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Modes of log gravity

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The physical modes of a recently proposed D-dimensional "critical gravity", linearized about its anti-de Sitter vacuum, are investigated. All "log mode" solutions, which we categorize as "spin-2" or "Proca", arise as limits of the massive spin-2 modes of the noncritical theory. The linearized Einstein tensor of a spin-2 log mode is itself a 'nongauge' solution of the linearized Einstein equations whereas the linearized Einstein tensor of a Proca mode takes the form of a linearized general coordinate transformation. Our results suggest the existence of a holographically dual logarithmic conformal field theory.

I. INTRODUCTION

When considered as a theory of interacting massless spin-2 particles in a 4-dimensional Minkowski background, Einstein's theory of gravity is nonrenormalizable. It can be made renormalizable by the addition to the standard Einstein-Hilbert (EH) action of curvature-squared terms, but the price is a loss of unitarity [1]. There are two exceptional cases. First, by adding a Ricci-scalar squared term (with an appropriate sign) one gets a theory equivalent to a scalar coupled to gravity, which is unitary but not renormalizable. Second, by adding a Weyl-tensor squared term one gets a theory that is neither unitary nor renormalizable. Renormalizability requires improved high-energy behavior for both the spin 0 and spin-2 projections of the graviton propagator, and this requires the presence of both Ricci-scalar squared and Weyl-squared terms [1].

The situation is different in three spacetime dimensions in the sense that one gets a unitary theory of gravitons, albeit massive ones, by the addition to the standard EH action of a particular curvature-squared term, obtained by contracting the Einstein tensor with the Schouten tensor; this has been dubbed "new massive gravity" (NMG) [2]. The extension to a "cosmological NMG" theory introduces a new dimensionless parameter λ, and it has been shown that a unitary theory of massive gravitons in an anti-de Sitter background is thus obtained for a certain range of λ [3]. There are similarities here to topologically-massive gravity [4], which involves the addition to the EH term of a Lorentz-Chern-Simons term, and this may also be added to NMG to yield a "general massive gravity" model. However such parity-violating terms have no natural extension to higher dimensions and so will not play a role here.

The properties of cosmological NMG are most easily understood by using a formulation in which the curvature-squared terms, of fourth order in derivatives, are replaced by terms of at most second order by introducing a symmetric tensor auxiliary field.[1] Linearizing about a maximally symmetric background one then finds a quadratic action for the metric perturbation and the auxiliary tensor field. For generic values of λ this action can be diagonalized to produce the sum of a linearized EH term, which propagates no degrees of freedom in three dimensions, and a Fierz-Pauli action for a massive spin-2 mode. The form of the Fierz-Pauli mass term, which is crucial for unitarity, is what requires the original curvature-squared term to be the contraction of the Einstein and Schouten tensors, and for a certain range of values of λ the overall sign of the action is also what is required for unitarity. The same analysis can be carried out in a higher spacetime dimension [6] but then the linearized EH term propagates a massless spin-2 mode, and either it or the massive spin-2 mode (depending on the overall sign of the action) is a ghost. This is why the construction of NMG only yields a unitary theory in three dimensions.

However, there is another feature of NMG, which works for *arbitrary* spacetime dimension. It turns out that for a critical[2] value of λ, at the boundary of the unitarity region, the linearized gravitational field becomes a Lagrange multiplier imposing a constraint on the linearized auxiliary field. This constraint implies (in three dimensions) that the linearized auxiliary field takes the form of a field-dependent general coordinate transformation. This does not mean, however, that this field can be gauged away and, indeed, it corresponds to an additional mode, the so-called logarithmic mode. In higher dimensions the constraint becomes a dynamical equation that allows for a

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[1]This formulation is also useful for other applications, see [5].
[2]A similar critical value was found earlier for cosmological topologically-massive gravity [7].

wider class of solutions, which we analyze here. At the critical point, the massive modes of the noncritical theory coincide with the massless modes, and new logarithmic modes appear to replace them. In three dimensions, these logarithmic modes were discussed in e.g. [8] and their existence led to the conjecture that three-dimensional critical gravity theories are dual to two-dimensional logarithmic CFT's (see e.g. [8–12]).

Logarithmic solutions in the context of the higher-dimensional critical gravity models were recently found in [13,14]. In this paper, we study the logarithmic modes in more detail. We show that they are of two types, which we dub "spin-2" and "Proca" log modes. The number of independent spin-2 log modes is given by the number of polarization states of a *massless* spin-2 field, while the number of independent Proca log modes is given by the number of polarization states of a *massive* spin-1 field. We present explicitly the logarithmic solutions of the linearized $D = 4$ critical gravity. We will show that these log modes have properties similar to those of the three-dimensional log modes that were crucial in conjecturing the logarithmic CFT duals of three-dimensional critical gravities.

II. THE MODEL

We consider the following D-dimensional gravity theory

$$S = \frac{1}{\kappa^2} \int d^D x \sqrt{-g} \left[\sigma R - 2\lambda m^2 \right.$$
$$\left. + \frac{1}{m^2} G^{\mu\nu} S_{\mu\nu} + \frac{1}{m'^2} \mathcal{L}_{\text{GB}} \right], \quad (1)$$

where

$$\mathcal{L}_{\text{GB}} \equiv (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2), \quad (2)$$

which is the Gauss-Bonnet combination. The parameter $\sigma = 0, \pm 1$ is a dimensionless constant, λ is a dimensionless cosmological parameter, and m^2, m'^2 are arbitrary parameters of dimension mass squared that may be positive or negative. Furthermore, $G_{\mu\nu}$ is the Einstein tensor and $S_{\mu\nu}$ is the D-dimensional Schouten tensor

$$S_{\mu\nu} = \frac{1}{D-2} \left(R_{\mu\nu} - \frac{1}{2(D-1)} R g_{\mu\nu} \right). \quad (3)$$

The reason that we have allowed, starting from higher than four dimensions, for the Gauss-Bonnet [15] term $\mathcal{L}_{\text{GB}}$ in (1) is that the linearization of this term around a maximally symmetric background only affects the coefficient of the Einstein term in the quadratic action (see Eq. (12) below) but does not lead to new fourth-order higher-derivative terms.

For $D = 3$ the term $\mathcal{L}_{\text{GB}}$ vanishes identically and the action (1) is that of cosmological NMG [2].[3] For $D = 4$ the

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[3]The special case of $D = 3$, $\lambda = \sigma = 0$ was discussed in [16].

term $\mathcal{L}_{\text{GB}}$ reduces to a total derivative. At this point it is convenient to use the identity [17]

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2 = W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma}$$
$$- 4(D-3)G^{\mu\nu} S_{\mu\nu}, \quad (4)$$

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor. This identity is valid for any $D \geq 3$, although both sides vanish identically for $D = 3$. For $D = 4$ this identity shows that the Einstein tensor times the Schouten tensor equals the square of the Weyl tensor, up to a total derivative, and the action (1), for $\sigma = 1$, reduces to the critical gravity theory considered in [18]. For general $D \geq 5$ and $\sigma = 1$ the same action reduces to the two-parameter family of theories recently considered in [19].

To discuss the quadratic approximation to the action (1) it is convenient to lower the number of derivatives in the action. For the $G^{\mu\nu} S_{\mu\nu}$ term this is achieved by introducing an auxiliary field $f_{\mu\nu}$ that is a symmetric two-tensor [2]. For the Gauss-Bonnet combination a similar trick does not work, at least not with a two-tensor auxiliary field, but it is also not needed here since, as we already mentioned above, this term does not lead to higher-derivative terms in the quadratic action. In terms of $f_{\mu\nu}$ the action (1) reads

$$S = \frac{1}{\kappa^2} \int d^D x \sqrt{-g} \left[\sigma R - 2\lambda m^2 + \frac{1}{(D-2)} f^{\mu\nu} G_{\mu\nu} \right.$$
$$\left. - \frac{m^2}{4(D-2)} (f^{\mu\nu} f_{\mu\nu} - f^2) + \frac{1}{m'^2} \mathcal{L}_{\text{GB}} \right]. \quad (5)$$

Elimination of the auxiliary field leads one back to the original formulation (1).

We now consider the linearization of the theory defined by (5) around a maximally symmetric vacuum with background metric $\bar{g}_{\mu\nu}$ and cosmological constant Λ. For such a background the Ricci tensor, Ricci scalar and Einstein tensor are given by

$$\bar{R}_{\mu\nu} = \frac{2\Lambda}{(D-2)} \bar{g}_{\mu\nu},$$
$$\bar{R} = \frac{2D\Lambda}{(D-2)}, \quad (6)$$
$$\bar{G}_{\mu\nu} = -\Lambda \bar{g}_{\mu\nu}.$$

In general the cosmological constant Λ is not equal to the parameter λ [20]. The two are related by the relation

$$\frac{(D-4)}{(D-1)(D-2)} \left[\frac{1}{2m^2} - \frac{2}{m'^2} (D-3) \right] \Lambda^2 - \Lambda \sigma$$
$$+ \lambda m^2 = 0. \quad (7)$$

This is a quadratic equation in Λ which, for given values of the parameters, has 0, 1, or 2 solutions, except for $D = 4$ and $\sigma \neq 0$ where Λ is uniquely fixed.

We next expand the metric $g_{\mu\nu}$ and the auxiliary field $f_{\mu\nu}$ around their background values:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu},$$

$$f_{\mu\nu} = \frac{2}{m^2(D-1)}[\Lambda(\bar{g}_{\mu\nu} + \kappa h_{\mu\nu}) + \kappa k_{\mu\nu}]. \tag{8}$$

The linearized Ricci tensor is given by

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \kappa R^{(1)}_{\mu\nu} + \kappa^2 R^{(2)}_{\mu\nu} + \mathcal{O}(\kappa^3), \tag{9}$$

where

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}(\Box h_{\mu\nu} - \nabla^\rho \nabla_\mu h_{\rho\nu} - \nabla^\rho \nabla_\nu h_{\rho\mu} + \nabla_\mu \nabla_\nu h), \tag{10}$$

and

$$\bar{g}^{\mu\nu} R^{(2)}_{\mu\nu} = \frac{1}{2} h^{\mu\nu} \left(R^{(1)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}^{\rho\sigma} R^{(1)}_{\rho\sigma} \right)$$

$$+ \text{total derivatives}. \tag{11}$$

Linearizing (5) one finds that the terms linear in $1/\kappa$ cancel as a consequence of (7). The quadratic κ-independent terms lead to the following linearized Lagrangian

$$\mathcal{L}_2 = -\frac{1}{2}\bar{\sigma} h^{\mu\nu} \mathcal{G}_{\mu\nu}(h) + \frac{2}{m^2(D-1)(D-2)} k^{\mu\nu} \mathcal{G}_{\mu\nu}(h)$$

$$- \frac{1}{m^2(D-2)(D-1)^2}(k^{\mu\nu} k_{\mu\nu} - k^2), \tag{12}$$

where

$$\bar{\sigma}(\Lambda) \equiv \sigma - \frac{\Lambda}{m^2}\frac{1}{D-1} + 4\frac{\Lambda}{m'^2}\frac{(D-3)(D-4)}{(D-1)(D-2)} \tag{13}$$

and where we have defined the Einstein operator

$$\mathcal{G}_{\mu\nu}(h) = R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}R^{(1)}_{\rho\sigma} - \frac{2\Lambda}{(D-2)}h_{\mu\nu}$$

$$+ \frac{\Lambda}{(D-2)}\bar{g}_{\mu\nu}h. \tag{14}$$

The linearized Lagrangian (12) is invariant under the linearized diffeomorphisms

$$\delta h_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu. \tag{15}$$

This may be verified using the relation

$$[\nabla_\mu, \nabla_\nu]V_\rho = \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}_{\mu\rho}V_\nu - \bar{g}_{\nu\rho}V_\mu) \tag{16}$$

for any vector V_μ. The expansion of $f_{\mu\nu}$ in (8) is defined such that $k_{\mu\nu}$ is gauge invariant. Note also that the Einstein operator (14) is gauge invariant. For $D = 3$ the above result agrees with the one given in [3].

For general values of the parameters the first term in (12) corresponds to a linearized Einstein-Hilbert term, the second term provides a coupling between the k- and h-fluctuation, while the last term provides a Fierz-Pauli mass term for the k-fluctuation. After a diagonalization of the second term, one deduces that the theory describes one massless graviton, described by the linearized Einstein term and one massive graviton, described by a Fierz-Pauli Lagrangian [21] with mass given by

$$M^2 = -m^2(D-2)\bar{\sigma}. \tag{17}$$

The kinetic terms of the massless and massive gravitons have opposite signs and therefore the theory is plagued with ghosts.

Following [18,19] we now observe that at the critical point defined by the following special value of the cosmological constant

$$\bar{\sigma}(\Lambda_{\text{crit}}) = 0 \tag{18}$$

the first term in (12), i.e. the linearized Einstein-Hilbert term, drops out. The resulting critical Lagrangian is given by

$$m^2(D-1)(D-2)\mathcal{L}_{\text{crit}} = 2h^{\mu\nu}\mathcal{G}_{\mu\nu}(k) - \frac{1}{(D-1)}$$

$$\times (k^{\mu\nu}k_{\mu\nu} - k^2). \tag{19}$$

The field equation for $h_{\mu\nu}$ is therefore

$$\mathcal{G}_{\mu\nu}(k) = 0, \tag{20}$$

while the k-equation of motion reads

$$\mathcal{G}_{\mu\nu}(h) - \frac{1}{(D-1)}(k_{\mu\nu} - \bar{g}_{\mu\nu}k) = 0. \tag{21}$$

By acting on (21) with ∇^μ and using the Bianchi identity $\nabla^\mu \mathcal{G}_{\mu\nu}(h) = 0$, we find

$$\nabla^\mu k_{\mu\nu} - \nabla_\nu k = 0. \tag{22}$$

Next, by taking the trace of (20) and using (22) one finds that $\Lambda_{\text{crit}} k = 0$ and hence that

$$k = 0, \tag{23}$$

provided that $\Lambda_{\text{crit}} \neq 0$, which we will assume to be the case from now on. Substituting $k = 0$ into Eq. (21) it follows that

$$k_{\mu\nu} = (D-1)\mathcal{G}_{\mu\nu}(h). \tag{24}$$

Finally, substituting (24) into (20), one finds that h obeys the following fourth-order equation

$$\mathcal{G}_{\mu\nu}(\mathcal{G}(h)) = 0, \tag{25}$$

together with the constraint

$$\bar{g}^{\mu\nu}\mathcal{G}_{\mu\nu}(h) = 0. \tag{26}$$

The equations of motion (25) which state that the square of the Einstein operator annihilates $h_{\mu\nu}$, can be further simplified by imposing the gauge condition

$$\nabla^\mu h_{\mu\nu} - \nabla_\nu h = 0. \tag{27}$$

Substituting this gauge condition into the constraint (26) one finds that

$$\Lambda_{\text{crit}} h = 0. \qquad (28)$$

Since we already assumed that $\Lambda_{\text{crit}} \neq 0$ we deduce that $h = 0$ and hence we find that

$$\nabla^\mu h_{\mu\nu} = h = 0. \qquad (29)$$

Using this, the linearized Einstein tensor reduces to

$$\mathcal{G}_{\mu\nu}(h) = -\frac{1}{2}\left(\Box - \frac{4\Lambda}{(D-1)(D-2)}\right)h_{\mu\nu}. \qquad (30)$$

It follows that the fourth-order operator appearing in the equation of motion (25) factorizes into two identical second-order operators [18,19]

$$\left(\Box - \frac{4\Lambda}{(D-1)(D-2)}\right)^2 h_{\mu\nu} = 0. \qquad (31)$$

III. MASSIVE, MASSLESS AND LOG MODES

We wish to analyze the solutions to the equations of motion (31) assuming that we have an AdS_D vacuum solution with $\Lambda_{\text{crit}} < 0$ and isometry algebra $SO(2, D-1)$. The case of $D = 4$ is of particular interest because of the improved short-distance behavior of curvature-squared theories in this dimension, as discussed in the introduction. We therefore focus on this case, for which we infer from (13) that

$$\Lambda_{\text{crit}}(D = 4) = 3m^2\sigma. \qquad (32)$$

The generalization to $D > 4$ will be apparent while the $D = 3$ case has been discussed in [3]. Since away from the critical point there are both massless and massive gravitons it is convenient to consider both types of solutions since this will facilitate the construction of the so-called log solutions at the critical point. To determine the massive solutions we follow the presentation of [7]. Next, the massless modes are obtained by taking the massless limit of the massive ones and the log modes, which are valid solutions at the critical point only, are obtained by applying the limiting procedure of [8]. These log modes are solutions to the equations of motion (31) that are not annihilated by the separate second-order Einstein operators.

In general, we expect for $D > 3$ three classes of solutions at the critical point.

(1) The first class of solutions are the massless gravitons, which correspond to solutions of the homogeneous equation

$$k_{\mu\nu} = 3\mathcal{G}_{\mu\nu}(h) = 0. \qquad (33)$$

(2) The second class of solutions will be called Proca log modes and solve the inhomogeneous equation

$$k_{\mu\nu} = 3\mathcal{G}_{\mu\nu}(h) = 2\nabla_{(\mu}A_{\nu)}, \qquad (34)$$

for some vector field A_μ. Written in terms of $k_{\mu\nu}$ they are solutions of the massless Einstein equations $\mathcal{G}_{\mu\nu}(k) = 0$ that take the form of a field-dependent general coordinate transformation.[4] By substituting (34) into (22), one finds that A_μ satisfies the equations of motion that follow from the following Proca Lagrangian [3]:

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4m^2}F^{\mu\nu}F_{\mu\nu} + 3\sigma A^\mu A_\mu, \qquad (35)$$
$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]},$$

which is why we dubbed the corresponding modes Proca modes.

(3) The third class of solutions will be denoted as spin-2 log modes and correspond to solutions of the inhomogeneous equation

$$\mathcal{G}_{\mu\nu}(h) = k_{\mu\nu}^\perp, \qquad k_{\mu\nu}^\perp \neq \nabla_{(\mu}A_{\nu)}. \qquad (36)$$

In terms of $k_{\mu\nu}^\perp$ they correspond to nontrivial solutions of the massless Einstein equations $\mathcal{G}_{\mu\nu}(k^\perp) = 0$. Strictly speaking, Eq. (36) defines an equivalence class of solutions since to every spin-2 log mode one can add a Proca mode.

We now study the solutions of the linearized equations of motion *away from the critical point*, following the group theoretical approach of [7]. Our starting point is the AdS_4 metric which in global coordinates $(\tau, \rho, \theta, \phi)$ is given by:

$$\mathrm{d}s^2 = L^2(-\mathrm{d}\tau^2\cosh(\rho)^2 + \mathrm{d}\rho^2 + \sinh(\rho)^2(\mathrm{d}\theta^2 + \mathrm{d}\phi^2\sin(\theta)^2)). \qquad (37)$$

Here L is related to the cosmological constant Λ by

$$\Lambda = -\frac{3}{L^2}. \qquad (38)$$

The isometry group of AdS_4 is given by $SO(2, 3)$ which is generated by 10 Killing vectors. These Killing vectors can be grouped into Cartan generators and positive and negative root generators of $SO(2, 3)$. The Killing vectors corresponding to the two Cartan generators are given by

$$H_1 = i\partial_\tau, \qquad H_2 = -i\partial_\phi. \qquad (39)$$

The Killing vectors corresponding to the four positive roots will be taken to be

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[4]This does not mean that $k_{\mu\nu}$ can be gauged away since $k_{\mu\nu}$ is gauge invariant.

$$E^{\alpha_1} = \frac{1}{2}e^{i(\tau+\phi)}\sin(\theta)\tanh(\rho)\partial_\tau - \frac{1}{2}ie^{i(\tau+\phi)}\sin(\theta)\partial_\rho$$
$$- \frac{1}{2}ie^{i(\tau+\phi)}\cos(\theta)\coth(\rho)\partial_\theta + \frac{1}{2}e^{i(\tau+\phi)}$$
$$\times \coth(\rho)\csc(\theta)\partial_\phi,$$

$$E^{\alpha_2} = -ie^{i\phi}\partial_\theta + e^{i\phi}\cot(\theta)\partial_\phi,$$

$$E^{\alpha_3} = \frac{1}{2}e^{i(\tau-\phi)}\sin(\theta)\tanh(\rho)\partial_\tau - \frac{1}{2}ie^{i(\tau-\phi)}\sin(\theta)\partial_\rho,$$
$$- \frac{1}{2}ie^{i(\tau-\phi)}\cos(\theta)\coth(\rho)\partial_\theta - \frac{1}{2}e^{i(\tau-\phi)}$$
$$\times \coth(\rho)\csc(\theta)\partial_\phi,$$

$$E^{\alpha_4} = e^{i\tau}\cos(\theta)\tanh(\rho)\partial_\tau - ie^{i\tau}\cos(\theta)\partial_\rho$$
$$+ ie^{i\tau}\coth(\rho)\sin(\theta)\partial_\theta. \tag{40}$$

The Killing vectors corresponding to the four negative roots are proportional to the complex conjugates of the above four Killing vectors:

$$E^{-\alpha_1} = (E^{\alpha_1})^*, \qquad E^{-\alpha_2} = -(E^{\alpha_2})^*,$$
$$E^{-\alpha_3} = (E^{\alpha_3})^*, \qquad E^{-\alpha_4} = (E^{\alpha_4})^*. \tag{41}$$

The root vectors corresponding to the above positive roots are given by

$$\alpha_1 = (-1, 1), \qquad \alpha_2 = (0, 1),$$
$$\alpha_3 = (-1, -1), \qquad \alpha_4 = (-1, 0). \tag{42}$$

The above Killing vectors are normalized in the Cartan-Weyl fashion, i.e. the following commutation relations hold

$$[H_i, H_j] = 0, \qquad i = 1, 2,$$
$$[H_i, E^{\alpha_x}] = \alpha_x^i E^{\alpha_x}, \qquad x = 1, \cdots, 4, \tag{43}$$
$$[E^{\alpha_x}, E^{-\alpha_x}] = \frac{2}{|\alpha_x|^2}\alpha_x \cdot H,$$

where

$$|\alpha_x|^2 = \sum_{i=1}^{2}(\alpha_x^i)^2. \tag{44}$$

The Casimir operator $\mathcal{C}$ can then be constructed as follows

$$\mathcal{C} = \sum_{i=1}^{2}H_iH_i + \sum_{x=1}^{4}\frac{|\alpha_x|^2}{2}(E^{\alpha_x}E^{-\alpha_x} + E^{-\alpha_x}E^{\alpha_x}). \tag{45}$$

When acting on a scalar field $S(\tau, \rho, \theta, \phi)$ the Casimir operator is given by

$$\mathcal{C}S = L^2\nabla^2 S. \tag{46}$$

Similarly, when acting on a metric perturbation $h_{\mu\nu}$ the Casimir operator is given by

$$(\mathcal{C} - 8)h_{\mu\nu} = L^2\nabla^2 h_{\mu\nu}. \tag{47}$$

We now consider the $D = 4$ linearized equations of motion away from the critical point

$$\left(\nabla^2 + \frac{2}{L^2} - M^2\right)\left(\nabla^2 + \frac{2}{L^2}\right)h_{\mu\nu} = 0, \tag{48}$$

where $M^2 = -m^2\bar{\sigma}$ is the mass of the graviton, see Eq. (17). These equations can be rewritten in terms of the Casimir operator as follows:

$$(\mathcal{C} - 6 - L^2 M^2)(\mathcal{C} - 6)h_{\mu\nu} = 0. \tag{49}$$

We now look for a metric perturbation $\psi_{\mu\nu}$ that forms a highest weight state, with $M^2 \neq 0$, of the SO(2, 3) isometry algebra. This state is an eigenstate of H_1 and H_2 (acting as Lie derivatives) with eigenvalues E_0 and s

$$H_1\psi_{\mu\nu} = E_0\psi_{\mu\nu}, \qquad H_2\psi_{\mu\nu} = s\psi_{\mu\nu}, \tag{50}$$

while it is annihilated by all positive roots E^{α_x}

$$E^{\alpha_x}\psi_{\mu\nu} = 0, \qquad x = 1, \cdots, 4. \tag{51}$$

Using the conditions (29), i.e.

$$\nabla^\mu\psi_{\mu\nu} = 0, \qquad \bar{g}^{\mu\nu}\psi_{\mu\nu} = 0, \tag{52}$$

we find that a solution for the highest weight state can be found for

$$s = 2. \tag{53}$$

The explicit expression of the solution reads[5]

$$\psi_{\tau\tau} = -\psi_{\tau\phi} = \psi_{\phi\phi}$$
$$= e^{-iE_0\tau + 2i\phi}\sin(\theta)^2\sinh(2\rho)^{(2-E_0)/2}\tanh(\rho)^{1+(E_0/2)},$$
$$\psi_{\tau\rho} = -\psi_{\rho\phi} = i\,\mathrm{csch}(\rho)\mathrm{sech}(\rho)\psi_{\tau\tau},$$
$$\psi_{\tau\theta} = -\psi_{\theta\phi} = i\cot(\theta)\psi_{\tau\tau}$$
$$\psi_{\rho\rho} = -4\,\mathrm{csch}(2\rho)^2\psi_{\tau\tau}, \tag{54}$$
$$\psi_{\rho\theta} = -2\cot(\theta)\mathrm{csch}(2\rho)\psi_{\tau\tau},$$
$$\psi_{\theta\theta} = -\cot(\theta)^2\psi_{\tau\tau}.$$

Using that on a highest weight state

$$\mathcal{C}\psi_{\mu\nu} = (E_0(E_0 - 3) + s(s + 1))\psi_{\mu\nu}, \tag{55}$$

we find from the equation of motion (49) that E_0 has to obey

$$(E_0(E_0 - 3) - L^2 M^2)E_0(E_0 - 3) = 0. \tag{56}$$

The descendant states of (54) can be obtained by acting with Killing vectors corresponding to the negative roots. There is an infinite number of descendant states, but they can be organized in representations of SO(3). Indeed, the negative root $E^{-\alpha_2}$ only lowers the s-eigenvalue, while it leaves E_0 untouched. H_2, E^{α_2} and $E^{-\alpha_2}$ thus form the

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[5]Similar anti-de Sitter wave solutions for the full nonlinear theory have recently been considered in [13].

algebra of the compact SO(3) subgroup of SO(2, 3) and the descendant states organize themselves in representations of this SO(3) subgroup. By acting with $E^{-\alpha_2}$ on (54) one thus obtains five solutions of the equations of motion (49), that form a spin-2 SO(3) multiplet, with $s = +2, +1, 0, -1$ and -2, respectively. In principle we can now determine all descendant solutions. In practice, it is often enough to restrict to the highest weight state and the above SO(3) descendants. This finishes our discussion of the massive solutions.

The massless solutions are obtained by taking the limit $M \rightarrow 0$ of the massive ones. The resulting massless solutions solve the equations

$$\left(\nabla^2 + \frac{2}{L^2}\right)h_{\mu\nu} = 0. \qquad (57)$$

In the massless limit we must have $E_0 = 0$ or $E_0 = 3$. From the first line in (54) we see that for $E_0 = 0$ the solution blows up for $\rho \rightarrow +\infty$, while for $E_0 = 3$ the solution is well-behaved in this limit. In the following we will mainly concentrate on solutions that fall off to zero in the $\rho \rightarrow +\infty$ limit. In the massless limit, with $E_0 = 3$, the five massive solutions, with $s = +2, \cdots, s = -2$, all become nonzero solutions of the Einstein Eq. (57). This happens for the $s = +2$ and $s = -2$ solutions, in particular, but also for the $s = +1, 0$ and -1 solutions. As the massless Einstein equations describe only two helicity-2 modes, it is to be expected that only two linear combinations of the above modes correspond to physical modes, belonging to the first class of solutions at the critical point described in (33). Three other linear combinations are then expected to correspond to infinitesimal general coordinate transformations.

Having discussed the massive and massless modes we now consider the log modes. As in the three-dimensional case, one expects logarithmic modes to show up that are solutions of the fourth-order equation of motion, but that do not solve (57). Applying the limiting procedure of [8] on the highest weight state, we find the following logarithmic mode:[6]

$$\psi_{\mu\nu}^{\log}(s = 2) = f(\tau, \rho)\psi_{\mu\nu}^{(2)}(E_0 = 3), \qquad (58)$$

with

$$f(\tau, \rho) = \frac{1}{2}(-2i\tau - \log(\sinh(2\rho)) + \log(\tanh(\rho))), \tag{59}$$

and where $\psi_{\mu\nu}^{(2)}(E_0 = 3)$ denotes the $s = 2$ solution (54) taken at the massless point $E_0 = 3$. One can check that (58) is traceless and has zero divergence. The Einstein tensor of this log mode can thus be calculated via (30) and we find that it is given by

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[6]An alternative expression for such a log mode has recently been given in [14].

$$\mathcal{G}_{\mu\nu}(\psi^{\log}(s = 2)) = -\frac{3}{2L^2}\psi_{\mu\nu}^{(2)}(E_0 = 3). \qquad (60)$$

The above features of the $s = 2$ log state persist for all five spin-2 states. In all cases the log mode solution is given by

$$\psi_{\mu\nu}^{\log}(s) = f(\tau, \rho)\psi_{\mu\nu}^{(s)}(E_0 = 3), \qquad (61)$$

with $f(\tau, \rho)$ given by (59) and where $\psi_{\mu\nu}^{(s)}(E_0 = 3)$ is the helicity s solution of the massless Einstein Eq. (57). These five log modes form a 5-plet under SO(3) and are related to each other by the raising and lowering operators of SO(3). We have checked that in all cases the Einstein tensor of $\psi_{\mu\nu}^{\log}(s)$ is proportional to the helicity s solution of the massless Einstein equation:

$$\mathcal{G}_{\mu\nu}(\psi^{\log}(s)) = -\frac{3}{2L^2}\psi_{\mu\nu}^{(s)}(E_0 = 3). \qquad (62)$$

One thus expects that linear combinations of the log modes can be divided in two classes. Two linear combinations are such that their Einstein tensor gives rise to nontrivial solutions of the Einstein equations. These are the so-called spin-2 log modes that belong to the third class of solutions, defined in (36). Three linear combinations are then expected to have an Einstein tensor that takes the form of an infinitesimal general coordinate transformation. These three log modes are therefore Proca log modes and belong to the second class of solutions described in (34).

Finally, we mention some properties of the logarithmic modes that we have found. The mode $\psi_{\mu\nu}^{\log}(s = 2)$ is annihilated by all four positive root generators:

$$E^{\alpha_x}\psi_{\mu\nu}^{\log}(s = 2) = 0, \qquad x = 1, \cdots, 4. \qquad (63)$$

The other log modes in the 5-plet are obtained by acting with the SO(3) lowering operator $E^{-\alpha_2}$. All log modes correspond to eigenstates of H_2

$$H_2\psi_{\mu\nu}^{\log}(s) = s\psi_{\mu\nu}^{\log}(s), \qquad (64)$$

but they do not correspond to eigenstates of H_1

$$H_1\psi_{\mu\nu}^{\log}(s) = 3\psi_{\mu\nu}^{\log}(s) + \psi_{\mu\nu}^{(s)}(E_0 = 3). \qquad (65)$$

This structure is reminiscent of the three-dimensional case [8–11]. In that case, the analogue of the properties (64) and (65) led to the conjecture that the dual CFT is a logarithmic one.

IV. CONCLUSIONS

We have studied the recently proposed D-dimensional critical gravity theories of [18,19]. The family of models considered contains, besides a dimensionless cosmological parameter λ and two mass parameters m^2, m'^2, an additional dimensionless parameter $\sigma = 0, \pm 1$ multiplying the Einstein-Hilbert term. After linearization about a maximally symmetric background, σ is replaced by an effective

EH coefficient $\bar{\sigma}$. The critical theory is defined by $\bar{\sigma} = 0$, which condition determines the cosmological constant Λ, which we assumed to be negative. The quadratic critical Lagrangian (19) depends only on the mass parameter m^2. This allows for different choices of σ. In particular, one could take a "wrong sign" Einstein-Hilbert term in the starting action or even no Einstein-Hilbert term at all.[7]

At the critical point, the linearized equation of motion is essentially given by the Einstein tensor of the Einstein tensor of the metric perturbation; in others words, one acts twice on the perturbation with the "Einstein operator" (defined by linearization of the Einstein tensor). Any solution of the linearized Einstein equations is therefore a solution, and these are the massless spin-2 modes. In addition, there are logarithmic solutions that are not annihilated by a single action of the Einstein operator. We subdivided these logarithmic solutions into two classes: the spin-2 and Proca modes. For $D = 4$, we used the SO(2, 3) isometry group of AdS$_4$ to explicitly calculate the massive and massless modes away from the critical point. We have shown how, at the critical point, the massive modes are replaced by spin-2 and Proca log modes.

So far, our analysis has been done without a careful consideration of the boundary conditions. As an example of how important boundary conditions can be, it is interesting to consider, for $D = 3$, the relation between the Proca modes, propagated by the Lagrangian (35) and the Proca log modes. This relationship is not one-to-one. It turns out that the highest weight state of the Proca log mode corresponds to a non-normalizable solution of the equations of motion that follow from (35). Its descendants, however, do give rise to normalizable solutions.[8]

The boundary conditions, when logarithmic modes are included, have been well-studied for the special case of

$D = 3$: it has been established [8] that the logarithmic bulk modes require weaker boundary conditions than the Brown-Henneaux ones [25]. These weaker boundary conditions were dubbed "logarithmic boundary conditions" and they play an essential role in the search for the two-dimensional CFT duals of the various three-dimensional massive gravities. The logarithmic modes for $D = 4$ critical gravity studied here exhibit an analogous group theoretical structure. In the $D = 3$ case, the existence and structure of these logarithmic modes lends support for the conjecture that the CFT dual of three-dimensional critical gravity theories is of the logarithmic type (see e.g. [9–11]). It would be of interest to see whether one could similarly define a consistent set of logarithmic boundary conditions in the higher-dimensional case and, if so, to see what one could say about the CFT duals of critical gravities in arbitrary dimensions.

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Note added:—Following submission of the original version of this paper to the arxiv, a revised version of Ref. [18] appeared in which the log modes of 4D critical gravity presented here were found to have positive energy (the massless Einstein modes have zero energy). Although this is encouraging, it appears likely that the log modes are not orthogonal to the Einstein modes, which would imply the existence of linear combinations of negative norm, as happens in critical topologically-massive gravity (see, e.g., sec. 4.1.2 of [26]).[9] This would imply nonunitarity, as is to be expected from the nonunitarity of the dual logarithmic CFT.

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[7] Note, however, that a pure Weyl-squared term is not possible, in view of (13).

[8] In the context of topologically-massive gravity such a descendant mode has been described in [22]. Different formulations at the linearized level also occur in the case of topologically-massive gravity, see e.g. the discussion in [23,24].

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[9] We thank Massimo Porrati for pointing this out to us.

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