DETERMINANTS OF ELLIPTIC OPERATORS

by

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ABSTRACT

Let A be a classical elliptic pseudodifferential operator of positive order on a closed manifold. We investigate some properties of $\det A$. The function det is not multiplicative on the algebra of elliptic operators. We evaluate the ratio $\det AB / \det A \cdot \det B$ if operators A and B are self-adjoint. If the inverse operator A^{-1} belongs to the trace class one can consider the Fredholm determinant $\det(I + \epsilon A^{-1})$. We derive the asymptotics of this Fredholm determinant when $\epsilon \to \infty$ and show that $\det A$ appears in this asymptotics as the coefficient against ϵ^0. Finally, we evaluate the determinant of Sturm–Liouville operator on a segment with Dirichlet's boundary conditions.

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This work has been done at MIT, and I was happy to work in such a great institution. But my first acknowledgements are to the people from a very different place, Moscow. My advisor and friend Professor M.S.Agranovich taught me many wonderful things in mathematics. Before my departure from the USSR we had spent hours discussing microlocal analysis and spectral theory. Even now when we live into almost opposite points of the Earth (if not geometrically, then politically) his letters are very helpful for me. I have always felt his support. This support was especially valuable during difficult and dangerous years.

I participated in Professor M.A.Shubin's seminar for many years. All my previous works were reported at this seminar. Professor Shubin is a man of great knowledge and erudition. It is not easy to talk at his seminar. Sometimes he can ask a question which seems to be trivial, if not foolish. But after a minute you understand that in fact this question is deep and essential.

I learned about determinants of elliptic operators in a very strange place. During a mountain trip, at a Pamir's glacier, Dr. P.Wiegmann told me about his paper with Polyakov on the determinant of a two–dimensional Dirac operator. We had several discussions later in Moscow, and I learned the variational technique from him.

I am very grateful to my thesis advisor Professor V.Guillemin. He gave me the opportunity to finish this work. Our discussions were extremely useful for me.

My last (but not the least) acknowledgements are to my wife Elena, my parents, Mura and Felix Friedlander, and to all my friends.

Chapter 1

Introduction

Let A be a classical elliptic pseudo-differential operator of order $n > 0$ which acts on sections of a vector bundle E over a compact manifold M of dimension d. Suppose that there exists a ray of minimal growth for A i.e. for some α and ϵ eigenvalues of $a_n(x, \xi)$ do not lie into the angle $\mid \arg s - \alpha \mid < \epsilon$, where a_n is the principal symbol of A. Then the spectrum of A is discreet and one can consider the ζ- function of A

$$\zeta_A(s) = \sum \lambda_j^{-s} \tag{1.1}$$

where λ_j are eigenvalues of A and every eigenvalue appears in (1.1) according to its multiplicity. For definition of λ_j^{-s} we cut the complex s- plane along the ray $\arg s = \alpha$ (or along a close ray if $\arg \lambda_j = \alpha$ for a finite number of λ_j). The series (1) is absolutely convergent if Re$s > d/n$ so the function $\zeta_A(s)$ is holomorphic in this half-plane. R.T.Seeley proved in [1] that $\zeta_A(s)$ admits analytical continuation to the meromorphic function on the complex plane with poles into the points $(d-k)/n, k = 1, 2, \ldots$. Non-positive integers are in fact regular points of $\zeta_A(s)$. Particularly the point $s = 0$ is regular. The residues of the ζ–function and its value in the point 0 one can evaluate if he knows the complete symbol of the operator. If the operator is differential it is possible to evaluate ζ–function in the negative integers [1–3].

The analogous results are valid for elliptic differential boundary value problems [4] and for elliptic pseudodifferential boundary value problems which satisfy the transmission property [5].

Ray and Singer defined the determinant of the operator A by the formula

$$\log \det A = -\zeta_A'(0). \tag{1.2}$$

This definition agrees with the usual definition of the finite-dimensional determinant.

Recently interest in determinants of elliptic operators has grown due to the applications to string theory. In string theory there arises the problem of the Feynmann integrals' evaluation

$$\int \exp\{-\epsilon(A\phi,\phi)\}\mathcal{D}\phi\mathcal{D}\overline{\phi} \tag{1.3}$$

where A is the Laplacian acting on k–forms or A is the Dirac operator. This integral equals $\det(\epsilon A)$ by definition.

In the case $n > d$ the inverse operator A^{-1} belongs to the trace class and there exists the Gaussian measure $d\mu[0,A^{-1}]$ with the average 0 and the correlation operator A^{-1}. It is known [6] that

$$\int e^{-\epsilon(x,x)}d\mu[0,A^{-1}] = \det(I+\epsilon A^{-1}). \tag{1.4}$$

In the right hand side of (1.4) the Fredholm determinant [7] stands. We shall prove (under a little bit more restrictive assumptions on A) that

$$\log\det(I+\epsilon A^{-1}) \sim \sum_{k=-d}^{\infty} p_k\epsilon^{-k/n} + \sum_{j=0}^{\infty} q_j\epsilon^{-j}\log\epsilon,\ \epsilon\to\infty, \tag{1.5}$$

and

$$p_0 = -\log\det A \tag{1.6}$$

This expansion justifies in some sense the definition of the integral (1.3): this integral is inverse to (1.4) after canceling the "divergent terms". Heuristically the integral (1.3) reduces to (1.4) by substitution $\phi = A^{-1/2}\psi$.

The determinant of elliptic operator does not satisfy the usual multiplicative property of finite-dimensional determinants [8]. There arises the problem of evaluation of the ratio

$$\frac{\det(AB)}{\det A\cdot\det B}. \tag{1.7}$$

We shall show that (under some assumptions on A and B) this ratio is a local invariant and it can be calculated if one knows $d+1$ terms in the asymptotic expansions of symbols of A and B. To compute the ratio (1.7) we use the technique of non–commutative residue [9].

The plan of our exposition is as follows. Chapter 2 contains some preliminary facts about determinants and non–commutative residue. In the chapter 3 we prove the asymptotics (1.5). Chapter 4 is about determinants of the product. In the chapter 5 we calculate the determinant of the Sturm–Liouville operator with the Dirichlet's conditions on the segment. This chapter follows section 1 from our paper [11]. The same result was proved by M.Wodzicki [10].

Chapter 2

Preliminaries

1. Non–commutative residue and holomorphic families of pseudo–differential operators

Let A be a classical pseudo–differential operator of order n which acts on sections of some vector bundle E over a closed manifold M, $\dim M = d$. In local coordinates the symbol of A admits the asymptotic expansion

$$a(x,\xi) \sim a_n(x,\xi) + a_{n-1}(x,\xi) + \cdots \tag{2.1}$$

where the matrices $a_\nu(x,\xi)$ are homogeneous of degree ν with respect to ξ. This means that

$$a_\nu(x,\tau\xi) = \tau^\nu a_\nu(x,\xi), \ \tau > 0.$$

The first result we shall use is

Thoerem 2.1 ([9], see also [12]). *Let $d \geq 2$ and $d'\xi$ be the standard measure on the unit sphere. Then the integral*

$$\mathrm{res}(A,x) = (2\pi)^{-n} \int_{|\xi|=1} a_{-d}(x,\xi) d'\xi dx \tag{2.2}$$

gives a well defined measure on M.

Let us consider now a holomorphic family $\Phi(s)$ of classical pseudo-differential operators such that

$$\mathrm{ord}\Phi(s) = \alpha s + \beta, \ \alpha \neq 0.$$

The term "holomorphic" means that the family of bounded operators

$$\Lambda^{-(\alpha s+\beta)}\Phi(s)$$

is holomorphic for some elliptic invertible operator Λ of order 1. Clearly, this definition does not depend on the choice of Λ.

Let $\Phi_s(x,x)$ be the restriction of the Schwartz kernel of the operator $\Phi(s)$ to the diagonal. This function is defined when

$$\mathrm{Re}(\alpha s+\beta) < -d. \tag{2.3}$$

In fact, in this case the operators $\Phi(s)$ are of the trace class. Moreover, the function $\Phi_s(x,x)$ is holomorphic in the half-plane (1.3). M. Wodzicki generalized the classical Seeley's theorem;

Theorem 2.2 [9]. *The function $\Phi_s(x,x)$ admits analytical continuation to a meromorphic function on the whole complex plane with simple poles into the points*

$$s_j = (j-d-\beta)/\alpha,\ j=0,1,\dots. \tag{2.4}$$

Moreover,

$$\mathrm{Res}_{s=s_j}\Phi_s(x,x)dx = -\frac{1}{\alpha}\mathrm{res}(\Phi(s),x). \tag{2.5}$$

In the case $\Phi(s)=\Lambda^s$ the theorem 2.2 gives us a part of Seeley's results about the ζ–function of an elliptic operator. Theorem 2.2 does not give a recipe for calculation of $\Phi_0(x,x)dx$ (and $\Phi_{-k}(x,x)dx$, $k=1,2,\dots$, in the case of differential operator Λ). It implies that the poles in these points drop. Nevertheless, it is possible to obtain the values of $\Phi_0(x,x)dx$ (and $\Phi_{-k}(x,x)dx$) in terms of the non-commutative residue [9].

We shall apply theorem 2.2 to the cases

$$\Phi(s) = PA^{-s}$$

and

$$\Phi(s) = P\frac{A^{-s/\mathrm{ord}A} - B^{-s/\mathrm{ord}B}}{s},$$

where P,A,B are pseudo-differential operators, A and B are elliptic. Of course, all the assertions about these operators can be derived from the properties of the complex powers of the elliptic operators, but the theorem 2.2 gives a convenient technique to treat such operators.

2. The ζ–function and the determinant of an elliptic operator

In this section we shall recall the formal construction of an elliptic operator's complex powers. Let A be an elliptic pseudo-differential operator of order n which satisfies the condition (1.1). We assume that the symbol of A admits an asymptotic expansion (2.1). Then one can construct the symbol

$$r(\lambda, x, \xi) \sim r_{-n}(\lambda, x, \xi) + r_{-n-1}(\lambda, x, \xi) + \cdots \tag{2.6}$$

of the resolvent $(\lambda I - A)^{-1}$ in a conic neighborhood of the half-line $\Gamma_\theta = \{te^{i\theta} : t > 0\}$ by the following recurrent formulas (e.g. see [3])

$$\begin{aligned} r_{-n} &= (\lambda - a_n)^{-1}, \\ r_{-n-j} &= (\lambda - a_n)^{-1} \sum_{\nu=0}^{j-1} \sum_{k+|\alpha|=j-\nu} \frac{1}{\alpha!} \partial_\xi^\alpha r_{-n-\nu} D_x^\alpha a_{-n-k},\ j = 1, 2, \ldots \end{aligned} \tag{2.7}$$

The operator A^{-s} is a pseudo-differential operator with the symbol

$$a^s \sim a^{(s)}_{-sn}(x, \xi) + a^{(s)}_{-sn-1}(x, \xi) + \cdots \tag{2.8}$$

and

$$a^{(s)}_{-sn-j}(x, \xi) = \frac{1}{2\pi i} \oint \lambda^{-s} r_{-n-j}(x, \xi, \lambda) d\lambda \tag{2.9}$$

The function λ^{-s} is defined on the complex plane without the half-line Γ_θ, and we integrate from $\infty \cdot e^{i\theta}$ to 0 along the left hand side of Γ_θ, then around the point 0 and, finally, from 0 to $\infty \cdot e^{i\theta}$ along the right hand side of Γ_θ. The complex powers A^{-s} and the ζ–function

$$\zeta_{A,\theta}(s) = \mathrm{Tr} A_\theta^{-s}$$

depend on the choice of Γ_θ. The dependence of $\zeta_A(s)$ on the choice of Γ_θ was treated in [8]. However, *the difference*

$$\zeta'_{A,\theta_1}(0) - \zeta'_{A,\theta_2}(0)$$

is a local invariant. Note that neither $\zeta'_{A,\theta_1}(0)$ nor $\zeta'_{A,\theta_2}(0)$ is a local invariant.

Chapter 3

Asymptotic expansion of the Fredholm determinant

In this chapter we shall assume that A is an invertible elliptic pseudo-differential operator on a closed manifold M, $n = \text{ord}A > \dim M = d$ and $| \arg(A\phi, \phi) | \leq \alpha,\ 0 < \alpha < \pi/2$. To define complex powers of A we shall cut the complex plane over the negative half-axis. The inverse operator A^{-1} belongs to the trace class, so the determinants

$$D(\epsilon) = \det(I + \epsilon A^{-1}) = \prod_{j=1}^{\infty}(1 + \epsilon\lambda_j) \quad (\lambda_j = \mu_j^{-1}) \tag{3.1}$$

are defined. The infinite product on the right hand side of (3.1) converges. It is more convenient to consider the function

$$d(\epsilon) = \log D(\epsilon) = \sum_{j=1}^{\infty} \log(1 + \epsilon\lambda_j) \tag{3.2}$$

We are going to investigate the asymptotic expansion of $d(\epsilon)$ when $\epsilon \to \infty$. In particular the determinant of the elliptic operator A will appear in this expansion.

The fact that $\det(I + \epsilon A^{-1})$ admits an asymptotic expansion for big ϵ is not surprising. Let us differentiate formally the equality (3.2) with respect to ϵ:

$$d'(\epsilon) = \sum_{j=1}^{\infty} \lambda_j(1 + \epsilon\lambda_j)^{-1} = \text{Tr}(A + \epsilon)^{-1}$$

The trace of the resolvent $\mathrm{Tr}(A+\epsilon)^{-1}$ admits an asymptotic expansion when $\epsilon \to \infty$ [13]. So one can justify the possibility of the last expansion's integration and obtain the asymptotics of $d(\epsilon)$. The only thing we can not get in this way is the constant of integration, i.e. the coefficient of ϵ^0. We shall see later that this term is of the most interest. So we shall derive the asymptotics of $d(\epsilon)$ directly.

In this chapter we shall prove

Theorem 3.1. *The function $d(\epsilon)$ admits an asymptotic expansion*

$$d(\epsilon) \sim \sum_{k=-d}^{\infty} p_k \epsilon^{-k/n} + \sum_{j=0}^{\infty} q_j \epsilon^{-j} \log \epsilon \tag{3.3}$$

with

$$p_0 = -\log \det A. \tag{3.4}$$

1. Connection between $d(\epsilon)$ and $\zeta(s)$. In this section we shall prove

Proposition 3.2. *Let $d/n < a < 1$. Then*

$$d(\epsilon) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=a} \epsilon^s b(s) \zeta(s) ds \tag{3.5}$$

where

$$b(s) = \frac{1}{s} \int_0^{\infty} \frac{t^{-s}}{1+t} dt \tag{3.6}$$

Proof. It is convenient to introduce a new variable $\omega = \log \epsilon$. Let $d^*(\omega) = d(e^\omega)$. Then formula (3.5) can be rewritten in the form

$$d^*(\omega) = \tfrac{1}{2\pi i} \textstyle\int_{\mathrm{Re}\, s=a} e^{s\omega} b(s) \zeta(s) ds \tag{3.5$'$}$$

Integrating by parts and then introducting the variable $t = e^\omega$ we derive

$$b(s) = \int_0^{\infty} t^{-s-1} \log(1+t) dt = \int_{-\infty}^{\infty} e^{-\omega s} \log(1+e^\omega) d\omega.$$

Note that the last formula gives the Fourier transform $\log(1+e^\omega) \to b(is)$. So by the inverse Fourier formula

$$\log(1+e^\omega) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=a} b(s) e^{\omega s} ds$$

After substitution of $\epsilon\lambda$ for e^ω we derive

$$\log(1+\epsilon\lambda) = \frac{1}{2\pi i}\int_{\mathrm{Re}s=a} \epsilon^s b(s)\lambda^s ds. \qquad (3.7)$$

Now to obtain (3.5) we have to sum identities (3.7) for $\lambda = \lambda_j$. Note that the summation and integration operations in the right hand side commute and $\zeta(s)$ is the sum of λ_j^s.

2. Analiytical properties of $b(s)$. We shall obtain the asymptotic expansion for $d^*(\omega)$ (or $d(\epsilon)$) by shifting the contour in (3.5′) to the left. So we must have the analytic continuation of $b(s)$ into the left half-plane and we must estimate $\mid b(s) \mid$ and $\mid \zeta(s) \mid$ for large $\mid \mathrm{Im}s \mid$. In this section we shall prove

Proposition 3.3. *The function $b(s)$ admits analytic continuation to a meromorphic function in the half-plane* Re$s < 1$. *It has a pole of order* 2 *at the point $s = 0$ and simple poles at the points* -1,-2,...;
$\mathrm{Res}_{s=0} b(s) = 0$, $\mathrm{Res}_{s=0} s b(s) = 1$. *The function $b(s)$ satisfies the following estimate:*

$$\mid b(\sigma + i\tau) \mid \le C(\sigma) \mid \tau \mid^{\sigma-1} \exp(-\frac{\pi}{2} \mid \tau \mid),\ \sigma < 1,\ \mid \tau \mid \ge 1. \qquad (3.8)$$

Proof. Let

$$J_n = \int_0^\infty t^{-s}(1+t)^{-n} dt,\ n \ge 1.$$

Integrating by parts we obtain

$$J_n = -\frac{n}{s-1}\int_0^\infty \frac{t^{1-s}}{(1+t)^{n+1}} dt = -\frac{n}{s-1} J_n + \frac{n}{s-1} J_{n+1}.$$

So

$$\frac{s+n-1}{s-1} J_n = \frac{n}{s-1} J_{n+1}$$

and

$$J_n = \frac{n}{s+n-1} J_{n+1}.$$

Thus

$$J_1 = \frac{k!}{s(s+1)\cdots(s+k-1)}\int_0^\infty t^{-s}(1+t)^{-(k+1)} dt. \qquad (3.9)$$

The integral in the right hand side of (3.9) is absolutely convergent in the strip $-k < \text{Re}s < 1$. So the assertion about analytic continuation of $b(s)$ is proved.

Applying (3.9) with $k = 1$ we get

$$\text{Res}_{s=0} s b(s) = \int_0^\infty \frac{dt}{(1+t)^2} = 1;$$

$$\text{Res}_{s=0} b(s) = \frac{d}{ds} \int_0^\infty \frac{t^{-s} dt}{(1+t)^2} \Big|_{s=0} = \int_0^\infty \frac{\log t}{(1+t)^2} dt.$$

Changing the argument $t \to 1/u$ in the last integral one can see that

$$\int_0^\infty \frac{\log t}{(1+t)^2} dt = -\int_0^\infty \frac{\log u}{(1+u)^2} du,$$

so

$$\text{Res}_{s=0} b(s) = 0.$$

To prove (3.8) we shall use the representation (3.9). The number k in (3.9) will be large for large $|\tau|$ (really $k \sim |\tau|^2$). Suppose that $-\infty < \sigma_0 < \sigma < \sigma_1 < 1$ with some σ_0 and σ_1. All constants in the following estimates will depend on σ_0 and σ_1 only. We shall not number these constants. They will be designated by the same letter C.

To begin with, let us estimate the integral in (3.9). This integral splits into

$$J = J_1 + J_2 = \int_0^{1/k} t^{-s}(1+t)^{-(k+1)} dt + \int_{1/k}^\infty t^{-s}(1+t)^{-(k+1)} dt.$$

The estimation of J_1 is very easy:

$$|J_1| \leq C \int_0^{1/k} t^{-\sigma} dt \leq C k^{\sigma-1}.$$

To estimate J_2 let us suppose that $0 \leq \sigma \leq 1$ (note that we shift s from the strip $\sigma_0 < \sigma < \sigma_1$!). The function $t^{-\sigma}$ decreases, so

$$|J_2| \leq k^\sigma \int_{1/k}^\infty (1+t)^{-(k+1)} dt \leq k^{\sigma-1}.$$

If $\sigma < 0$ we shall integrate by parts $n = |[\sigma]|$ times ($[x]$ denotes the entire part of the number x).

$$\begin{aligned}
|J_2| &\leq \int_{1/k}^{\infty} t^{-\sigma}(1+t)^{-(k+1)}dt \\
&= k^{\sigma-1}(1+\frac{1}{k})^{-k} - \frac{\sigma}{k}\int_{1/k}^{\infty} t^{-\sigma}(1+t)^{-(k+1)}dt = \cdots \\
&= k^{\sigma-1}(1+\frac{1}{k})^{-k} - \sigma\frac{k^{\sigma}}{k-1}(1+\frac{1}{k})^{-(k-1)} + \cdots \\
&+ (-1)^{n-1}\frac{\sigma(\sigma+1)\cdots(\sigma+n-1)}{k(k-1)\cdots(k-n+1)}\int_{1/k}^{\infty} t^{-\{\sigma\}}(1+t)^{-k+n-1}dt.
\end{aligned}$$

Every term except the last is obviously estimated by $Ck^{\sigma-1}$. The integral in the last term is of the form we have just investigated ($k \to k-n$; $\sigma \to \{\sigma\}$), so it is estimated by $k^{\{\sigma\}-1}$ ($\{\sigma\} = \sigma - [\sigma]$). The product which stands before the integral is of order $k^{-n} = k^{-[\sigma]}$. Thus the last term is also estimated by $k^{\sigma-1}$. Finally,

$$|J_2| \leq Ck^{\sigma-1}$$

and

$$\left| \int_0^{\infty} t^{-s}(1+t)^{-(k+1)}dt \right| \leq Ck^{\sigma-1}. \tag{3.10}$$

Now we intend to estimate the product which stands before the integral in the right hand side of (3.9). Denote

$$r(k) = \log \left| \frac{k!}{s^2(s+1)\cdots(s+k-1)} \right|$$

and

$$l = |[\sigma_0]| + 1.$$

We have

$$\begin{aligned}
r(k) &= \log k! - 2\log|s| - \sum_{j=1}^{l-1}\log|s+j| - \sum_{j=l}^{k-1}\log|s+j| \\
&\leq \log k! - (l+1)\log|\tau| - \frac{1}{2}\sum_{j=l}^{k-1}\log((\sigma+j)^2+\tau^2) \\
&= \log k! - (l+1)\log|\tau|
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=l}^{k-1} \log(\sigma+j) - \frac{1}{2}\sum_{j=l}^{k-1} \log(1+\frac{\tau^2}{(\sigma+j)^2}) \\
&= \log l! + \sum_{j=0}^{k-l-1} \log(\frac{k-j}{k-j+\sigma-1}) - \frac{1}{2}\sum_{j=l}^{k-1} \log(1+\frac{\tau^2}{(\sigma+j)^2}) \\
&- (l+1)\log|\tau|. \qquad (3.11)
\end{aligned}
$$

The first term in the right hand side of (3.11) is constant. The second term is bounded by $C + (1-\sigma)\log k$. Indeed,

$$
\begin{aligned}
\sum_{j=0}^{k-l-1} \log(\frac{k-j}{k-j+\sigma-1}) &= \sum_{j=0}^{k-l-1} \log(1+\frac{1-\sigma}{k-j+\sigma-1}) \\
&= \sum_{j=0}^{k-l-1} \log(1+\frac{1-\sigma}{l+j+\sigma}) \\
&\leq \log(1+\frac{1-\sigma}{l+\sigma}) + \int_1^{k-l} \log(1+\frac{1-\sigma}{x})dx \\
&= C + x\log(1+\frac{1-\sigma}{x}) \,|_1^{k-l} \\
&+ (1-\sigma)\int_1^{k-l} \frac{dx}{x+1-\sigma} \\
&\leq C + (1-\sigma)\log k.
\end{aligned}
$$

To estimate the third term in the right hand side of (3.11) we note that the function

$$
\log(1+\frac{\tau^2}{(\sigma+j)^2})
$$

is decreasing with respect to j. Thus

$$
\begin{aligned}
\frac{1}{2}\sum_{j=l}^{k-1} \log(1 &+ \frac{\tau^2}{(\sigma+j)^2}) \geq \frac{1}{2}\int_l^k \log(1+\frac{\tau^2}{(\sigma+x)^2})dx \\
&= \frac{1}{2}\int_{l+\sigma}^{k+\sigma} \log(1+\frac{\tau^2}{x^2})dx \\
&= \frac{1}{2}(k+\sigma)\log(1+\frac{\tau^2}{(\sigma+k)^2}) - \frac{1}{2}(l+\sigma)\log(1+\frac{\tau^2}{(\sigma+l)^2}) \\
&+ \tau \arctan\frac{k+\sigma}{\tau} - \tau\arctan l + \sigma\tau
\end{aligned}
$$

$$
\begin{aligned}
&\geq \; C+\frac{1}{2}(k+\sigma)\log(1+\frac{\tau^2}{(\sigma+k)^2})-(l+\sigma)\log|\tau| \\
&+ \; |\tau|\arctan\frac{k+\sigma}{|\tau|}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
r(k) \; &\leq \; C+(1-\sigma)\log k-\frac{1}{2}(k+\sigma)\log(1+\frac{\tau^2}{(\sigma+k)^2}) \\
&+ \; (l+\sigma)\log|\tau|-|\tau|\arctan\frac{k+\sigma}{|\tau|}-(l+1)\log|\tau| \\
&= \; C+(1-\sigma)\log k+(\sigma-1)\log|\tau| \\
&- \; \frac{1}{2}(k+\sigma)\log(1+\frac{\tau^2}{(\sigma+k)^2})-|\tau|\arctan\frac{k+\sigma}{|\tau|}. \qquad (3.12)
\end{aligned}
$$

Now we take $k\sim\tau^2$. Then the second term in the right hand side of (3.12) equals to $2(1-\sigma)\log|\tau|$ up to an additive constant. The fourth term is bounded. The fifth term equals

$$
|\tau|\arctan(|\tau|+O(1))=\frac{\pi}{2}|\tau|+O(1).
$$

So

$$
r(k)\leq-\frac{\pi}{2}|\tau|+(1-\sigma)\log|\tau|+c;\; k\sim\tau^2. \qquad (3.13)
$$

Therefore after substitution $[|\tau|^2]$ instead of k into (3.10) we obtain

$$
|b(s)|\leq C|\tau|^{2(\sigma-1)}\exp(-\frac{\pi}{2}|\tau|+(1-\sigma)\log|\tau|)=C|\tau|^{\sigma-1}\exp(-\frac{\pi}{2}|\tau|).
$$

3. Estimation of $|\zeta(\sigma+i\tau)|$ for large $|\tau|$.

Proposition 3.4. *If operator A satisfies the assumptions of this chapter then*

$$
|\zeta(\sigma+i\tau)|\leq C|\tau|^{\sigma-1/2}\exp(\frac{\pi}{2}|\tau|),\; C=C(\sigma),\; |\tau|\geq1.
$$

Proof. We shall use the representation

$$
\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\theta(t)dt=\frac{J(s)}{\Gamma(s)} \qquad (3.14)
$$

with

$$\theta(t) = \mathrm{Tr}e^{-tA} = \sum_{j=1}^{\infty} e^{-t\mu_j}. \tag{3.15}$$

It is well known (e.g. see [2]) that

$$\theta(t) \sim e^{-at},\ t \to \infty \tag{3.16}$$

and (Minakshisundaram–Plejel expansion)

$$\theta(t) \sim \sum_{j=-d}^{0} a_j t^{j/n} + \sum_{j=1}^{\infty}(a_j t^{j/n} + b_j t^{j/n}\log t). \tag{3.17}$$

Moreover one can differentiate both expansions (3.16) and (3.17). Split the integral in the right hand side of (3.14):

$$J(s) = J_1(s) + J_2(s) = \int_0^1 t^{s-1}\theta(t)dt + \int_1^{\infty} t^{s-1}\theta(t)dt.$$

To estimate J_2 we use integration by parts

$$\begin{aligned} |J_2(s)| &= |-\frac{1}{s}\theta(1) - \frac{1}{s}\int_1^{\infty} t^s\theta'(t)dt| \leq \frac{1}{|s|}[\theta(1) + C\int_1^{\infty} t^{\sigma}e^{-at}dt] \\ &\leq \frac{C}{|\tau|};\ C = C(\sigma). \end{aligned}$$

To estimate J_1 we take $l = 1 + |[\sigma]|$. Then

$$\theta(t) = \sum_{j=-d}^{l}(a_j t^{j/n} + b_j t^{j/n}\log t) + r(t)$$

($b_j = 0$ if $j \leq 0$) with

$$|r(t)| \leq Ct^{(l+1)/n}|\log t|\ \mathit{and}\ |r'(t)| \leq Ct^{l/n}|\log t|;\ 0 \leq t \leq 1.$$

So

$$\begin{aligned} J_1(s) &= \sum_{j=-d}^{l} a_j\int_0^1 t^{j/n+s-1}dt + \sum_{j=1}^{l} b_j\int_0^1 t^{j/n+s-1}\log t dt + \int_0^1 t^{s-1}r(t)dt \\ &= \sum_{j=-d}^{l}\frac{a_j}{s+j/n} - \sum_{j=1}^{l}\frac{b_j}{(s+j/n)^2} + \frac{1}{s}r(1) - \frac{1}{s}\int_0^1 t^s r'(t)dt \end{aligned}$$

and

$$|J_1(s)| \le C/|s| \le C/|\tau|.$$

Finally,

$$|J(s)| \le C/|\tau|;\ C = C(\sigma). \tag{3.18}$$

To estimate $1/\Gamma(s)$ we use the Stirling's asymptotics

$$\log\Gamma(z) = (z-1/2)\log z - z + (1/2)\log(2\pi) + o(1);\ |z| \to \infty,\ |\arg z| < \pi$$

(e.g., see [14]). We can write

$$\begin{aligned}\log\Gamma(\sigma+i\tau) &= [(\sigma-1/2)+i\tau][(1/2)\log(\sigma^2+\tau^2)+i\arg(\sigma+i\tau)] \\ &- \sigma - i\tau + (1/2)\log(2\pi) + o(1)\end{aligned}$$

and

$$|\Gamma(\sigma+i\tau)| \sim \sqrt{2\pi}e^{-\sigma}\exp((1/2)(\sigma-1/2)\log(\sigma^2+\tau^2)-\tau\arg(\sigma+i\tau));\ |\tau| \to \infty.$$

Note that

$$\arg(\sigma+i\tau) = \frac{\pi}{2}\mathrm{sgn}\tau - \frac{\sigma}{\tau} + o(1/|\tau|)$$

and

$$\log(\sigma^2+\tau^2) = 2\log|\tau| + O(1/|\tau|^2).$$

Therefore

$$|\Gamma(\sigma+i\tau)| \sim \sqrt{2\pi}|\tau|^{\sigma-1/2}e^{-\pi|\tau|/2};\ |\tau| \to \infty. \tag{3.19}$$

Now the assertion of the proposition 3.4 follows from (3.14), (3.18) and (3.19).

4. Proof of Theorem 3.1. Propositions 3.3 and 3.4 show us that the function which is integrated in (3.5′) is bounded by $C|\tau|^{-3/2}$. So one san shift the path of integration to the left in the complex plane as far as he wants. The poles of the function $b(s)\zeta(s)$ give us terms in the asymptotics of $d^*(\omega)$. Denote

$$\begin{aligned}\beta_j = \mathrm{Res}b(s)\,|_{s=-j},\ j = 0,1,\ldots;\ \hat\beta_0 = \mathrm{Res}sb(s)\,|_{s=0};\\ \hat\beta_j = (b(s) - \beta_j/(s+j))\,|_{s=-j};\ j = 1,2,\ldots;\\ \alpha_k = \mathrm{Res}\zeta(s)\,|_{s=-k/n},\ k = -d,-d+1,\ldots;\\ \gamma_j = (\zeta(s) - \alpha_{nj}/(s+j))\,|_{s=-j},\ j = 1,2,\ldots.\end{aligned} \tag{3.20}$$

Note that $\alpha_0 = 0$ and if A is differential operator or it is a fractional power of a differential operator then

$$\alpha_{nj} = 0,\ \gamma_j = \zeta(-j).$$

By Proposition 3.3

$$\beta_0 = 0 \ and\ \hat{\beta}_0 = 1.$$

The points $s = -k/n,\ k \neq jn\ (j = 0, 1, \ldots)$ are simple poles of the function

$$F(s) = e^{s\omega} b(s)\zeta(s)$$

and

$$\mathrm{Res}_{s=-k/n} F(s) = \alpha_k e^{-k\omega/n} b(-k/n).$$

The points $s = jn$ are poles of the second order of $F(s)$. It is easy to evaluate residues in these points:

$$\mathrm{Res}_{s=0} F(s) = \hat{\beta}_0 \zeta(0)\omega + \beta_0 \zeta(\omega) + \hat{\beta}_0 \zeta'(0) = \zeta'(0) + \omega\zeta(0),$$
$$\mathrm{Res}_{s=-j} F(s) = \alpha_{nj} \beta_j \omega e^{-j\omega} + (\alpha_{nj} \hat{\beta}_j + \gamma_j \beta_j) e^{-j\omega}.$$

Thus we have obtained

$$d^*(\omega) \sim \sum_{k=-d}^{\infty} p_k e^{-k\omega/n} + \sum_{j=0}^{\infty} q_j \omega e^{-j\omega},\ \omega \to \infty \tag{3.21}$$

with

$$\begin{aligned} p_k = \alpha_k b(-k/n),\ k \neq jn\ (j = 0, 1, \ldots); \\ p_0 = \zeta'(0); \\ p_{jn} = \alpha_{nj} \hat{\beta}_j + \gamma_j \beta_j,\ j = 1, 2, \ldots; \\ q_0 = \zeta(0);\ q_j = \alpha_{nj} \beta_j. \end{aligned} \tag{3.22}$$

After substitution $e^{-\omega} = \epsilon$ into (3.21) one obtains (3.3). In particular (3.4) is fulfilled.

Chapter 4

Determinant of the product

1. Variations of $\log\det$. In this section we shall recall well known variational properties of the $\log\det$(e.g., see [15]). Let $A(\epsilon)$ be a differentiable family of elliptic pseudodifferential operators, and suppose there exists a common ray of minimal growth for all these operators. Complex powers are defined with respect to this ray.

Proposition 4.1. [15]

$$\frac{d}{d\epsilon}\log\det A(\epsilon) = (1 + s\frac{d}{ds})\mathrm{Tr}(A'(\epsilon)A(\epsilon)^{-s-1})\mid_{s=0} \tag{4.1}$$

where $A'(\epsilon)$ is the derivative of $A(\epsilon)$ with respect to ϵ.

Note that the right hand side of (4.1) has sense. Indeed,

$$A'(\epsilon)A(\epsilon)^{-s-1}$$

is a holomorphic operator-function;

$$\mathrm{ord}A'(\epsilon)A(\epsilon)^{-s-1} = \mathrm{ord}A'(\epsilon) - \mathrm{ord}A(\epsilon) - s\mathrm{ord}A(\epsilon)$$

so one can apply theorem 2.2. The point $s = 0$ is either a regular point of the meromorphic function

$$\mathrm{Tr}(A'(\epsilon)A(\epsilon)^{-s-1}) \tag{4.2}$$

or a simple pole. The right hand side of (4.1) equals the value of (4.2) in the point $s = 0$ (the first case) or the finite part of (4.2) in this point (the second case).

Formula (4.1) plays the crucial role in almost all papers where determinants of elliptic operators are evaluated.

2. Determinant of the product $S(I+T)$, $T \in \Sigma_1$.

Proposition 4.2. *Let A be an elliptic pseudo-differential operator with a ray of minimal growth and T be a pseudo-differential operator from the trace class Σ_1 ($\mathrm{ord}T < -d$). Then*

$$\log\det S(I+T) = \log\det S + \log\det(I+T). \tag{4.3}$$

Proof. Let us consider the family

$$A(\epsilon) = S(I+\epsilon T).$$

The function

$$\mathrm{Tr}(A'(\epsilon)A(\epsilon)^{-s-1}) = ST(S(I+\epsilon T))^{-s-1}$$

is holomorphic in the half-plane

$$\mathrm{Re}s > \frac{\mathrm{ord}T + d}{\mathrm{ord}S}.$$

Thus the point $s=0$ is regular, and by (4.1)

$$\begin{aligned}\frac{d}{d\epsilon}\log\det A(\epsilon) &= \mathrm{Tr}ST(I+\epsilon T)^{-1}S^{-1}\\ &= \mathrm{Tr}T(I+\epsilon T)^{-1}.\end{aligned}$$

On the other hand if $\{\mu_j\}$ is the set of eigenvalues of T then

$$\begin{aligned}\frac{d}{d\epsilon}\log\det(I+\epsilon T) &= \frac{d}{d\epsilon}\sum\log(1+\epsilon\mu_j)\\ &= \sum\frac{\mu_j}{1+\epsilon\mu_j} = \mathrm{Tr}T(I+\epsilon T)^{-1}.\end{aligned}$$

So

$$\log\det A(\epsilon) - \log\det(I+\epsilon T) = \mathrm{const}.$$

This constant equals $\log\det S$ (put $\epsilon = 0$).

□

The formula (4.3) can be proved for abstract operators in Hilbert space under some conditions. The following theorem and corollary are taken from our paper [11]. Assume that A is a positive operator in Hilbert space H and $A^{-\sigma} \in \Sigma_1$ for some positive σ. One can define the function $\zeta_A(z) = \mathrm{Tr} A^{-z}$ which is regular in the half-plane $\mathrm{Re} z > \sigma$. In some cases (e.g. if A is a pseudodifferential operator) this function has a meromorphic continuation. It may happen that 0 is a regular point of this ζ–function. In this case we say that A has a determinant and $\det A = \exp(-\zeta_A'(0))$.

Theorem 4.3. *Let $S \geq c_0 > 0$ be a positive operator in a separable Hilbert space $\mathcal{H}$,* let $S^{-\sigma} \in \Sigma_1$ *for some σ, $0 < \sigma < 1$ and* let $\det S$ *be defined. Let T be a bounded operator. Then there exists a constant C which depends upon c_0 and $\| T \|$ only, such that* $\det A(\epsilon) = \det(S + \epsilon T)$ *is defined when $|\epsilon| < C$ and is equal to* $\det S \det(I + \epsilon S^{-1} T)$.

Proof. One has the following integral representation in the strip $0 < \mathrm{Re} s < 1$, see [16]:

$$\begin{aligned} A^{-z}(\epsilon) &= \frac{\sin \pi z}{\pi} \int_0^\infty t^{-z} (tI + A(\epsilon))^{-1} dt \\ &= S^{-z} + \frac{\sin \pi z}{\pi} \int_0^\infty t^{-z} \sum_{k=1}^{\infty} (-1)^k \epsilon^k [(tI + S)^{-1} T]^k (tI + S)^{-1} dt. \end{aligned}$$

If $\epsilon < c_0 / \| T \|$ we can change the order of summation and integration:

$$A^{-z}(\epsilon) - S^{-z} = \frac{\sin \pi z}{\pi} \sum_{k=1}^{\infty} (-1)^k \epsilon^k \int_0^\infty t^{-z} [(tI + S)^{-1} T]^k (tI + S)^{-1} dt. \quad (4.4)$$

Let us show that all terms on the right-hand side of (4.4) are nuclear operators and estimate their Σ_1-norms which will be denoted by $\| \cdot \|_1$. One has

$$\begin{aligned} \| [& (tI + S)^{-1} T]^k (tI + S)^{-1} \|_1 \\ \leq & \|S^{-\sigma}\|_1 \cdot \|[(tI + S)^{-1} T]^k (tI + S)^{-1} S^{\sigma}\| \\ \leq & \|S^{-\sigma}\|_1 \cdot \|T\|^k (t + c_0)^{-k} \begin{cases} \sigma^\sigma (1 - \sigma)^{1-\sigma} t^{\sigma - 1} & \text{if } t \geq c_0 (1 - \sigma) \sigma \\ c_0^\sigma / (t + c_0) & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} \| \int_0^\infty & t^{-z} [(tI + S)^{-1} T]^k (tI + S)^{-1} dt \|_1 \\ \leq & \|S^{-\sigma}\|_1 \|T\|^k \{ (1 - \sigma) \sigma^{-1} (1 - \mathrm{Re} z)^{-1} c_0^{-k+\sigma+1-\mathrm{Re} z} \\ + & \sigma^\sigma (1 - \sigma)^{1-\sigma} (\mathrm{Re} z + k - \sigma)^{-1} c_0^{-k+\sigma+\mathrm{Re} z} \}. \end{aligned}$$

Thus the series (4.4) is Σ_1-convergent when $\epsilon < c_0/\|T\|$ and it defines a Σ_1-valued regular function on the strip $\sigma - 1 < \mathrm{Re} z < 1$. Hence $\zeta_{A(\epsilon)}(z)$ has a meromorphic extension to the half-plane $\mathrm{Re} z > \sigma - 1$ and 0 is a regular point of this function;

$$\zeta'_{A(\epsilon)}(0) - \zeta'_s(0) = \sum_{k=1}^{\infty}(-1)^k \epsilon^k \int_0^{\infty} \mathrm{Tr}[(tI + S)^{-1}T]^k (tI + S)^{-1} dt.$$

Note that

$$\frac{d}{dt}[(tI + S)^{-1}T]^k = -\sum_{i-1}^{k-1}[(tI + S)^{-1}T]^i (tI + S)^{-1}(tI + S)^{-1}[(tI + S)^{-1}T]^{k-i}.$$

Hence

$$\mathrm{Tr}[(tI + S)^{-1}T]^k (tI + S)^{-1} = -\frac{1}{k}\frac{d}{dt}\mathrm{Tr}[(tI + S)^{-1}T]^k,$$

and

$$\begin{aligned} \zeta'_{A(\epsilon)}(0) - \zeta'_S(0) &= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\epsilon^k}{k}\int_0^{\infty}\frac{d}{dt}\mathrm{Tr}[(tI + S)^{-1}T]^k dt \\ &= \sum_{k=1}^{\infty}(-1)^k \epsilon^k k^{-1} \mathrm{Tr}(S^{-1}T)^k = -\mathrm{Tr}\log(I + \epsilon S^{-1}T). \end{aligned}$$

□

Corollary. *Let S be the same operator as in Theorem 4.3 and let T be a non-negative bounded operator. Then* $\det(S + T)$ *is defined and*

$$\det(S + T) = \det S \det(I + S^{-1}T).$$

Proof. Note that $S + \epsilon T \geq c_0$ for every $\epsilon \geq 0$. So we can apply Theorem 4.3 N times if N is sufficiently large and obtain

$$\det(S + T) = \det S \prod_{j=0}^{N-1} \det(I + N^{-1}(S + jN^{-1}T)^{-1}T).$$

The product $\prod$ in the last formula equals to $\det(I + S^{-1}T)$, as follows from the identity

$$\det(I + \epsilon_1 S^{-1}T)\det(I + \epsilon_2(S + \epsilon_1 T)^{-1}T) = \det(I + (\epsilon_1 + \epsilon_2)S^{-1}T). \quad (4.5)$$

In order to prove this identity we introduce $R = S^{-1}T$ and obtain

$$\begin{aligned}(I &+ \epsilon_1 S^{-1}T)(I+\epsilon_2(S+\epsilon_1 T)^{-1}T) \\ &= I+\epsilon_1 R+\epsilon_2(I+\epsilon_1 R)^{-1}R+\epsilon_1\epsilon_2 R(I+\epsilon_1 R)^{-1}R \\ &= I+\epsilon_1 R+\epsilon_2(I+\epsilon_1 R)^{-1}R \\ &+ \epsilon_2(I+\epsilon_1 R)(I+\epsilon_1 R)^{-1}R\epsilon_2(I+\epsilon_1 R)^{-1}R \\ &= I+(\epsilon_1+\epsilon_2)R.\end{aligned}$$

Now (4.5) follows from the well-known formula

$$\det(I+A_1)\det(I+A_2) = \det(I+A_1)(I+A_2)$$

with $A_1, A_2 \in \Sigma_1$ (e.g., see [7]).
□

3. Non-multiplicativity of the determinant. Now we intend to demonstrate that the function det is not multiplicative. Let A and B be elliptic operators. Let us consider variations

$$\delta A = A\delta\phi, \quad \delta B = -\delta\phi B$$

with a function $\delta\phi$. The difference

$$A(1+\delta\phi)(1-\delta\phi)B - AB$$

is of order 2 with respect to $\delta\phi$, so

$$\delta(AB) = 0$$

and applying (4.1) we get

$$\begin{aligned}\delta[\log\det(AB) &- \log\det A - \log\det B] \\ &= -\mathrm{Tr}[A\delta\phi A^{-s-1}]\,|_{s=0} + \mathrm{Tr}[\delta\phi B\cdot B^{-s-1}]\,|_{s=0} \\ &= \mathrm{Tr}[\delta\phi(B^{-s}-A^{-s})]\,|_{s=0} \\ &= \int[\mathrm{tr}K_B(x,x)-\mathrm{tr}K_A(x,x)]\delta\phi(x)dx \qquad (4.6)\end{aligned}$$

where $K_A(x,x)$ and $K_B(x,x)$ are non-commutative residues of the holomorphic families A^{-s} and B^{-s} in the point $s=0$. The values of $\mathrm{tr}K_A(x,x)$ and

$\mathrm{tr} K_B(x,x)$ are not equal for every A and B so the integral in the right-hand side of (4.6) does not vanish for every A, B and $\delta\phi$.

4. Symmetry property of $\det AB$. In this section we shall prove

Proposition 4.4. *Let elliptic operators AB and BA have the same ray of minimal growth. Then*

$$\det AB = \det BA.$$

Proof. Suppose that the common ray of minimal growth for AB and BA is the negative half-axis. Let k be an integer, such that

$$k(\mathrm{ord} A + \mathrm{ord} B) > d.$$

Then one has the representation [16]

$$\begin{aligned} \mathrm{Tr}(AB)^{-s} &= \mathrm{Tr}((AB)^k)^{-s/k} = \\ &= \frac{\sin \pi s}{\pi} \int_0^\infty t^{-s/k} \mathrm{Tr}((AB)^k + t)^{-1} dt \end{aligned}$$

in the strip

$$\frac{d}{\mathrm{ord} A + \mathrm{ord} B} < \mathrm{Re} s < k.$$

For small values of t

$$\begin{aligned} \mathrm{Tr}((AB)^k + t)^{-1} &= \sum_{j=0}^{\infty} (-1)^j t^j \mathrm{Tr}(AB)^{-k(j+1)} \\ &= \sum_{j=0}^{\infty} (-1)^j t^j \mathrm{Tr}(BA)^{-k(j+1)} \\ &= \mathrm{Tr}((BA)^k + t)^{-1}. \end{aligned}$$

Both sides of the last equality are analytic with respect to t; hence it holds for all $t \geq 0$. Therefore

$$\zeta_{AB}(s) = \zeta_{BA}(s).$$

Particularly,

$$\det AB = \det BA.$$

5. Determinant of the product of elliptic operators. From this point we restrict ourselves to the case of positive operators A and B. Let

$$\nu[A, B] = \log\det AB - \log\det A - \log\det B.$$

Note that the operator

$$AB = A^{1/2}(A^{1/2}BA^{1/2})A^{-1/2}$$

is conjugated to a positive operator $A^{1/2}BA^{1/2}$. Hence every ray except the positive half-axis is a ray of minimal growth for AB. We denote by $a_j(x,\xi)$ (resp., $b_j(x,\xi)$) the term in the asymptotic expansion of the complete symbol of A (resp., B) which is positively homogeneous of degree j with respect to ξ.

Theorem 4.5. *Let A and B be positive elliptic pseudo-differential operators of positive orders n and m respectively. Then $\nu[A,B]$ depends on $a_n,\dots,a_{n-d}; b_m,\dots,b_{m-d}$ only. The value of $\nu[A,B]$ is the integral of a local expression involving $a_n,\dots,a_{n-d}; b_m,\dots,b_{m-d}$:*

$$\nu[A,B] = \int_M \int_{|\xi|=1} P[a_n,\dots,a_{n-d}; b_m,\dots,b_{m-d}](x,\xi)dxd'\xi \tag{4.7}$$

where P is a polynomial in its arguments and their derivatives.

First we shall prove

Lemma 4.6 *Let C and F be elliptic pseudo-differential operators of orders $\gamma > 0$ and $\delta > 0$ respectively and P be a pseudo- differential operator of order k. Let C and F have a common ray of minimal growth. Then the function*

$$g(s) = g(P;C;F;s) = \mathrm{Tr}P(C^{-s/\gamma} - F^{-s/\delta}) \tag{4.8}$$

which is holomorphic in the half-plane $\mathrm{Re}s > d+k$ admits analytic continuation to a meromorphic function on the whole complex plane; the point $s=0$ is a regular point of this function, and

$$g(0) = \int_M \int_{|\xi|=1} Q[p_k,\dots,p_{-d}; c_\gamma,\dots,c_{\gamma-d-k}; f_\delta,\dots,f_{\delta-d-k}]dxd'\xi \tag{4.9}$$

where Q is a polynomial in its arguments and their derivatives; p, c and f are the symbols of P, C and F respectively.

Proof. To calculate $g(s)$ we shall consider the operator-valued function

$$\Phi(s) = P\frac{C^{-s/\gamma} - F^{-s/\delta}}{s}$$

Obviously

$$g(s) = \mathrm{Res}_{s=0}\mathrm{Tr}\Phi(s).$$

The function $\Phi(s)$ is holomorphic, so by (2.5)

$$g(s) = -(2\pi)^{-n} \int_{|\xi|=1} \mathrm{tr}\sigma_{-n}[\Phi(0)](x,\xi)dxd\xi' \tag{4.10}$$

To calculate $\sigma_{-n}[\Phi(0)](x,\xi)$ we must calculate the symbol

$$l(x,\xi) \sim l_0(x,\xi) + l_{-1}(x,\xi) + \cdots$$

of the operator

$$L = \frac{C^{-s/\gamma} - F^{-s/\delta}}{s}$$

and then apply the composition formula (e.g. see [3]).

1. Calculation of l_0. The principal symbols $c_{-s}^{(s)}(x,\xi)$ and $f_{-s}^{(s)}(x,\xi)$ of operators $C^{-s/\gamma}$ and $F^{-s/\delta}$ equal to $c_\gamma^{-s/\gamma}$ and $f_\delta^{-s/\delta}$ respectively. Therefore

$$l_0(x,\xi) = \frac{c_\gamma^{-s/\gamma} - f_\delta^{-s/\delta}}{s}|_{s=0} = \frac{1}{\delta}\log f_\delta - \frac{1}{\gamma}\log c_\gamma. \tag{4.11}$$

2. Calculation of $l_{-j}, j = 1, 2, \ldots$. First one has to determine the terms $c_{-s-j}^{(-s)}$ and $f_{-s-j}^{(-s)}$ in the asymptotic expansions of the symbols of $C^{-s/\gamma}$ and $F^{-s/\delta}$ respectively. Let us consider $C^{-s/\gamma}$. The recipe is to build the symbol

$$r(\lambda; x, \xi) \sim r_{-\gamma} + r_{-\gamma-1} + \cdots$$

of the resolvent $(\lambda I - C)^{-1}$, and then

$$c_{-s-j}^{(-s)} = \frac{1}{2\pi i}\oint \lambda^{-s/\gamma} r_{-\gamma-j}(\lambda; x, \xi)d\lambda. \tag{4.12}$$

The terms $r_{-\gamma-j}$ are determined by the following recurrent formulas (e.g. see [3])

$$\begin{aligned} r_{-\gamma} &= (\lambda - c_\gamma(x,\xi))^{-1} \\ r_{-\gamma-j} &= \sum_{\nu=0}^{j-1} \sum_{k+|\alpha|=j-\nu} \frac{1}{\alpha!}\partial_\xi^\alpha r_{-\gamma-\nu} D_x^\alpha c_{\gamma-k}(\lambda - c_\gamma)^{-1}. \end{aligned}$$

This formulas imply that $r_{-\gamma-j}$ is the sum of terms which are of the form

$$(\lambda - c_\gamma)^{-1}G_1(x,\xi)\cdots(\lambda - c_\gamma)^{-1}G_p(x,\xi)(\lambda - c_\gamma)^{-1} \tag{4.13}$$

with some matrices G_μ. The entities of these matrices are local expressions from $r_{-\gamma}, \ldots, c_{-\gamma-j}$. If $j \geq 1$ then $1 \leq p \leq 2j$. Every entity of the matrix (4.13) is of the form

$$G(x,\xi)(\lambda-\lambda_1)^{-p_1}\cdots(\lambda-\lambda_k)^{-p_k} \tag{4.14}$$

with $\sum p_l \geq 2$; $\lambda_l = \lambda_l(x,\xi)$ are eigenvalues of c_γ. The integral

$$\frac{1}{2\pi i}G(x,\xi)\oint \lambda^{-s/\gamma}(\lambda-\lambda_1)^{-p_1}\cdots(\lambda-\lambda_k)^{-p_k}d\lambda =$$
$$\frac{1}{2\pi i}G(x,\xi)\oint(\lambda-\lambda_1)^{-p_1}\cdots(\lambda-\lambda_k)^{-p_k}d\lambda +$$
$$\frac{1}{2\pi i}G(x,\xi)\oint[\lambda^{-s/\gamma}-1](\lambda-\lambda_1)^{-p_1}\cdots(\lambda-\lambda_k)^{-p_k}d\lambda =$$
$$\frac{1}{2\pi i}G(x,\xi)s\oint\frac{\lambda^{-s/\gamma}-1}{s}(\lambda-\lambda_1)^{-p_1}\cdots(\lambda-\lambda_k)^{-p_k}d\lambda$$

is an entire function with respect to s which vanishes in the point $s = 0$. Thus $s^{-1}c^{(-s)}_{-s-i}$ is an entire function with respect to s. One can calculate $s^{-1}c^{(-s)}_{-s-i}$ as a local expression from $c_{\gamma-l}$, $l = 0, \ldots, j$. The term (4.14) in the matryx (4.13) generates the term

$$-\frac{1}{2\pi i\gamma}G(x,\xi)\oint\log\lambda(\lambda-\lambda_1)^{-p_1}\cdots(\lambda-\lambda_k)^{-p_k}d\lambda$$

in $s^{-1}c^{(-s)}_{-s-i}\,|_{s=0}$. The same procedure is applicable to the operator F.
□

Remark. The calculation of $s^{-1}c^{(-s)}_{-s-i}\,|_{s=0}$ is much more simple in the scalar case. For example in this case

$$s^{-1}c^{(-s)}_{-s-i}\,|_{s=0} = -\frac{i}{2}c_\gamma^{-2}\sum\frac{\partial c_\gamma}{\partial \xi_j}\frac{\partial c_\gamma}{\partial x_j} + c_\gamma^{-1}c_{\gamma-1}$$

and

$$l_{-1}(x,\xi) = c_\gamma^{-1}c_{\gamma-1} - f_\delta^{-1}f_{\delta-1} - \frac{i}{2}\{c_\gamma^{-2}\sum\frac{\partial c_\gamma}{\partial \xi_j}\frac{\partial c_\gamma}{\partial x_j} - f_\delta^{-2}\sum\frac{\partial f_\delta}{\partial \xi_j}\frac{\partial f_\delta}{\partial x_j}\}$$

Proof of Theorem 4.5. Let $B(\tau) = (1-\tau)A^{m/n} + \tau B$. Then

$$\nu[A, B(0)] = \log\det A^{m/n+1} - \log\det A^{m/n} - log\det A = 0$$

since

$$\log\det A^{\alpha} = \alpha \log\det A.$$

Then by (7)

$$\frac{d}{d\tau}\nu[A, B(\tau)] =$$
$$(1 + s\frac{d}{ds})\mathrm{Tr}[(A\dot{B}(\tau)(AB(\tau))^{-s-1} - \dot{B}(\tau)B(\tau)^{-s-1}] \mid_{s=0}=$$
$$(1 + s\frac{d}{ds})[\mathrm{Tr}(A\dot{B}(\tau)(AB(\tau))^{-s}B(\tau)^{-1}A^{-1} - \mathrm{Tr}\dot{B}(\tau)B(\tau)^{-s-1}] \mid_{s=0}=$$
$$(1 + s\frac{d}{ds})\mathrm{Tr}B^{-1}(\tau)\dot{B}(\tau)[(AB(\tau))^{-s} - B(\tau)^{-s}] \mid_{s=0}=$$
$$\mathrm{Tr}B^{-1}(\tau)\dot{B}(\tau)[(AB(\tau))^{-s/(m+n)} - B(\tau)^{-s/m}] \mid_{s=0} \quad (4.15)$$

since

$$(1 + s\frac{d}{ds})E(s) \mid_{s=0}= (1 + s\frac{d}{ds})E(\alpha s) \mid_{s=0}$$

for every function E which is regular or has a simple pole in the point $s = 0$. By Lemma 3 the function on the right hand side of (4.16) is regular in the point $s = 0$. Here $P = B^{-1}(\tau)\dot{B}(\tau)$, $C = AB(\tau)$ and $F = B(\tau)$. Lemma 4.6 tells us that the right hand side of (4.16) is the integral of a local expression $P_\tau[a_j, b_j]$. One can easily verify that only the terms $a_n, \ldots, a_{n-d}; b_m, \ldots, b_{m-d}$ appear in this expression. Thus

$$\nu[A, B] = \int_0^1 d\tau \int_M \int_{|\xi|=1} P_\tau[a_n, \ldots, a_{n-d}; b_m, \ldots, b_{m-d}](x, \xi)dxd\xi'.$$

□

Chapter 5

The determinant of a Sturm–Liouville operator with Dirichlet conditions

In this chapter we consider the Sturm–Liouville operator

$$S = -\frac{d^2}{dx^2} + F(x)$$

on the segment $[0, a]$ with the Dirichlet boundary conditions.

Theorem 5.1 *Let* $F(x) \in C^2[0, a]$ *and* $A(x)$ be *the solution of the equation*

$$A''(x) = F(x)A(x)$$

with the initial conditions

$$A(0) = 0, \;\; A'(0) = 2.$$

Then

$$\det S = A(a).$$

We shall investigate properties of functional determinants by finite-dimensional approximations. The key lemma is

Lemma 5.2. *Let* $F(x) \in C^\rho[0, a]$, $\rho > 0$, *and let* Δ_0 *be the operator*

$-d^2/dx^2$ with Dirichlet conditions. Consider $(N-1)\times(N-1)$ matrices

$$\delta_N = \frac{N^2}{a^2}\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

and

$$f_N = \|f_{N;ij}\|,$$

where

$$f_{N;ij} = \begin{cases} \alpha F(j/N) + r_{jj}^{(N)} & \text{if } i = j \\ \beta F(j/N) + r_{j+1,j}^{(N)} & \text{if } i = j+1 \\ \beta F((j-1)/N) + r_{j-1,j}^{(N)} & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

$\alpha + 2\beta = 1$ and

$$\lim_{N\to\infty} \max_{i,j} |r_{ij}^{(N)}| = 0.$$

Then

$$\det(I + \Delta_0^{-1}F) = \lim_{N\to\infty} \det(I + \delta_N^{-1} f_N).$$

Proof. Consider the orthonormal basis

$$E_k(x) = \sqrt{2/a}\,\sin(\pi kx/a)$$

of the eigenfunctions of the operator Δ_0: $\Delta_0 E_k = \lambda_k E_k$ with $\lambda_k = \pi k^2/a^2$, $k = 1, 2, \dots$. Denote by H_*^s the scale of Sobolev spaces which are generated by $\Delta_0^{-1/2}$: $\|E_k(x)\|_s = \lambda_k^{s/2}$. The operator δ_N is defined on C^{N-1}; its eigenvalues

$$\lambda_k^{(N)} = \frac{4N^2}{a^2}\sin^2\frac{\pi k}{2N}, \qquad k = 1, \dots, N-1,$$

the corresponding eigenvectors

$$e_k^{(N)} = (e_{k,1}^{(N)}, \dots, e_{k,N-1}^{(N)}) \text{with } e_{ks}^{(N)} = \sqrt{2/a}\,\sin(\pi ks/N), k, s = 1, \dots, N-1.$$

We normalize $e_k^{(N)}$ by the condition

$$|e_k^{(N)}|^2 = \frac{a}{N}\sum_{s=1}^{N-1} |e_{ks}^{(N)}|^2 = 1.$$

Let l^2_N be the space C^{N-1} with the norm $|\cdot|$ and let h^s_N be the same space with the norm $|y|_s = |\delta_N^{s/2} y|$. Now we introduce the interpolation operator $i_N : l^2_N \to L^2[0,a]$ and the restriction operator $j_N : L^2[0,a] \to l^2_N$:

$$\begin{aligned} i_N e_k^{(N)} &= E_k(x), \qquad k = 1, \ldots, N-1, \\ j_N E_k(x) &= \begin{cases} e_k^{(N)} & \text{if } k = 1, \ldots, N-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We split the segment $[0,a]$ into N equal parts by the points $0 = x_0 < x_1 < \cdots < x_N = a$; $x_j = ja/N$. Then i_N is the operator of trigonometrical interpolation of the values at x_j; $j_N = r_N P_N$, where P_N is the ortho-projector onto the space spanned by $E_1, \ldots, E_{N-1}$ and

$$r_N G = (G(a/N), \ldots, G((N-1)a/N)).$$

First of all we notice that the norms of i_N and j_N as operators which map h^s_N into H^s_* and H^s_* onto h^s_N correspondingly are bounded by constants which do not depend on N because

$$1 \le \frac{\lambda_k}{\lambda_k^{(N)}} = \frac{(\pi k/2N)^2}{\sin^2(\pi k/2N)} \le \frac{\pi^2}{4}; \; k = 1, \ldots, N-1.$$

Consider the finite-dimensional operator

$$T_N = i_N \delta_N^{-1} f_N j_N : \; L^2[0,a] \to L^2[0,a].$$

So the convergence of T_N to $T = \Delta_0^{-1} F$ in the space Σ_1 of nuclear operators implies the assertion of the lemma, see [7]. We split the proof of convergence into the following steps. The operators

(*i*) T_N are uniformly bounded in the space $\mathcal{L}(L_2, H^s_*)$ of linear operators $L_2 \to H^s_*$.

(*ii*) $T_N \to T$ in the space $\mathcal{L}_s(L_2, L_2)$ with the strong topology. Let ϕ be a trigonometrical polynomial. Then

$$\begin{aligned} T_N\phi - T\phi &= i_N\delta_N^{-1}[f_N j_N - j_N F(x)]\phi + i_N \delta_N^{-1} j_N (I - P_k) F(x)\phi \\ &+ (i_N \delta_N^{-1} j_N - \Delta_0^{-1}) P_k F(x)\phi - \Delta_0^{-1}(I - P_k)F(x)\phi. \quad (5.1) \end{aligned}$$

The second and the fourth terms on the right-hand side of (5.1) converge to 0 uniformly with respect to N when $k \to \infty$. Operators $(i_N\delta_N^{-1} j_N -$

$\Delta_0^{-1})P_k$ have orthonormal basis of eigenfunctions $E_j(x)$. The corresponding eigenvalues are equal to

$$a^2/(4N^2\sin^2(\pi j/2N)) - a^2/(\pi^2 j^2) \to 0 \quad \text{if } j \leq k-1 \text{ and}$$
$$0 \quad \text{otherwise;}$$

therefore the third term in (5.1) converges to 0 when $N \to \infty$ and k is fixed. Let

$$[f_N r_N - r_N F(x)]\phi(x) = (y_1^{(N)}, \ldots, y_{N-1}^{(N)}).$$

Then

$$\begin{aligned} y_j^{(N)} &= \beta F((j-1)a/N)\phi((j-1)a/N) + r_{j,j-1}^{(N)}\phi((j-1)a/N) \\ &+ \alpha F(ja/N)\phi(ja/N) + r_{j,j}^{(N)}\phi(ja/N) + \beta F(ja/N)\phi((j+1)a/N) \\ &+ r_{j,j+1}^{(N)}\phi((j+1)a/N) - F(ja/N)\phi(ja/N) \end{aligned}$$

and

$$\lim_{N\to\infty} \max_j |y_j^{(N)}| = 0.$$

Thus $|(f_N r_N - r_N F)| \to 0$ in l_2^N. Further, $(r_N - j_N)F\phi \to 0$ when $N \to \infty$ and $r_N\phi = j_N\phi$ if N is sufficiently large. So the first term on the right-hand side of (5.1) converges to 0 when $N \to \infty$. Combining the results above we obtain that $T_N\phi \to T\phi$. The set $\{T_N\}$ is bounded and trigonometrical polynomials are dense in L_2; hence $T_N \to T$ in strong topology.

(*iii*) $T_N \to T$ in the space $\mathcal{L}(L_2, H_*^2)$, by virtue of (*i*), (*ii*) and the Banach-Steinhaus theorem.

(*iv*) $T_N \to T$ in the space $\mathcal{L}(H_*^s, H_*^2)$, $s > 0$ by virtue of (*iii*) and compactness of the embedding $H_*^s \hookrightarrow L^2$.

The space $\mathcal{L}(H_*^s, H_*^2)$ belongs to $\Sigma_1(H_*^s)$ when $s < 1$, see [3]. Hence $T_N \to T$ in $\Sigma_1(H_*^s)$: $0 < s < 1$.

□

Lemma 5.3. *Let $F(x) \in C^2[0,a]$ and let $A(x)$ be the solution of the equation*

$$A''(x) = F(x)A(x)$$

with the boundary conditions

$$A(a) = 0, \qquad A'(a) = -k/a,$$

where $A_\nu^{(N)}$, $\nu = 0, 1, \ldots, N$, $N = 2, 3, \ldots$, satisfies the difference equation

$$\frac{N^2}{a^2}(A_{\nu+1}^{(N)} - 2A_\nu^{(N)} + A_{\nu-1}^{(N)}) = F(\frac{(N-\nu)a}{N})$$

with

$$A_0^{(N)} = 0, \qquad A_1^{(N)} = k/N.$$

Then

$$A(0) = \lim_{N\to\infty} A_N^{(N)}.$$

Proof. Let

$$R_\nu^{(N)} = A_\nu^{(N)} - A((N-\nu)a/N)$$

and

$$C_\nu^{(N)} = R_\nu^{(N)} - R_{\nu-1}^{(N)}.$$

Then

$$\frac{N^2}{a^2}(R_{\nu+1}^{(N)} - 2R_\nu^{(N)} + R_{\nu-1}^{(N)}) = F(\frac{(N-\nu)a}{N})R_\nu^{(N)} + b_\nu^{(N)}$$

and

$$\frac{N^2}{a^2}(C_{\nu+1}^{(N)} - C_\nu^{(N)}) = F(\frac{(N-\nu)a}{N})\sum_{j=1}^{\nu} C_j^{(N)} + b_\nu^{(N)} \tag{5.2}$$

with $R_0^{(N)} = 0$, $R_1^{(N)} = C_1^{(N)} + O(N^{-3})$ and $b_\nu^{(N)} = O(N^{-2})$ uniformly with respect to nu. Clearly, $C_\nu^{(N)}$ are bounded by the solution of the equation (5.2) with $F((N-\nu)a/N)$, $b_\nu^{(N)}$ and $C_1^{(N)}$ replaced by $C_1 = \sup|F(x)|$, C_2/N^2 and C_3/N^3, respectively. Hence $R_\nu^{(N)}$ are bounded by the solution of the following difference equation

$$\frac{N^2}{a^2}(r_{\nu+1}^{(N)} - 2r_\nu^{(N)} + r_{\nu-1}^{(N)}) = C_1 r_\nu^{(N)} + \frac{C_2}{N^2};\ r_0^{(N)},\ r_1^{(N)} = \frac{C_3}{N^2}.$$

The general solution of this equation is

$$r_\nu^{(N)} = -\frac{C_2}{C_1 N^2} + \alpha^{(N)}[\lambda_+^{(N)}]^\nu + \beta^{(N)}[\lambda_-^{(N)}]^\nu$$

with $\lambda_\pm^{(N)} = 1 \pm C_4/N + \cdots$ and $\lambda_+^{(N)}\lambda_-^{(N)} = 1$. According to the initial conditions

$$\alpha^{(N)} + \beta^{(N)} = C_5/N^2,\ \ \alpha^{(N)}\lambda_+^{(N)} + \beta^{(N)}\lambda_-^{(N)} = C_3/N^2.$$

Hence

$$\alpha^{(N)} = \frac{(C_3\lambda_+^{(N)})/N^3 - C_5/N^2}{[\lambda_+^{(N)}]^2 - 1} = O(\frac{1}{N}), \;\; \beta^{(N)} = O(\frac{1}{N}).$$

Therefore,

$$r_N^{(N)} \le \frac{C_5}{N^2} + C_6\frac{(1+C_7/N)^N}{N} \le \frac{C_8}{N}$$

and

$$R_N^{(N)} = O(1/N).$$

□

Proposition 5.4. *Let* $F(x) \in C^2[0,a]$ *and let* $A(x)$ *be the solution of the equation*

$$A''(x) = F(x)A(x) \tag{5.3}$$

with the boundary conditions

$$A(a) = 0, \;\; A'(a) = -1/a.$$

Then

$$\det(I + \Delta_0^{-1}F) = A(0).$$

Proof. Let

$$f_N = \operatorname{diag}(F(a/N), \ldots, F((N-1)a/N))$$

be the diagonal matrix. By Lemma 5.2

$$\begin{aligned} \det(I + \Delta_0^{-1}F) &= \lim_{N\to\infty} \det(I + \delta_N^{-1} f_N) \\ &= \lim_{N\to\infty} \left[\det\begin{pmatrix} 2 & -1 & \ldots & 0 \\ -1 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{pmatrix}\right]^{-1} \end{aligned}$$

$$\cdot \; \det\begin{pmatrix} 2 + a^2F(a/N)N^2 & -1 & \ldots & 0 \\ -1 & & \ldots & \\ \ldots & \ldots & \ldots & \ldots \\ 0 & & & -1 \\ 0 & \ldots & -1 & 2 + a^2F((n-1)a/N)/N^2 \end{pmatrix}$$

$$= \lim_{N\to\infty} N^{-1} \det D_N.$$

We have used the relation

$$\det\begin{pmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 2 \end{pmatrix} = N$$

which can be proved easily.

By elementary transformations the matrix D_N can be transformed into

$$\begin{pmatrix} v_1 & -1 & 0 & \dots & 0 \\ 0 & v_2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & v_{N-1} \end{pmatrix}$$

with

$$v_j = 2 + \frac{a^2F(ja/N)}{N^2} - \frac{1}{v_{j-1}}, \quad v_1 = 2 + a^2F(a/N)N^2. \tag{5.4}$$

Our aim is to find $N^{-1}v_1 \cdots v_{N-1}$. Let

$$N^{-1}v_{N-\nu}\cdots v_{N-1} = A_\nu^{(N)}v_{N-\nu} + B_\nu^{(N)}; \;\; A_1^{(N)} = N^{-1},\; B_1^{(N)} = 0.$$

It follows from (5.4) that

$$\begin{aligned} (N^2/a^2)(A_{\nu+1}^{(N)} - 2A_\nu(N) + A_{\nu-1}(N)) &= F((N-\nu)a/N)A^{(N)}, \\ A_0^{(N)} = 0, \qquad & A_1^{(N)} = 1/N. \end{aligned}$$

The value of

$$N^{-1}\det D_N = A_{N-1}^{(N)}v_1 + B_{N-1}^{(N)} = A_N^{(N)}$$

converges to $A(0)$ when $N \to \infty$ by Lemma 5.3. The proposition is proved. □

Proof of Theorem 5.1. The operator $\Delta_0^{-1}F$ belongs to the trace class. Thus, by Proposition 4.2 (see also Theorem 4.3 with the Corollary)

$$\det S = \det \Delta_0 \cdot \det(I + \Delta_0^{-1}F).$$

Note that

$$\zeta_{\Delta_0}(z) = (\pi/a)^{-2s}\zeta(2z)$$

and

$$\det \Delta_0 = \frac{a}{\pi} e^{-2\zeta'(0)} = 2a$$

where $\zeta(z)$ is the Riemann's ζ-function. Therefore,

$$\det S = -2A'(0)$$

where $A(x)$ is the solution of (5.3).

After the substitution $F(x) \longmapsto F(a-x)$ (this substitution does not change the determinant) we get the expression for $\det S$ in the form of Theorem 5.1.

□

Remark. We can explain why the constant 2 appears in Theorem 5.1. Let $D(\lambda)$ be the determinant of the operator

$$-d^2/dx^2 + F(x) - \lambda$$

with Dirichlet boundary conditions. The function $D(\lambda)$ vanishes when λ is an eigenvalue of S. So does the function $D_1(\lambda) = B(a, \lambda)$ where $B(x, \lambda)$ is the solution of

$$B''(x, \lambda) = (F(x) - \lambda) B(x, \lambda)$$

with the initial data

$$B(0, \lambda) = 0, \quad B'(0, \lambda) = 1.$$

So $D_0(\lambda)/D_1(\lambda)$ is a non-vanishing entire function. Let us assume that $D_0(\lambda)/D_1(\lambda) = \text{const}$ and this constant is independent of F;

$$D(\lambda) = cB(a, \lambda).$$

Consider the case $F(x) = 0$. Then

$$B(a, -\lambda) = \frac{\sinh a\sqrt{\lambda}}{\sqrt{\lambda}}$$

and

$$\log B(a, -\lambda) = a\sqrt{\lambda} - \frac{1}{2}\log \lambda - \log 2 + o(1), \quad \lambda \to \infty.$$

On the other hand Theorem 3.1 and Proposition 4.2 imply that the constant term in the asymptotics of $D(\lambda)$, $\lambda \to \infty$, equals 0. Hence $c = 2$.

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