later a second correction for Poisson's ratio.

Equation (10) is inconvenient because, containing only active stresses in the diagonals, it requires a separation of stresses into active and passive parts during the distribution, which is not ordinarily done. This difficulty, however, is absent if $\nu = \frac{1}{3}$, because in that case the active stress X_a becomes the total stress X. Substituting $f_\alpha = \frac{1}{4}$ the equation (10) becomes for $\nu = \frac{1}{3}$.

$$\Delta X = \Delta Y = (4\,f_\alpha' - 1)\,(X + Y); \qquad (10a)$$

Thus, reduction of a solution made for $\nu = \frac{1}{3}$ to some other close value of ν presents no difficulty. All that is necessary is to load the framework ν' with stresses in the diagonals according to (10a), do the distribution, which should not be long because the unbalanced forces are small, and superimpose the result on the original distribution in the framework.

15. Doubling the Framework.

A. General.

As the size of the framework unit is decreased the framework solution approaches that of the theory of elasticity. Although the precision of such solution is generally not certain it may be tested by cutting the units in half, re-solving the problem and comparing the results. A close agreement should indicate that the stresses found by framework are not far from the truth. This doubling of mesh, however, nearly quadruples the number of joints and lengthens greatly the distribution. Any method, therefore, that may shorten the procedure would be welcome.

It has been noted that while in the process of distribution the framework, whose continuity is always preserved, passes through a number of configurations approaching closer and closer the state of equilibrium. It stands to reason that if the joints of the doubled framework are brought into the positions of equilibrium of the previously solved structure with twice larger mesh, the unbalanced joint forces of the doubled framework would be not large, and the lengthy distribution may be shortened.

B. Simple Square Framework.

ABCD in Figure 53 represents an interior unit of a simple square framework, whose stresses have been found N_1, N_2, N_3, N_4 and D_1 and D_2, the two latter being the H or V components of stresses in diagonals. The diagonals are assumed pinned at midpoint O, and E and F are the midpoints of the bars AB and AD. The problem is to find stresses in the members of doubled framework whose joints are held in positions of equilibrium of the points A, B, C, D, E, F, O etc. of the original mesh. It should be understood that the small mesh structure is not in equilibrium in this configuration.

Since on doubling the units the cross-section areas of members are halved, the stresses in the new mesh members lying along the old ones are also halved.

In order to find the stresses in the members shown dotted, the changes in distances OE and FE brought about by stresses in the large mesh framework will be determined by the method of virtual work.

Change in distance EO. Since the areas of the members in the large mesh framework according to (1), Art. 5, are $A = \frac{3}{4}$ at and $A_1 = \frac{3}{4\sqrt{2}}$ at, the elastic distortions of OA, OB and AB are as shown in Figure 54 (a). The equation of virtual work, done by loads and stresses of Figure 54 (b) on deformations of Figure 54 a gives:

$$\text{Increase in length OE} = \delta_{OE} = \frac{4}{3tE}\left(D_1 + D_2 - \frac{N_1}{2}\right); \quad (a)$$

Change in distance between E and AO. The elastic distortions of the sides of the triangle AEO are shown in Figure 55 (a). From the equation of virtual work done by stresses of Figure 55 (b) on deformations of Figure 55 (a):

$$\Delta_1 = \frac{2\sqrt{2}\, D_2}{3tE} \quad .$$

Therefore, the increase in length EF = $2\Delta_1 = \frac{4\sqrt{2}\, D_2}{3tE}$ = one half of the deformation of the diagonal DB. This result is evident, if it is realized that EF is a median line in ABD (Figure 53) and its length does not depend on AD and AB, but only on BD.

Since the new bar areas are twice smaller than the corresponding old ones, stress in EF $= \frac{D_2}{2}$, and stress in OE $= \frac{A'E\Delta}{a'} =$

$$= \frac{3/4\, a'tE}{a'} \, \frac{4}{3tE}\left(D_1 + D_2 - \frac{N_1}{2}\right) = \left(D_1 + D_2 - \frac{N_1}{2}\right). \qquad \text{(b)}$$

Here A' and a' refer to small mesh, and N_1, D_1 and D_2 to the large mesh.

C. Square Framework for an Arbitrary ν.

Figure 56 represents a large mesh unit with stresses N_1, N_2, N_3, N_4 and S in its horizontal and vertical members and the components of stresses in the diagonals: D_1, D_2, C_1 and C_2. The diagonals of the heart are pinned at O.

Before determining the relative movements of the midpoints E, F and O brought about by stress distortions, the stresses C_1, C_2 and S will be expressed in terms of D_1 and D_2 with reference to Figure 57 (a).

From statics: $C_1 = D_1 - S$ (c)

$$C_2 = D_2 - S \qquad \text{(d)}$$

According to (4), (5), (6) of Art. 7 the bar areas are:

$$A = \frac{at}{1+\nu} \quad ; \quad A_1 = \frac{at}{\sqrt{2}\,(1+\nu)} \quad ; \quad A_2 = \frac{(3\nu-1)\,at}{2(1+\nu)(1-2\nu)} .$$

The third equation for finding C_1, C_2 and S may be written by virtual work. The bar distortions in Figure 57 (a) are:

$$\delta_{GH} = \frac{S\,\frac{a}{2}}{A_2 E} = \frac{(1+\nu)\,(1-2\nu)}{(3\nu-1)\;tE}\,S \qquad \text{(e)}$$

$$\delta_{GK} = \frac{C_1\sqrt{2}\,\frac{a}{\sqrt{2}}}{A_1 E}$$, which by using (c) becomes

$$\delta_{GK} = \frac{1+\nu}{tE}\sqrt{2}\,(D_1 - S) \quad ; \qquad (f)$$

similarly, $$\delta_{HL} = \frac{1+\nu}{tE}\sqrt{2}\,(D_2 - S) \quad . \qquad (g)$$

Virtual work of stresses of Figure 57 (b) on deformations of 57 (a) gives

$$\frac{4(1+\nu)(1-2\nu)}{(3\nu-1)\,tE}\,S + 2\,\frac{1+\nu}{tE}\left[(S-D_1)+(S-D_2)\right] = 0 \;;$$

From which $S = \frac{3\nu-1}{2\nu}\,(D_1 + D_2)$. (h)

Then from (c) and (d)

$$C_1 = D_1 + \frac{1-3\nu}{2\nu}\,(D_1 + D_2) \quad , \qquad (k)$$

$$C_2 = D_2 + \frac{1-3\nu}{2\nu}\,(D_1 + D_2) \quad . \qquad (l)$$

Following the procedure of the previous sub-article, changes in distances OE and EF will be found now.

Change in distance OE is affected only by the deformations of the members AB, AGO and BHO (Figure 56). Other members, except GH cause only rotation of the triangle OGH about O, which does not influence the distance OE, while a change in GH also leaves point O unmoved.

Lengthenings of the members affecting OE are as follows:

$$\delta_{AB} = \frac{N_1 a}{AE} = \frac{1+\nu}{tE}\,N_1 \quad , \qquad (m)$$

$$\delta_{AO} = \frac{D_1+C_1}{\sqrt{2}}\;\frac{a}{\sqrt{2}\,A_1 E} \quad ,$$ which on substitution of proper values for C_1 and A_1 becomes:

$$\delta_{AO} = \frac{1+\nu}{2\sqrt{2}\,\nu tE}\left[(1+\nu)D_1 + (1-3\nu)D_2\right] \;; \qquad (n)$$

similarly, $$\delta_{BO} = \frac{1+\nu}{2\sqrt{2}\,\nu tE}\left[(1-3\nu)D_1 + (1+\nu)D_2\right] \quad . \qquad (o)$$

Equation of virtual work, done by the same stresses as in Figure 54 (b) on the deformations (m), (n) and (o), gives the value of δ_{oE} the lengthening of the distance EO.

$$\delta_{oE} = \frac{1}{\sqrt{2}} (\delta_{Ao} + \delta_{Bo}) - \frac{1}{2} \delta_{AB} \; ; \qquad (p)$$

which on substitution of the values of δ^s becomes:

$$\delta_{oE} = \frac{1+\nu}{2tE} \left[\frac{1-\nu}{\nu} (D_1 + D_2) - N_1 \right] \; ; \qquad (q)$$

Knowing δ_{oE}, δ_{Ao} and $\delta_{4E} = \frac{1}{2} \delta_{AB}$ the distance Δ_1 by which point E moves away from the straight line AO (joint G does not in general lie on this straight line) can be found by virtual work of stresses of Figure 55 (b):

$$\Delta_1 = \frac{1-\nu^2}{2\sqrt{2}\,\nu tE} (D_1 + D_2) - \frac{1+\nu}{4\sqrt{2}\,\nu tE} \left[(1+\nu) D_1 + (1-3\nu) D_2 \right] =$$

$$= \frac{1+\nu}{4\sqrt{2}\,\nu tE} \left[(1 - 3\nu)\, D_1 + (1+\nu)\, D_2 \right] \; . \qquad (r)$$

Since this expression does not depend on N_1,

$$\delta_{EF} = 2\Delta_1 = \frac{1+\nu}{2\sqrt{2}\,\nu tE} \left[(1 - 3\nu)\, D_1 + (1+\nu)\, D_2 \right] ; \qquad (s)$$

Using expressions for C_1 and A_1 as well as (s), it is easy to show that $\delta_{EF} = \delta_{OB}$.

Suppose now the units are doubled (Figure 58), while the areas of the members are halved, the new principal joints being placed at the mid-points such as E, F and O. The new stresses in the members along the periphery of the old units, such as AEB, are evidently equal to half of the old stresses.

Stresses in the outside parts of the diagonals of the new unit AEOF depend only on δ_{Ao} and δ_{EF}, not depending on δ_{Eo} or δ_{Fo}. Since, however, $\delta_{Ao} = \frac{1}{2} \delta_{AL}$ and $\delta_{EF} = \frac{1}{2} \delta_{BK}$, the stresses in the outside parts of new diagonals are respectively equal to one half of stresses in old diagonals parallel to them.

$$\text{Stress in OE} = \frac{A' E \delta_{OE}}{a'} = \frac{1}{2}\left[\frac{1-\nu}{\nu}(D_1 + D_2) - N_1\right]. \quad (t)$$

This expression becomes indeterminate for $\nu = 0$, and in order to find stress in OE in that case it is necessary to go back a few steps. It will be recalled that for such case $D_1 = -D_2$ and from Figure 57 it is clear that $S = 0$ and $C_1 = D_1$ and $C_2 = D_2$. From this it follows that $\delta_{AO} = -\delta_{BO}$, and the equation of virtual work will give instead of (p) a simpler expression $\delta_{OE} = -\frac{1}{2}\delta_{AB}$, which will result in:

$$\text{stress in OE} = -\frac{N_1}{2} \quad . \qquad (u)$$

The previous discussion has been dealing with an interior unit, but it holds also for the marginal unit with the following single modification caused by twice smaller area of the marginal members. If a side of the unit, for example AB, is a part of the margin, and its stress is N_1 as before, then the expression for the stress OE of the doubled framework should contain the term $-N_1$ instead of $-\frac{N_1}{2}$.

All these deductions concerning doubling of units in the square frameworks with $\nu = \frac{1}{3}$, ν = any value, except zero, and $\nu = 0$ may be summarized by means of diagrams of Figures 59, 60 and 61. It will be remembered that the large mesh framework is in equilibrium under the stresses shown, while the small mesh structure is not, although its continuity is everywhere preserved. The probability is that the unbalance of the small mesh is not large, and that time may be saved by using first this doubling scheme and then proceeding with the distribution, instead of starting it from the original unbalanced forces.

It may be added, that the experience with doubling procedure has not proved beyond doubt its time saving value.

16. Principle of Symmetry and Antisymmetry.

As the units become more numerous, the labour involved in distribution increases greatly, much faster than in proportion to their number, and the method soon becomes unworkable. Fifty or sixty degrees of freedom of the joints, corresponding to an equivalent of a 4 x 5 framework, is probably all that can be handled without an excessive expenditure of time.

It is possible, however, at least in some cases, to push this limit further by utilization of the principle of symmetry. The necessary and sufficient conditions for applicability of this principle are as follows:

a. The framework should be symmetrical about one or two or more axes.

b. It should be either unrestrained at all, or restrained similarly, but not necessarily in identical manner, at the symmetrical points.

The second condition is explained by reference to Figure 62. If any point, such as A, is restrained by being given a known displacement δ_1 (which may be zero), in a certain direction, for example, along Y axis, then the symmetrically located points B, C and D should also be given some known, but not necessarily the same, displacements along the same Y axis. A known displacement of point D along X instead of Y axis or an absence of any restraint there would destroy the applicability of the principle in relation to the X axis.

In order to have the principle still applicable to Y axis points A and B should be restrained similarly.

The principle does not qualify in any way the manner of action of any known forces applied to the framework.

If the two above stated conditions are satisfied in relation to two axes of symmetry, the framework problem may be broken up into four symmetrical and antisymmetrical cases each of which involves consideration only of a quadrant of the plate. The four cases must be solved separately and then combined. The problem of distribution in the given framework is thus replaced by four distributions in the frameworks of one quarter of the size of the given one. With large number of units this substitution is bound to result in a saving of time.

The method of forming symmetrical and antisymmetrical cases out of the given problem is illustrated, in the presence of two axes of symmetry, in Figure 63 (a) to (e). The first of these five figures shows a plate or framework, symmetrical about X and Y axes, with one of the forces P_x acting on it at point M parallel to X axis. The four other figures show the four cases into which the problem is broken up. Case 1, Figure 63 (b) is symmetrical about both axes; case 2 is symmetrical about X axis and antisymmetrical about Y axis; case 3 is antisymmetrical about X axis and symmetrical about Y axis; while in case 4 both axes are the axes of antisymmetry.

It is convenient in its suggestiveness to designate the axes of symmetry by capital letters S with subscripts x or y, as the case may be, and the axes of antisymmetry by letters A with the same subscripts. This has been done in Figure 63 and everywhere in the following discussion.

It may be observed that the plate in each of these four component cases is acted upon by four forces $\frac{P_x}{4}$ applied at point M and at three other symmetrical points. The force applied at M acts in all four cases in the same direction as the original force P_x in Figure 63 (a), while the forces applied at three other points act sometimes in the same direction and sometimes in the opposite one, depending on whether they are located on the other side of S axis or of A axis in relation to point M. Thus, forces applied at M_1 act in the opposite direction to the one applied at M, i.e., symmetrically with respect to it, if Y axis is the S axis, and in the same direction, i.e., antisymmetrically, if the Y axis is the A axis.

It may be noticed, that if the four component cases are superimposed on each other the original case with a single force P_x, acting at M, results.

Forces acting not in the direction of the axes must first be broken up into components parallel to the axes.

A special case arises if a force is applied at one of the axes, as shown in Figure 64 (a) to (e). A horizontal force P_x, applied at point M on Y axis, when broken up into the component cases, appears only in cases 2 and 4, both antisymmetrical about Y axis. It is convenient to consider, that two forces $\frac{P_x}{4}$ are acting in each of these two cases, either at point M or at M_1 each applied to its respective quadrant of the plate as shown.

The force Q_x applied at point N on X axis, appears only in cases 1 and 2, where one force $\frac{Q_x}{4}$ is applied to each quadrant.

A more restricted special case of a single force P_x acting horizontally at the centre of symmetry is shown in Figure 65. The only component case affected by this force is case 2, S_x, A_y, in which four forces $\frac{P_x}{4}$ appear at the corner of each respective quadrant.

A similar discussion is applicable to breaking up of the known restraints, Figure 62, at the symmetrical points. The four cases are illustrated in Figure 66.

It may be seen that in any of the four component cases the state of stress and deformation in any quadrant resembles closely the state in the three other quadrants. A little thought shows that in two quadrants located perpendicularly opposite across an S axis the situation at corresponding points is as follows: the displacements are symmetrical, the normal stresses and strains along the axes are identical and the shearing stresses and strains are equal and opposite in sign, while in quadrants located oppositely across an A axis the displacements are antisymmetrical, the normal stresses are equal and opposite in sign, and the shearing stresses are identical. This explains why it is sufficient to consider only one quarter of the whole plate in any of the constituent cases, and by doing that to help greatly the distribution.

The conditions of stress and deformation at the axes, when a quadrant of the plate is separated out of the whole plate, are shown in Figure 67. Joints on S axes have no displacements normal to them being restrained by reactions perpendicular to S axes. Joints belonging to A axes move only perpendicularly to them and are restrained by reactions in the directions of A axes; at the same time, members lying along the A axes are unstressed. All these points are illustrated in Figure 67 (a) to (d), showing quadrants, separated from the original plate, for the four component cases.

The unknown reactions at the axes are found from distributions. The bars along the axes are considered as ordinary marginal bars.

It may be stated that further subdivision of each quadrant of plate into four smaller quadrants is generally impossible since the second condition necessary for applicability of symmetry principle does not hold, in view of the axis joints being restrained, while the joints on the opposite sides are not.

The method of successive movements in conjunction with the principle of symmetry presents an exceedingly powerful tool in framework analysis. One of the problems discussed later involves 8 x 12 framework with 404 members, out of which 173 are redundant, and with 117 joints. Its solution by the method of Least Work would require 173 simultaneous equations, and by the Joint Displacements, 231 simultaneous equations. Solution of such number of equations, even in the problems of research, seems to be entirely out of the question as a practical possibility, and the use of a slide rule is worthless from the viewpoint of accuracy of the results. The problem has been solved by successive movements of the four component cases. Each case has taken between forty and fifty hours with additional twenty hours at the end for combining the results, altogether some two hundred hours of work. The unbalanced joint forces, of the order of several thousand pounds at a joint, have been reduced in distribution to a fraction of a pound.

It may be added that each of the four component cases contains between 38 and 48 redundants and 57 or 58 degrees of freedom. Even solution of four sets of 38 to 48 simultaneous equations would be practically impossible.

17. Interpretation of Framework.

A. General.

The problems coming under this heading are of two kinds: one - how to apply the forces acting on the plate prototype to its framework analogue and the other - how to convert the bar stresses into the continuous plate stresses.

Some of the questions belonging here are difficult to answer and this part of the framework theory, more than any other, requires additional thought and investigation. Minor disagreements with the theory of elasticity, where they exist, are mostly traceable to this source.

When dealing with framework of infinitesimal units no interpretation difficulties arise. Any force acting on the plate can be applied to the infinitesimal framework at the proper place, since joints are available everywhere, and again the definition of unit stress in such framework is in no way different from the definition of unit stress in the plate, and such framework stress varies continuously from point to point.

The finite framework is, however, different. The forces must be applied at the joints only, but the joints are few and far between, and so there is generally an error in the point of application. On the other hand, if the same definition of framework unit stress is to be used with finite units as with infinitesimal, the value of the stress will depend on the manner of spreading the individual bar stresses over the tributory areas.

For the sake of convenience, the unit stresses should be computed first on the planes of the framework axes, and only after that reduced, if necessary, to other planes by usual formulae.

B. Conversion of Bar Stresses Into Plate Stresses.

a. Normal Stresses.

In order to find the normal stresses in plate on the line AE, Figure 68 (a), from the bar stresses, a section MM is passed and the stresses in the bars cut are converted into the normal joint concentrations, Figure 68 (b), by summing up the normal components, for example, $N_B = S_1 + S_2 + S_3$. In converting intermediate concentrations N into stresses, the stress diagram, Figure 68 (c), is assumed polygonal in shape, and each joint ordinate is found from $\sigma = \frac{N}{at}$, (a) t being the thickness of the plate. This is equivalent to transformation of each N into a triangle of base 2a with maximum ordinate σ, which is in agreement with both conditions of statics ΣH and ΣM.

For determining the end ordinate, such as σ_A, it would seem in line with the above procedure to spread N_A over the area of the triangle ab 1 ; then $\sigma_A = \frac{2N_A}{at}$. (b)

This, however, would in general violate statics since the center of gravity of the triangle and N_A are not opposite each other. For this reason three cases will be distinguished:

1. Concentrations N symmetrical about the centre C, as happens in any of the loading cases, symmetrical about the horizontal axis, discussed in previous article. Here σ_A is taken by formula (b) and no violation of the moment equation for the whole section results in view of the symmetrical situation on the other side.

2. Concentrations N antisymmetrical about C as in Figure 69 (a) and (b). In order to satisfy statics the moment of the triangle ab 1 about C must be equal to the moment of N_A, while the area of ab 1 need not be equal to N_A, since ΣH condition for the whole section is satisfied automatically on account of antisymmetry. Therefore,

$$\sigma_A = \frac{2}{(2-\frac{1}{3})} \frac{2N_A}{at} . \qquad \text{(c)}$$

In general, with 2n units in the section

$$\sigma_A = \frac{n}{n-\frac{1}{3}} \frac{2N_A}{at} . \qquad \text{(d)}$$

3. General case. The end concentrations N_A and N_E are broken up into symmetric and antisymmetric parts:

$$N_{Sym.} = \frac{N_A+N_E}{2} \text{ either at A or at E,}$$

$$N_{Ant.} = \frac{N_A-N_E}{2} \text{ at A and } \frac{N_E-N_A}{2} \text{ at E,}$$

after which each part is treated as in the two above cases.

This method of stress interpretation, referred to below as Method 1, besides being in agreement with statics and with the definition of stress in the infinitesimal framework, has been corroborated fairly well on the example discussed in the next article.

Other methods have been tried. In one of them, Method 2, the ordinates of the stress polygon σ in Figure 68 (c) have been computed as follows. Parts ab, bc, etc. have been assumed to be simply supported beams loaded with polygonal load σ. The key ordinates σ_A, σ_B, etc. have been found from the condition that the simple beam reactions at a, b, etc. are equal to the actual joint concentrations N. The results turned out to be unsatisfactory in addition to being fairly complicated.

Method 3 for interpreting normal stresses is based not on statics but on considerations of deformability.

By well-known formulae of elasticity for plane stress:

$$E\epsilon_x = \sigma_x - \nu\sigma_y \,,$$

$$E\epsilon_y = \sigma_y - \nu\sigma_x \,, \text{ from which}$$

$$\sigma_x = \frac{1}{1-\nu^2}(E\epsilon_x + \nu E\epsilon_y); \qquad \text{(e)}$$

Considering first an infinitesimal framework, let the stresses in horizontal and vertical bars at any point of the framework be S_x and S_y. Since the area of such bars is $A = \frac{at}{1+\nu}$, the strain in the framework is

$$E\epsilon_x = \frac{S_x}{A} = \frac{(1+\nu)S_x}{at} \qquad \text{and} \qquad E\epsilon_y = \frac{(1+\nu)S_y}{at} \,,$$

which on substitution into (e) give

$$\sigma_x = \frac{1}{at}\,\frac{S_x + \nu S_y}{1-\nu} \qquad \text{(f)}$$

and a similar expression for σ_y.

Stresses S_x and S_y must refer to the same point of the framework or plate. In finite framework, however, no such stresses are available, and the equation (f) may be used only approximately. Thus, the point, where the plate stresses are being determined, may be taken as the mid-square point O (Figure 70). Then the most logical values for S_x and S_y should be the averages of the corresponding bar stresses on the two sides: $S_x = \frac{1}{2}(S_1 + S_5)$ and $S_y = \frac{1}{2}(S_3 + S_6)$.

If the point of stress is taken at midbar, point 1, then $S_x = S_1$ and $S_y = \frac{1}{4}(S \;\; + S \;\; + S \;\; + S \;\;)$. For the stresses at the joint point A, $S_x = \frac{1}{2}(S_1 + S_2)$; and $S_y = \frac{1}{2}(S_3 + S_4)$.

When using stresses of marginal bars in these expressions they should be doubled.

This method for interpreting normal stresses has proved less satisfactory than the first one.

In the Methods 1 and 2, it has been assumed that no body forces are applied to the joints on the line AE (Figure 68), and as a result, the concentrations N are not any different if section M_1M_1 is used instead of MM. If, however, there are body forces, such as P (Figure 71), coming from load distributed over the plate all around point A, the normal concentrations at A on MM and M_1M_1 are different, and their average value should be taken for determination of normal stress at A.

If the body force P comes from load distributed over the plate on the right side of A only, stress at A should be calculated on the basis of concentration on M_1M_1 rather than MM.

The force P may also be caused by a single concentration, rather than by a distributed load. What happens in that case is fully discussed later in connection with the gusset plate problem.

Method 1 is fully applicable also to determination of the normal stresses in the plate at the restrained periphery. The normal joint reactions at the periphery, found from distribution, play evidently the part of the internal joint concentrations N.

b. Shear Stresses.

In finding shear stresses in the plate on the plane AE, Figure 72 (a), from the framework stresses, the first step is to find tangential concentrations at the joints A, B, etc. However, unlike the normal concentrations, the tangential ones are different for the planes MM and M_1M_1. Thus, at the joint B the concentration on the left section is $(S_2 - S_3)$ and on the right $(S_3' - S_2')$, which differs from $(S_2 - S_3)$ by $(S_{AB} - S_{BC})$.

This difference of the two concentrations is non-existant in infinitesimal framework, where $(S_{AB} - S_{BC})$ is an infinitesimal of a higher order than any of the bar stresses, and, consequently, $S_2 - S_3 = S_3' - S_2'$, disregarding higher order infinitesimals.

The inequality of the two tangential concentrations at B is thus directly traceable to the finite size of the unit, and neither of the two concentrations may be considered as corresponding to the shear in the plate at B.

It seems reasonable to think that the left concentration corresponds to an average shear condition on some length to the left of B, and the right concentration represents an average shear on some length to the right of this point. From this a conclusion may be reached that, unless the shear curve in plate runs irregularly between the points B_1 and B_2, the shear at point B should be represented by the average of the two tangential concentrations on the right and on the left of this point. Therefore,

$$T_B = \frac{1}{2}\left[(S_2 - S_3) + (S_3' - S_2')\right] \quad . \tag{g}$$

It may be pointed out, that using the same rule for calculation of tangential concentration on the horizontal plane at B an equal value is obtained: $T_B' = \frac{1}{2}\left[(S_3' - S_3) + (S_2 - S_2')\right]$, which is in agreement with the law of equality of shears on two perpendicular planes.

The concentration at A should be taken

$$T_A = \frac{1}{2}(S_1' - S_1) \quad . \tag{h}$$

It is clear that in the plate shear stress at A is zero either on horizontal or on vertical plane. The concentration T_A, however, corresponds not to shear at A, but to an average shear on some length from point A down, so that there is nothing inconsistent in T_A not being zero.

In order to satisfy the requirement of statics the sum $T_A + T_B + T_C + T_D + T_E$ must be equal to the shearing force on line AE. That this is the case may be easily proved in the following manner. Assuming no body forces at the joints on line AE,

Shear on AE = $-S_1 + S_2 - S_3 + S_4 - S_5 + S_6 - S_7 + S_8$;

and also, Shear on AE = $S_1' - S_2' + S_3' - S_4' + S_5' - S_6' + S_7' - S_8'$;

Adding and dividing by 2,

$$\text{Shear on AE} = \tfrac{1}{2}(S_1{}' - S_1) + \tfrac{1}{2}\left[(S_2 - S_3) + (S_3{}' - S_2{}')\right] + \tfrac{1}{2}\left[(S_4 - S_5) + (S_5{}' - S_4{}')\right] + \dots\dots\dots$$

The terms on the right are recognized as the expressions for T_A, T_B, etc. Therefore,

$$\text{Shear on AE} = T_A + T_B + T_C + T_D + T_E, \qquad \text{Q.E.D.}$$

The intermediate shear concentrations are converted into stresses in a manner similar to the one used with normal stresses:

$$\tau_B = \frac{T_B}{at} \quad . \tag{k}$$

The end concentration, however, cannot be handled as in normal stress, because the edge shear is zero. The way of taking care of T_A has been to convert it into an area K2f (Figure 72 c) bounded by a parabola. The mid-panel ordinate is

$$\tau_1 = \frac{\tau_B}{2} + \frac{3}{2}\,\frac{T_A}{at} = \frac{T_B + 3T_A}{2at} \quad . \tag{l}$$

This method of handling T_A is not fully satisfactory, and often results in an unreasonable diagram with a sharp reversal of slope at point 2, in other words, T_A is too large to be applied wholly in the end panel. A modification of the method, with extension of the influence of T_A to the second panel, is quite possible without any algebraic difficulties, but the ensuing increase in τ_B is hard to justify from the viewpoint of conditions of statics on the horizontal plane.

This difficulty with T_A is however local, and it affects the region where the shear is small, while its influence on shear at points farther in is apparently negligible.

A method of shear interpretation believed to be better than the previous one, although still not fully satisfactory at the edge, is as follows: The shear diagram (Figure 72 c) is drawn polygonal or curved in such a manner that its area from f to g, that is for a distance $\frac{a}{2}$ each way from B, is equal to T_B; and its area for a distance $\frac{a}{2}$ from A is equal to T_A. The key ordinates cannot be found at once, but only in two steps and approximately, so the method is more laborious.

Shears can also be determined at mid-squares, although those points are less desirable than the joints, since the normal stresses at them are unknown, unless found by interpolation of the values at the joints.

This calculation is demonstrated in Figure 73. The shear concentration in each panel is equal to the difference of the stress components in its two diagonals, i.e., $T_A = S_2 - S_1$. The stress at the middle of an interior panel is found by $\tau_B = \frac{T_B}{at}$. (m)
The stress in the outside panel is assumed to be parabolic on one half of the length and constant on the other half, as shown. Therefore, $\tau_A = \frac{6T_A}{5at}$; (n). Having found in this manner the key ordinates at mid-panels the diagram of shearing stress is drawn, as shown either by solid or by dotted lines, modifying somewhat, if necessary, the shape in the two outside panels.

Presence of external forces arising from a load distributed over the surface of the plate would not change the procedure for determination of shear stresses. Concentrated loads will be discussed later in connection with the gusset plate problem.

In problems involving plates with restrained edges the tangential joint reactions, found from the distribution procedure, are converted into continuous shearing stresses by the same method as the internal tangential concentrations.

A peculiar difficulty arises at the corner where two fully restrained edges meet, see Figure 74. Although the two corner framework reactions R_X and R_Y are known, there seems to be no way of finding what part of R_X is due to normal reactions on edge AC near point A, and what part is caused by shear on the edge AB, the same difficulty being applicable to R_Y.

Some assistance may be derived sometimes from the general principles of the theory of elasticity. Thus, a plate fully fixed at both edges will have no stresses at the very corner, which allows to draw the stress curves along the edges right to the corner. Exterpolation of the stress curves toward the point A may also be useful, but the framework method as such seems to fail here.

The same difficulty is present, if the restraint is partial, with one edge being restrained only normally, and the other edge only tangentially. On the other hand, if both edges are held only normally or only tangentially, no trouble arises, since it is evident to which edges the reactions R_X and R_Y should be attributed. Of course, no complication is present if only one edge is restrained.

C. Application of Loads to the Framework.

The question, how to apply forces acting on the plate to the corresponding framework, so as to obtain a truly equivalent effect, is answered largely by judgment, founded on common sense.

The theory of framework is based on the presence of only direct stresses in its members, therefore, all forces must be applied at the joints.

Preservation of statics is important and so the forces acting on the framework must have the same statical effect as the loads applied to the plate prototype.

Load distributed over the surface of the plate should be applied at the nearest joint; thus, joint A (Figure 75) will receive the load from the area 1,2,3,4, and if the resultant of load on this area does not pass through A, an additional force will be applied at some other point to correct the statical effect.

If there is a load over an area 5-6-7-8 acting in Y direction the proper location for the point of its application to the framework should be its centre C, but there is no joint available there and, consequently, the load is applied at D. Even though such displacement of the load is consistent with statics, it cannot be perfectly equal in effect.

A concentrated force P (Figure 75) should be broken up into parts, using the law of the lever, and these parts applied at the nearest joints.

A complete equivalence in these various adjustments is, of course, impossible, and the errors may be rightly charged to the framework method.

The question of applying to the framework, loads acting at the edges of the plate is the reverse of the problem of stress interpretation, since it requires determination of joint concentrations when the stress diagram is given.

With polygonal shape of stress diagram the normal concentrations N are found from formulae (a), (b), (d) of the previous sub-article. A curved shape of the diagram requires some modification of values obtained by these formulae, so as to preserve the static effect of the applied load.

The shear concentrations may preferably be found by calculating the shear stress area for a length $\frac{a}{2}$ each way from the joint. The corner joint concentration may be taken from the length $\frac{a}{2}$ near the corner.

The concentrated edge loads, if not applied exactly at the joint, should be divided between two nearest joints by the law of the lever.

18. Bending of a Wide Beam by Framework and Elasticity.

A. General.

In order to check precision of the framework method it has been applied to a problem for which there is a known solution of the theory of elasticity, namely, the problem of bending of a wide beam of a rectangular cross-section, loaded with uniform load and supported by shears at both ends, Figure 76. The shears are distributed over the ends in a parabolic manner, as demanded by theory and the uniform load is applied one half at the top and the other half at the bottom. There are also normal stresses following the law of cubic parabola, applied at the ends, whose static effect on each end is zero. They are necessary for a rigourous solution of the problem. The problem is manifestly symmetrical about yy axis and antisymmetrical about xx axis, therefore, only one quarter of the beam need be considered. The ratio of span 2 l to depth 2 c has been taken 4:3, and the thickness, unity.

The exact solution is found in Theory of Elasticity by Professor Timoshenko, page 38, with a slight modification, the whole load being applied at the top of the beam, instead of half at the top and half at the bottom. The necessary minor changes in formulae may be easily accomplished. In view of the absence of body forces, the state of stress is independent of Poisson's ratio.

The framework solution has been done three times: using 4 x 3 framework with $\nu = \frac{1}{3}$ for one quarter of the beam, then again, using 8 x 6, $\nu = \frac{1}{3}$ framework in order to test convergence of the solution, and finally utilizing 4 x 3 framework, but only with $\nu = 0$.

The stress formulae of the elasticity solution of the problem are as follows:

$$\sigma_x = \frac{q}{2I}(l^2 - x^2)y + \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right) \quad , \qquad \text{(a)}$$

$$\sigma_y = \frac{q}{2I}\left(c^2y - \frac{1}{3}y^3\right) \quad , \qquad \text{(b)}$$

$$\tau_{xy} = -\frac{q}{2I}(c^2 - y^2)x \quad , \qquad \text{(c)}$$

where $I = \frac{2}{3}c^3$.

B. <u>Framework 4 x 3 with $\nu = \frac{1}{3}$.</u>

Call the size of the square unit a, then c = 3a and l = 4a. For convenience, let $x = \alpha l$ and $y = \beta c$, where α and β are dimensionless coordinates of the points on the beam. Substitution of these expressions into (a), (b) and (c) gives:

$$\sigma_x = \left[\tfrac{4}{3}(1 - \alpha^2)\beta + \tfrac{1}{2}(\beta^3 - \tfrac{3}{5}\beta)\right] q \;; \qquad \text{(d)}$$

$$\sigma_y = \tfrac{1}{4}(3\beta - \beta^3)\, q \;; \qquad \text{(e)}$$

$$\tau_{xy} = -(1 - \beta^2)\alpha q \;. \qquad \text{(f)}$$

The shear and normal edge stresses found from (d) and (f) by using $\alpha = 1$ are shown respectively in Figures 77 and 78 at $\frac{a}{2}$ intervals. These stresses are considered as loads for the framework, and they are converted into joint concentrations. The methods of conversion have been different from the recommended ones, since the latter have been evolved only as a result of experience gained on this problem. The difference, however, is small and its effects are confined almost exclusively to the stress conditions at the edge.

In this inferior method the load concentrations at the edge joints have been found as end reactions of simply supported beams of spans a, loaded with the load diagrams of Figures 77 and 78, similar method being used for both the normal and the tangential concentrations.

The setup of the framework problem may be seen in Figure 79, representing one quarter of the beam with the acting loads applied at the edges y = 3a and x = 4a. In connection with the tangential concentrations applied at the latter edge it may be said that it is important in later calculations to realize from which side these concentrations are contributed. The load 0.491 is contributed from below, and 0.102 from above the corresponding joints. The intermediate concentrations 0.870 and 0.537 come partly from above and partly from below, which explains the meaning of the figures 0.471, 0.399 and others, the upper figure being a contribution from above.

The X and Y axes are respectively the axes of antisymmetry and symmetry of the beam, and the joints lying along them are prevented from movement in X direction, while no restraint is placed on movements in Y direction. The restraint of the axis joints is accomplished by horizontal joint reactions which are found from distribution. Members along the X axis are evidently unstressed.

The resultant framework stresses are given in Figure 80 and they are converted into continuous plate stresses by the methods explained in the preceding article and illustrated on a few examples below.

The results of these calculations are presented in the form of several diagrams, showing a quarter of the beam with stresses of one particular kind, like σ_x , σ_y or τ_{xy} , computed at several points by the theory of elasticity, formulae (d), (e), (f) and by the framework method, the first figure being everywhere the one found by elasticity. Percent errors are often given too by third figures at different points.

Figure 81 gives σ_x stresses at the joint points by the Method 1, which is considered the best.

Figures 82, 83 and 84 give σ_x stresses by Method 3 at mid-square points, mid-bar points and the joint points respectively. The percent errors show that the method is not very good, especially for the joint points, which are the most logical places, where the stresses should be determined.

Figure 85 presents σ_y stresses at the joint points, while Figures 86 and 87 show τ_{xy} stresses at the joints and at the mid-squares respectively.

Typical computations below illustrate how the stresses by framework have been arrived at. See also Figure 80 in this connection.

σ_x at $x = a$, $y = a$; by Method 1:

Concentration $N = (+\,0.209 + 0.152 - 0.038)\,q\,a = 0.323\,q\,a$.

By eq-n (a), Art. 17: $\quad \sigma_x = \dfrac{N}{a(1)} = 0.323\,q$.

σ_x at $x = a$, $y = 3a$; by Method 1.

Concentration $N = (0.238 + 0.405)\,q\,a = 0.643\,q\,a$.

By eq-n (d), Art. 17: $\quad \sigma_x = \dfrac{3}{3-\frac{1}{3}}\;\dfrac{2(0.643)qa}{a} = 1.445\,q$.

σ_y is determined similarly, except for a slight peculiarity at the edge, where the shearing load must also be taken into consideration.

σ_y at $x = 4a$, $y = a$, by Method 1:

Concentration $N = (0.376 + 0.152 - 0.399)\,q\,a = 0.129\,q\,a$.

By eq-n (b), Art. 17: $\quad \sigma_y = \dfrac{2(0.129)\,q\,a}{a} = 0.258\,q$.

When using Method 3 for computing normal stresses, eq-n (f), Art. 17 becomes for $\nu = \frac{1}{3}$: $\quad \sigma_x = \dfrac{1.5}{at}(S_x + \frac{1}{3}S_y)$.

σ_x at $x = 0.5a$, $y = 1.5a$, Method 3:

$S_x = \frac{1}{2}(0.189 + 0.469)\,q\,a = 0.329\,q\,a$,

$S_y = (0.054 + \frac{0.116}{2})\,q\,a = 0.112\,q\,a$, 0.054 bar being marginal.

$\sigma_x = 1.5\,(0.329 + \frac{0.112}{3})\,q = 0.550\,q$.

σ_x at $x = 0.5a$, $y = 2a$, Method 3:

$S_x = 0.469\,q\,a$,

$S_y = \frac{1}{4}\left[2(0.054) + 2(0.035) + 0.116 + 0.087\right]q\,a = 0.095\,q\,a$

$\sigma_x = 1.5\,(0.469 + \frac{0.095}{3})\,q = 0.751\,q$.

<u>σ_x at x = a, y = a, Method 3:</u>

$S_x = \frac{1}{2}(0.189 + 0.152)\ q\, a = 0.1705\ q\, a$,

$S_y = \frac{1}{2}(0.116 + 0.050)\, q\, a = 0.083\, q\, a$,

$\sigma_x = 1.5\ (0.1705 + \frac{0.083}{3})\ q = 0.297\ q\, a.$

<u>τ_{xy} at x = 2a, y = a;</u>

Concentration T = $\frac{1}{2}\left[-0.144 - 0.320 - 0.149 - 0.239\right] q\, a = -0.426\ q\ a.$

By eq-n (a), Art. 17, $\tau_{xy} = -0.426\ q$.

At x = 1.5a, y = 0.5a, $\tau_{xy} = (-0.144 - 0.209)q = -0.353\, q.$

These diagrams of stresses show a good agreement of the framework results with those of elasticity. The greatest errors which, by the way, never reach 8%, are observed in proximity to the edge x = 4a, undoubtedly an account of the referred to above irregularity of the load application at that edge. Most of the errors are, however, under 4%. Attention is called to the extreme simplicity of the stress computations, once the framework stresses are known.

It is interesting also to compare shear stress curves determined by framework and by elasticity on one of the vertical lines of joints, for example, on the plane x = 2a. These curves are shown in Figure 89. The shear concentration at y = 3a is T = $\frac{1}{2}(0.119 - 0.245)q a =$ $= -0.063\ q$ a and it is superimposed as a parabolic area on the triangular shear curve in the bottom panel, so that the ordinate at y = 2.5a becomes $\left[0.5\ (0.272) + 1.5\ (0.063)\right] q = 0.230 q$.

This makes apparent the irregularity of the shear curve, determined by framework, near the edge, with too high ordinates in that area and too small ordinates elsewhere. One may feel that the edge shear concentration should be spread over the whole section, but no rational reason for such procedure is apparent.

C. Framework 8 x 6 with $\nu = \frac{1}{3}$.

The setup of the problem in this framework is given in Figure 90 and the results are presented in diagrams of Figures 91, 92, 93 and 94. All computations have followed the routine of the previous framework. The object of this calculation has been to investigate the convergence of stress values, found by framework, toward their true values, as the size of the mesh decreases. The results reveal a great degree of precision obtained in 8 x 6 framework, the majority of stresses being within $1\frac{1}{4}$ % of their true values. It appears that doubling of the framework improves the accuracy roughly four times, i.e., in proportion to the number of units, although at some unfavourably located points the errors are somewhat larger. Minor irregularities, especially in the values of small stresses, are due to working only with three decimals.

It may be of interest to state that the distribution in this case has been a most formidable problem in view of 110 degrees of freedom of the joints. The work has been considerably reduced by distorting the structure into the equilibrium position of the plate prototype found by elasticity, and then distributing in an ordinary manner the comparatively small remaining unbalanced joint forces.

D. Framework 4 x 3 with $\nu = 0$.

The same problem has also been solved using 4 x 3 square framework with $\nu = 0$. The loads have been applied in the same manner as in the two previously discussed cases. The results appear in Figures 95 to 99.

For reasons explained earlier, the plate stresses in this problem, according to the theory of elasticity, should be independent of ν, but, this has not been found to be exactly the case with the framework results. It is, of course, natural that the individual corresponding bar stresses should be entirely different for the two ν's, but it is peculiar that, after they have been duly interpreted, the resultant plate stresses have come out also somewhat different for $\nu = \frac{1}{3}$ and $\nu = 0$. The results show that the accuracy of $\nu = 0$ framework is poorer than that of $\nu = \frac{1}{3}$ for the same number of units. Quite a few stresses are as much as 6 - 7% out, with the maximum error reaching 11%. The accuracy of $\nu = 0$ framework is thus roughly 70% worse than that of $\nu = \frac{1}{3}$ type, however, in spite of this, the former framework is just as practicable as the latter, and the labour involved in its use is only slightly higher in the distribution part, than the labour necessary to solve a $\nu = \frac{1}{3}$ type.

19. Gusset Plate Problem.

A. Statement of the Problem and General Remarks.

The main utility of the framework method and its main claim for a place among the tools of structural analysis is its applicability to unsolved problems. Any problem of two-dimensional stress in bodies of rectangular outline may be handled by it, and, judging from the results of the previous article, the errors ensuing from the finite size of the framework are small, even if the size of the unit is large.

Among the unsolved problems of structural engineering of considerable practical value, for which the framework method shows a good promise, is the problem of stress analysis in the gusset plate. Although it is quite true that this problem contains a number of features, for the explanation of whose effects the framework method is powerless, such as the presence of the rivet holes, stress distribution among the rivets, the manner of action between the rivets and the plate, including friction under heads, and the influence of plastic deformation at the points of stress concentration, still on the whole, apart from these secondary features, the problem is essentially the one of plane stress, and may in simpler cases be handled satisfactorily by the method of this treatise.

In making this statement it is not implied that the framework method is considered suitable for commercial use in design of gusset plates, but rather, that the commonly used in designing offices beam formula method, having very little theoretical justification, may be checked and modified by it.

The problem is stated by means of Figure 100 representing the gusset plate at the top chord joint of a truss. Two top chord members and three web members, indicated by dotted lines, meet at this joint. The plate is divided into 4 x 6 square units, and later into 8 x 12 of such units. The member stresses are assumed distributed uniformly over the lengths of their attachment to the plate along their axes; this explains twice smaller values of the extreme concentrations on each member compared to the intermediate concentrations, on account of twice smaller tributary length. The loaded joints need not be considered as representing individual rivets, but they may be looked upon as representing a whole group of rivets. This means that the state of stress in the plate is somewhat generalized, the local effect of each rivet being lost and replaced by the local effect of a whole group of them.

The assumed value of Poisson's ratio is $\frac{1}{3}$, which is not far from the commonly used for the mild steel value of 0.3.

After solution of 4 x 6 units' structure the framework has been doubled and re-solved in order to detect tendencies in stresses on decreasing the mesh and by that to judge of the degree of accuracy.

B. Solution of 4 x 6 Framework.

In order to simplify distribution the loading of the framework is broken up into four symmetrical and antisymmetrical cases, as explained in Art. 16. The first quadrants of these cases, represented in Figure 101, are distributed separately, and the results of the procedure are shown in Figures 102 to 105.

In order to give an idea of an actual distribution, Figure 105 presents not only the result, but also the whole process for one of these cases, namely, the doubly antisymmetrical case, $A_x A_y$. Although the separate movements are difficult to recognize, one can see how they gradually die down starting with the ones causing thousands and hundreds of pounds in stresses and finally ending with fractions of a pound. The remaining unbalanced forces are everywhere less than a pound.

C. Interpretation of Concentrated Loads.

It may be noticed that in view of the existence of the active joint forces the normal joint concentrations on two sides of the joint are different, and this feature brings up the question of proper stress interpretation in the presence of concentrated loads.

According to the theory of elasticity, Timoshenko, page 111, a concentrated force, acting on an infinite plate, causes infinite normal and shear stresses at the point of application, which, however, quickly decrease away from the point of action. On the other hand, in a mild steel specimen, high stresses are relieved by plastic yielding and lose their significance as far as the actual strength is concerned, so that while a large finite stress distributed over a large area is dangerous to the member, an infinite stress over an infinitesimal area is apparently immaterial in static loading. Furthermore, true concentration of a finite force at a point is impossible, and such a force in actual practice is represented by a rivet or a weld of a finite size.

It is also easy to see that a finite framework may have only finite stresses in its members, whose interpretation by the rules explained above may lead only to finite plate stresses.

All these somewhat contradictory considerations, arising from the peculiarities of the theory of elasticity, structural design, and the framework method, must be reconciled in the proper method of stress interpretation in the presence of concentrated loads.

Let Figure 106 represent a plate acted upon by three concentrated forces, which are in equilibrium. Imagine now the same plate under the same forces, to be infinitely extended in all directions, and let the stresses existing along the obliterated now former boundaries be as shown in Figure 107.

The state of stress of Figure 106 may evidently be considered as a superposition of the following four states shown in Figures 108 (a) to (d).

a. Infinite plate acted upon by a single force P_1.

b. Ditto with a single force P_2.

c. Ditto with P_3.

d. The original finite plate under the action of the boundary forces equal and opposite to the ones in Figure 107. These forces, by the way, are in equilibrium.

Cases (a), (b) and (c) may be solved by the referred to above article of the theory of elasticity.

Suppose, that the stresses at point O_1 are being investigated. Contributions of parts (b), (c) and (d) to stresses at O_1 are evidently finite, but that of (a) is infinite, and it is this latter part whose effect on strength at point O_1 is questionable, and may perhaps be best taken care of by a value assigned by judgment.

Applying these considerations to the framework, it is first noted that its solutions show the effect of all the above mentioned factors (a) to (d). Therefore, the proper way of interpreting its stresses at the joints with concentrated loads should be to separate the effect (a), to interpret the remaining stresses into the plate stresses and to augment the latter by some reasonable value, to take care of the disregarded effect (a). Of course, no special measures need be taken with regard to the joints where no concentrated forces occur.

Effect (a), which should be subtracted from the framework solution at each joint carrying a concentrated load, is nothing but a state of stress in an infinite framework under the action of a single concentrated force P (Figure 109a).

Although the individual bar stresses for this load condition can be found only by a distribution, the resultant joint concentrations are apparent directly from symmetry. Thus, the normal concentrations at point O on vertical plane are: $-\frac{P}{2}$ on the right and $+\frac{P}{2}$ on the left of O, while normal concentrations on horizontal planes are zero. The average of two tangential concentrations on horizontal plane above and below O is $T = \frac{1}{2}\left[(S_1 - S_2) + (S_3 - S_4)\right] = 0$.

Expressed differently, elimination of the concentration effect amounts to removal from the vicinity of O of the load P and of the stresses shown in Figure 109 (b), while the same stresses remain acting on the adjacent joints, in other words, force P is transferred to the adjacent joints, one-half ahead and one-half behind.

From this it follows that the exclusion of part (a) from the framework solution causes no change in tangential concentration, and as to the normal concentrations, it amounts to nothing more than to taking an average of the two unequal values on two sides of the joint with a concentrated load. This disposes in an exceedingly simple manner of the first part of the problem.

The question arising now is what should be added to stresses in order to take care of the disregarded effect. When considering a physically impossible abstraction of a point force, it is rigourously correct to augment the plate stresses, computed from the modified framework stresses, in the four units adjacent to the concentration point, by the values u (Figure 109b) which are the differences between the elasticity stresses in an infinite plate and the ordinates of the dotted closing line AB, which is used rather than the base line, because the concentration effect (a) has not been disregarded at the adjacent joints. This procedure applies naturally to all stresses near O, σ_x, σ_y and τ_{xy}.

In an actual case, when the concentrated force is applied by means of a rivet, the described procedure would be without significance. A practically reasonable way to take account of the normal effect of concentration is to add the value of the rivet bearing stress $\frac{P}{td}$ on the plane normal to the force P, on compression side of it, leaving the tension side intact. This, of course, presupposes that the rivet force is transmitted by bearing and not by friction under the head, which is in line with the conventional method of rivet analysis.

The normal stress on the plane parallel to force P need not be affected, but the shear stress should also be modified as is made clear in the following discussion.

Figure 110 (a) represents a framework near the joint O with a concentrated force P acting on it. The true shear stresses, either horizontal or vertical, in the plate prototype along the planes MM, NN and TT are shown in Figures (b), (c) and (d). While the dotted line corresponds to the shear caused by the factors referred to above as (b), (c) and (d), the difference of τ ordinates between the full and the dotted lines is caused by the action of the concentrated force P on the infinite plate, and this latter effect is infinite at the point 0. Although this sharply concentrated infinite stress is of no significance for strength of an actual gusset plate the average horizontal shear, corresponding to it, taken over a reasonable area, must not be disregarded. Assuming the size of the unit a as the reasonable length for averaging shear effect the resultant diagram of average shear on horizontal planes, taken in different locations along the line MM, is shown in Figure (e). As may be seen, there is a finite discontinuity at the point 0 whose amount can be found from Figure 111 as $\tau_{1\,av.} - \tau_{2\,av} = \frac{P}{at}$, taking d as an infinitesimal, and, therefore, the average significant shear stresses just above and just below 0 are respectively $\tau_o + \frac{P}{2at}$ and $\tau_o - \frac{P}{2at}$, where τ_o is the value found by an ordinary interpretation of the framework. Taking d in Figure 111 as a finite gradually increasing quantity, the difference $(\tau_{1\,av.} - \tau_{2\,av.})$ gradually diminishes in view of the effect of the end stresses σ_1 and σ_2, which means that the average shear stresses on horizontal planes gradually decrease, giving away from P in vertical direction.

Instead of assuming that the discontinuity $\frac{P}{at}$ of the average shear persists for a length a and then suddenly disappears, it has been thought more reasonable to assume it extending for a distance 2a petering out to nothing at the adjacent joints, as shown in Figure 110 (f) and (g).

Although the above reasoning has been dealing with shear on horizontal planes, it is equally applicable to shear on vertical planes, since the two shears are equal.

The use of length a for averaging the shear effect of P seems arbitrary only in case of a single concentrated force; if on the other hand, there is a line loading of concentrated forces P acting in the direction of this line the use of length a is logical.

This discussion explains the recommended modification of the shear interpretation in the vicinity of a concentrated force. It is emphasized here that the stresses so found are not the true ones but the averages of true stresses, believed in a way to be more significant than the true ones for the riveted plates of mild steel under static loading.

Although the explanation of the shear interpretation in the vicinity of a concentrated load is somewhat involved the interpretation itself is very simple. First, shear τ_o is found from the average tangential concentration at the joint by eq-ns (g) and (k) of Art. 17, paying no regard for the concentrated load P, and then, secondly, this τ_o is augmented at the joint in question by $\pm \frac{P}{2at}$ on two sides of P, this extra addition being gradually petered out to nothing at the adjacent joints.

That this interpretation is consistent with statics can be easily proved by repeating the reasoning of Art. 17 in connection with Figure 72 (a). Assume a vertical force P acting downward on one of the joints, for example B, although this force is not shown in Figure 72 (a). Then by definition:

Shear, left of AE $= S_L = -S_1 + S_2 - S_3 + S_4 - S_5 + S_6 - S_7 + S_8$;

Shear, right of AE $= S_R = S_L - P = S_1' - S_2' + S_3' - S_4' + S_5' - S_6' + S_7' - S_8'$;

Adding, dividing by 2 and transposing $\frac{P}{2}$;

$$S_L = \frac{1}{2}(S_1' - S_1) + \left\{\frac{P}{2} + \frac{1}{2}\left[(S_2 - S_3) + (S_3' - S_2')\right]\right\} + \frac{1}{2}\left[(S_4 - S_5) + (S_5' - S_4')\right] + \ldots \text{(a)}$$

The individual terms on the right are recognized as the tangential concentrations from which the plate shear stresses are determined by spreading the concentrations over appropriate areas. The second term may be seen to be the concentration at B modified in the manner explained on account of presence of the load P. This proves the static consistency of the method for a section to the left of AE, while an expression for S_R, similar to (a) above, proves it for a section right of AE.

The same principles may be extended to concentrated loads acting at marginal joints, as in case of force P in Figure 112 (a). Unlike the condition where there are no edge forces the shear stress at the edge in Figure 112 is not zero, but may be taken as

$$\tau_A = \frac{S_1' - S}{2\ at} + \frac{P}{at} \qquad \text{(b) on a plane just above AB}$$

and $\tau_A' = \frac{S_1' - S}{2\ at} - \frac{P}{at}$ (c) just below AB, as shown in Figure 112 (b) and (c). These stresses are again not the true ones but only the averages.

As to the normal stress on a vertical plane at A it should be taken equal to the bearing stress produced by P, while the normal stress on the horizontal plane at A may be disregarded as insignificant in a mild steel plate. No other stresses at A need be considered.

These recommendations concerning normal stresses near a marginal joint such as A have not been followed in the gusset plate problem, but instead of them, the horizontal and vertical normal concentrations have been used and interpreted in the ordinary manner. As a result of this, some interesting increase in σ_y at A has been produced on doubling the framework, as mentioned later.

Similar recommendations can be made when a concentrated load acts parallel to the margin, however, the actual manner of its application must also be considered.

It remains to consider now the case of a concentrated force acting at an angle to the framework axes, Figure 113. In line with the previous statements, the plate stresses at O, obtained by using average concentrations, should be increased by bearing stress $\sigma' = -\frac{P}{td}$; (d) in the direction OF on compression side of P, and by additional shearing stress $\tau' = \pm \frac{P}{2(BO)t} = \pm \frac{P_x}{2at}$;(e), acting parallel to OF and being positive on one side of this line and negative on the other, so that there is a discontinuity in shear stress on line OF of the amount $\frac{P_x}{at}$ at point O, tapering to zero at points B and C.

Since all stress calculation is done for the framework axes, it is necessary to convert these additional oblique stresses into their σ_x, σ_y and τ_{xy} equivalents. Using well known conversion formulae [5),]

Equivalents of $\sigma' = -\frac{P}{td}$ are:

$$\sigma_x = \sigma' \operatorname{Cos}^2\alpha \; ; \; \sigma_y = \sigma' \operatorname{Sin}^2\alpha \text{ and } \tau_{xy} = \tfrac{1}{2}\sigma' \operatorname{Sin} 2\alpha \; ; \text{(f)}$$

---

5) Page 16.

Equivalents of $\tau' = \pm\frac{P_x}{2at}$ are:

$$\sigma_x = -\sigma_y = \tau' \text{ Sin } 2\alpha \quad \text{and} \quad \tau_{xy} = \tau' \text{ Cos } 2\alpha \quad . \qquad \text{(g)}$$

The signs of these stresses are evident from Figure 113 (b) and (c).

For $\alpha = 45^o$ the equations (f) and (g) become as follows.

Equivalents of σ' : $\quad \sigma_x = \sigma_y = \tau_{xy} = \frac{\sigma'}{2}$. (h)

Equivalents of τ' : $\quad \sigma_x = -\sigma_y = \tau'$ and $\tau_{xy} = 0$. (k)

From expressions (g) and (k) it follows that the shear discontinuity on the inclined plane OF results in discontinuity of σ_x , σ_y and τ_{xy} stresses on two sides of that plane. Only when $\alpha = 45^o$ the break in τ_{xy} disappears.

Some of the items among the ~~ones~~ presented here recommendations for stress interpretation in the vicinity of concentrated loads, may undoubtedly raise objections, valid reasons may possibly be given for their modification, but it is felt that the basic principle of applicability of the framework method to the plane stress problems involving concentrated loads, will stand intact in spite of possible change in detail.

D. Computation of Plate Stresses from 4 x 6 Framework.

The average joints concentrations of the first quadrants of the four component framework cases are properly combined and converted into the plate stresses. The numerical procedure in relation to normal stresses is illustrated on the following example referring to σ_x stresses on plane $X = 2a$.

Figure 114 (a) presents the sum of the average normal concentrations in the first quadrant of the two symmetrical about

X axis cases, while Figure (c) shows a similar sum for the two antisymmetrical cases. These concentrations are converted into stresses in Figures (b) and (d). By adding (d) to (b) and then by subtracting (d) from (b), plate stresses respectively below and above X axis are found, which, however, need to be modified in view of the presence of two concentrated forces. This modification at the point $y = -a$ amounts to addition of the bearing stress of the rivet, while the one at $x = +a$, where the line loading crosses the plane of stress at 45^o, expresses itself in two features; first, in an addition of a compression stress equal to one half of the bearing value of the rivet, and secondly, in introduction of a break in the stress diagram equal to $\frac{5000}{at}$. The stress diagram of σ_x on the plane $x = 2a$ is shown in Figure 115 (a). The numbers on this and the following diagrams give the values of $\sigma_x a't$, where $a' = \frac{a}{2}$, for which reason they are twice smaller than the values found in Figure 114.

The rivet bearing stresses cannot be added numerically to the ordinates of the curve because they are expressed in terms different from the ordinates; for this reason they are indicated on the diagram by arrows.

A similar procedure is followed in computing normal stresses on other planes. There is also nothing special to add to what already has been said about computing shearing stresses.

The results of stress calculation are presented in the form of numerous diagrams. σ_x stresses on five different planes are given in Figure 115; σ_y stresses, in Figure 116; while τ_{xy} stresses appear in Figures 117 and 118, distributed respectively over the horizontal and vertical planes.

Before discussing various theoretical and structural features of the results obtained, a brief reference will be made to a solution of the same problem by means of an 8 x 12 framework.

E. Solution by 8 x 12 Framework.

This solution has been made in order to compare the results with the previous ones and through that to form an opinion regarding convergence of stresses toward their true values on decrease of the size of the mesh.

It is this problem that has led to some exceedingly laborious distributions, referred to at the end of Art. 16. The problem itself and its four component cases are given in Figures 119 and 120 (a) to (d), while the resultant framework stresses for the four cases appear in Figure 121 (a) to (d).

Figure 122 gives a copy of an actual distribution sheet for the case S_X, A_Y, and shows clearly the amount of arithmetic involved in such distribution. Both the original and the final unbalanced forces are shown on the sheet. The partial summations of the member stresses correspond to several current checks of the distribution explained in Art. 12.

Interpretation of the bar stresses into the plate stresses has followed the usual routine, and the resultant stress curves have been drawn for comparison alongside the ones determined earlier in the same Figures 115 to 118.

The rectilinear diagrams of normal stresses corresponding to conventional beam formula method have also been recorded in the same pictures. These stresses have been calculated on the basis of load condition of 8 x 12 framework; the normal loads that happen to act at the plane, whose stresses are calculated, have been split into two halves and pushed one half back and the other forward.

F. Gusset Plate Stresses.

Normal stresses found by framework in many cases are not completely dissimilar from the ones determined by the beam formula; this resemblance is particularly close for σ_x stresses on plane $x = -a$ (Figure 115d). The cause of this resemblance lies in the fact that both sets of stresses must satisfy numerically the same two conditions of statics $\sum N$ and $\sum M$, and this requirement often does not permit the two curves to diverge much.

For the same reason the maximum σ_x has been found in all sections nearly the same in magnitude by both methods.

On the other hand, there are also some differences. Thus, the true maximum stress has a tendency to occur on the line of heavy loads, rather than at the extreme fiber. Sudden breaks characterizing the true curves, caused by shear discontinuities of the line loadings, are absent in the conventional diagrams.

The diagrams of σ_y stresses bear little resemblance to what they appear when using beam formula; this is partly the result of too small values of N and M on the horizontal planes.

In this comparison of normal stresses determined by two methods, the additional effect of bearing stress, shown by arrows, on the stresses determined by framework has been left out, not because it is immaterial, but because it is expressed in terms of different variables than the primary part of the stresses. This arrangement, however, puts the more correct stresses on the same basis as the conventional ones, because in usual analysis of gusset plates by the beam formula the local effect of the rivet bearing is also disregarded.

As to the shear stresses found by framework, their distribution over the section is entirely different from parabolic.

Comparing the stresses found by two framework solutions with each other, one is struck by the close agreement of each two curves of stresses on any of the planes, especially in view of representation of the plate in one of these solutions by the framework as crude as only 4 x 6 units. The agreement becomes even more impressive if one considers the discontinuous character of the load and the lack of mathematical rigour in the rules of stress interpretation. Only in a few details is the difference more than negligible, and where it exists it is mostly susceptible to a rational explanation.

The only large disagreement of normal stresses found by two frameworks occurs in σ_y stresses on plane y = -a at the two edges where concentrated forces are applied to the plate, Figure 116c. The explanation lies in the fact that the concentration effect of these two edge forces has not been removed as in the case of all interior concentrations. The result of it is, that the decrease in the size of mesh increases σ_y stresses at the edges without limit.

The latter statement will not be demonstrated here, although the proof is possible by means of a lengthy derivation based on Theory of Elasticity. In line with previous discussion, the high edge stress should be discarded as insignificant.

The same cause explains the sudden turn of the dotted curve of σ_x stress on plane x = 0 (Figure 115c) near the bottom of the plate. Since the concentrated force applied at this point is comparatively small, the difference in the two ordinates is also only minor.

Figures 115 (b) and (c) show a fairly large increase in σ_x stress at y = -a and an accompanying increase in area under the curve, which occur on doubling of the framework. The reason for this discrepancy is that the two curves are not statically equivalent, since their resultant normal forces are not equal. It will be recalled that elimination of concentration from a joint amounts to removal of the load to the adjacent joints, so that one half of the load is moved forward and the other half backward. Therefore, on section x = 0, for example, out of 84 kips of stress of the left top chord member (Figures 100 and 119) there will be in 4 x 6 framework 77 kips on the left of the section and 7 kips on the right of it, while in case of 8 x 12 framework there will be 80.5 kips on the left and 3.5 kips on the right. The larger normal force in 8 x 12 solution accounts for the larger area under the curve than in case of 4 x 6 framework. The same applies to Figure 115 (b) which presents normal stresses on x = a.

Similar effects, only to a smaller degree, are noticed at $y = 0$ in σ_x stress on the plane $x = -a$ (Figure 115d) and in σ_y stress on the plane $y = 0$ at $x = -a$ and zero (Figure 116b).

The same phenomenon causes a similar discrepancy in shearing stresses on the plane $y = 0$ (Figure 117e). The negative area of shear curve is evidently larger for 8 x 12 framework than for 4 x 6 mesh. On the other hand, considering the two loaded frameworks, Figures 100 and 119, it is easy to see that elimination of concentrations from the joints on line $y = 0$ results in a numerically larger negative shearing force on the plane $y = 0$ in case of 8 x 12 mesh than in case of 4 x 6 mesh.

Some minor differences may be traced to dislocations of the concentrated forces from their true positions. Thus, the end concentrations of all members are slightly displaced. They should be applied at the centres of their tributary areas, but this is not done because no joints are available there. That causes error, and since the dislocations are different in the two frameworks, some minor discrepancies between their solutions are bound to arise.

Small differences are also produced as a result of removal of concentrations. Since such removal affects only the joint at which the force is applied, and since the joints in the small mesh framework are spaced closer together, the effect of removal of concentration is more localized in a small mesh than in a large mesh, which produces some difference in the corresponding states of stress.

Most of the shear diagrams have unsightly looking portions in the end panels, caused by the manner of transforming the tangential concentrations at the edges into shear stresses by adding parabolic swellings. The only virtue of this arbitrary method of handling the end shear is its agreement with statics and numerical simplicity. It is, however, recommended to modify the curves near the ends by sketching, thus improving their appearance but retaining their areas.

Stresses determined by 8 x 12 framework have been made the basis for calculation of principal stresses at various points of the plate as shown in Figure 123. The two principal stresses and the maximum shearing stress are stated at each point, and the directions of principal stresses are indicated approximately by the two mutually perpendicular lines, the longer line corresponding to the numerically larger stress. It is reminded that the stress discontinuities exist all along the working lines of the members. The effect of local bearing stresses has been left out. The absolute maximum compression stress occurs at the point $x = 0$; $y = -a$, and is equal to 18300 units against 17813 units found by the conventional method, the latter however, occurring at a different point.

II.

Framework Method Applied to Bending of Plates.

1. Differential Equations of Bent Plate.

The general theory of small bending of thin plates of constant thickness made of homogeneous elastic material will be discussed here briefly, following the presentation of Professor Timoshenko in his book on Elastic Stability.[6)]

A small element of plate in the form of a rectangular parallelopiped is presented in Figure 124. The moments and shears per unit length of section are shown on two positive faces of the element, and the following may be said about their sign convention. The shears Q_x and Q_y are positive, if they are acting in the positive direction of z axis, i.e., downward, on the positive faces of the element. The bending moments M_x and M_y are positive if they produce a concavity of the plate upward, while the torsional moments M_{xy} and M_{yx} are positive if they act on the element in the clockwise direction, when looking from inside of the element. From the equality of shearing stresses on two perpendicular planes it follows that $M_{xy} = -M_{yx}$.

The following equations, relating to these various functions, are obtained by applying to the element the equations of statics: $\sum Z$, $\sum M_x$ and $\sum M_y$.

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \quad ; \qquad \text{(a)}$$

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0 \quad ; \qquad \text{(b)}$$

$$\frac{\partial M_{XY}}{\partial x} - \frac{\partial M_Y}{\partial y} + Q_y = 0 \qquad , \qquad \text{(c)}$$

which after elimination of Q_x and Q_Y give:

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{XY}}{\partial x \partial y} + \frac{\partial^2 M_Y}{\partial y^2} = -q \quad . \qquad \text{(d)}$$

It may be pointed out that this equation is derived exclusively from statics, and does not depend on the material of the plate. By introduction of the laws of deformability it is transformed into the partial differential equation for plate bending.

The laws of deformability may be stated in the form:

$$M_X = D\left(\frac{1}{r_x} + \nu\frac{1}{r_y}\right) \qquad , \qquad \text{(e)}$$

$$M_Y = D\left(\frac{1}{r_y} + \nu\frac{1}{r_x}\right) \qquad , \qquad \text{(f)}$$

where $D = \dfrac{Eh^3}{12(1-\nu^2)}$. (g)

In these expressions r_x and r_Y are the radii of curvature of the bent plate. The deformations are assumed small, the material elastic, and the cross-sections plane after the deformation. Replacing the curvatures by their expressions in terms of deflection w the equations (e) and (f) become:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad , \qquad \text{(h)}$$

$$M_Y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \qquad . \qquad \text{(k)}$$

A similar deformability expression for M_{XY} will be derived now. Figure 125 represents a triangular element of the plate cut out parallel to the axes N and T at 45° to the X,Y axes. It is assumed that the element is acted upon on the planes N and T by equal and opposite in sign bending moments M_o, while no torsional moments are present on these planes. It is required to find the moments on planes X and Y.

Moments acting on the triangular element are indicated by vector arrows drawn in accordance with the commonly used rule of a right-handed screw.

From statics $M_X = 0$; $M_{XY}\, l\sqrt{2} = -2\, M_o\, l\, \frac{\sqrt{2}}{2}$; or $M_{XY} = -M_o$. From this it is easy to see that $M_Y = 0$ and $M_{YX} = M_o$. Thus, the state of pure bending in opposite directions on two perpendicular planes N and T is acompanied by the state of pure torsion on the planes X and Y at 45° to the former planes, and the intensities of bending and torsion are the same. The converse is also true. This result is utilized below.

Applying the equations (h) and (k) to the planes N and T;

$$M_n = M_o = -D\left(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial t^2}\right) \quad , \qquad \text{(l) and}$$

$$M_t = -M_o = -D\left(\frac{\partial^2 w}{\partial t^2} + \nu\frac{\partial^2 w}{\partial n^2}\right) \quad . \qquad \text{(m)}$$

From Figure 125:

$$x = \frac{n}{\sqrt{2}} - \frac{t}{\sqrt{2}} \quad ,$$
$$y = \frac{n}{\sqrt{2}} + \frac{t}{\sqrt{2}} \quad .$$

By partial differentiation of these:

$$\frac{\partial x}{\partial n} = \frac{1}{\sqrt{2}}\ , \quad \frac{\partial x}{\partial t} = -\frac{1}{\sqrt{2}}\ ; \quad \frac{\partial y}{\partial n} = \frac{1}{\sqrt{2}} \ \text{ and } \ \frac{\partial y}{\partial t} = \frac{1}{\sqrt{2}} \qquad \text{(n)}$$

By double partial differentiation of w with respect to n and t and by the use of equations (n) the following expressions are obtained:

$$\frac{\partial^2 w}{\partial n^2} = \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2} + 2\,\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\right),$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2} - 2\,\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\right),$$

which on subtracting give:

$$\frac{\partial^2 w}{\partial n^2} - \frac{\partial^2 w}{\partial t^2} = 2\,\frac{\partial^2 w}{\partial x \partial y} \quad . \qquad \text{(p)}$$

Subtracting (m) from (l), dividing by 2, substituting (p) and replacing M_o by an equal value $-M_{XY}$ gives the third deformability expression, relating the torsional moment with the derivative of the deflection,

$$M_{XY} = D\,(1 - \nu)\,\frac{\partial^2 w}{\partial x \partial y} \quad . \qquad \text{(q)}$$

Substitution of the expressions for moments (h), (k) and (q) into the equation (d), derived by statics, results in well known differential equation of the bent plate:

$$\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^2} = \frac{q}{D} \quad . \qquad \text{(r)}$$

In order to be a true solution for the deflection of the plate the expression for w must satisfy not only the differential equation (r), but also the boundary conditions. The mathematical difficulties in the way of finding w are just as great as in solving problems of plane stress, and many plate problems of considerable practical interest are still waiting their solution.

Once w is known, the moments and shears may be found, at least theoretically, by differentiations: moments from the expressions (h), (k) and (q) and shears from the following equations, which are easily derived from (b) and (c).

$$Q_x = -D \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \; ; \qquad (s)$$

$$Q_Y = -D \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \; ; \qquad (t)$$

Rigourous mathematical solutions, when they exist, usually give expressions for w in terms of an infinite or even of a doubly infinite series of terms consisting of trigonometric or hyperbolic functions. While the convergence of the series for w is usually very good, so that only few terms are required to get an accurate value of the deflection, the expressions for moments and shears, as a rule, converge slowly, or even do not converge at all, which lowers the value of mathematical solution, because it is not the deflections, but the moments and partly the shears that are required by engineers.

Many approximate mathematical solutions, while quite good as far as the deflections are concerned, become inaccurate in their expressions for moments.

2. Boundary Conditions and Influence of ν.

Some of the simple boundary conditions will now be considered.

Simply supported edge. Taking this edge as y = 0, the conditions which the expression for w must satisfy along it are:

$$(w)_{y=0} = 0 \;\; ; \;\; \text{and} \;\; (M_y)_{y=0} = 0.$$

The latter expression gives: $\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)_{y=0} = 0$, and since $\frac{\partial^2 w}{\partial x^2}$ is automatically zero along the edge, the two boundary conditions become:

$$(w)_{y=0} = 0 \quad ; \quad \left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0 \quad . \tag{u}$$

One might think that there is a third condition pertaining to this edge, namely $(M_{YX})_{y=0} = 0$, but, as explained in the book of Professor Timoshenko, this condition is satisfied automatically by transformation of the edge torsional moment into the statically equivalent to it additional edge shear, so that the combined shear at the edge becomes $\left(Q_Y - \frac{\partial M_{YX}}{\partial x}\right)_{y=0}$.

Clamped Edge. Assuming the edge in question again to be $y = 0$, the boundary conditions at it are:

$$(w)_{y=0} = 0 \quad ; \quad \left(\frac{\partial w}{\partial y}\right)_{y=0} = 0 \, . \tag{v}$$

By differentiating the second condition with respect to x, $\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{y=0} = 0$, from which it follows that $(M_{YX})_{y=0} = 0$, i.e., the torsional moment along the clamped edge is absent.

Free Edge. If this edge is $y = 0$, the following two conditions pertain to it:

Combined shear $\left(Q_Y - \frac{\partial M_{YX}}{\partial x}\right)_{y=0} = 0$, and $(M_y)_{y=0} = 0$.

Shear and moments in these expressions are replaced by their values in terms of derivatives of w, and the following results are obtained for the two boundary conditions:

$$\left[\frac{\partial^3 w}{\partial y^3} + (2 - \nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]_{y=0} = 0 \; ; \text{ and } \left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)_{y=0} = 0 ; \tag{x}$$

A question which has an important bearing on the following work of this treatise is, how much deflections, moments and shears are dependent on the value of ν, and whether it is possible to transform a set of numerical values of these functions, obtained for a plate with certain ν, into a corresponding set for a plate with the same loading and edge conditions, but with a different ν.

Consider first a plate whose edges are all either simply supported or clamped. The Poisson's ratio of the plate is ν, rigidity D and the loading q (x;y). Suppose a function $w = f(x;y)$ has been found satisfying the differential equation (r) and the boundary conditions (u) or (v) on all edges. The question is what happens to w, if ν changes to ν', or D to D', everything else remaining the same.

It does not take long to see that the new value of the deflection is

$$w' = \frac{D}{D'} w \quad , \qquad \text{(aa)}$$

since it satisfies both, the new differential equation and the boundary conditions, which means that w varies in inverse proportion to D, or

$$w' = w \frac{1-\nu'^2}{1-\nu^2} \quad . \qquad \text{(11)}$$

The new shear is

$$Q_x' = -D' \left(\frac{\partial^3 w'}{\partial x^3} + \frac{\partial^3 w'}{\partial x \partial y^2} \right),$$ which on substitution of (aa) for w' gives,

$$\left.\begin{aligned} Q_x' &= Q_x \\ \text{similarly } Q_y' &= Q_y \end{aligned}\right\} \qquad \text{(ac)}$$

This shows that shears are not affected by the change in ν or in D.

From (h) and (aa) the new bending moment comes out,

$$M_x' = -D' \left(\frac{\partial^2 w'}{\partial x^2} + \nu' \frac{\partial^2 w'}{\partial y^2} \right) = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu' \frac{\partial^2 w}{\partial y^2} \right); \text{(ad)}$$

Substituting into this expression second derivatives of the old w in terms of the old bending moments, the required expression for the new moment in terms of old moments and the old and new ν^s is obtained.

Solving together (h) and (k)

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{(1-\nu^2)D} \left(-M_x + \nu M_Y\right) \quad ,$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{(1-\nu^2)D} \left(-M_Y + \nu M_x\right) \quad .$$

Substituting these into (ad)

$$\left.\begin{aligned} M_x' &= \frac{1}{1-\nu^2}\left[(1-\nu\nu')\,M_x + (\nu' - \nu)\,M_Y\right] \quad , \\ \text{by analogy } M_Y' &= \frac{1}{1-\nu^2}\left[(\nu' - \nu)\,M_x + (1-\nu\nu')\,M_Y\right] . \end{aligned}\right\} \quad (12)$$

The new torsion moment

$$M_{xY}' = D'(1 - \nu')\frac{\partial^2 w'}{\partial x \partial y} = \frac{1-\nu'}{1-\nu} M_{xY} \quad . \qquad (13)$$

Should change in D be produced by a modification of E or h the deflection w still varies in inverse proportion to D, while moments and shears remain unchanged. Likewise, a change of the load q (x;y) into a proportional value Kq(x;y) alters all the plate functions in the same ratio K.

This discussion shows that the following relations exist between various functions in plates with simply supported or clamped edges, having different ν^s or D^s but being otherwise identical with respect to shape, edge conditions and loading: the deflections are inversely proportional to D, the shears are the same, the torsional moments are proportional to $(1 - \nu)$, while the bending moments follow a more complicated relationship of eq-n (12). It is therefore quite possible for such plates to reduce numerical values of moments, shears and deflections from one value of ν to another.

This important conclusion may be extended further. So far it has been tacitly assumed that the plate is supported, in the manner stated, only on the periphery, while no intermediate supports are present. The latter qualification may be removed now without changing the above conclusion. The intermediate supports may be in the form of immovable line supports, transforming the structure considered into a continuous plate of several spans, or in the form of immovable point supports, and in both these cases the effect of Poisson's ratio is still the same as stated above, as is evident from the following reasoning.

Suppose there is an immovable intermediate line support at X = 0. The conditions at it are:

$(w)_{x=0}$ left $= (w)_{x=0}$ right $= 0.$

$(\frac{\partial w}{\partial x})_{x=0}$ in left span $= (\frac{\partial w}{\partial x})_{x=0}$ in right span.

$(M_x)_{x=0}$ left $= (M_x)_{x=0}$ right,

the latter reducing to $(\frac{\partial^2 w}{\partial x^2})_{x=0}$ left $= (\frac{\partial^2 w}{\partial x^2})_{x=0}$ right. All these conditions are evidently satisfied by the expression (aa) for w' corresponding to changed ν, which proves the point.

Should there be immovable intermediate point supports, the reasoning is as follows. Imagine the supports removed, then the deflections δ_1, δ_2 .... occurring at their points are inversely proportional to D. Let now the plate be acted upon by the reactions of the point supports alone. The deflections δ_1', δ_2'... produced now are again inversely proportional to D. If the reactions have been found correctly for the value of Poisson's ratio ν, then for that ν, $\delta_1 + \delta_1' = 0$; $\delta_2 + \delta_2' = 0$..... (ae)

The corresponding equations for a new value of ν will be $\frac{D}{D'}\delta_1 + \frac{D}{D'}\delta_1' = 0$ etc. They are evidently satisfied by virtue of eq-ns (ae) being satisfied, therefore, the values of the reactions found for ν are good also for ν'. Having thus proved the invariance of the reactions with ν, it is possible to regard them as active forces, which view reduces the plate to the one supported only at the periphery, for which the stated dependence of various plate functions on ν has already been demonstrated.

These important deductions are utilized in the last article of this chapter in the use of the framework corresponding to $\nu = \frac{1}{3}$ whose solution is considerably more simple than that of a framework for any other value of Poisson's ratio.

It is interesting to note that these conclusions regarding ν do not hold when any of the outside edges of the plate are free, since the value of w, inversely proportional to D, although satisfying the differential equation, does not satisfy the boundary conditions (x) in view of their dependence on ν. A change in ν for such plate thus produces a drastic change in the shape of deflection surface and no simple relation between w and w', or between other corresponding plate functions, seems to exist. It is possible, however, that the lack of resemblance is mostly confined to the vicinity of the free edge.

Simple relations between various plate functions, corresponding to different values of ν, are also absent in the case of elastic supports, or when, in addition to action by known forces, some points of the plates are subjected to known deformations.

3. <u>Framework Analogy</u>.

Mathematical difficulties standing in the way of a purely analytical solution of plate problem present an opportunity for methods which, like the framework method, do not depend on highly intricate mathematical procedures.

The variety of the framework method used for solution of problems of bending of plates has a great resemblance with the one used in two-dimensional stress. Here again a plane structure is composed of articulated bars, joined by means of pins normal to the plane of the structure, the bars being arranged according to some definite pattern, repeating itself everywhere in the structure, whose external outline and dimensions in plan are made identical with the given plate. In addition to that, the same loads and boundary conditions are imposed on the frame analogue as on the plate prototype, all the loads and the restraints of the frame being applied at the principal joints. With proper type of pattern, proper sections of the members and infinitesimal size of unit the framework is again rigourously equivalent to the plate prototype in its structural behaviour. In order, however, to solve the problem the size of the unit must be taken finite and fairly large, which changes the method from an exact to an approximate one. This latter procedure is again justified by a hypothesis assuming that the error involved in the use of a finite unit is small.

One may notice that all these details are identical with the ones stated in connection with the plane stress framework. The feature which distinguishes the framework now considered from the previous one is that its bars are assumed to be endowed with flexural property instead of the property of extensibility of the earlier variety. This flexural property or rigidity is limited to bending out of the plane of the framework, while no resistance is offered to bending in the plane of the structure, and none is necessary in view of articulation of the joints. Furthermore, the bars do not possess any resistance against torsion. As to the change in length, none occurs since deflections are small, and the central plane of the plate neither stretches nor shrinks.

The definition of the terms moment and shear in the framework, as distinct from the moment and shear in an individual member, is made similar on the one hand to the same terms in the bent plate, and on the other hand to the term unit stress in the plane stress framework. This means, for example, that bending moment per unit length of plane AA (Figure 1) is equal to the sum of corresponding components of bar moments on length dx, divided by this length. Should the plane of stress cut the bars in such a way that the manner of their cutting repeats itself all along the plane, the repeating length λ may be taken in place of dx.

In order that the infinitesimal framework could faithfully reproduce the mechanical behaviour of the plate, as manifested either in stresses or in deformations, it should possess the same deformability as the plate, as expressed by any of the equations (e), (f), (h) or (k) of Art. 1, Chap. II, referred to any direction in the framework. If this requirement is satisfied, the three deformability expressions (h), (k) and (q), Art. 1, hold for the framework as well as for the plate, with w in them signifying the same deflection in either one of the two structures; at the same time the equation of statics (d) is likewise equally applicable to both the framework and the plate. This means that the basic differential equation (r) is valid for the framework no less than for the plate, and owing to the same boundary conditions, the expression for w comes out the same in both cases. This constitutes the proof of the statement that the problem of bending of a plate is exactly equivalent to the problem of bending of a corresponding infinitesimal framework of a proper pattern.

It has been just mentioned that a framework whose flexural deformability in any direction is identical with that of the plate prototype must satisfy the three equations of deformability (h), (k) and (q), or their equivalents (e), (f) and (q), in which X and Y are the axes of symmetry of the framework. In other words, the framework must deform properly under the following conditions:

1. When subjected to pure bending in X direction, with no bending deformation in Y direction.
2. When in pure bending in Y direction, with no deformation in X direction.

3. When subjected to pure torsion on X and Y planes.

Equations derived from these conditions provide the means for checking suitability of each particular pattern and for determination of its characteristics, such as the rigidities of individual bars. When the pattern has the same form in directions of both axes X and Y, the condition 2, naturally, falls out, and only the two other conditions remain. The only patterns that have been investigated are the two square ones, similar in appearance to the types used in plane stress.

4. <u>Square Pattern With Simple Intersecting Diagonals</u>.

Before proceeding with the derivation, a new vector notation for moments, used throughout the remaining part of this treatise, will be explained here. According to this notation, a moment, acting on the plate or on the framework, is designated by an arrow perpendicular to the axis of rotation, i.e., lying in the plane, in which the turning takes place. This method allows to record the moment in a framework bar by means of an arrow coincident with the bar, a notation which has a decisive advantage in the process of distribution described later wherein moments appearing in different members are recorded as numbers written along the same members, which arrangement is essential in order to avoid confusion.

It is needless to say that the ordinary rules concerning various operations on vectors, such as addition, subtraction, breaking into components and projecting on different lines, hold with respect to these vectors.

The following sign convention is adopted; the arrow points in the direction which has a tendency to rise, on account of rotation produced by the moment. This means that if a bar is bent with a concavity upward, its moment arrow points away from the member, resembling an arrow indicating tension in a member with direct stress. Similarly, a moment producing downward concavity is shown by an arrow as if it was a compression stress.

The feature which makes the convenient vector notation, explained here, possible, is that all the moments associated with flexure of a plate or a framework have their axes coincident with the plane of the structure. It is evident that the method would be unsuitable for designation of moments located at random in three-dimensional space.

Returning now to framework, shown in Figure 126 (a), it is assumed that all horizontal and vertical members have moments of inertia I, while all diagonals have them equal to I_1 and the two diagonals of each unit are not connected at intersection. All members of the framework are made of the same material as the plate prototype. Thickness of plate is h.

<u>Condition 1</u>. Let the plate and the framework, Figure 126, be bent in X direction to a radius r_x by moments M_o per length a, while the curvature in perpendicular direction remains zero. This necessitates application of moments in Y direction of the magnitude νM_o per length a. Let angle change in plate or in horizontal bar per length a be θ. The corresponding angle change between the two ends of a diagonal is evidently $\frac{\theta}{\sqrt{2}}$.

Moment in the vertical bar is evidently zero.

Moment in the horizontal bar $M = \frac{EI}{r_x} = \frac{EI\theta}{a}$; (a)

Moment in the diagonal bar $M_1 = \frac{EI_1 \frac{\theta}{\sqrt{2}}}{a\sqrt{2}} = \frac{EI_1\theta}{2a}$; (b)

Moment in the plate in X direction per length a is

$$M_o = aD\frac{1}{r_x} = D\theta = \frac{Eh^3\theta}{12(1-\nu^2)} \quad . \qquad \text{(c)}$$

The two equations for determination of the pattern characteristics I and I_1 are obtained from the moment equilibrium of the joints in X and Y directions, illustrated in Figure 127 (a) and (b) by means of newly defined vectors.

From equilibrium in Y direction:

$$\nu M_o = 2\frac{M_1}{\sqrt{2}} \; ; \qquad \text{or} \quad \frac{EI_1\theta}{\sqrt{2}\,a} = \frac{Eh^3\theta\nu}{12(1-\nu^2)} \quad ,$$

from which $I_1 = \frac{\sqrt{2}}{12}\frac{\nu}{(1-\nu^2)}\,ah^3$. (d)

From equilibrium in X direction:

$$M_o = 2\frac{M_1}{\sqrt{2}} + M \; ; \qquad \text{or} \quad \frac{EI_1\theta}{\sqrt{2}\,a} + \frac{EI\theta}{a} = \frac{Eh^3\theta}{12(1-\nu^2)} \quad ,$$

from which, using (d): $I = \frac{ah^3}{12(1+\nu)}$. (e)

<u>Other Condition</u>. Pure torsion on X and Y planes is assumed here, which has been proved to be equivalent to equal and opposite flexures on two planes at 45° to the co-ordinate planes, Figure 128.

In order to forestall a possible question at this stage, it may be pointed out that a simultaneous uniform curvature of the framework in two opposite directions along the lines at 90° with each other is not physically inconceivable, because the curvature is small, and no warping results.

It is easy to see that horizontal and vertical bars are unstressed, which follows from symmetry. All diagonals going in one direction have moments $+M_1$ and those going in the opposite directions have moments $-M_1$. Moment in the plate per length $\frac{a}{\sqrt{2}}$ is $M_o = M_1$ which gives the third equation of the framework. All that is necessary now is to substitute for M_o and M_1 their expressions.

Using an equivalent of equation (e), Art. 1,

$$M_o = \frac{a}{\sqrt{2}} \frac{Eh^3}{12(1-\nu^2)}\left(\frac{1}{r_n} + \nu\frac{1}{r_t}\right) = \frac{a}{\sqrt{2}} \frac{Eh^3}{12(1-\nu^2)}\left(\frac{\theta}{a\sqrt{2}} - \frac{\nu\theta}{a\sqrt{2}}\right) = \frac{Eh^3\theta}{24(1+\nu)} ;$$

$$M_1 = \frac{EI_1\theta}{a\sqrt{2}} .$$

Equating M_o and M_1, $I_1 = \dfrac{ah^3}{12\sqrt{2}(1+\nu)}$. (f)

Thus, there are obtained three equations, (d), (e) and (f) for determination of only two pattern characteristics, I and I_1. Evidently, the pattern is suitable only for one value of Poisson's ratio, determined by equating (e) and (f), which gives $\nu = \frac{1}{3}$.

Substituting this value into (e) and (f)

$$\left.\begin{aligned} I &= \frac{ah^3}{16} , \\ I_1 &= \frac{ah^3}{16\sqrt{2}} . \end{aligned}\right\} \quad (14)$$

Marginal members have their moments of inertia twice smaller than the interior members.

A curious identity with the value of ν for the same pattern in plane stress is noted here.

5. Square Pattern With Auxiliary Members.

A pattern similar in all respects to the one used in plane stress for any arbitrary ν is adopted here, see Figure 129. The diagonals are assumed to be not connected at the intersection.

Condition 1. The plate is assumed to be bent to a radius r_x in X direction while remaining straight in Y direction. The deformation of the framework is such that the horizontal bars are bent to the same radius r_x, the vertical bars remain straight, while the other members are bent into shapes determined by their conditions of equilibrium.

It is pointed out in this connection that the deformability of the framework is considered to be characterized by the positions of the main joints, while locations of the secondary joints inside the squares are regarded as insignificant, even if those joints project one way or another from the general surface determined by the primary joints.

Moments acting in X direction are M_o per length a, while those acting in a perpendicular direction are νM_o per length a. Calling angle change in plate or in a main horizontal bar θ, the values of M_o and of M, the moment in the main horizontal bar, are found as in the previous article:

$$M_o = \frac{Eh^3\theta}{12(1-\nu^2)} \quad ; \qquad (a)$$

$$M = \frac{EI\theta}{a} \quad ; \qquad (b)$$

while the moment in the main vertical bar is zero.

The problem requiring solution is to express the moments M_1 acting in the outside parts of the diagonals in terms of the angle θ.

However, before this is undertaken, it is necessary to demonstrate that all the members inside the squares are in a state of pure bending, unaccompanied by any shear, since it is quite conceivable that, although the framework as a whole is in a shearless state, the individual members may be carrying shears, which would change their moments from end to end.

Considering first horizontal and vertical members large or small, one by one, it may be concluded from conditions of symmetry that if their ends are acted upon by any transverse forces, those forces must be equal and in the same direction at their both ends, which, however, is impossible from considerations of statics. Therefore, all these members are in a shearless state. The same conclusion may be reached in a similar manner with respect to the inner halves of the diagonals. In connection with the outer parts of these members, it will be noticed from symmetry that, if the main framework joints act on them at all, with any transverse forces, they act in the same direction on all the diagonals. Such a situation, however, contradicts statics of any main joint or of any heart of the unit. Therefore, all members of the framework are in a shearless state and carry constant moments from end to end of each member.

It will be noticed from equilibrium of a secondary joint that all auxiliary members have moments M_2 equal in magnitude and sign, i.e., they have the concavity either all up or all down. Moments corresponding to upward concavity will be considered positive.

As a preliminary step to finding M_1, the angle changes in various interior members are expressed in terms of their moments.

The total angle change in two outer portions of the same diagonal is,

$$\theta_1 = \frac{M_1}{EI_1}\,\frac{a}{\sqrt{2}} \quad . \tag{c}$$

A similar angle change in the central portion of the diagonal is,

$$\theta_{10} = \frac{M_{10}}{EI_1}\,\frac{a}{\sqrt{2}} \quad , \tag{d}$$

and in an auxiliary member:

$$\theta_2 = \frac{M_2}{EI_2}\,\frac{a}{2} \quad . \tag{e}$$

The three equations which are necessary for expressing the moments M_1, M_{10} and M_2 in terms of the angle θ are as follows:

1. Equation of equilibrium of a secondary joint $M_1 = M_{10} + \sqrt{2}\,M_2$, (f), assuming all the moments positive.

2. Equation of deformation of the diagonal stating, total angle change $= \frac{\theta}{\sqrt{2}} = \theta_1 + \theta_{10}$, which after replacing θ_1 and θ_{10} by their expressions (c) and (d) gives, $M_1 + M_{10} = \frac{EI_1\theta}{a}$. (g).

3. Equation of deformation of the heart of the unit. The curvatures of the heart in X and Y directions are equal; the curvatures along the two diagonals are also equal; therefore, the curvature of the heart is spherical, all lines on it have equal curvatures and, consequently, all angle changes are proportional to respective lengths.

$$\therefore \quad \theta_{10} = \sqrt{2}\,\theta_2 \quad . \tag{h}$$

The same result may also be obtained by superposition. First give the heart a uniform cylindrical curvature in X direction with angle change θ_2 on the length of the auxiliary horizontal member. The angle change of either diagonal inside the heart is evidently $\frac{\theta_2}{\sqrt{2}}$. Now superimpose an equal cylindrical curvature in Y direction. This doubles the curvature of the diagonal, wherefrom the relation (h) follows. Substituting for θ_{10} and θ_2 their expressions (d) and (e) the following relation is obtained:

$$\frac{M_{10}}{I_1} = \frac{M_2}{I_2} \quad . \tag{k}$$

Simultaneous solution of the three equations (f), (g) and (h) containing three unknown moments gives,

$$M_1 = \frac{EI_1}{a} \frac{I_1 + \sqrt{2}\, I_2}{2I_1 + \sqrt{2}\, I_2} \theta \quad , \tag{l}$$

while the two other moments are not required.

Expressions (a), (b) and (l) of the moments M_o, M and M_1 allow to set up now two equations for determination of the pattern characteristics I, I_1 and I_2.

Referring again to Figure 127, the two equations of joint equilibrium, identical with previous article, are

$$\sqrt{2}\, M_1 = \nu M_o \; , \tag{m}$$

$$M + \sqrt{2}\, M_1 = M_o \quad . \tag{n}$$

After replacing the moments by their expressions and some simplification the following two equations are obtained:

$$I = \frac{ah^3}{12(1+\nu)} \quad , \tag{p}$$

$$I_1 \frac{I_1 + \sqrt{2}\, I_2}{2I_1 + \sqrt{2}\, I_2} = \frac{\nu a h^3}{12\sqrt{2}(1-\nu^2)} \quad . \tag{q}$$

The third equation for determination of I_1 and I_2 follows from the torsional stress condition.

Other Condition. Pure torsion on X and Y planes which is just the same as equal and opposite flexures on two planes at 45°.

The state of stress here considered may be imagined to be brought about in the following manner. First, let the framework be bent slightly to any constant radius, in a direction at 45° with the axes, so that all members become coincident with the cylindrical surface so formed. All members, with the exception of unstressed diagonals, are bent to circular shapes and, consequently, are in a state of pure bending, which, by the way, may be preserved only by the presence of some necessary external moments at the secondary joints. Now superimpose on this distortion a similar one, but in an opposite direction and along the line at right angles to the first one. The new state of stress created in all members is again that of pure bending. The second deformation unbends all horizontal and vertical members, and the only members that remain bent are the diagonal ones. The constant moments present in the diagonals are $+M_1$ in the members going in one direction and $-M_1$ in those at 90° with the first.

Repeating the reasoning of the previous article with reference to Figure 128, the equation $M_1 = M_0$, arising from this condition, leads to eq-n (f), Art. 4, which, in conjunction with (p) and (q) gives the following results for the framework characteristics:

$$I = \frac{ah^3}{12(1+\nu)} , \tag{15}$$

$$I_1 = \frac{ah^3}{12\sqrt{2}\,(1+\nu)} , \tag{16}$$

$$I_2 = \frac{3\nu-1}{(1+\nu)(1-2\nu)} \frac{ah^3}{24} . \tag{17}$$

Again it must be remembered that marginal members have their I^s twice smaller than according to these formulae.

This derivation shows that the framework of this article is quite suitable for imitating flexural behaviour of a plate with any value of ν. The most remarkable feature is that the necessary moments of inertia of the bars, equations (15), (16) and (17), can be obtained from the necessary areas of the bars in the similar plane stress framework, equations (4), (5), (6) of Art. 7, Chap. I, by replacing at in the latter expressions with $\frac{ah^3}{12}$.

Similarly to the plane stress framework, $I_2 > 0$ when $\frac{1}{2} > \nu > \frac{1}{3}$, $I_2 = 0$ when $\nu = \frac{1}{3}$ and $I_2 < 0$ when $\frac{1}{3} > \nu > 0$. The negative value of I_2 in the latter case has no effect on the validity of calculations.

It is interesting to mention that the torsional state of stress, just considered in connection with the third equation for determination of I^s, may also be brought about by a load condition causing shears in the interior members of the units, as is evident from the following discussion.

Let the boundary conditions be represented by the vertical loads $\pm$ P acting up and down at alternate joints, Figure 130, wherein +P corresponds to a downward load, and -P to an upward one. The overall effect of these forces is evidently a pure uniform bending with a moment $\pm \frac{P}{2}$ per unit length in two directions at 45^o with X and Y axes, bending about one of the axes being positive and about the other negative. From considerations made clear in previous discussions, the main horizontal and vertical members are totally unstressed, while the central parts of the diagonals have no shear.

If shear exists in the outer portions of the diagonals then from considerations of symmetry it must be such that all diagonals at $+45^{o}$ with X axis act on the main joints in the same direction, say upward, while all the diagonals at -45^{o} with X axis act on the same joints in the opposite direction, i.e., downward, which keeps these joints in balance. Considering equilibrium of one joint after the other the boundary joints are finally reached, and from their consideration it is concluded that the $+45^{o}$ diagonals do act on the joints with forces P in the upward direction, as has been assumed.

The state of stress in the members inside each square may be explained on Figure 131 (a). The ends A and F are acted upon by the upward forces P, while the ends K and M by downward forces P. The shear P in the member AB divides equally between the members BC and BD, since the central parts of the diagonals are in pure bending. The shear $\frac{P}{2}$ in BC is augmented by an equal shear coming from EC to form shear P in CK.

In order to find the distribution of moments in different members a section MM (Figure 130) is passed through the framework just touching the corners of the central squares. Since all horizontal and vertical members are unstressed and since all diagonals have equal moments at the points where they are cut by the section MM the moment in the diagonal at any of these points, such as point B, in Figure 131 (a) is $\frac{Pa}{2\sqrt{2}}$ since the tributory length is $\frac{a}{\sqrt{2}}$ and the moment per unit length $\frac{P}{2}$. The moment at the outer end of every diagonal is then $\frac{Pa}{2\sqrt{2}} - P\,\frac{a}{2\sqrt{2}} = 0$.

This allows to construct the moment diagram along the diagonal inside of each square, Figure 131 (b). While parts ab and ef represent moments in the sections of the diagonal AB and EF, the part of the diagram be is due to a combined effect of the diagonal BE and of the auxiliary members BD, BC etc. In order to separate these two contributions, a section GH is passed through the mid-points of the auxiliary members. Since moments in the auxiliary members at G and H are zero from symmetry the constant moment in the central portion of the diagonal is found to be $\frac{3}{4}\frac{Pa}{\sqrt{2}}$.

The full line in Figure 131 (c) represents the moment diagram in the diagonal member, taken by itself. The dotted line, shown in the same figure, corresponds to the shearless state of flexure in the diagonal, discussed earlier in this article. A comparison of these two diagrams shows that they have equal areas and equal statical moments about the outer ends of the diagonal. This observation leads to an important conclusion that both the linear and the angular distortions of one end of the diagonal, with respect to the other, are not affected by non-uniformity of the moment along the diagonal, when shears are present.

It is peculiar that the flexural rigidities of the members, and particularly the rigidity of the auxiliary members, depending to a great extent on the value of ν, have not entered the discussion. The only effect of Poisson's ratio on deformation in the loading considered here is the relative displacement in vertical direction of the adjacent diagonal lines, brought about by bending of the auxiliary members while transmitting shears from one diagonal to another. Owing to this effect, all joints, situated on the diagonals terminating in the boundary joints, acted upon by forces +P, Figure 130, are depressed with respect to the other joints. In making this statement it is reminded that the diagonals are not joined together at intersections.

The relative vertical displacement of the adjacent diagonal lines, as any deflection of a beam caused by a concentrated load, is proportional to $\frac{Pa^3}{I_2}$. As the size of the unit is decreased, I_2 is decreased in proportion to a, while P remains constant in order to retain the same intensity of moment per unit length. This causes a decrease in this relative vertical displacement in proportion to a^2, so that this displacement is insignificant for small a.

This discussion shows that in determination of the framework characteristics no deformation other than shearless need be considered.

6. Distribution Factors in a Framework with $\nu = \frac{1}{3}$.

A. Definitions.

Distribution factors play most important part in the solution of the flexural framework. They are defined as quantities proportional to moments and shears at the ends of different members radiating from a joint undergoing one of the following two kinds of deformation: either a vertical displacement without a rotation or a rotation about one of the framework axes, unaccompanied by any displacement. The joints adjacent to the one undergoing such movement are considered to be stationary.

The following sign convention regarding displacements, shears and moments in the framework members is adopted. Displacement is positive if directed downward, i.e., in positive direction of Z axis. Shear at the end of a member is positive when the member acts on the joint in the downward direction. As to the moment at the end of the member, it is positive if the member is concave upward. It is reminded in this connection that a positive moment in the bar is indicated by a tension arrow, according to the vector notation of this treatise.

When investigating rotational equilibrium of a joint, moments along the framework axes are added up. For this reason, it is not the moment in any diagonal that is significant, but its X or Y component; consequently, it is such components that will be considered as the distribution factors of the diagonals.

B. <u>Displacement Factors</u> .

Let a joint, A, (Figure 132) move downward through a distance Δ, while the adjacent joints remain fixed. The moments produced at the ends of any member are equal and opposite in sign. The end moment in any horizontal or vertical member is:

$$M = 2\frac{EI}{a}\,3\,\frac{\Delta}{a} = 6\,\frac{EI\Delta}{a^2} \quad . \tag{a}$$

The H or V component of the end moment in any diagonal is:

$$\text{Comp. } M_1 = 2\,\frac{EI_1}{a\sqrt{2}}\,3\,\frac{\Delta}{a\sqrt{2}}\;\frac{1}{\sqrt{2}} = \frac{3EI_1\Delta}{a^2\sqrt{2}} \quad . \tag{b}$$

Substituting for I and I_1 their expressions (14) of Art. 4

$$M = \frac{3}{8}\ \frac{Eh^3\Delta}{a}\ ; \tag{c}$$

$$\text{Comp. } M_1 = \frac{3}{32}\ \frac{Eh^3\Delta}{a}\ . \tag{d}$$

Shear in the horizontal or vertical member is:

$$V = \frac{2M}{a} = \frac{3}{4}\ \frac{Eh^3\Delta}{a^2}\ . \tag{e}$$

Shear in the diagonal member is:

$$V_1 = \frac{2\,(\text{Comp.} M_1)}{a} = \frac{3}{16}\ \frac{Eh^3\Delta}{a^2}\ . \tag{f}$$

The displacement distribution factors are shown in Figure 132 (a) and (b), (a) giving the shear factors, and (b) the moment factors. All these factors are numbers proportional to the expressions (c), (d), (e) and (f), with the following proportionality factors.

Factor of proportionality for moments $= \frac{3}{4}\ \frac{Eh^3\Delta}{a}$; (g) .

Factor of proportionality for shears $= \frac{3}{4}\ \frac{Eh^3\Delta}{a^2}$; (h) .

The displacement, like the one represented in Figure 132, which causes a unit shear in horizontal or vertical members, when their far ends remain stationary, will be referred to later as "unit displacement".

It is scarcely necessary to remind that the units of the two kinds of distribution factors have different dimensionalities, which necessitates some later adjustment in the values obtained by using these factors. Either the moment values will have to be multiplied by a, or the shear values divided by a.

It will be noticed that the signs of distribution factors agree with the above sign convention, assuming a downward deflection Δ. The arrows in Figure 132 (b) represent the moments acting on various joints according to the vector notation of this paper.

C. Rotation Factors. Let a joint, A, rotate about Y axis through an angle θ, as shown in Figure 133. Vertical members remain unstressed, while all other members get stressed. Moment on the near end of a horizontal member, i.e., on the end where the movement takes place, is,

$$M_N = \frac{4E\bar{I}\theta}{a} \quad . \tag{k}$$

The H or V component of the moment on near end of the diagonal is,

$$\text{Comp. } M_{1,N} = \frac{4E\bar{I}_1 \frac{\theta}{\sqrt{2}}}{a\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{\sqrt{2}\,E\bar{I}_1\theta}{a} \quad . \tag{l}$$

Substituting for I and I_1 their expressions

$$M_N = \frac{1}{4} Eh^3\theta \quad , \tag{m}$$

$$\text{Comp. } M_{1,N} = \frac{1}{16} Eh^3\theta \quad . \tag{n}$$

The corresponding moments on far ends of the members are twice smaller. Shear at either end of a horizontal member is,

$$V = \frac{3}{8}\frac{Eh^3\theta}{a} \quad , \tag{p}$$

while shear on the end of a diagonal is,

$$V_1 = \frac{3}{32}\frac{Eh^3\theta}{a} \quad . \tag{q}$$

The rotation distribution factors appear in Figure 133 (a) and (b), the former showing the shear factors and the latter the moment factors. All these factors are proportional to the expressions (m) to (q), with the following proportionality factors.

Factor of proportionality for moments = $\frac{1}{4} Eh^3\theta$; (r) .

Factor of proportionality for shears = $\frac{1}{4}\frac{Eh^3\theta}{a}$; (s) .

A rotation causing stresses of Figure 133 will be called "unit rotation".

The signs of distribution factors in Figure 133 are in agreement with the convention stated, and so are the arrows representing the joint moments.

An important relation between the shear and the moment factors either in case of displacement or in case of rotation is as follows. Shear at any end of any member taken with its sign equals the moment on the other end minus the moment on this end, both moments being taken with their signs. This relation is used in checking moments and shears, as will be explained later.

Distribution factors of marginal members are twice smaller than those of interior members.

Similar distribution factors exist also in the flexural square framework corresponding to any arbitrary γ. Their determination is quite laborious and has not been carried out for lack of time.

7. <u>Methods of Framework Solution</u>.

Much of what has been said about the plane stress framework in Art. 9, Chap. I regarding the use of finite units of uniform and different sizes, suitability of the method for different shapes of plates in the plan and high statical indeterminacy holds equally well for the flexural framework. For practical reasons, however, the square pattern is scarcely applicable to any plates but rectangular in plan.

It is reminded that forces acting on a flexural framework are limited by the nature of the problem to transverse loads and to moments about axes lying in the plane of the framework.

In discussing questions involved in solution of framework it is important to realize structural features of the pattern related to statical indeterminacy and to freedom of joints to move.

Each framework member may be considered as possessing two unknowns, i.e., moment and shear, on one of the ends, while the values of similar functions on the other end follow from statics of the member. Three equations of statics may be written for each joint, one equation for shears and two others for moments in the directions of the axes.

From these considerations it is easy to deduce that the introduction of auxiliary members in the framework with arbitrary γ, compared with $\gamma = \frac{1}{3}$ framework, does not increase statical indeterminacy of the structure since the number of new members added in each unit is 6, corresponding to 12 new unknowns, while the number of new equations of statics is also 12, owing to 4 new joints. Therefore, the more complicated type of pattern is not different in its indeterminacy from the type with simple diagonals.

Each simply supported joint, whether standing by itself or forming a part of a boundary, has one reaction associated with it. A joint, other than a corner one, forming a part of a clamped edge possesses two reactions, i.e., a vertical reaction and a moment about an axis parallel to the edge. It is peculiar that such a joint is not restricted to turn about an axis normal to the edge. A corner joint belonging to two clamped edges possesses three reactions, one vertical reaction and two moments.

Calling the number of members in the framework, with $\gamma = \frac{1}{3}$,
m, the number of joints j, the number of statical unknowns u, and the total number of reaction values r, it may be seen that if $r \geqslant 3$,

$$u = 2m + r - 3j \quad , \qquad \text{(a)}$$

and if $r \leqslant 3$,

$$u = 2m + 3 - 3j \quad , \qquad \text{(b)}$$

the difference between these two relations being due to the fact that the active forces corresponding to the latter case are not fully independent.

Each unrestrained main joint of the framework has three degrees of freedom, i.e., one vertical displacement and two rotations about the framework axes. A simply supported joint has two degrees of freedom, i.e., two rotations. An intermediate joint belonging to a clamped edge has one degree of freedom, as has been stated, while a corner joint at intersection of two such edges cannot move at all.

The following methods may theoretically be used for solution of flexural framework.

1. Method of Least Work. This method would be extremely labourious for the reason of the large number of unknowns. Thus, a 4 x 4 framework, (Figure 134), simply supported at the edges has the following number of elements: m = 72, j = 25, r = 16, and the number of statical unknowns comes out 85.

2. Method of Slope-deflection. Joint movements may be taken as unknowns, end moments and shears expressed through them and equations of equilibrium of joints set up to find these unknowns. This

method applied to the problem just mentioned leads to 59 simultaneous equations, which, although better than Least Work, is still prohibitive. Introduction of more restraints, for example, by making the edges clamped, is favourable to this method and unfavourable to Least Work.

3. Method of seccessive joint movements is the one that has been found the best for solving flexural frameworks. The idea of this method is similar to the one used in connection with plane stress framework. Different joints are given successive movements consisting of displacements and rotations. The end moments and shears, caused by these movements in the members of framework, are computed by means of distribution factors of the previous article, simply by multiplying these factors by the number of units of movement given to the joint. Summation of end moments and shears at each joint gives the unbalanced functions. The purpose of successive movements is to reduce to zero these unbalanced joint mements and shears. The record of current values of functions during distribution is kept on two diagrams of framework, one for shears and the other for moments. Shears and moments at the ends of members are recorded by means of figures written on the members. The total joint shears and moments are also stated near joints, the moment figures being accompanied by two arrows, indicating the directions of X and Y moment components. The method of recording is much similar to the one used in plane stress. Furthermore, it is very convenient, when combining the moment arrows into the resultant joint moments, to think of them as if they were tensions or compressions in a plane stress framework.

By consideration of Figures 132 (b) and 133 (b) it may be seen that a joint displacement results in moment arrows radiating from this joint and directed in all the members either all away or all toward the displaced joint. In case of rotation, the moment arrows are all streaming in the same direction through the rotated joint. In rotating the joint toward the balance it is not even necessary to visualize the direction of rotation. All that is required is to apply to the unbalanced joint and to its neighbours, moment arrows directed in accordance with the above made observations.

This explanation presents only a brief outline of a rather elaborate procedure. The difficulty encountered in application of the method is that, unlike the plane stress problem, the movements which would lead to equilibrium of joints are not apparent. It is easy enough to eliminate the moment unbalance by proper rotations of joints, but this leads to shear unbalance; on the other hand, a removal of shear unbalance by the necessary displacements leads to unbalanced moments; at the same time it seems impossible to foresee directly the true combination of the two kinds of movements.

In order to overcome this difficulty an indirect procedure has been adopted, in which an important part is played by the so-called reaction factors, whose meaning may be explained with reference to Figure 135 representing any arbitrary framework.

Suppose, that transverse forces are applied to all the joints of the framework so that one of the joints, for example A, gets a unit transverse displacement, while all other joints remain in their original locations; at the same time all the joints are left free to rotate to equilibrium. These joint forces are called the reaction factors corresponding to movement of joint A. The values of these factors quickly decrease as the distance from joint A increases, approaching zero at the joints more than three framework units away from A.

These factors may be computed either exactly by using the actual framework of the problem, or approximately, by using an imaginary framework, extending from the displaced joint not more than three units on each side. The computation is comparatively simple, since the displacements are known, and the distribution of rotations can be easily performed.

Sets of reaction factors must be computed for displacements of all the joints of the framework. Approximate factors should be preferred since their computation is easier, and since they have similar values for several sets as long as the displaced joints are not less than three units distant from the periphery. Actual boundary conditions should be taken into account for those sets whose displaced joints are, like joint B, only one or two units distant from the edge.

There is only a limited number of sets of approximate reaction factors; they may be calculated beforehand and used in all framework problems. On the other hand, the number of sets of the exact factors is larger, and they are good only for one particular framework.

Figures 136 to 140 give five sets of reaction factors, utilized later. In these figures, frameworks extend for one, two or three units from the displaced joints. All edges are assumed clamped. Since the frameworks are symmetrical about one or two axes, reaction factors repeat themselves in corresponding parts, and only one quarter or one half of each framework is shown on the drawing.

Rotational distributions used in determination of these factors have been carried out only approximately, and the values of the factors are not extremely accurate.

Reaction factors are made the basis for an approximate determination of joint displacements, which is done on a separate form. In this part of the problem joints are displaced one at a time through a number of units of displacement estimated on the basis of unbalanced joint forces; new joint forces, caused by the movements, are added up and the operation is repeated until the unbalanced joint forces become small.

The joint displacements so found are, however, only approximate, in view of the approximate nature of the reaction factors. The next step in the procedure is to introduce these displacements into the actual framework, and to distribute all unbalanced moments. If the displacements were correct, no unbalanced joint forces would be present. In actual problem, however, unbalanced forces do show up, although they are of the order of some twenty times smaller than the original unbalances. Two or three additional cycles of the same procedure are necessary to make the unbalances negligible.

This explanation gives a general idea of the process of solving a flexural framework. Some minor features will be added later when an actual problem will be solved.

4. Finding joint displacements by equations. This is a modification of the method just described. In this method the exact reaction factors are found by rotational distributions in the actual framework. In order to find joint displacements necessary for equilibrium a system of simultaneous equations is set up by means of reaction factors, each equation expressing equilibrium of a free joint. After the equations are solved, the framework is given the necessary displacements and the unbalanced moments are distributed by means of rotations. The number of simultaneous equations is equal to the number of free joints. This method is believed to be more laborious than the previous one.

The principle of symmetry is no less applicable in flexural framework than in plane stress. The necessary and sufficient requirement for its application is symmetry of the structure, including its restraints, about one, two or more axes. The manner of breaking up of the problem into its component cases, similarity of stress conditions in different portions of the plate and some other features of the principle discussed in the chapter on plane stress, are equally applicable in flexural framework and will not be repeated here.

8. Principles Used in Checking. Routine of the Distribution. Stress Interpretation.

As has been explained in the chapter on plane stress, current checks are imperative in any lengthy arithmetical procedure, such as distribution of unbalanced joint forces in a framework.

Consideration of Figures 132 (a) and 133 (a) shows that the algebraic sum of all the joint shears springing up after either a displacement or a rotation is zero, which is the direct result of the fact that the shears on two ends of any member are numerically equal and opposite in sign. It is needless to say that the same statement holds for the movements of marginal joints.

This principle is made the basis for checking the unbalanced joint shears (joint forces). At any stage in the distribution the algebraic sum of the unbalanced joint shears, including the shears at the boundary joints, before they are neutralized by the reactions, is equal to the algebraic sum of the original joint loads. The purpose of distribution is to push these unbalanced joint forces to the points of support, where they become neutralized by the reactions.

The situation with the moments is somewhat more complicated. While in case of joint displacement, Figure 132 (b), the algebraic sum of the joint vectors, either horizontal or vertical, is zero, in case of rotation, Figure 133 (b), the sum of vertical moment vectors is zero, but the sum of horizontal ones is $4\frac{1}{2}\rightarrow$ of which vector $3\rightarrow$ is applied at the rotated joint, and the remainder comes from the adjacent joints. Thus, when a rotation of a joint takes place, adding a certain moment vector to the rotated joint, the framework, as a whole, receives a moment vector $1\frac{1}{2}$ times larger than the one applied at the joint in question.

Since it is convenient to base a check on equality of certain values to zero, an arrangement will be agreed upon to accompany any joint rotation by entering in a special column a moment vector $1\frac{1}{2}$ times larger than the one applied to the rotated joint, in the direction opposite to it. This vector will be referred to as "balance" vector. The algebraic sum of the joint vectors and of the balance vectors, either in a horizontal or in a vertical direction, remains constant at all stages of distribution, and, unless the original loading of the framework is effected by means of couples, this sum stays zero. Two columns are used for the balance vectors, one, indicated in Figure 133 (c), for the horizontal vectors, and the other for vertical ones.

So far, only the movements of the interior joints have been considered. Balance vectors corresponding to movements of marginal joints are somewhat different, and will be discussed now.

In Figure 141, rotation of a marginal joint about an axis perpendicular to the margin is shown. The balance vector is again $1\frac{1}{2}$ times larger than the moment applied to the rotated joint, which ratio is preserved in all cases of rotation. No vertical balance vector is needed.

Figures 142 and 143 show rotations of an intermediate and corner marginal joints about horizontal axes, and the balance vectors that result from these movements. In case of corner joint two balance vectors are needed.

Balance vectors are also required in case of a vertical displacement of a marginal joint as demonstrated in Figures 144 and 145.

Unlike rotation, the balance factors accompanying joint displacement are twice larger than the moments applied to the displaced joint.

In the framework problem solved below, the portion of structure subjected to distribution is separated from the framework by means of cuts along two axes of symmetry, one of which follows the direction of a diagonal and forms a margin at 45° to the framework axes. Although this feature does not introduce any new principles, it becomes necessary to state the values of distribution and balance factors for movements of joints adjacent to the diagonal margin. The section of members along such margin is naturally twice smaller than that of an ordinary diagonal. Since the loading of the problem is symmetrical about this margin, rotations of marginal joints will be considered only about axes normal to the margin.

In Figure 146, moment factors and balance vectors corresponding to displacement of the marginal joint A are shown. In Figure 147, a joint B, which is a short distance away from the margin is displaced. Since, in view of the symmetrical nature of the loading, this displacement is assumed to be accompanied by a similar movement of a joint located symmetrically on the other side of the margin, the diagonal joining these two joints remains straight. This accounts for the moment unbalance of the joint B and for the presence of the balance vectors.

Figure 148 represents rotation of the joint A about an axis perpendicular to the margin. The values of the distribution factors may be obtained by superposition of two rotations about the horizontal and vertical axes. In Figure 149, joint B rotates about a horizontal axis, while another joint symmetrically located on the other side of the margin, performs a symmetrical movement, i.e., rotates about a vertical axis. The half-member BB' is in a state of pure bending with moment equal to one half of the moments of the other diagonal members at point B. In recording the balance vectors, first the two vectors $-\frac{3}{8}$ each are entered, which transforms joint B into a regular interior joint, then the vertical vector $+ 4\frac{1}{2}$ is entered in order to neutralize the moments corresponding to rotation of an ordinary interior joint. When checking moments, the vector arrows at the joints similar to B' must also be included in the summation.

Since joints situated on the inclined margin are permitted, in view of the load symmetry, to rotate only about the axes perpendicular to the margin it is necessary to break up their moments into components perpendicular and parallel to the margin. While the former arrows are neutralized by the moments coming from the other side of the margin, the latter constitute the true unbalanced moments.

The rule for transforming horizontal and vertical moment arrows is very simple and becomes evident from consideration of Figure 150, in which the horizontal and vertical moment vectors X and Y are changed into vectors N and T parallel to lines at 45° with X and Y. In line with previous convention, numbers N and T stand for the values of the horizontal or vertical components of these vectors. Therefore,

$$\left.\begin{aligned} N &= \frac{X+Y}{2}\,, \\ T &= \frac{X-Y}{2}\,, \end{aligned}\right\} \qquad \text{(a)}$$

or

$$\left.\begin{aligned} X &= N + T\,, \\ Y &= N - T\,. \end{aligned}\right\} \qquad \text{(b)}$$

and

Another idea utilized in current checks of a framework solution has been alluded to in the article on distribution factors, and should be made clear from comparison of the shear and moment factors in Figures 132 and 133. At any stage of distribution, the shear at the end A of a framework member AB is equal to the algebraic difference of the moments at the ends B and A.

The following routine in working out each cycle of distribution has been found convenient.

1. Determination of joint displacements by means of approximate reaction factors.

2. Introduction of these displacements into the framework, accompanied by recording of the moments and shears at the ends of the members, of the balance vectors, and of the unbalanced joint moments and shears.

3. Distribution of moment unbalances by joint rotations, one by one, accompanied by recording of the same functions as in 2. The unbalanced joint forces now decrease some fifteen to twenty times compared to the original values.

4. Current check, consisting in the following procedure.

   a. Separate additions of X components of joint moments and balance vectors, ditto Y components and, thirdly, unbalanced joint shears. Each sum must be equal to the original value of it before the distribution.

b. If any of these sums do not check, the end moments and shears in all members are found by adding up individual items on the members. The shears are checked against the differences in end moments, as has been explained.

c. Unbalanced joint moments and shears are found by addition of corresponding functions at the ends of the members. Results are compared with the original figures of joint unbalances, which are thus checked.

After the summations referred to in 4 a,are found to check, another cycle of distribution is carried out. It is believed that no error can escape detection in this procedure, but on a few occasions the discrepancy has been found to be due to errors in the balance vectors or even in a failure to enter these quantities at all. For this reason special care should be exercised in dealing with belance vectors, and it may even be necessary to go over them once more for check.

The simplicity of the ratios between different values of distribution factors makes possible to perform distribution in round figures. In order to avoid undesirable fractions the displacements must be taken in numbers of units multiple of 8 and 16, and the rotations in multiples of 24 or 12.

After completion of distribution a final check, similar to the one used in plane stress, may be easily carried out, although it is unnecessary if current checks have been properly followed. In this final check the framework is given due displacements after which the joints are rotated, one by one, through the angles necessary to bring about the moments found at the ends of horizontal and vertical members. The state of deformation, resulting from these rotations, must be the true state of equilibrium, if the original solution is correct, therefore, the end moments in the diagonals must agree with the originally found values, which constitutes the check. There is no need to check shears, since they have been found in the last current check to agree with the moments.

Interpretation of framework stresses into continuous moments and shears of the bent plate follows the principles explained in the chapter on plane stress and will be mentioned here only briefly. All the plate functions per unit length of plate, i.e., bending and torsional moments M and shears Q will be found from the corresponding framework joint concentrations K by formulae:

$$\left.\begin{aligned} M &= K_M , && \text{(c)} \\ Q &= \frac{K_Q}{a} && \text{(d)} \end{aligned}\right\} \text{for intermediate joints,}$$

and

$$\left.\begin{aligned} M &= 2\,K_M && \text{(e)} \\ Q &= \frac{2K_Q}{a} && \text{(f)} \end{aligned}\right\} \text{for edge joints.}$$

Concentrations K for shears and torsional moments are the averages of the values on two sides of the joint. The absence of a in the expressions for moments follows from the difference in dimensions of the proportionality factors for shears and moments as has been mentioned in Art. 6. It is assumed here that the plate is loaded with vertical loads and not with the load couples. Should the opposite be true, the right hand sides of the expressions (c) to (f) would have to be divided by a.

Some minor qualifications in using these formulae will be mentioned later.

9. Problem of Bending of a Clamped Edged Square Plate by a Concentrated Load at the Centre.

A. Solution of the Framework.

An 8 x 8 square framework with $\nu = \frac{1}{3}$, Figure 154, Plate 18, is used in solution of this problem. Since the framework is symmetrical about four axes a triangular octant of the structure is all that is necessary to consider. Being axes of symmetry the two cuts, separating the octant from the rest of the structure, do not rotate about the axes parallel to them, so that the corresponding components of the joint moments are balanced by the moments coming from the other side. Joints located along the clamped edge do not displace and do not rotate about the axes parallel to the edge. Due to an oversight, they were at first not allowed to rotate about the axes perpendicular to the edge, which left some torsional moments at the edge. This error was corrected by an additional cycle of distribution at the end of the procedure.

The standard distribution routine has been modified in the first cycle, and the problem has been commenced by distorting the framework according to Figure 151 (a). Numbers stated at the joints represent their units of displacement, arrived at by rough visual estimation of the expected deflected shape of the plate. They are all made devisable by 8, while the numbers along the inclined cut are made devisable by 16 in order to avoid fractional shears and moments. Numbers written along the members represent relative displacements of the two ends. It must be realized that the actual values of displacements are immaterial in this problem, and all the numbers might have been taken ten times larger or smaller.

Shears and moments, corresponding to these displacements, are now entered on the distribution diagram. It may be observed that the shear values in the horizontal and vertical interior members are equal to the displacement numbers with proper signs, shears in the horizontal marginal members equal to one half of the displacement numbers, shears in the interior diagonal members are obtained by dividing the displacement numbers by 4, while those in the marginal diagonal members by dividing by 8. End moments caused by these displacements are found by dividing the corresponding shears by -2.

Horizontal and vertical balance vectors are now found and duly entered. There is no need to find them for each member separately since their total may be found more easily. This is best done by adding up the displacement numbers, separately for horizontal, vertical and diagonal members going each way, wherein the numbers on the marginal members are divided by two, and by obtaining contributions to the balance vectors separately from each sum.

X and Y components of the unbalanced joint moments are now determined and recorded. Unbalanced joint shears may also be calculated, although this may be waived.

The next step is to distribute unbalanced moments by rotations about X and Y axes except for the joints along 45° margin, which are rotated only about the lines normal to the edge. Rotations are carried out one by one beginning with larger unbalances. It is recommended to take into consideration, at least approximately, the future influence of rotations of the adjacent joints, even though such influence is not great.

If an interior joint (see Figure 133c) is allowed to rotate to equilibrium, one third of the unbalanced moment goes into the near end of the horizontal or vertical member parallel to rotation and $\frac{1}{12}$ into each diagonal. The far ends of the members get moments twice smaller. This consideration allows to estimate the number of the necessary units of rotation, and the influence of rotation of the neighbouring joints. In order to avoid fractions the number of units must be made a multiple of 24 for interior joints and of 12 for marginal joints.

The added end moments, end shears and balance vectors must be immediately recorded after each joint rotation. All affected joints are gone over and the unbalanced moments modified by the new additions, the old figures being simply rubbed out. If the unbalanced joint shears have been recorded, they are also modified now. It is desirable to use coloured pencil for those joint moments and shears which are balanced by restraints.

When the joint moments are all reduced to small values they are checked as has been already explained in the previous article, and this completes the first cycle. The end moments and shears, resulting from the first cycle, may be seen in Figure 152 (a) and (b) as the first partial sums recorded on all the members nearest the joint centres.

The unbalanced joint shears remaining after the first cycle are stated in Figure 153 (a), while the unbalanced joint moments, which are small, are not shown. These shears are quite large, which indicates that the assumed displacement curve is not the true one. If the guess at the first cycle displacements had been correct the joint shears would have been everywhere zero with the exception of the edge joints and of the joint at the centre.

Joint displacements used in the second cycle are determined by means of the reaction factors, Figures 136 to 140. These factors, however, have to be adjusted in order to suit the peculiarities of the problem. Since the plate and the loading conditions are symmetrical about the four axes shown in Figure 154, a displacement of any intermediate joint, such as A, must be accompanied by similar displacements of seven other symmetrical joints A_1 to A_7. Similarly, a displacement of a marginal joint B is co-existent with similar displacements of B_1, B_2 and B_3. Therefore, in order to find a reaction factor at some joint, such as M, caused by translation of the joint A, it is necessary to superimpose the effects of unit translations of the joints A, A_1, A_2, A_3, A_6 and A_7 on M. Translations of A_4 and A_5 are too far distant to be felt at M. These factors will be referred to as "influence factors", and the one at M due to displacements at A, A_1...may be found from Figure 139 to be

$f_{MA} = \frac{1}{2}\left[2\ (0.228 + 0.029 + 0.002)\right] = 0.259$, the reason for the coefficient $\frac{1}{2}$ being that the joint M is a part of an interior margin.

It may be mentioned that the influence factors are subject to the law of reciprocity, which may be proved by Betti's theorem, whereby $f_{MA} = f_{AM}$. This has been used as a check of the determined values of the factors, and in case of a minor disagreement the average of the two may be taken as the true value.

The five sets of reaction factors, Figures 136 to 140, are theoretically incomplete for determination of all the influence factors, but the deficiency is immaterial and may be supplemented by judgement. Furthermore, the reaction factors themselves are not only approximate by their nature, but have also been determined approximately. All this accounts for certain lack of precision on the part of the influence factors. Although they have done a good work by cutting down unbalanced joint forces on each cycle some fifteen to twenty times, it is felt now that it probably would pay to have them calculated with greater accuracy.

Influence factors are shown in Table 3, Plate 24. The second cycle displacements, found by means of these factors, are presented in Figure 151 (b), while the results of two other cycles appear in Figure 151 (c) and (d).

Each set of displacements is handled in the distribution diagrams of Figure 152 (a) and (b) in exactly the same way as has just been explained. The total unbalanced joint forces remaining after each cycle are shown in Figures 153 (a) to (c) and in Figure 152 (b). These figures show how the joint forces everywhere, with the exception of the centre and the edge, decrease with each cycle until in Figure 152 (b) they become negligible.

The technique of finding the displacements of each cycle by means of influence factors is demonstrated in Figure 155, in relation to the third cycle displacements. The procedure amounts to a solution of nine simultaneous equations by approximations. Figures at the heads of third columns may be recognized from Figure 153 (b) as the joint unbalances left after the second cycle. Successive movements, recorded in the first columns and decided upon from inspection of unbalanced forces, are introduced at various joints; the corresponding joint forces, found by multiplication of the number of units of displacement by the influence factors, are stated in the second columns, while the resultant joint forces appear in the third columns. New displacements are added until the remaining joint forces are small. It is not an easy matter to decide what should be the values of the displacements, and it may be seen from Figure 155 that most of the movements have been underestimated on the first trial, while the movement of the joint 1,0 has even been taken originally in wrong direction. Proficiency in this procedure depends greatly on experience. The central joint has been left stationary which simplified somewhat the work. In other problems this joint may have also to be moved.

The final sums in Figure 152 (a) and (b) represent the solution of the framework problem, which, however, is erroneous insofar as the edge joints have not been allowed to rotate about axes perpendicular to the edge, and, therefore, some torsional moments remain at the edge. This oversight has been corrected by allowing these joints to rotate and by adding one more cycle of distribution. Final displacements, shears and moments are given in Figure 156 (a), (b) and (c).

B. Plate Stresses by Framework and by Other Methods.

In connection with interpretation of results of framework solution into the plate stresses, a few words must be said about the signs. It is clear that the sign convention of the framework, determined by the needs of distribution procedure, does not agree with the theory of elasticity convention of Art. 1; this necessitates some sign conversion before the results may be presented in form of stresses. How this operation should be done on shears is quite evident; in case of moments, however, some explanation is necessary, and it is done by means of Figure 157. The arrows K represent joint concentrations of the moments external to the framework which correspond to positive M_x, M_y, M_{xy} and M_{yx} according to the elasticity convention.

The numerical results of the distribution just accomplished may be reduced to general form, corresponding to the action of a force P at the centre of the framework, in the following manner.

Let letter K, with an appropriate subscript, stand for a joint concentration of bending or torsional moment, then, since the size of the unit of moments, or the proportionality factor of Art.6 is $\frac{3}{4}Eh^3\frac{\Delta}{a}$,

$$M_x a = \frac{3}{4}\frac{Eh^3\Delta}{a}K_x \qquad \text{or} \qquad M_x = \frac{3}{4}\frac{Eh^3\Delta}{a^2}K_x \quad . \qquad (a)$$

Since the shear concentration at the centre joint of the octant is 347.29 and since the size of the unit of shears is $\frac{3}{4}Eh^3\frac{\Delta}{a^2}$,

$$P = 8\ (347.29)\ \frac{3}{4}\ Eh^3\ \frac{\Delta}{a^2} \quad . \qquad (b)$$

Eliminating Δ from (a) and (b),

$$M_x = \frac{K_x}{2778.32} P \;, \tag{c}$$

which is the required expression.

Replacing K_x with K_Y or K_{XY}, similar expressions for the other unit moments result. Corresponding expression for shear per unit length of plate is,

$$Q_x = \frac{K_x'}{2778.32} \frac{P}{a} = \frac{K_x'}{694.58} \frac{P}{4a} \;, \tag{d}$$

where 4a is half side of the square plate.

Let k be the number of units of deflection corresponding to the given joint (Figure 156a), the size of the unit being Δ, which agrees with the size of the unit for shears. Then deflection at that joint is,

$$\delta = k\Delta \;. \tag{e}$$

Eliminating Δ from (b) and (e),

$$\delta = \frac{k}{2083.74} \frac{Pa^2}{Eh^3} \;. \tag{f}$$

It will be reminded that the deflections and the moments found in this problem apply only to a plate with $\nu = \frac{1}{3}$, although they may be easily reduced to any other value of ν by formulae (11) and (12) of Art. 2. The shears, on the other hand, are good for any ν.

In order to check the method, the results have been compared with those derived by Marcus[6] and by Professor Timoshenko.[7] A few words about their methods are in order.

The method of Marcus, referred to below as Method No.2, is an approximate one, based on the replacement of the differential equations of the bent plate by the corresponding difference equations obtained by using ratios of finite quantities for the derivatives, like

$$\frac{\partial w}{\partial x} = \frac{w_2 - w_1}{a} , \quad \text{(g)}$$

or
$$\frac{\partial^2 w}{\partial x^2} = \frac{w_3 - 2w_2 + w_1}{2a} \quad \text{(h) etc.}$$

Marcus divided the plate into the same number of squares as has been done in the framework solution, namely 8 x 8, and used the corners of those squares for the key ordinates $w_1, w_2 \dots$. This makes the degree of his approximation appear comparable to the one used in this problem. Since the results of the comparison of two solutions have been found not very satisfactory, an additional computation has been made by the equations derived by Professor Timoshenko [7].

The latter method, called below No.3, is an exact one, and its solution is expressed in the form of a combination of several infinite series of trigonometric and hyperbolic terms. In order to make the solution workable, terms after a certain number, have been discarded by Professor Timoshenko, which made his final formulae also somewhat approximate. However, from a practical viewpoint they will be considered as exact.

Some of the results below are given by all three methods, while others by only two or even one.

Figure 158 presents deflections by all three methods. The framework figures are almost indistinguishable from those of Professor Timoshenko, while Marcus' figures are everywhere considerably out. At the centre, for $\nu = \frac{1}{3}$

$$\delta_c = 3.839 \frac{Pa^2}{Eh^3} = 0.240 \frac{P(4a)^2}{Eh^3} , \quad \text{(k)}$$

where 4a is one half of the side of the square plate.

Moments M_x, M_{xy} and shears Q_x are stated in Figures 159, 160 and 161 respectively. Bending moments are given also by two other methods, and torsional moments by the Method No.3, while the figures of shears are supported by comparison with Method No.2 at the periphery points only.

Agreement of the framework moments M_x with those by Method No.3 is good, except near the centre and at a few points where the moment values are very low. Method No.2 again compares unfavourably with No.1, although its disadvantage is not so great as in deflections.

At the centre M_x is theoretically infinite. The framework method by its very nature cannot produce infinity, since it averages the effect over a unit of framework. Therefore, the value given by it for M_x at the centre should not be taken as an indication of the true moment at that point. The same applies to the method of Marcus; the flattened shape of his moment curve near the centre is not typical for the moment conditions near a point of concentration. It appears that he has assumed the load P distributed uniformly over a certain area near the centre.

It is the proximity of the point of application of load P that is believed to be responsible for somewhat poorer precision of the framework values of M_x at points 1,0 and 0,1.

The showing of torsional moments determined by framework in comparison with the results of Method No.3 (Figure 160) is unexpectedly poor, the difference being mostly around 15-20% with as much as nearly 40% of the true value at the worst point.

The values of shear Q_x by the methods 1 and 2 show an excellent agreement at the edge parallel to Y axis (Figure 161), while similar values at the other edge are badly out. Although the method of Marcus does not appear very reliable, the error in this particular case probably lies in the framework method. No check has been made on shears at intermediate points.

The same data on moments and shears by these various methods are again presented in the form of curves in Figures 162 to 166. The curves of M_y and Q_y are plotted from the equal values of M_x and Q_x at corresponding points, since M_x at point x, y is equal to M_y at the point y, x. At the edge perpendicular to X axis $\frac{M_y}{M_x} = \nu = \frac{1}{3}$. This relation is satisfied by the framework values fairly closely.

The poor showing of the framework method in torsional moments and in shears Q_y at the edge x = 4a has been most disappointing and unexpected in view of the excellent results with deflections and good results with bending moments and other shears. No explanation of the discrepancy has been found, although it is believed that imperfect interpretation is at the root of the trouble.

In Figures 167 to 169 are given by the same three methods some of the functions for the plate with $\nu = 0$. The curves by Marcus are borrowed from his book. There is a fair agreement of all three methods in moments M_x and M_y on the plane x = 0, except near the centre of the plate. Peculiarly enough, moment M_x at the edge x = 4a does not depend on the value of ν, which may be seen from the eq-n (12), Art. 2, therefore, Figure 162 (e) may be referred to for the curve of this moment for $\nu = 0$. Figure 169 presents three different functions, taken at the points along the diagonal of the plate.

These functions are: M_{xy}; M_1, which is the bending moment on the plane normal to the diagonal, and M_2, bending moment on the plane of the diagonal. Moments M_1 and M_2 are computed from the formulae:

$$\left.\begin{aligned} M_1 &= M_x - M_{xy} \,, \\ M_2 &= M_x + M_{xy} \,. \end{aligned}\right\} \qquad \text{(m)}$$

The presumably true curves determined by the method of Professor Timoshenko split nearly evenly the differences in ordinates determined by the two other methods. It is evidently the error in M_{xy} which makes the framework values of M_1 and M_2 disagree with the true values. The true value of M_1 or M_2 at the centre of the plate is infinity.

It may be concluded from these curves and diagrams that, in the absence of adequate explanations of the discrepancies, the framework method cannot be considered fully satisfactory at least for determination of torsional moments and, possibly, shears, but even with that it compares favourably in accuracy with the method of Marcus. The results of the deflections and bending moments show that the method is correct in its basic idea, and that the source of error lies in some of its secondary features.

The framework method is of course very laborious, but so is the method of Marcus and on the strength of its presentation by its author one would not expect it to be less time consuming than the method of this treatise.

As to the method of Professor Timoshenko, it has taken the writer approximately nine days to compute the data presented in this article from the apparently explicit formulae with all coefficients given. The formulae of this method are bulky and the series long, on account of slow convergence, which, by the way, has been the reason for not having calculated shears by this method. In some cases it has been necessary to include as many as twenty terms, and even after that, a special investigation has been required of the error involved in disregarded part. Difficulty has also been experienced in devising any adequate check. The method, however, is the least laborious of all three, if the values of moments or shears are required only at a few definite points, and if the values of the basic coefficients are known beforehand.

III.

Conclusion

The most important peculiarity of the framework method is its applicability to a variety of unsolved problems of elasticity in which the shape of the stressed body conforms to the rectangular form of the framework pattern, and the points of application of loads may be adjusted to the locations of the joints. The following examples will be cited.

The gusset plate problem of Chapter I may be modified in such a manner that all truss members will be assumed as deformed equally with the gusset plate all along their lengths of attachment, instead of delivering to the gusset plate a uniform line loading, as assumed in the problem solved. The framework bars coincident with the attached parts of the truss members will have to be increased in area, and the corresponding distribution factors modified. In this form the gusset plate problem will, probably, be closer to the truth, even though the deformability of the rivets will be left out of consideration.

In another problem, the plate, acted upon by forces in its plane, may have different thicknesses in its different parts, with a qualification, that the boundaries between them must coincide with the framework bars.

The plate may also have rectangular openings of the sizes multiple of the size of the unit. A few half size units will, probably, be necessary near the re-entrant corners.

The plate under plane stress may be a reinforced concrete plate with reinforcing bars running in directions of the framework members. The areas of framework bars will have to be modified to allow for reinforcing. It will be necessary to assume that no cracking occurs. It may be pointed out that the basic differential equation of deformability will not hold in this case without some modification. As an application of this problem, stresses set up by shrinkage of concrete at the junction of a wall and a floor slab may be investigated.

It seems feasible to extend the method into the realm of three dimensions in application to rectangular elastic blocks, although the manner of keeping record of bar stresses during distribution may prove clumsy.

A variety of problems on bending of plates may also be attacked by the framework method. Some of the problems, however, cannot be solved by the pattern with $\nu = \frac{1}{3}$, and so, before the extension of the method, the distribution factors for other values of ν must be computed. All possible edge conditions may be handled, although fixed edges are the simplest in distribution. Increased plate thickness over part of the area, as in dropped panels of flat slabs, presence of rectangular openings, also common in these structures, strengthening of free edges by beams, presence of reinforcement, all these features and some others, although increasing considerably the labour of solution, may, nevertheless, be taken care of by the method of this treatise.

Another field into which this method may possibly be extended is the analysis of cylindrical shells with determination of both, the membrane stresses and the bending stresses caused by the edge restraints.

As an interesting by-product of theoretical investigation, application of the framework method to model analysis seems also feasible, at least for the material with $\nu = \frac{1}{3}$. Such framework model of a plate in the conditions of plane stress may probably be more easily loaded, especially at the interior points, than the bakelite or rubber models, and the strains of the bar members may be investigated in ordinary ways by strain gauges or by a microscope.

It will be remembered that the stresses in the plate subjected to plane stress and loaded at interior points depend on ν. For this reason, when investigating a metal plate so loaded, a model of it in the form of a framework with $\nu = \frac{1}{3}$ has an advantage before photoelastic or rubber models with $\nu = 0.4 - 0.5$; in view of proximity of the figure $\frac{1}{3}$ to the actual values of Poisson's ratio for metals, ranging within the limits $\frac{1}{4} - \frac{1}{3}$.

The distribution procedure, especially in the flexural framework, seems quite involved to a beginner, but once experience has been gained, this feeling disappears, and all the necessary steps are taken in a stride. A thorough familiarity with the values of distribution factors, kinds of block movements and with the manner of keeping record is, however, essential in this kind of work.

The following remarks summarize the most important features of the framework method.

Advantages.

1. Applicability to a wide range of unsolved problems from two-dimensional stress to bending of plates and shells.

2. Suitability to any value of ν and to all boundary conditions.

3. Availability of good checks.

4. Absence of any higher mathematics.

Disadvantages.

1. Great amount of labour involved in solution.

2. Impossibility of solving a part of the framework without solving all of it.

3. Defects of stress interpretation.

4. Intricacy of the arithmetical procedure.

5. Large degree of dependence of proficiency on the experience of the computor.

6. Limitation of applicability of square pattern to rectangular plates.

Most of the work on the framework method, if this method is going to become an established tool of structural analysis, lies ahead, since the present treatise is merely its introduction. Among the important tasks standing now before the investigator may be mentioned further comparison of the framework solutions with the available results of the theory of elasticity, research into the question of stress interpretation, further improvement of the distribution procedure, especially in flexural framework, and investigation into new types of patterns. Then, there is a wide room for extension of the method to new problems and into new fields, some of which have been mentioned. It is not an idle dream to suggest that should a fraction of time and energy used now in some kindred lines of endeavour, like the photoelasticity, be diverted to the study and research in this new field, some very important results will ensue.

BIBLIOGRAPHY

1. W. Riedel.

On plane stress by framework.

Zeitschrift für angewandte Mathematik und Mechanik, 1927.

See also discussion by C. Weber, Z.f.a.M.M. 1928.

2. K. Wieghardt.

On plane stress by framework.

Verhandlungen des Vereins z. Beförderung des Gewerbefleißes.

Bd. 85, 1906.

3. D. G. Christopherson and R. V. Southwell.

On torsion of a triangular shaft by relaxation method.

Proceedings of the Royal Society of London. Series A. Vol.168,1938.

4. H. Marcus .

Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten.

Springer, Berlin, 1924.

5. S. Timoshenko.

"Theory of Elasticity".

McGraw-Hill, New York, 1934.

6. S. Timoshenko.

"Theory of Elastic Stability".

McGraw-Hill, New York, 1936.

7. S. Timoshenko

On plates with clamped edges.

Proceedings, Fifth International Congress for Applied Mechanics, Cambridge, Mass., U.S.A. 1938.

BIOGRAPHICAL NOTE.

1. Education.

   a. Graduate of Institute of Communication Engineering (College of Civil Engineering), Moscow, Russia, 1920.

   b. B.A.Sc., University of British Columbia, Vancouver, Canada, 1930.

   c. M.A.Sc., University of British Columbia, 1933.

2. Engineering Experience.

   a. Assistant Chief of Party, Railway Surveys, Kotlas-Soroka, Northern Russia, September 1920-November 1921.

   b. Engineer in the main office of Maintenance of Way and Assistant Division Engineer of Marituy and Irkutsk Divisions, Zabaikalskaya Railway, Eastern Siberia, December 1921-May 1923.

   c. Engineer on construction of buildings, with Szelezniak and Krichinsky Construction Company, Harbin, Manchuria, June 1923-November 1923.

   d. Engineer in charge of surveys and construction of a seven-mile logging railroad, with Day-Jay-Tan Logging Company, Harbin, Manchuria, November 1923-February 1924.

   e. Engineer on construction of buildings, with Russian-Chinese Polytechnic Institute, Harbin, Manchuria, April 1924-February 1925.

   f. Transitman, Maintenance of Way, Cranbrook Division, B.C.District, Canadian Pacific Railway, May 1928-September 1928.

g. Structural Detailer, Dominion Bridge Company, Vancouver, B.C., Canada, May 1929-September 1929 and May 1930-August 1933.

3. <u>Teaching Experience</u>.

a. Instructor in Survey School, Russian-Chinese Polytechnic Institute, Harbin, Manchuria, June 1925.

b. Instructor in Civil Engineering, University of British Columbia, September 1933-May 1938. On leave of absence since latter date.

4. <u>Publications</u>.

a. Work of Rivets in Riveted Joints.
Transactions, A.S.C.E., Vol. 99, 1934.

b. Analysis of Multiple Arches.
Transactions, A.S.C.E., Vol. 101, 1936.

c. Elastic Stability of a Pony Truss.
Publications of International Association for Bridge and Structural Engineering, Zurich, Switzerland. Vol. 3, 1935.

d. Discussion of paper "Pressures Beneath a Spread Foundation"
Transactions, A.S.C.E., Vol. 103, 1938.

5. <u>Membership of Engineering Societies</u>

a. Associate Member, American Society of Civil Engineers.

b. Member, International Association for Bridge and Structural Engineering, Zurich, Switzerland.

c. Prior to coming to Boston: Member, Association of Professional Engineers of the Province of British Columbia, Canada.

FIGURE PAGES

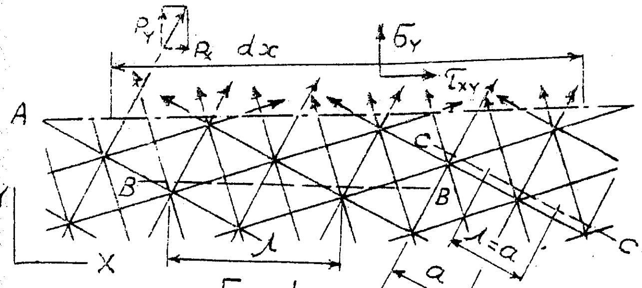


Fig. 1

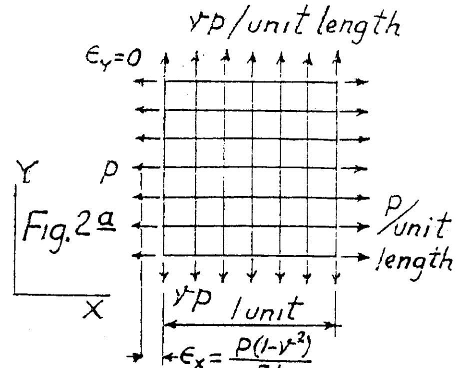


Fig. 2a

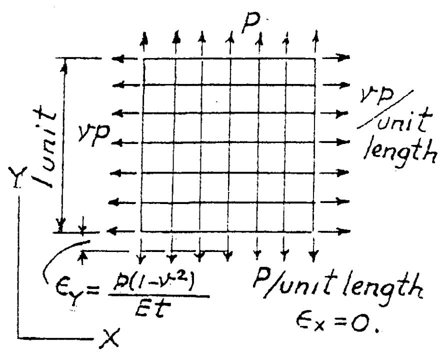


Fig. 2b

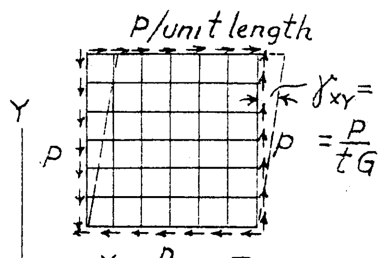


Fig. 2c

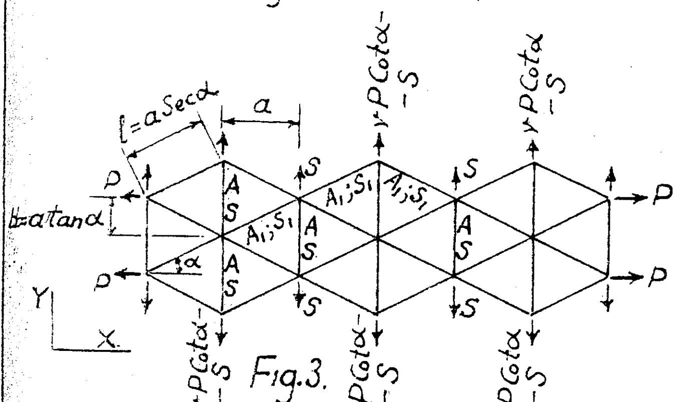


Fig. 3.

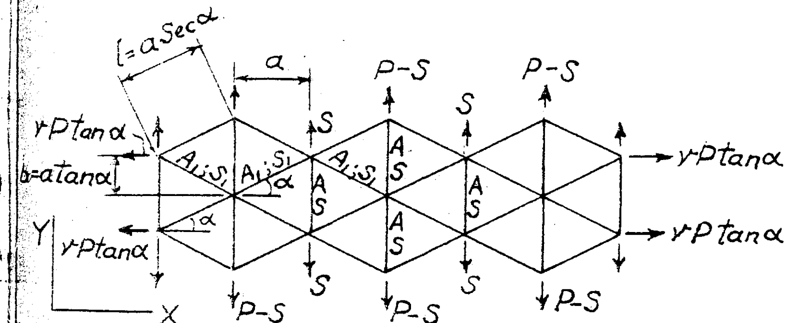


Fig. 4

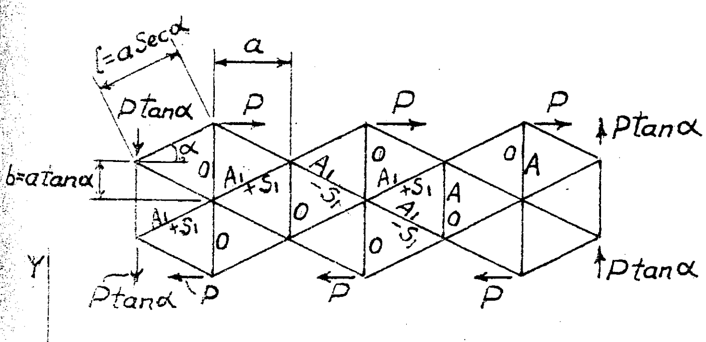


Fig. 5

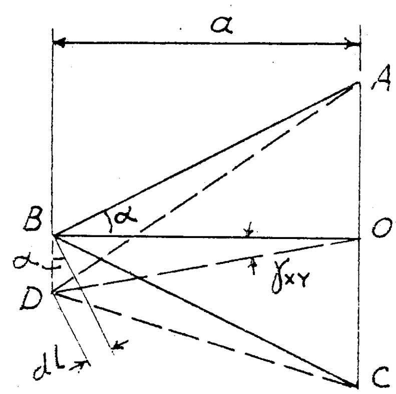


Fig. 6.

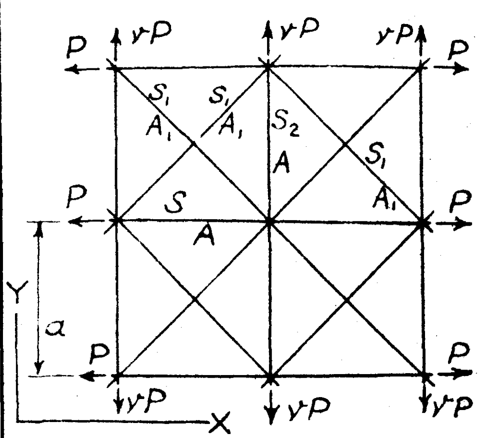


Fig. 7

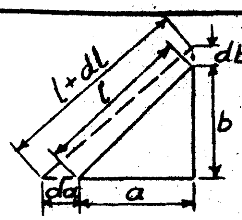


Fig. 8

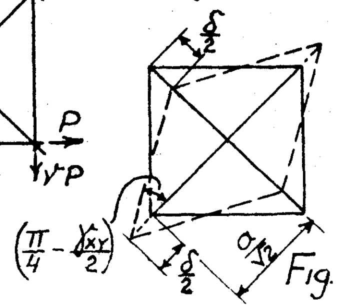


Fig. 10.

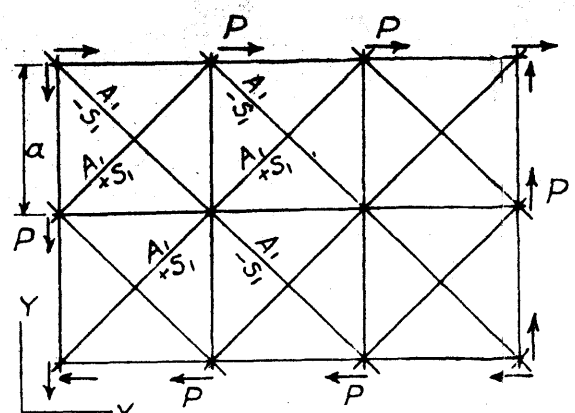


Fig. 9

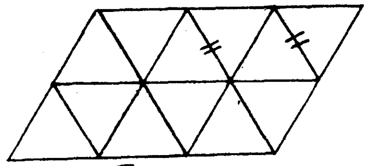


Fig. 20

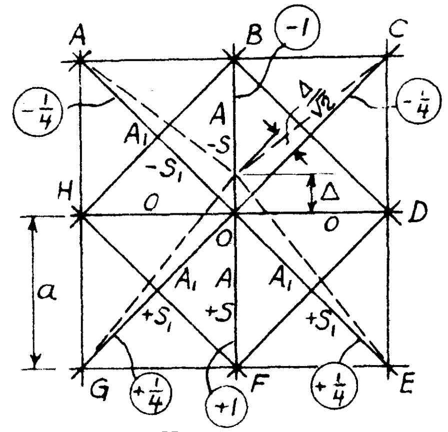


Fig. 21

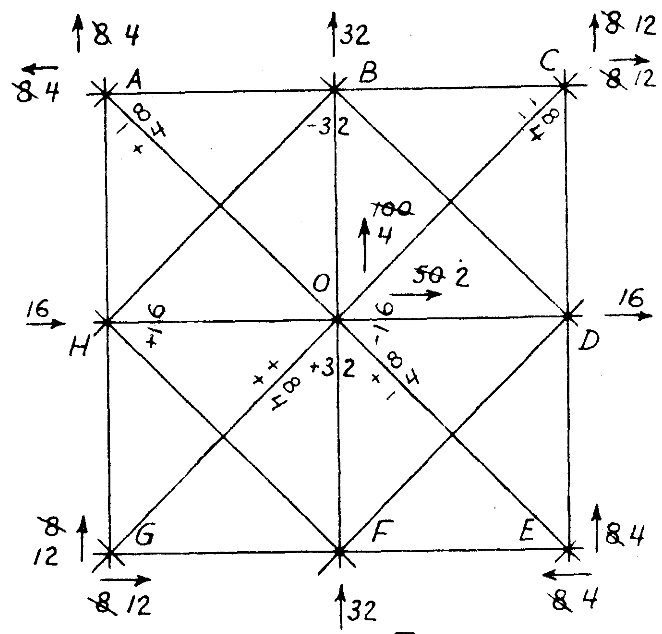


Fig. 22

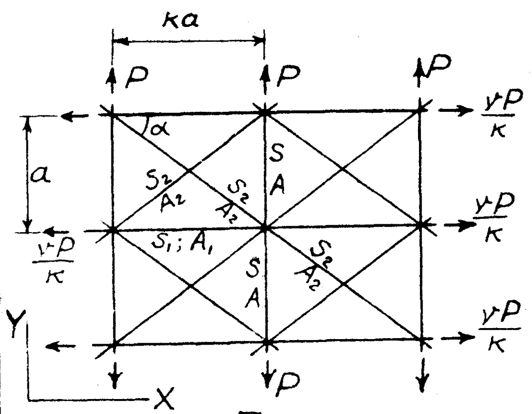


Fig. 11

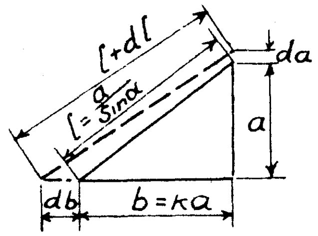


Fig. 12.

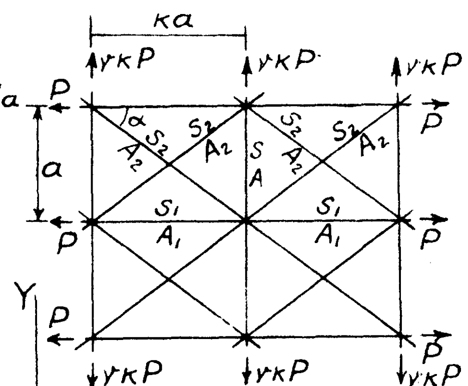


Fig. 13.

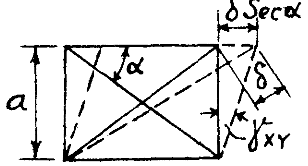


Fig 15.

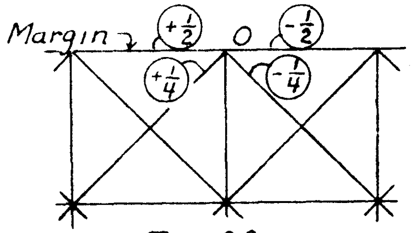


Fig. 23.

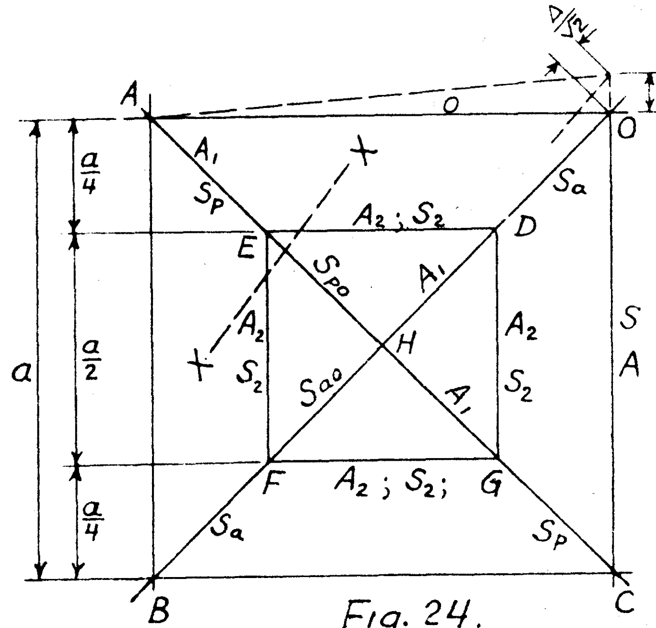


Fig. 24.

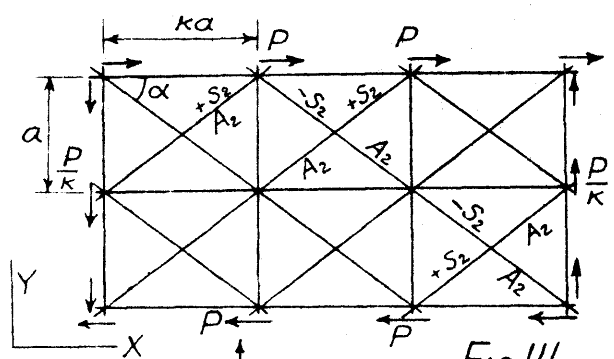


Fig. 14.

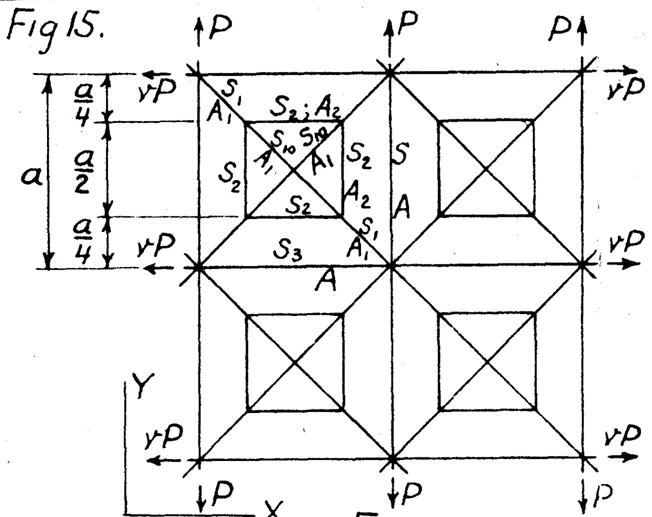


Fig. 16

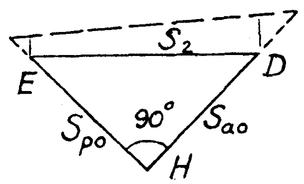


Fig. 25

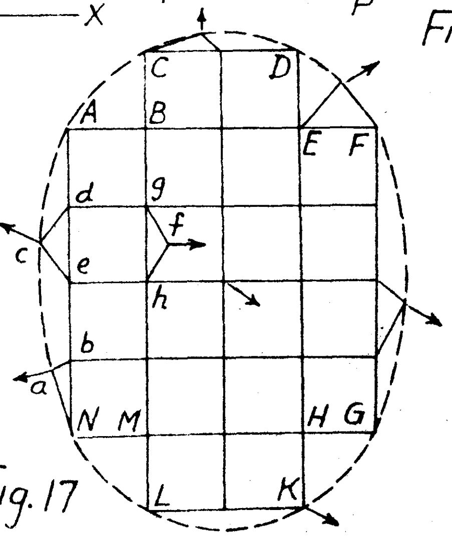


Fig. 17

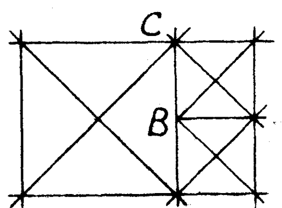


Fig. 18

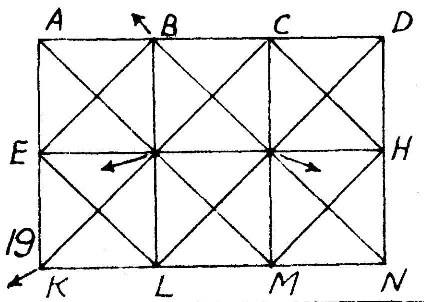


Fig. 19

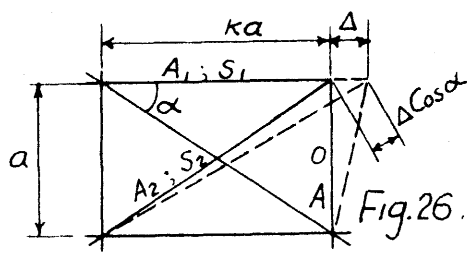


Fig. 26.

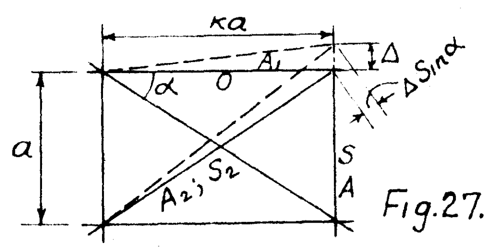


Fig. 27.

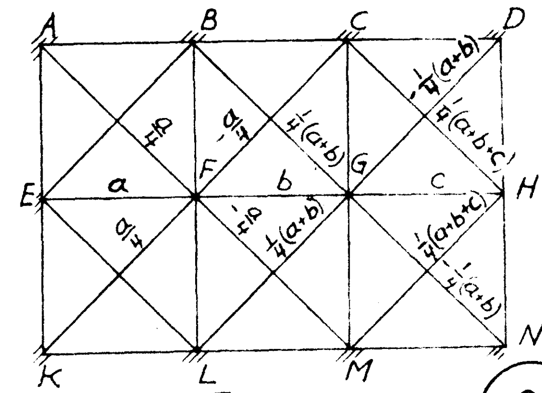


Fig. 28.

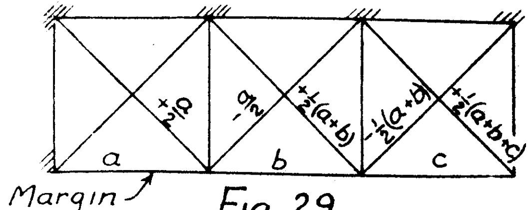


Fig. 29

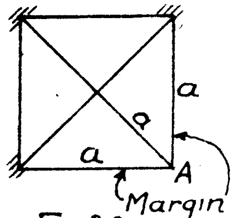


Fig. 30

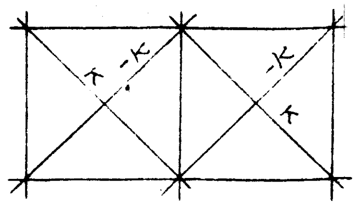


Fig. 31

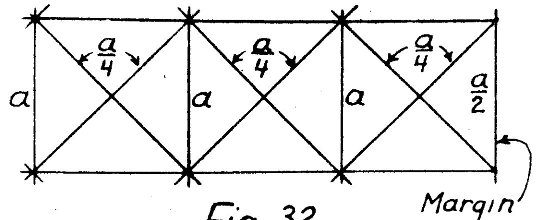


Fig. 32.

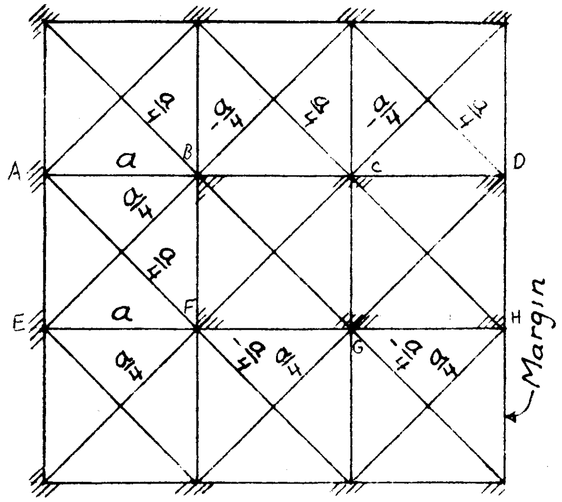


Fig. 33

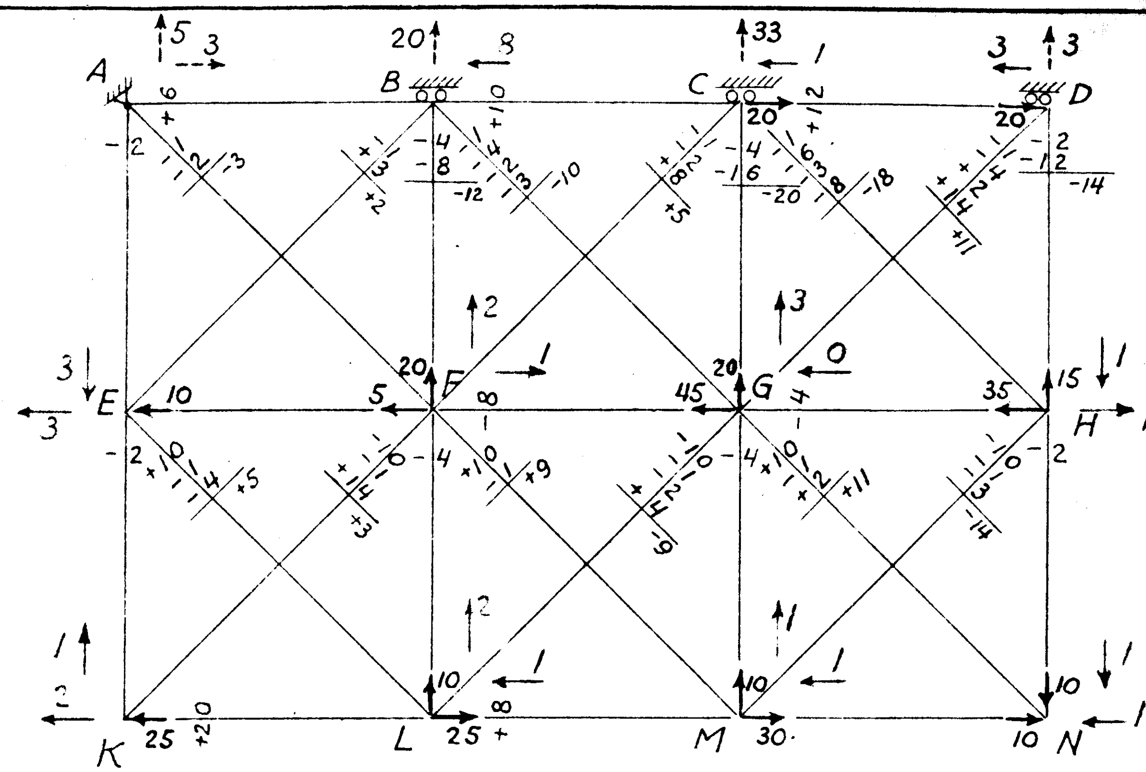


Fig. 38. $r = \frac{1}{3}$. Original: $\Sigma V = 65 \uparrow$ $\Sigma H = 15 \leftarrow$

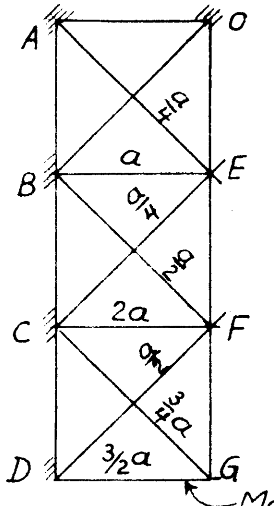


Fig. 34

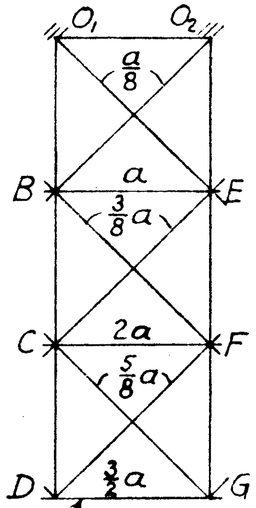


Fig. 35

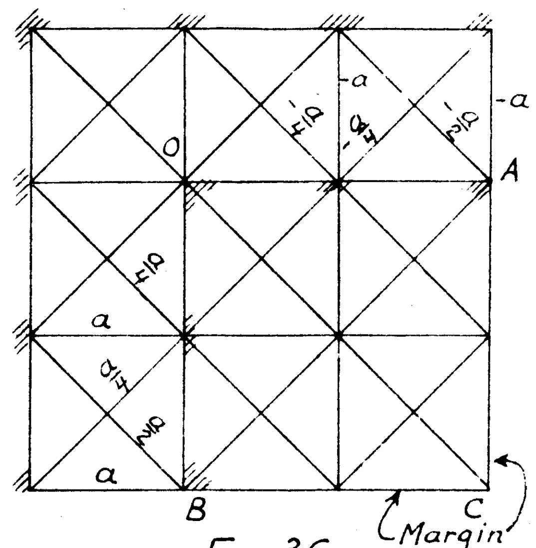


Fig 36

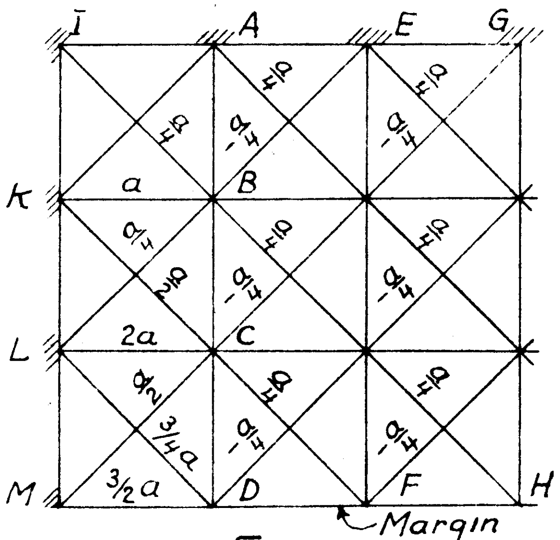


Fig. 37.

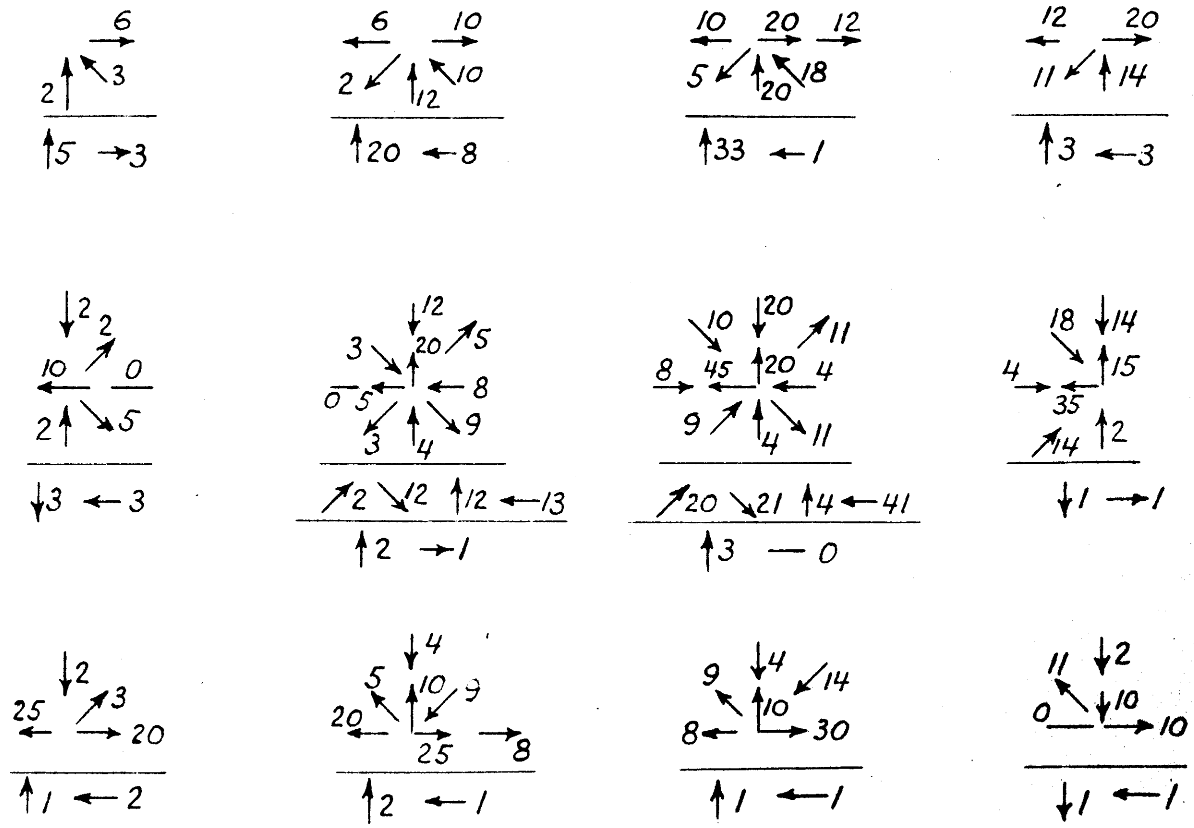


Fig. 39.

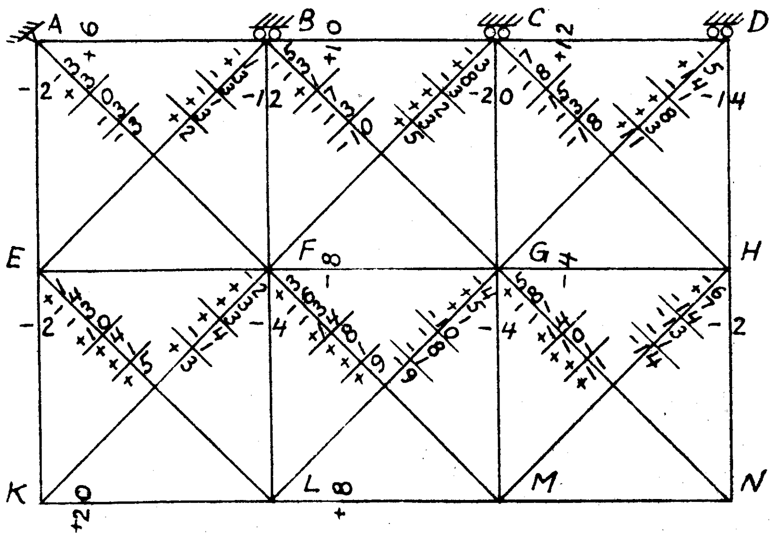


Fig. 40.

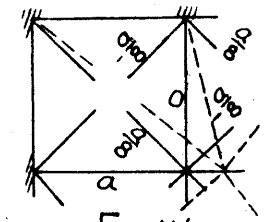


Fig. 41

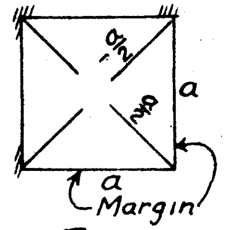


Fig. 44.

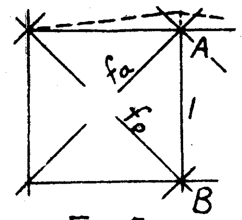


Fig. 50

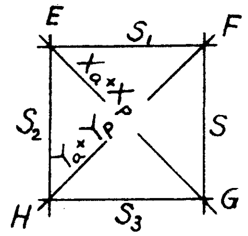


Fig. 51

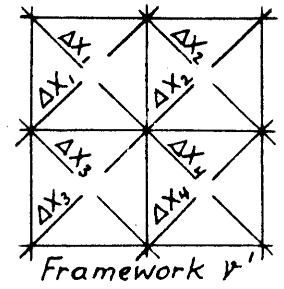


Fig. 52

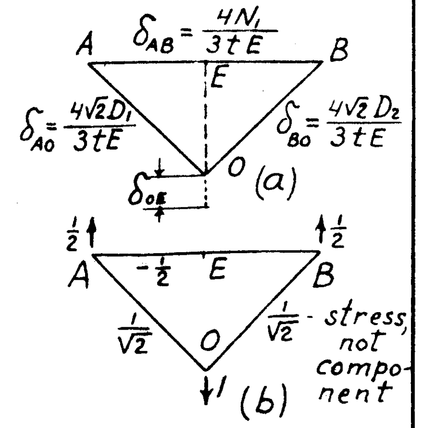


Fig. 54 (a) & (b)

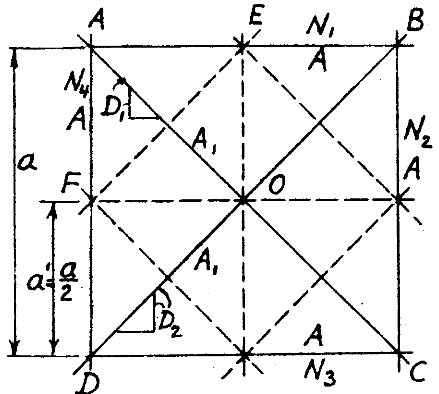


Fig. 53.

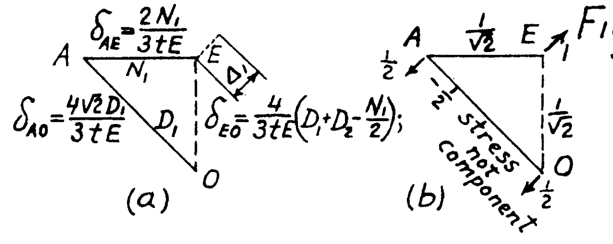


Fig. 55 (a) & (b)

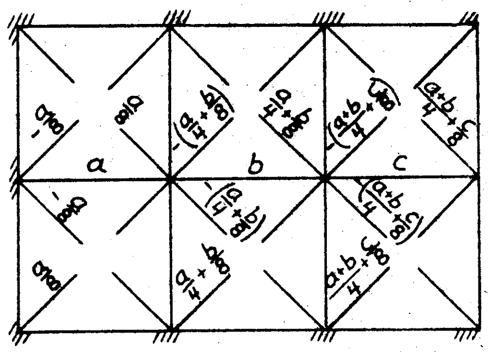


Fig. 42

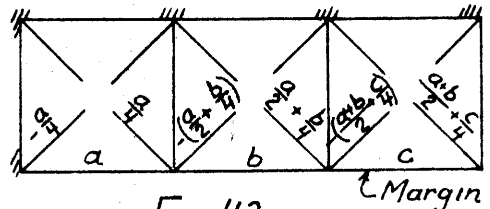


Fig. 43.

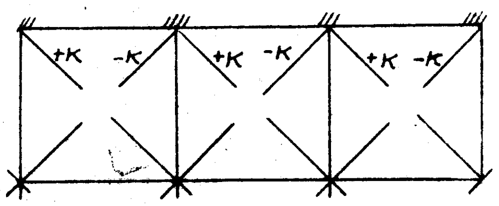


Fig. 45

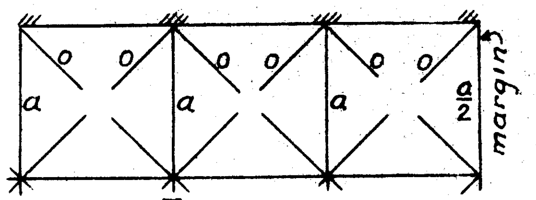


Fig. 46

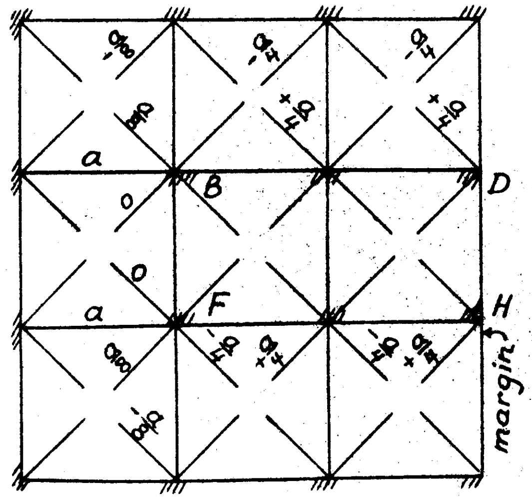


Fig 47.

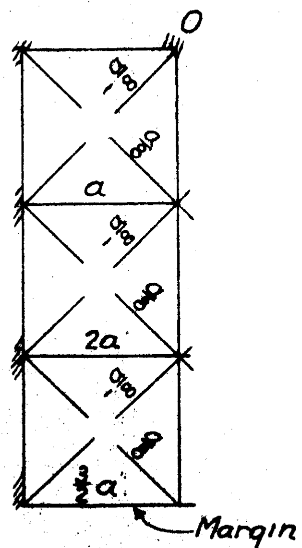


Fig. 48

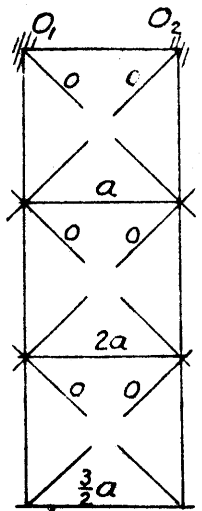


Fig. 49.

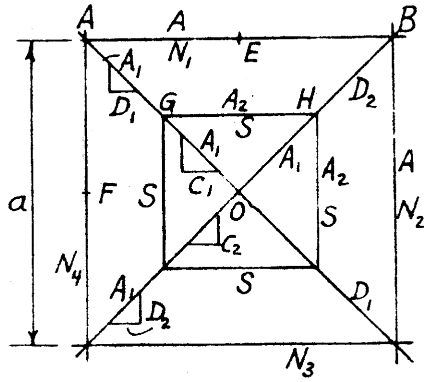


Fig 56

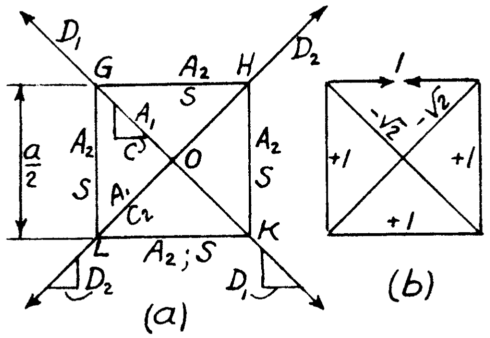


Fig. 57 (a) & (b)

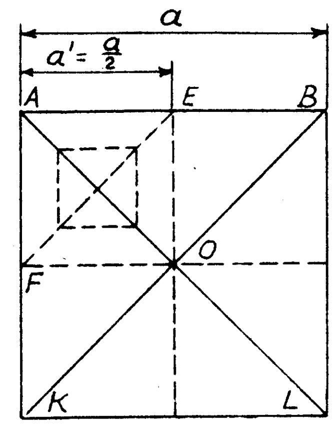


Fig. 58.

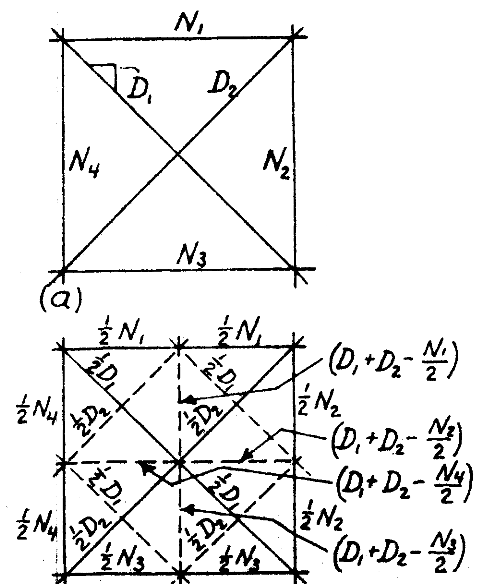


Fig. 59 (a) & (b)

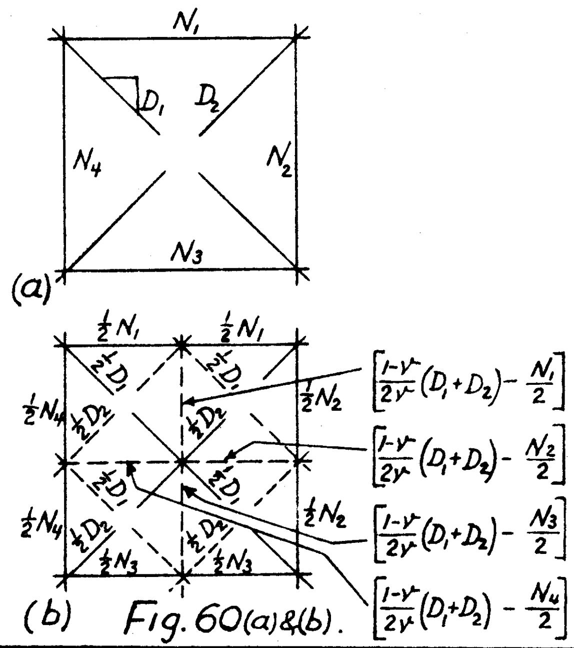


Fig. 60 (a) & (b).

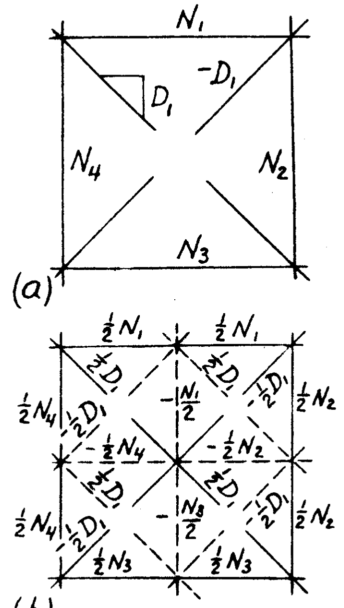


Fig. 61(a)&(b).

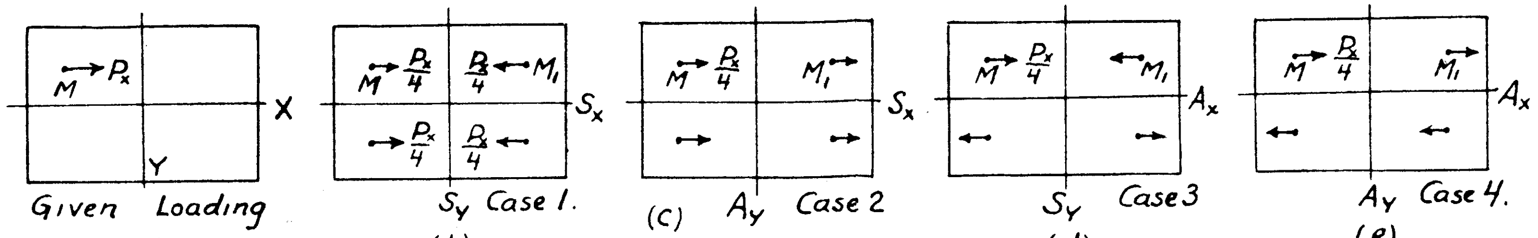


Fig. 63(a) to (e)

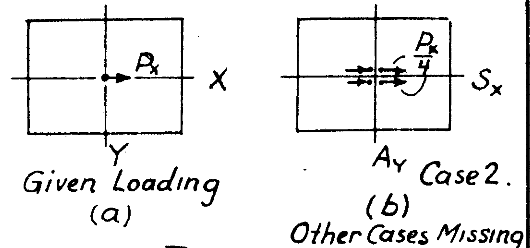


Fig. 65 (a) & (b)

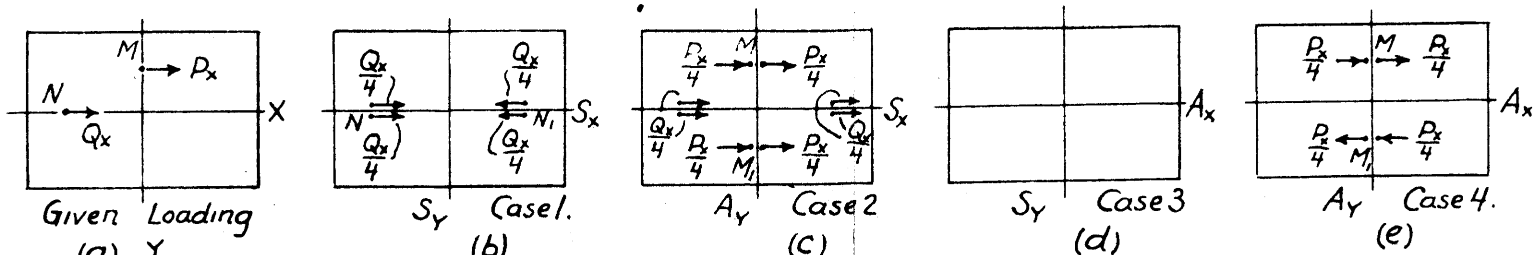


Fig. 64 (a) to (e).

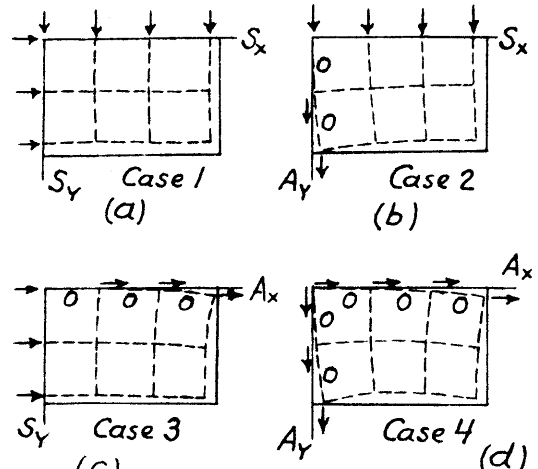


Fig. 67 (a) to (d)

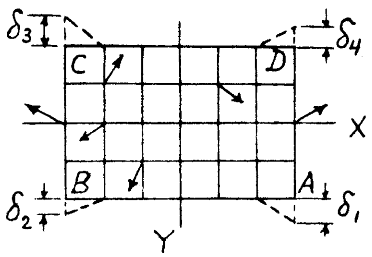


Fig. 62

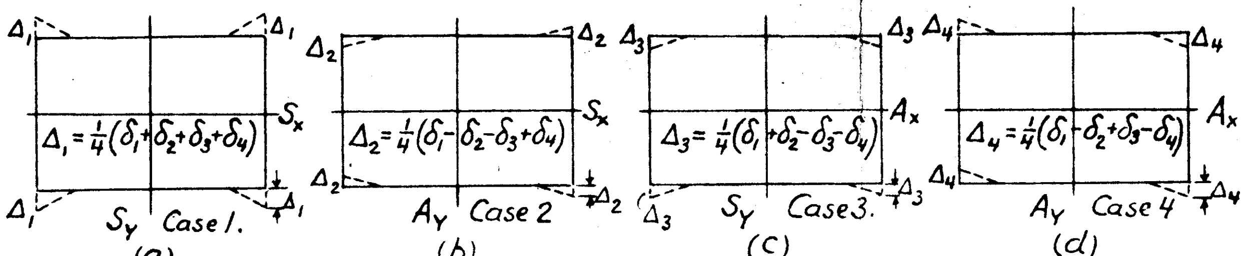


Fig. 66 (a) to (d)

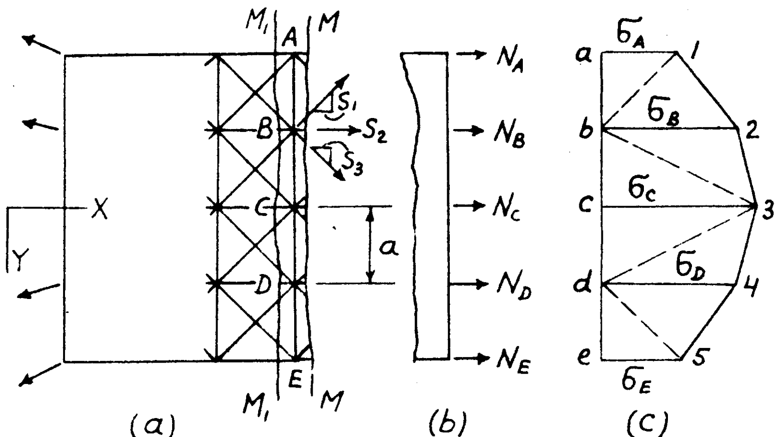


Fig. 68 (a), (b) & (c)

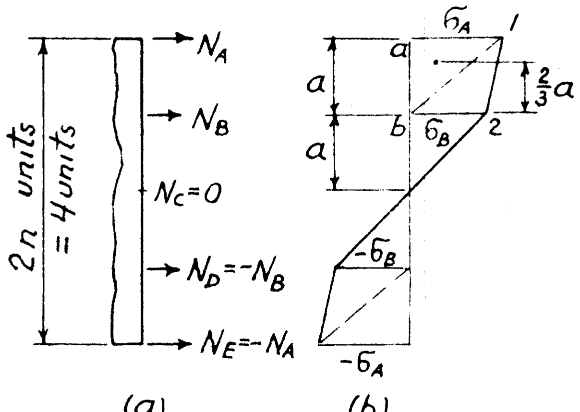


Fig. 69 (a) & (b)

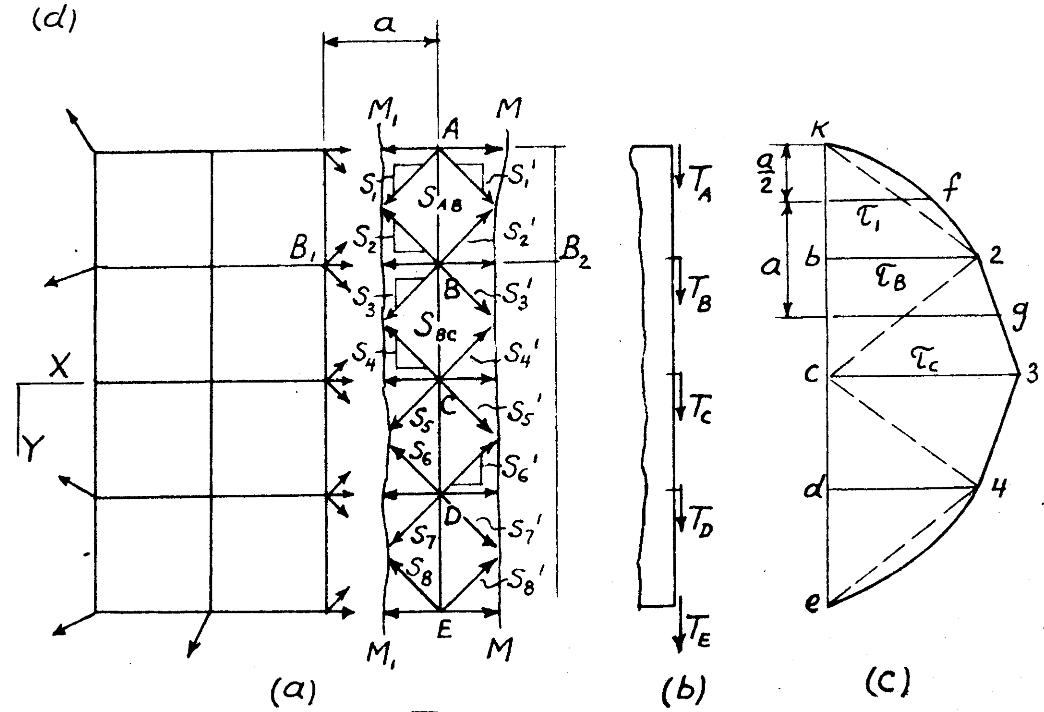


Fig. 72 (a), (b) & (c)

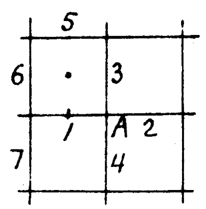


Fig. 70

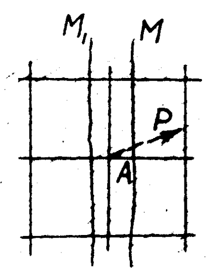


Fig. 71

All figures to be multiplied by qa.
Figures on diagonals - their H or V components.

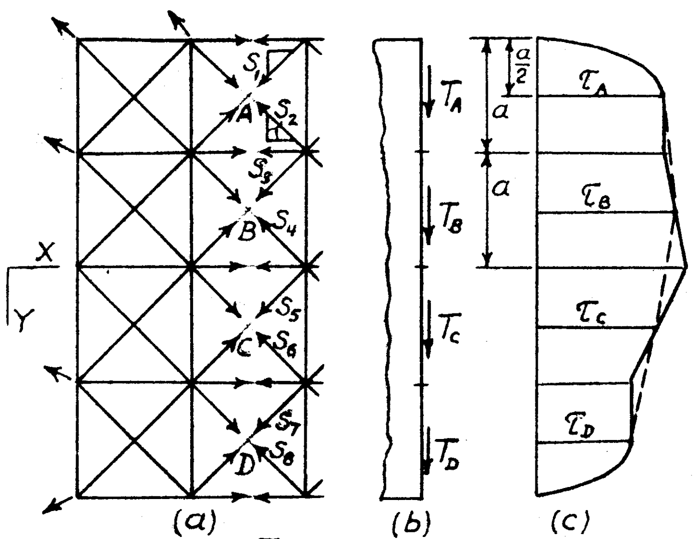


Fig. 73 (a),(b) & (c).

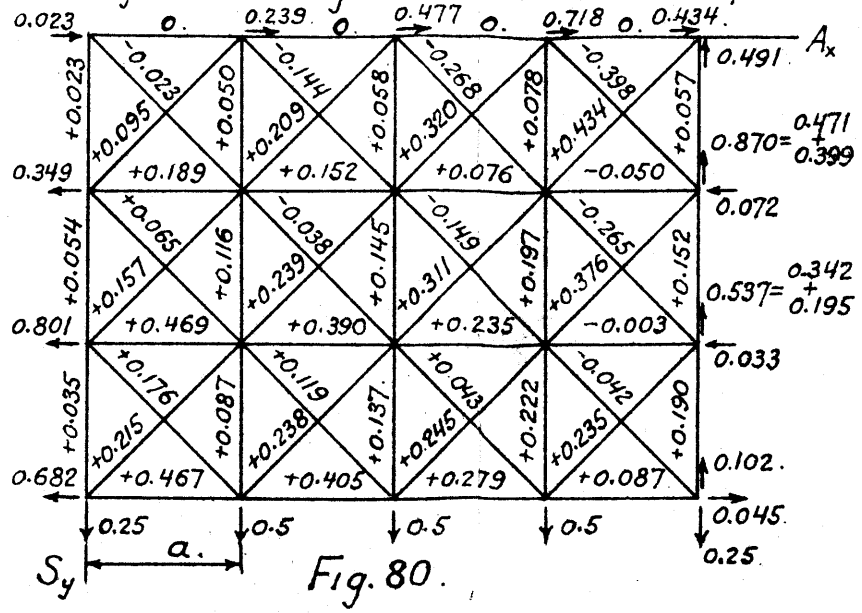


Fig. 80.

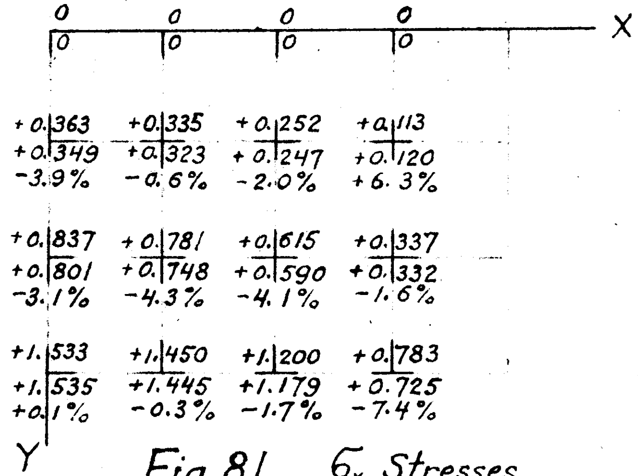


Fig. 81. σ_x Stresses by Method 1.

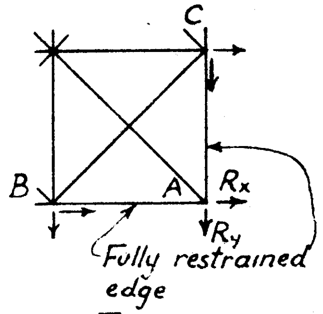


Fig. 74

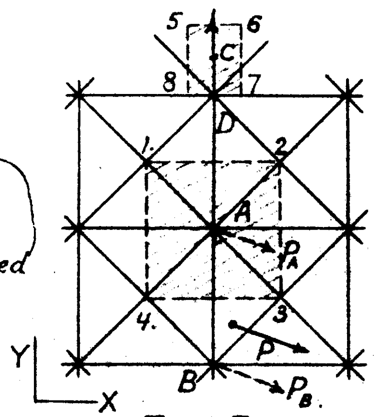


Fig. 75.

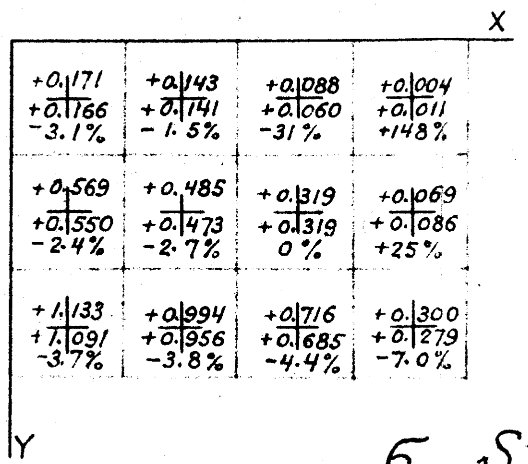


Fig. 82.

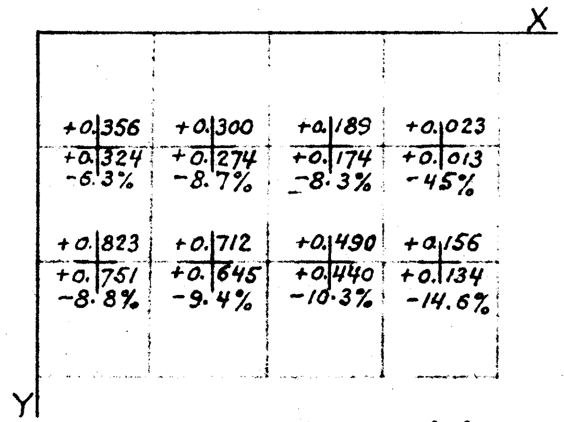


Fig. 83.

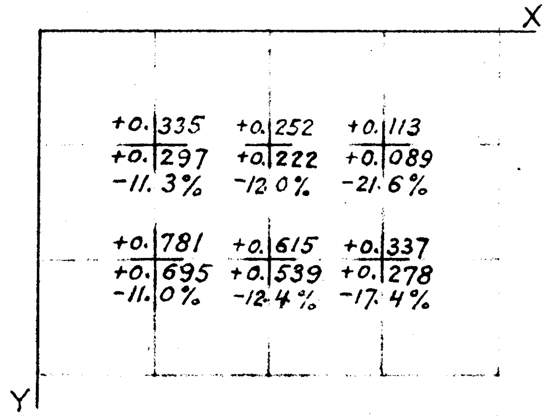


Fig. 84.

σ_x Stresses by Method 3.

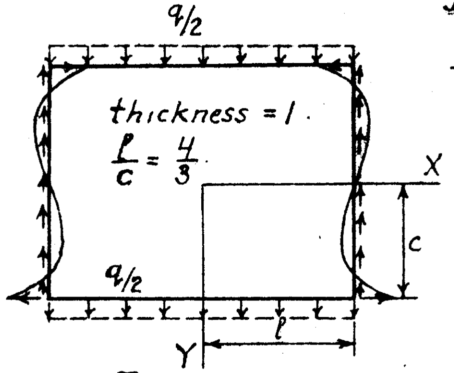


Fig. 76.

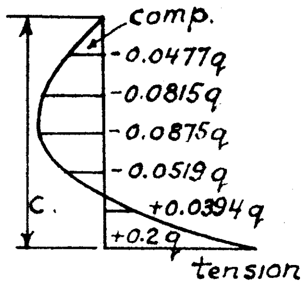


Fig. 78.

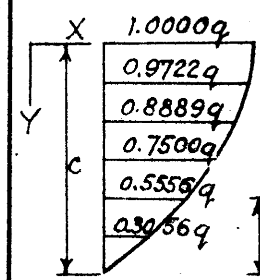


Fig. 77.

Figures to be multiplied by qa.

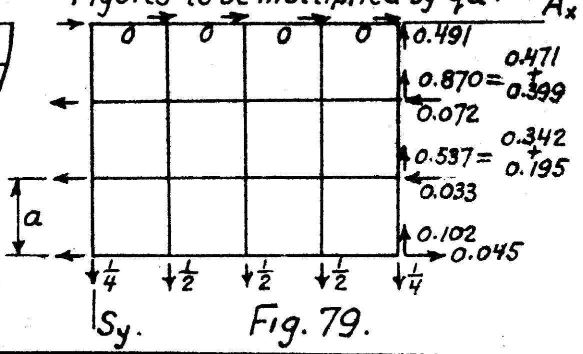
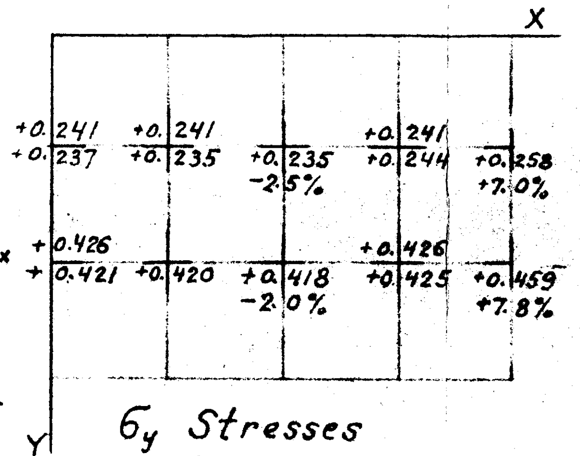


Fig. 79.



σ_y Stresses
Fig. 85

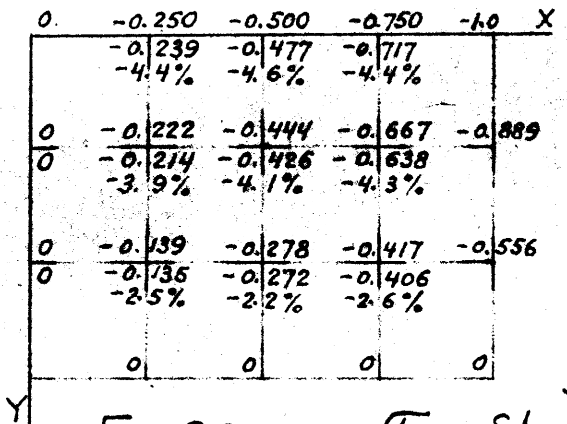


Fig. 86.

τ_{xy} Stresses

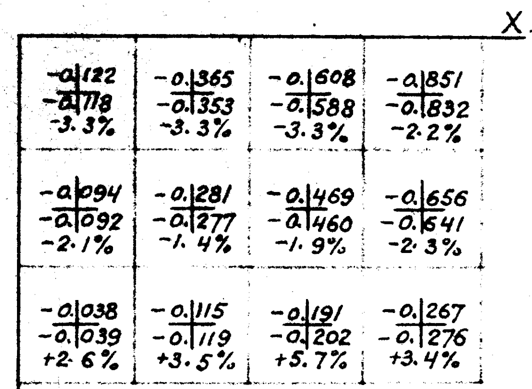


Fig. 87.

Note Re. Fig. 81-87.
All figures to be multiplied by q.
1st fig - by Th of El., 2nd - by 3×4, $\nu = 1/3$ Framework, 3rd - % error.

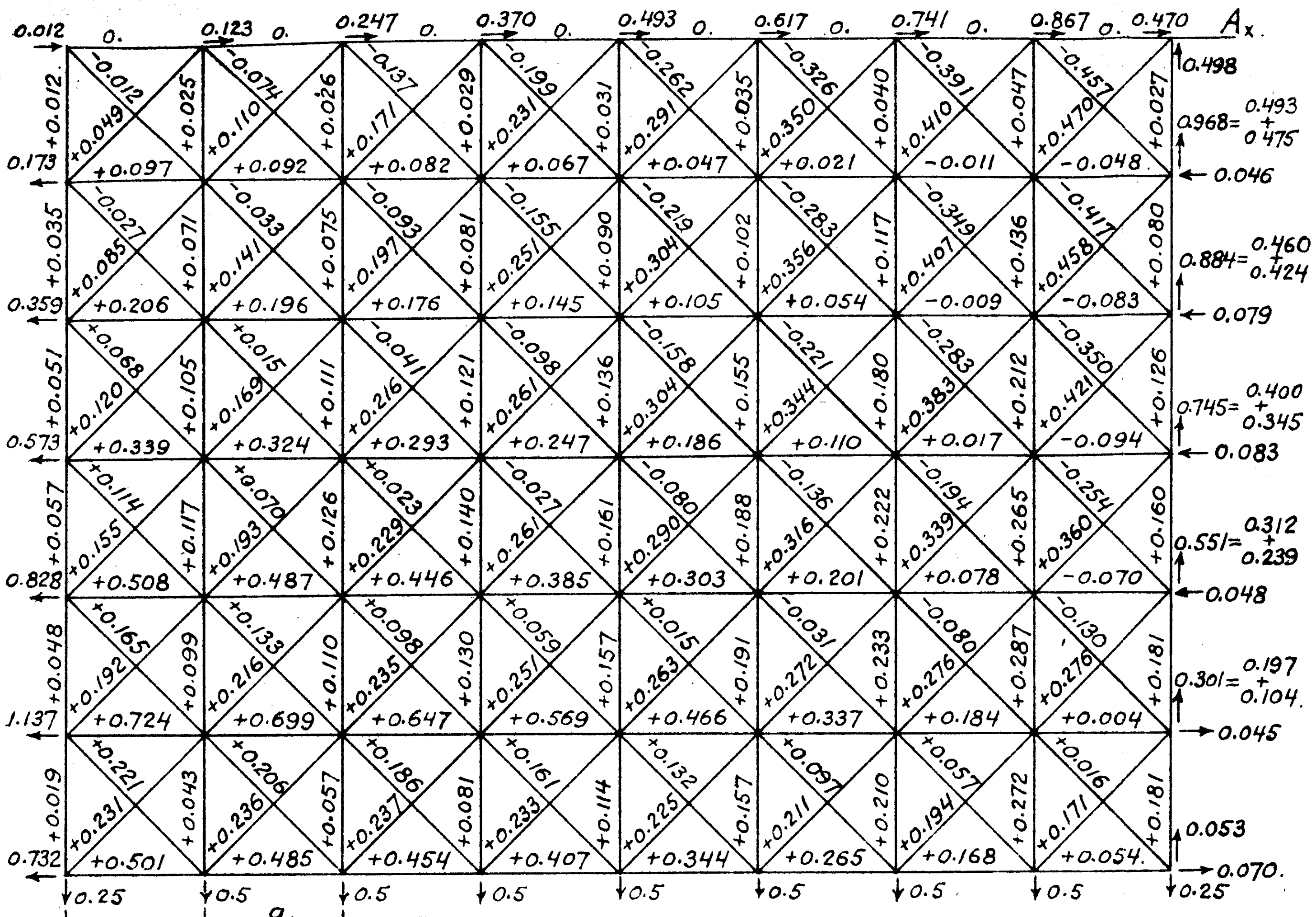


Fig. 90. Framework stresses, $\gamma = 1/3$. To be multiplied by qa. Figures on diagonals – their H or V components.

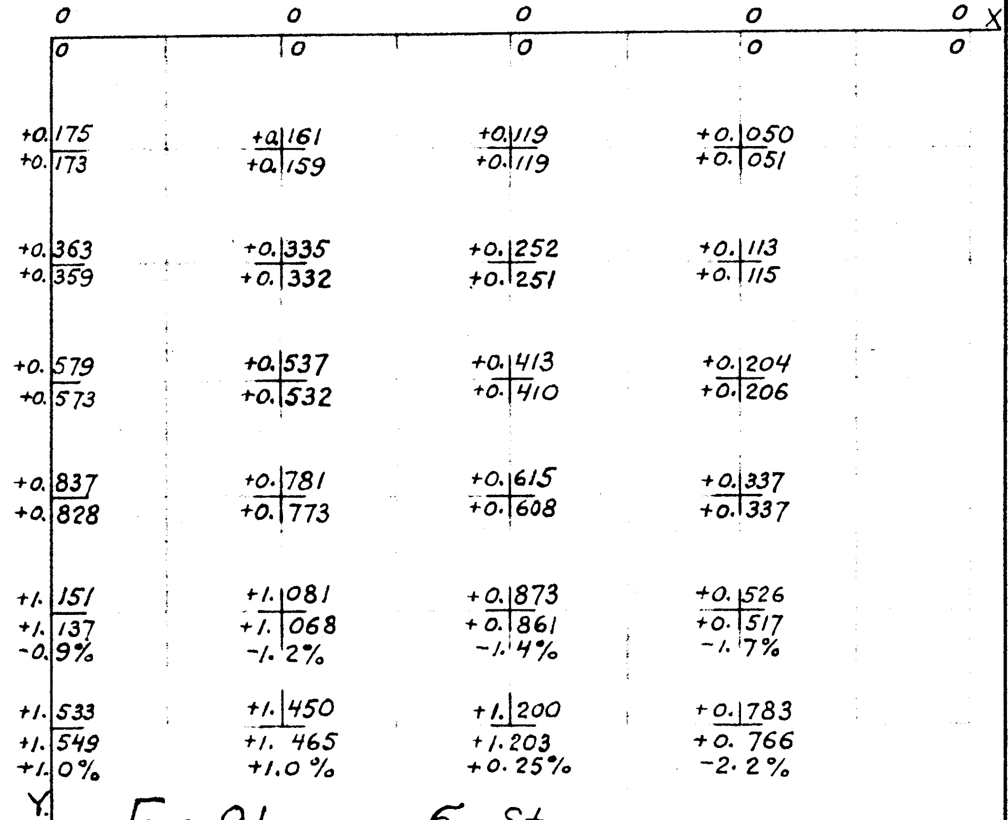


Fig. 91. σ_x Stresses.

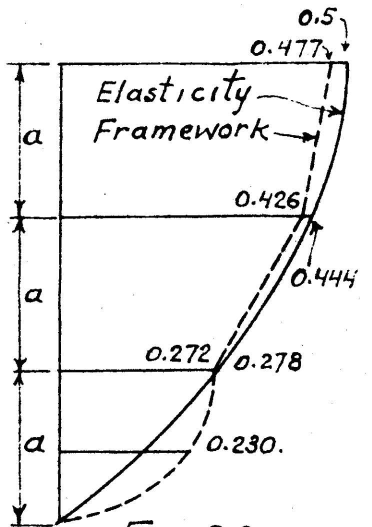


Fig. 89. Shear on Plane $x = 2a$.

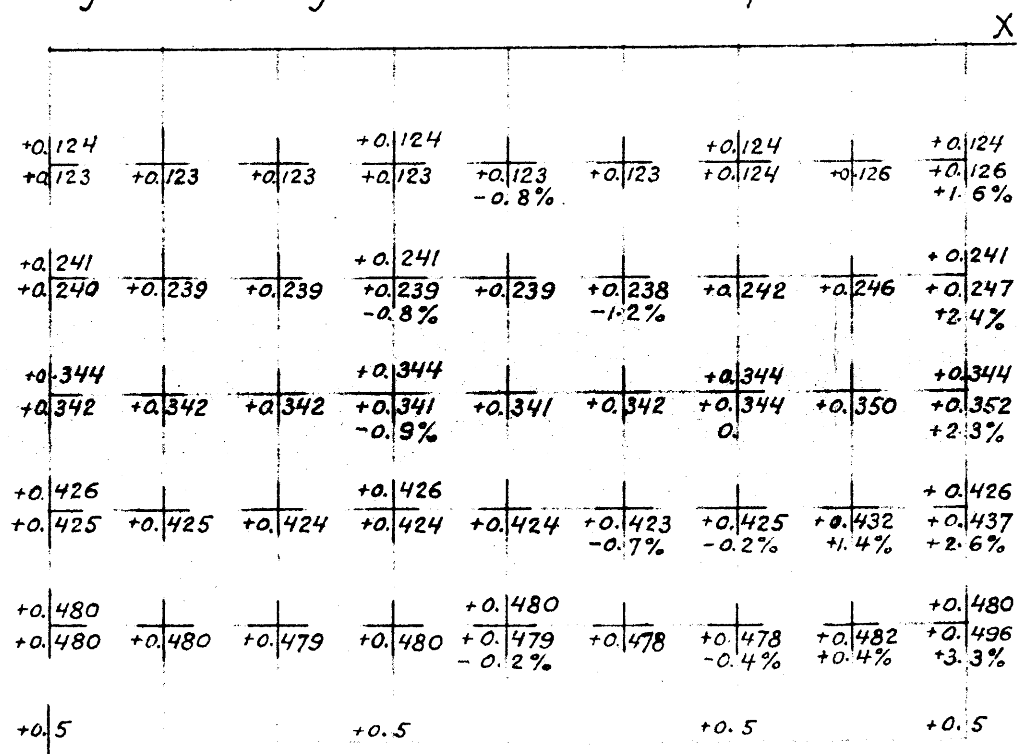


Fig. 92. σ_y Stresses.

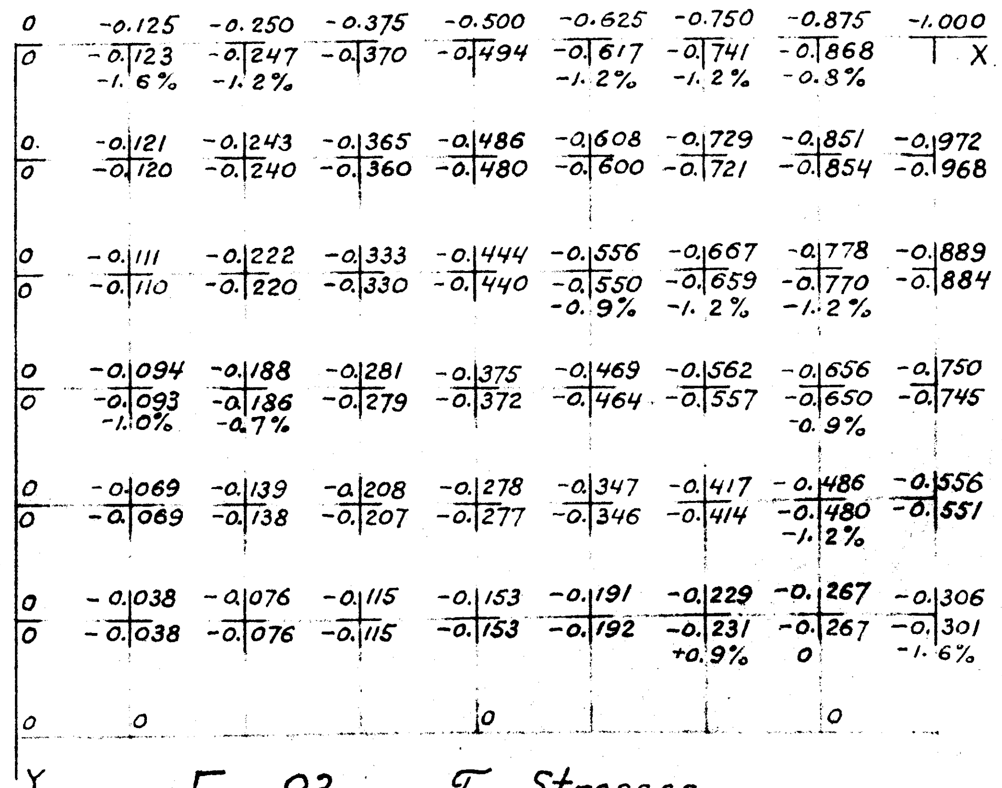


Fig. 93. τ_{xy} Stresses.

Note Re. Fig. 91–93.
All figures to be multiplied by q.
1st fig. – by T. of E., 2nd – by 6×8, $\gamma = 1/3$, Framework, 3rd – % error.

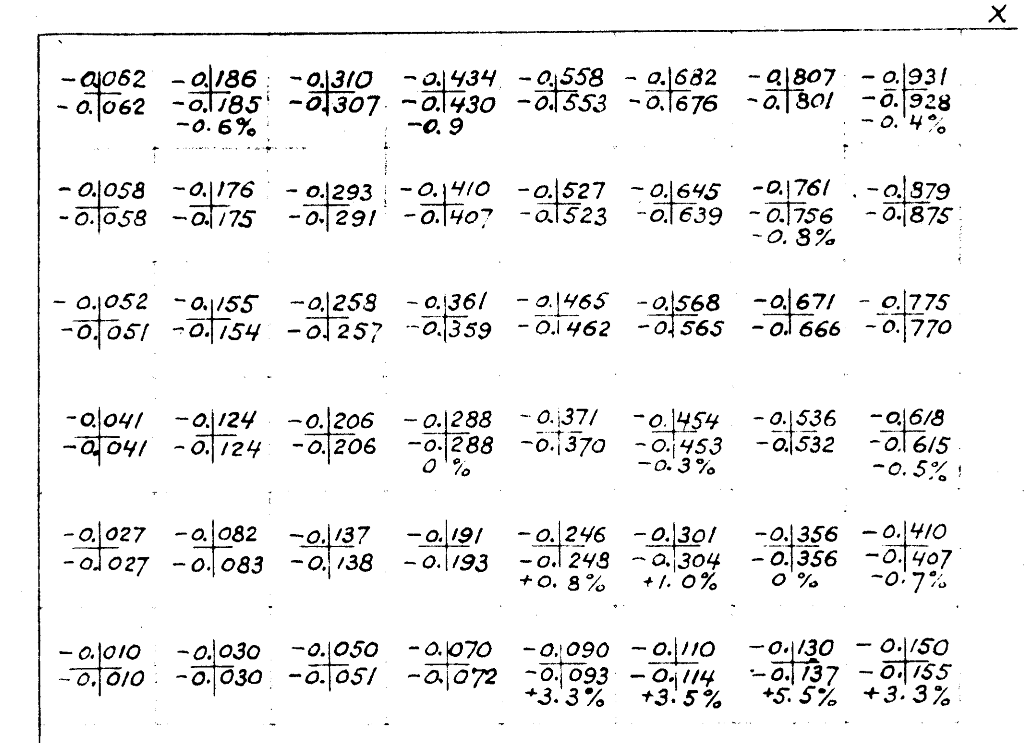


Fig. 94. τ_{xy} Stresses. $\nu = \frac{1}{3}$.

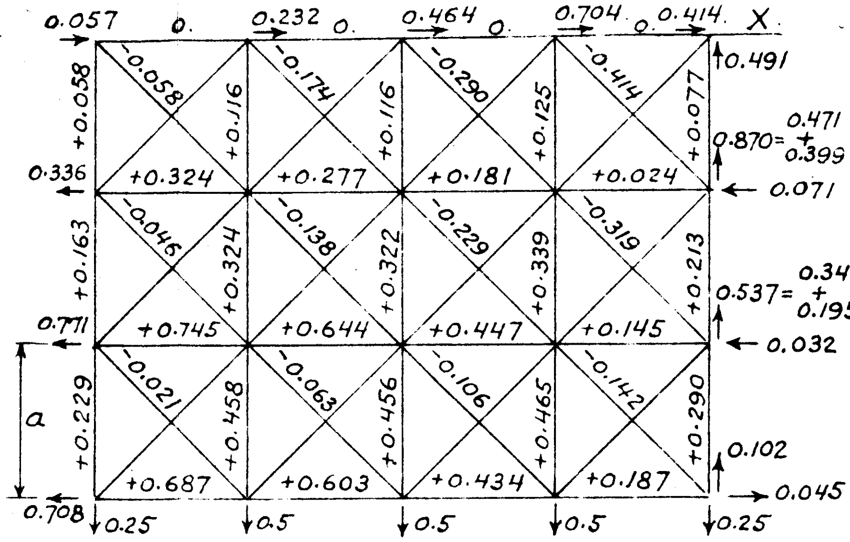


Fig. 95. Framework stresses, $\nu = 0$.
To be multiplied by qa.
Figures on diagonals - their H or V components.
In each square, diagonals have equal & opposite stresses.

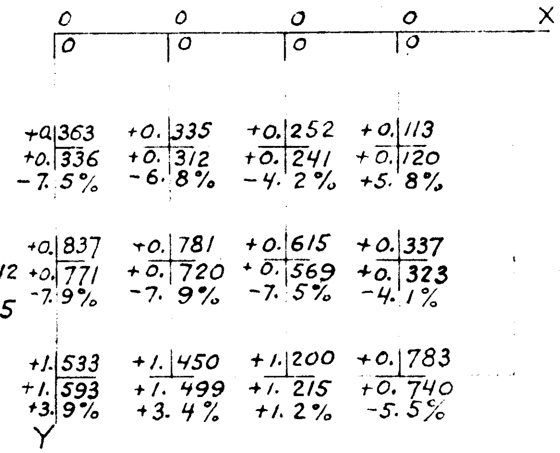


Fig. 96. σ_x Stresses, $\nu = 0$.

Note Re. Fig. 94 & 96-99.
All figures to be multiplied by q
1st figure - by Th. of El.
2nd figure - by Framework.
3rd figure - % error.

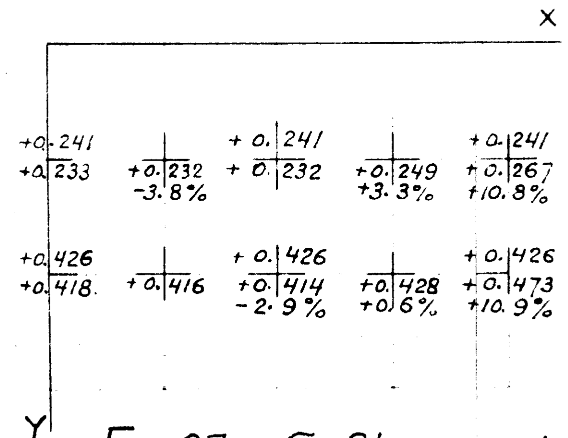


Fig. 97. σ_y Stresses, $\nu = 0$.

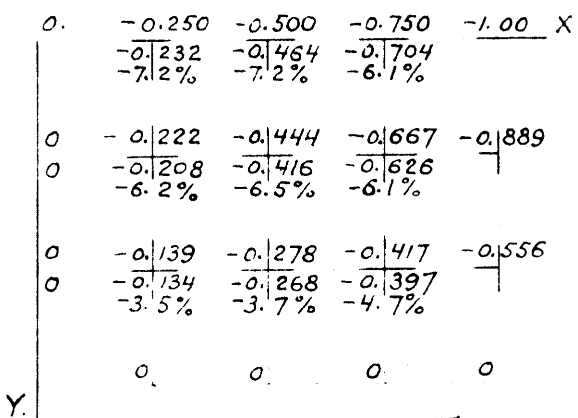
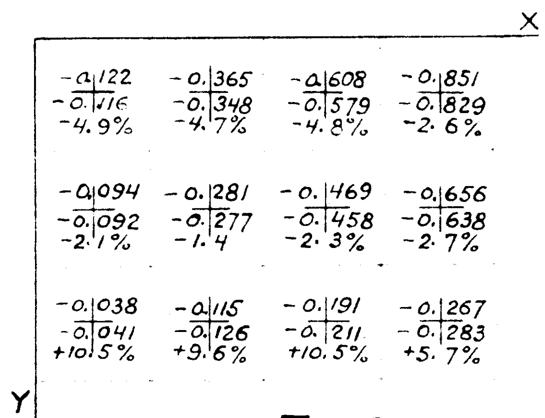


Fig. 98.



τ_{xy} Stresses, $\nu = 0$. Fig. 99.

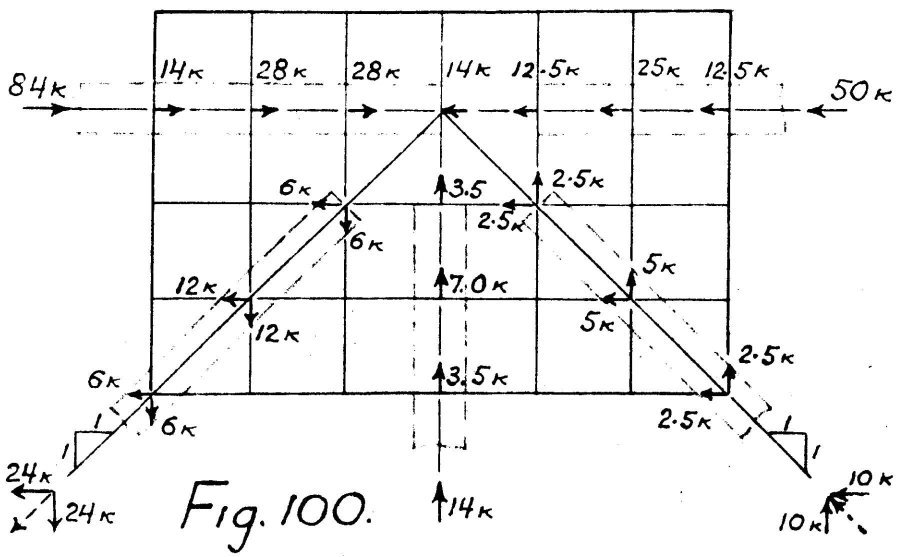


Fig. 100.

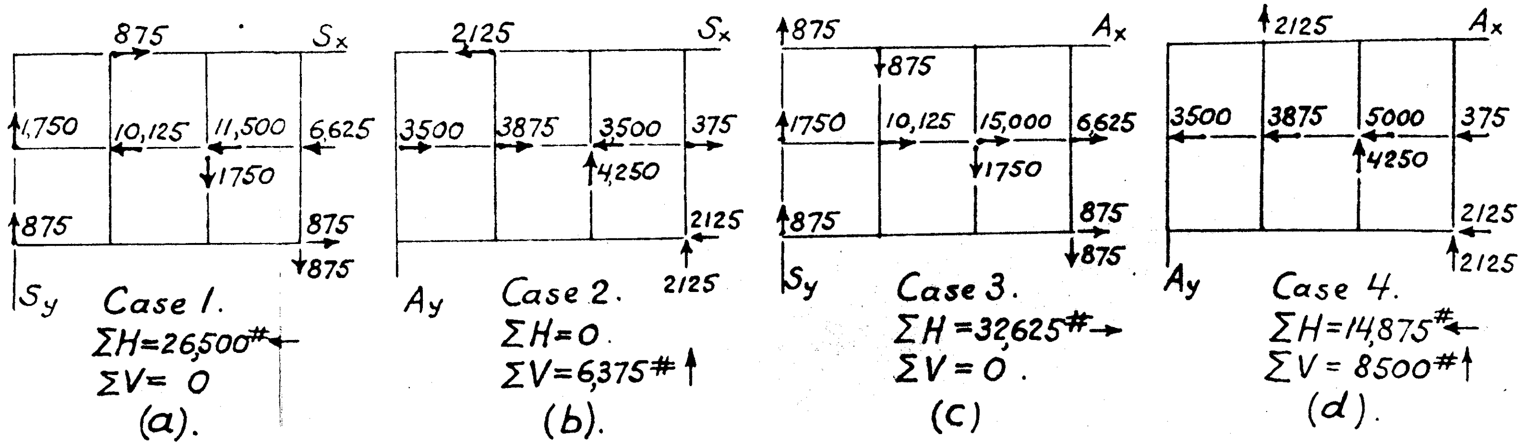


Fig. 101 (a) to (d).

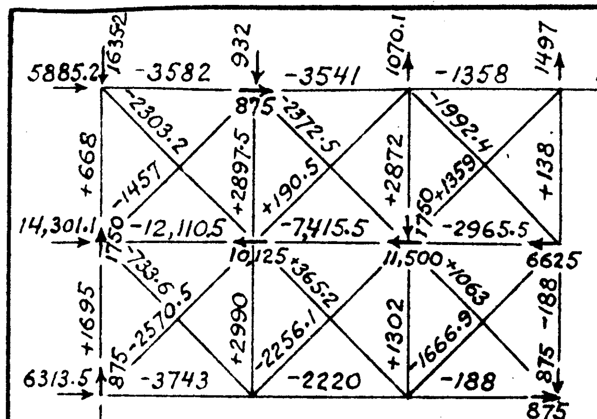


Fig. 102. Case 1, $\nu = \frac{1}{3}$

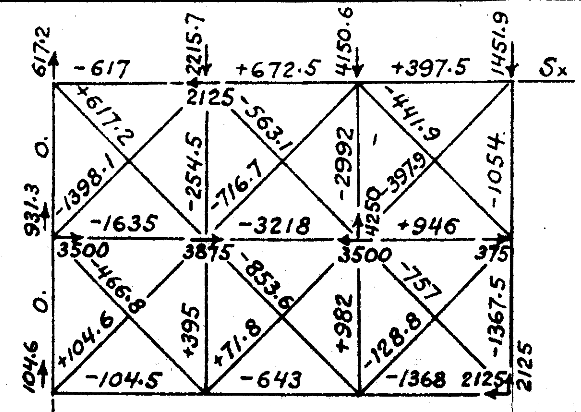


Fig. 103. Case 2, $\nu = \frac{1}{3}$.

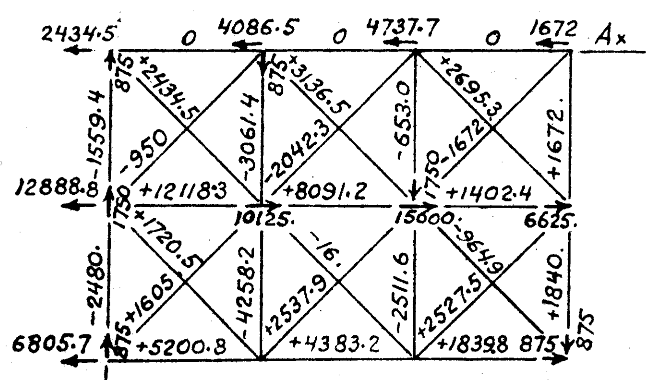


Fig. 104 Case 3, $\nu = \frac{1}{3}$

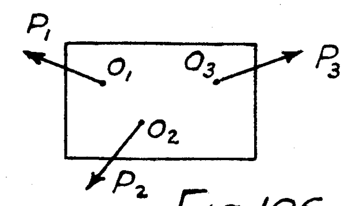


Fig. 106.

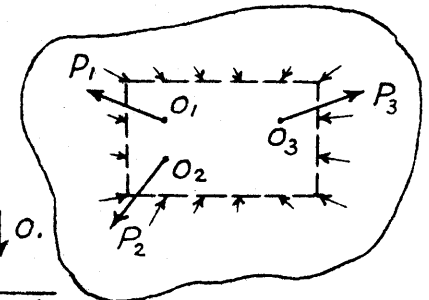


Fig. 107

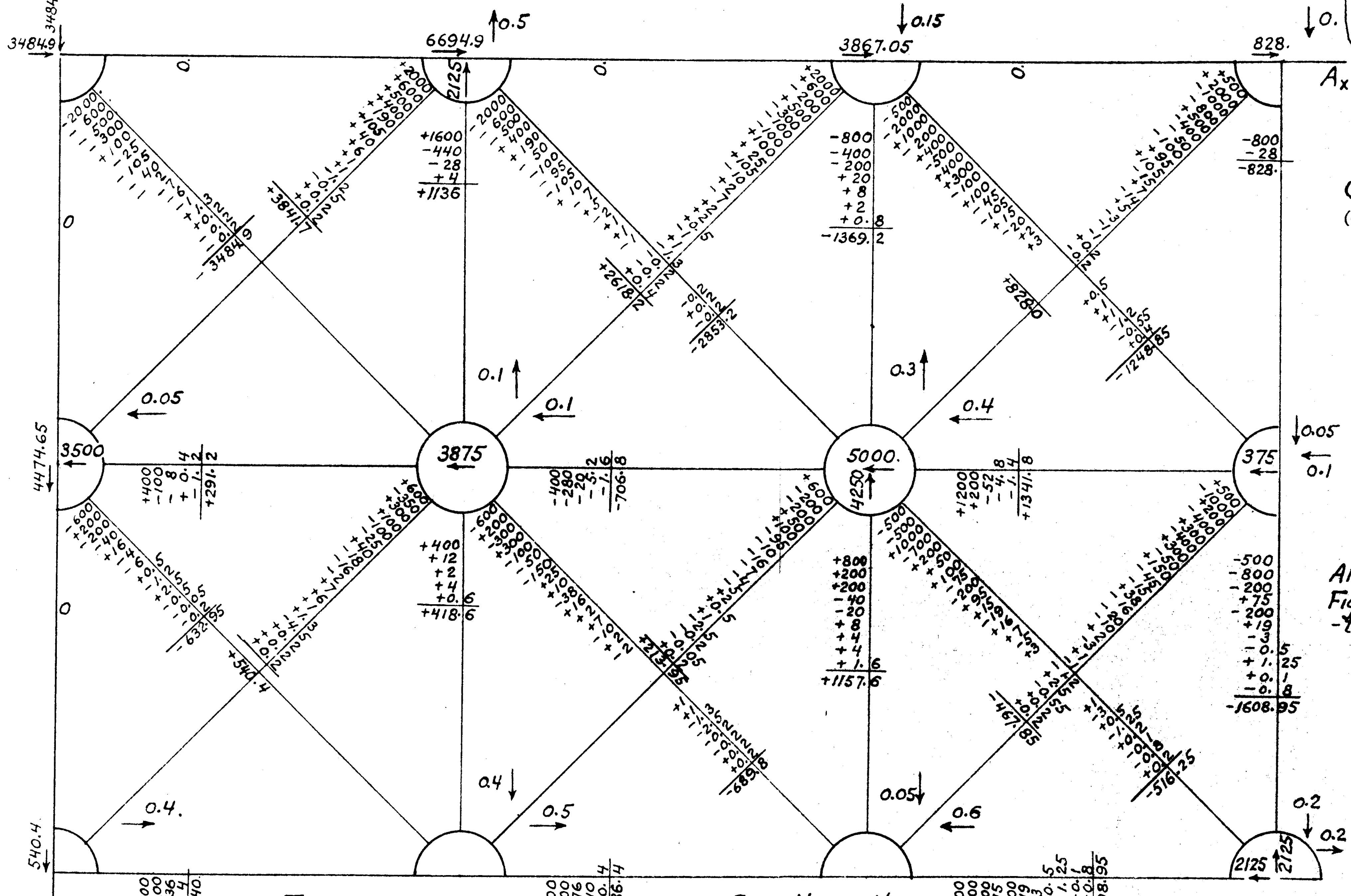


Fig. 105. Case 4, $\nu = 1/3$.

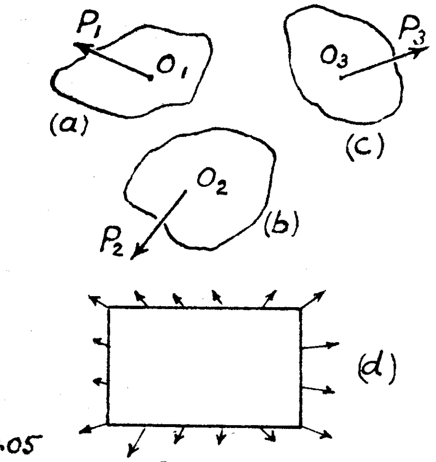


Fig. 108 (a) to (d).

All figures - in lbs.
Figures on diagonals - their H or V components.

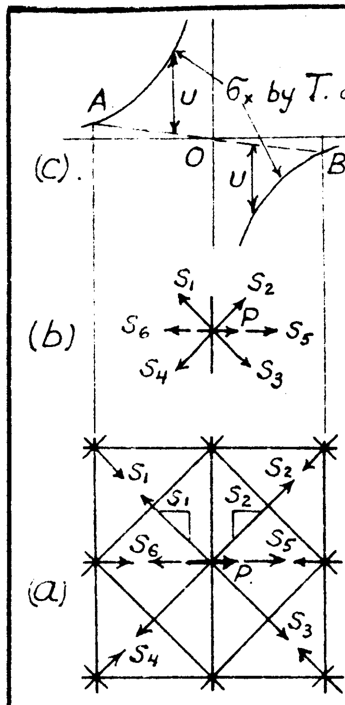


Fig. 109 (a), (b) & (c)

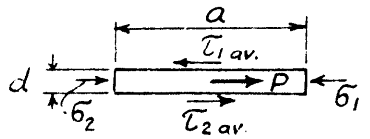


Fig. 111.

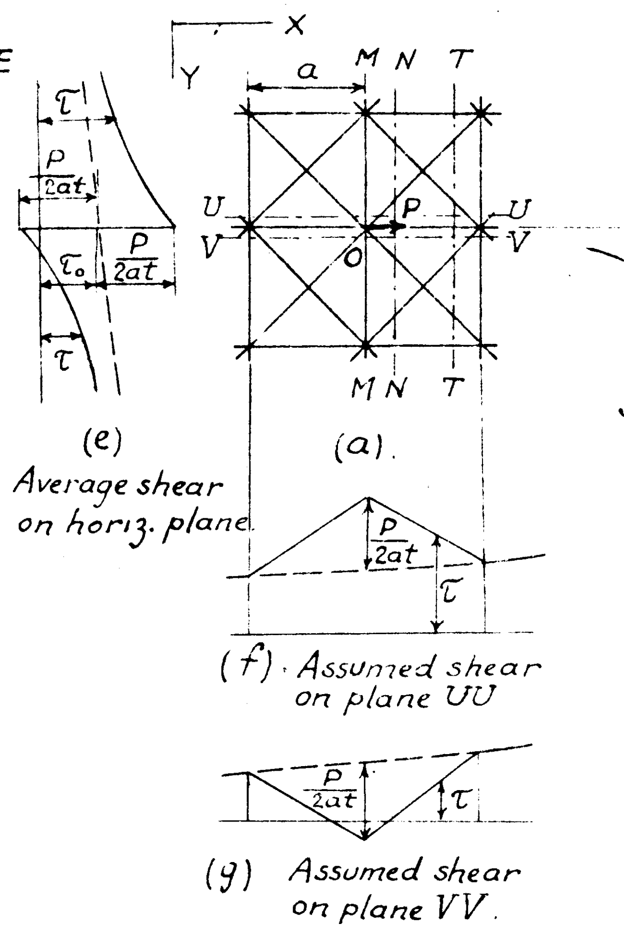
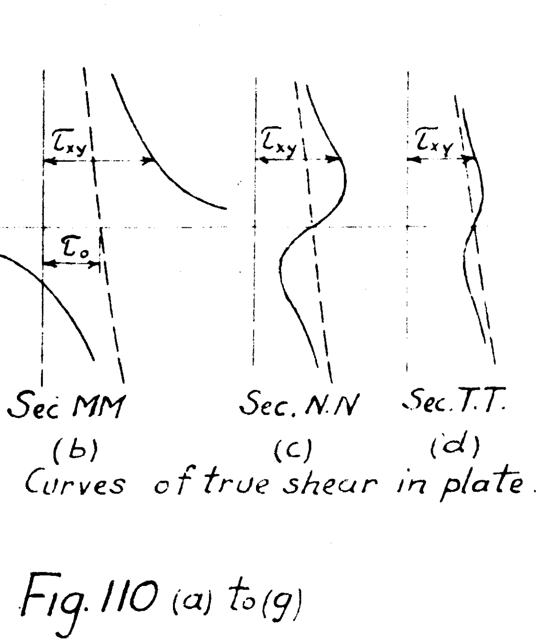



Fig. 110 (a) to (g)

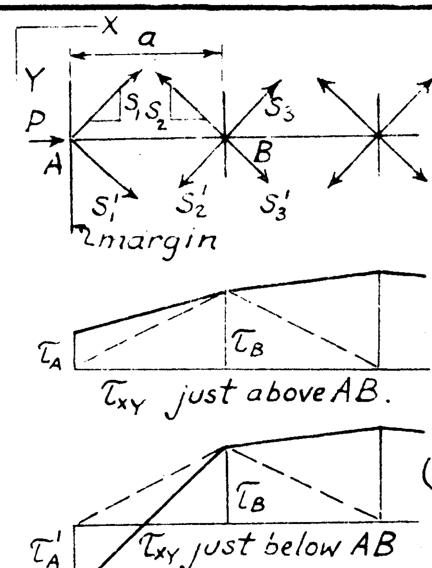


Fig. 112 (a) to (c)

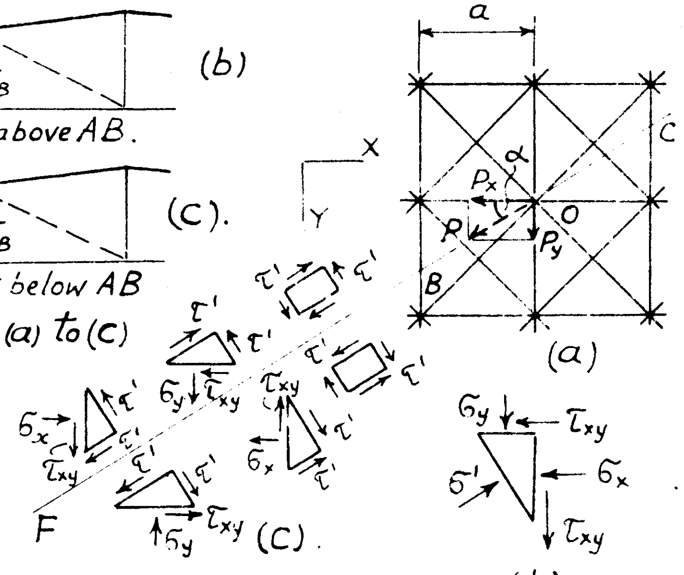


Fig. 113 (a) to (c).

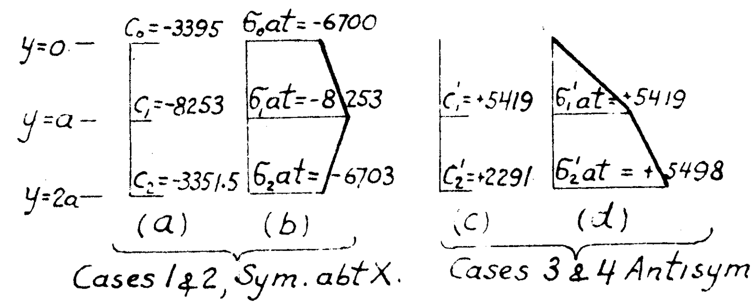


Fig. 114 (a) to (d).

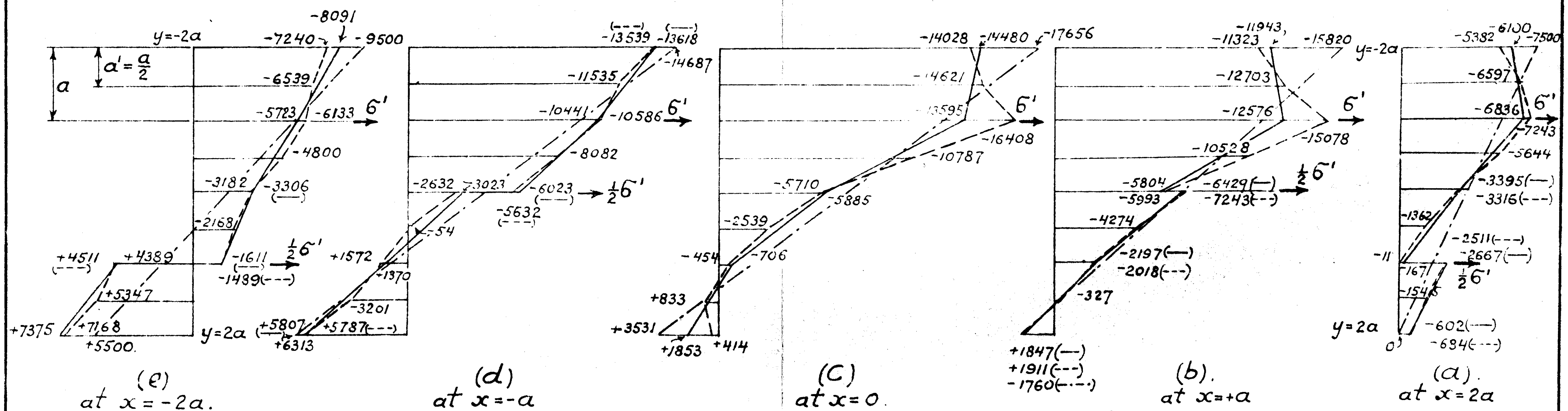


Fig. 115 (a) to (e). σ_x Stresses: —— by 4×6 Framework. ---- by 8×12 Framework. -·-·- by Beam Formula.

Note Re. Fig. 115.
All figures are in lbs. They should be divided by a't.
Arrows σ' signify rivet bearing stress to be added to σ_x.

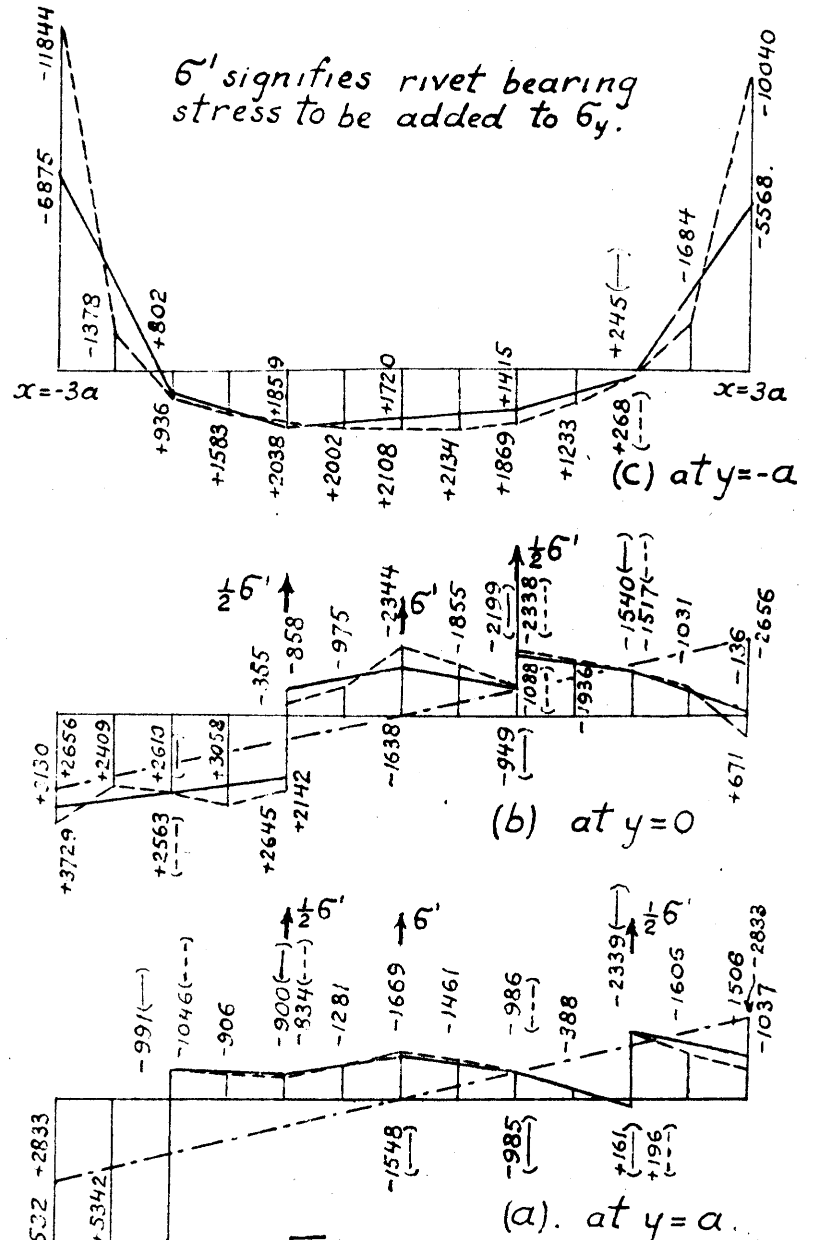


Fig. 116 (a), (b) & (c)
σ_y Stresses.

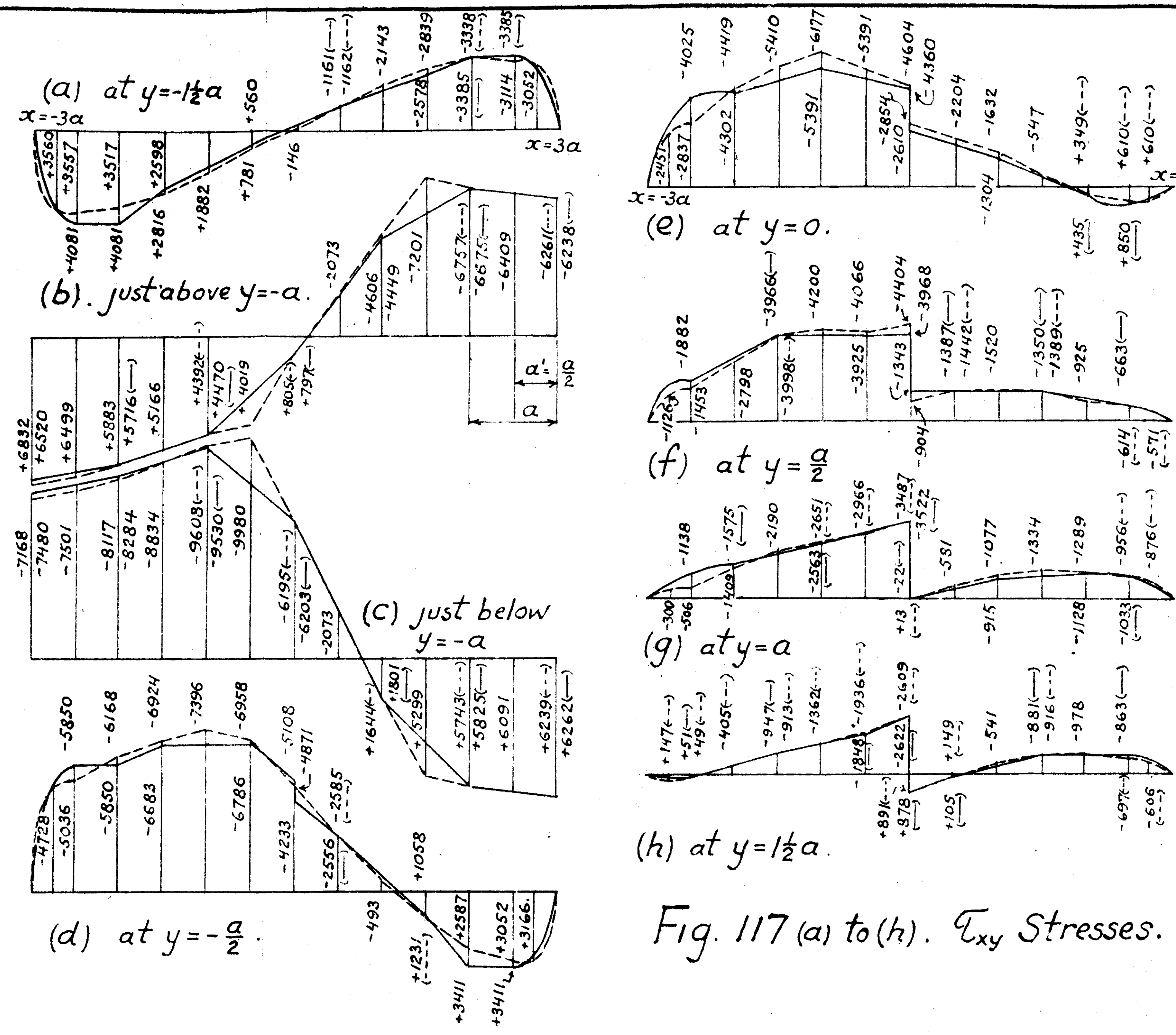


Fig. 117 (a) to (h). τ_{xy} Stresses.

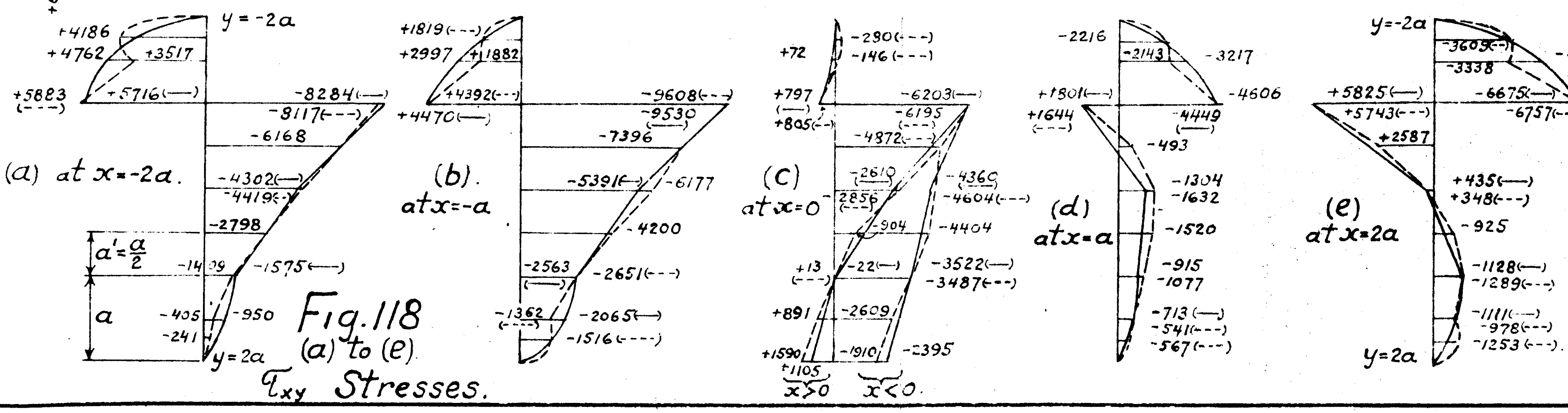


Fig. 118 (a) to (e) τ_{xy} Stresses.

General Note.
All figures are in lbs.
They should be divided by a't.
—— 4×6 Framework
- - - 8×12 Framework
-·-·- Beam F-L

11.

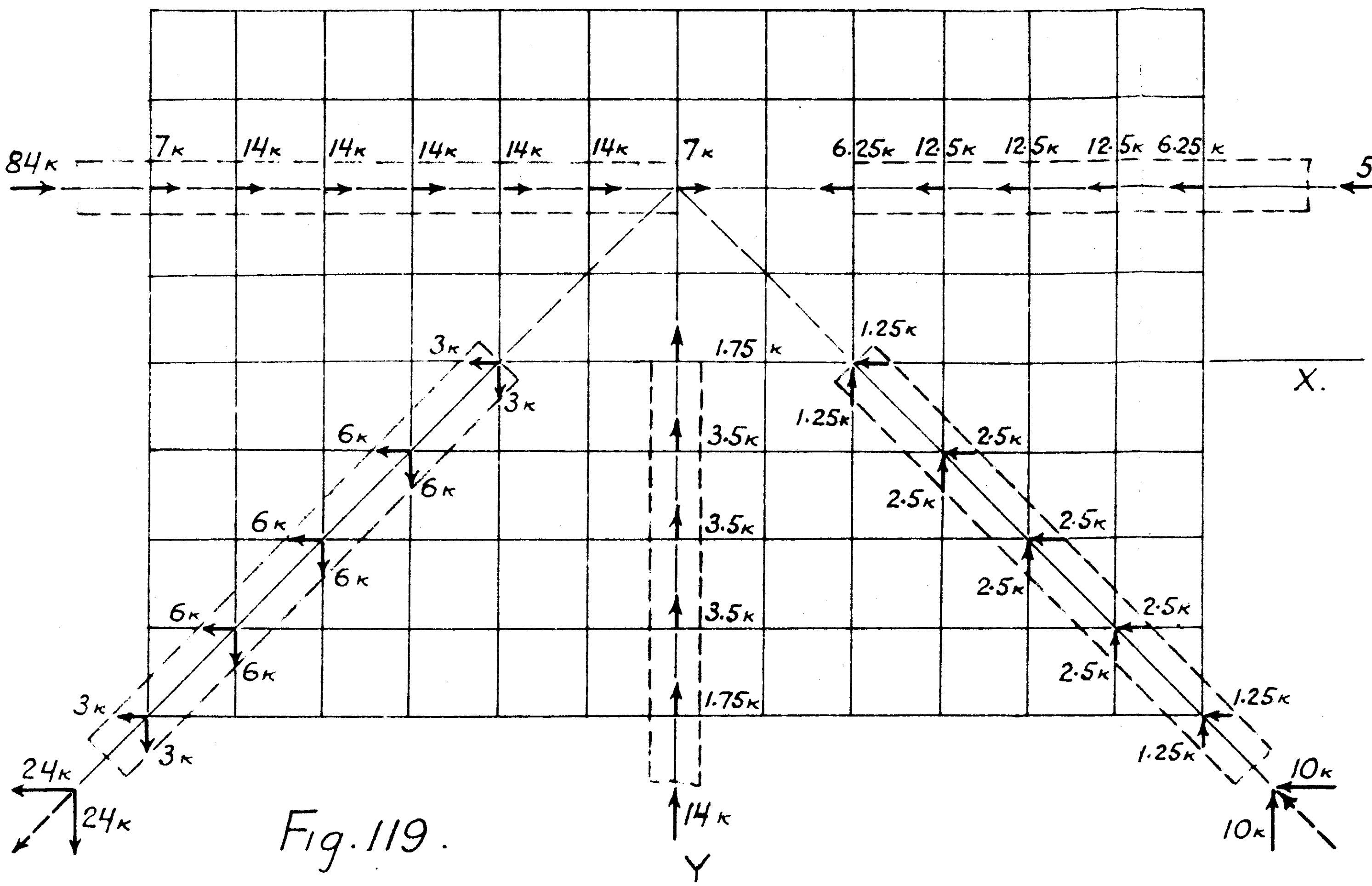
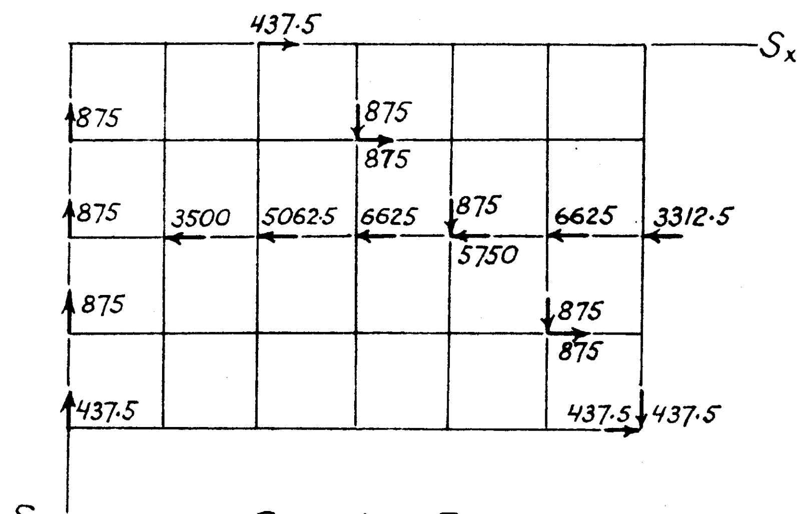
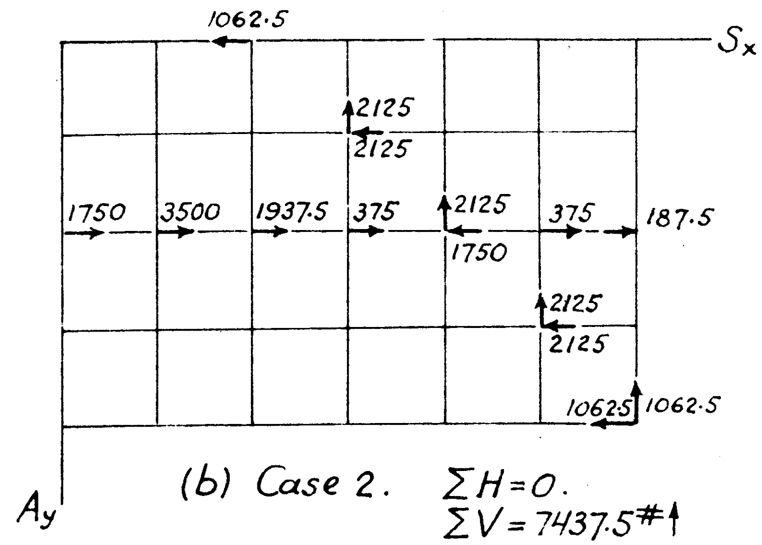


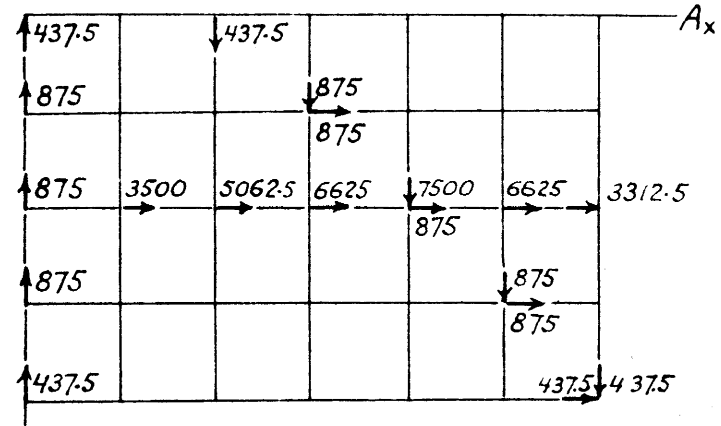
Fig. 119.



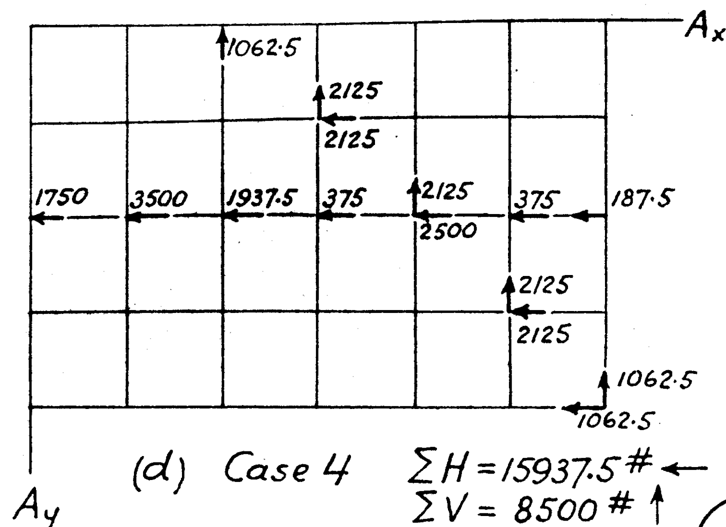
(a) Case 1. $\Sigma H = 28250^{\#} \leftarrow$
$\Sigma V = 0.$



(b) Case 2. $\Sigma H = 0.$
$\Sigma V = 7437.5^{\#} \uparrow$



(c) Case 3. $\Sigma H = 34812.5^{\#} \rightarrow$
$\Sigma V = 0.$



(d) Case 4 $\Sigma H = 15937.5^{\#} \leftarrow$
$\Sigma V = 8500^{\#} \uparrow$

Fig. 120 (a) to (d).

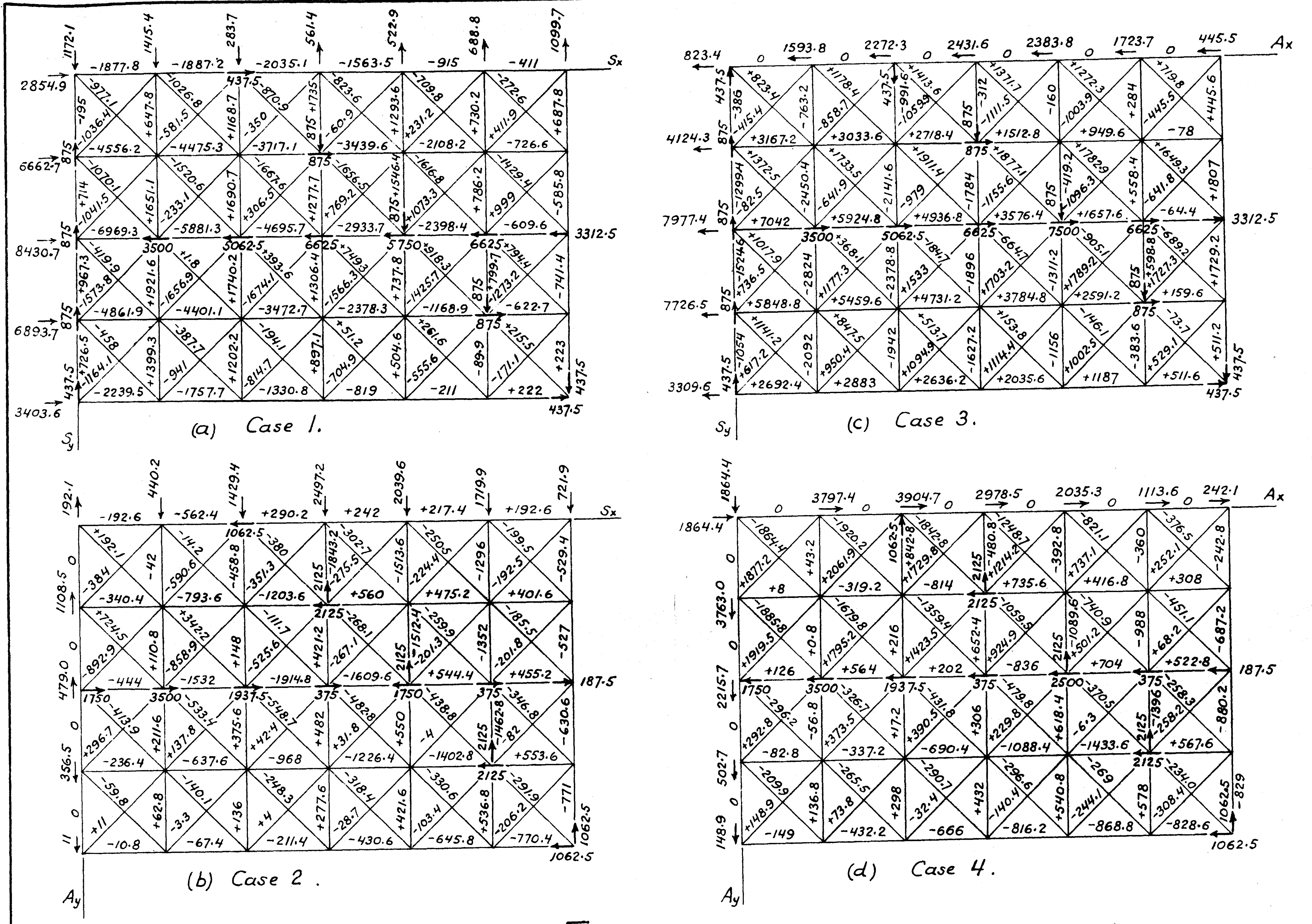
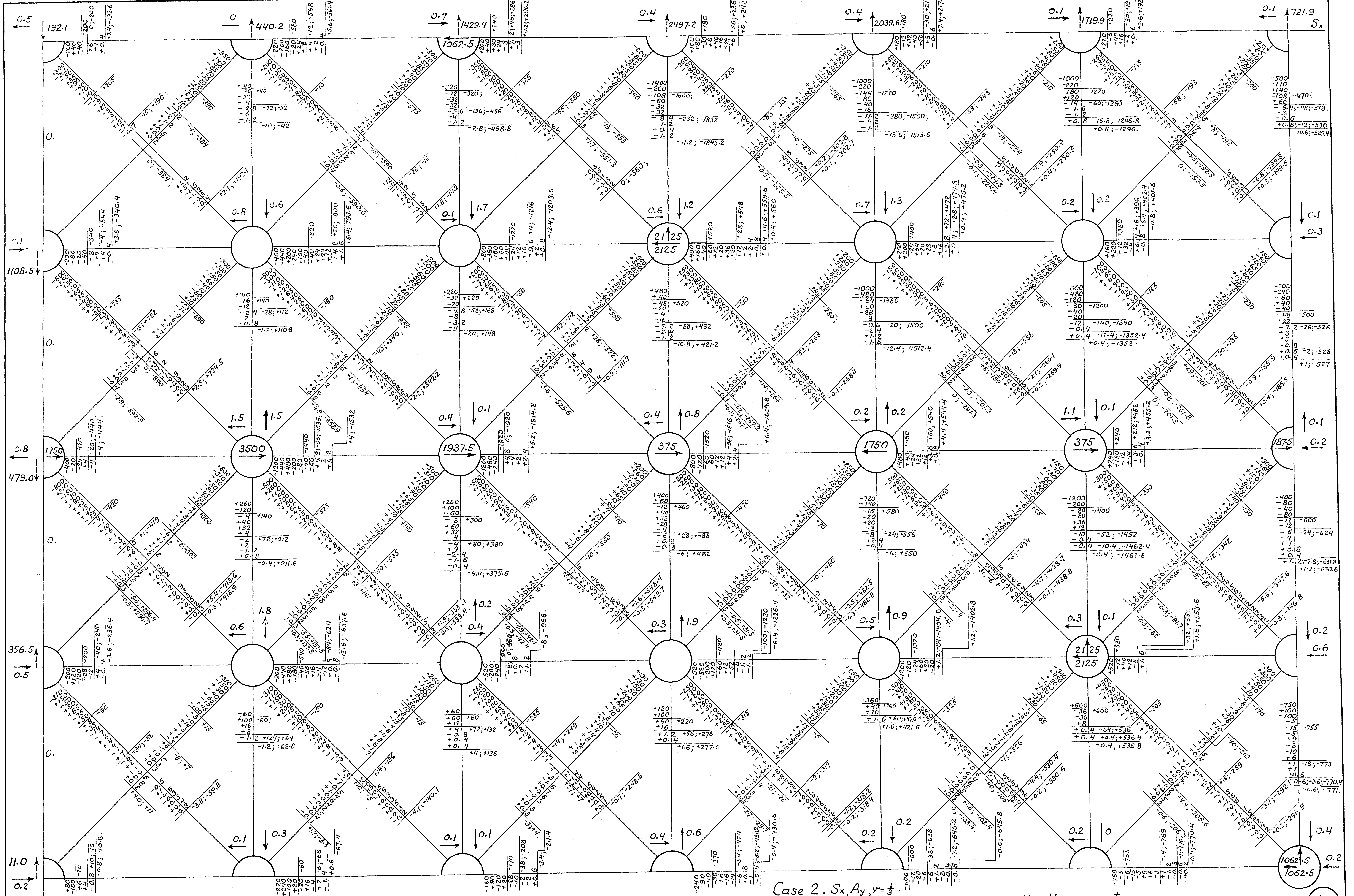


Fig 121 (a) to (d).

Framework Stresses. $\nu = \frac{1}{3}$.

All stresses in lbs.
Figures on diagonals - their H or V components.



Case 2. $S_x, A_y, \gamma = \frac{1}{3}$.
All figures - lbs. Figures on diagonals - their H or V components.
Acting forces - in circles. Figures at joints on S_x and A_y - reversed reactions.
Remaining unbalances - in faint (red) figures.

Fig. 122.

14.

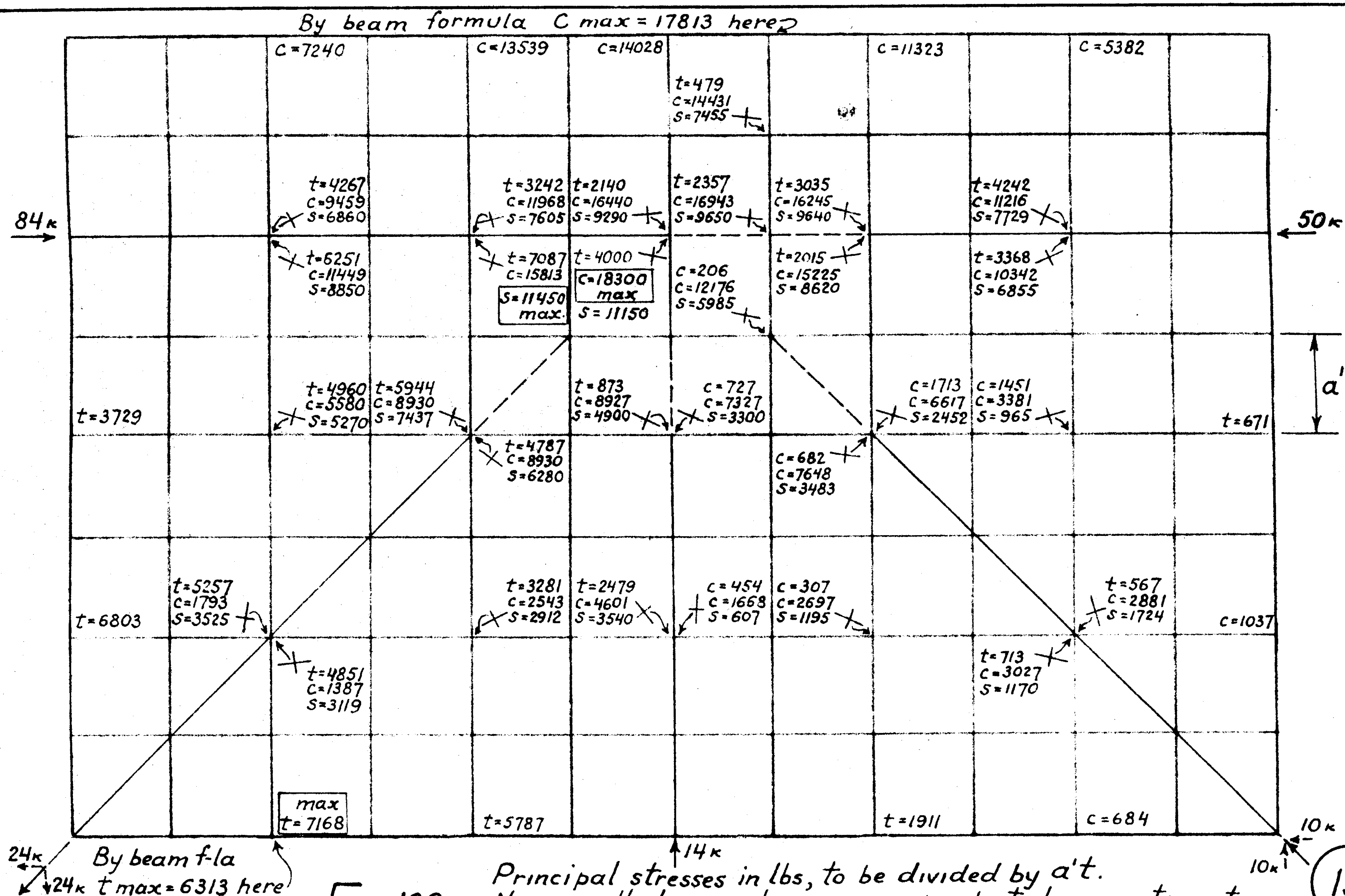


Fig 123.

Principal stresses in lbs, to be divided by a't.
Numerically larger stress corresponds to larger intercept.
Stress discontinuities along lines of loading.

15

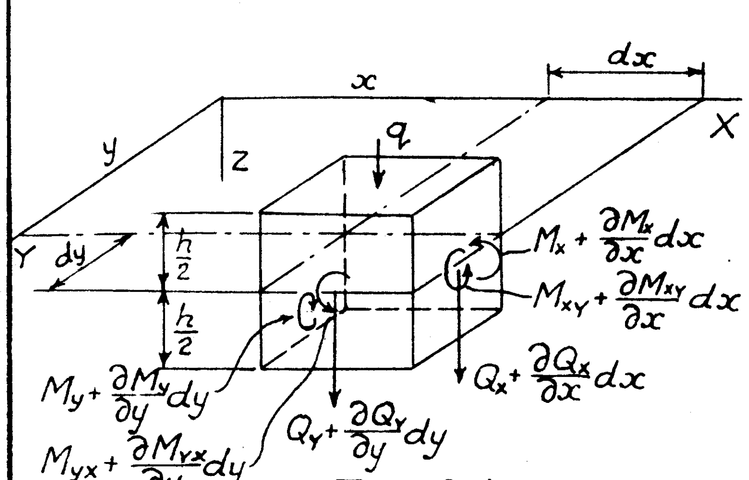


Fig. 124.

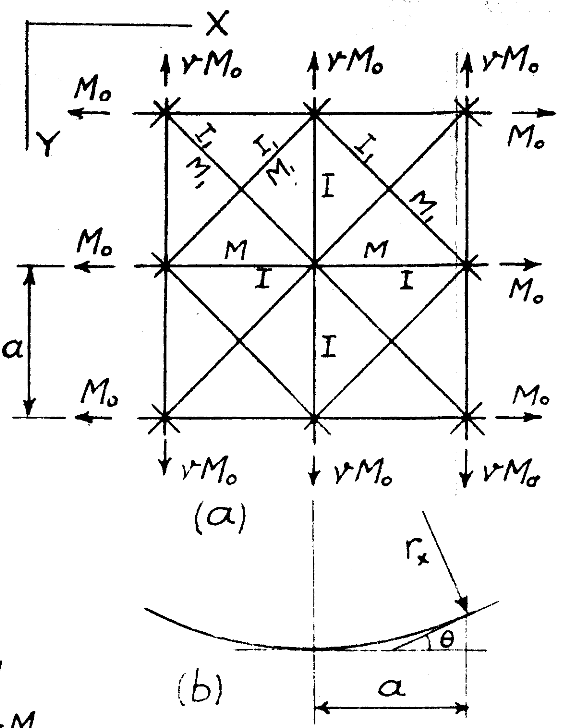


Fig. 126 (a) & (b)

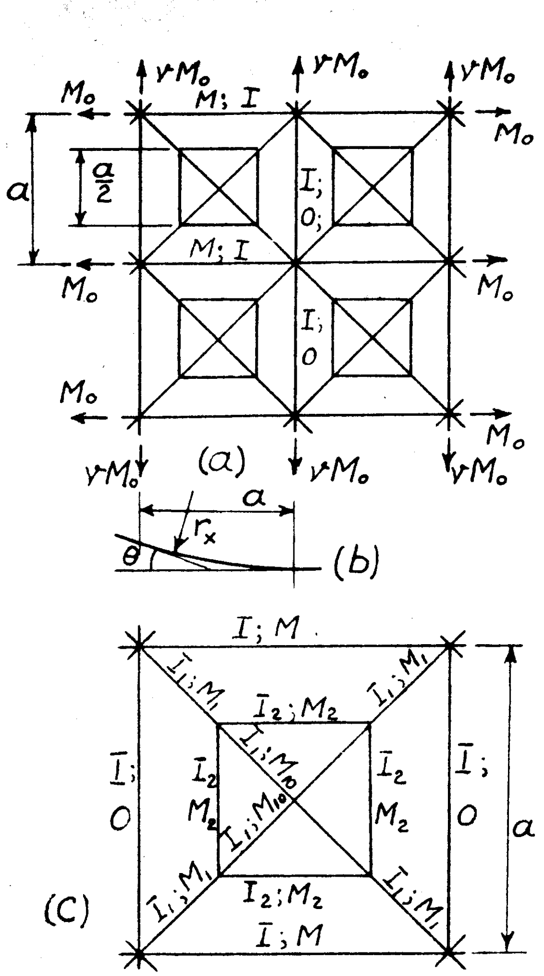


Fig. 129 (a) to (c)

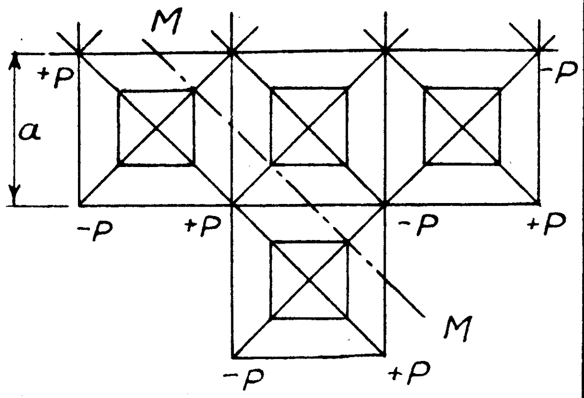


Fig. 130.

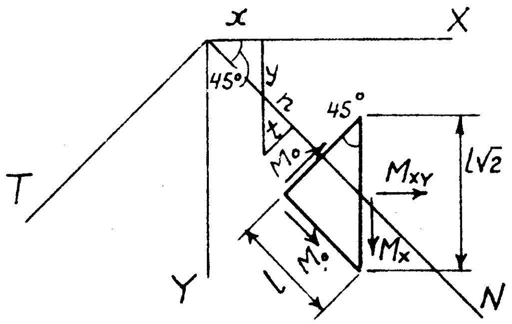


Fig. 125.

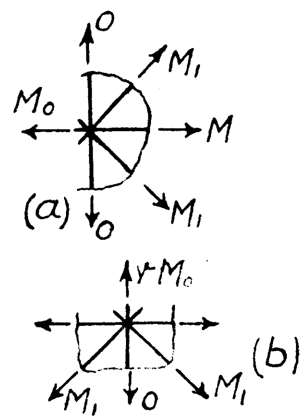


Fig. 127 (a) & (b)

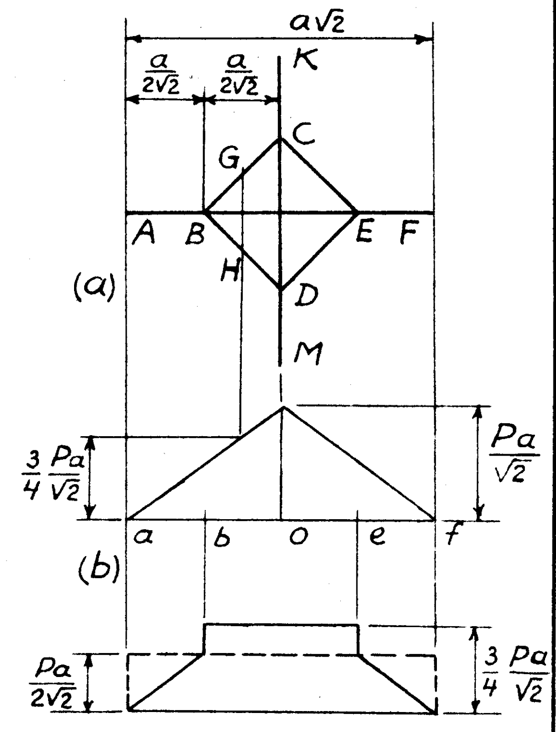


Fig. 131 (a) to (c)

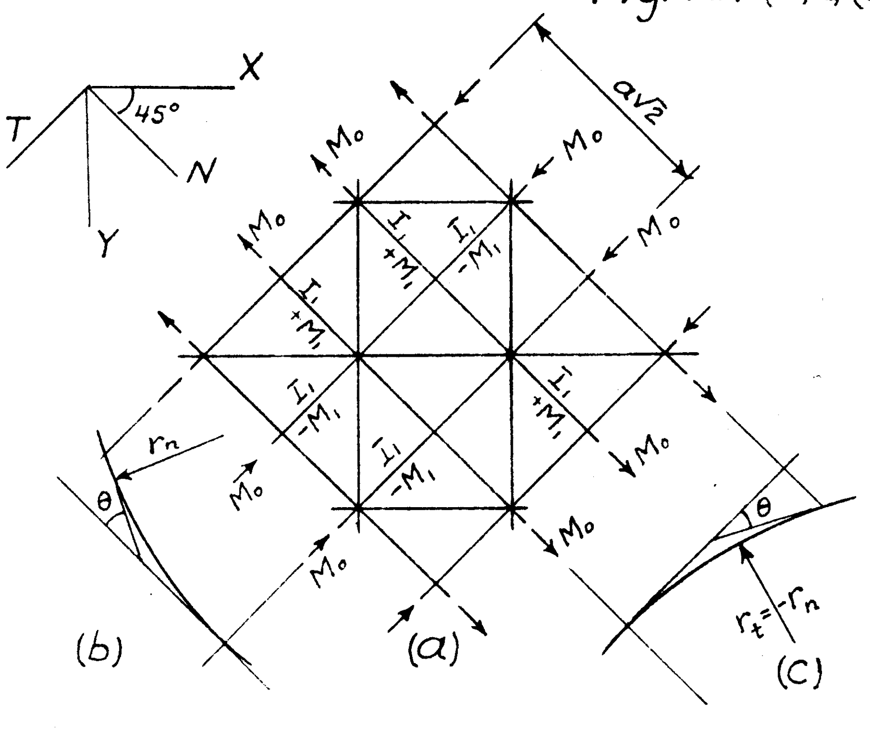
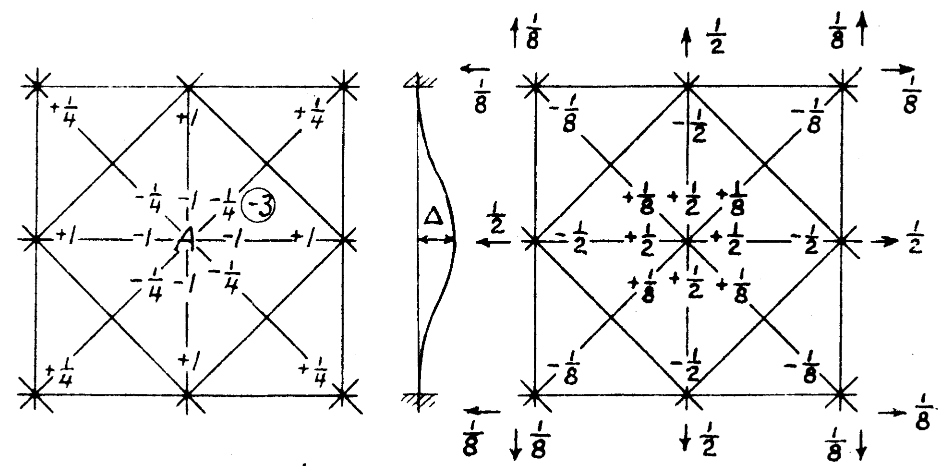


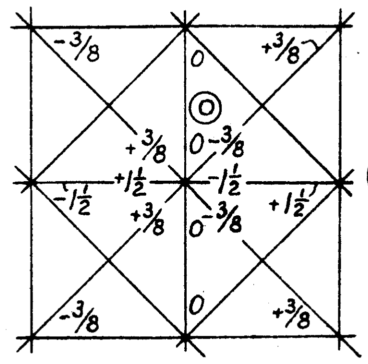
Fig. 128 (a) to (c)



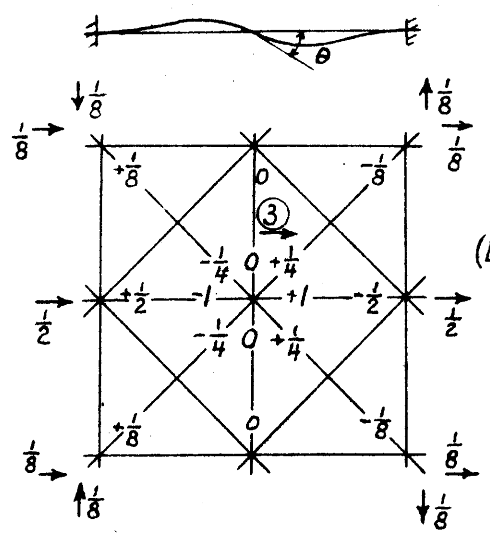
(a) Displacement Shear Factors $\gamma = \frac{1}{3}$.

(b) Displacement Moment Factors $\gamma = \frac{1}{3}$.

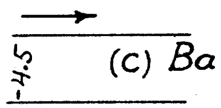
Fig. 132 (a) & (b).



(a) Rotation Shear Factors. $\gamma = \frac{1}{3}$.

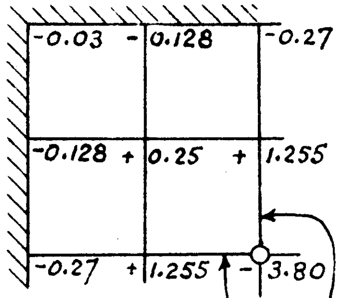


(b) Rotation Moment Factors. $\gamma = \frac{1}{3}$.



(c) Balance Vectors.

Fig. 133 (a) to (c).



Axes of Sym

Fig. 136.

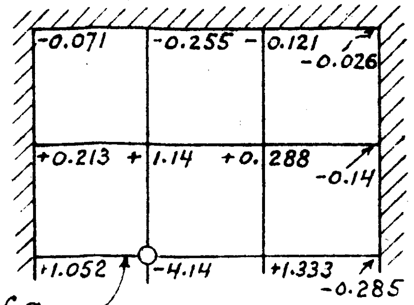


Fig. 138

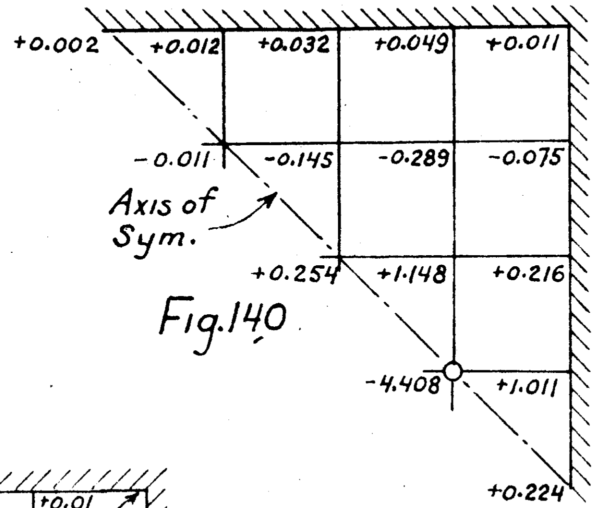
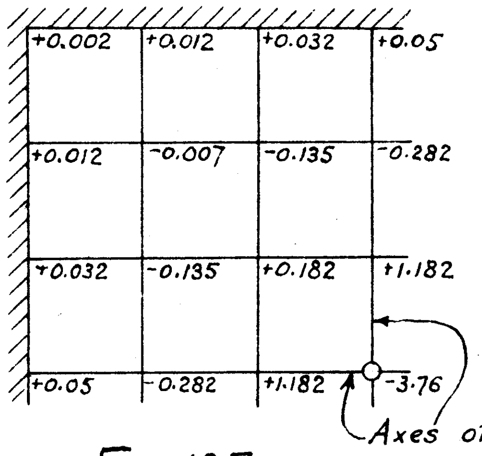


Fig. 140.

Fig. 136 to 140 — Reaction Factors.



Axes of Sym.

Fig. 137.

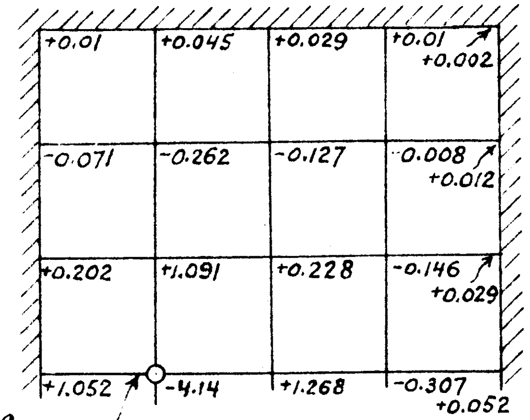
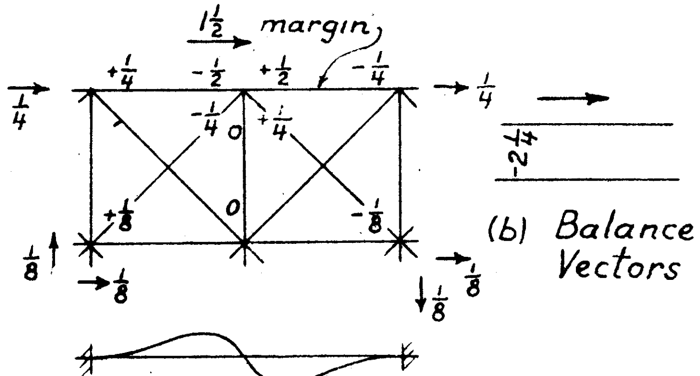


Fig. 139



(a) Rotation Moment Factors.

(b) Balance Vectors

Fig. 141 (a) & (b).

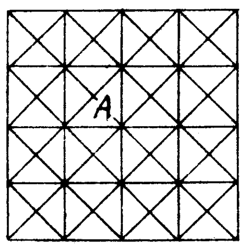


Fig. 134

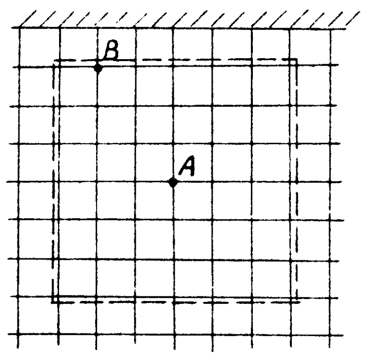
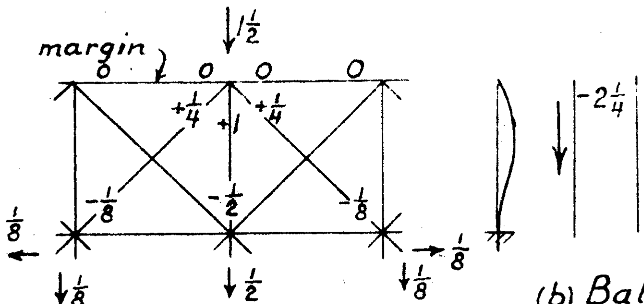


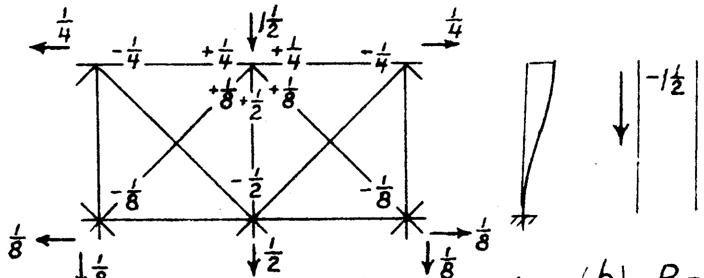
Fig. 135



(a) Rotation Moment Factors

(b) Balance Vectors

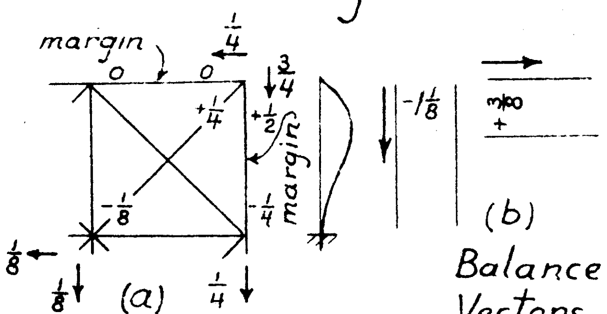
Fig. 142 (a) & (b).



(a) Displacement Moment Factors

(b) Balance Vectors

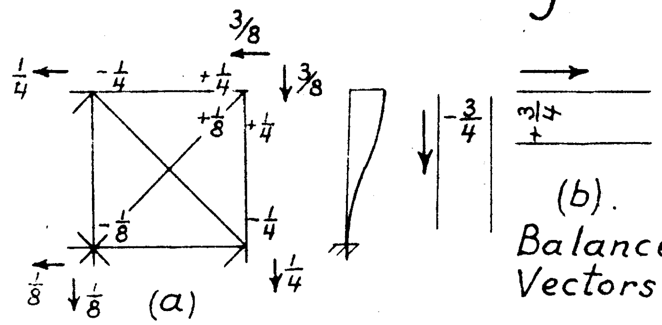
Fig. 144 (a) & (b).



(a) Rotation Moment Factors

(b) Balance Vectors.

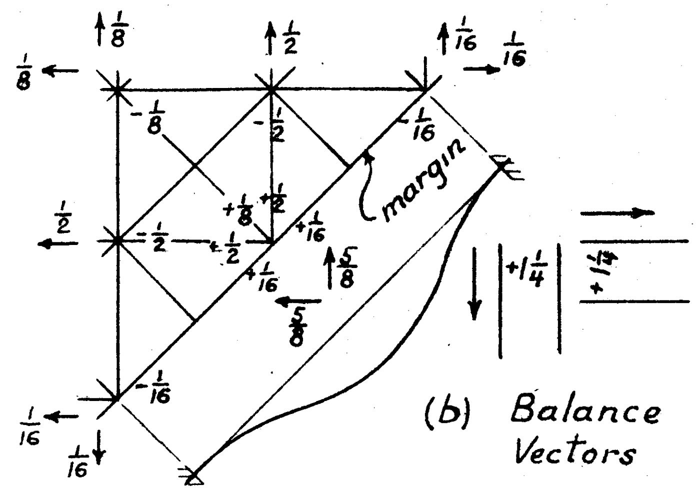
Fig. 143 (a) & (b).



(a) Displacement Moment Factors

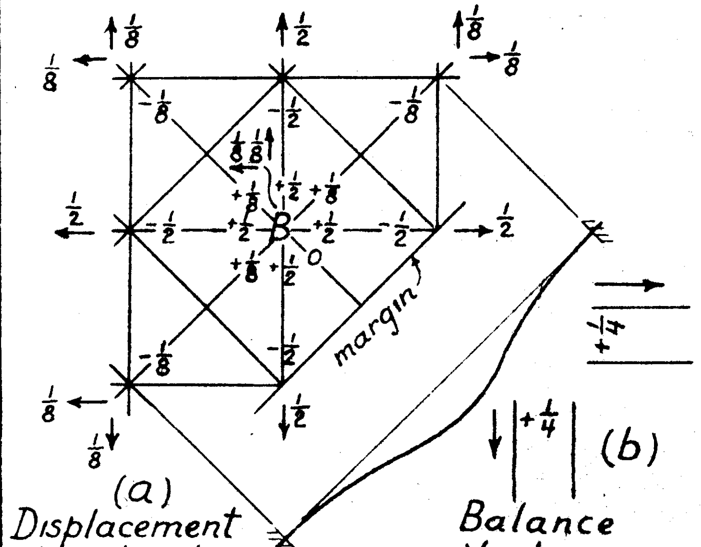
(b). Balance Vectors.

Fig. 145 (a) & (b).



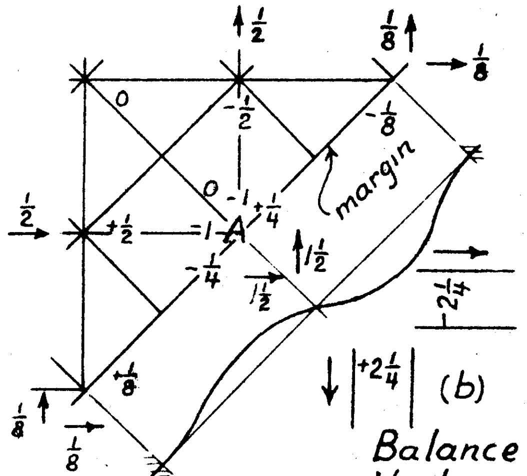
(a) Displacement Moment Factors (b) Balance Vectors

Fig. 146 (a) & (b)



(a) Displacement Moment Factors (b) Balance Vectors

Fig 147 (a) & (b)



(a) Rotation Moment Factors. (b) Balance Vectors

Fig. 148 (a) & (b)

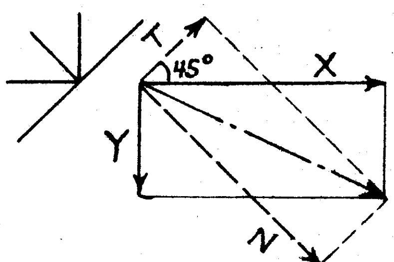
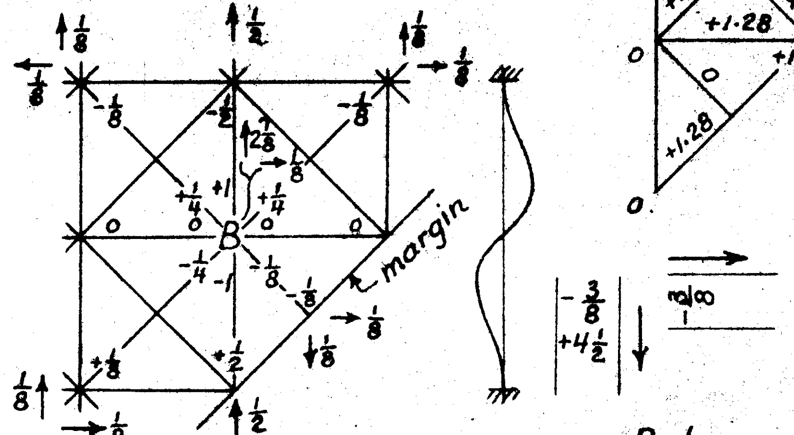
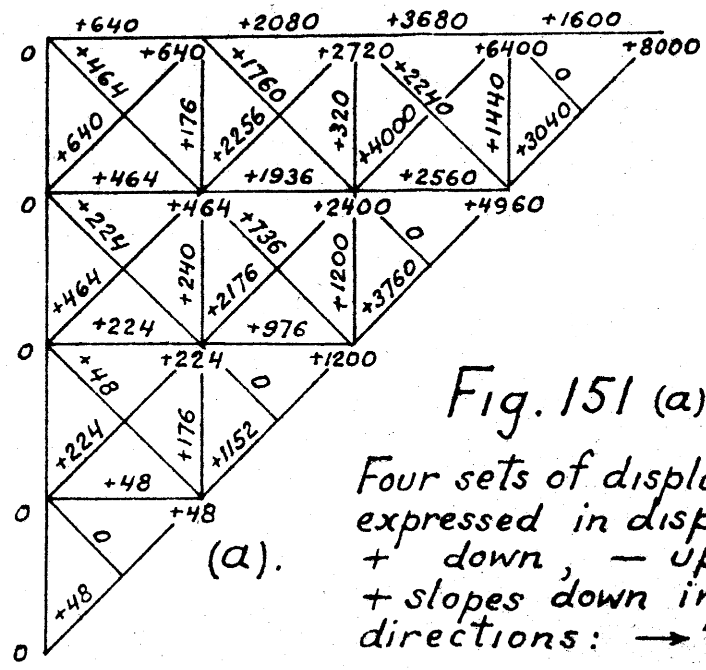


Fig. 150.

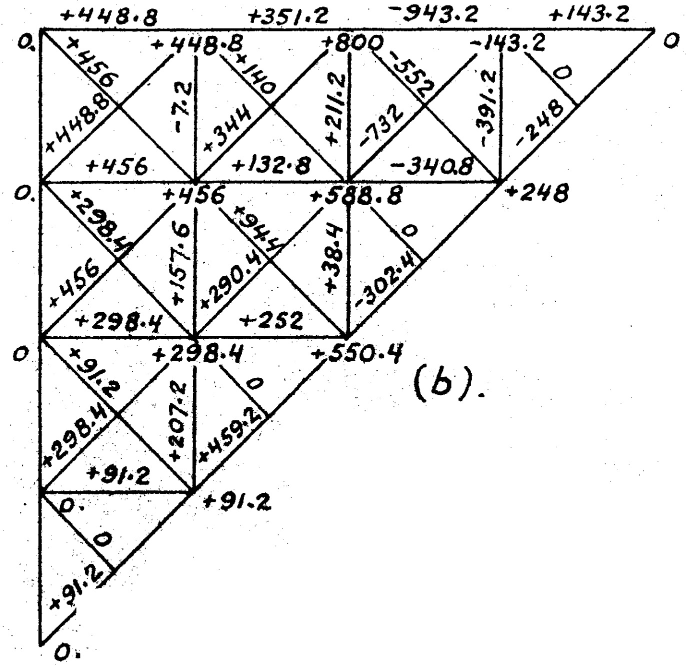


(a) Rotation Moment Factors. (b) Balance Vectors.

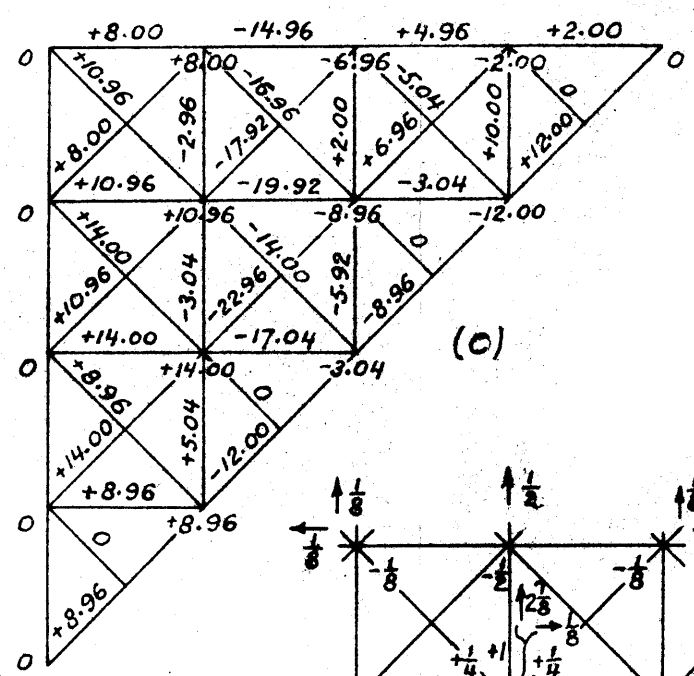
Fig. 149 (a) & (b).



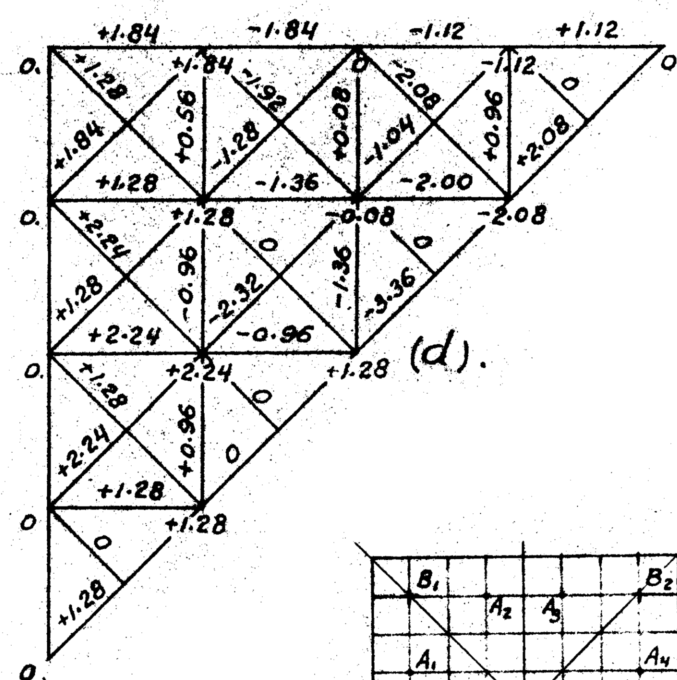
(a).



(b).



(c)



(d).

Fig. 151 (a) to (d).

Four sets of displacements expressed in displ. units. + down, − up. + slopes down in these directions: → ↑ ↗ ↘

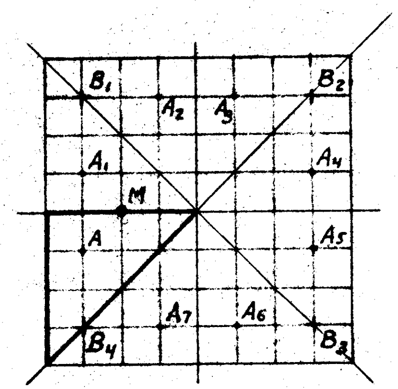


Fig. 154.

a	b	c	a	b	c
		+11.6		+0.14	+2.74
	-0.75	+10.85		+0.59	+3.33
	-2.33	+8.52		-0.86	+2.47
	+8.75	+17.27		-1.93	+0.54
+7	-14.48	+2.79		+0.44	+0.98
	+0.23	+3.02		+0.50	+1.48
	-1.07	+1.95		-0.26	+1.22
	+0.58	+2.53		+0.15	+1.37
	+0.65	+3.18		-0.23	+1.14
	-2.57	+0.61		+1.10	+2.24
	-0.70	-0.09	+1	-2.07	+0.17
	+2.19	+2.10		+0.05	+0.22
	+0.50	+2.60	+8		

a	b	c	a	b	c
		-9.1		+0.10	+0.85
	+2.89	-6.21		-0.85	0
	-0.78	-6.99		-4.38	-4.38
	+2.03	-4.96	-3	+5.7	+1.32
	+4.51	-0.45		-0.64	+0.68
	+0.16	-0.29		-1.92	-1.24
	-5.47	-5.76		-0.57	-1.81
	-0.86	-6.62		-0.21	-2.02
	+1.41	-5.21		-0.08	-2.10
-4	+7.6	+2.39		+0.25	-1.85
	-0.23	+2.16		+0.64	-1.21
	+0.51	+2.67		+0.03	-1.18
	-1.92	+0.75	-7		

a	b	c	a	b	c
		+13.1		+0.04	+3.01
+6	-11.04	+2.06		-4.19	-1.18
	+0.24	+2.3		+0.30	-0.88
	-0.64	+1.66		-1.44	-2.32
	-0.88	+0.78		-3.14	-5.46
	+0.06	+0.84	-4	+7.35	+1.89
	+0.37	+1.21		-0.25	+1.64
	-4.19	-2.98		-1.05	+0.59
	+0.61	-2.37		+0.02	+0.61
	-1.93	-4.30		-0.08	+0.53
	+0.08	-4.22		-0.12	+0.41
	-0.16	-4.38		+0.01	+0.42
-4	+7.35	+2.97	-2		

		+35.4		-0.85	+3.54
	-0.48	+34.92		+1.22	+4.76
	+9.95	+44.87		-3.87	+0.89
+8	-35.3	+9.57		-0.76	+0.13
	+7.66	+17.23		+0.91	+1.04
	-1.42	+15.81		+0.32	+1.36
	-4.83	+10.98		+0.51	+1.87
	+1.22	+12.20		+0.31	+2.18
	-1.27	+10.93		+1.00	+3.18
	-1.02	+9.91	+1	-4.41	-1.23
	+2.99	+12.90		+1.10	-0.13
+2	-8.83	+4.07		-0.28	-0.41
	+0.32	+4.39	+11		

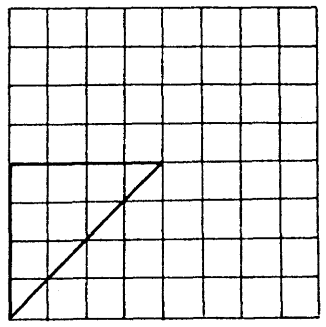
		-19.		-0.43	-5.92
	-0.45	-19.45		-4.43	-10.33
	-0.51	-19.96	-4	+15.20	+4.87
	+7.73	-12.23		-3.28	+1.59
	+1.50	-10.73		-3.32	-1.73
	-0.71	-11.44		+0.30	-1.43
-5	+19.0	+7.56		+2.53	+1.10
	-4.44	+3.12		-1.11	-0.01
	-6.31	-3.19		-0.05	-0.06
	-4.38	-7.57		+0.97	+0.91
	-0.15	-7.72		+0.21	+1.12
	+1.93	-5.79		-0.14	+0.98
	+0.30	-5.49	-9		

		-5.8		-0.02	-7.93
	+6.29	+0.49	-4	+8.66	+0.73
	-1.34	-0.85		-4.44	-3.71
	-2.44	-3.29		-0.64	-4.35
	-1.02	-4.31	-3	+6.5	+2.15
	-0.04	-4.35		-4.19	-2.04
	-5.54	-9.80		+0.25	-1.79
-4	+8.66	-1.23	-1	+2.17	+0.38
	-0.62	-1.85		-0.13	+0.25
	-0.86	-2.71		-0.31	-0.06
	-0.40	-3.11		-0.15	-0.21
	-0.61	-3.72		-0.01	-0.22
	-4.19	-7.91	-12		

		+39.5		+3.40	-0.41
	+0.14	+39.64		+0.53	+0.12
+10	-40.12	-0.48		+0.20	+0.32
	+7.96	+7.48		+0.23	+0.55
	-1.63	+5.85		+0.40	+0.95
	+5.67	+11.52		-0.10	+0.85
	+0.25	+11.77		+2.51	+3.36
	+0.54	+12.31		+0.13	+3.49
	-6.28	+6.03	+1	-4.01	-0.52
	+0.31	+6.34		+1.00	+0.48
+3	-12.04	-5.70		-0.23	+0.25
	+1.99	-3.71		+1.13	+1.38
	-0.10	-3.81	+14		

		-15.3		+0.38	+7.33
	-0.74	-16.04		-0.50	+6.83
	+12.65	-3.39		-5.05	+1.78
	+2.02	-1.37		+0.85	+2.63
	-0.90	-2.27		-0.37	+2.26
	+0.63	-1.64		+0.49	+2.75
	-6.31	-7.95	+2	-3.80	-1.05
	-0.50	-8.45		-0.12	-1.17
-5	+9.5	+1.05		+1.26	+0.09
	+1.13	+2.18		+0.25	+0.34
	+3.77	+5.95		-0.13	+0.21
	+0.51	+6.46		+0.13	+0.34
	+0.49	+6.95	-3		

		+5.6	+3	-6.61	+0.19
	+0.07	+5.67		+0.03	+0.22
	+11.34	+17.01		+0.57	+0.79
	-2.28	+14.73		-0.10	+0.69
	+0.33	+15.06		+0.02	+0.71
+5	-11.02	+4.04		-0.05	+0.66
	+0.71	+4.75		+0.25	+0.91
	+0.03	+4.78		+0.01	+0.92
	-0.63	+4.15		+1.13	+2.05
	-0.13	+4.02		-0.28	+1.77
	+3.40	+7.42		+0.05	+1.82
	-0.57	+6.85	+1	-2.20	-0.38
	-0.05	+6.80	+9		

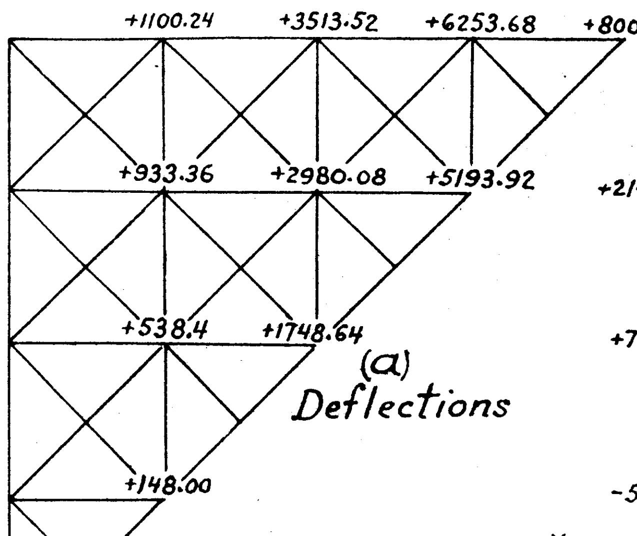


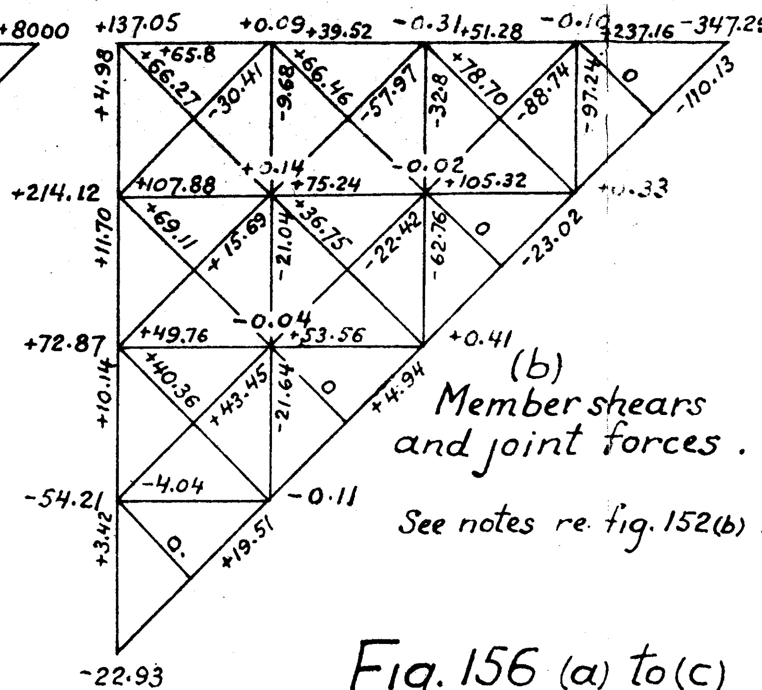
Key Diagram.

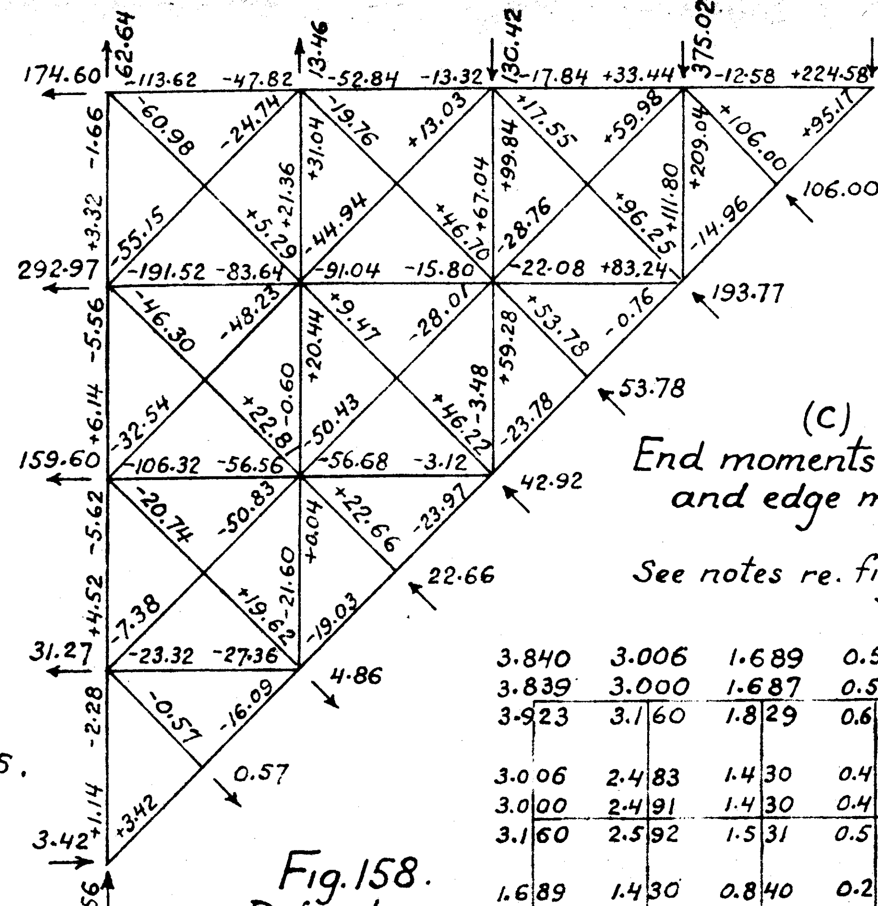
Fig. 155

Determination of displacements of 3rd cycle.

a - Units of displacement.
b - Joint forces added.
c.- Total joint forces.







(c)
End moments in members and edge moments.

See notes re. fig. 152(a).

Fig. 156 (a) to (c)
Resultant Functions.

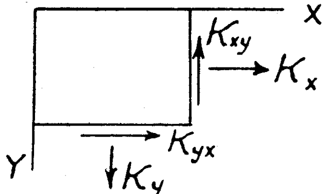


Fig. 157.

3.840 3.839 3.923	3.006 3.000 3.160	1.689 1.687 1.829	0.528 0.527 0.625
3.006 3.000 3.160	2.483 2.491 2.592	1.430 1.430 1.531	0.445 0.447 0.525
1.689 1.687 1.829	1.430 1.430 1.531	0.840 0.839 0.911	0.262 0.258 0.303
0.528 0.527 0.625	0.445 0.447 0.525	0.262 0.258 0.303	0.071 0.072 0.090

Fig. 158.
Deflections.
$\delta : \frac{Pa^2}{Eh^3}$; $(\nu = \frac{1}{3})$
1st. fig. - Timosh.
2nd " - Framewk
3rd " - Marcus.

	+0.0746 +0.0673 +0.0892	+0.0016 -0.0002 +0.0066	-0.0520 -0.0523 -0.0481	-0.1257 -0.1257 -0.1171
+0.1251 +0.1351 +0.1246	+0.0637 +0.0641 +0.0616	+0.0021 +0.0011 +0.0048	-0.0457 -0.0456 -0.0412	-0.1050 -0.1055 -0.0985
+0.0473 +0.0469 +0.0519	+0.0292 +0.0306 +0.0284	-0.0019 -0.0017 -0.0014	-0.0303 -0.0304 -0.0289	-0.0578 -0.0574 -0.0568
-0.0033 -0.0048 +0.0005	-0.0066 -0.0065 -0.0035	-0.0102 -0.0101 -0.0104	-0.0146 -0.0144 -0.0154	-0.0105 -0.0113 -0.0169
-0.0419 -0.0451 -0.0391	-0.0350 -0.0374 -0.0328	-0.0193 -0.0190 -0.0190	-0.0035 -0.0021 -0.0056	0. 0.

Fig. 159.
Bending Moments $\frac{M_x}{P}$. $(\nu = \frac{1}{3})$.
1st fig - Timoshenko.
2nd " - Framework.
3rd " - Marcus.

0 0	0 0	0 0	0 0	0 0
0 0	+0.0289 +0.0403	+0.0244 +0.0283	+0.0162 +0.0194	0 0
0 0	+0.0244 +0.0283	+0.0299 +0.0338	+0.0221 +0.0264	0 0
0 0	+0.0162 +0.0192	+0.0221 +0.0264	+0.0151 +0.0197	0 0
0 0	0 0	0 0	0 0	0 0

Fig. 160.
Torsional Moments $\frac{M_{xy}}{P}$. $(\nu = \frac{1}{3})$
1st. fig. - Timoshenko.
2nd " - Framework.

0	-0.5437	-0.3263	-0.2912	-0.3900 -0.4004
0	-0.3375	-0.2579	-0.2364	-0.3084 -0.3072
0	-0.1009	-0.1097	-0.1091	-0.1049 -0.1051
0	+0.0218	+0.0342	+0.0224	+0.0781 +0.0705
0	+0.0798 +0.1209	+0.1122 +0.1634	+0.0821 +0.1135	+0.0660 +0.0676

4a

Fig 161
Shears
$Q_x : \frac{P}{4a}$
By framework
2nd fig. - Marcus.
Any ν.

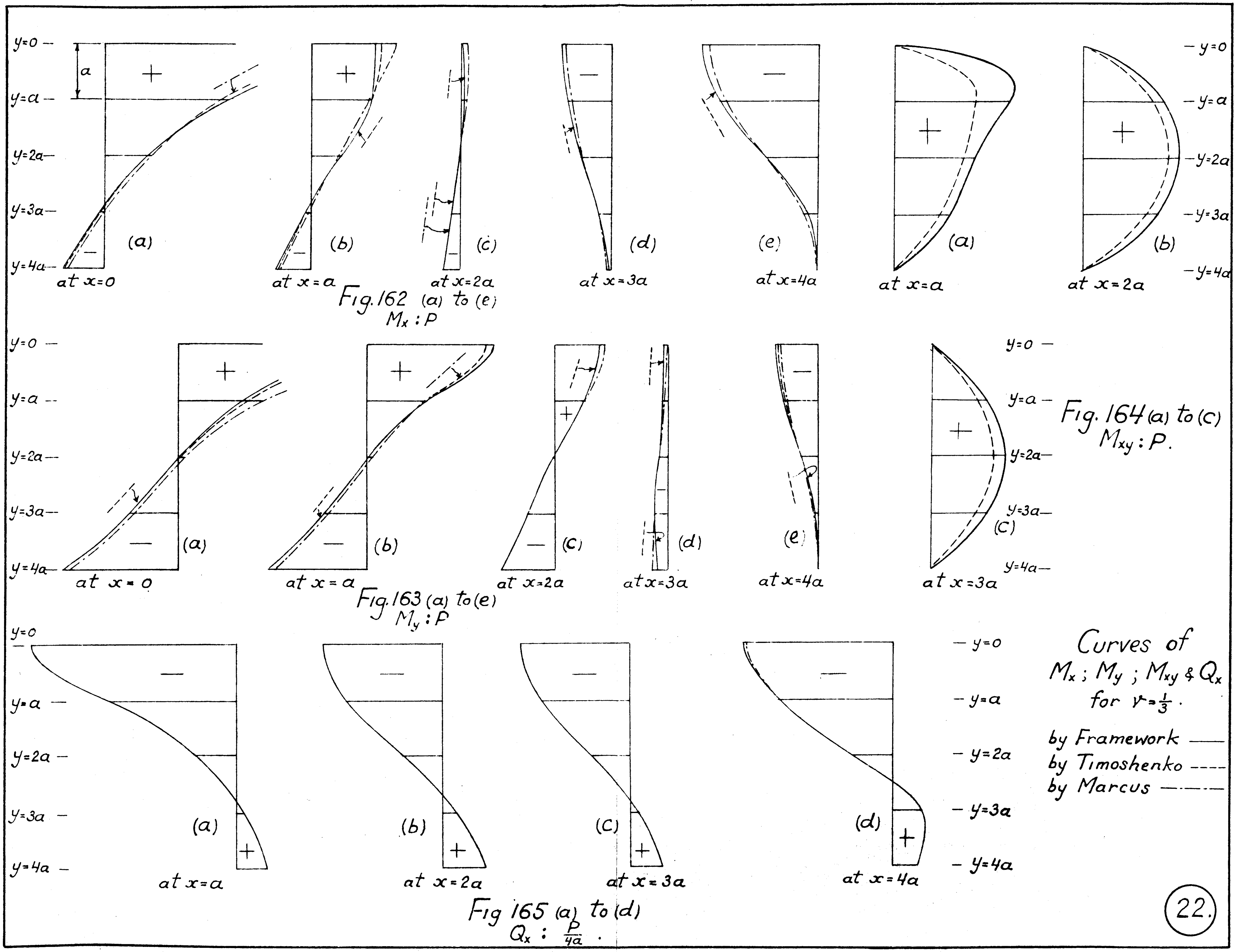


Fig. 162 (a) to (e) $M_x : P$

Fig. 163 (a) to (e) $M_y : P$

Fig. 164 (a) to (c) $M_{xy} : P$.

Fig 165 (a) to (d) $Q_x : \frac{P}{4a}$.

Curves of M_x ; M_y ; M_{xy} & Q_x for $\nu = \frac{1}{3}$.

by Framework ———

by Timoshenko - - - -

by Marcus —·—·—

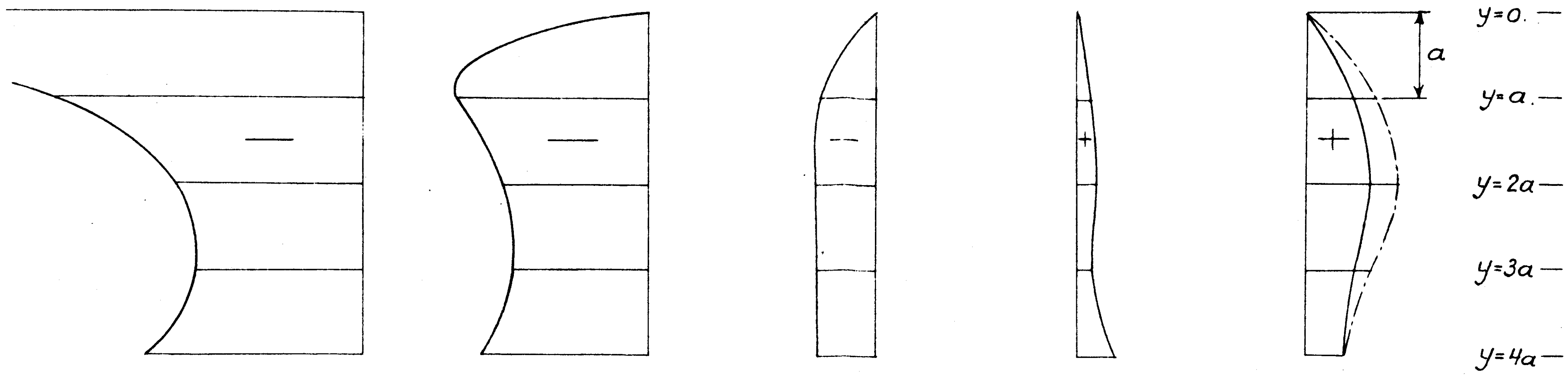


Fig. 166. Shears $Q_y : \frac{P}{4a}$. Any ν.

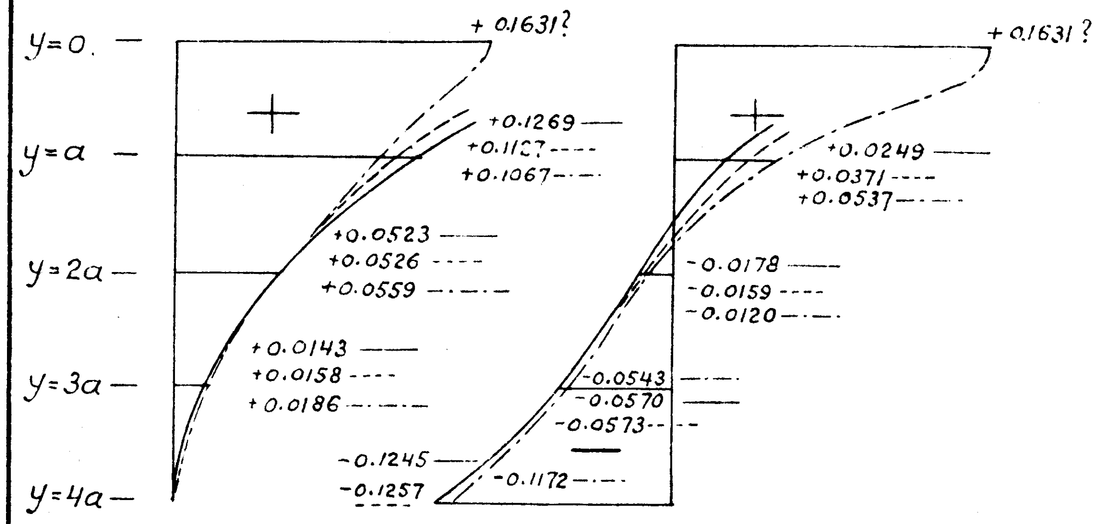


Fig. 167. M_x at $x=0$; $\nu=0$.

Fig. 168. M_y at $x=0$; $\nu=0$;

Notation.

Framework ———

Timoshenko - - - -

Marcus —·—·—

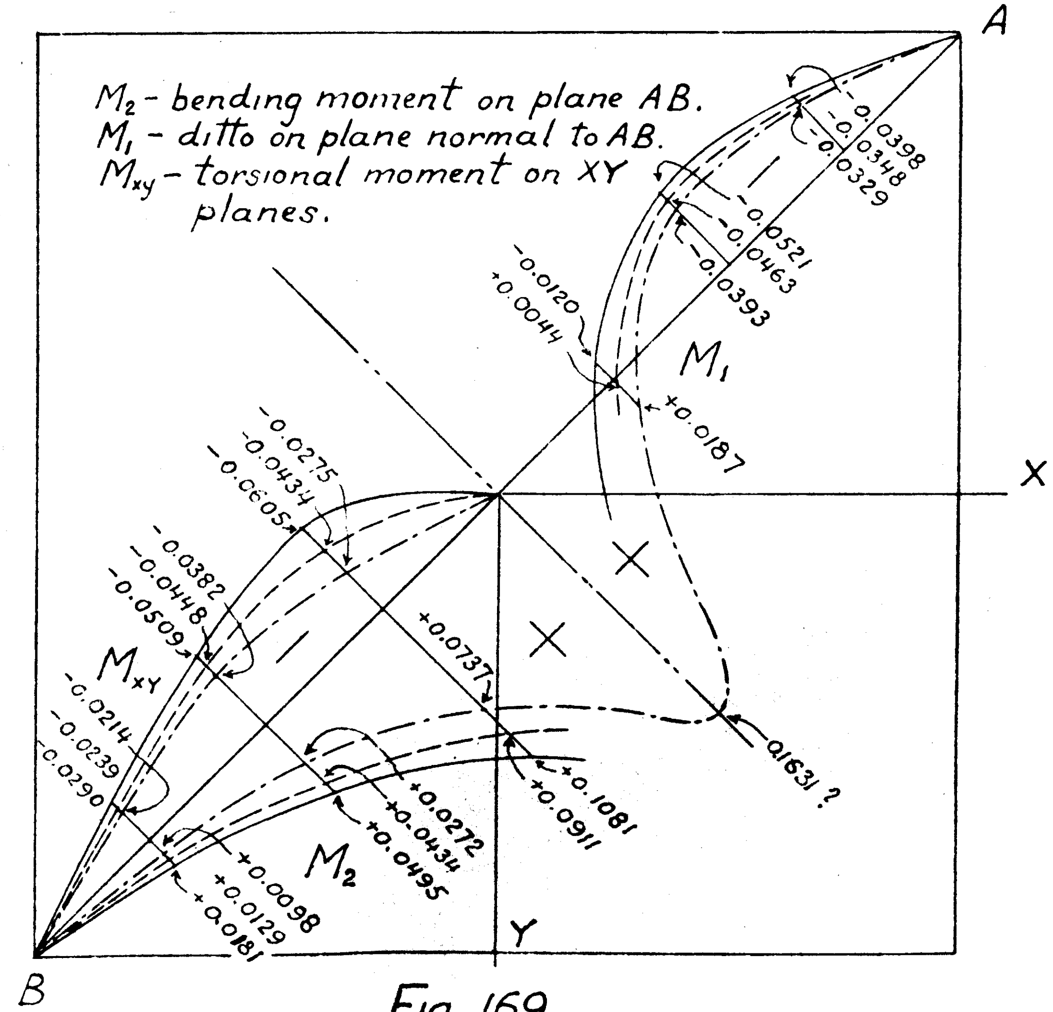


Fig. 169.
Moments at points along diagonal AB; $\nu=0$.

$m = \frac{1}{\nu}$	2.	2.5	3	$3\frac{1}{3}$	4	5	6	7	8	9	10	11	12	15	19	20	∞
f_a	$3/8$ 0.375	$7/24$ 0.2917	$1/4$ 0.25	$13/56$ 0.2321	$5/24$ 0.2083	$3/16$ 0.1875	$7/40$ 0.175	$1/6$ 0.1667	$9/56$ 0.1607	$5/32$ 0.1562	$11/72$ 0.1527	$3/20$ 0.15	$13/88$ 0.1477	$1/7$ 0.1428	$5/36$ 0.1389	$21/152$ 0.1381	$1/8$ 0.125
f_p	$1/8$ 0.125	$1/24$ 0.0417	0	$-1/56$ -0.01785	$-1/24$ -0.0417	$-1/16$ -0.0625	$-3/40$ -0.075	$-1/12$ -0.0833	$-5/56$ -0.0893	$-3/32$ -0.0937	$-7/72$ -0.0972	$-1/10$ -0.1	$-9/88$ -0.1022	$-3/28$ -0.1072	$-1/9$ -0.1111	$-17/152$ -0.1119	$-1/8$ -0.125

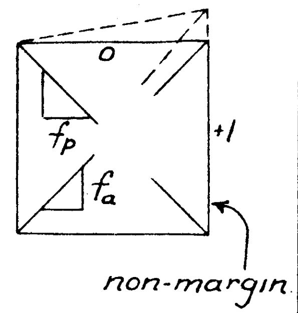
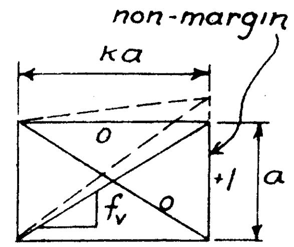


Table 1. Distribution Factors of Square Framework for any ν.

K.		1	1.1	1.2	1.25	$4/3$	$\sqrt{2}$	1.5	$\sqrt{3}$	1.75	2	2.5	3	4	5	10
Vertical Movement	Stress in Vert. Memb.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	Vert Comp. of Diag. f_v	0.25	0.1898	0.1507	0.1355	0.1154	0.1	0.0870	0.0625	0.0611	0.04545	0.02817	0.0192	0.01064	0.00675	0.00167
Horizontal Movement	Stress in Horiz. Memb.	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1
	Horiz. Comp. of Diag. f_h	0.25	0.338	0.461	0.543	0.727	1	1.5	1	-24.5	-2	-0.962	-0.75	-0.615	-0.5685	-0.5155



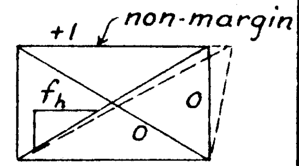
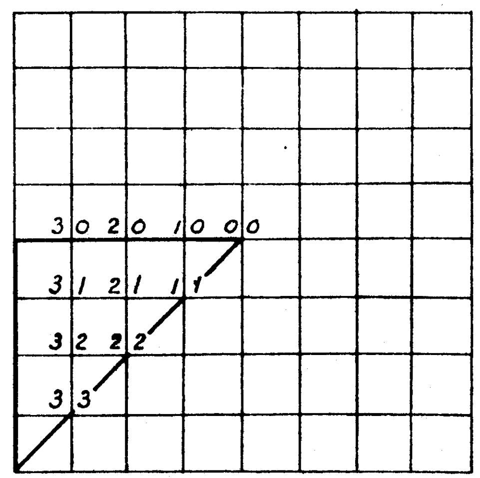


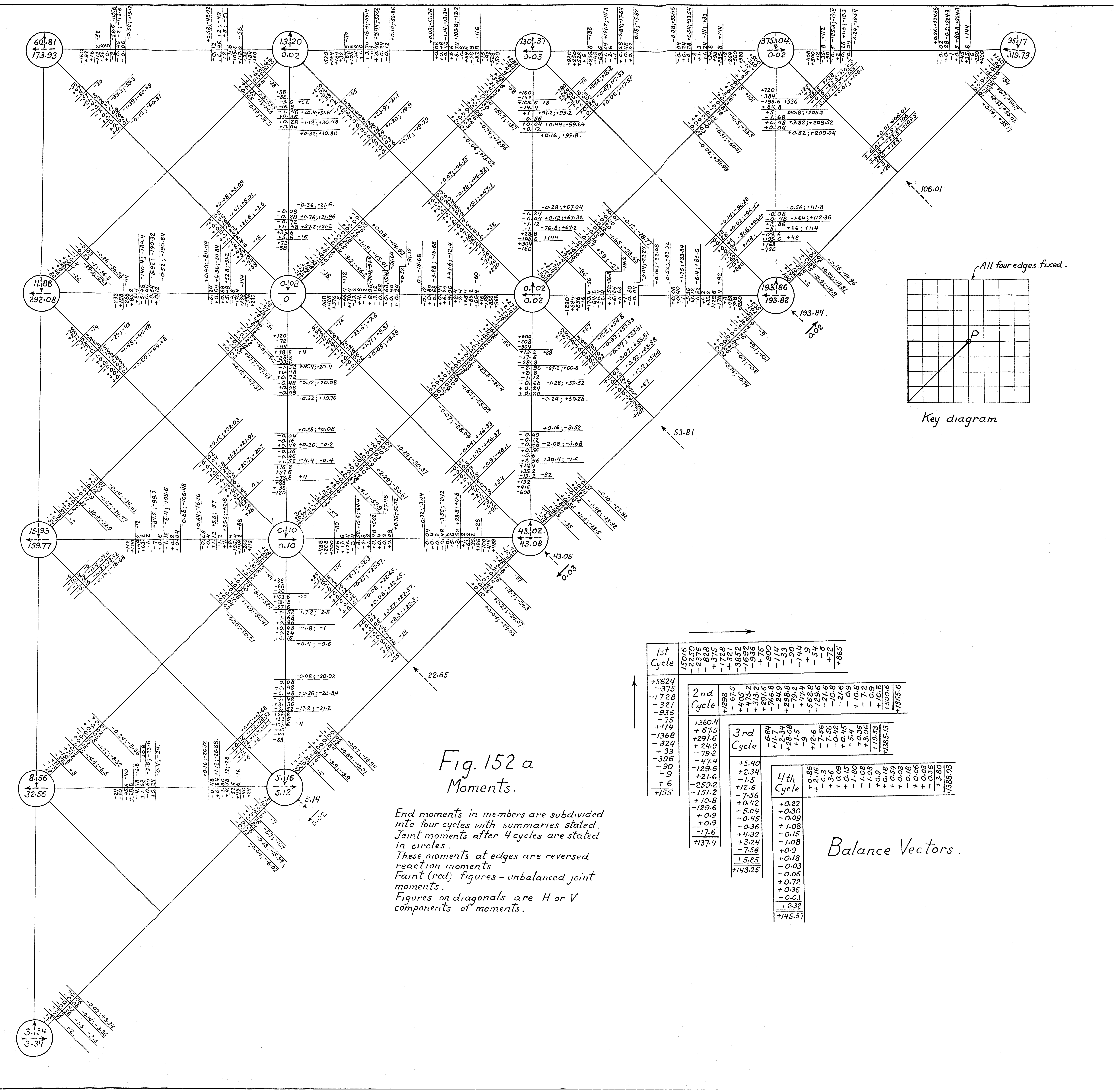
Table 2. Distribution Factors of Oblong Framework, $\nu = \frac{1}{3}$.

Displaced Joint.	Influence Factors at the Joints. 0.0	1.0	2.0	3.0	1.1	2.1	3.1	2.2	3.2	3.3
0.0	-0.470	+0.591	-0.141	+0.026	+0.091	-0.135	+0.030	-0.004	+0.012	+0.001
1.0	+0.591	-1.839	+0.481	-0.125	+1.047	-0.075	-0.080	-0.123	+0.024	+0.012
2.0	-0.141	+0.481	-1.900	+0.643	+0.214	+1.094	+0.254	-0.283	-0.078	+0.032
3.0	+0.026	-0.125	+0.643	-2.068	-0.146	+0.214	+1.095	-0.129	-0.233	+0.047
1.1	+0.091	+1.047	+0.214	-0.146	-2.166	+1.109	-0.305	+0.124	-0.134	-0.007
2.1	-0.135	-0.075	+1.094	+0.214	+1.109	-3.801	+0.966	+1.263	-0.051	-0.143
3.1	+0.030	-0.080	+0.254	+1.095	-0.305	+0.966	-4.414	+0.253	+0.995	-0.285
2.2	-0.004	-0.123	-0.283	-0.129	+0.124	+1.263	+0.253	-1.900	+1.256	+0.126
3.2	+0.012	+0.024	-0.078	-0.233	-0.134	-0.051	+0.995	+1.256	-4.012	+1.134
3.3	+0.001	+0.012	+0.032	+0.047	-0.007	-0.143	-0.285	+0.126	+1.134	-2.204



Numbering of Joints.

Table 3. Influence Factors.
for Square Plate with Clamped Edges.



Key diagram

Fig. 152 a
Moments.

End moments in members are subdivided into four cycles with summaries stated.
Joint moments after 4 cycles are stated in circles.
These moments at edges are reversed reaction moments
Faint (red) figures - unbalanced joint moments.
Figures on diagonals are H or V components of moments.

1st Cycle →	↑
15016	+5624
-2250	-375
-2376	-1728
-828	-321
+375	-936
-1728	-75
+321	+114
-3852	-1368
-1692	-324
-936	+33
+75	-396
-900	-90
-114	-9
-33	+6
-90	+155
-144	
+9	
-54	
-6	
+72	
+865	

2nd Cycle →	↑
+1298	+360.4
-67.5	+67.5
+405	+291.6
-475.2	+24.9
+313.2	-79.2
+291.6	-47.4
-766.8	-129.6
-24.9	+21.6
+298.8	-259.2
-79.2	-151.2
+47.4	+10.8
-568.8	-129.6
-129.6	+0.9
-21.6	+0.9
-10.8	-17.6
-21.6	+137.4
-0.9	
+10.8	
-7.2	
-0.9	
+10.8	
+500.6	
+1365.6	

3rd Cycle →	↑
-6.84	+5.40
+11.7	+2.34
-2.34	-1.5
+28.08	+12.6
+1.5	-7.56
-9	+0.42
+12.6	-5.04
-7.56	-0.45
-7.56	-0.36
-0.42	+4.32
+0.45	+3.24
-5.4	-7.56
+0.36	+5.85
+3.96	+143.25
+19.53	
+1385.13	

4th Cycle →	↑
+0.86	+0.22
+2.16	+0.30
-0.3	-0.09
+3.6	+1.08
+0.09	-0.15
+0.15	-1.08
-1.80	+0.9
-1.08	+0.18
-1.08	-0.03
+0.9	-0.06
+0.18	+0.72
+0.54	+0.36
+0.03	-0.03
-0.18	+2.32
+0.06	+145.57
+0.03	
-0.36	
+3.80	
+1388.93	

Balance Vectors.

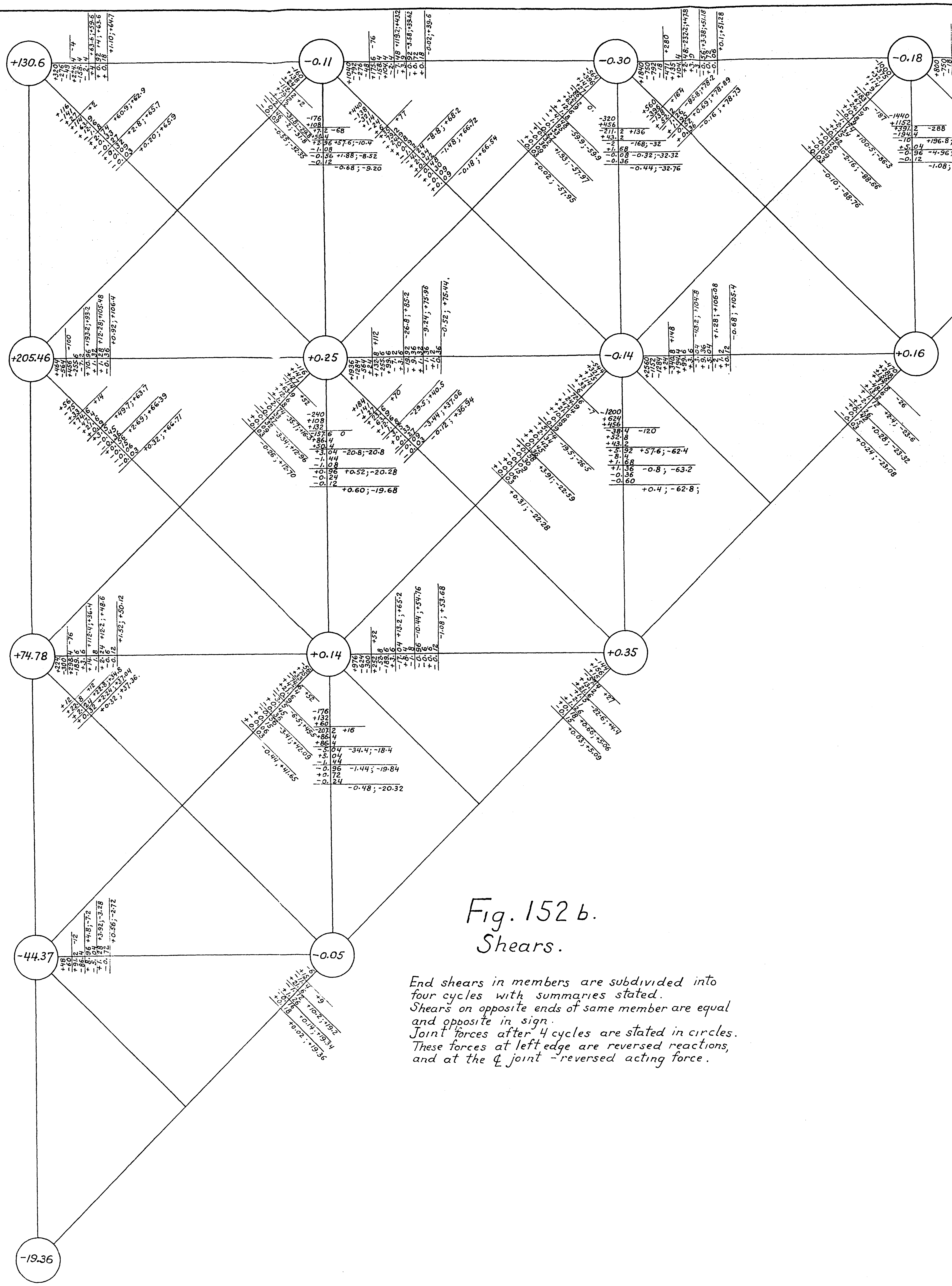


Fig. 152 b.
Shears.

End shears in members are subdivided into four cycles with summaries stated.
Shears on opposite ends of same member are equal and opposite in sign.
Joint forces after 4 cycles are stated in circles.
These forces at left edge are reversed reactions, and at the ℄ joint – reversed acting force.

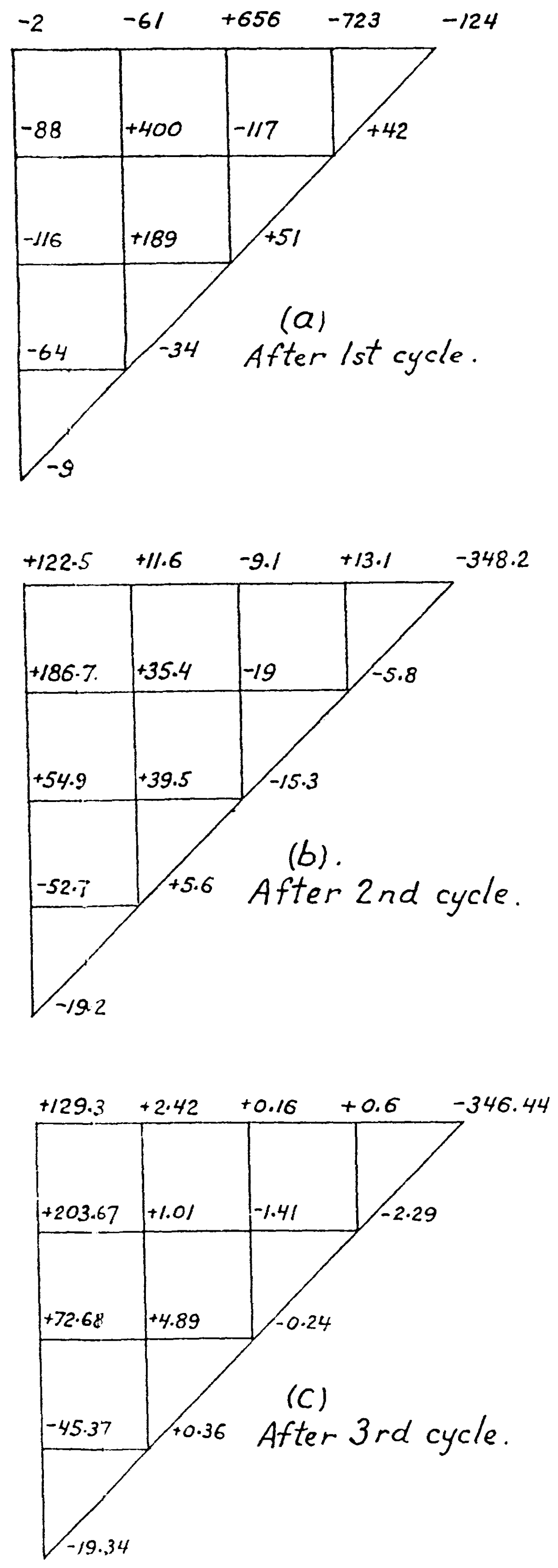


Fig. 153 (a) to (c).
Joint forces.