Essays on Bargaining and Repeated Games

by

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Submitted to the Department of Economics on May 13, 2011 in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Economics

ABSTRACT

The thesis consists of four essays on bargaining and repeated games.

The first essay studies whether allowing players to sign binding contracts governing future play leads to reputation effects in repeated games with long-run players. Given any prior over behavioral types, a modified prior is constructed with the same total weight on behavioral types and a larger support under which almost all efficient, feasible, and individually rational payoffs are attainable in perfect Bayesian equilibrium. Thus, whether reputation effects emerge in repeated games with contracts depends on details of the prior distribution over behavioral types other than its support.

The second essay studies reputational bargaining under the assumption of first-order knowledge of rationality. The share of the surplus that a player can guarantee herself is determined, as is the bargaining posture that she must announce in order to guarantee herself this much. It is shown that this maxmin share of the surplus is large relative to the player’s initial reputation, and that the corresponding bargaining posture simply demands this share plus compensation for any delay in reaching agreement.

The third essay studies the maximum level of cooperation that can be sustained in sequential equilibrium in repeated games with network monitoring. The foundational result is that the maximum level of cooperation can be sustained in grim trigger strategies. Comparative statics on the maximum level of cooperation are shown to be highly tractable. For the case of fixed monitoring networks, a new notion of network centrality is introduced, which characterizes which players have greater capacities for cooperation and which networks can support more cooperation.

The fourth essay studies the price-setting problem of a monopoly that in each time period has the option of failing to deliver its good after receiving payment. Optimal equilibrium pricing and profits are characterized. For durable goods, a lower bound on optimal profit for any discount factor is provided. The bound converges to the optimal static monopoly profit as the discount factor converges to one, in contrast to the Coase conjecture.

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1 Indeterminacy of Reputation Effects in Repeated Games with Contracts

1.1 Introduction

Does game theory make strong predictions about the outcomes of long-run relationships? It has been known since the seminal papers of Kreps et al (1982), Kreps and Wilson (1982), and Milgrom and Roberts (1982) that reputation effects have important consequences for equilibrium selection in many dynamic games. Fudenberg and Levine (1989) famously showed that a patient long-run player facing a series of short-run opponents receives at least her Stackelberg payoff in any Nash equilibrium, if her “Stackelberg type” has positive prior probability, and similar results hold in two-player repeated games in the limit where one player becomes infinitely more patient than the other. However, reputation effects are elusive in two-player games with comparably patient players: indeed, it is not obvious what outcome one would expect reputation effects to select in such games. For this reason, the reputation result of Abreu and Pearce (2007, henceforth AP) is striking: AP show that, in two-player repeated games with common discounting in which players may offer each other binding commitments to future divisions of the surplus, all perfect Bayesian equilibrium (PBE) payoffs converge to the Nash bargaining with threats payoffs as the probability of behavioral types converges to zero, so long as the “Nash bargaining with threats type” has positive prior probability and different commitment types are distinguishable from each other from the start of the game. Thus, AP’s results suggest that allowing players to sign binding contracts in repeated games—which seems very plausible in many applications, such as employer-employee and union-firm relationships—leads to extremely strong equilibrium

---


selection results in the presence of an arbitrarily small amount of incomplete information. The current paper investigates whether this intuition is correct, or whether the ability to make such strong predictions about long-run relationships requires additional assumptions about the nature of the incomplete information in the model.

Formally, I extend AP’s model by allowing that different commitment types may not be immediately distinguishable, and show that whether or not reputation effects emerge depends on the relative probabilities of different behavioral types, rather than on only the support of the prior distribution over behavioral types. In particular, given any prior over behavioral types, I construct a modified prior with a larger support under which almost all efficient, feasible, and individual rational payoffs are perfect Bayesian equilibrium payoffs (Theorem 1). Furthermore, the weight on any behavioral type under the original prior is at most $K$ times its weight under the modified prior, where $K$ is a constant that does not depend on the original prior and is non-decreasing in the discount rate; thus, there is a uniform bound on the extent to which any original prior must be modified to yield a new prior for which a folk theorem holds. Therefore, if the only assumption that a researcher is willing to make about the prior distribution of behavioral types is that some types have positive prior probability, she cannot rule out any efficient, feasible, individually rational payoffs. This stands in stark contrast with the case of one long-run player facing a series of short-run players (Fudenberg and Levine (1989, 1992)), where assumptions of this form lead to strong conclusions about equilibrium payoffs.

The essential intuition for the result is that, when different behavioral types are initially indistinguishable, imitating a “tough” behavioral type may not be profitable for a normal player ($i$, say), because doing so may lead her opponent ($j$) to believe that she is a “soft” behavioral type, at least for a long time. This is the key difference between my model and AP’s, in which if player $i$ imitates a tough behavioral type, player $j$ believes that player $i$ is either tough or normal, since in AP’s model different behavioral types are immediately distinguishable. In particular, in my model there may be soft types of player $i$ that play like tough types with some probability, but also concede to player $j$ with high enough probability.

I do, however, assume that normal types have the ability to distinguish themselves from behavioral types. The role of this assumption is discussed in Section 1.4.
that player \( j \) will keep playing against an apparently tough type in the hope that it will turn out to be a soft type. As long as soft types continue to concede on the equilibrium path, player \( j \) will eventually become convinced that she is facing a tough type and concede. But if the prior probability of soft types is high enough relative to the prior probability of tough types, this will take long enough that player \( i \) will not be tempted to imitate a tough type.

Note that these soft types reward one's opponent for failing to concede in much the same way as Evans and Thomas' (1997, 2001) commitment types punish one's opponent for failing to play a prescribed action. The reason why allowing complicated commitment types leads to multiplicity in my model and uniqueness in Evans and Thomas is the difference in patience: with equal patience, the fact that player \( j \) thinks that player \( i \) may be a complicated commitment type may limit player \( i \)'s ability to manipulate player \( j \)'s beliefs quickly enough for her to benefit from doing so, while if player \( i \) is infinitely more patient than player \( j \) she can only benefit from player \( j \)'s attributing to her a wide range of possible commitment types. This line of argument shows why a player cannot guarantee herself a high payoff in my model even if she is the only reputation-builder (i.e., if her opponent is known to be normal), despite her potentially useful ability to offer binding contracts. It also provides an intuition for why existing reputation results with equal patience rely on strong restrictions on the prior distribution over commitment types, even in the limited class of games for which such results apply, while reputation results for games in which one player is infinitely more patient than the other do not require such restrictions.

Finally, there is an interesting connection—suggested to me by an anonymous referee—between my results and the failure of reputation effects in some repeated games with a patient reputation-builder and a relatively impatient long-run opponent. Reputation effects may fail to obtain in that setting because the normal reputation-builder may punish her opponent for

\footnote{Chan's (2000) uniqueness result depends on there being only one commitment type; Cripps et al (2005) obtain uniqueness only in the limit as the weight on commitment types other than the Stackelberg type converges to zero; Atakan and Ekmekci (2009a) assume that non-Stackelberg types distinguish themselves from the Stackelberg type at a uniform rate; Atakan and Ekmekci (2009b) assume that there is only one commitment type; Atakan and Ekmekci (2009c) assume that all commitments types are finite-automata, which in their model is a similar restriction to that in Atakan and Ekmekci (2009a); and Aumann and Sorin (1989) assume that every commitment type that follows a pure strategy with finite memory has positive probability, but that no other commitment types have positive probability.}

\footnote{For example, none of the papers cited in Footnote 1 relies on upper bounds on the prior probability of any type.}
best-responding to her Stackelberg action (see chapter 16 of Mailath and Samuelson (2006) for an informative discussion of this point). However, the reputation-builder can circumvent this problem when she is allowed to offer binding contracts, as in AP, which makes AP’s uniqueness result possible. Introducing additional behavioral types, as in my model, can restore the opponent’s incentive to fail to best-respond to the reputation builder’s action, leading to the failure of reputation effects.

The remainder of the paper proceeds as follows: Section 1.2 introduces the model, which is very similar to AP’s model, with the modification that distinct behavioral types are not immediately distinguishable from each other. Section 1.3 presents the main idea of the paper in the context of a simple example: the prisoner’s dilemma with a single behavioral type on each side. It serves to build intuition and to contrast my results with AP’s. Section 1.4 presents the main result, Theorem 1. Section 1.5 offers brief concluding remarks.

### 1.2 Model

I begin with the hybrid discrete-time/continuous-time model developed by AP. There are two players. At each integer time \( n = 0, 1, 2, \ldots \), players choose actions in a finite stage game \( G = (S_i, U_i)_{i=1}^2 \) and also make demands (“contracts,” “offers”) \( u_i \in \Pi_i \), where \( \Pi_i \) is the convex hull of the set of player \( i \)’s feasible payoffs in \( G \), and \( u_i \) is interpreted as the lowest payoff that player \( i \) is willing to accept in the continuation game. Actions determine flow payoffs until the next integer time, assuming neither player accepts the other’s contract offer. That is, if players use actions \( (s_1, s_2) \), player \( i \)’s period payoff is \( U_i(s_1, s_2) \int_0^1 e^{-rt} dt \), where \( r \) is the common discount rate. Assume also, as in AP, that players can select mixed actions \( (m_i, m_j) \) at integer times, in which case mixing occurs continuously throughout the period, so it is as if mixed actions are observable; let \( M_i \) be the set of player \( i \)’s mixed actions. At any time (not just integer times), either player \( j (= -i) \) can accept the other player’s standing offer \( u_i \) (“concede”), in which case the players receive \( (u_i, \phi_j(u_i)) \), where \( \phi_j(u_i) \) is

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6Indeed, the ability to offer binding contracts makes reputation-building easier in many settings, which provides another motivation for our indeterminacy result. Games with a patient reputation-builder facing a relatively impatient opponent is one example. Another is common-interest games with two equally patient players (Cripps and Thomas, 1997), where it is again easy to see that allowing binding contracts leads to reputation effects.
the highest feasible payoff for \( j \) consistent with \( i \) getting \( u_i \), and the game ends (each player only has one standing offer at a time—these may change on the integers). As in AP, there is a first and last date at which player \( j \) can accept each offer of player \( i \)'s (i.e., “just after \( n \)” and “just before \( n + 1 \)”), and the players move sequentially in an arbitrary, pre-specified order at each integer time \( n \); see AP for more details of this formulation of time. I assume that the function \( \phi_j \) is strictly decreasing, which rules out common-interest games, and use \( \phi_j^{-1} \) and \( \phi_i \) interchangeably. The game ends immediately if the standing offers ever satisfy \((u_1, u_2) \in \Pi\), in which case both players get their demands. Thus, the game can be thought of as a “repeated game with contracts” or as “bargaining with payoffs as you go.” At time \( t \), the (disagreement) history \( h' \) of mixed actions \((m_i, m_j)\) and demands \((u_i, u_j)\) is publicly observed.

At the beginning of the game, there is a chance that each player is one of a number of behavioral types, which are simply repeated game strategies (i.e., arbitrary automata that may condition their player on the entire history \( h' \)). Player \( i \) is of behavioral type \( \gamma_i \) (i.e., is committed to strategy \( \gamma_i \)) with prior probability \( \pi_i(\gamma_i) \), and \( \pi_i \) is assumed have countable support; since I do not assume that \( \pi_i(\gamma_i) \) is positive for any \( \gamma_i \), this formulation allows for both one-sided and two-sided reputation-formation.\(^7\) I assume that each \( \gamma_i \) plays a pure strategy over \((\Pi_i, M_i)\) but may mix over accepting or rejecting \( j \)'s offer. This restriction is made to simplify notation, and is without significant loss of generality, since a mixed strategy over \((\Pi_i, M_i)\) can be approximated by a lottery over countably many pure strategies over \((\Pi_i, M_i)\); in addition, an element of \( M_i \) is already a lottery over \( S_i \), so the only restriction here is that behavioral types do not mix over uncountably many elements of \( \Pi_i \).\(^8\) Players’ types are drawn independently. Let \( z_i \) be the probability that \( i \) is one of the behavioral types, i.e., \( z_i = \sum_{\gamma_i \epsilon \text{supp } \pi_i} \pi_i(\gamma_i) \). Let \((G, \pi)\) describe the stage game together with the

\(^7\)In AP, \( \pi_i(\gamma_i) \) is the probability of player \( i \)'s being of type \( \gamma_i \) conditional on being a behavioral type. We let \( \pi_i(\gamma_i) \) be the unconditional probability of player \( i \)'s being of type \( \gamma_i \).

\(^8\)One difference between the model and AP is that AP do not allow behavioral types to play mixed strategies or concede at non-integer times. The assumption that behavioral types can mix and concede at non-integer times is not crucial, as each type I consider that mixes and concedes at non-integer times can be replaced by a set of types, each one of which concedes with probability 1 at a different integer time without substantially affecting the results. Furthermore, AP’s results do not rely on their assumption that behavioral types do not mix or concede at non-integer times. Thus, this difference in assumptions—which substantially simplifies our exposition—does not drive the difference in results between the current paper and AP.
common prior over behavioral types.

AP assume that, before play over \((\Pi, M)\) begins, there is an initial “announcement” stage, where each player simultaneously announces a behavioral type \(\gamma_i\). AP assume that behavioral types (but not normal types) announce their types truthfully; this is why behavioral types are instantly distinguishable from each other in their model. I dispense with the announcement stage almost entirely: I assume only that there is an initial “revelation” stage, in which each normal player has the option to “reveal rationality”, i.e., to costlessly and certifiably reveal to the other player that she is normal. Formally, I assume that before players choose their initial \((\Pi, M)\), they publicly announce an element of the set \(\{0, 1\}\), and that all behavioral types announce 0; I refer to announcing 1 as “revealing rationality”. I also assume that behavioral types do not condition their play on whether their opponents reveal rationality.\(^9\)^10

Unlike the announcement stage of AP, the revelation stage is included in my model essentially for convenience, and indeterminacy of reputation effects persists without the revelation stage; see the discussion preceding the proof of Theorem 1 for a discussion of the role of the revelation stage in the model. In addition, the analysis goes through if behavioral types also have the ability to certifiably reveal their types, because a normal type cannot mimic a behavioral type that certifiably reveals itself. Hence, the revelation stage can also be given a positive justification if players can exhibit hard information that reveals their types. For example, an incumbent firm may be able to publicly exhibit its production costs by letting potential entrants tour its factories and look at its financial records, and an employee may be able to publicly exhibit her outside option by producing job offers from rival employers. Thus, even with the revelation stage, the analysis does not rely on normal and behavioral players having different abilities to reveal their types (as it is as if every player can either reveal her true type or reveal “nothing”), in contrast to the analysis of AP (as in their model behavioral players are forced to reveal their true types but normal players

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\(^9\)The assumption that normal players may “reveal rationality” to each other is also present, roughly speaking, in AP. Technically, AP require normal players to announce a behavioral type, rather than allowing them to announce that they are normal, but they show in their Footnote 17 that this assumption is immaterial in their model. More substantively, AP also assume that behavioral types do not condition their play on announcements.

\(^10\)One can check that Theorem 1 continues to hold if behavioral types can condition their play on whether their opponents reveal rationality, provided that the prior probability that each player is behavioral is sufficiently small (proof available upon request).
1.3 Example

In this section, I illustrate the main idea of the paper in the context of a simple example. Let $G$ be the prisoner's dilemma:

\[
\begin{array}{cc}
C & D \\
C & 1,1 & -1,3 \\
D & 3,-1 & 0,0 \\
\end{array}
\]

I first consider this stage game with a single behavioral type (the Nash bargaining with threats type, analyzed by AP) on each side, and note that a uniqueness result applies as in AP. I then add an additional “soft” behavioral type on each side and show that, for suitably chosen priors, almost all efficient, feasible, and individually rational payoffs can be attained in PBE. The argument for this fact contains many of the ideas of the proof of the main result (Theorem 1) in a much simpler context.

First, consider the case where each player is normal with probability $1-z$, and with probability $z$ is the behavioral type that in every period plays $D$ and demands 1, and never accepts an offer of less than 1. Call this behavioral type $\gamma$; note that $\gamma$ is the Nash bargaining with threats type. AP show that, when $z$ is small, a normal player’s expected payoff in any PBE is close to her Nash bargaining with threats payoff. In this simple example, an even stronger result applies: for any $z > 0$, both normal players receive payoff 1 in any PBE. This follows from an argument similar to the proof of Lemma 1 of AP, which I sketch here. Suppose there exists a PBE in which normal player $i$’s payoff is less than 1. Then she must receive payoff less than 1 from imitating $\gamma$, i.e., from not revealing rationality and then playing $D$ and demanding 1 in every period. Player $i$ receives 1 from this strategy when player $j$ is of type $\gamma$, so she must receive less than 1 from this strategy when player $j$ is normal. However, if player $i$ imitates $\gamma$, there exists some finite time $T$ such that normal player $j$ accepts her demand with probability 1 by $T$.\footnote{This follows by a fairly standard reputation-building argument. For the details, see the proof of Lemma 1 of AP.} Let $T_0$ be the infimum over all $T$.\footnote{This follows by a fairly standard reputation-building argument. For the details, see the proof of Lemma 1 of AP.}
such that this is the case, and suppose towards a contradiction that $T_0 > 0$. Then there exists $\varepsilon > 0$ such that player $j$’s offer at $T_0 - \varepsilon$, $\phi_i(u_j)$, is less than 1 (as otherwise the two demands would be compatible and the game would have ended), and \( e^{-\varepsilon} (1) > \phi_i(u_j) \).

Therefore, upon reaching time $T_0 - \varepsilon$ player $i$ will not concede until after time $T_0$ (as this yields payoff at least $e^{-\varepsilon} (1)$, whereas conceding yields payoff $\phi_i(u_j)$). This implies that normal player $j$ must concede at time $T_0 - \varepsilon$, contradicting the hypothesis that $T_0 > 0$.

Hence, player $i$’s offer of 1 must be accepted with probability 1 by normal player $j$ at time 0, which implies that player $i$ can guarantee herself a payoff of 1 in any PBE by imitating $\gamma$. And, of course, the same argument applies to player $j$.

Next, suppose that each player is still normal with probability $1 - z$, but is now of type $\gamma$ with probability $z \left( \frac{1}{K} \right)$ for some $K \geq 1$, and with probability $z \left( \frac{K-1}{K} \right)$ is of type $\tilde{\gamma}$, defined as follows: $\tilde{\gamma}$ plays $D$ and demands 1 in every period, but also accepts any non-negative offer at time $t$ at hazard rate $r/\chi(t)$, where $\chi(t)$ is the probability that player $i$ is of type $\tilde{\gamma}$ at time $t$ conditional on her being behavioral (i.e., of type $\gamma$ or $\tilde{\gamma}$) and having played $D$, demanded 1, and not conceded until time $t$. That is,

\[
\chi(t) \equiv \frac{e^{-rt} - \frac{1}{K}}{e^{-rt}}
\]

for all $t$ such that this is nonnegative. This is illustrated in Figure 1, where $\chi(t)$ is the ratio (at time $t$) of the distance from the curve $e^{-rt}$ (which is the probability that player $i$ does not concede before time $t$ conditional on her being behavioral) to the dotted line at $\frac{1}{K}$ to the distance from the curve to the $x$-axis. $\tilde{\gamma}$ always rejects negative offers, and also rejects any offer at any time $t$ such that $e^{-rt} \leq \frac{1}{K}$. I claim that, for any $u^*_i \in \left( \frac{1}{K}, 2 - \frac{1}{K} \right)$, there is now a PBE in which player $i$ receives payoff $u^*_i$ when both players are normal.

To see this, consider the following strategy profile: Normal player $j$ reveals rationality. If player $i$ reveals rationality, player $j$ plays $D$ and demands $\phi_j(u^*_i)$ in every period, and accepts player $i$’s demand if and only if $\phi_j(u_i) \geq \phi_j(u^*_i)$. If player $i$ does not reveal rationality, player $j$ plays $D$ and demands 2 in every period up to time $\frac{1}{t} \log K$, and never accepts an offer of less than 2 until time $\frac{1}{t} \log K$. At time $\frac{1}{t} \log K$, player $j$ continues playing $D$ and demanding 2 in every period, but switches to accepting player $i$’s demand if and only
if \( \phi_j(u_i) \geq 1 \). To complete the description of on-path play, I specify that normal player \( i \)'s strategy is identical to player \( j \)'s, except that in the subgame after both players reveal rationality player \( i \) demands \( u_i^* \) and accepts player \( j \)'s demand if and only if \( \phi_i(u_j) \geq u_i^* \). Without going into the details, I specify that off-path behavior is as in the proof of Theorem 1, and assert that this behavior is sequentially rational. Thus, to check that this strategy profile is a PBE, one must only check that there are no profitable one-shot deviations at on-path histories.

Clearly, both players' play is optimal after both players reveal rationality. It remains only to check that their play is optimal (at on-path histories) after one's opponent fails to reveal rationality, and that it is optimal to reveal rationality initially. I first verify that player \( j \)'s play is optimal after player \( i \) does not reveal rationality (the argument for player \( i \) is symmetric). If player \( i \) does not reveal rationality, then she must be one of the behavioral types, which implies that it is optimal for player \( j \) to play \( D \) in every period. Also, since player \( j \) always has the option of accepting her opponent's offer of 1, and only type \( \tilde{\gamma} \) ever accepts a demand of more than 1, player \( j \) can do no worse than always demanding 2, the

Figure 1: Evolution of Beliefs Conditional on Facing a Behavioral Opponent
highest demand that offers player \( i \) a non-negative payoff. Furthermore, note that if player \( j \)’s assessment that player \( i \) is of type \( \gamma \) is \( \chi(t) > 0 \), then player \( j \) expects player \( i \) to accept her demand of 2 at rate \( \frac{r}{\chi(0)} \chi(t) = r \). Therefore, for any \( \tau \) such that \( \chi(\tau) > 0 \), player \( j \)’s expected payoff from rejecting player \( i \)’s offer until time \( \tau \) and then accepting is

\[
\int_0^{\tau} e^{-\left(r+r\right)t} \left(r(2) + 0\right) dt + e^{-(r+r)\tau} (1) = 1,
\]

which is the same as player \( j \)’s payoff from accepting player \( i \)’s offer immediately. Therefore, player \( j \)’s decision to reject player \( i \)’s offer whenever \( \chi(t) > 0 \) is (weakly) optimal. Next, observe that \( \chi(t) \) reaches 0 at time \( T \) satisfying

\[
e^{-rT} = \frac{1}{K},
\]

or

\[
T = \frac{1}{r} \log K.
\]

At this time, player \( j \) becomes certain that player \( i \) is of type \( \gamma \), and therefore must accept player \( i \)’s offer of 1. Thus, player \( j \)’s continuation strategy is optimal after player \( i \) does not reveal rationality.

Finally, one must verify that revealing rationality is optimal for both players. Consider player \( i \) first. If player \( j \) is behavioral, then player \( i \)’s payoff is not affected by whether or not she reveals rationality, so it is optimal for her to reveal rationality if and only if it is optimal for her to do so conditional on the event that player \( j \) is normal. If player \( j \) is normal and player \( i \) reveals rationality, player \( i \) receives payoff \( u^*_i \). If player \( j \) is normal and player \( i \) does not reveal rationality, player \( i \) can never receive a positive flow payoff and cannot accept a positive offer \( \phi_i(u_j) \) or have a positive demand of her own accepted until time \( T \). Furthermore, the highest offer she ever receives is 1, and the highest demand of hers that player \( j \) ever accepts is also 1. Therefore, in the event that player \( j \) is normal, player \( i \)’s payoff in the subgame after she does not reveal rationality is no more than

\[
e^{-rT} (1) = \frac{1}{K}.
\]
Since \( u_i^* > \frac{1}{K} \), it follows that player \( i \)'s decision to reveal rationality is optimal. Finally, the same argument applies to player \( j \), and the fact that \( u_j^* < 2 - \frac{1}{K} \) implies that \( \phi_j(u_j^*) > \frac{1}{K} \), so player \( j \)'s decision to reveal rationality is optimal as well. This completes the argument that each player can receive any payoff in \( \left( \frac{1}{K}, 2 - \frac{1}{K} \right) \) when her opponent is normal in some PBE.

I make four brief remarks to conclude the analysis of this example: First, the fact that each player can receive any payoff in \( \left( \frac{1}{K}, 2 - \frac{1}{K} \right) \) when her opponent is normal in a PBE implies that she can receive any ex ante expected payoff in this range when the probability that her opponent is behavioral \( (z) \) is sufficiently small. Second, taking \( K \) large yields a single prior distribution over behavioral types under which almost any efficient, feasible, and individually rational payoff vector is attainable in PBE; that is, there is no need to tailor the prior distribution to the target payoff vector. Third, the above argument does not require type \( \tilde{\gamma} \)'s concession rate to be exactly \( r/\chi(t) \); all that is needed is that type \( \tilde{\gamma} \) concedes at least this quickly.\(^{12}\) Fourth, the smallest \( K \) required for a given payoff vector to be attainable in PBE with the above prior is independent of the discount rate, \( r \). All but the last of these observations also apply to the general model, as will become clear in the following section. Furthermore, the constant \( K \) used in the construction of the modified prior in the general analysis is non-decreasing in \( r \), so \( K \) remains bounded as the players become more patient.

### 1.4 Indeterminacy of Reputation Effects

This section contains the formal statement and proof of the main result. The analysis is complicated by the possibility that players may imitate behavioral types with arbitrary repeated game strategies, rather than only the stationary type considered in the above example. However, the idea that a player will not imitate a given behavioral type if there is a high prior probability on a particular “soft” type whose play initially resembles that type carries over from the example.

Let \( \bar{u}_i \) be \( i \)'s (mixed action) minmax payoff, let \( \bar{u}_i = \phi_i(\bar{y}_i) \), and let \( \hat{u}_i \) and \( \hat{\hat{u}}_i \) be \( i \)'s lowest and highest feasible payoffs, respectively. Let \( \bar{m}_i \) be a mixed action of \( i \)'s that minmaxes \( j \).

\(^{12}\)However, if type \( \tilde{\gamma} \) concedes faster than this, then \( K \) must be larger to support the same range of target payoffs in PBE.
Say that \( u \) is a “PBE payoff of \((G, \pi)\) when both players are normal” if there is a PBE of \((G, \pi)\) that yields expected payoff \( u \) conditional on both players’ being normal. Of course, if \( z_1 \) and \( z_2 \) are small, then \( u \) is a PBE payoff of \((G, \pi)\) when both players are normal if and only if \( u \) is close to an ex ante expected PBE payoff of the normal players in \((G, \pi)\) (since \( G \) is finite), but one need not assume that \( z_1 \) and \( z_2 \) are small. Write \( u \geq (\geq) u' \) if \( u_i \geq (\geq) u_i' \) for \( i \in \{1, 2\} \).

The main result is the following:

**Theorem 1** For any finite game \( G \), vector \( \bar{u} > u \), and number \( \bar{r} > 0 \), there exists a number \( K \geq 1 \) such that, for every prior \( \pi \), there exists a modified prior \( \pi' \) with the following three properties:

1. \( z'_i = z_i \) for \( i \in \{1, 2\} \).
2. \( \pi'_i(\gamma_i) \geq \frac{1}{K} \pi_i(\gamma_i) \) for all \( \gamma_i \in \text{supp} \pi_i \) and \( i \in \{1, 2\} \).
3. For any discount rate \( r \in (0, \bar{r}) \), the set of PBE payoffs of \((G, \pi')\) when both players are normal contains any efficient \( u^* \in \Pi \) such that \( u^* \geq \bar{u} \).

Theorem 1 says that there exists a single modified prior for which the PBE set contains almost any efficient, feasible, and individually rational payoff, for any discount rate below an arbitrary fixed number (in particular, the constant \( \bar{r} \) is chosen freely, and need not be “small”). Also, the extent to which the original prior must be modified to yield such a new prior, measured by \( K \), does not depend on the original prior, \( \pi \), but only on \( G, \bar{u}, \) and \( \bar{r} \). Finally, Theorem 1 does not show that the uniqueness result of AP is sensitive to the addition of a “small” mass of behavioral types that initially pool with another behavioral type. Rather, it shows that their result is sensitive to the addition of a “large” mass of such types, where “largeness” is determined only by \( G \) and \( \bar{u} \).

A noteworthy consequence of Theorem 1 is the existence of a bound on the extent to which the original prior must be modified (\( K \)) that is uniform over discount rates below \( \bar{r} \); in particular, this bound does not explode as the discount rate goes to zero. This contrasts with the results of Fudenberg and Levine (1989), which imply that the prior probability of the Stackelberg type must converge to zero if payoff multiplicity is to persist as \( r \) goes to
The key difference is the presence of equal discounting in my model. In particular, the rate at which player $i$ must be conceding to player $j$ for player $j$ to be willing to reject her demand scales with $r$. This implies that the time $T$ required for player $i$ to convince player $j$ that she is a “tough” type scales with $1/r$. Hence, the resulting cost of delay to player $i$, $e^{-rT}$, is independent of $r$. This argument is exactly as in the example of Section 1.3. Indeed, $K$ only depends on the discount rate at all in Theorem 1 due to a technical issue resulting from the hybrid discrete-time/continuous-time nature of the model. I now outline the proof of Theorem 1. The first step is constructing the modified prior $\pi'$ for given $\pi$, $G$, $\tilde{u}$, and $\tilde{r}$. The goal is constructing a “soft” (henceforth, “offsetting”) type $\gamma_i$ for every $\gamma_i \in \text{supp} \pi_i$ such that $\gamma_i$ has the following properties: On-path, $\gamma_i$ follows the same strategy over $(\Pi_i, M_i)$ as $\gamma_i$ does for a long time; $\gamma_i$ concedes to player $j$ quickly enough that player $j$ does not accept any offer less than $\phi_j(\bar{u}_i)$ when she is not confident whether she is facing type $\gamma_i$ or type $\tilde{\gamma}_i$, but slowly enough that it takes a long time for player $j$ to learn whether she is facing type $\gamma_i$ or type $\tilde{\gamma}_i$; and $\gamma_i$ induces player $j$ to play either $(\bar{u}_j, \bar{m}_j)$ or some other “tough” action for a long time. These “tough” actions of player $j$ that are induced by type $\tilde{\gamma}_i$ are called admissible in the proof of Theorem 1. The point of this construction is that, if player $j$ assigns sufficient weight to his facing type $\tilde{\gamma}_i$, then player $j$ will play an admissible action and reject all offers of less than $\phi_j(\bar{u}_i)$ until some distant time $T_i$. Therefore, if normal player $i$ receives at least $\bar{u}_i$ in some strategy profile and $T_i$ is sufficiently large, then she does not want to pretend to be of type $\gamma_i$, since

---

13 It is easy to see that Fudenberg and Levine's results continue to apply if contracts are allowed, as a patient long-run player could simply imitate the Stackelberg type who rejects all contract offers and demands her highest feasible payoff every period. This relates to the observation in Footnote 6 that allowing contracts often makes reputation results easier to obtain.

14 In contrast, if in the current model player $i$ were made much more patient than player $j$, she could guarantee herself nearly $\bar{u}_i$ in any PBE by making a demand close to $\bar{u}_i$ and playing $m_i$ every period, as long as the behavioral type that follows this strategy is present with positive probability.

15 The issue is that, if player $j$ acts second at an integer time $t$, it is impossible to punish player $j$ for deviating from his prescribed action until time $t + 1$. This friction is larger when $r$ is larger, which necessitates a larger $K$ in the construction in the proof of Theorem 1. This friction would be entirely absent if the continuous-time model were replaced by the limit of discrete-time models as actions become frequent, which is why the fact that $K$ depends on $\tilde{r}$ may be viewed as an artifact of the hybrid discrete time/continuous time nature of the model.

16 An exception to this is that player $j$ may play an inadmissible action in response to an offer by player $i$ of at least $\phi_j(\bar{u}_i)$. However, it can be shown that player $i$ also receives a low payoff in this case, essentially because either player $j$ accepts her generous offer or she accepts a correspondingly aggressive demand of player $j$'s.
she is guaranteed to receive a low flow payoff until time $T_i$. Finally, the modified prior $\pi'$ is defined so that player $j$'s beliefs after player $i$ fails to reveal rationality are that, whichever strategy $\gamma_i \in \text{supp } \pi_i$ player $i$ follows, player $i$ is initially very likely to be an offsetting type.

The second step is verifying that it is indeed optimal for player $j$ to play an admissible action and reject any offer less than $\phi_j (\tilde{u}_i)$ until time $T_i$. This might at first appear to be difficult, because it is very difficult to determine player $j$'s entire optimal strategy under prior $\pi'$. However, to show that it is optimal for player $j$ to play an admissible action and reject any offers less than $\phi_j (\tilde{u}_i)$ at some history $h'$, it suffices to exhibit a single continuation strategy of player $j$'s that involves playing an admissible action and rejecting any offer less than $\phi_j (\tilde{u}_i)$ and yields a higher payoff than any continuation strategy that involves playing any inadmissible action or accepting an offer less than $\phi_j (\tilde{u}_i)$. And, if type $\tilde{\gamma}_i$ concedes at a high enough rate until time $T_i$, then it can be verified that playing the best admissible action in the current period, then playing $(\tilde{u}_j, m_i)$ and rejecting player $i$'s offer until time $T$, and finally accepting player $i$'s offer just after time $T$ yields a higher payoff for player $j$ than does any continuation strategy involving playing any inadmissible action or accepting an offer less than $\phi_j (\tilde{u}_i)$ at history $h'$.

Finally, I construct a PBE in which normal players attain the target payoffs $(u_i^*, u_j^*)$. In this PBE, when both players reveal rationality, they then demand their target payoffs, ending the game immediately with the target payoffs. If player $i$ deviates by failing to reveal rationality, then, as we have seen, she faces only admissible actions until time $T_i$ and her offer is rejected unless she demands less than $\tilde{u}_i$ (with the exception described in Footnote 16), so, regardless of her continuation strategy, she receives a payoff below her target payoff. Hence, both players reveal rationality. The specification of off-path play supporting this behavior is somewhat complicated, and builds on a construction in AP.

The role of the assumption that normal players can "reveal rationality" to each other is as follows: Suppose that, at the beginning of the game, normal player $i$ is supposed to demand $u_i^*$ and normal player $j$ is supposed to demand $\phi_j (u_i^*)$, ending the game. If some behavioral player $j$ offers player $i$ more than $u_i^*$, player $i$ is tempted to demand more than $u_i^*$ in the hope that player $j$ is of this type. If player $j$ turns out to make the normal demand $\phi_j (u_i^*)$, player $i$ can simply accept this offer an instant after it is made and thus
go unpunished for her experimentation. Allowing normal players to reveal rationality to each other before the beginning of play eliminates this problem, since, if player \(j\) reveals rationality, normal player \(i\) has no reason to experiment with higher initial demands. Since this is the only point in the proof where the ability of normal players to reveal rationality matters, any modification of the game that prevents normal player \(i\) from experimenting in this way allows us to eliminate the revelation stage.\(^{17}\) The revelation stage can also be eliminated without such a modification if \(z_1, z_2,\) and \(r\) are sufficiently small. The idea is that the strategy profile constructed in the proof of Theorem 1 can be modified by prescribing that normal player 1 initially demands \(\hat{u}_1\) and normal player 2 initially demands \(\phi_2(u^*_1),\) player 1 immediately accepts player 2’s demand if it equals \(\phi_2(u^*_1),\) and the first player who deviates receives her minmax payoff if agreement is not reached by time 1. Under this strategy profile, neither player has an incentive to experiment in the above manner, because player 1 is already demanding her highest feasible payoff, and, when \(z_1\) is small, player 2 receives less than her minmax payoff if she deviates and then accepts when player 1 demands \(\hat{u}_1.\) The requirement that \(r\) is also small is needed to ensure that a player is willing to wait and receive her highest feasible and individually rational at the next integer time when her opponent deviates. The details of this construction are available upon request.

**Proof of Theorem 1. Step 1: Construction of \(\pi'\)**

I begin by constructing, for any type \(\gamma_i,\) an offsetting type \(\hat{\gamma}_i\) that at every instant either follows the strategy of type \(\gamma_i\) or concedes. The motivation for the details of the specification of \(\hat{\gamma}_i\) will become clear in Step 2 of the proof.

First, observe that the theorem is trivial if there is no vector \(u^* \in \Pi\) such that \(u^* \geq \hat{u},\) so assume that such a vector exists. Since \(\hat{u} > u\) and \(\phi_j\) is decreasing, this implies that \(\tilde{u}_j > \phi_j(\tilde{u}_i) > u_j.\) Let

\[
\tilde{u}_i^0 = \frac{\tilde{u}_i + u_i}{2},
\]

\(^{17}\)For example, rather than allowing normal players to reveal rationality one could impose an \(\varepsilon\) penalty on both players for failing to come to agreement immediately, and assume that \(z_1\) and \(z_2\) are sufficiently small that this penalty outweighs any incentive to experiment, i.e., that \(\varepsilon (1 - z_j) > (\hat{u}_i - u_i) z_j\) for all \(i\) (proof available upon request).
let
\[ \rho_i \equiv \max \left\{ \frac{\phi_j(u_i^0) - \tilde{u}_j}{\tilde{u}_j - \tilde{u}_j - \phi_j(u_i^0) \tilde{u}_j - \phi_j(u_i^0)}, \frac{\nu_j - \phi_j(u_i^0) / \nu_j - \phi_j(u_i^0)}{\nu_j - \phi_j(u_i^0)} \right\}, \]

and let
\[ K \equiv \frac{\phi_j(u_i^0) - \tilde{u}_j}{\tilde{u}_j - \phi_j(u_i^0) / \nu_j - \phi_j(u_i^0)}. \]

Observe that \( \rho_i < 1 \), so \( K \) is finite. This number \( K \) will suffice for the proof.

Fix a discount rate \( r \in (0, \bar{r}) \). Let
\[ \lambda_i \equiv \frac{\phi_j(u_i^0) - \tilde{u}_j}{\tilde{u}_j - \phi_j(u_i^0)}, \]

and let
\[ \chi_i(t) \equiv 1 - \frac{e^{\lambda_i t}}{K}. \]

To understand the definition of \( \chi_i(t) \), suppose that initially player \( i \) plays strategy \( \gamma_i \) with probability \( \frac{1}{K} \), and with probability \( \frac{K-1}{K} \) plays a different strategy that up to time \( t \) plays the same \( (\pi_i, m_i) \) as does \( \gamma_i \) and also accepts player \( j \)'s offer at a hazard rate that makes the unconditional hazard rate that player \( i \) accepts player \( j \)'s offer equal \( \lambda_i \). Then \( \chi_i(t) \) is a lower bound on the probability that player \( i \) is not playing strategy \( \gamma_i \) at time \( t \) conditional on the event that she has not accepted player \( j \)'s offer by time \( t \) (indeed, \( \chi_i(t) \) is exactly this probability if and only if strategy \( \gamma_i \) never accepts player \( j \)'s offer before time \( t \)).

Next, let \( T \) be the time at which \( \chi_i(t) \) reaches \( \rho_i \); that is,
\[ T_i \equiv \frac{1}{\lambda_i} \log (K (1 - \rho_i)). \]

Finally, let
\[ \lambda_i(t) \equiv \frac{\lambda_i(t)}{\chi_i(t)} \]
if \( \chi_i(t) > 0 \), and let \( \lambda_i(t) = 0 \) otherwise. If \( \lambda_i(t) \) is player \( i \)'s acceptance rate conditional on not playing \( \gamma_i \), and \( \chi_i(t) \) is the probability that player \( i \) is not playing \( \gamma_i \), then \( \lambda_i(t) \) is player \( i \)'s unconditional acceptance rate.

I now define admissible actions, as previewed in the outline of the proof:
Definition 1 At an integer time $t$ at which player $j$ acts first, the action $(\tilde{u}_j, m_j)$ is admissible and all other actions are inadmissible. At an integer time $t$ at which player $j$ acts second, the action $(\tilde{u}_j, m_j)$ is admissible, as is the action $(\max \{\tilde{u}_j, U_j(m_i(t), m_j)\}, m_j)$ for any $m_j$ that satisfies $m_j \in \arg \max_{m_j} U_j(m_i(t), m_j)$ and $U_j(m_i(t), m_j) > \phi_j(w_0)$, where $m_i(t)$ is player $i$'s realized time-$t$ action; and all other actions are inadmissible. A history $h^t$ is admissible if player $j$ has never played an inadmissible action, and is inadmissible otherwise.

We are ready to define type $\tilde{\gamma}_i$:

Definition 2 Given any type $\gamma_i$, the $\gamma$-offsetting type, $\tilde{\gamma}_i$, is the strategy defined as follows: If history $h^t$ is admissible, $\tilde{\gamma}_i$ plays the same $(u_i, m_i)$ as does $\gamma_i$, and $\tilde{\gamma}_i$ accepts player $j$’s demand at hazard rate $\lambda_i(t)$; if in addition $t = T_i$, then $\tilde{\gamma}_i$ accepts player $j$’s demand with probability 1. If $h^t$ is inadmissible, $\tilde{\gamma}_i$ plays $(\tilde{u}_i, m_i)$ and rejects player $j$’s demand.

Finally, I define the modified prior $\pi'_i$. In this definition, $\tilde{\gamma}'_i$ is the $\gamma'_i$-offsetting type defined above.

Definition 3 The modified prior $\pi'_i$ is given by $\pi'_i(\gamma_i) = \frac{1}{K} \pi_i(\gamma_i) + \sum_{\gamma'_i \in \text{supp } \pi_i : \tilde{\gamma}'_i = \gamma'_i} \frac{K-1}{K} \pi_i(\gamma'_i)$ for all types $\gamma_i$.

Observe that $z'_i = z_i$ and $\pi'_i(\gamma_i) \geq \frac{1}{K} \pi_i(\gamma_i)$ for all $\gamma_i \in \text{supp } \pi_i$. Thus, to prove the theorem it suffices to show that the set of PBE payoffs of $(G, \pi')$ when both players are normal contains any efficient $u^* \in \Pi$ such that $u^* \geq \tilde{u}$. This is done in Steps 2 and 3 of the proof.

Step 2: Behavior under $\pi'_i$

I now establish the key property of the modified prior $\pi'_i$. Under a strategy profile in the game $(G, \pi')$ in which normal players reveal rationality, player $j$’s optimal continuation strategy at any history $h^t$ that is consistent with player $i$ following some strategy $\gamma_i \in \text{supp } \pi'_i$ is determined up to indifference by sequential rationality. Say that an action of player $j$’s is (weakly) optimal at such a history if it is part of an optimal continuation strategy.
Lemma 1  Fix a strategy profile in the game \((G, \pi')\) in which normal players reveal rationality. Suppose that history \(h^t\) is admissible, player \(i\)'s past play at \(h^t\) is consistent with her following some strategy \(\gamma_i \in \text{supp} \pi'_i\), and \(t < T_i\). Then the following two statements hold at \(h^t\):

1. Suppose that player \(i\)'s demand is at least \(\tilde{u}_i\). Then it is optimal for player \(j\) to reject player \(i\)'s demand. If in addition \(t\) is an integer, then it is optimal for player \(j\) to play an admissible action.

2. If \(t\) is an integer at which player \(j\) acts second and it is optimal for player \(j\) to reject player \(i\)'s demand and play an inadmissible action, then under any optimal continuation strategy agreement is reached by time \(t + 1\) and player \(j\) receives continuation payoff at least \(\phi_j (u^0_j)\).

Proof of Lemma 1. First, suppose that player \(i\)'s time-\(t\) demand is at least \(\tilde{u}_i\). I show that rejecting player \(i\)'s demand at time \(t\) is optimal by exhibiting a strategy that involves rejected player \(i\)'s demand at time \(t\) and yields a weakly higher payoff than accepting this demand. In particular, suppose that player \(j\) plays \((\bar{u}_j, m_j)\) from the next integer time onward and rejects player \(i\)'s demand until just after time \(T_i\), and then accepts player \(i\)'s demand. When player \(j\) follows this strategy, player \(i\) concedes at unconditional rate at least \(\lambda_i\) at all times earlier than \(T_i\) (since at such times there is probability at least \(\chi_i(t)\) that she is conceding at rate \(\dot{\lambda}_i(t)\), as player \(j\) is always playing an admissible action), and concedes at time \(T_i\) with unconditional probability at least

\[
\chi_i(T_i) = \rho_i \geq \frac{\phi_j (u^0_i) - \tilde{u}_j}{\bar{u}_j - \hat{u}_j}.
\]

Therefore, player \(j\)'s continuation payoff from this strategy is at least

\[
\int_0^{T_i - t} e^{-\left(r + \lambda_i\right)t} (\lambda_i \tilde{u}_j + r \hat{u}_j) dt + e^{-\left(r + \lambda_i\right)(T_i - t)} \left( \frac{\phi_j (u^0_i) - \tilde{u}_j}{\bar{u}_j - \hat{u}_j} (\tilde{u}_j) + \frac{\tilde{u}_j - \phi_j (u^0_j)}{\bar{u}_j - \hat{u}_j} (\hat{u}_j) \right) \\
= \left(1 - e^{-\left(r + \lambda_i\right)(T_i - t)}\right) \frac{1}{r + \lambda_i} \left( \lambda_i \tilde{u}_j + r \hat{u}_j \right) + e^{-\left(r + \lambda_i\right)(T_i - t)} \phi_j (u^0_i) \\
= \left(1 - e^{-\left(r + \lambda_i\right)(T_i - t)}\right) \phi_j (u^0_i) + e^{-\left(r + \lambda_i\right)(T_i - t)} \phi_j (u^0_i) \\
= \phi_j (u^0_i).
\]
On the other hand, player $j$'s continuation payoff from accepting player $i$'s demand at time $t$ is at most $\phi_j(\hat{u}_i)$. This is less than $\phi_j(u_i^0)$, so it is optimal for player $j$ to reject player $i$'s demand at time $t$.

Second, suppose that $t$ is an integer at which player $j$ acts first. Player $j$’s continuation payoff from rejecting player $i$’s demand and playing any action other than $(\hat{u}_j, m_j)$ (the unique admissible action) is at most

$$\chi_i(t)\hat{u}_j + (1 - \chi_i(t))\hat{u}_j,$$

because if player $i$ is of type $\tilde{\gamma}_i$ (which is the case with probability at least $\chi_i(t)$) she immediately minmaxes player $j$. Since $\chi_i(t)$ is decreasing and $t < T$, this is less than

$$\chi_i(T)\hat{u}_j + (1 - \chi_i(T))\hat{u}_j = \phi_j(\hat{u}_i).$$

As we have seen, $\phi_j(\hat{u}_i)$ is a lower bound on player $i$’s continuation from rejecting player $i$’s demand and following (at least) one strategy that involves playing $(\hat{u}_j, m_j)$. Therefore, if player $i$’s time-$t$ demand is at least $\tilde{u}_i$, then it is optimal for player $j$ to reject this demand and play $(\hat{u}_j, m_j)$.

Third, suppose that $t$ is an integer at which player $j$ acts second. Let $U_j^* = \max_{m_j} U_j(m_i(t), m_j)$ and let $m_j^* \in \arg\max_{m_j} U_j(m_i(t), m_j)$. Then playing $(\max\{\tilde{u}_j, U_j^*\}, m_j^*)$ at time $t$ and subsequently playing $(\tilde{u}_j, m_j)$ and rejecting player $i$’s demand until just after time $T_i$ (and then accepting) yields continuation payoff at least

$$(1 - e^{-\tau}) \max\{U_j^*, \phi_j(u_i^0)\} + e^{-\tau}\phi_j(u_i^0).$$

(1)

This follows because, if $U_j^* \geq \phi_j(u_i^0)$, then player $j$ receives flow payoff $U_j^*$ and demands $U_j^*$.
from time $t$ to time $t + 1$; and we have already seen that player $j$'s continuation payoff from playing $(\bar{u}_j, m_j)$ and rejecting player $i$'s demand until time $T_i$ is at least $\phi_j (u^0_i)$. On the other hand, playing any inadmissible action at time $t$ yields continuation payoff at most

\[
(1 - \chi_i (t)) \hat{u}_j + \chi_i (t) \left( (1 - e^{-r}) \max \{ U_j^*, \phi_j (u_i) \} + e^{-r} \phi_j (\bar{u}_i) \right),
\]

where $u_i$ is player $i$'s time-$t$ demand, because type $\hat{\gamma}_i$ responds to an inadmissible action by always rejecting player $j$'s demand and minmaxing player $j$ starting at time $t + 1$. If $u_i \geq \bar{u}_i$, then (2) is at most

\[
(1 - \chi_i (t)) \hat{u}_j + \chi_i (t) \left( (1 - e^{-r}) \max \{ U_j^*, \phi_j (\bar{u}_i) \} + e^{-r} \phi_j (\bar{u}_i) \right),
\]

and therefore the difference between (1) and (2) is at least

\[
(1 - \chi_i (t)) (1 - e^{-r}) \phi_j (u^0_i) + e^{-r} \left( \phi_j (u^0_i) - \chi_i (t) \phi_j (\bar{u}_i) \right) - (1 - \chi_i (t)) \hat{u}_j.
\]

This expression is non-negative if and only if

\[
\chi_i (t) \geq \frac{\hat{u}_j - \phi_j (u^0_i)}{\hat{u}_j - ((1 - e^{-r}) \phi_j (u^0_i) + e^{-r} \phi_j (\bar{u}_i))}.
\]

Since $r \leq \bar{r}$, a sufficient condition for this inequality is

\[
\chi_i (t) \geq \frac{\hat{u}_j - \phi_j (u^0_i)}{\hat{u}_j - ((1 - e^{-\bar{r}}) \phi_j (u^0_i) + e^{-\bar{r}} \phi_j (\bar{u}_i))}
\]

Now $\chi_i (t) \geq \chi_i (T) = \rho_i \geq \frac{\bar{u}_j - \phi_j (u^0_i)}{\bar{u}_j - ((1 - e^{-\bar{r}}) \phi_j (u^0_i) + e^{-\bar{r}} \phi_j (\bar{u}_i))}$, so this sufficient condition holds. Hence, it is optimal for player $j$ to play an admissible action at time $t$ whenever player $i$'s demand is at least $\bar{u}_i$.

Finally, if player $j$ plays an inadmissible action at time $t$, then just before time $t + 1$ his continuation payoff from rejecting player $i$'s demand is at most

\[
\chi_i (t) \bar{u}_j + (1 - \chi_i (t)) \hat{u}_j,
\]
because the probability that player $i$ is of type $\tilde{\gamma}_i$ at this time is at least $\chi_i(t)$, and type $\tilde{\gamma}_i$ minmaxes player $j$ starting at time $t + 1$. Since $\chi_i(t) \leq \chi_i(T) \leq \frac{\hat{u}_j - \phi_j(\tilde{u}_i)}{\hat{u}_j - \underline{u}_j}$, this is no more than
\[
\frac{\hat{u}_j - \phi_j(\tilde{u}_i)}{\hat{u}_j - \underline{u}_j} \underline{u}_j + \left(1 - \frac{\hat{u}_j - \phi_j(\tilde{u}_i)}{\hat{u}_j - \underline{u}_j}\right) \hat{u}_j = \phi_j(\tilde{u}_i).
\]
Hence, if $u_i < \hat{u}_i$, then agreement is reached by time $t + 1$ under any optimal continuation strategy. Finally, as we have seen, it is optimal for player $j$ to play an admissible action at time $t$ if $u_i \geq \hat{u}_i$, and this yields continuation payoff at least $\phi_j(u^0_i)$. Therefore, at time $t$ it is optimal for player $j$ to reject player $i$’s demand and play an inadmissible action only if, under any optimal continuation strategy, agreement is reached by time $t + 1$ and he receives continuation payoff at least $\phi_j(u^0_i)$. ■

Step 3: Equilibrium Construction

I now construct strategy profiles for the normal types in $(G, \pi')$ that yield the desired range of PBE payoffs. The construction builds on that in Lemma 24 of AP. The next paragraph specifies on-path play, and the three paragraphs after it specify off-path play.

Fix $u^*_i \in [\hat{u}_i, \phi_i(\tilde{u}_j)]$. Normal players reveal rationality. If both players reveal rationality, normal player $i$ initially plays $(u^*_i, m_i)$ and normal player $j$ initially plays $(\phi_j(u^*_i), m_j)$. Thus, if both players are normal and follow their equilibrium strategies, the game ends immediately with payoffs $(u^*_i, \phi_j(u^*_j))$. If player $i$ does not reveal rationality and at history $h'$ her play is consistent with some type $\gamma_i \in \text{supp} \pi_i$, then normal player $j$ is certain that her opponent is behavioral, and her on-path continuation play is pinned down up to indifference by sequential rationality. At any history at which player $j$ is certain that player $i$ is behavioral and is indifferent between accepting and rejecting player $i$’s offer, specify that she rejects; and at any history at which player $j$ is certain that player $i$ is behavioral and is indifferent between playing $(\tilde{u}_j, m_j)$ and playing any other action, specify that she plays $(\tilde{u}_j, m_j)$. Player $j$’s play at other histories at which he is certain that player $i$ is behavioral and is indifferent between any two actions is irrelevant and can therefore be specified arbitrarily.

Off-path play in the subgame after both players reveal rationality is as in Lemma 24 of AP. In particular, if player $i$ deviates to an incompatible demand, then player $i$ plays $(u_i, m_i)$ and player $j$ plays $(\tilde{u}_j, m_j)$ at the next integer time, and in the interim player $i$
accepts player \( j \)'s demand and player \( j \) rejects player \( i \)'s demand. The same continuation play follows any single-player deviation. Next, suppose that both players deviate from their prescribed play (to incompatible demands) at integer time \( t \). If player 1's flow payoff given the realized time-\( t \) actions is weakly less than player 2's offer to her, then she accepts player 2's demand, and player 2 rejects her demand. If this condition fails for player 1 but holds for player 2, then player 2 accepts and player 1 rejects. If both players' flow payoffs are strictly greater than their opponents' offers to them, then neither player accepts until the next integer time, at which point continuation play is as at the beginning of the subgame after both players reveal rationality, with \( u_i^* \) replaced by player \( i \)'s flow payoff.

Next, consider the subgame after player \( j \) reveals rationality and player \( i \) does not. Suppose that at time \( t \) player \( i \)'s play becomes inconsistent with all types \( \gamma_i \in \text{supp } \pi_i \) (i.e., at time \( t \) player \( i \) either makes or rejects a demand that makes the history inconsistent with all \( \gamma_i \in \text{supp } \pi_i \)). Then player \( i \) plays \((u_i, m_i)\) and player \( j \) plays \((\bar{u}_j, m_j)\) at the next integer time. In the interim, if player \( i \)'s flow payoff given the realized time-\( t \) actions is weakly less than player \( j \)'s offer to her, then she accepts player \( j \)'s demand, and player \( j \) rejects her demand. If this condition fails for player \( i \), then player \( j \) accepts player \( i \)'s demand if and only if this yields a higher payoff than receiving his flow payoff until the next integer time and then receiving \( \bar{u}_j \). If both of these conditions fail, then neither player accepts until just before the next integer time, at which point player \( i \) accepts player \( j \)'s demand, if this demand is no more than \( \bar{u}_j \). Continuation play following the next integer time is as at the beginning of the subgame after both players reveal rationality, with \( u_i^* \) replaced by \( u_i \). Now, as long as player \( i \)'s play is consistent with some type \( \gamma_i \in \text{supp } \pi_i \), player \( j \)'s play is pinned down by sequential rationality and an arbitrary rule for breaking indifferences. Hence, the above specification of play after player \( i \) deviates from any type \( \gamma_i \in \text{supp } \pi_i \) determines player \( i \)'s entire optimal continuation strategy, again up to indifference.

Finally, consider the subgame after neither player reveals rationality. If at time \( t \) player \( j \)'s play becomes inconsistent with all types \( \gamma_j \in \text{supp } \pi_j \), then continuation strategies at the resulting history \( h^t \) are identical to continuation strategies at history \( h'^t \), defined to equal \( h^t \) with the modification that player \( j \) initially revealed rationality (which are specified in the previous paragraph). That is, continuation play is "as if" player \( j \) had revealed rationality.
Similarly, if at time $t$ both players’ play simultaneously becomes inconsistent with all of their types in $\text{supp} \pi'$, then continuation play is specified to be the continuation play at the corresponding history where both players have revealed rationality.

It is clear that each player’s strategy is optimal at every history except the null history before the players do or do not reveal rationality. Therefore, to check that the above strategy profile is a PBE, it suffices to check that revealing rationality is optimal for both players. This in turn requires only checking that revealing rationality is optimal for player $i$ conditional on player $j$’s being normal, as revealing rationality has no effect on play if player $j$ is behavioral, by the assumption that behavioral types do not condition their play on whether their opponents reveal rationality.

Suppose that player $j$ is normal and normal player $i$ does not reveal rationality. At any admissible history, if player $i$ takes an action that is not consistent with any type $\gamma_i \in \text{supp} \pi'_i$, then her continuation payoff equals $u_i$ (by the above specification of off-path play). Thus, suppose that player $i$’s play remains consistent with some type $\gamma_i \in \text{supp} \pi'_i$ at all admissible histories. By Lemma 1, continuation play prior to time $T_i$ falls into one of two categories: either player $j$ always plays an admissible action and only accepts demands of no more than $\tilde{u}_i$; or, at some integer time $t$ at which player $j$ acts second, player $j$ play an inadmissible action and agreement is reached by the next integer time. I now show that in either case player $i$’s payoff is no more than $u_i^*$, regardless of her strategy.

In the former case, the fact that $u_i^* \geq \tilde{u}_i > u_i$ implies that player $i$ may receive a payoff strictly above $u_i^*$ only if the game does not end before $T_i$. Since in this case player $j$ plays either $m_j$ or some action $m_j$ such that $U_j(m_i(t), m_j) > \phi_j(u_i^0)$ (and thus $U_i(m_i(t), m_j) \leq u_i^0$) at every time $t < T_i$, player $i$’s payoff is at most

$$(1 - e^{-rT_i}) u_i^0 + e^{-rT_i} \tilde{u}_i.$$
Now
\[ e^{-rT_i} = (K(1 - \rho_i))^{-r/\lambda_i} \]
\[ = \left( \frac{\tilde{u}_i - u_i}{u_i^0 - u_i} \right)^{\lambda_i/\rho} \]
\[ = \frac{\tilde{u}_i - u_i}{u_i^0 - u_i}. \]

Hence,
\[ (1 - e^{-rT_i}) u_i + e^{-rT_i} \hat{u}_i \leq \left( \frac{\tilde{u}_i - u_i}{u_i^0 - u_i} \right) u_i + \left( \frac{\tilde{u}_i - u_i}{u_i^0 - u_i} \right) \hat{u}_i = \tilde{u}_i \leq u_i^*. \]

In the latter case, recalling that agreement is reached before time \( t + 1 \) if player \( j \) plays an inadmissible action at time \( t \), leaving player \( j \) with payoff at least \( \phi_j(u_i^0) \), player \( i \)'s continuation payoff cannot be higher than the maximum of player \( i \)'s time-\( t \) demand, denoted \( u_i \), and the continuation payoff of type \( \tilde{\gamma}_i \), denoted \( u_i^\tilde{\gamma}_i \) (this is because player \( i \)'s time-\( t \) action \( m_i \) is the same as the time-\( t \) action of types \( \gamma_i \) and \( \tilde{\gamma}_i \), because player \( j \) acts second at time \( t \)). I claim that both of these values are weakly less than \( \tilde{u}_i \) (and thus weakly less than \( u_i^* \)).

First, the fact that player \( j \)'s continuation payoff is at least \( \phi_j(u_i^0) \) implies that
\[ (1 - \chi_i(t)) \hat{u}_j + \chi_i(t) \left( (1 - e^{-r}) \hat{u}_j + e^{-r} \phi_j(u_i) \right) \geq \phi_j(u_i^0), \]
and therefore
\[ \phi_j(u_i) \geq \hat{u}_j - \frac{e^r}{\chi_i(t)} \left( \hat{u}_j - \phi_j(u_i^0) \right). \]

Hence,
\[ u_i \leq \phi_i \left( \hat{u}_j - \frac{e^r}{\chi_i(t)} \left( \hat{u}_j - \phi_j(u_i^0) \right) \right) \]
\[ \leq \phi_i \left( \hat{u}_j - \frac{e^r}{\chi_i(T)} \left( \hat{u}_j - \phi_j(u_i^0) \right) \right) \]
\[ \leq \tilde{u}_i. \]

Second, because player \( i \) cannot receive continuation payoff greater than \( \phi_j \left( u_i^\gamma_i \right) \) when
player $i$ is of type $\hat{x}_i$,

$$(1 - \chi_i(t)) \hat{u}_j + \chi_i(t) \phi_j \left( \hat{u}_{i}^{\hat{x}_i} \right) \geq \phi_j \left( u_{i}^{0} \right).$$

Hence,

$$\phi_j \left( u_i^{\hat{x}_i} \right) \geq \hat{u}_j - \frac{1}{\chi_i(t)} \left( \hat{u}_j - \phi_j \left( u_{i}^{0} \right) \right),$$

which implies that $u_i^{\hat{x}_i} \leq \hat{u}_i$ by the same argument as above.

We conclude that player $i$ receives payoff weakly below $u_i^*$ if she does not reveal rationality, which implies that failing to reveal rationality is not a profitable deviation for player $i$. The same argument applies to player $j$, because the fact that $u_i^* \leq \phi_j \left( \hat{u}_j \right)$ implies that $\phi_j \left( u_i^* \right) \geq \hat{u}_j$. Therefore, the above strategy profile is a PBE. ■

1.5 Conclusion

This paper shows that allowing players to sign binding contracts governing future play does not lead to reputation effects in the absence of assumptions on the relative probabilities of different behavioral types. This suggests that equilibrium selection due to reputation effects is substantially weaker in games with two long-run players than in games with a single long-run player, even in the presence of contracts, and that existing results do not provide a completely convincing equilibrium selection argument for applications in which different behavioral types may not be immediately distinguishable.

However, I reiterate that AP’s uniqueness result is robust to introducing a small mass of behavioral types that initially pool with other behavioral types; in particular, AP’s result continues to hold when behavioral types are not immediately distinguishable if the prior probability of the Nash bargaining with threats is high enough relative to the prior probability of “softer” types whose early play resembles that of the Nash bargaining with threats type. This raises the intriguing question of where the boundary between AP’s uniqueness result and my multiplicity results lies. That is, for what prior distributions of behavioral types do repeated games with contracts have unique equilibria, and for what priors does the folk theorem apply? What happens in the transitional region between these regimes? Relatedly,
my arguments suggest that some behavioral types may be more profitably imitated for a wide range of prior distributions than others; for example, a player must be very confident that her opponent is really a soft type for her to keep playing when her opponent imitates a type that gets “tougher” over time, as this behavior penalizes her for failing to concede. The next essay in this thesis suggests that this approach may be tractable: there, I characterize the behavioral type that is most profitably imitated in bargaining by a player who holds “worst-case” beliefs about her opponent’s prior belief about her own strategy, and show that this type does indeed get “tougher” over time. I view these ideas as interesting directions for future research.
2 Reputational Bargaining Under Knowledge of Rationality

2.1 Introduction

Economists have long been interested in how individuals split gains from trade. The division of surplus often determines not only equity, but also efficiency, as it affects individuals' ex ante incentives to make investments; this effect of surplus division on efficiency is a major theme of, for example, property rights theories of the firm (Grossman and Hart, 1986), industrial organization models of cumulative innovation (Green and Scotchmer, 1995), and search-and-matching models of the labor market (Hosios, 1990). Recently, "reputational" models of bargaining have been developed that make sharp prediction about the division of the surplus independently of many details of the bargaining procedure (Myerson, 1991; Abreu and Gul, 2000; Kambe, 1999; Compte and Jehiel, 2002). In these models, players may be committed to a range of possible bargaining strategies, or "postures," before the start of bargaining, and bargaining consists of each player attempting to convince her opponent that she is committed to a strong posture. These models assume that the probabilities with which the players are committed to various bargaining postures (either ex ante or after a stage where players strategically announce bargaining postures) are common knowledge, and that play constitutes a (sequential) equilibrium. In this paper, I study reputational bargaining while assuming only that the players know that each other is rational, and show that each player can guarantee herself a relatively large share of the surplus—even if her probability of being committed is small—by announcing the posture that simply demands this share plus compensation for any delay in reaching agreement. Furthermore, announcing any other posture does not guarantee as much.

A key feature of my model is the existence of a positive number $\varepsilon$ such that, if a player announces any bargaining posture (i.e., any infinite path of demands) at the beginning of the game, she then becomes committed to that posture with probability at least $\varepsilon$ (or, equivalently, she convinces her opponent that she is committed to that posture with probability at least $\varepsilon$). I derive the highest payoff that a player can guarantee herself by announcing
any posture, regardless of her opponent’s beliefs about her bargaining strategy, *so long as her opponent is rational and believes that she is committed to her announced posture with probability at least* $\varepsilon$. More precisely, player 1’s “highest guaranteed,” or “maxmin,” payoff is the highest payoff $u_1$ with the property that there exists a corresponding posture (the “maxmin posture”) and bargaining strategy such that player 1 receives at least $u_1$ whenever she announces this posture and follows this strategy and player 2 plays *any* best-response to *any* belief about player 1’s strategy that assigns probability at least $\varepsilon$ to player 1 following her announced posture. In particular, player 2 need not play a best-response to player 1’s actual strategy, or vice versa; thus, player 1’s maxmin payoff is below her lowest Nash equilibrium payoff.

The main result of this paper characterizes the maxmin payoff and posture when only one player may become committed to her announced posture; as discussed below, a very similar characterization applies when both players may become committed. While the maxmin payoff may be very small when $\varepsilon$ is small in general two-person games, it is relatively large in my model: in particular, it equals $1/(1 - \log \varepsilon)$. This equals 1 when $\varepsilon = 1$ (i.e., when the player makes a take-it-or-leave-it offer) and, more interestingly, goes to 0 very slowly as $\varepsilon$ goes to 0 (more precisely, it goes to 0 at a logarithmic rate, which is slower than any polynomial rate). For example, a bargainer can guarantee herself approximately 30% of the surplus if her commitment probability is 1 in ten; 13% if it is 1 in 1 thousand; and 7% if it is 1 in 1 million. The second part of the main result is that the unique bargaining posture that guarantees this share of the surplus simply demands this share in addition to compensation for any delay; that is, it demands a share of the surplus that increases at rate equal to the discount rate, $r$. This compensation amounts to the entire surplus after a long enough delay, so the unique maxmin posture demands

$$\min \left\{ e^{rt} / (1 - \log \varepsilon), 1 \right\}$$

at every time $t$. This posture is depicted in Figure 2, for commitment probability $\varepsilon = 1/1000$ and discount rate $r = 1$.

The intuition for the result that the unique maxmin posture demands compensation for
delay involves two key ideas. First, when player 2’s beliefs are those that lead him to reject player 1’s demand for as long as possible, player 1’s demand is accepted sooner when it is lower. This is analogous to the argument in the existing reputational bargaining literature that player 1 builds reputation more quickly in equilibrium when her current demand is lower, though my analysis is not based on equilibrium. Second, the maxmin posture can never make demands that would give player 1 less than her maxmin payoff if they were accepted, because player 2 could simply accept some such demand and give player 1 a payoff below her maxmin payoff, which was supposed to be guaranteed to player 1 (though it must be verified that such behavior by player 2 is rational). Combining these ideas implies that player 1 must always demand at least her maxmin level of utility (hence, compensation for delay), but no more.

Three distinctive features of my approach are the timing of commitment (players freely choose which bargaining postures to announce, but may become bound by their announcements), the range of bargaining postures players may announce (all possible paths of demands for the duration of bargaining), and the solution concept (first-order knowledge of rationality). The timing of commitment is appropriate if players are rational but may credibly announce bargaining postures. This assumption has many precedents in the literature,
starting with Schelling (1956), who discusses observable factors that make announced postures more credible, corresponding to a higher value of \( \varepsilon \) in my model. The assumption that a player may announce any path of demands and that all such announcements are equally credible seems unappealing a priori, because announcing a "simple" posture may be more credible than announcing a "complicated" posture. Fortunately, the unique maxmin posture is simply announcing, "I want a certain share of the surplus, and if you make me wait to get it then you must compensate me for the delay." Thus, allowing players to credibly announce complicated postures ensures that my characterization of the maxmin posture does not depend on ad hoc restrictions on the range of credible postures, but my characterization would still apply if only "simple" postures were credible. In addition, the techniques I develop would allow one to characterize the maxmin payoff and posture in a more general model where credibility varies across announcements.

A player's maxmin payoff is her lowest payoff consistent with (first-order) knowledge of rationality (at the start of the game). In bargaining, a player cannot guarantee herself any positive payoff without knowledge of her opponent's rationality, as, for example, she receives payoff 0 if her opponent always rejects her offer and demands the entire surplus. Thus, knowledge of rationality is the weakest solution concept consistent with positive guaranteed payoffs. Furthermore, I show that any feasible payoff greater than the maxmin payoff is consistent with knowledge of rationality, which implies that the maxmin payoff and posture are the key objects of interest under knowledge of rationality.

Imposing only knowledge of rationality rather than a stronger solution concept, such as rationalizability or equilibrium, leads to more robust predictions. In particular, predictions

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19 For example, an announcement is more credible if the stakes in the current negotiation are small relative to the stakes in potential future negotiations; if the announcement is observable to a large number of third parties; if the bargainer can side-contract with third parties to bind herself to her announcements; if the bargainer may be acting as an agent for a third party and does not have independent authority to change her posture; or if the bargainer displays emotions that suggest an unwillingness to modify her posture.

20 To my knowledge, this is the first bargaining model that predicts that such a posture will be adopted, though it seems like a reasonable bargaining position to stake out. For example, in most U.S. states defendants must pay "prejudgment interest" on damages in torts cases, which amounts to plaintiffs demanding the initial damages in addition to compensation for any delay (e.g., Knoll, 1996); similarly, unions sometimes include payment for strike days among their demands.
under knowledge of rationality (such as the prediction that each player receives at least her maxmin payoff) do not depend on each player’s beliefs about her opponent’s strategies, so long as each player believes that her opponent is playing a best-response to some belief about her own play; and also do not depend on unmodelled strategic considerations that do not affect a player’s payoffs or her beliefs about her opponent’s payoffs, but may affect higher-order beliefs about payoffs. Such considerations arise naturally in bargaining: for example, a local union that strategically announces that it will strike until offered a wage of at least $25 an hour may be concerned that the firm’s management may believe that it is actually required by the national union to strike until offered a wage of at least $20 an hour. The maxmin payoff and posture may also be viewed as the predicted payoff and posture of a positive theory of bargaining in which each player is either maximally pessimistic (in a Bayesian sense) about her opponent’s strategy or expects her opponent to play her “worst-case” strategy (in a maxmin sense), given her knowledge of her opponent’s rationality. These two approaches are equivalent in my model, though they differ in general games.

I consider two main extensions of the model. First, I characterize the maxmin payoffs and postures when both players may become committed to their announced postures. I find that each player’s maxmin posture is exactly the same as in the one-sided commitment model, and that each player’s maxmin payoff is close to her maxmin payoff in the one-sided commitment model as long as her opponent’s commitment probability is small. Thus, the one-sided commitment analysis applies to each player separately.

Second, I consider the role of the (continuous-time or discrete-time) bargaining procedure. Here, I provide a result showing that the maxmin payoff and posture are robust to details of the bargaining procedure such as the order and relative frequency of offers, so long as both players have the opportunity to make offers frequently.\(^{21}\)

The paper proceeds as follows: Section 2.2 relates this paper to the literature. Section 2.3 presents the model and defines maxmin payoffs and postures. Section 2.4 analyzes the baseline case with one-sided commitment and presents the main characterization of maxmin payoffs and postures. Section 2.5 presents four brief extensions. Section 2.6 considers

\(^{21}\)Abreu and Gul (2000) prove an analogous “independence of procedures” result in an equilibrium reputational bargaining model. No such result holds in complete-information bargaining models in the tradition of Rubinstein (1982).
two-sided commitment. Section 2.7 considers discrete-time bargaining games with frequent offers. Section 2.8 concludes. Omitted proofs for Sections 2.3 and 2.4 are in the appendix, and omitted proofs for Sections 2.5 and 2.7 are in the supplementary appendix.

2.2 Related Literature

The seminal paper on reputational bargaining is Abreu and Gul (2000), which generalizes Myerson (1991). In their model, there is a vector of probabilities for each player corresponding to the probability that she is committed to each of a variety of behavioral types (which are analogous to bargaining postures in my model), and these vectors are common knowledge. In the (effectively unique) sequential equilibrium, players randomize over mimicking different behavioral types, with mixing probabilities determined by the prior, and play proceeds according to a war of attrition, where each player hopes that her opponent will concede. A player’s equilibrium payoff is higher when she is more likely to be committed to strong behavioral types and when she is more patient. Thus, Abreu and Gul present a complete and elegant bargaining theory in which the bargaining procedure is unimportant and sharp predictions are driven by the vector of prior commitment probabilities.

The main difference between my analysis and Abreu and Gul’s is that I characterize maxmin payoffs and postures rather than sequential equilibria. My approach entails weaker assumptions on knowledge of commitment probabilities (i.e., second-order knowledge that each player is committed to her announced posture with probability at least $\varepsilon$, rather than common knowledge of a vector of commitment probabilities) and on behavior (i.e., first-order knowledge of rationality, rather than sequential equilibrium), and does not yield unique predictions about the division of surplus or about the details of how bargaining will proceed. One motivation for this complementary approach is that behavioral types are sometimes viewed as “perturbations” reflecting the fact that a player (or an outside observer) cannot be sure that the model captures all of the other player’s strategic considerations. Thus, it seems reasonable to assume that players realize that their opponents’ type may be perturbed in

\[^{22}\text{Other important antecedents of Abreu and Gul (2000) include Kreps and Wilson (1982) and Milgrom and Roberts (1982), who pioneered the incomplete information approach to reputation-formation, and Chatterjee and Samuelson (1987, 1988), who study somewhat simpler reputational bargaining models.}\]
some manner (e.g., that a rich set of types have positive prior weight), but assuming that the
distribution over perturbations is common knowledge goes against the spirit of introducing
derturbations.\textsuperscript{23}

The paper most closely related to mine is Kambe (1999). In Kambe’s model, each player
first strategically announces a posture and then becomes committed to her announced pos-
ture with probability $\varepsilon$, as in my model. Thus, Kambe endogenizes the behavioral types
of Abreu and Gul. The structure of equilibrium and the determinants of the division of
the surplus are similar to those in Abreu and Gul’s model. There are two differences be-
tween Kambe’s model and mine. First, Kambe requires that players announce postures
that demand a constant share of the surplus (as do Abreu and Gul), while I allow players to
announce non-constant postures (and players do benefit from announcing non-constant pos-
tures in my model). Second, and more fundamentally, Kambe studies sequential equilibria,
while I study maxmin payoffs and postures. These differences lead my analysis and results
to be quite different from Kambe’s, with the exception that Kambe’s calculation of bounds
on the set of sequential equilibrium payoffs resembles my calculation of the maxmin payoff
in the special case where players can only announce constant postures (Section 2.5.1).

There are also a number of earlier bargaining models in which players try to commit
themselves to advantageous postures. Crawford (1982) studies a two-stage model in which
players first announce demands and then learn their private costs of changing these demands,
and shows that such a model can lead to impasse. Fershtman and Seidmann (1993) show
that agreement is delayed until an exogenous deadline if each player is unable to accept an
offer that she has previously rejected. Muthoo (1996) studies a two-stage model related to
Crawford (1982), with the feature that making a larger change to one’s initial demand is
more costly, and shows that a player’s equilibrium payoff is increasing in her marginal cost
of changing her demand. Ellingsen and Miettinen (2008) point out that if commitment is

\textsuperscript{23}In games with a long-run player facing a series of short-run players, Watson (1993) and Battigalli and
Watson (1997) show that common knowledge of the mere fact that the long-run player is committed to a
certain strategy with probability bounded away from 0 determines the division of the surplus. However, the
first essay in this thesis shows that common knowledge of the relative probabilities with which each player
is committed to each strategy is needed for equilibrium selection in games with two long-run players, even
when binding contracts are available (as is the case in bargaining).
equilibrium and do not involve reputation formation.

Finally, this paper is also related to the literature on bargaining with incomplete information either without common priors (Yildiz 2003, 2004; Feinberg and Skrzypacz, 2005) or with rationalizability rather than equilibrium (Cho, 1994; Watson, 1998), in that players may disagree about the distribution over outcomes of bargaining. I briefly discuss a connection with the literature on reputation in repeated games in the conclusion.

2.3 Model and Key Definitions

This section describes the model and defines maxmin payoffs and postures, which are the main objects of analysis.

2.3.1 Model

Two players ("she," "he") bargain over one unit of surplus in two phases: a "commitment phase" followed by a "bargaining phase." I describe the bargaining phase first. It is intended to capture a continuous bargaining process where players can change their demands and accept their opponents' demands at any time, but in order to avoid well-known technical issues that emerge when players can condition their play on "instantaneous" actions of their opponents (Simon and Stinchcombe, 1989; Bergin and MacLeod, 1993) I assume that players can revise their paths of demands only at integer times (while letting them accept their opponents' demands at any time).

Time runs continuously from $t = 0$ to $\infty$. At every integer time $t \in \mathbb{N}$ (where $\mathbb{N}$ is the natural numbers), each player $i \in \{1, 2\}$ chooses a path of demands for the next length-1 period of time, $u_i^t : [t, t+1) \to [0,1]$, which is required to be the restriction to $[t, t+1)$ of a continuous function on $[t, t+1]$. Let $\mathcal{U}^t$ be the set of all such functions. The interpretation is that $u_i^t(\tau)$ is the demand that player $i$ makes at time $\tau$ (this is simply denoted by $u_i(\tau)$ when $t$ is understood; note that $u_i(\tau)$ can be discontinuous at integer times but is right-continuous everywhere\footnote{A function $f : \mathbb{R} \to \mathbb{R}$ is right-continuous if, for every $x \in \mathbb{R}$ and every $\eta > 0$, there exists $\delta > 0$ such that $|f(x) - f(x')| < \eta$ for all $x' \in (x, x + \delta)$.}). Even though player $i$'s path of demands for $[t, t+1)$ is decided at $t$, player $j$ only observes demands as they are made. Intuitively, each
player $i$ may accept her opponent’s demand $u_j(t)$ at any time $t$, which ends the game with payoffs $(e^{-rt}(1 - u_j(t)), e^{-rt}u_j(t))$, where $r \in \mathbb{R}_+$ is the common discount rate (throughout, $j = -i$). Formally, every instant of time $t$ is divided into three dates, $(t, -1)$, $(t, 0)$, and $(t, 1)$, with the following timing: First, at date $(t, -1)$, each player $i$ announces accept or reject. If both players reject, the game continues; if only player $i$ accepts, the game ends with payoffs $(e^{-rt}(1 - \lim_{\tau \uparrow t} u_j(\tau)), e^{-rt}\lim_{\tau \downarrow t} u_j(\tau))$; and if both players accept, the game ends with payoffs determined by the average of the two demands, $\lim_{\tau \uparrow t} u_1(\tau)$ and $\lim_{\tau \downarrow t} u_2(\tau)$. Next, at date $(t, 0)$, both players simultaneously announce their time-$t$ demands $(u_1(t), u_2(t))$ (which were determined at the most recent integer time); if $t$ is an integer, this is also the date where each player chooses a path of demands for the next length-1 period. Finally, at date $(t, 1)$, each player $i$ again announces accept or reject. If both players reject, the game continues; if only player $i$ accepts, the game ends with payoffs $(e^{-rt}(1 - u_j(t)), e^{-rt}u_j(t))$; and if both players accept, the game ends and the demands $u_1(t)$ and $u_2(t)$ are averaged.

This timing ensures that there is a first and last date at which each player can accept each of her opponent’s demands. In particular, at integer time $t$, player $i$ may accept either her opponent’s “left” demand, $\lim_{\tau \downarrow t} u_j(\tau)$, or her time-$t$ demand, $u_j(t)$.

The public history up to time $t$ excluding the time-$t$ demands is denoted by $h^t- = (u_1(\tau), u_2(\tau))_{\tau < t}$, and the public history up to time $t$ including the time-$t$ demands is denoted by $h^t+ = (u_1(\tau), u_2(\tau))_{\tau \leq t}$ (with the convention that this corresponds to all offers having been rejected, as otherwise the game would have ended). A generic time-$t$ history is denoted by $h^t$. Since $\lim_{\tau \downarrow t} u_j(\tau) = u_j(t)$ if $t$ is not an integer, I generally distinguish between $h^t-$ and $h^t+$ only for integer $t$. Formally, a bargaining phase (behavior) strategy for player $i$ is a pair $\sigma_i = (F_i, G_i)$ such that $F_i$ maps histories into $[0, 1]$ with the property that $F_i(h^t) \leq F_i(h^{t*})$ whenever $h^{t*}$ is a successor of $h^t$, and $G_i$ maps histories of the form $h^t-$ with $t \in \mathbb{N}$ into $\Delta(U^t)$. Let $\Sigma_i$ be the set of player $i$’s bargaining phase strategies. The interpretation is that $F_i(h^t-)$ is the probability that player $i$ accepts player $j$’s demand at or before date $(t, -1)$, $F_i(h^{t+})$ is the probability that player $i$ accepts player $j$’s demand at or before date $(t, 1)$, and $G_i(h^t-)$ is the probability distribution over paths of demands $u_i : [t, t + 1) \rightarrow [0, 1]$ chosen by player $i$ at date $(t, 0)$. This formalism implies

\footnote{This is similar to the notion of date introduced by Abreu and Pearce (2007).}
that player i's hazard rate of acceptance at history \( h^t \), \( f_i(h^t) / (1 - F_i(h^t)) \), is well-defined at any time \( t \) at which the realized distribution function \( F_i \) admits a density \( f_i \) (in which case \( F_i(h^t-)=F_i(h^t+) \)); and in addition player i's probability of acceptance at history \( h^t+ \) (resp., \( h^t- \)), \( F_i(h^t+) - F_i(h^t-) \) (resp., \( F_i(h^t-) - \lim_{t\to t} F_i(h^t-) \)), is well-defined for all times \( t \). However, so long as one bears in mind these formal definitions, it suffices for the remainder of the paper to omit the notation \( (F_i, G_i) \) and instead simply view a (bargaining phase) strategy \( \sigma_i \) as a function that maps every history \( h^t \) to a hazard rate of acceptance, a discrete probability of acceptance, and (if \( h^t = h^t_0 \) for some \( t \in \mathbb{N} \)) a probability distribution over paths of demands \( u_i^t \). I say that agreement is reached at time \( t \) if the game ends at time \( t \) (i.e., at date \((t,-1)\) or \((t,1)\)). Both players receive payoff 0 if agreement is never reached.

At the beginning of the bargaining phase, player i has an initial belief \( \pi_i \) about the behavior of her opponent. That is, \( \pi_i \in \Delta(\Sigma_j) \), so \( \pi_i \) is a probability distribution over behavior strategies \( \sigma_j \); note that \( \pi_i \) can alternatively be viewed as an element of \( \Sigma_j \) by reducing compound lotteries over pure strategies. Let \( \text{supp}(\pi_i) \subseteq \Sigma_j \) be the support of \( \pi_i \), let \( u_i(\sigma_i, \sigma_j) \) be player i’s expected utility given strategy profile \( (\sigma_i, \sigma_j) \), let \( u_i(\sigma_i, \pi_i) \) be player i’s expected utility given strategy \( \sigma_i \) and belief \( \pi_i \), and let \( \Sigma_i^* (\pi_i) \equiv \arg\max_{\sigma_i} u_i(\sigma_i, \pi_i) \) be the set of player i’s best-responses to belief \( \pi_i \).

At the beginning of the game (prior to time 0), player 1 (but not player 2) publicly announces a bargaining posture \( \gamma : [0, \infty) \to [0, 1] \), which must be continuous at non-integer times \( t \), be right-continuous everywhere, and have well-defined left limits everywhere. Slightly abusing notation, a posture \( \gamma \) is identified with the strategy of player 1’s that demands \( \gamma(t) \) for all \( t \in \mathbb{R}_+ \) and always rejects player 2’s demand; with this notation, \( \gamma \in \Sigma_1 \). In other words, a posture is a pure bargaining phase strategy that does not condition on player 2’s play or accept player 2’s demand. After announcing posture \( \gamma \), player 1 becomes committed to \( \gamma \) with some probability \( \varepsilon > 0 \), meaning that she must follow strategy \( \gamma \) in the bargaining phase. With probability \( 1 - \varepsilon \), she is free to play any strategy in the bargaining phase. Whether or not player 1 becomes committed to \( \gamma \) is observed only by player 1.
2.3.2 Defining the Maxmin Payoff and Posture

This subsection defines player 1’s maxmin payoff and posture. Intuitively, player 1’s maxmin payoff is the highest payoff she can guarantee herself when all she knows about player 2 is that he is rational (i.e., maximizes his expected payoff given his belief about player 1’s behavior, and updates his belief according to Bayes’ rule when possible) and that he believes that player 1 follows her announced posture $\gamma$ with probability at least $\varepsilon$.

Formally, that player 2 is rational and assigns probability at least $\varepsilon$ to player 1 following her announced posture $\gamma$ means that his strategy satisfies the following condition:

**Definition 4** A strategy $\sigma_2$ of player 2’s is rational given posture $\gamma$ if there exists a belief $\pi_2$ of player 2’s such that $\pi_2(\gamma) \geq \varepsilon$ and $\sigma_2 \in \Sigma_2^*(\pi_2)$.

A belief $\pi_1$ of player 1’s is consistent with knowledge of rationality given posture $\gamma$ if every strategy $\sigma_2 \in \text{supp}(\pi_1)$ is rational given posture $\gamma$. In other words, the set of beliefs $\pi_1$ that are consistent with knowledge of rationality given posture $\gamma$ is $\Pi_1^\gamma = \Delta \{\sigma_2 : \sigma_2 \text{ is rational given posture } \gamma\}$.

Given that her belief is consistent with knowledge of rationality, the highest payoff that player 1 can guarantee herself after announcing posture $\gamma$ is the following:

**Definition 5** Player 1’s maxmin payoff given posture $\gamma$ is

$$u_1^*(\gamma) = \sup_{\sigma_1} \inf_{\pi_1 \in \Pi_1^\gamma} u_1(\sigma_1, \pi_1).$$

A strategy $\sigma_1^*(\gamma)$ of player 1’s is a maxmin strategy given posture $\gamma$ if

$$\sigma_1^*(\gamma) \in \arg\max_{\sigma_1} \inf_{\pi_1 \in \Pi_1^\gamma} u_1(\sigma_1, \pi_1).$$

Equivalently, $u_1^*(\gamma)$ is the highest payoff player 1 can receive when she chooses a strategy $\sigma_1$ and then player 2 chooses a rational strategy $\sigma_2$ that minimizes $u_1(\sigma_1, \sigma_2)$; that is,

$$u_1^*(\gamma) = \sup_{\sigma_1} \inf_{\sigma_2, \sigma_2 \text{ is rational given posture } \gamma} u_1(\sigma_1, \sigma_2).$$
In particular, to guarantee herself a high payoff, player 1 must play a strategy that does well against *any* rational strategy of player 2's.26

Finally, I define player 1’s maxmin payoff, the highest payoff that player 1 can guarantee herself before announcing a posture, as well as the corresponding maxmin posture.

**Definition 6** Player 1’s maxmin payoff is

\[ u_1^* \equiv \sup_\gamma u_1^* (\gamma). \]

A posture \( \gamma^* \) is a maxmin posture if there exists a sequence of postures \( \{ \gamma_n \} \) such that \( \gamma_n (t) \to \gamma^* (t) \) for all \( t \in \mathbb{R}_+ \) and \( u_1^* (\gamma_n) \to u_1^* \).

I sometimes emphasize the dependence of \( u_1^* \) and \( \gamma^* \) on \( \varepsilon \) by writing \( u_1^* (\varepsilon) \) and \( \gamma_\varepsilon^* \).27 Both the set of maxmin strategies given any posture \( \gamma \) and the set of maxmin postures are non-empty, though at this point this is not obvious.

The reason why \( \gamma^* \) is defined as a limit of postures \( \{ \gamma_n \} \) such that \( u_1^* (\gamma_n) \to u_1^* \), rather than as an element of \( \arg\max_{\gamma} u_1^* (\gamma) \), is that the latter set may be empty because of an openness problem that is standard in bargaining models. To see the problem, consider the ultimatum bargaining game, where player 1 makes a take-it-or-leave-it demand in \([0, 1]\) to player 2. By knowledge of rationality, player 1 knows that any demand strictly below 1 will be accepted, but demanding 1 does not guarantee her a positive payoff because it is a best-response for player 2 to reject. Definition 6 is analogous to specifying that in this game player 1’s maxmin payoff is 1 and her maxmin strategy is demanding 1.

Note that Definitions 5 and 6 are “non-Bayesian” in the sense that they characterize the largest payoff that player 1 can guarantee herself, rather than the maximum payoff that she can obtain given some belief. The following is the “Bayesian” version of these definitions:

26 A potential criticism of the concept of the maxmin payoff given posture \( \gamma \) is that it appears to neglect the fact that, in the event that player 1 does become committed to posture \( \gamma \), she is guaranteed only \( \inf_{\gamma_1 \in \Pi_1} E^{y_1} [u_1 (\gamma, \sigma_2)] \) in the bargaining phase, rather than \( \sup_{\sigma_1} \inf_{\gamma_1 \in \Pi_1} E^{y_1} [u_1 (\sigma_1, \sigma_2)] \). However, I show in Section 2.4.3 that these two numbers are actually identical.

27 The notation \( \gamma^* (\cdot) \) is already taken by the time-\( t \) demand of posture \( \gamma^* \). I apologize for abusing notation in writing \( u_1^* (\gamma) \) and \( u_1^* (\varepsilon) \) for different objects and hope that this will not cause confusion.
**Definition 7** Player 1’s pessimistic payoff given posture $\gamma$ is

$$u_1^{\text{pess}} (\gamma) \equiv \inf_{\pi_1 \in \Pi_1^\gamma} \sup_{\sigma_1} u_1 (\sigma_1, \pi_1).$$

Player 1’s pessimistic payoff is $u_1^{\text{pess}} \equiv \sup_{\gamma} u_1^{\text{pess}} (\gamma)$. A posture $\gamma^{\text{pess}}$ is a pessimistic posture if there exists a sequence of postures $\{\gamma_n\}$ such that $\gamma_n (t) \to \gamma^{\text{pess}} (t)$ for all $t \in \mathbb{R}^+$ and $u_1^{\text{pess}} (\gamma_n) \to u_1^{\text{pess}}$.

Player 1’s pessimistic payoff is the worst payoff she can receive by best-responding to a fixed rational strategy of player 2’s. Player 1’s maxmin payoff is weakly lower than her pessimistic payoff, because in the definition of the pessimistic payoff player 1 “knows” the distribution over player 2’s strategies when she chooses her strategy, while in the definition of the maxmin payoff her strategy is evaluated with respect to the worst-case response of player 2’s. However, I show in Section 2.4.3 that these payoffs are in fact identical in my model. For expositional consistency, I focus on maxmin payoffs and strategies.

Another reason for studying player 1’s maxmin payoff is that it determines the entire range of payoffs that are consistent with knowledge of rationality, as shown by the following proposition.

**Proposition 1** For any posture $\gamma$ and any payoff $u_1 \in [u_1^* (\gamma), 1)$, there exists a belief $\pi_1 \in \Pi_1^\gamma$ such that $\max_{\sigma} u_1 (\sigma_1, \pi_1) = u_1$.

### 2.4 Characterization of the Maxmin Payoff and Posture

This section states and proves Theorem 2, the main result of the paper, which solves for player 1’s maxmin payoff and posture. Section 2.4.1 states and discusses Theorem 2, and Sections 2.4.2 through 2.4.4 provide the proof. The approach is as follows: In Section 2.4.2, I fix a posture $\gamma$ and find a belief of player 2’s, $\pi_2^\gamma$ (satisfying $\pi_2^\gamma (\gamma) \geq \varepsilon$), and corresponding best-response, $\sigma_2^\gamma \in \Sigma_2^* (\pi_2^\gamma)$, that minimize $u_1 (\gamma, \sigma_2)$, player 1’s payoff from mimicking $\gamma$ in the bargaining phase; $\pi_2^\gamma$ and $\sigma_2^\gamma$ are called the $\gamma$-offsetting belief and $\gamma$-offsetting strategy, respectively, and play a key role in the analysis. Section 2.4.3 shows that $\gamma$ itself is a maxmin strategy given posture $\gamma$, for any $\gamma$, which implies that $u_1^* (\gamma) = u_1 (\gamma, \sigma_2^\gamma)$ for any posture $\gamma$. 
That is, player 1’s maxmin payoff given posture $\gamma$ is the payoff she receives from mimicking $\gamma$ when her opponent follows his $\gamma$-offsetting strategy. Section 2.4.4 maximizes $u_1(\gamma, \sigma_2^\gamma)$ over $\gamma$ to prove Theorem 2.

2.4.1 Main Result

The main result is the following:

**Theorem 2** Player 1’s maxmin payoff is

$$u_1^*(\varepsilon) = \frac{1}{(1 - \log \varepsilon)},$$

and the unique maxmin posture $\gamma_\varepsilon^*$ is given by

$$\gamma_\varepsilon^*(t) = \min \{ e^{rt} / (1 - \log \varepsilon), 1 \} \text{ for all } t \in \mathbb{R}_+.$$

A priori, one might have expected player 1’s maxmin payoff to be very small when $\varepsilon$ is small (because player 2’s beliefs and strategy may be chosen quite freely in the definition of the maxmin payoff), and might have expected player 1’s maxmin posture to be complicated (as player 1 is not restricted to announcing monotone, continuous, or otherwise well-behaved postures). Theorem 2 shows that, on the contrary, player 1’s maxmin payoff is relatively high for even very small commitment probabilities $\varepsilon$, as shown by Table 1, and that player 1’s unique maxmin posture is simply demanding the maxmin payoff plus compensation for any delay in reaching agreement.\(^{28}\)

---

\(^{28}\)The importance of non-constant postures is a difference between this paper and existing reputational bargaining models, where it is usually assumed that players may only be committed to strategies that demand a constant share of the surplus (as in Abreu and Gul (2000), Compte and Jehiel (2002), and Kambe (1999)). A notable exception is Abreu and Pearce (2007), where players may be committed to non-constant postures that can also condition their play on their opponents’ behavior. However, Abreu and Pearce’s main result is that a particular posture that demands a constant share of the surplus is approximately optimal in their model, when commitment probabilities are small.
Table 1: The Maxmin Payoff $u_1^*(\varepsilon)$ for Different Commitment Probabilities $\varepsilon$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$u_1^*(\varepsilon)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.25</td>
<td>.42</td>
</tr>
<tr>
<td>.1</td>
<td>.30</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>.13</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>.07</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>.05</td>
</tr>
</tbody>
</table>

The intuition for why the unique maxmin posture is given by $\gamma_1^*(t) = \min \{ e^{rt} u_1^*(\varepsilon), 1 \}$ was outlined in the introduction. The most basic intuition for why $u_1^*(\varepsilon)$ is large relative to $\varepsilon$ is that, when player 1 announces a posture that offers player 2 a large share of the surplus and then mimics this posture, player 2 must accept player 1’s offer unless he believes that he will be rewarded with high probability for rejecting. In the latter case, if player 1 does not reward player 2 for rejecting, then player 2 quickly updates his belief toward player 1’s being committed to her announced posture (i.e., player 1 builds reputation quickly), and player 2 accepts player 1’s offer when he becomes convinced that she is committed. Thus, player 1 builds reputation quickly when her demand is small, so a small commitment probability need not lead to much delay before her demand is accepted.

2.4.2 Offsetting Beliefs and Strategies

In this subsection, I fix an arbitrary posture for player 1, $\gamma$, and find a rational strategy of player 2’s, $\sigma_2^\gamma$ (the “$\gamma$-offsetting strategy”), that minimizes player 1’s payoff when she announces posture $\gamma$ and then mimics $\gamma$ in the bargaining phase. That is, I solve the problem

$$\inf_{(\gamma, \sigma_2) : \gamma \in \Sigma, \sigma_2 \in \Pi_2(\gamma)} u_1(\gamma, \sigma_2).$$

The resulting strategy $\sigma_2^\gamma$ always demands the entire surplus and rejects player 1’s demand until some time $t^*$, and accepts player 1’s smaller time-$t^*$ demand, $\min \{ \lim_{\tau \uparrow t^*} \gamma(\tau), \gamma(t^*) \}$ (henceforth denoted by $\gamma(t^*)$), if player 1 follows $\gamma$ until time $t^*$. If player 1 ever deviates
from \( \gamma \), then \( \sigma_2' \) rejects player 1’s demand forever. The corresponding belief \( \pi_2' \) (the “\( \gamma \)-offsetting belief”) is that player 1 plays \( \gamma \) with probability \( \varepsilon \), and with probability \( 1 - \varepsilon \) plays a particular strategy \( \tilde{\gamma} \) that mixes between mimicking \( \gamma \) and accepting player 2’s demand up until time \( t^* \), and subsequently mimics \( \gamma \).

The key step in solving (3) is computing the smallest time \( T \) by which agreement must be reached under strategy profile \( (\gamma, \sigma_2') \). I then show that the value of (3) is simply \( \min_{t \leq T} e^{-rt} \gamma(t) \), and that the time \( t^* \) at which the strategy \( \sigma_2' \) accepts \( \gamma(t) \) is a time before \( T \) that minimizes \( e^{-rt} \gamma(t) \).

Toward computing \( T \), let \( v(t, -1) \) be the continuation value of player 2 from best-responding to \( \gamma \) starting from date \( (t, -1) \), and let \( v(t) \) be the corresponding continuation value starting from date \( (t, 1) \):

\[
v(t, -1) \equiv \max_{\tau \geq t} e^{-r(\tau - t)} (1 - \gamma(\tau)),
\]

\[
v(t) \equiv \max \left\{1 - \gamma(t), \sup_{\tau > t} e^{-r(\tau - t)} (1 - \gamma(\tau))\right\},
\]

where \( \gamma(\tau) \equiv \min \{\lim_{s \uparrow \tau} \gamma(s), \gamma(\tau)\} \). Thus, the difference between \( v(t, -1) \) and \( v(t) \) is that only \( v(t, -1) \) gives player 2 the opportunity to accept the demand \( 1 - \lim_{\tau \uparrow t} \gamma(\tau) \); in particular, \( v(t, -1) = v(t) \) if \( \gamma(t) \) (or \( v(t) \)) is continuous at \( t \). Note that \( \max_{\tau \geq t} e^{-r(\tau - t)} (1 - \gamma(\tau)) \) is well-defined because \( \gamma(\tau) \) is lower semi-continuous and \( \lim_{\tau \to \infty} e^{-r(\tau - t)} (1 - \gamma(\tau)) = 0 \), and that \( v(t) \) is continuous at all non-integer times \( t \); let \( \{s_1, s_2, \ldots\} \equiv S \subseteq \mathbb{N} \) be the set of discontinuity points of \( v(t) \). Finally, note that \( v(t) \) can increase at rate no faster than \( r \). That is, \( v(t) \geq e^{-r(t' - t)} v(t') \) for all \( t' \geq t \), because if \( v(t') = e^{-r(\tau - t')} (1 - \gamma(\tau)) \) for some \( \tau \geq t' \), then \( v(t) \geq e^{-r(\tau - t)} (1 - \gamma(\tau)) = e^{-r(t' - t)} v(t') \). This implies that \( v(t) \) is continuous but for downward jumps,\(^{29}\) and that \( v(t) \) is differentiable almost everywhere.\(^{30}\) These are but two of the useful properties of the function \( v(t) \) (which are not shared by \( \gamma(t) \)) that reward working with \( v(t) \) rather than \( \gamma(t) \) in the subsequent analysis.

Next, I introduce two functions \( \lambda(t) \) and \( p(t) \) with the property that if player 1 mixes

\(^{29}\) A function \( f : \mathbb{R} \to \mathbb{R} \) is continuous but for downward jumps if \( \liminf_{x, x' \uparrow} f(x) \geq f(x') \geq \limsup_{x, x' \downarrow} f(x) \) for all \( x \in \mathbb{R} \).

\(^{30}\) Proof: Let \( f(t) = e^{-rt} v(t) \). Then \( f(t) \) is non-increasing, which implies that \( f(t) \) is differentiable almost everywhere (e.g., Royden, 1988, p. 100). Hence, \( v(t) \) is differentiable almost everywhere.
between mimicking \( \gamma \) and conceding the entire surplus to player 2, then \( \lambda (t) \) (resp., \( p(t) \)) is the smallest non-negative hazard rate (resp., discrete probability) at which player 1 must concede in order for player 2 to be willing to reject player 1’s time-\( t \) demand, \( \gamma (t) \). Let

\[
\lambda (t) = \frac{rv(t) - v'(t)}{1 - v(t)} \tag{5}
\]

if \( v(t) \) is differentiable at \( t \) and \( v(t) < 1 \), and let \( \lambda (t) = 0 \) otherwise; note that \( \lambda (t) \geq 0 \) for all \( t \), because \( v(t) \) cannot increase at rate faster than \( r \). Also, let

\[
p(t) = \frac{v(t) - v(t-1)}{1 - v(t)} \tag{6}
\]

if \( v(t) < 1 \), and let \( p(t) = 0 \) otherwise. To see intuitively why the aforementioned property holds, note that accepting player 1’s time-\( t \) demand gives player 2 flow payoff \( rv(t) \), while rejecting gives player 2 flow payoff \( \lambda (t) (1 - v(t)) + v'(t) \), and equalizing these flow payoffs yields (5);\(^{31, 32}\) similarly, accepting player 1’s demand at date \( (t, -1) \) gives player 2 payoff \( v(t, -1) \), while delaying acceptance until date \( (t, 1) \) gives player 2 payoff \( p(t) (1) + (1 - p(t)) v(t) \), and equalizing these payoffs yields (6).

When player 2 expects player 1 to accept his demand at rate (resp., probability) \( \lambda (t) \) (resp., \( p(t) \)), he becomes convinced that player 1 is committed to posture \( \gamma \) at the time \( \hat{T} \) defined in the following lemma, which leads him to accept player 1’s demand no later than the time \( T \) defined in the lemma. In the lemma, and throughout the paper, maximization or minimization over times \( t \) should be read as taking place over \( t \in \mathbb{R}_+ \cup \{\infty\} \) (i.e., as allowing \( t = \infty \), with the convention that \( e^{-r\infty} \gamma (\infty) \equiv 0 \) for all postures \( \gamma \)).

\(^{31}\)This intuition is correct when \( v(t) = 1 - \gamma (t) \). When \( v(t) > 1 - \gamma (t) \), player 2 prefers to reject player 1’s time-\( t \) demand even when player 1 concedes at rate 0. At these times, \( rv(t) = v'(t) \), which implies that \( \lambda (t) = 0 \). Hence, \( \lambda (t) \) is always the smallest non-negative hazard rate at which player 1 must concede in order for player 2 to be willing to reject \( \gamma (t) \).

\(^{32}\)If \( v'(t) = 0 \), then \( \lambda (t) \) becomes the concession rate that makes player 2 indifferent between accepting and rejecting the constant offer \( v(t) \), which is familiar from the literatures on wars of attrition and reputational bargaining. However, in these literatures \( \lambda (t) \) is the rate at which player 1 concedes in equilibrium, while here is the rate at which player 1 concedes according to player 2’s offsetting beliefs, as will become clear.
Lemma 2 Let

\[ \tilde{T} \equiv \sup \left\{ t : \exp \left( -\int_0^t \lambda(s) \, ds \right) \prod_{s \in S^t(0, t)} (1 - p(s)) > \varepsilon \right\}, \]

and let

\[ T \equiv \max \arg\max_{t \geq \tilde{T}} \begin{cases} e^{-rt} (1 - \gamma(t)) & \text{if } t = \tilde{T} \\ e^{-rt} (1 - \gamma(t)) & \text{if } t > \tilde{T} \end{cases}. \]

Then, for any \( \pi_2 \) such that \( \pi_2(\gamma) \geq \varepsilon \) and any \( \sigma_2 \in \Sigma^*_2(\pi_2) \), agreement is reached no later than time \( T \) under strategy profile \( (\gamma, \sigma_2) \). In particular,

\[ \inf_{(\pi_2, \sigma_2) : \pi_2(\gamma) \geq \varepsilon, \sigma_2 \in \Sigma^*_2(\pi_2)} u_1(\gamma, \sigma_2) \geq \min_{t \leq T} e^{-rt} \tilde{\gamma}(t). \]  

(7)

Lemma 2 amounts to the statement that \( \tilde{T} \) is the latest time at which it is possible that agreement has not yet been reached and player 2 is not certain that player 1 is playing \( \gamma \), when player 2’s initial belief is some \( \pi_2 \) such that \( \pi_2(\gamma) \geq \varepsilon \), under strategy profile \( (\gamma, \sigma_2) \) for some \( \sigma_2 \in \Sigma^*_2(\pi_2) \). The first step of the proof (of Lemma 2) shows that, in computing this time, one can restrict attention to beliefs \( \pi_2 \) that assign probability 1 to player 1’s accepting player 2’s demand whenever player 1 deviates from strategy \( \gamma \), and to strategies \( \sigma_2 \) that always demand the entire surplus. This is because giving more surplus to player 2 in the event that player 1 deviates from \( \gamma \) makes player 2 more willing to reject player 1’s demand, without changing player 2’s beliefs about the probability that player 1 is playing \( \gamma \) at any history. The second step shows that, with beliefs of this form, if \( v(t) \) is always equal to player 2’s continuation payoff from delaying acceptance until he becomes convinced that player 1 is playing \( \gamma \), then player 1’s concession rate and probability must be given by \( \lambda(t) \) and \( p(t) \), and player 2 becomes convinced that player 1 is playing \( \gamma \) at time \( \tilde{T} \); this formalizes the motivation for \( \lambda(t) \) and \( p(t) \) given above. The proof is completed by showing that if \( v(t) \) is ever strictly less than player 2’s continuation payoff from delaying acceptance until he becomes convinced that player 1 is playing \( \gamma \), then player 2 becomes convinced that player 1 is playing \( \gamma \) no later than \( \tilde{T} \).

The remainder of this subsection is devoted to showing that (7) holds with equality,
which proves that (3) equals \( \min_{t \leq T} e^{-rt} \gamma(t) \). The idea is that player 2 may hold a belief that induces him to demand the entire surplus until time \( t^* \equiv \min \arg\min_{t \leq T} e^{-rt} \gamma(t) \) and then accept player 1’s offer; this is the \( \gamma \)-offsetting belief.\(^{33}\) I first define the \( \gamma \)-offsetting belief, and then show that (7) holds with equality.

I begin by introducing a strategy, \( \tilde{\gamma} \), which is used in defining the \( \gamma \)-offsetting belief.\(^{34}\) Let

\[
\chi(t) = \max \left\{ \frac{\exp \left( -\int_0^t \lambda(s) \, ds \right) \prod_{s \in S \cap [0,t]} (1 - p(s)) \cdot \varepsilon}{\exp \left( -\int_0^t \lambda(s) \, ds \right) \prod_{s \in S \cap [0,t]} (1 - p(s))}, 0 \right\}; \tag{8}
\]

let

\[
\hat{\lambda}(t) = \frac{\lambda(t)}{\chi(t)} \tag{9}
\]

if \( \chi(t) > 0 \), and let \( \hat{\lambda}(t) = 0 \) otherwise; and let

\[
\hat{p}(t) = \min \left\{ \frac{p(t)}{\chi(t)}, 1 \right\} \tag{10}
\]

if \( \chi(t) > 0 \), and let \( \hat{p}(t) = 0 \) otherwise. Intuitively, \( \chi(t) \) is the posterior probability that player 2 assigns to player 1’s playing a strategy other than \( \gamma \) at time \( t \) when player 1’s unconditional concession rate and probability are \( \lambda(t) \) and \( p(t) \), and \( \hat{\lambda}(t) \) and \( \hat{p}(t) \) are the conditional (on not playing \( \gamma \)) concession rate and probability needed for the unconditional concession rate and probability to equal \( \lambda(t) \) and \( p(t) \).

**Definition 8** \( \tilde{\gamma} \) is the strategy that demands \( u_1(t) = \gamma(t) \) for all \( t \in \mathbb{R}_+ \), accepts with hazard rate \( \hat{\lambda}(t) \) for all \( t < t^* \), accepts with probability \( \hat{p}(t) \) at date \( (t,1) \) for all \( t \leq t^* \), and rejects for all \( t > t^* \), for all histories \( h' \).

I now define the \( \gamma \)-offsetting belief.

**Definition 9** The \( \gamma \)-offsetting belief, denoted \( \pi_2^\gamma \), is given by \( \pi_2^\gamma(\gamma) = \varepsilon \) and \( \pi_2^\gamma(\tilde{\gamma}) = 1 - \varepsilon \).

The \( \gamma \)-offsetting strategy, denoted \( \sigma_2^\gamma \), is the strategy that demands \( u_2(t) = 1 \) for all \( t \) and accepts or rejects player 1’s demand as follows:

\(^{33}\)Note that \( \min \arg\min_{t \leq T} e^{-rt} \gamma(t) \) is well-defined, because \( \gamma(t) \) is lower semi-continuous (though it may equal \( \infty \), if \( T = \infty \)). This particular choice of \( t^* \) is for concreteness; any element of \( \arg\min_{t \leq T} e^{-rt} \gamma(t) \) would suffice for the analysis.

\(^{34}\)This approach is related to a construction in the first essay in this thesis.
1. If \( h^t \) is consistent with \( \gamma \), then reject if \( t < t^* \); accept at date \( (t^*, -1) \) if and only if \( \lim_{\tau \to t^*} \gamma(\tau) \leq \gamma(t^*) \); accept at date \( (t^*, 1) \) if and only if \( \lim_{\tau \to t^*} \gamma(\tau) > \gamma(t^*) \); and reject if \( t > t^*. \)

2. If \( h^t \) is not consistent with \( \gamma \), then reject.

Finally, I show that (7) holds with equality, and also that the \( \gamma \)-offsetting (belief, strategy) pair \( (\pi_2^\gamma, \sigma_2^\gamma) \) is a solution to (3). If \( t^* = \infty \), then the following statement that agreement is reached at time \( t^* \) means that agreement is never reached.

**Lemma 3** Agreement is reached at time \( t^* \) under strategy profile \( (\gamma, \sigma_2^\gamma) \), and \( \sigma_2^\gamma \in \Sigma_2^* (\pi_2^\gamma) \). In particular, the pair \( (\pi_2^\gamma, \sigma_2^\gamma) \) is a solution to (3), and \( u_1 (\gamma, \sigma_2^\gamma) = \min_{t \leq T} e^{-rt} \gamma(t) \).

**Proof.** It is immediate from Definition 9 that agreement is reached at \( t^* \) under strategy profile \( (\gamma, \sigma_2^\gamma) \), which implies that \( u_1 (\gamma, \sigma_2^\gamma) \) equals \( \min_{t \leq T} e^{-rt} \gamma(t) \), the right-hand side of (7). Since \( \pi_2^\gamma (\gamma) \geq \varepsilon \), it remains only to show that \( \sigma_2^\gamma \in \Sigma_2^* (\pi_2^\gamma) \).

If \( t < \min \{ \hat{t}, t^* \} \) and \( h^t \) is consistent with \( \gamma \), then, by construction of \( \tilde{\gamma} \), player 1 accepts player 2’s demand of 1 with unconditional hazard rate \( \lambda(t) \) and unconditional discrete probability \( p(t) \) under \( \pi_2^\gamma \). The proof of Lemma 2 implies that it is optimal for player 2 to demand \( u_2(t) = 1 \) and reject at any time \( t < \min \{ \hat{t}, t^* \} \) when player 1 accepts player 2’s demand of 1 at rate \( \lambda(t) \) and probability \( p(t) \) until time \( \hat{t} \); and that in addition if \( t^* < \hat{t} \) then player 2 is indifferent between between accepting and rejecting at time \( t^* \) when player 1 accepts with this rate and probability until time \( \hat{t} \). Therefore, it is optimal for player 2 to demand \( u_2(t) = 1 \) and reject at time \( t \) when player 1 accepts with this rate and probability only until time \( \min \{ \hat{t}, t^* \} \).

If \( t \in [\hat{t}, t^*] \) and \( h^t \) is consistent with \( \gamma \), then under \( \pi_2^\gamma \) player 2 is certain that player 1 is playing \( \gamma \) at \( h^t \). Since \( t^* \leq T \), this implies that it is optimal for player 2 to reject. If a history \( h^t \) is not reached under strategy profile \( (\pi_2^\gamma, \sigma_2^\gamma) \) (as is the case if \( t > t^* \)), then any continuation strategy of player 2’s is optimal. Finally, to see that accepting \( \gamma(t^*) \) (i.e., accepting at the more favorable of dates \( (t^*, -1) \) and \( (t^*, 1) \)) is optimal, note that \(^{35}\)History \( h^t \) is consistent with strategy \( \sigma_2 \) if there exists a strategy \( \sigma_2 \) such that \( h^t \) is reached under \( (\sigma_1, \sigma_2) \). In particular, history \( h^T^\gamma \) (resp., \( h^T \)) is consistent with \( \gamma \) if and only if \( u_1 (\tau) = \gamma(\tau) \) for all \( \tau < t \) (resp., \( \tau \leq t \)).
the fact that \( t^* \in \text{argmin}_{t \leq T} e^{-rt} \gamma(t) \) implies that \( \gamma(t) \geq \gamma(t^*) \) for all \( t \in [t^*, T] \). Hence, \( t^* \in \text{argmax}_{t \in [t^*, T]} e^{-rt} (1 - \gamma(t)) \). Because \( \gamma(t^*) \) coincides with \( \gamma \) after time \( t^* \), it follows that, conditional on having reached time \( t^* \), player 2 receives at most \( \sup_{t \in [t^*, T]} e^{-rt} (1 - \gamma(t)) \) if he rejects, and receives \( e^{-rt^*} (1 - \gamma(t^*)) \) if he accepts, which is weakly more. Therefore, \( \sigma_2^* \in \Sigma_2^*(\pi_2^*) \). □

2.4.3 Maxmin Strategies

This subsection shows that \( \gamma \) itself is a maxmin strategy given posture \( \gamma \), and that in particular \( u_1^*(\gamma) = u_1(\gamma, \sigma_2^*) = \min_{t \leq T(\gamma)} e^{-rt} \gamma(t) \), where I have made the dependence of \( T \) on \( \gamma \) explicit. The intuition is that player 1 is not guaranteed a positive payoff in any continuation game following a deviation from her announced posture, because at such histories player 2's beliefs and strategy are unrestricted.

The key result of this subsection is the following:

**Lemma 4** For any posture \( \gamma \), \( u_1^*(\gamma) = \min_{t \leq T(\gamma)} e^{-rt} \gamma(t) \).

**Proof.** By Lemma 3, \((\pi_2^*, \sigma_2^*)\) is a solution to (3), so

\[ \sigma_2^* \in \text{argmin}_{\pi_1 \in \Pi_1^*} u_1(\gamma, \pi_1) \tag{11} \]

Under strategy \( \sigma_2^* \), player 2 always demands \( u_2(t) = 1 \) and only accepts player 1's demand if player 1 conforms to \( \gamma \) through time \( t^* \). Hence, \( \sup_{\sigma_1} u_1(\sigma_1, \sigma_2^*) = e^{-rt^*} \gamma(t^*) = u_1(\gamma, \sigma_2^*) \), and therefore

\[ \gamma \in \text{argmax}_{\sigma_1} u_1(\sigma_1, \sigma_2^*) \tag{12} \]

(11) and (12) imply the following chain of inequalities:

\[
\begin{align*}
\sup_{\sigma_1} \inf_{\pi_1 \in \Pi_1^*} u_1(\sigma_1, \pi_1) & \geq \inf_{\pi_1 \in \Pi_1^*} u_1(\gamma, \pi_1) \\
& = u_1(\gamma, \sigma_2^*) \quad \text{(by (11))} \\
& = \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*) \quad \text{(by (12))} \\
& \geq \max_{\sigma_1} \min_{\pi_1 \in \Pi_1^*} u_1(\sigma_1, \pi_1).
\end{align*}
\]
This is possible only if both inequalities hold with equality (and the supremum and infimum in the first line are attained at $\gamma$ and $\sigma_2^*$, respectively). Therefore, $u_1^*(\gamma) = u_1(\gamma, \sigma_2^*) = \min_{t \leq T(\gamma)} e^{-rt} \gamma(t)$.

As an aside, a similar argument establishes the equivalence between the maxmin approach of Definitions 5 and 6 and the Bayesian approach of Definition 7.

**Corollary 1** $u_1^{\text{pess}}(\gamma) = u_1^*(\gamma)$ for all $\gamma$; $u_1^{\text{pess}} = u_1^*$; and $\gamma$ is a maxmin posture if and only if it is a pessimistic posture.

**Proof.** (11) and (12) imply that $\inf_{\pi \in \Pi_1^*} \sup_\sigma u_1(\sigma, \pi_1) = u_1(\gamma, \sigma_2^*)$, by the same chain of inequalities that proves that $\sup_\sigma \inf_{\pi \in \Pi_1^*} u_1(\sigma, \pi_1) = u_1(\gamma, \sigma_2^*)$. Hence, $u_1^{\text{pess}}(\gamma) = u_1(\gamma, \sigma_2^*) = u_1^*(\gamma)$ for all $\gamma$, and the remainder of the result follows from Definition 7.

### 2.4.4 Proof of Theorem 2

I now sketch the remainder of the proof of Theorem 2. The details of the proof are deferred to the appendix.

The first part of the proof is constructing a sequence of postures $\{\gamma_n\}$ such that $\lim_{n \to \infty} u_1^*(\gamma_n) = 1/(1 - \log \varepsilon)$ and $\{\gamma_n(t)\}$ converges pointwise to $\gamma^*(t) \equiv \min \{e^{rt}/(1 - \log \varepsilon), 1\}$. Define $\gamma_n$ by

$$
\gamma_n(t) = \min \left\{ \left( \frac{n}{n+1} \right) \frac{e^{rt}}{1 - \log \varepsilon}, 1 \right\} \text{ for all } t \in \mathbb{R}_+.
$$

Let $T_n^1$ be the time where $\gamma_n(t)$ reaches 1. It can be shown that $T_n^1 > T(\gamma_n)$ for all $n \in \mathbb{N}$, where $T$ is defined as in Lemma 1 and I have emphasized the dependence of $T$ on $\gamma$. This implies that $\gamma_n(t) = \left( \frac{n}{n+1} \right) \frac{e^{rt}}{1 - \log \varepsilon}$ for all $t \leq T(\gamma_n)$, and that $\gamma_n(T(\gamma_n)) < 1$. Since $\gamma_n(t)$ is non-decreasing and $\gamma_n(T(\gamma_n)) < 1$, it follows from the definition of $T(\gamma_n)$ that $T(\gamma_n) = T(\gamma_n)$. Thus, by Lemma 4,

$$
u_1^*(\gamma_n) = \min_{t \leq T(\gamma_n)} e^{-rt} \gamma_n(t)$$

$$= \min_{t \leq T(\gamma_n)} \left( \frac{n}{n+1} \right) \frac{1}{1 - \log \varepsilon}$$

$$= \left( \frac{n}{n+1} \right) \frac{1}{1 - \log \varepsilon}.$$
Therefore, \( \lim_{n \to \infty} u_1^*(\gamma_n) = 1/(1 - \log \varepsilon) \).

The second part is showing that no posture \( \gamma \) guarantees more than \( 1/(1 - \log \varepsilon) \). Here, the crucial observation is that any posture \( \gamma \) such that \( \gamma(t) \geq e^{rt}/(1 - \log \varepsilon) \) for all \( t \leq T(\gamma) \) satisfies \( \tilde{T}(\gamma) \geq T^1 \), where \( T^1 \) is the time at which \( \gamma^*(t) \) reaches 1. Since any posture that guarantees at least \( 1/(1 - \log \varepsilon) \) must satisfy \( \gamma(t) \geq e^{rt}/(1 - \log \varepsilon) \) for all \( t \leq T(\gamma) \) (by Lemma 4), and \( T(\gamma) \geq \tilde{T}(\gamma) \) for any posture \( \gamma \), this implies that \( u_1^*(\gamma) = \min_{t \leq T(\gamma)} e^{-rt}\gamma(t) \leq e^{-rT^1} = 1/(1 - \log \varepsilon) \) for any posture \( \gamma \). The appendix shows that in addition no sequence of postures \( \{\gamma'_n\} \) converging pointwise to any posture other than \( \gamma^* \) can correspond to a sequence of maxmin payoffs \( \{u_1^*(\gamma'_n)\} \) converging to \( 1/(1 - \log \varepsilon) \).

### 2.5 Extensions

This section presents four extensions of Theorem 2. Section 2.5.1 characterizes the maxmin payoff when player 1 can only announce constant postures; Section 2.5.2 extends Theorem 2 to general convex bargaining sets; Section 2.5.3 considers heterogeneous discounting; and Section 2.5.4 extends Theorem 2 to higher-order knowledge of rationality.

#### 2.5.1 Constant Postures

Theorem 2 shows that the unique maxmin posture is non-constant. In this subsection, I determine how much lower a player’s maxmin payoff is when she is required to announce a constant posture. The purpose of this study is, first, to establish that announcing non-constant postures allows a player to guarantee herself a significantly higher payoff; second, to determine the share of the surplus that a player can guarantee herself in settings where announcing a non-constant posture might not be credible; and, third, to facilitate comparison with the existing reputational bargaining literature, in which typically players can only announce constant postures.

A posture \( \gamma \) is constant if \( \gamma(t) = \gamma(0) \) for all \( t \). If \( \gamma \) is constant, I slightly abuse notation by writing \( \gamma \) for the constant demand \( \gamma(t) \) in addition to the posture itself. The constant posture \( \gamma \) that maximizes \( u_1^*(\gamma) \) is the maxmin constant posture, denoted \( \tilde{\gamma}^* \),

\[ \text{[36]} \text{Such a posture exists for all } \varepsilon < 1, \text{ so there is no need for a limit definition like Definition 6. When } \varepsilon = 1, \text{ such a definition would imply that the maxmin constant posture equals 1.} \]
the corresponding payoff is the maxmin constant payoff, denoted $\bar{u}_1^\ast$. These can be derived using Lemmas 2 through 4, leading to the following:

**Proposition 2** For all $\varepsilon < 1$, the unique maxmin constant posture is $\bar{\gamma}_\varepsilon^\ast = \frac{2 - \log \varepsilon - \sqrt{(\log \varepsilon)^2 - 4 \log \varepsilon}}{2}$, and the maxmin constant payoff is $\bar{u}_1^\ast (\varepsilon) = \exp \left( - (1 - \bar{\gamma}_\varepsilon^\ast) \right) \bar{\gamma}_\varepsilon^\ast$.

Proposition 2 solves for $\bar{\gamma}_\varepsilon^\ast$ and $\bar{u}_1^\ast (\varepsilon)$, but it does not yield a clear relationship between the maxmin constant payoff, $\bar{u}_1^\ast (\varepsilon)$, and the (overall) maxmin payoff, $u_1^\ast (\varepsilon)$. Therefore, I graph the ratio of $u_1^\ast (\varepsilon)$ to $\bar{u}_1^\ast (\varepsilon)$ in Figure 3. In addition, the following analytical result regarding the ratio of $u_1^\ast (\varepsilon)$ to $\bar{u}_1^\ast (\varepsilon)$ is straightforward:

**Corollary 2** $u_1^\ast (\varepsilon) / \bar{u}_1^\ast (\varepsilon)$ is decreasing in $\varepsilon$, $\lim_{\varepsilon \to 1} u_1^\ast (\varepsilon) / \bar{u}_1^\ast (\varepsilon) = 1$, and $\lim_{\varepsilon \to 0} u_1^\ast (\varepsilon) / \bar{u}_1^\ast (\varepsilon) = e$.

The most interesting part of Corollary 2 is that a player’s maxmin payoff is approximately $e$ times greater when she can announce non-constant postures than when she can only announce constant postures, when her commitment probability is small. Thus, there is a large advantage to announcing non-constant postures. However, a player can still guarantee herself a substantial share of the surplus when she can only announce constant postures, and her maxmin payoff goes to 0 with $\varepsilon$ at the same rate in either case.
2.5.2 General Convex Bargaining Sets

This subsection shows that the maxmin payoff derived in Theorem 2 is a lower bound on the maxmin payoff with general convex bargaining sets, normalized so that each player’s lowest and highest feasible payoffs are 0 and 1. More generally, taking a concave transformation of the Pareto frontier of the bargaining set weakly increases the maxmin payoff.

Formally, a decreasing function $\phi : [0, 1] \rightarrow [0, 1]$ is the Pareto frontier (of the bargaining set) if the game ends with payoffs $(e^{-rt}u_1(t), e^{-rt}\phi(u_1(t)))$ when player 2 accepts player 1’s demand $u_1(t)$, and ends with payoffs $(e^{-rt}\phi^{-1}(u_2(t)), e^{-rt}u_2(t))$ when player 1 accepts player 2’s demand $u_2(t)$. Note that the definition of player 1’s maxmin payoff is valid for any bargaining set. The result is the following:

**Proposition 3** Suppose that $\phi : [0, 1] \rightarrow [0, 1]$ is a decreasing and concave function satisfying $\phi(0) = 1$ and $\phi(1) = 0$, and that $\psi : [0, 1] \rightarrow [0, 1]$ is an increasing and concave function satisfying $\psi(0) = 0$ and $\psi(1) = 1$. Let $u_1^{\phi}$ be player 1’s maxmin payoff when the Pareto frontier is $\phi$, and let $u_1^{\psi \circ \phi}$ be player 1’s maxmin payoff when the Pareto frontier is $\psi \circ \phi$. Then $u_1^{\psi \circ \phi} \geq u_1^{\phi}$.

Proposition 3 shows that taking any concave transformation $\psi$ of a Pareto frontier $\phi$ weakly increases player 1’s maxmin payoff. The intuition is that any fixed demand of player 1’s leaves more for player 2 when the Pareto frontier is $\psi \circ \phi$ than when it is $\phi$, which implies that player 2 must believe that player 1 is conceding more rapidly in order for him to reject player 1’s demand when the Pareto frontier is $\psi \circ \phi$. This in turn lets player 1 build reputation more quickly and thus guarantee herself a higher payoff.

2.5.3 Heterogeneous Discounting

I have assumed that the players have the same discount rate. This simplified notation and led to simple formulas for $u_i^*(\varepsilon)$ and $\gamma^*_\varepsilon$ in Theorem 2. However, it is straightforward to generalize the model to the case where player $i$ has discount rate $r_i$ and $r_i \neq r_j$; one must only keep track of whose discount rate “$r$” stands for in the above analysis. Introducing heterogeneous discounting yields interesting comparative statics with respect to the players’ relative patience, $r_1/r_2$ (as will become clear, $u_i^*$ depends on $r_1$ and $r_2$ only through $r_1/r_2$).
First, the standard result in the reputational bargaining literature (Abreu and Gul, 2000; Compte and Jehiel, 2002; Kambe, 1999) that player 1's sequential equilibrium payoff converges to 1 as $r_1/r_2$ converges to 0, and converges to 0 as $r_1/r_2$ converges to $\infty$, also applies to player 1's maxmin payoff. Thus, this important comparative static result continues to hold under knowledge of rationality, and in particular does not rely on equilibrium. This is analogous to the results on reputation in repeated games under knowledge of rationality of Watson (1993) and Battigalli and Watson (1997). However, I also derive player 1’s maxmin payoff for fixed $r_1/r_2$ (rather than only in the limit). This leads to a second comparative static result, which indicates that a change in relative patience has a larger effect on the maxmin payoff than a much larger change in commitment probability. An analogous result holds in equilibrium in existing reputational bargaining models.

I first present the analog of Theorem 2 for heterogeneous discount rates, and then state the two comparative statics results as corollaries.

**Proposition 4** If player $i$’s discount rate is $r_i$, then player 1’s maxmin payoff, $u_1^*(\varepsilon)$, is the unique number $u_1^*$ that solves

$$ u_1^* = \frac{1}{1 - \frac{r_1}{r_2} \log \varepsilon - \left(\frac{r_1}{r_2} - 1\right) \log u_1^*}. $$

**Corollary 3** shows that the standard limit comparative statics on $r_1/r_2$ in reputational bargaining models require only first-order knowledge of rationality.

**Corollary 3** $\lim_{r_1/r_2 \to 0} u_1^*(\varepsilon) = 1$. If $\varepsilon < 1$, then in addition $\lim_{r_1/r_2 \to \infty} u_1^*(\varepsilon) = 0$.

**Corollary 4** shows that the commitment probability $\varepsilon$ must decrease exponentially to (approximately) offset a geometric increase in relative patience $(r_1/r_2)^{-1}$. The result is stated for the case $r_1/r_2 \leq 1$, where even an exponential decrease in $\varepsilon$ does not fully offset a geometric increase in $(r_1/r_2)^{-1}$. If $r_1/r_2 > 1$, then an exponential decrease in $\varepsilon$ more than offsets a geometric increase in $(r_1/r_2)^{-1}$.

**Corollary 4** Suppose that $r_1/r_2 \leq 1$ and that $r_1/r_2$ and $\varepsilon$ both decrease while $(r_1/r_2) \log \varepsilon$ remains constant. Then $u_1^*(\varepsilon)$ increases.

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2.5.4 Rationalizability

Theorem 2 derives the highest payoff that player 1 can guarantee herself under first-order knowledge of rationality, the weakest epistemic assumption consistent with the possibility of reputation-building. I now show that player 1 cannot guarantee herself more than this under the much stronger assumption of rationalizability (or under any finite-order knowledge of rationality), which reenforces Theorem 2 substantially. The intuition is that the γ-offsetting belief—and thus the γ-offsetting strategy—is not only rational but also rationalizable, and player 1 receives payoff $u_1^*$ when she best-responds to the γ-offsetting strategy.

I consider the following definition of rationalizability:

**Definition 10** A set of bargaining phase strategy profiles $\Omega = \Omega_1 \times \Omega_2 \subseteq \Sigma_1 \times \Sigma_2$ has the best-response property given posture $\gamma$ if for all $\sigma_1 \in \Omega_1$ there exists some belief $\pi_1 \in \Delta(\Omega_2)$ such that $\sigma_1 \in \Sigma_1^* (\pi_1)$; and for all $\sigma_2 \in \Omega_2$ there exists some belief $\pi_2 \in \Delta(\Omega_1 \cup \{\gamma\})$ such that $\pi_2(\gamma) \geq \varepsilon$, with strict inequality only if $\gamma \in \Omega_1$, and $\sigma_2 \in \Sigma_2^* (\pi_2)$. The set of rationalizable strategies given posture $\gamma$ is

$$\Omega^{\text{RAT}}(\gamma) \equiv \bigcup \{\Omega : \Omega \text{ has the best-response property given posture } \gamma\}.$$

Player 1’s rationalizable maxmin payoff given posture $\gamma$ is

$$u_1^{\text{RAT}}(\gamma) \equiv \sup_{\sigma_1} \inf_{\sigma_2 \in \Omega^{\text{RAT}}(\gamma)} u_1(\sigma_1, \sigma_2).$$

Player 1’s rationalizable maxmin payoff is

$$u_1^{\text{RAT}} \equiv \sup \ u_1^{\text{RAT}}(\gamma).$$

A posture $\gamma^{\text{RAT}}$ is a rationalizable maxmin posture if there exists a sequence of postures $\{\gamma_n\}$ such that $\gamma_n(t) \to \gamma^{\text{RAT}}(t)$ for all $t \in \mathbb{R}_+$ and $u_1^{\text{RAT}}(\gamma_n) \to u_1^{\text{RAT}}$.

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37While consistent with this paper’s focus on normal-form rationality, this normal-form definition of rationalizability is weak in that it does not eliminate strategies that are dominated “off-path.” However, I conjecture that Proposition 5 also holds under extensive-form rationalizability (Pearce, 1984; Battigalli and Siniscalchi, 2003).
The result is the following:

**Proposition 5** Player 1’s rationalizable maxmin payoff equals her maxmin payoff, and the unique rationalizable maxmin posture is the unique maxmin posture. That is, $u_1^{RAT} = u_1^*$, and the unique rationalizable maxmin posture is $\gamma^{RAT} = \gamma^*$.

Any rationalizable strategy given posture $\gamma$ is also rational given posture $\gamma$. Therefore, Lemma 2 applies under rationalizability. The only additional fact used in the proof of Theorem 2 is that $u_1^*(\gamma) = \min_{t \leq T(\gamma)} e^{-rt}\gamma(t)$ for any posture $\gamma$ (Lemma 4). Supposing that the analogous equation holds under rationalizability (i.e., that $u_1^{RAT}(\gamma) = \min_{t \leq T(\gamma)} e^{-rt}\gamma(t)$), the proof of Theorem 2 goes through as written. Hence, to prove Proposition 5 it suffices to prove the following lemma, the proof of which shows that the $\gamma$-offsetting belief and strategy are rationalizable:

**Lemma 5** For any posture $\gamma$, $u_1^{RAT}(\gamma) = \min_{t \leq T(\gamma)} e^{-rt}\gamma(t)$.

### 2.6 Two-Sided Commitment

This section introduces the possibility that both players may announce—and become committed to—postures prior to the start of bargaining. I show that each player $i$’s maxmin payoff is close to that derived in Section 2.4 when her opponent’s commitment probability, $\varepsilon_j$, is small in absolute terms (even if $\varepsilon_j$ is large relative to $\varepsilon_i$). In addition, each player’s maxmin posture is exactly as in Section 2.4. This shows that the analysis of Section 2.4 provides a two-sided theory of reputational bargaining. The results of this section contrast with the existing reputational bargaining literature, which emphasizes that relative commitment probabilities are crucial for determining equilibrium behavior and payoffs.

Formally, modify the model of Section 2.3 by assuming that in the announcement stage players simultaneously announce postures $(\gamma_1, \gamma_2)$, to which they become committed with probabilities $\varepsilon_1$ and $\varepsilon_2$, respectively. The bargaining phase is unaltered. Thus, at the beginning of the bargaining phase, player $i$ believes that player $j$ is committed to posture $\gamma_j$ with probability $\varepsilon_j$ and is rational with probability $1 - \varepsilon_j$ (though this fact is not common

\[38\] The events that player 1 and player 2 become committed need not be independent.
knowledge). The following definitions are analogs of Definitions 4 through 6 that allow for the fact that both players may become committed to the postures they announce:

**Definition 11** A belief $\pi_i$ of player $i$'s is consistent with knowledge of rationality given postures $(\gamma_i, \gamma_j)$ if $\pi_i(\gamma_j) \geq \varepsilon_j$ and $\pi_i(\gamma_i) > \varepsilon_i$ only if there exists $\pi_j$ such that $\pi_j(\gamma_i) \geq \varepsilon_i$ and $\gamma_j \in \Sigma_j^*(\pi_j)$; and, for all $\sigma_j \neq \gamma_j$, $\sigma_j \in \text{supp}(\pi_i)$ only if there exists $\pi_j$ such that $\pi_j(\gamma_i) \geq \varepsilon_i$ and $\sigma_j \in \Sigma_j^*(\pi_j)$. Let $\Pi_{i(\gamma_i, \gamma_j)}$ be the set of player $i$'s beliefs that are consistent with knowledge of rationality given postures $(\gamma_i, \gamma_j)$. Player $i$'s maxmin payoff given postures $(\gamma_i, \gamma_j)$ is

$$u_i^*(\gamma_i, \gamma_j) = \sup_{\sigma_i} \inf_{\pi_i \in \Pi_i(\gamma_i, \gamma_j)} u_i(\sigma_i, \pi_i).$$

Player $i$'s maxmin payoff is

$$u_i^* = \sup_{\gamma_i} \inf_{\gamma_j} u_i^*(\gamma_i, \gamma_j).$$

A posture $\gamma_i^*$ is a maxmin posture (of player $i$'s) if there exists a sequence of postures $\{\gamma_n\}$ such that $\gamma_n(t) \to \gamma_i^*(t)$ for all $t \in \mathbb{R}_+$ and $\inf_{\gamma_j} u_i^*(\gamma_n, \gamma_j) \to u_i^*.$

Note that if $\varepsilon_j = 0$ then all of these definitions (for player $i$) reduce to the corresponding definitions in the one-sided commitment model. Thus, writing $u_i^*(\varepsilon_i, \varepsilon_j)$ for player $i$'s maxmin payoff in the two-sided commitment model when the commitment probabilities are $\varepsilon_i$ and $\varepsilon_j$, it follows that $u_i^*(\varepsilon_i, 0) = u_i^*(\varepsilon_i)$, player $i$'s maxmin payoff in the one-sided commitment model.

I now show that $u_i^*(\varepsilon_i, \varepsilon_j)$ is approximately equal to $u_i^*(\varepsilon_i)$ whenever $\varepsilon_j$ is small, and that the maxmin posture is exactly as in the one-sided commitment model. This is simply because player $i$ cannot guarantee herself anything in the event that player $j$ is committed (e.g., if player $j$'s announced posture always demands the entire surplus), which implies that player $i$ guarantees herself as much as possible by conditioning on the event that player $j$ is not committed. In this event, which occurs with probability $1 - \varepsilon_j$, player $i$ can guarantee herself $u_i^*(\varepsilon_i)$, and the only way she can guarantee herself this much is by announcing $\gamma_i^*.

**Theorem 3** Player $i$'s maxmin payoff is $u_i^*(\varepsilon_i, \varepsilon_j) = (1 - \varepsilon_j) u_i^*(\varepsilon_i)$, and player $i$'s unique maxmin posture is $\gamma_{i(\varepsilon_i, \varepsilon_j)} = \gamma_{\varepsilon_i}$.
**Proof.** Let $\gamma_j^0$ be the posture of player $j$'s given by $\gamma_j^0(t) = 1$ for all $t$. Note that $u_i(\sigma_i, \gamma_j^0) = 0$ for all $\sigma_i$. Therefore, $\inf_j u_i(\sigma_i, \gamma_j) = 0$ for all $\sigma_i$.

Next, let $\Pi_i^{\gamma_i\gamma_j} (\varepsilon_i, \varepsilon_j)$ be the set of beliefs $\pi_i$ that are consistent with knowledge of rationality for commitment probabilities $(\varepsilon_i, \varepsilon_j)$, and let $\Pi_i^\varepsilon (\varepsilon_i)$ be the analogous set in the one-sided commitment model. I claim that if $\pi_i \in \Pi_i^{\gamma_i\gamma_j} (\varepsilon_i, \varepsilon_j)$, then there exists $\pi_i' \in \Pi_i^{\gamma_i} (\varepsilon_i)$ such that $\pi_i = (1 - \varepsilon_j)\pi_i' + \varepsilon_j\gamma_j$, where $(1 - \alpha)x \oplus \alpha y$ is the compound lottery that puts weight $1 - \alpha$ on $x$ and $\alpha$ on $y$. To see this, note that $\pi_i(\gamma_j) \geq \varepsilon_j$, so there exists a probability distribution $\pi_i'$ such that $\pi_i = (1 - \varepsilon_j)\pi_i' + \varepsilon_j\gamma_j$. Furthermore, by definition of $\Pi_i^{\gamma_i\gamma_j} (\varepsilon_i, \varepsilon_j)$, $\sigma_j \in \text{supp}(\pi_i')$ only if there exists $\pi_j$ such that $\pi_j(\gamma_i) \geq \varepsilon_i$ and $\sigma_j \in \Sigma_j^*(\pi_j)$ (whether or not $\sigma_j$ equals $\gamma_j$).  

Combining the above observations,

$$\inf_{\gamma_j} u_i^* (\gamma_i, \gamma_j) = \inf_{\gamma_j} \sup_{\sigma_i} \inf_{\pi_i' \in \Pi_i^{\gamma_i\gamma_j} (\varepsilon_i, \varepsilon_j)} u_i(\sigma_i, \pi_i)$$

$$= \inf_{\gamma_j} \sup_{\sigma_i} \inf_{\pi_i' \in \Pi_i^{\gamma_i} (\varepsilon_i)} (1 - \varepsilon_j)u_i(\sigma_i, \pi_i') + \varepsilon_ju_i(\sigma_i, \gamma_j).$$

$$= \sup_{\sigma_i} \inf_{\pi_i' \in \Pi_i^{\gamma_i} (\varepsilon_i)} (1 - \varepsilon_j)u_i(\sigma_i, \pi_i') + \varepsilon_j(0)$$

$$= (1 - \varepsilon_j)u_i^* (\gamma_i).$$

Therefore, the definitions of $u_i^* (\varepsilon_i, \varepsilon_j)$ and $u_i^* (\varepsilon_i)$ imply that $u_i^* (\varepsilon_i, \varepsilon_j) = \sup_{\gamma_i} (1 - \varepsilon_j)u_i^* (\gamma_i) = (1 - \varepsilon_j)u_i^* (\varepsilon_i)$. Similarly, the definition of a maxmin posture in the one-sided commitment model implies that $\gamma_i^{\varepsilon_i (\varepsilon_i, \varepsilon_j)}$ is a maxmin posture in the two-sided commitment model if and only if it is a maxmin posture in the one-sided commitment model with $\varepsilon = \varepsilon_i$.  

Theorem 3 implies that the qualitative insights of Theorem 2 also apply with two-sided commitment. For example, fixing any $\varepsilon_2$ bounded away from 0, $u_i^* (\varepsilon_1, \varepsilon_2)$ goes to 0 at a logarithmic rate in $\varepsilon_1$. Thus, Theorem 3 says much more than that $u_i^* (\varepsilon_1, \varepsilon_2)$ is continuous in $\varepsilon_2$ at $\varepsilon_2 = 0$. Table 2 displays the maxmin payoff for both the one-sided commitment

---

Footnote 39: Here, the weaker statement that $\pi_i' (\gamma_j) > 0$ only if there exists $\pi_j$ such that $\pi_j(\gamma_i) \geq \varepsilon_i$ and $\sigma_j \in \Sigma_j^*(\pi_j)$ is immediate, and this can be strengthened to the statement that $\gamma_j \in \text{supp}(\pi_i')$ only if there exists such a $\pi_j$ because the best-response correspondence is upper semi-continuous in beliefs.
model and the two-sided commitment model in the case where $\varepsilon_1 = \varepsilon_2 = \varepsilon$:

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$u_i^*(\varepsilon)$</th>
<th>$(1 - \varepsilon) u_i^*(\varepsilon)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.25</td>
<td>.42</td>
<td>.31</td>
</tr>
<tr>
<td>.1</td>
<td>.30</td>
<td>.27</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>.13</td>
<td>.13</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>.07</td>
<td>.07</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>.05</td>
<td>.05</td>
</tr>
</tbody>
</table>

Table 2: The Maxmin Payoff for Different Commitment Probabilities $\varepsilon$ with One- and Two-Sided Commitment

Finally, Definition 11 specifies that player $i$’s belief is consistent with knowledge of rationality only if it assigns probability exactly $\varepsilon_j$ to the event that player $j$ is committed to $\varepsilon_j$. If this were relaxed by specifying that a belief is consistent with knowledge of rationality if it assigns any probability $\varepsilon'_j \leq \varepsilon_j$ to the event that player $j$ is committed to $\varepsilon_j$ (and assigns probability $1 - \varepsilon'_j$ to player $j$’s being rational), Theorem 3 and its proof would go through with trivial modifications. Thus, Theorem 3 requires only that player $i$ believes that player $j$’s commitment probability is not more than $\varepsilon_j$, not that player $i$ believes that player $j$’s commitment probability is exactly $\varepsilon_j$.

2.7 Discrete-Time Bargaining with Frequent Offers

This section considers discrete-time bargaining procedures in which both players can make offers frequently. This includes procedures with any order and relative frequency of offers. I show that, with one-sided commitment, for any sequence of discrete-time bargaining games that converges to continuous time (in the sense that each player may make an offer close to any given time), the corresponding sequence of maxmin payoffs and postures converges to the continuous-time maxmin payoff and posture given by Theorem 2 (the analogous result with two-sided commitment is immediate and is omitted to simplify the exposition). Abreu and Gul (2000) provide a similar independence-of-procedures result for sequential equilibrium outcomes of reputational bargaining. Because my result concerns maxmin payoffs and
postures rather than equilibria, my proof is very different from Abreu and Gul's.

Formally, replace the (continuous time) bargaining phase of Section 2.3 with the following procedure: There is a (commonly known) function \( g : \mathbb{R}_+ \to \{0, 1, 2\} \) that specifies who makes an offer at each time. If \( g(t) = 0 \), no player takes an action at time \( t \). If \( g(t) = i \in \{1, 2\} \), then player \( i \) makes a demand \( u_i(t) \in [0, 1] \) at time \( t \), and player \( j \) immediately accepts or rejects. If player \( j \) accepts, the game ends with payoffs \( (e^{-rt}u_i(t), e^{-rt}(1 - u_i(t))) \); if player \( j \) rejects, the game continues. Let \( I_i^n = \{ t : g(t) = i \} \), and assume that \( I_i^n \cap [0, t] \) is finite for all \( t \) and that \( I_i^n \) is infinite. The announcement phase is correspondingly modified so that player 1 announces a posture \( \gamma : I_1^n \to [0, 1] \), and if player 1 becomes committed to posture \( \gamma \) (which continues to occur with probability \( \varepsilon \)), she demands \( \gamma(t) \) at time \( t \) and rejects all of player 2's demands. I refer to the function \( g \) as a discrete-time bargaining game.

I now define convergence to continuous time. This definition is very similar to that of Abreu and Gul (2000), as is the above model of discrete-time bargaining and the corresponding notation.

**Definition 12** A sequence of discrete-time bargaining games \( \{g_n\} \) converges to continuous time if for all \( \Delta > 0 \), there exists \( N \) such that for all \( n \geq N \), \( t \in \mathbb{R}_+ \), and \( i \in \{1, 2\} \), \( I_i^{g_n} \cap [t, t + \Delta] \neq \emptyset \).

The maxmin payoff and posture in a discrete-time bargaining game are defined exactly as in Section 2.3. Let \( u_1^{*g} \) be player 1's maxmin payoff in discrete-time bargaining game \( g \), and let \( u_1^{*g} (\gamma) \) be player 1's maxmin payoff given posture \( \gamma \) in \( g \). The independence-of-procedures results states that, for any sequence of discrete-time bargaining games converging to continuous time, the corresponding sequence of maxmin payoffs \( \{u_1^{*g_n}\} \) converges to \( u_1^* \), and any corresponding sequence of postures \( \{\gamma^{g_n}\} \) such that \( u_1^{*g_n} (\gamma^{g_n}) \to u_1^* \) "converges" to \( \gamma^* \), where \( u_1^* \) and \( \gamma^* \) are the maxmin payoff and posture identified in Theorem 2. The nature of the convergence of the sequence \( \{\gamma^{g_n}\} \) to \( \gamma^* \) is slightly delicate. For example, there may be (infinitely many) times \( t \in \mathbb{R}_+ \) such that \( \lim_{n \to \infty} \gamma^{g_n}(t) \) exists and is greater than \( \gamma^*(t) \), because these demands may be "non-serious" (in that they are followed immediately by lower
Thus, rather than stating the convergence in terms of \( \{y^n\} \) and \( \gamma^* \), I state it in terms of the corresponding continuation values of player 2, which are the economically more important variables. Formally, given a posture \( y^n \) in discrete-time bargaining game \( g^n \), let

\[
v^n(t) \equiv \max_{\tau \geq t: \tau \in t^n} e^{-r(t-\tau)} (1 - \gamma^n(\tau)).
\]

Let \( v^*(t) \equiv \max \{1 - e^{rt}/(1 - \log \varepsilon), 0\} \), the continuation value corresponding to \( \gamma^* \) in the continuous-time model of Section 2.3. The independence-of-procedures result is as follows:

**Theorem 4** Let \( \{g_n\} \) be a sequence of discrete-time bargaining games converging to continuous time. Then \( u^*_n \to u^*_1 \), and if \( \{y^n\} \) is a sequence of postures with \( y^n \) a posture in \( g_n \) and \( u^*_1(y^n) \to u^*_1 \), then \( v^n(t) \to v^*(t) \) for all \( t \in \mathbb{R}_+ \).

The key fact behind the proof of Theorem 4 is that for any sequence of discrete-time postures \( \{y^n\} \) converging to some continuous-time posture \( \gamma \), \( \lim_{n \to \infty} u^*_1(y^n) = \lim_{n \to \infty} u^*_1(\gamma^n) \) (where \( u^*_1(\gamma^n) \) is the maxmin payoff given a natural embedding of \( \gamma^n \) in continuous time, defined formally in the supplementary appendix). This fact is proved by constructing a belief that is similar to the \( \gamma^n \)-offsetting belief in each discrete-time game \( g_n \) and then showing that these beliefs converge to the \( \gamma \)-offsetting belief in the limiting continuous-time game.

### 2.8 Conclusion

This paper analyzes a model of reputational bargaining in which players initially announce postures to which they may become committed and then bargain over a unit of surplus. It characterizes the highest payoff that a player can guarantee herself under first-order knowledge of rationality, along with the bargaining posture that she must announce in order to guarantee herself this much. A key step in the characterization is showing that this maxmin payoff is the payoff a player receives when her opponent holds the “offsetting belief” that she mixes between following her announced posture and accepting her opponent’s demand at a specific rate. Technically, this intermediate result lets one evaluate a posture in terms of its performance against an opponent who holds the corresponding offsetting belief, rather
than having to check it’s performance against every rational opposing strategy. Conceptually, it shows that the maxmin payoff is also the lowest payoff that can be obtained by a player who knows her opponent’s strategy, establishing an equivalence between “maxmin” and “Bayesian” definitions of the highest guaranteed payoff.

I find that a player can guarantee herself a relatively high share of the surplus even if her probability of becoming committed is very small, and that the unique bargaining posture that guarantees this much is simply demanding this share of the surplus in addition to compensation for any delay in reaching agreement. These insights apply for one- or two-sided commitment, for any bargaining procedure with frequent offers, for general bargaining sets, for heterogeneous discount factors, and for any level of knowledge of rationality. In addition, if a player could only announce postures that always demand the same share of the surplus (as in most of the existing literature), her maxmin payoff would be approximately $e$ times lower.

These results are intended to complement the existing equilibrium analysis of reputational bargaining models. Consider the fundamental question, “What posture should a bargainer stake out?” In equilibrium analysis, the answer to this question depends on her opponent’s beliefs about her continuation play following every possible announcement. Yet it may be impossible for either the bargainer or an outside observer to learn these beliefs, especially when bargaining is one-shot. Hence, an appealing alternative approach is to look for a posture that guarantees a high payoff against any belief of one’s opponent, and for the highest payoff that each player can guarantee herself. This paper shows that this approach yields sharp and economically plausible results, while addressing important concerns about robustness.

The results of this paper are particularly applicable in models where the division of the surplus is of primary importance (rather than the details of how bargaining proceeds, which depend on additional behavioral assumptions). A leading example is the class of models where two parties make costly ex ante investments and then bargain over the resulting surplus. It is a direct consequence of Theorem 3 that, if the players’ commitment probabilities are small, both players benefit from comparable increases in their commitment probabil-
ties.\textsuperscript{41} Note that this is distinct from the idea that both players benefit from reducing delay; rather, both players benefit because higher commitment probabilities reduce the scope for pessimism about how bargaining will proceed (which would not be possible in an equilibrium analysis). This implies that, for example, comparably increasing both players' commitment powers increases investments whenever investments are complementary. It seems likely that additional insights could be derived by further studying non-equilibrium models of bargaining both in this class of models and in other applied theory models involving bargaining.

Finally, I discuss two additional interesting issues for future research. First, an earlier version of this paper extends the model to multilateral bargaining, where \( n \geq 3 \) players must unanimously agree on the division of the surplus. In such a model, player \( j \) may reject any proposal if he expects player \( k \) to do so as well, and vice versa. Hence, a player with commitment power cannot guarantee herself a positive payoff under knowledge of rationality. It therefore remains to be seen whether reputational models can make sharp and robust predictions in multilateral bargaining.

Second, it would be interesting to analyze commitment and reputation-building under knowledge of rationality in dynamic games other than bargaining, noting that the definition of a player's maxmin payoff and posture extends to general games. One intriguing observation is that the rate at which the reputation-builder's maxmin payoff converges to her lowest feasible payoff as her commitment probability converges to 0 is slower in my model than in existing repeated game models. In particular, the reputation-builder's (player 1's) maxmin payoff converges to her minimum payoff of 0 at a logarithmic rate in \( \varepsilon \) in my model, while in repeated game models this convergence is at a polynomial rate in \( \varepsilon \).\textsuperscript{42} To understand this difference, recall from the proof of Proposition 2 that if player 1 announces a constant posture \( \gamma \), then her maxmin payoff given posture \( \gamma \) equals \( \varepsilon^{\gamma/(1-\gamma)} \gamma \), which is polynomial in

\textsuperscript{41}To see this somewhat more formally, recall that Theorem 3 states that player i's maxmin payoff is \( u_i^* (\varepsilon_i, \varepsilon_j) = \frac{1-\varepsilon_j}{1-\log \varepsilon_i} \). Therefore, \( \frac{\partial u_i^* (\varepsilon_i, \varepsilon_j)}{\partial \varepsilon_i} = \frac{1-\varepsilon_j}{\varepsilon_i (1-\log \varepsilon_i)^2} \), while \( \frac{\partial u_i^* (\varepsilon_i, \varepsilon_j)}{\partial \varepsilon_j} = -\frac{1}{1-\log \varepsilon_i} \). Since \( \lim_{\varepsilon_i \to 0} \varepsilon_i (1-\log \varepsilon_i) = 0 \), it follows that \( \lim_{\varepsilon_i \to 0} \frac{\partial u_i^* (\varepsilon_i, \varepsilon_j)}{\partial \varepsilon_i} (\varepsilon_i, \varepsilon_j) = \infty \). In this sense, an increase in \( \varepsilon_i \) increases player i's maxmin payoff by much more than an increase in \( \varepsilon_j \) decreases it.

\textsuperscript{42}Fudenberg and Levine (1989) show that if player 1 is committed to her Stackelberg action with probability \( \varepsilon \) and player 2 is myopic, then player 1's payoff in any Nash equilibrium is at least \( \varepsilon^{1-\alpha}_{\gamma} \hat{u}_1 + (1 - \varepsilon^{1-\alpha}_{\gamma}) \hat{u}_2 \), for some constant \( \alpha > 0 \), where \( \hat{u}_1 \) is player 1's Stackelberg payoff and \( \hat{u}_2 \) is her lowest feasible payoff. This bound is the basis for most of the subsequent literature; for example, convergence to \( \hat{u}_1 \) is also polynomial in \( \varepsilon \) in Schmidt (1993), Cripps, Schmidt, and Thomas (1996), and Evans and Thomas (1997).
However, the maxmin constant posture is increasing in $\varepsilon$ (and goes to 0 as $\varepsilon \to 0$), and Corollary 2 shows that player 1 can guarantee herself a payoff that goes to 0 at a logarithmic rate in $\varepsilon$ by appropriately recalibrating her announced posture as $\varepsilon \to 0$. Thus, roughly speaking, the reason why reputation bounds in the repeated games literature converge to 0 more quickly than in my model is that there is generally no way to continuously moderate one's posture as one's commitment probability decreases in repeated games.

2.9 Appendix A: Omitted Proofs for Sections 2.3 and 2.4

Proof of Proposition 1. The proof uses results from Section 2.4, and therefore should not be read before reading Section 2.4.

Fix a posture $\gamma$ and payoff $u_1 \in [u_1^*(\gamma), 1)$. If $u_1 \neq \gamma(0)$, then let $\delta_2^\gamma$ be identical to the $\gamma$-offsetting strategy defined in Definition 9, with the modification that player 1's demand is accepted at any history $h'$ at which player 1 has demanded $u_1$ at all previous dates. If $u_1 = \gamma(0)$, then let $\delta_2^\gamma$ be identical to the $\gamma$-offsetting strategy defined in Definition 9, with the modification that player 1's demand is accepted at date $(-\log(u_1)/r, -1)$ if player 1 has demanded 1 at all previous dates. In either case, let $\pi_2^\gamma$ be as in Definition 9, and note that $\pi_2^\gamma(\gamma) \geq \varepsilon$. If $u_1 \neq \gamma(0)$, no strategy under which $u_1(0) = u_1$ is in the support of $\pi_2^\gamma$; similarly, if $u_1 = \gamma(0)$, no strategy under which $u_1(0) = 1$ is in the support of $\pi_2^\gamma$ (since $u_1 < 1$). Therefore, the same argument as in the proof of Lemma 3 shows that $\delta_2^\gamma \in \Sigma_2^*(\pi_2^\gamma)$. Hence, the belief $\pi_1$ given by $\pi_1(\delta_2^\gamma) = 1$ is an element of $\Pi_1^\gamma$. Furthermore, under strategy $\delta_2^\gamma$, player 2 always demands 1 and only accepts player 1's demand if player 1 has either conformed to $\gamma$ through time $t^*$ (defined in Section 2.4.2) or has always demanded $u_1$ (in the $u_1 \neq \gamma(0)$ case) or 1 (in the $u_1 = \gamma(0)$ case). Note that $\exp(-r(-\log(u_1)/r)) = u_1$. Hence, in either case, $u_1(\sigma_1, \pi_1) \in \{0, u_1^*(\gamma), 1\}$ for every strategy $\sigma_1$. Let $\delta_1$ be the strategy of player 1's that always demands $u_1$ (if $u_1 \neq \gamma(0)$) or 1 (if $u_1 = \gamma(0)$) and never accepts player 2's demand. Then $u_1(\delta_1, \pi_1) = u_1 = \max_{\sigma_1} u_1(\sigma_1, \pi_1)$, completing the proof.

Proof of Lemma 2. I prove the result for pure strategies $\sigma_2$, which immediately implies the result for mixed strategies.

Fix $\pi_2$ such that $\pi_2(\gamma) \geq \varepsilon$ and pure strategy $\sigma_2 \in \Sigma_2^*(\pi_2)$. The plan of the proof is
to show that if $\pi_2(\gamma) \geq \varepsilon$ and agreement is not reached by $\hat{T}$ under strategy profile $({\gamma, \sigma_2})$, then player 2 must be certain that player 1 is playing $\gamma$ at any time $t > \hat{T}$. This suffices to prove the lemma, because $\sigma_2 \in \Sigma_2^*({\pi_2})$ implies that player 2 accepts $\gamma(t)$ no later than time $t = T$ if at any time $t > \hat{T}$ agreement has not been reached and he is certain that player 1 is playing $\gamma$.

Let $\chi^{(\pi_2,\sigma_2)}(t)$ be the probability that player 2 assigns to player 1 not playing $\gamma$ at date $(t, -1)$ when his initial belief is $\pi_2$ and play up until date $(t, -1)$ is given by player 1’s following strategy $\gamma$ and player 2’s following (pure) strategy $\sigma_2$; this is determined by Bayes’ rule, because $\pi_2(\gamma) \geq \varepsilon > 0$. By convention, if agreement is reached at time $\tau$, let $\chi^{(\pi_2,\sigma_2)}(t) = \chi^{(\pi_2,\sigma_2)}(\tau)$ for all $t > \tau$. Let $t(\gamma, \sigma_2)$ be the time at which agreement is reached under strategy profile $({\gamma, \sigma_2})$ (with the convention that $t(\gamma, \sigma_2) \equiv \infty$ if agreement is never reached under $({\gamma, \sigma_2})$); and let

$$\hat{t}(\gamma, \sigma_2) \equiv \sup \{ t : \chi^{(\pi_2,\sigma_2)}(t) > 0 \},$$

the latest time at which player 2 is not certain that player 1 is playing $\gamma$ under strategy profile $({\gamma, \sigma_2})$ with belief $\pi_2$. Let

$$\hat{T} \equiv \sup_{(\pi_2, \sigma_2) : \pi_2(\gamma) \geq \varepsilon, \sigma_2 \in \Sigma_2^*(\pi_2), t(\gamma, \sigma_2) \geq \hat{t}(\gamma, \sigma_2)} \hat{t}(\gamma, \sigma_2).$$

That is, $\hat{T}$ is the latest possible time $t$ at which player 2 is not certain that player 1 is following $\gamma$ and agreement is not reached by $t$. I will show that $\hat{T} = \hat{T}$, which completes the proof.

I first claim that in the definition of $\hat{T}$ it is without loss of generality to restrict attention to $(\pi_2, \sigma_2)$ such that $\sigma_2$ always demands $u_2(t) = 1$, $\pi_2$ puts probability 1 on player 1 conceding at any history $h^{i+}$ at which $u_1(t) \neq \gamma(t)$, and $\pi_2$ puts probability 0 on player 1 conceding at any history $h^{i-}$; that is, that the right-hand side of (14) continues to equal $\hat{T}$ when this additional constraint is imposed. To see this, suppose that $(\pi_2', \sigma_2')$ satisfies $\pi_2'(\gamma) \geq \varepsilon$, $\sigma_2' \in \Sigma_2^*(\pi_2')$, and $t(\gamma, \sigma_2') \geq \hat{t}(\gamma, \sigma_2')$ (the constraints of (14)). Let $\pi_2$ be the belief under which player 1 demands $u_1(t) = \gamma(t)$ for all $t \in \mathbb{R}^+$; accepts player 2’s demand at every
history of the form \((\gamma (\tau), 1)_{\tau \leq t}\) at the same rate and probability at which player 1 deviates from \(\gamma\) at time \(t\) (i.e., at date \((t, -1), (t, 0),\) or \((t, 1)\)) under strategy profile \((\pi'_2, \sigma'_2)\) (viewing \(\pi'_2\) as a mixed strategy of player 1’s); and rejects player 2’s demand at every other history. Clearly, there exists a strategy \(\sigma_2 \in \Sigma'_2(\pi_2)\) that always demands \(u_2(t) = 1\). Note that player 1’s rate and probability of deviating from \(\gamma\) at history \((\gamma (\tau), 1)_{\tau \leq t}\) under belief \(\pi_2\) is the same as at time \(t\) under strategy profile \((\pi'_2, \sigma'_2)\), and that player 2’s continuation payoff after such a deviation is weakly higher in the former case. Recall that strategy \(\gamma\) never accepts player 2’s demand, so agreement is reached only if player 2 accepts player 1’s demand or if player 1 has deviated from \(\gamma\). Therefore, since rejecting player 1’s demand \(\gamma (t)\) under strategy profile \((\pi'_2, \sigma'_2)\) is optimal for all \(t < t(\gamma, \sigma'_2)\), it follows that rejecting player 1’s demand \(\gamma (t)\) at history \((\gamma (\tau), 1)_{\tau \leq t}\) is optimal under belief \(\pi_2\), for all \(t < t(\gamma, \sigma'_2)\). This implies that \(t(\gamma, \sigma_2) \geq t(\gamma, \sigma'_2)\). Furthermore, \(\chi^{(\tau_2, \sigma_2)}(t) = \chi^{(\tau'_2, \sigma'_2)}(t)\) for all \(t \in \mathbb{R}_+\), so \(\hat{t}(\gamma, \sigma_2) = \hat{t}(\gamma, \sigma'_2)\). Hence, \(t(\gamma, \sigma_2) \geq \hat{t}(\gamma, \sigma_2)\). Finally, \(\pi_2(\gamma) \geq \varepsilon\). Therefore, \((\pi_2, \sigma_2)\) satisfies the constraints of (14); \(\sigma_2\) always demands \(u_2(t) = 1\); \(\pi_2\) puts probability 1 on player 1 conceding at any history \(h^{t+}\) at which \(u_1(t) \neq \gamma(t)\); \(\pi_2\) puts probability 0 on player 1 conceding at any history \(h^{t-}\); and \(\hat{t}(\gamma, \sigma_2) \geq \hat{t}(\gamma, \sigma'_2)\); so the right-hand side of (14) continues to equal \(\hat{T}\) when the additional constraint is imposed.

Thus, fix a belief \(\pi_2\) that puts probability 1 on player 1 conceding at any history \(h^{t+}\) at which \(u_1(t) \neq \gamma(t)\), and puts probability 0 on player 1 conceding at any history \(h^{t-}\). Let \(\lambda^{\pi_2}(t)\) and \(p^{\pi_2}(t)\) be the concession rate and probability of player 1 at history \((\gamma (\tau), 1)_{\tau \leq t}\) when her strategy is given by \(\pi_2\); let \(S^{\pi_2}\) be the (countable) set of times \(s\) such that \(p^{\pi_2}(s) > 0\); and let \(\hat{t}(\pi_2) \equiv \hat{t}(\gamma, \sigma_2^0)\), where \(\sigma_2^0\) is the strategy that always demands \(u_2(t) = 1\) and always rejects player 1’s demand. Fixing a strategy \(\sigma_2 \in \Sigma'_2(\pi_2)\) that always demands \(u_2(t) = 1\), note that \((\gamma, \sigma_2)\) and \((\gamma, \sigma_2^0)\) induce the same path of play until time \(t(\gamma, \sigma_2)\), and therefore \(t(\gamma, \sigma_2) \geq \hat{t}(\gamma, \sigma_2)\) if and only if \(t(\gamma, \sigma_2) \geq \hat{t}(\pi_2)\). Hence, \(t(\gamma, \sigma_2) \geq \hat{t}(\gamma, \sigma_2)\) if and only if it is optimal for player 2 to reject player 1’s offer until time \(\hat{t}(\pi_2)\), when his
initial belief is $\pi_2$ and player 1 plays $\gamma$. I claim that this holds if and only if

$$1 - \gamma(t) \leq \int_t^{i(\pi_2)} \exp \left( -r(\tau - t) - \int_t^{\tau} \lambda^\pi_2(s) \, ds \right) \left( \prod_{s \in S^\pi_2 \cap (t, \tau)} (1 - p^\pi_2(s)) \right) \lambda^\pi_2(\tau) \, d\tau$$

$$+ \sum_{s \in S^\pi_2 \cap (t, i(\pi_2))} \exp \left( -r(s - t) - \int_t^s \lambda^\pi_2(q) \, dq \right) \left( \prod_{q \in S^\pi_2 \cap (t, s)} (1 - p^\pi_2(q)) \right) p^\pi_2(s)$$

$$+ \exp \left( -r(i(\pi_2) - t) - \int_t^{i(\pi_2)} \lambda^\pi_2(s) \, ds \right) \left( \prod_{s \in S^\pi_2 \cap (t, i(\pi_2))} (1 - p^\pi_2(s)) \right) v(i(\pi_2))$$

for all $t < i(\pi_2)$.

The left-hand side of (15) is player 2’s payoff from accepting player 1’s demand at date $(t, 1)$ when $p^\pi_2(t) = 0$. The right-hand side of (15) is player 2’s continuation payoff from rejecting player 1’s demand until time $i(\pi_2)$ when $p^\pi_2(t) = 0$. Thus, (15) must hold if $t(\gamma, \sigma_2) \geq i(\pi_2)$. It remains to show that (15) implies that it is optimal for player 2 to reject at times where $p^\pi_2(t) > 0$. Suppose that $p^\pi_2(t) > 0$. At date $(t, -1)$, the fact that $S^\pi_2$ is countable and (15) holds at all times before $t$ that are not in $S^\pi_2$ implies that $\lim_{\tau \uparrow t} (1 - \gamma(\tau))$ is weakly less than player 2’s continuation payoff from rejecting playing 1’s demand until time $i(\pi_2)$. Furthermore, the fact that player 1 concedes with probability 0 at date $(t, -1)$ implies that $\lim_{\tau \uparrow t} (1 - \gamma(\tau))$ is indeed player 2’s payoff from accepting at date $(t, -1)$. Thus, rejecting is optimal at date $(t, -1)$. At date $(t, 1)$, player 2’s payoff from accepting is $(1 - p^\pi_2(t)/2)(1 - \gamma(t)) + (p^\pi_2(t)/2)(1)$, while his continuation payoff from rejecting until time $i(\pi_2)$ is $1 - p^\pi_2(t)$ times the right-hand side of (15) plus $p^\pi_2(t)(1)$. Hence, (15) implies that rejecting is optimal at date $(t, 1)$ as well.
In addition, (15) holds if and only if

\[
v(t) \leq \int_t^{i(\pi_2)} \exp \left( -r (\tau - t) - \int_t^\tau \lambda^{\pi_2} (s) \, ds \right) \left( \prod_{s \in S^{\pi_2} \cap (t, \tau)} (1 - p^{\pi_2} (s)) \right) \lambda^{\pi_2} (\tau) \, d\tau \\
+ \sum_{s \in S^{\pi_2} \cap (t, i(\pi_2))} \exp \left( -r (s - t) - \int_t^s \lambda^{\pi_2} (q) \, dq \right) \left( \prod_{q \in S^{\pi_2} \cap (t, s)} (1 - p^{\pi_2} (q)) \right) p^{\pi_2} (s) \\
+ \exp \left( -r (i(\pi_2) - t) - \int_t^{i(\pi_2)} \lambda^{\pi_2} (s) \, ds \right) \left( \prod_{s \in S^{\pi_2} \cap (t, i(\pi_2))} (1 - p^{\pi_2} (s)) \right) v(i(\pi_2))
\]

for all \( t < i(\pi_2) \).

(16)

To see this, note that (16) immediately implies (15) because \( v(t) \geq 1 - \gamma(t) \) for all \( t \). For the converse, suppose that (15) holds. If \( v(t) > 1 - \gamma(t) \) then \( v(t) = e^{-r(t-\tau)} (1 - \gamma(\tau)) \) for some \( \tau > t \) such that \( v(\tau, -1) = 1 - \gamma(\tau) \), which implies that \( v(\tau, -1) \) is weakly less than the limit as \( s \uparrow \tau \) of the right-hand side of (16) evaluated at time \( s \) (with the convention that the right-hand side of (16) equals \( v(s) \) if \( s \geq i(\pi_2) \)). Now the right-hand side of (16) at time \( t \) is at least \( e^{-r(t-\tau)} \) times as large as is this limit, which implies that the right-hand side of (16) at time \( t \) is at least \( e^{-r(t-\tau)} v(\tau, -1) = v(t) \). Hence, (16) holds.

By the previous two paragraphs, (14) may be rewritten as

\[
\hat{T} = \sup_{\pi_2: \pi_2(\gamma) \geq \varepsilon, \ (16) \ holds} \sup \left\{ t : \chi^{\pi_2}(t) \equiv \frac{\exp \left( - \int_0^t \lambda^{\pi_2} (s) \, ds \right) \prod_{s \in S^{\pi_2} \cap (0, t)} (1 - p^{\pi_2} (s)) - \varepsilon}{\exp \left( - \int_0^t \lambda^{\pi_2} (s) \, ds \right) \prod_{s \in S^{\pi_2} \cap (0, t)} (1 - p^{\pi_2} (s))} > 0 \right\}.
\]

(17)

I first show that there exists some belief \( \pi_2 \) that both attains the (outer) supremum in (17) (with the convention that the supremum is attained at \( \pi_2 \) if \( i(\pi_2) = \hat{T} = \infty \)) and also maximizes \( \lim_{t\uparrow \hat{T}} \chi^{\pi_2}(t) \) over all beliefs \( \pi_2 \) that attain the supremum (note that this limit exists for all \( \pi_2 \), because \( \chi^{\pi_2}(t) \) is non-increasing). I also show that (16) must hold with equality (at all \( t < \hat{T} \)) under any such belief \( \pi_2 \), which implies that (17) may be solved under the additional constraint that (16) holds with equality.

First, fix a sequence \( \{\chi^{\pi_2}_n\} \) such that \( i(\pi_2^n) \uparrow \hat{T}, \pi_2^n(\gamma) \geq \varepsilon \) for all \( n \), and (16) holds for all
Note that \( \chi^{\pi_n}(t) \) is non-increasing in \( t \), for all \( n \). Since the space of monotone functions from \( \mathbb{R}_+ \) to \([0, 1]\) is sequentially compact (by Helly’s selection theorem (see, e.g., Billingsley (1995) Theorem 25.9)), there exists a subsequence \( \{ \chi^{\pi_n} \} \) that converges pointwise to some (non-increasing) function \( \chi^{\pi_2}. \)

Furthermore, \( \chi^{\pi_n}(0) = 1 - \varepsilon \), because \( \chi^{\pi_{\infty}}(0) = 1 - \varepsilon \) for all \( m \). Combined with the fact that \( \chi^{\pi_n} \) is non-increasing, this implies that there exists a pair of functions \( (\lambda, p) : \mathbb{R}_+ \to \mathbb{R}_+, p^{\pi_n} : \mathbb{R}_+ \to [0, 1] \) such that \( p^{\pi_n}(t) = 0 \) for all \( t \) outside of a countable set \( S^{\pi_n} \) and

\[
\chi^{\pi_2}(t) = \frac{\exp \left( - \int_0^t \lambda^{\pi_2}(s) \, ds \right) \prod_{s \in S^{\pi_2} \cap [0,t]} (1 - p^{\pi_2}(s))}{\exp \left( - \int_0^t \lambda^{\pi_2}(s) \, ds \right) \prod_{s \in S^{\pi_2} \cap [0,t]} (1 - p^{\pi_2}(s))}
\]

for all \( t \). Therefore, there exists a belief \( \pi_2 \in \Delta(S_1) \) corresponding to concession rate (resp., probability) \( \lambda^{\pi_2}(t) \) (resp., \( p^{\pi_2}(t) \)) such that \( \pi_2(\gamma) \geq \varepsilon \). Finally, the fact that \( \chi^{\pi_n}(t) \to \chi^{\pi_2}(t) \) for all \( t \) implies that

\[
\exp \left( - \int_0^t \lambda^{\pi_2}(s) \, ds \right) \prod_{s \in S^{\pi_2} \cap [0,t]} (1 - p^{\pi_2}(s)) \to \exp \left( - \int_0^t \lambda^{\pi_2}(s) \, ds \right) \prod_{s \in S^{\pi_2} \cap [0,t]} (1 - p^{\pi_2}(s))
\]

for all \( t \). Since for all \( t < \hat{T} \), there exists \( M > 0 \) such that (16) holds at time \( t \) under \( \pi_n \) for all \( m > M \), this implies that (16) holds at all times \( t < \hat{T} \) under \( \pi_2 \).

I now show that if there exists a time \( t < \hat{T} \) at which (16) holds with strict inequality under belief \( \pi_2 \), then there exists an alternative belief \( \pi'_2 \) that attains the supremum in (17). Suppose such a time \( t \) exists. I claim that it follows that there exists a time \( t_1 < \hat{t}(\pi_2) \) at which (16) holds with strict inequality and in addition either \( \int_{t_1}^{t_1 + \Delta} \lambda^{\pi_2}(s) \, ds > 0 \) for all \( \Delta > 0 \) or \( \sum_{s \in S^{\pi_2} \cap [t_1,t_1 + \Delta]} p^{\pi_2}(s) > 0 \) for all \( \Delta > 0 \). To see this, note that there must exist a time \( t' \in (t, \hat{t}(\pi_2)) \) such that either \( \int_{t'}^{t' + \Delta} \lambda^{\pi_2}(s) \, ds > 0 \) for all \( \Delta > 0 \) or \( p^{\pi_2}(t') > 0 \) (because otherwise (16) could not hold with strict inequality at \( t \)). Let \( t_1 \) be the infimum of such times.
times \( t' \), and note that either \( \int_{t_1}^{t_1+\Delta} \lambda^{\pi_2}(s) \, ds > 0 \) for all \( \Delta > 0 \) or \( \sum_{s \in S^{\pi_2} \cap [t_1, t_1+\Delta]} p^{\pi_2}(s) > 0 \) for all \( \Delta > 0 \). Then the fact that (16) holds with strict inequality at time \( t \) implies that (16) holds with strict inequality at time \( t_1 \), because otherwise the fact that \( \int_1^{t_1} \lambda^{\pi_2}(s) \, ds = 0 \) and \( p^{\pi_2}(t'') = 0 \) for all \( t'' \in [t, t_1) \) would imply that (16) could not hold with strict inequality at time \( t \).

Thus, let \( t_0 < \hat{T} \) be such that (16) holds with strict inequality at time \( t_0 \) and in addition \( \int_{t_0}^{t_0+\Delta} \lambda^{\pi_2}(s) \, ds > 0 \) for all \( \Delta > 0 \) (the case where \( \sum_{s \in S^{\pi_2} \cap [t_0, t_0+\Delta]} p^{\pi_2}(s) > 0 \) is similar, and thus omitted). Since \( v(t) \) is continuous but for downward jumps, there exist \( \eta > 0 \) and \( \Delta > 0 \) such that (16) holds with strict inequality at \( t \) for all \( t \in [t_0, t_0+\Delta) \) when \( \lambda^{\pi_2}(t) \) is replaced by \( (1-\eta)\lambda^{\pi_2}(t) \) for all \( t \in [t_0, t_0+\Delta) \). Define \( \lambda^{\pi_2'}(t) \) by \( \lambda^{\pi_2'}(t) \equiv \lambda^{\pi_2}(t) \) for all \( t \not\in [t_0, t_0+\Delta) \) and \( \lambda^{\pi_2'}(t) \equiv (1-\eta)\lambda^{\pi_2}(t) \) for all \( t \in [t_0, t_0+\Delta) \). Next, I claim that at time \( t_0 \) player 2's continuation payoff from rejecting \( \gamma \) until \( \hat{i}(\pi_2) \) is strictly lower when player 1’s concessions are given by \( (\lambda^{\pi_2}(t), p^{\pi_2}(t)) \) than when they are given by \( (\lambda^{\pi_2'}(t), p^{\pi_2'}(t)) \), where \( p^{\pi_2'}(t) \) is defined by \( p^{\pi_2'}(t) \equiv p^{\pi_2}(t) \) for all \( t \neq t_0 \), and \( p^{\pi_2'}(t_0) \equiv 1 - \exp \left(-\eta \int_{t_0}^{t_0+\Delta} \lambda^{\pi_2}(s) \, ds \right) \left(1-p^{\pi_2}(t_0)\right) > 0 \). This follows because the total probability with which player 1 concedes in the interval \([t_0, t_0+\Delta)\) is the same under \((\lambda^{\pi_2}(t), p^{\pi_2}(t))\) and under \((\lambda^{\pi_2'}(t), p^{\pi_2'}(t))\), and some probability mass of concession is moved earlier to \( t_0 \) under \((\lambda^{\pi_2'}(t), p^{\pi_2'}(t))\). Therefore, there exists \( \zeta > 0 \) such that at time \( t_0 \) player 2’s continuation payoff from rejecting \( \gamma \) until \( \hat{i}(\pi_2) \) is the same when player 1’s concessions are given by \((\lambda^{\pi_2}(t), p^{\pi_2}(t))\) and when they are given by \((\lambda^{\pi_2'}(t), p^{\pi_2'}(t))\), where \( p^{\pi_2'}(t) \) is defined by \( p^{\pi_2'}(t) \equiv p^{\pi_2}(t) \) for all \( t \neq t_0 \), and \( p^{\pi_2'}(t_0) \equiv (1-\zeta)p^{\pi_2'}(t_0) < p^{\pi_2'}(t_0) \). The fact that (16) holds at all \( t < \hat{T} \) when player 1’s concessions are given by \((\lambda^{\pi_2}(t), p^{\pi_2}(t))\) now implies that (16) holds at all \( t < \hat{T} \) when player 1’s concessions are given by \((\lambda^{\pi_2'}(t), p^{\pi_2'}(t))\). Furthermore, \( \exp \left(-\int_0^{\hat{T}} \lambda^{\pi_2'}(t) \, dt \right) \prod_{s \in S^{\pi_2} \cap [0, \hat{T}]} (1-p^{\pi_2'}(s)) > \exp \left(-\int_0^{\hat{T}} \lambda^{\pi_2}(t) \, dt \right) \prod_{s \in S^{\pi_2} \cap [0, \hat{T}]} (1-p^{\pi_2}(s)) \geq \varepsilon \). Therefore, \( \sup \{ t : \chi^{\pi_2'}(t) > 0 \} \geq \hat{T} \), so by the definition of \( \hat{T} \) it must be that \( \sup \{ t : \chi^{\pi_2}(t) > 0 \} = \hat{T} \).

Next, suppose that (16) holds with equality under belief \( \pi_2 \) (defined above), and that in addition \( v(t) \) is differentiable at some time \( t < \hat{i}(\pi_2) \). Then the derivative of the right-hand side of (16) at \( t \) must exist and equal \( v'(t) \). This implies that \( p^{\pi_2}(t) = 0 \), and, by Leibniz's
rule, the derivative of the right-hand side of (16) equals $-\lambda^\pi_{\tau_{2}}(t) + (\tau + \lambda^\pi_{\tau_{2}}(t)) v(t)$. Hence,

$$\lambda^\pi_{\tau_{2}}(t) = \frac{rv(t) - v'(t)}{1 - v(t)}.$$ 

Since $v(t)$ is differentiable almost everywhere, this implies that

$$\int_0^\tau \lambda^\pi_{\tau_{2}}(s) \, ds = \int_0^\tau \lambda(s) \, ds$$

for all $\tau < \hat{t}(\pi_{2})$, where $\lambda(s)$ is defined by (5). Similarly, if (16) holds with equality then the difference between the limit as $s \uparrow t$ of the right-hand side of (16) evaluated at $s$ and the limit as $s \downarrow t$ of the right-hand side of (16) evaluated at $s$ must equal $v(t, -1) - v(t)$, for all $t < \hat{t}(\pi_{2})$. By inspection, this difference equals $p^\pi_{\tau_{2}}(t) - p^\pi_{\tau_{2}}(t) v(t)$. Hence,

$$p^\pi_{\tau_{2}}(t) = \frac{v(t, -1) - v(t)}{1 - v(t)}$$

for all $t < \hat{t}(\pi_{2})$. Therefore,

$$\prod_{s \in S^\pi_{\tau_{2}} \cap [0, \tau)} (1 - p^\pi_{\tau_{2}}(s)) = \prod_{s \in S \cap [0, \tau)} (1 - p(s))$$

for all $\tau < \hat{t}(\pi_{2})$, where $S$ is the set of discontinuity points of $v(t)$, and $p(s)$ is defined by (6). Combining (18) and (19), I conclude that if (16) holds with equality under belief $\pi_{2}$, then

$$\hat{t}(\pi_{2}) = \sup \left\{ t : \exp \left( - \int_0^t \lambda(s) \, ds \right) \prod_{s \in S \cap [0, t)} (1 - p(s)) > \varepsilon \right\},$$

which equals $\hat{T}$. In addition, $\chi^\pi_{\tau_{2}}(t) < 0$ for all $t \in (\hat{T}, \hat{\pi})$, so $\hat{T} = \hat{\pi}$ and the supremum in (17) is attained at $\pi_{2}$.

Combining the previous three paragraphs, it follows that the supremum in (17) is always attained at some belief $\pi_{2}$. I now show that there exists a belief that both attains the supremum in (17) and maximizes $\lim_{t \uparrow T} \chi^\pi_{\tau_{2}}(t)$ over all beliefs $\pi_{2}$ that attain the supremum in (17). Let $\chi \in [0, 1]$ be the supremum of $\lim_{t \uparrow T} \chi^\pi_{\tau_{2}}(t)$ over all beliefs $\pi_{2}$ that attain the supremum in (17). If $\chi = 0$, then any belief $\pi_{2}$ that attains the supremum in (17) also
satisfies \( \lim_{t \uparrow T} X_{\pi^2}(t) = \chi \). Thus, suppose that \( \chi > 0 \). Let \( \{\pi^2_n\} \) be a sequence of beliefs that all attain the supremum in (17) such that \( \lim_{t \uparrow T} X_{\pi^2}(t) \uparrow \chi \). The above sequential compactness argument implies that there exists a subsequence \( \{\pi^2_{m_n}\} \subseteq \{\pi^2_n\} \) and a belief \( \pi^2 \) satisfying the constraints of (17) such that \( X_{\pi^2}(t) \rightarrow X_{\pi^2}(t) \) for all \( t \). Furthermore, \( X_{\pi^2}(t) \) is non-increasing, so \( \lim_{t \uparrow T} X_{\pi^2}(t) \) exists. Because \( \pi^2 \) satisfies the constraints of (17), \( \lim_{t \uparrow T} X_{\pi^2}(t) \leq \chi \). Now suppose, toward a contradiction, that \( \lim_{t \uparrow T} X_{\pi^2}(t) < \chi \). Then there exists \( \eta > 0 \) and \( t' \leq \hat{T} \) such that \( X_{\pi^2}(t') < \chi - \eta \). Since \( \lim_{m \rightarrow \infty} \lim_{t \uparrow T} X_{\pi^2_m}(t) = \chi \), there exists \( M > 0 \) such that, for all \( m > M \), \( \lim_{t \uparrow T} X_{\pi^m}(t) > \chi - \eta \). And \( X_{\pi^2_m}(t) \) is non-increasing for all \( m \), so this implies that \( X_{\pi^2_m}(t') > \chi - \eta \) for all \( m > M \). Now \( X_{\pi^2_m}(t') \rightarrow X_{\pi^2}(t') \) implies that \( X_{\pi^2}(t') \geq \chi - \eta \), a contradiction. Therefore, \( \lim_{t \uparrow T} X_{\pi^2}(t) = \chi \). Furthermore, \( \lim_{t \uparrow T} X_{\pi^2}(t) > 0 \) implies that \( \pi^2 \) attains the supremum in (17).

Finally, if (16) holds with strict inequality at some time \( t < \hat{T} \) under a belief \( \pi^2 \) such that \( \hat{t}(\pi^2) = \hat{T} \), the same procedure for modifying \( \pi^2 \) described above yields a belief \( \pi^2_{\pi^2} \) such that \( \hat{t}(\pi^2_{\pi^2}) = \hat{T} \) and \( \lim_{t \uparrow T} X_{\pi^2_{\pi^2}}(t) > \lim_{t \uparrow T} X_{\pi^2}(t) \). This implies that the only beliefs \( \pi^2 \) that both attain the supremum in (17) and maximize \( \lim_{t \uparrow T} X_{\pi^2}(t) \) (over all beliefs that attain the supremum in (17)) satisfy the additional constraint that (16) holds with equality. Since I have proved that such a belief exists, the value of (17) equals the value of (17) under this additional constraint, which I have shown to equal \( \hat{T} \).

**Proof of Theorem 1.** Let \( \gamma_n \) and \( \gamma^* \) be defined as in Section 2.4.4. Note that \( \{\gamma_n\} \) converges pointwise to \( \gamma^* \). To show that \( \lim_{n \rightarrow \infty} u^*_v(\gamma_n) = 1/(1 - \log \varepsilon) \), it remains only to show that \( T^1_n > \hat{T}(\gamma_n) \) for all \( n \in \mathbb{N} \). To see this, note that \( T^1_n = \frac{1}{r} \log \left( \frac{n+1}{n} (1 - \log \varepsilon) \right) \).

Since \( \gamma_n(t) = \left( \frac{n}{n+1} \right) \frac{e^t}{1-\log \varepsilon} \) for all \( t \leq T^1_n \) and \( \gamma_n(t) \) is non-decreasing, it follows that \( v(t) = \)
1 - (\frac{n}{n+1}) \frac{e^{-rt}}{1 - \log e} \quad \text{for all } t \leq T^1_n. \quad \text{Therefore,}

\begin{align*}
\exp \left( - \int_0^{T^1_n} r v(t) - v'(t) dt \right) \prod_{s \in S \cap [0,T^1_n]} \left( \frac{1 - v(s, -1)}{1 - v(s)} \right) \\
= \exp \left( - \int_0^{T^1_n} \frac{r}{n+1} (n+1) (1 - \log \varepsilon) e^{-rt} dt \right) \\
= \exp \left( - \frac{n+1}{n} (1 - \log \varepsilon) \left( 1 - e^{-r T^1_n} \right) \right) \\
= \exp \left( - \frac{n+1}{n} (1 - \log \varepsilon) \left( 1 - \left( \frac{n}{n+1} \right) \frac{1}{1 - \log \varepsilon} \right) \right) \\
= \exp \left( - \frac{1}{n} (1 - \log \varepsilon) \right) \varepsilon \\
< \varepsilon.
\end{align*}

Hence, by the definition of \( \tilde{T}(\gamma_n) \), \( T^1_n \geq \tilde{T}(\gamma_n) \). Furthermore, the fact that \( \exp \left( - \int_0^T \frac{rv(t) - v'(t)}{1 - v(t)} dt \right) \) is strictly decreasing in \( \tau \) for all \( \tau \in [0,T^1_n] \) implies that \( T^1_n > \tilde{T}(\gamma_n) \).

To complete the proof of Theorem 2, I must show that if \( \{\gamma_n\} \) is any sequence of postures converging pointwise to some posture \( \gamma \) satisfying \( u^*_1(\gamma_n) \rightarrow u_1 \geq 1/(1 - \log \varepsilon) \), then \( \gamma = \gamma^* \).\footnote{Technically, I must also show that \( u^*_1(\gamma^*_1) \leq 1/(1 - \log \varepsilon) \). In fact, \( u^*_1(\gamma^*_2) = 0 \), by Lemma 4 and the observation that \( T(\gamma^*_2) = \infty \) (which follows because \( \gamma^*_2(t) = 1 \) for all \( t \geq T(\gamma^*_2) \)).} There are two steps. First, letting \( \{v_n\} \) be the continuation value functions corresponding to the \( \{\gamma_n\} \), and letting \( v^* \) be the continuation value function corresponding to \( \gamma^* \), I show that \( \sup_{t \in \mathbb{R}_+} e^{-rt} |v^*(t) - v_n(t)| \rightarrow 0 \). Second, I show that this implies that \( \gamma' = \gamma^* \).

**Step 1:**

Suppose that \( u^*_1(\gamma) \geq 1/(1 - \log \varepsilon) - \zeta \) for some posture \( \gamma \) and some \( \zeta \in (0, 1/(1 - \log \varepsilon)) \). Let \( T^1 \equiv (1/r) \log (1 - \log \varepsilon) \) (which equals \( \lim _{n \to \infty} T^1_n \)). Then it must be that \( \tilde{T}(\gamma) \leq T^1 - (1/r) \log (1 - \zeta (1 - \log \varepsilon)) \), for otherwise it would follow from \( T(\gamma) \geq \tilde{T}(\gamma) \) that

\begin{align*}
u^*_1(\gamma) &= \min_{t \leq \tilde{T}(\gamma)} \exp (-r e^{-(r \tilde{T}(\gamma))} (1) = \frac{1}{1 - \log \varepsilon} - \zeta.
\end{align*}

Furthermore, if \( u^*_1(\gamma) \geq 1/(1 - \log \varepsilon) - \zeta \), it must also be that \( \gamma(t) \geq e^{rt} (1/(1 - \log \varepsilon) - \zeta) \).
for all $t \leq T(\gamma)$, for otherwise $\min_{t \leq T(\gamma)} e^{-rt} \gamma(t)$ would be strictly less than $1/(1 - \log \varepsilon) - \zeta$. I will show that, for all $\delta > 0$, there exists $\zeta > 0$ such that, if both $\gamma(t) \geq e^{-rt} (1/(1 - \log \varepsilon)) - \zeta$ for all $t \leq T(\gamma)$ and $\tilde{T}(\gamma) \leq T_1 - (1/r) \log (1 - \zeta (1 - \log \varepsilon))$, then $\sup_{s \leq \tilde{T}(\gamma)} |u^*(t) - v(t)| \leq \delta$.

If $\tilde{T}(\gamma) \leq T_1 - (1/r) \log (1 - \zeta (1 - \log \varepsilon))$ then $\tilde{T}(\gamma)$ is finite, and therefore

$$\exp \left( - \int_0^{\tilde{T}(\gamma)} \frac{rv(t) - v'(t)}{1 - v(t)} dt \right) \prod_{s \in S \cap [0, \tilde{T}(\gamma)]} \left( \frac{1 - v(s, -1)}{1 - v(s)} \right) \leq \varepsilon.$$  

It is straightforward to check that $\gamma(t) \geq e^{-rt} (1/(1 - \log \varepsilon)) - \zeta$ for all $t \leq T(\gamma)$ only if $v(t) \leq 1 - e^{-rt} (1/(1 - \log \varepsilon)) - \zeta$ for all $t \leq T(\gamma)$. Recall that $e^{-rt} v(t)$ is non-increasing. Thus, if $\gamma(t) \geq e^{-rt} (1/(1 - \log \varepsilon)) - \zeta$ for all $t \leq T(\gamma)$ and $\tilde{T}(\gamma)$ is finite, then

$$\inf_{\text{v(t): } e^{-rtv(t)} \text{ non-increasing, } v(t) \leq 1 - e^{-rt(1/(1-\log\varepsilon)-\zeta)}} \exp \left( - \int_0^{\tilde{T}(\gamma)} \frac{rv(t) - v'(t)}{1 - v(t)} dt \right) \prod_{s \in S \cap [0, \tilde{T}(\gamma)]} \left( \frac{1 - v(s, -1)}{1 - v(s)} \right) \leq \varepsilon. \quad (20)$$

I first show that any attainable value of the program on the left-hand side of (20) can be arbitrarily closely approximated by the value attained by a continuous function $v(t)$ satisfying the constraints of (20); hence, in calculating the infimum over such values, attention may be restricted to continuous functions. To see this, fix $\eta \in (0, 1)$ and let

$$S^\eta \equiv \bigcup_{s \in S \cap [0, \tilde{T}(\gamma)]} [s - \eta, s].$$

Define the function $v^\eta(t)$ by $v^\eta(t) \equiv v(t)$ for all $t \notin S^\eta$, and

$$v^\eta(t) \equiv v(s - \eta) - \frac{t - (s - \eta)}{\eta} (v(s - \eta) - v(s)) \text{ for all } t \in S^\eta.$$  

Observe that $v^\eta(t)$ is continuous. Furthermore, for all $s \in S$,

$$\exp \left( \int_{s-\eta}^s \frac{v^\eta(t)}{1 - v^\eta(t)} dt \right) = \frac{1 - v^\eta(s - \eta)}{1 - v^\eta(s)} = \frac{1 - v(s - \eta)}{1 - v(s)}.$$
Also, since \( v^n(t) \leq 1 - (1/(1 - \log \varepsilon) - \zeta) < 1 \) for all \( t \in [0, \tilde{T}(\gamma)] \), and the measure of \( S^n \) goes to 0 as \( \eta \to 0 \),

\[
\lim_{\eta \to 0} \exp \left( - \int_{S^n} \frac{r v^n(t)}{1 - v^n(t)} dt \right) = 1.
\]

Therefore,

\[
\lim_{\eta \to 0} \exp \left( - \int_0^{\tilde{T}(\gamma)} \frac{r v^n(t) - v^n'(t)}{1 - v^n(t)} dt \right) \exp \left( - \int_{S^n} \frac{r v^n(t)}{1 - v^n(t)} dt \right) \prod_{s \in S^n \cap [0, \tilde{T}(\gamma)]} \left( 1 - \frac{1 - v(s - \eta)}{1 - v(s)} \right).
\]

I now derive a lower bound on the left-hand side \((20)\) under the additional constraint that \( v(t) \) is continuous. Using the fact that \( v(s, -1) = v(s) \) for all \( s \) when \( v \) is continuous and integrating the \( v'(t)/(1 - v(t)) \) term, this constrained program may be rewritten as

\[
\inf_{\substack{v(t) \text{ continuous} \\ e^{-rt} v(t) \text{ non-increasing,} \\ v(t) \leq 1 - e^{rt}(1/(1 - \log \varepsilon) - \zeta)}} \exp \left( - \int_0^{\tilde{T}(\gamma)} \frac{r v(t)}{1 - v(t)} dt \right) \left( 1 - \frac{1 - v(0)}{1 - v(\tilde{T}(\gamma))} \right).
\]

Since \( v(t) \geq 0 \) for all \( t \), the value of this program is bounded from below by the value of the program:

\[
\inf_{\substack{v(t) \text{ continuous} \\ e^{-rt} v(t) \text{ non-increasing,} \\ v(t) \leq 1 - e^{rt}(1/(1 - \log \varepsilon) - \zeta)}} \exp \left( - \int_0^{\tilde{T}(\gamma)} \frac{r v(t)}{1 - v(t)} dt \right) (1 - v(0)). \tag{21}
\]

Note that \((21)\) decreases whenever the value of \( v(t) \) is increased on a subset of \([0, \tilde{T}(\gamma)]\) of positive measure, so the unique solution to \((21)\) is \( v(t) = 1 - e^{rt}(1/(1 - \log \varepsilon) - \zeta) \) for all \( t \leq \tilde{T}(\gamma) \). With this function \( v(t) \), it can be checked that the value of \( \tilde{T}(\gamma) \) such that \((21)\) equals \( \varepsilon \) is

\[
T^1 = \frac{1}{r} \log (1 - \zeta(1 - \log \varepsilon) \log \varepsilon). \tag{22}
\]

This value is a lower bound on \( \tilde{T}(\gamma) \) for any posture \( \gamma \) such that \( \gamma(t) \geq e^{rt}(1/(1 - \log \varepsilon) - \zeta) \).
for all \( t \leq T(\gamma) \). Thus, as \( \zeta \to 0 \), the unique solution to (21) converges to \( v^*(t) = 1 - e^{rt}/(1 - \log \varepsilon) \) for all \( t \leq \tilde{T}(\gamma) \), and the corresponding lower bound on \( \tilde{T}(\gamma) \) (i.e., (22)) converges to \( T^1 \). Furthermore, by the condition that \( e^{-rt}v(t) \) is non-increasing, any function \( v(t) \) satisfying the constraints of (21) yields a lower bound on \( \tilde{T}(\gamma) \) that is greater than (22) by at least an amount proportional to \( \sup_{t \leq \tilde{T}(\gamma)} |v^*(t) - v(t)| \). Therefore, for any fixed \( \delta > 0 \), there exists \( \zeta > 0 \) such that if both \( \gamma(t) \geq e^{rt} (1/ (1 - \log \varepsilon) - \zeta) \) for all \( t \leq T(\gamma) \) and \( \tilde{T}(\gamma) \leq T^1 - (1/ r) \log (1 - (1 - \log \varepsilon)) \) (which converges to \( T^1 \) as \( \zeta \to 0 \)), then \( \sup_{t \leq \tilde{T}(\gamma)} |v^*(t) - v(t)| \leq \delta \).

Thus, I have shown that, for any \( \delta > 0 \) and \( K > 1 \), there exists \( \zeta(K) > 0 \) such that if \( u_1^*(\gamma) \geq 1/ (1 - \log \varepsilon) - \zeta(K) \), then \( \sup_{t \leq \tilde{T}(\gamma)} e^{-rt} |v(t) - v^*(t)| \leq \delta/K \). I now argue that, for \( K \) sufficiently large, there exists \( \zeta' \in (0, \zeta(K)) \) such that if \( u_1^*(\gamma) \geq 1/ (1 - \log \varepsilon) - \zeta' \), then in addition \( \sup_{t \geq \tilde{T}(\gamma)} e^{-rt} |v(t) - v^*(t)| \leq \delta \). To see this, note that as \( K \to \infty \), \( \tilde{T}(\gamma) \to T^1 \) uniformly over all postures \( \gamma \) such that \( \sup_{t \leq \tilde{T}(\gamma)} e^{-rt} |v(t) - v^*(t)| \leq \delta/K \). Choose \( K^* > 1 \) such that \( e^{-r\tilde{T}(\gamma)} - e^{-rT^1} < \delta/2 \) and \( v^*(\tilde{T}(\gamma)) \leq e^{-\tilde{T}(\gamma)} \delta \) for any such posture \( \gamma \), and suppose that a posture \( \gamma \) is such that \( \sup_{t \leq \tilde{T}(\gamma)} e^{-rt} |v(t) - v^*(t)| \leq \delta/K \) but \( e^{-r0} |v(t_0) - v^*(t_0)| > \delta \) for some \( t_0 > \tilde{T}(\gamma) \). Then \( v^*(t_0) \leq e^{r0} \delta \), so it follows that \( e^{-r00} v^*(t_0) + \delta < e^{-r00} v(t_0) \). Therefore,

\[
\max_{t \geq \tilde{T}(\gamma)} e^{-rt} (1 - \gamma(t)) \geq e^{-r00} v(t_0) \geq \delta.
\]

By the definition of \( T(\gamma) \), this implies that there exists \( t_1 \in \left[ \tilde{T}(\gamma), T(\gamma) \right] \) such that \( e^{-rt_1} (1 - \gamma(t_1)) \geq \delta \), or equivalently \( \gamma(t_1) \leq 1 - e^{rt_1} \delta \). Hence,

\[
u_1^*(\gamma) = \min_{t \leq \tilde{T}(\gamma)} e^{-rt} \gamma(t) \leq e^{-rt_1} (1 - e^{rt_1} \delta) \leq e^{-r\tilde{T}(\gamma)} \left( 1 - e^{-rT(\gamma)} \delta \right) = e^{-r\tilde{T}(\gamma)} - \delta \leq e^{-rT^1} - \delta/2 = 1/ (1 - \log \varepsilon) - \delta/2.
\]

Therefore, taking \( \zeta' \equiv \min \{ \zeta(K^*), \delta/2 \} \) completes the first step of the proof.

**Step 2:**

I show that if \( \gamma_n(t) \to \gamma(t) \) for all \( t \in \mathbb{R}_+ \) for some posture \( \gamma \), and \( \sup_{t \in \mathbb{R}_+} e^{-rt} |v^*(t) - v_n(t)| \to \)
0, then \( \gamma = \gamma^* \). First, note that if \( \gamma(t) < \gamma^*(t) \) for some \( t \in \mathbb{R}_+ \), then there exist \( N > 0 \) and \( \eta > 0 \) such that \( \gamma_n(t) < \gamma^*(t) - \eta \) for all \( n > N \). Since \( v_n(t) \geq 1 - \gamma_n(t) \), this implies that \( v_n(t) \geq 1 - \gamma^*(t) + \eta = v^*(t) + \eta \) for all \( n > N \), a contradiction.

It is more difficult to rule out the possibility that \( \gamma(t) > \gamma^*(t) \) for some \( t \in \mathbb{R}_+ \). Suppose that this is so. Since \( \gamma \) and \( \gamma^* \) are right-continuous, there exist \( \eta > 0 \) and an open interval \( I_0 \subseteq \mathbb{R}_+ \) such that \( \gamma(t) > \gamma^*(t) + \eta \) for all \( t \in I_0 \). If it were the case that \( \gamma_n(t) \geq \gamma^*(t) + \eta/2 \) for all \( t \in I_0 \) and \( n \) sufficiently large, then the condition \( \sup_{t \in \mathbb{R}_+} e^{-rt} |v^*(t) - v_n(t)| \to 0 \) would fail, so this is not possible.\(^{15}\) Hence, there exists \( t_1 \in I_0 \) and \( n_1 \geq 0 \) such that \( \gamma_{n_1}(t_1) < \gamma^*(t_1) + \eta/2 \). Since \( \gamma_{n_1} \) and \( \gamma^* \) are right-continuous, there exists an open interval \( I_1 \subseteq I_0 \) such that \( \gamma_{n_1}(t) < \gamma^*(t) + \eta/2 \) for all \( t \in I_1 \). Next, it cannot be the case that \( \gamma_n(t) \geq \gamma^*(t) + \eta/2 \) for all \( t \in I_1 \) and \( n > n_1 \) (by the same argument as above), so there exists \( t_2 \in I_1 \) and \( n_2 > n_1 \) such that \( \gamma_{n_2}(t_2) < \gamma^*(t_2) + \eta/2 \). As above, this implies that there exists an open interval \( I_2 \subseteq I_1 \) such that \( \gamma_{n_2}(t) < \gamma^*(t) + \eta/2 \) for all \( t \in I_2 \). Proceeding in this manner yields a sequence of open intervals \( \{I_m\} \) and integers \( \{n_m\} \) such that \( I_{m+1} \subseteq I_m \), \( n_{m+1} > n_m \), and \( \gamma_{n_m}(t) < \gamma^*(t) + \eta/2 \) for all \( t \in I_m \) and \( m \in \mathbb{N} \). Let \( I \equiv \bigcap_{m \in \mathbb{N}} I_m \), a non-empty set (possibly a single point), and fix \( t \in I \). Then \( \gamma_{n_m}(t) < \gamma^*(t) + \eta/2 \) for all \( m \in \mathbb{N} \), and since \( n_{m+1} > n_m \) for all \( m \in \mathbb{N} \) this contradicts the assumption that \( \gamma_n(t) \to \gamma(t) \).

\(^{15}\)To see this, fix \( N > 0 \), suppose that \( \gamma_n(t) \geq \gamma^*(t) + \eta/2 \) for all \( t \in I_0 \) and \( n > N \), and denote the length of \( I_0 \) by \( 2\Delta \) and the midpoint of \( I_0 \) by \( t_0 \). Noting that \( \gamma^*(t_0) < 1 - \eta/2 \), there exists \( N' > 0 \) such that \( \gamma_n(t) \geq 1 - v_n(t) > 1 - v^*(t) - e^{-rt} (1 - e^{-r\Delta}) (1 - \gamma^*(t_0) - \eta/2) = \gamma^*(t) - e^{-rt} (1 - e^{-r\Delta}) (1 - \gamma^*(t_0) - \eta/2) \) for all \( t \in \mathbb{R}_+ \) and \( n > N' \). Therefore, \( v_n(t_0) \leq \max \{1 - \gamma^*(t_0) - \eta/2, e^{-r\Delta} (1 - \gamma^*(t_0 + \Delta)) + (1 - e^{-r\Delta}) (1 - \gamma^*(t_0) - \eta/2)\} \leq \max \{1 - \gamma^*(t_0) - \eta/2, 1 - \gamma^*(t_0) - (1 - e^{-r\Delta}) \eta/2\} = v^*(t_0) - (1 - e^{-r\Delta}) \eta/2 \) for all \( n > \max \{N, N'\} \), a contradiction.
2.10 Appendix B: Omitted Proofs for Sections 2.5 and 2.7

**Proof of Proposition 2.** Lemmas 2 through 4 apply to any posture, whether or not it is constant. In addition, if $\gamma$ is constant then $T(\gamma) = \tilde{T}(\gamma)$. Thus, Lemma 4 implies that $u^*_1(\gamma) = \min_{\epsilon \leq T(\gamma)} e^{-\epsilon} \gamma = e^{-r\tilde{T}(\gamma)\gamma}$. Furthermore, $\lambda(t) = r(1-\gamma)/\gamma$ and $p(t) = 0$ for all $t$, so, by the definition of $\tilde{T}(\gamma)$,

$$\exp \left( -r \left( \frac{1-\gamma}{\gamma} \right) \tilde{T}(\gamma) \right) = \epsilon,$$

or

$$\tilde{T}(\gamma) = -\frac{1}{r} \left( \frac{\gamma}{1-\gamma} \right) \log \epsilon$$

if $\gamma < 1$, and $\tilde{T}(\gamma) = \infty$ if $\gamma = 1$. Therefore,

$$\tilde{u}^*_1 = \max_{\gamma \in [0,1]} e^{-r\tilde{T}(\gamma)} \gamma = \max_{\gamma \in [0,1]} \exp \left( \frac{\gamma}{1-\gamma} \log \epsilon \right) \gamma.$$

Note that (23) is concave in $\gamma$. Hence, the first-order condition

$$1 = -\frac{\tilde{\gamma}^*_\epsilon}{(1-\tilde{\gamma}^*_\epsilon)^2} \log \epsilon,$$

which has a solution if $\epsilon < 1$, is both necessary and sufficient. Solving this quadratic equation yields

$$\tilde{\gamma}^*_\epsilon = \frac{2 - \log \epsilon - \sqrt{(\log \epsilon)^2 - 4 \log \epsilon}}{2}.$$

Finally, substituting (24) into (23) yields $\tilde{u}^*_1 = \exp \left( -(1-\tilde{\gamma}^*_\epsilon) \right) \tilde{\gamma}^*_\epsilon$. □

**Proof of Corollary 2.** By (24), $-\tilde{\gamma}^*_\epsilon \log \epsilon = (1-\tilde{\gamma}^*_\epsilon)^2$ for all $\epsilon$. Therefore,

$$\frac{u^*_1(\epsilon)}{\tilde{u}^*_1(\epsilon)} = \frac{1}{(1-\log \epsilon) \exp \left( -(1-\tilde{\gamma}^*_\epsilon) \right) \tilde{\gamma}^*_\epsilon} = \frac{\exp (1-\tilde{\gamma}^*_\epsilon)}{\tilde{\gamma}^*_\epsilon + (1-\tilde{\gamma}^*_\epsilon)^2}.$$

The derivative of (25) with respect to $\tilde{\gamma}^*_\epsilon$ is negative for all $\tilde{\gamma}^*_\epsilon \in [0,1]$. Since $\tilde{\gamma}^*_\epsilon$ is an
increasing function of \( \varepsilon \), this implies that \( u_1^*(\varepsilon) / \tilde{u}_1^*(\varepsilon) \) is decreasing in \( \varepsilon \). In addition, by (24), \( \lim_{\varepsilon \to 1} \tilde{u}_1^*(\varepsilon) = 1 \) and \( \lim_{\varepsilon \to 0} \tilde{u}_1^*(\varepsilon) = 0 \). Therefore, (25) implies that \( \lim_{\varepsilon \to 1} u_1^*(\varepsilon) / \tilde{u}_1^*(\varepsilon) = 1 \) and \( \lim_{\varepsilon \to 0} u_1^*(\varepsilon) / \tilde{u}_1^*(\varepsilon) = e \).

Proof of Proposition 3. The proof of Theorem 2 goes through for any decreasing Pareto frontier \( \phi \), with the modifications that \( v(t, -1) \) now equals \( \max_{r \geq t} e^{-r(t-t)} \phi (r(t)) \) rather than \( \max_{r \geq t} e^{-r(t-t)} (1 - \gamma(t)) \) (with the analogous modification for \( v(t) \)), and that the maximin posture is now given by \( \gamma^*(t) = \min \{ e^{rt} u_1^*, 1 \} \) for a value of \( u_1^* \) that may differ from \( 1 / (1 - \log \varepsilon) \). Therefore, it suffices to show that, for any \( u_1 \in [0, 1] \), the posture \( \gamma^*(t) = \min \{ e^{rt} u_1^*, 1 \} \) corresponds to a weakly higher concession rate \( \lambda(t) \), for all \( t \), when the Pareto frontier is \( \psi \circ \phi \) than when it is \( \phi \). Note that the set of times at which \( \lambda(t) \) is given by (5) is the same for either Pareto frontier, because \( \psi \circ \phi (u) = 1 \) if and only if \( \phi (u) = 1 \).

Hence, since \( \phi \) and \( \psi \) are concave and thus differentiable almost everywhere, it suffices to show that

\[
\frac{r \psi (\phi (e^{rt} u_1)) - (1 - \psi (\phi (e^{rt} u_1))) \phi' (e^{rt} u_1) r e^{rt} u_1}{1 - \psi (\phi (e^{rt} u_1))} \geq \frac{r \psi (\phi (e^{rt} u_1)) - (1 - \psi (\phi (e^{rt} u_1))) \phi' (e^{rt} u_1) r e^{rt} u_1}{1 - \phi (e^{rt} u_1)}
\]

for all \( t \in \mathbb{R}_+ \) and \( u_1 \in [0, 1] \); or, dividing both sides by \( r \) and writing \( u \) for \( e^{rt} u_1 \),

\[
\frac{\psi (\phi (u)) - (1 - \phi (u)) \phi' (\phi (u)) u}{1 - \psi (\phi (u))} \geq \frac{\phi (u) - (1 - \phi (u)) \phi' (\phi (u)) u}{1 - \phi (u)}
\]

for all \( u \in [0, 1] \). This inequality may be rearranged as

\[
\psi (\phi (u)) \phi (u) - \phi (u) - ((1 - \phi (u)) \psi' (\phi (u)) - (1 - \psi (\phi (u))) \phi' (u)) u \geq 0.
\] (26)

The maintained assumptions on \( \psi \) imply that \( \psi (x) \geq x \) and \( (1 - x) \psi' (x) \geq 1 - \psi (x) \) for all \( x \in [0, 1] \), so \( \psi (\phi (u)) - \phi (u) \geq 0 \) and \( (1 - \phi (u)) \psi' (\phi (u)) - (1 - \psi (\phi (u))) \geq 0 \). Since \( \phi' (u) \leq 0 \) and \( u \geq 0 \), it follows that (26) holds.

Proof of Proposition 4. Lemmas 2 through 4 continue to hold, replacing \( r \) with \( r_1 \) or \( r_2 \) as appropriate. In particular, \( \lambda(t) = \frac{r_2 u(t) - u(t)}{1 - u(t)} \); and the same argument as in the proof of Theorem 2 implies that the unique maximin posture \( \gamma^* \) satisfies \( \gamma^* (t) = \min \{ e^{rt} u_1^*, 1 \} \), where \( u_1^* \) is the (unique) number such that the time at which \( \gamma^* (t) \) reaches 1 equals \( \tilde{T} (\gamma^*) \).
Thus, given posture \( y^* \), it follows that \( \lambda(t) = r_2 \frac{1-e^{r_1 t}u_2^*+r_1 e^{r_1 t}u_1^*}{e^{r_1 t}u_1^*} = r_2 e^{-r_1 t} + r_1 - r_2 \). Now

\[
\exp \left( -\int_0^{\tilde{T}(y^*)} \left( r_2 \frac{e^{-r_1 t}}{u_1^*} + r_1 - r_2 \right) dt \right) = \exp \left( -\frac{1}{u_1^*} \left( \frac{r_2}{r_1} \right) \left( 1 - e^{-r_1 \tilde{T}(y^*)} \right) + (r_1 - r_2) \tilde{T}(y^*) \right).
\]

Setting this equal to \( \varepsilon \) and rearranging implies that \( \tilde{T}(y^*) \) is given by

\[
\exp \left( -\frac{r_1}{r_2} u_1^* \log (1 + \frac{r_1}{r_2} \varepsilon) \right) + \left( \frac{r_1}{r_2} - 1 \right) u_1^* r_1 \tilde{T}(y^*) = 1. \tag{27}
\]

Using the condition that \( e^{r_1 \tilde{T}(y^*)} u_1^* = 1 \), this can be rearranged to yield (13). Finally, there is a unique pair \( (u_1^*, \tilde{T}(y^*)) \) that satisfies both (27) and \( e^{r_1 \tilde{T}(y^*)} u_1^* = 1 \), because the curve in \( (u_1^*, \tilde{T}(y^*)) \) space defined by (27) is upward-sloping, while the curve defined by \( e^{r_1 \tilde{T}(y^*)} u_1^* = 1 \) is downward-sloping. \( \blacksquare \)

**Proof of Corollary 3.** As \( r_1/r_2 \to 0 \), (13) becomes \( u_1^* (1 + \log u_1^*) = 1 \), which has unique solution \( u_1^* = 1 \). Therefore, \( \lim_{r_1/r_2 \to 0} u_1^* (\varepsilon) = 1 \).

Suppose that \( \varepsilon < 1 \). For any sequence of relative discount rates \( \{r_1/r_2\}_n \), the sequence of corresponding values of \( u_1^* (\varepsilon) \) has a convergent subsequence. Suppose that \( \{r_1/r_2\}_m \to \infty \) and the corresponding values of \( u_1^* (\varepsilon) \) converge to some \( u_1^* \). Then (13) becomes \( u_1^* = 0 \), because \( \varepsilon < 1 \) and \( u_1^* \leq 1 \). Therefore, \( \lim_{r_1/r_2 \to \infty} u_1^* (\varepsilon) = 0 \). \( \blacksquare \)

**Proof of Corollary 4.** After such a decrease in \( r_1/r_2 \) and \( \varepsilon \), the right-hand side of (13) increases if \( u_1^* \) is held constant. If \( r_1/r_2 \leq 1 \), then the left-hand side of (13) is increasing in \( u_1^* \) and the right-hand side of (13) is non-increasing in \( u_1^* \). Therefore, if \( r_1/r_2 \leq 1 \), such a decrease in \( r_1/r_2 \) and \( \varepsilon \) leads to an increase in \( u_1^* (\varepsilon) \). \( \blacksquare \)

**Proof of Lemma 5.** The fact that \( \Omega_2^{\text{RAT}} (\gamma) \subseteq \Pi_1^t \) immediately implies that \( u_1^{\text{RAT}} (\gamma) \geq u_1^* (\gamma) = \min_{t \leq \tau(\gamma)} e^{-r_1 t} \gamma(t) \). Therefore, it suffices to show that \( u_1^{\text{RAT}} (\gamma) \leq \min_{t \leq \tau(\gamma)} e^{-r_1 t} \gamma(t) \).

Let \( \hat{T} \equiv \min \arg \max \ e^{-r_1 t} \left( 1 - \gamma(t) \right) \). Note that \( \hat{T} \) is well-defined and finite because \( \gamma(t) \) is lower semi-continuous and \( \lim_{t \to \infty} e^{-r_1 t} (1 - \gamma(t)) = 0 \). In addition, \( v(t) = e^{-r_1 (T - t)} \left( 1 - \gamma\left( T^{\tau} \right) \right) \) for all \( t < \hat{T} \), which implies that \( \lambda(t) = p(t) = 0 \) for all \( t < \hat{T} \). Hence, the mixed strategy \( \pi_2^\gamma \) coincides with \( \gamma \) for all \( t < \hat{T} \).

Let \( \sigma_2 \in \Sigma_2 \) be identical to the \( \gamma \)-offsetting strategy \( \sigma_2^\gamma \) with the exception that player 2 accepts at date \( (\hat{T}, -1) \) if player 1 follows \( \gamma \) until time \( \hat{T} \). Then, under strategy \( \hat{\sigma}_2 \),
player 2 always demands \( u_2(t) = 1 \) and only accepts player 1’s demand if player 1 follows \( \gamma \) until time \( \hat{T} \). Since the mixed strategy \( \pi_2^\gamma \) coincides with \( \gamma \) for all \( t \leq \hat{T} \), it follows that 
\[
\sup_{\sigma_1} \ u_1(\sigma_1, \hat{\sigma}_2) = e^{-r\hat{T}} \left( \hat{T} \right) = u_1(\pi_2^\gamma, \hat{\sigma}_2),
\]
and therefore \( \pi_2^\gamma \in \Sigma^*_2(\hat{\sigma}_2) \). In addition, it is clear that \( \hat{\sigma}_2 \in \Sigma^*_2(\gamma) \), and furthermore \( \gamma \in \Sigma^*_1(\sigma_2^\gamma) \) (by Lemma 4), and \( \sigma_2^\gamma \in \Sigma^*_2(\pi_2^\gamma) \) (by Lemma 3). Summarizing, I have established that the arrows in the following diagram may be read as “is a best-response to”:

\[
\begin{align*}
\gamma & \rightarrow \sigma_2^\gamma \\
\uparrow & \downarrow \\
\hat{\sigma}_2 & \leftarrow \pi_2^\gamma
\end{align*}
\]

Therefore, the set \( \{\gamma, \pi_2^\gamma\} \times \{\sigma_2^\gamma, \hat{\sigma}_2\} \) has the best-response property given posture \( \gamma \), which implies that \( \{\gamma, \pi_2^\gamma\} \times \{\sigma_2^\gamma, \hat{\sigma}_2\} \subseteq \Omega^{RAT} \). Hence, \( u_1^{RAT}(\gamma) \leq \sup_{\sigma_1} \ u_1(\sigma_1, \sigma_2^\gamma) = u_1(\gamma, \sigma_2^\gamma) = \min_{t \leq T(\gamma)} e^{-rt} \). \( \blacksquare \)

**Proof of Theorem 4.** Observe that a posture \( \gamma \) in discrete-time bargaining game \( g \) induces a “continuous-time posture” \( \hat{\gamma} \) (i.e., a map from \( \mathbb{R}_+ \rightarrow [0, 1] \)) according to \( \hat{\gamma}(t) = \gamma(\min\{\tau \geq t : \tau \in I^g_t\}) \). That is, \( \hat{\gamma} \)'s time-\( t \) demand is simply \( \gamma \)'s next demand in \( g \). I henceforth refer to a posture \( \gamma \) in \( g \) as also being a continuous-time posture, with the understanding that I mean the posture \( \hat{\gamma} \) defined above.

However, \( \gamma \) may not be a posture in the continuous-time bargaining game of Section 2.3, because it may be discontinuous at a non-integer time. To avoid this problem, I now introduce a modified version of the continuous-time bargaining game of Section 2.3. Formally, let the continuous-time bargaining game \( g^{cts} \) be defined as in Section 2.3, with the following modifications: Most importantly, omit the requirement that player \( i \)'s demand path \( u_i^t : [t, t+1) \rightarrow [0, 1] \) (which is still chosen at integer times \( t \)) is continuous. Second, specify that the payoffs if player \( i \) accepts player \( j \)'s offer at date \((t,-1)\) are \( (e^{-rt}(1 - \lim_{r \uparrow t} u_j(\tau)), e^{-rt}\lim_{r \uparrow t} u_j(\tau)) \) (because \( \lim_{r \uparrow t} u_j(\tau) \) may now fail to exist). Third, add a fourth date, \((t,2)\) to each instant of time \( t \). At date \((t,2)\), each player \( i \) announces **accept** or **reject**, and, if player \( i \) accepts player \( j \)'s offer at date \((t,2)\), the game ends with payoffs \( (e^{-rt}(1 - \lim_{r \uparrow t} u_j(\tau)), e^{-rt}\lim_{r \uparrow t} u_j(\tau)) \). Adding the date \((t,2)\) ensures that each player has a well-defined best-response to her belief, even though \( u_j(t) \)
may now fail to be right-continuous. One can check that the analysis of Sections 2.3 and 2.4, including Lemmas 2 through 4 and Theorem 2, continue to apply to the game \( g^{cts} \), with the exception that in \( g^{cts} \) the maxmin posture \( \gamma^* \) is not in fact unique; however, every maxmin posture corresponds to the continuation value function \( v^* (t) \) (by the same argument as in Step 1 of the proof of Theorem 2).\(^{46}\) Because of this, for the remainder of the proof I slightly abuse notation by writing \( u^*_1 (\gamma) \) for player 1’s maxmin payoff given posture \( \gamma \) in the game \( g^{cts} \), rather than in the model of Section 2.3. Importantly, \( u^*_1 (\gamma) \) equals player 1’s maxmin payoff given \( \gamma \) in both \( g^{cts} \) and in the model of Section 2.3 when \( \gamma \) is a posture in the model of Section 2.3, but \( u^*_1 (\gamma) \) is well-defined for all \( \gamma : \mathbb{R}_+ \to [0, 1] \). Similarly, I write \( u^*_1 (v) \) for player 1’s maxmin payoff given continuation value function \( v : \mathbb{R}_+ \to [0, 1] \). This is well-defined because \( u^*_1 (\gamma) = \min_{t \leq T} e^{-rt} \gamma (t) \) by Lemma 4, \( T \) depends on \( \gamma \) only through \( v \) (by Lemma 2), and it can be easily verified that \( \min_{t \leq T} e^{-rt} \gamma (t) = \min_{t \leq T} e^{-rt} (1 - v(t)) \) (and thus depends on \( \gamma \) only through \( v \)). A similar argument, which I omit, implies that one may write \( u^*_1 (\gamma) \) for player 1’s maxmin payoff given continuation value function \( v \) in discrete-time bargaining game \( g_n \).

With this notation, I may state the following lemma, from which Theorem 4 follows:

**Lemma 6** Let \( \{ g_n \} \) be a sequence of discrete-time bargaining games converging to continuous time. There exists a sequence of postures \( \{ \gamma^{g_n} \} \) with \( \gamma^{g_n} \) a posture in \( g_n \) and \( \lim_{n \to \infty} u^*_1 (\gamma^{g_n}) \geq u^*_1 \). In addition, for any sequence of functions \( \{ v^{g_n} \} \) such that \( v^{g_n} \) is a continuation value function in \( g_n \) and \( \lim_{n \to \infty} v^{g_n} (t) \) exists for all \( t \in \mathbb{R}_+ \), it follows that \( \lim_{n \to \infty} u^*_1 (v^{g_n}) \) exists and equals \( \lim_{n \to \infty} u^*_1 (v^{g_n}) \).

**Proof.** I first introduce some additional notation. Let \( \Sigma^0_i \) be the set of player \( i \)'s strategies in \( g^{cts} \) with the property that player \( i \)'s demand only changes at times \( t \in I^0_i \), player \( i \) only accepts player \( j \)'s offer at times \( t \in I^0_j \), and player \( i \)'s action at time \( t \) only depends on past play at times \( \tau \in I^0_i \cup I^0_j \). One can equivalently view \( \Sigma^0_i \) as player \( i \)'s strategy set in \( g \) itself. Thus, any belief \( \pi_2 \) in \( g \) may also be viewed as a belief in \( g^{cts} \) (with \( \text{supp} (\pi_2) \subseteq \Sigma^0_i \)).

\(^{46}\)The reason I did not use the game \( g^{cts} \) in Sections 2.3 and 2.4 is that it is difficult to interpret the assumption that player \( i \) can accept the demand \( \lim \inf_{\tau \to t} u_j (\tau) \) at time \( t \), since the demand \( u_j (\tau) \) has not yet been made at time \( t \) for all \( \tau > t \). Thus, I view the game \( g^{cts} \) as a technical construct for analyzing the limit of discrete-time games, and not as an appealing model of continuous-time bargaining in its own right.
Let $\gamma^{g_n'}$ be given by

$$
\gamma^{g_n'}(t) = \left(\frac{n}{(n+1)}\right) \gamma^*(\max\{\tau \leq t : \tau \in I^n_1\})
$$

for all $t \in \mathbb{R}_+$, with the convention that $\max\{\tau < t : \tau \in I^n_1\} \equiv 0$ if the set $\{\tau < t : \tau \in I^n_1\}$ is empty. I first claim that $\lim_{n \to \infty} u_1^*(\gamma^{g_n'}) \geq u_1^*$,\(^\text{17}\) To show this, I first establish that $T (\gamma^{g_n'}) \leq \min\{\tau > T^1 : \tau \in I^n_1\}$ for all $n$, where $T^1$ is defined as in the proof of Theorem 2. Since $\gamma^*$ (and thus $\gamma^{g_n'}$) are non-decreasing, $\sup_{t \geq t} e^{-r(\tau-t)} \left(1 - \gamma^{g_n'}(\tau)\right) = 1 - \gamma^{g_n'}(t)$. Therefore, by Lemma 2, $\hat{T} (\gamma^{g_n'})$ satisfies

$$
\exp\left(-\int_0^{\hat{T} (\gamma^{g_n'})} \frac{r \left(\frac{n+1}{n} - \gamma^*(\max\{\tau \leq t : \tau \in I^n_1\})\right)}{\gamma^*(\max\{\tau \leq t : \tau \in I^n_1\})} dt\right) \prod_{t \in I^n_1 \cap [0, \hat{T} (\gamma^{g_n'})]} \frac{\gamma^*(\max\{\tau < t : \tau \in I^n_1\})}{\gamma^*(t)} \geq \varepsilon.
$$

(28)

Now

$$
\exp\left(-\int_0^{\hat{T} (\gamma^{g_n'})} \frac{r \left(1 - \gamma^*(t)\right)}{\gamma^*(t)} dt\right) \frac{\gamma^*(0)}{\gamma^*(\max\{\tau < \hat{T} (\gamma^{g_n'}) : \tau \in I^n_1\})} \leq \exp\left(-\int_0^{\max\{\tau < \hat{T} (\gamma^{g_n'}) : \tau \in I^n_1\}} \frac{r \left(1 - \gamma^*(t)\right) + \gamma^{*'}(t)}{\gamma^*(t)} dt\right).
$$

(29)

If $\hat{T} (\gamma^{g_n'}) > \min\{\tau > T^1 : \tau \in I^n_1\}$, then $\max\{\tau < \hat{T} (\gamma^{g_n'}) : \tau \in I^n_1\} > T^1$ and therefore (29) is less than $\varepsilon$, which contradicts (28). Hence, $\hat{T} (\gamma^{g_n'}) \leq \min\{\tau > T^1 : \tau \in I^n_1\}$ for all $n$. In addition, $\gamma^{g_n'}(t)$ is non-decreasing and $\gamma^{g_n'}(t) < 1$ for all $t$, which implies that $T (\gamma^{g_n'}) = \hat{T} (\gamma^{g_n'})$. Hence, by Lemma 4, $u_1^*(\gamma^{g_n'}) = \min_{t \leq \hat{T} (\gamma^{g_n'})} e^{-rt}\gamma^{g_n'}(t)$. Since $\hat{T} (\gamma^{g_n'}) \leq \min\{\tau > T^1 : \tau \in I^n_1\}$ for all $n$, and $\{g_n\}$ converges to continuous time, $\lim_{n \to \infty} \hat{T} (\gamma^{g_n'}) \leq T^1$. In addition, $\lim_{n \to \infty} \sup_{t \in \mathbb{R}_+} |\gamma^{g_n'}(t) - \gamma^*(t)| = 0$, so it follows that $\lim_{n \to \infty} u_1^*(\gamma^{g_n'}) = \min_{t \leq T^1} e^{-rt}\gamma^*(t) = u_1^*$.

Next, I claim that $u_1^*(\gamma^{g_n'}) \geq u_1^*(\gamma^{g_n})$ for any posture $\gamma^{g_n}$ in discrete-time bargaining game $g_n$. To see this, note that if $\text{supp}(\pi_2) \subseteq \Sigma_1^{g_n}$ and $\sigma_2 \in \Sigma_2^{*g_n}(\pi_2)$, then $\sigma_2 \in \Sigma_2^{*}(\pi_2)$ as well (i.e., there is no benefit to responding to a strategy in $\Delta(\Sigma_1^{g_n})$ with a strategy outside of $\Sigma_2^{*}(\pi_2)$). Therefore, if $\pi_1 \in \Pi_1^{*g_n}$ (i.e., if $\pi_1$ is consistent with knowledge of rationality in

\(^{17}\)Theorem 2 implies that $\lim_{n \to \infty} u_1^*(\gamma^{g_n'}) \leq u_1^*$, so this inequality must hold with equality. But only the inequality is needed for the proof.
$g_n)$, then $\pi_1 \in \Pi_1^{\gamma_n, g_n}$; that is, $\Pi_1^{\gamma_n, g_n} \subseteq \Pi_1^{\gamma_n, g^c_{cts}}$. Now

$$u_1^{*, g_n} (\gamma_n) = \sup_{\sigma_1 \in \Sigma_1^{\gamma_n}} \inf_{\pi_1 \in \Pi_1^{\gamma_n, g_n}} u_1 (\sigma_1, \pi_1) \geq \sup_{\sigma_1 \in \Sigma_1^{\gamma_n}} \inf_{\pi_1 \in \Pi_1^{\gamma_n, g^c_{cts}}} u_1 (\sigma_1, \pi_1) = u_1 (\gamma_n, \sigma^{\gamma_n}) = u_1 (\gamma_n),$$

where $\sigma^{\gamma_n}$ is defined in Definition 9, and the second line follows because $\Pi_1^{\gamma_n, g_n} \subseteq \Pi_1^{\gamma_n, g^c_{cts}}$; the third line follows because $u_1 (\gamma_n, \sigma^{\gamma_n}) = \sup_{\sigma_1 \in \Sigma_1^{\gamma_n}} \inf_{\pi_1 \in \Pi_1^{\gamma_n, g^c_{cts}}} u_1 (\sigma_1, \pi_1)$ by Lemma 4, and $\gamma_n \in \Sigma_1^{\gamma_n} \subseteq \Sigma_1^{g^c_{cts}}$; and the fourth line follows by Lemma 4.

Combining the above claims, it follows that $\lim_{n \to \infty} u_1^{*, g_n} (\gamma_n') \geq \lim_{n \to \infty} u_1^{*} (\gamma_n') \geq u_1^{*} (\gamma)$. This proves the first part of the lemma.

For the second part of the lemma, fix a sequence of continuation value functions $\{v_n\}$ (with $v_n$ a continuation value function in discrete-time game $g_n$) converging pointwise to some function $v : \mathbb{R}_+ \to [0, 1]$. I have already shown that $u_1^{*, g_n} (\gamma_n) \geq u_1^{*} (\gamma_n)$ for any posture $\gamma_n$ in game $g_n$, or equivalently $u_1^{*, g_n} (v_n) \geq u_1^{*} (v_n)$. This immediately implies that $\lim_{n \to \infty} u_1^{*, g_n} (v_n) \geq \limsup_{n \to \infty} u_1^{*} (v_n)$ for every convergent subsequence of $\{u_1^{*, g_n} (v_n)\}$. Hence, I must show that $\lim_{n \to \infty} u_1^{*, g_n} (v_n) \leq \liminf_{n \to \infty} u_1^{*} (v_n)$ for every convergent subsequence of $\{u_1^{*, g_n} (v_n)\}$. I establish this inequality by assuming that there exists $\eta > 0$ such that $\lim_{n \to \infty} u_1^{*, g_n} (v_n) > \liminf_{n \to \infty} u_1^{*} (v_n) + \eta$ for some convergent subsequence of $\{u_1^{*, g_n} (v_n)\}$ and then deriving a contradiction.

Let $t_{\text{next}} (i) = \min \{\tau > t : \tau \in I_i^{g_n}\}$ be the time of player $i$'s next demand at $t$. Given continuation value function $v_n$, fix any corresponding posture $\gamma_n$, let $\gamma^n_{\text{max}}$ be defined as follows: First, $\gamma^n_{\text{max}}$ demands $\gamma^n_{\text{max}} (h^t) = \gamma_n (h^t)$ for all $t \in I_i^{g_n}$. Second, $\gamma^n_{\text{max}}$ accepts player 2's demand at time $t \in I_2^{g_n}$ with probability

$$\hat{p}^n (t) \equiv \min \left\{ \frac{p^n (t)}{\chi^n (t)}, 1 \right\},$$

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where

\[ p^n(t) \equiv \max_{\tau < t : \tau \in I^n_1, \tau_{\text{next}}(2) = t} \frac{e^{r(t-\tau)v^n(\tau)} - v^n(\tau)}{1 - v^n(\tau)} \]

if \( \{ \tau < t : \tau \in I^n_1, \tau_{\text{next}}(2) = t \} \) is non-empty and \( v^n(\tau) < 1 \) for all time \( \tau \) in this set, and \( p^n(t) \equiv 0 \) otherwise; and

\[ \chi^n(t) \equiv \max \left\{ \frac{\Pi_{\tau < t : \tau \in I^n_1} (1 - p^n(\tau)) - \varepsilon}{\Pi_{\tau < t : \tau \in I^n_1} (1 - p^n(\tau))}, 0 \right\} \]

Let \( \tilde{T}^n \) be the supremum over times \( t \) at which \( \chi^n(t_{\text{next}}(2)) \hat{p}^n(t_{\text{next}}(2)) = p^n(t_{\text{next}}(2)) \), and let

\[ T^n \equiv \sup_{t \geq \tilde{T}^n} \arg\max_{t \in I^n_1} e^{-rt} v^n(t) \]

By an argument similar to the proof of Lemma 3, if \( \gamma^n(t) < \eta \) for some \( t \leq T^n \), then there exists a belief \( \tilde{\pi}_2 \in \Delta(\Sigma^n_2) \) and strategy \( \sigma_2 \in \Sigma^n_2 \) such that \( \tilde{\pi}_2(\gamma^n) \geq \varepsilon, \sigma_2(\Sigma^n_2(\tilde{\pi}_2)) \), and the demand \( \gamma^n(t) \) is accepted under strategy profile \( (\gamma^n, \sigma_2) \). In particular, \( u^n_1(\gamma^n, \sigma_2) < \eta \). Thus, by the hypothesis that \( \lim_{n \to \infty} u^n_1(\gamma^n) > \liminf_{n \to \infty} u^n_1(\gamma^n) + \eta \), there must exist \( N > 0 \) such that \( \gamma^n(t) \geq \eta \) for all \( t \leq T^n \) and all \( n > N \), and hence \( v^n(t) \leq 1 - \eta \) for all \( t \leq T^n \) and all \( n > N \).

Let \( \tilde{\pi}_2^n \) assign probability \( \varepsilon \) to \( \gamma^n \) and probability \( 1 - \varepsilon \) to \( \tilde{\gamma}^n \), and fix \( \sigma_2^n \in \Sigma^n_2(\tilde{\pi}_2^n) \) with the property that \( \sigma_2^n \) always demands 1 and rejects player 1’s demand at any history at which player 1 has deviated from \( \gamma^n \) (which is possible because \( \tilde{\pi}_2^n \) assigns probability 0 to such histories, except for terminal histories), as well as at any history at which player 2 is indifferent between accepting and rejecting player 1’s demand, given belief \( \tilde{\pi}_2^n \). Note that \( \gamma^n \) is a best-response to \( \tilde{\sigma}_2^n \) in \( g^n \). This implies that \( u^n_1(\gamma^n) \leq u^n_1(\gamma^n, \sigma_2^n) \) for all \( n \).

Thus, to show that \( \lim_{n \to \infty} u^n_1(\gamma^n) \leq \liminf_{n \to \infty} u^n_1(\gamma^n) + \eta \) (the desired contradiction), it suffices to show that \( \lim_{n \to \infty} u^n_1(\gamma^n, \sigma_2^n) \leq \liminf_{n \to \infty} u^n_1(\gamma^n) + \eta \).

Observe that \( p^n(t) \) satisfies

\[ \exp(-r(t-T))(p^n(t)(1) + (1-p^n(t))v^n(t)) \geq v^n(t) \]
for all $\tau \leq t$ such that $\tau \in I^n$ and $\tau_{next}^{n} = t$. Hence, it is optimal for player 2 to reject player 1’s demand $\gamma$ at any time $\tau$ at which $\chi^n (\tau_{next}^{n}) \bar{p}^{n} (\tau_{next}^{n}) = p^n (\tau_{next}^{n})$, given belief $\pi^n_2$. Therefore, $u_1^n (\gamma^n, \sigma^n_2) = \min_{t \leq T} e^{-r t} (1 - \gamma^n (t))$. Now $u^*_1 (\gamma^n) = \min_{t \leq T (\nu^n_2)} e^{-r t} (1 - \gamma^n (t))$, and $\lim_{n \to \infty} T (\nu^n_2) = \bar{T} (v)$. Hence, showing that $\lim_{n \to \infty} T^n = \bar{T} (v) \equiv \bar{T}$ would imply that $\lim_{n \to \infty} u_1^n (\gamma^n, \sigma^n_2) = \lim_{n \to \infty} u^*_1 (\gamma^n)$, yielding the desired contradiction.

To see that $\lim_{n \to \infty} T^n = \bar{T}$, first fix $t_0 \leq \bar{T}$ and note that for all $\delta > 0$ there exists $N' > 0$ such that, for all $t \leq t_0$ and all $n \geq N'$, if $g^n (t) = 2$ then $\min \{ \tau \leq t : \tau \in I^n, \tau_{next}^{n} = t \} \geq t - \delta$ (if this set is non-empty). Next, since both $e^{-r t} v(t)$ and $e^{-r t} v^n (t)$ are non-increasing (by the same argument that showed that $e^{-r t} v(t)$ is non-increasing) and $v^n (t) \to v(t)$ for all $t \in \mathbb{R}_+$, it follows that for all $\delta' > 0$ there exists $\delta > 0$ such that $t \leq t_0$ and $\tau \in [t - \delta, t]$ implies that $|e^{(\tau - t)} v^n (\tau) - v (t, -1)| < \delta'$. Since $1 - v(t) \geq \eta$ for all $t \leq \bar{T}$, combining these observations and letting $S$ be the (countable) set of discontinuity points of $v(t)$, for all $\delta' > 0$ there exists $N''$ such that if $t = s_{next}^{n} (2)$ for some $s \in S \cap [0, t_0]$, and $n \geq N''$, then $\left| p^n (t) - \frac{v(t, -1) - v(t)}{1 - v(t)} \right| < \delta'$.\(^{18}\) Hence,

$$
\lim_{n \to \infty} \prod_{s \in S \cap [0, t_0]} (1 - p^n (s_{next}^{n} (2))) = \prod_{s \in S \cap [0, t_0]} (1 - p(s)) \quad \text{(30)}
$$

for all $t_0 \leq \bar{T}$.

Finally, I establish that, whenever $v$ is continuous on an interval $[t_0, t_{\infty}]$ with $t_{\infty} \leq \bar{T}$,

$$
\lim_{n \to \infty} \prod_{t \in I^n \cap [t_0, t_{\infty}]} (1 - p^n (t)) = \exp \left( - \int_{t_0}^{t_{\infty}} \frac{r v(t) - v'(t)}{1 - v(t)} \, dt \right) = \exp \left( - \int_{t_0}^{t_{\infty}} \lambda(t) \, dt \right). \quad \text{(31)}
$$

I will prove this fact by showing that the limit as $n \to \infty$ of a first-order approximation of the logarithm of $\prod_{t \in I^n \cap [t_0, t_{\infty}]} (1 - p^n (t))$ equals $- \int_{t_0}^{t_{\infty}} \frac{r v(t) - v'(t)}{1 - v(t)}$.

Let \( \{t_{1,g_0}, t_{2,g_0}, \ldots, t_{K(n),g_0}\} = \{t \in [t_0, t_{\infty}] : p^n (t) > 0\} \), with $t_{k,g_0} < t_{k+1,g_0}$ for all $k \in \{1, \ldots, K(n) - 1\}$ and all $n \in \mathbb{N}$; note that $K(n)$ is finite because $I^n \cap [t_0, t_{\infty}]$ is finite, and that in addition $t_{K(n) + 1,n} (1) = t_{K(n) + 1, g_0}$ for all $k$ (where $t_{K(n) + 1,n} (1) \equiv t_{K(n) + 1, g_0}$ (1) to avoid redundant

\(^{18}\) $S$ is countable because $e^{-r t} v(t)$ is non-increasing, and monotone functions have at most countably many discontinuity points. Unlike in Section 2.4, $S$ need not be a subset of $\mathbb{N}$ here.
notation). Furthermore, since \( e^{-\tau v^{g_n}(\tau)} \) is non-increasing,

\[
 t_{k,g_n}^{\text{next}}(1) \in \arg\max_{\tau < t_{k+1,g_n}; \tau \in (t_{k,g_n}^{\text{next}}(1), t_{k+1,g_n})} e^{r(t_{k+1,g_n} - \tau)} v^{g_n}(\tau)
\]

for all \( k \in \{0, 1, \ldots, K(n) - 1\} \). Therefore,

\[
\begin{align*}
\prod_{k=1}^{K(n)} (1 - p^n(t)) &= \prod_{k=1}^{K(n)} \min_{\tau \in (t_{k,g_n}^{\text{next}}(1), t_{k+1,g_n})} \frac{1 - e^{r(t_{k,g_n} - \tau)} v^{g_n}(\tau)}{1 - v^{g_n}(t_{k,g_n})} \\
&= \prod_{k=1}^{K(n)} \min_{\tau \in (t_{k,g_n}^{\text{next}}(1), t_{k+1,g_n})} \frac{1 - e^{r(t_{k,g_n} - \tau)} v^{g_n}(\tau)}{1 - e^{-r(t_{k,g_n}^{\text{next}}(1) - t_{k,g_n})} v^{g_n}(t_{k,g_n}^{\text{next}}(1))} \\
&= \left( \prod_{k=1}^{K(n)-1} \frac{1 - e^{r(t_{k+1,g_n} - t_{k,g_n})} v^{g_n}(t_{k,g_n}^{\text{next}}(1))}{1 - e^{-r(t_{k,g_n}^{\text{next}}(1) - t_{k,g_n})} v^{g_n}(t_{k,g_n}^{\text{next}}(1))} \right) \frac{1 - e^{r(t_{1,g_n} - t_{0,g_n})} v^{g_n}(t_{0,g_n}^{\text{next}}(1))}{1 - e^{-r(t_{K(n),g_n}^{\text{next}}(1) - t_{K(n),g_n})} v^{g_n}(t_{K(n),g_n}^{\text{next}}(1))} \quad (32)
\end{align*}
\]

Next, taking a first-order Taylor approximation of \( \log(1 - e^{r x v^{g_n}(t)}) \) at \( x = 0 \) yields

\[
\log(1 - e^{r x v^{g_n}(t)}) = \log(1 - v^{g_n}(t)) - \frac{r x v^{g_n}(t)}{1 - v^{g_n}(t)} + o(x^2).
\]

Therefore, a first-order approximation of the logarithm of (32) equals

\[
\begin{align*}
&\left( \prod_{k=1}^{K(n)-1} - (t_{k+1,g_n} - t_{k,g_n}) \frac{r v^{g_n}(t_{k,g_n}^{\text{next}}(1))}{1 - v^{g_n}(t_{k,g_n}^{\text{next}}(1))} \right) \\
&\quad \log \left( 1 - e^{r(t_{1,g_n} - t_{0,g_n})} v^{g_n}(t_{0,g_n}^{\text{next}}(1)) \right) - \log \left( 1 - e^{-r(t_{K(n),g_n}^{\text{next}}(1) - t_{K(n),g_n})} v^{g_n}(t_{K(n),g_n}^{\text{next}}(1)) \right) \\
&+ \log \left( 1 - e^{r(t_{1,g_n} - t_{0,g_n})} v^{g_n}(t_{0,g_n}^{\text{next}}(1)) \right) - \log \left( 1 - e^{-r(t_{K(n),g_n}^{\text{next}}(1) - t_{K(n),g_n})} v^{g_n}(t_{K(n),g_n}^{\text{next}}(1)) \right).
\end{align*}
\]

I now show that

\[
\lim_{n \to \infty} \sum_{k=1}^{K(n)-1} - (t_{k+1,g_n} - t_{k,g_n}) \frac{r v^{g_n}(t_{k,g_n}^{\text{next}}(1))}{1 - v^{g_n}(t_{k,g_n}^{\text{next}}(1))} = - \int_{t_0}^{t_{\infty}} \frac{r v(t)}{1 - v(t)} dt \quad (33)
\]
and

\[
\lim_{n \to \infty} \left( \log \left( 1 - e^{r(t_1 - t^\text{ext}(1))_{t_0, g_n}} (t^\text{ext}(1)) \right) - \log \left( 1 - e^{-r(t^\text{ext}(1))_{t_0, g_n} (t^\text{ext}(1))} \right) \right) = \int_{t_0}^{t_{\infty}} \frac{v'(t)}{1 - v(t)} dt,
\]

which completes the proof of (31). Equation (34) is immediate, because, since \( v \) is continuous on \([t_0, t_{\infty}]\), both the left- and right-hand sides equal

\[
\log (1 - v(t_0)) - \log (1 - v(t_{\infty})).
\]

To establish (33), let

\[
f^n(t) \equiv \exp \left( -r \left( \frac{1 + \eta}{\eta} \right) t \right) \frac{r v^{g_n}(t)}{1 - v^{g_n}(t)}
\]

and

\[
f(t) \equiv \exp \left( -r \left( \frac{1 + \eta}{\eta} \right) t \right) \frac{r v(t)}{1 - v(t)}.
\]

For all \( n > N \), it can be verified that both \( f^n(t) \) and \( f(t) \) are non-increasing on the interval \([t_0, t_{\infty}]\), using the facts that \( e^{-r t v^{g_n}(t)} \) and \( e^{-r t v(t)} \) are non-increasing, and that \( v^{g_n}(t) \leq 1 - \eta \) for all \( n > N \) and \( t \leq t_{\infty} \leq \tilde{T} \). Fix \( \zeta > 0 \) and \( m \in \mathbb{N} \). Because \( v^{g_n}(t) \to v(t) \) for all \( t \in \mathbb{R}_+ \), there exists \( N'' \geq N \) such that, for all \( n > N'' \), \(|f^n(t) - f(t)| < \zeta \) for all \( t \) in the set

\[
\left\{ t_0, \frac{(m - 1) t_0 + t_{\infty}}{m}, \frac{(m - 2) t_0 + 2 t_{\infty}}{m}, \ldots, t_{\infty} \right\}.
\]

Since both \( f^n \) and \( f \) are non-increasing on \([t_0, t_{\infty}]\), this implies that

\[
|f^n(t) - f(t)| < \zeta + \max_{k \in \{1, \ldots, K(n) - 1\}} \left( f \left( \frac{(m - k) t_0 + k t_{\infty}}{m} \right) - f \left( \frac{(m - k - 1) t_0 + (k + 1) t_{\infty}}{m} \right) \right)
\]

for all \( t \in [t_0, t_{\infty}] \). Since \( f \) is continuous on \([t_0, t_{\infty}]\), taking \( m \to \infty \) implies that \(|f^n(t) - f(t)| < 2 \zeta \) for all \( t \in [t_0, t_{\infty}] \), and therefore

\[
\left| \frac{r v^{g_n}(t)}{1 - v^{g_n}(t)} - \frac{r v(t)}{1 - v(t)} \right| \leq 2 \zeta \exp \left( r \left( \frac{1 + \eta}{\eta} \right) t_{\infty} \right)
\]

for all \( t \in [t_0, t_{\infty}] \).
\[ [t_0, t_\infty]. \] Hence,

\[
\lim_{n \to \infty} \sum_{k=1}^{K(n)-1} - (t_{k+1,g_n} - t_{k,g_n}) \frac{r v_{g_n} \left(t_{k,g_n}^{\text{next}} (1)\right)}{1 - v_{g_n} \left(t_{k,g_n}^{\text{next}} (1)\right)} = \lim_{n \to \infty} \sum_{k=1}^{K(n)-1} - (t_{k+1,g_n} - t_{k,g_n}) \frac{r v \left(t_{k,g_n}^{\text{next}} (1)\right)}{1 - v \left(t_{k,g_n}^{\text{next}} (1)\right)}
\]

\[ = \lim_{n \to \infty} \sum_{k=1}^{K(n)-1} - (t_{k+1,g_n} - t_{k,g_n}) \frac{r v \left(t_{k,g_n}\right)}{1 - v \left(t_{k,g_n}\right)}
\]

\[ = - \int_{t_0}^{t_\infty} f(t) \, dt,
\]

where the first equality follows because \( \sum_{k=1}^{K(n)-1} (t_{k+1,g_n} - t_{k,g_n}) \leq t_\infty - t_0 \) for all \( n \in \mathbb{N} \), the second follows because \( t_{k,g_n}^{\text{next}} (1) \in [t_{k,g_n}, t_{k+1,g_n}] \) and \( v \) is continuous on \([t_0, t_\infty] \), and the third follows by definition of the (Riemann) integral.

Combining (30) and (31), it follows that

\[
\lim_{n \to \infty} \prod_{s \in \mathbb{T}_g \cap [0,t]} (1 - p^g (s)) = \exp \left(- \int_0^t \lambda (s) \, ds\right) \prod_{s \in \mathbb{T} \cap [0,t]} (1 - p (s))
\]

for all \( t \leq \bar{T} \). This implies that \( \lim_{n \to \infty} \bar{T}^g = \bar{T} \), completing the proof of the lemma. \( \blacksquare \)

I now complete the proof of Theorem 4.

Let \( \{g_n\} \) be a sequence of discrete-time bargaining games converging to continuous time. Recall that \( u_1^{*,g_n} = \sup_{\gamma^g_n} u_1^{*,\gamma^g_n} (\gamma^g_n) \). Thus, there exists a sequence of postures \( \{\gamma^{g_n}\} \), with \( \gamma^{g_n} \) a posture in \( g_n \), such that \( \lim_{n \to \infty} |u_1^{*,g_n} - u_1^{*,\gamma^{g_n}} (\gamma^{g_n})| = 0 \). Let \( \{v^{g_n}\} \) be the corresponding sequence of continuation value functions. Because \( e^{-rt} v^{g_n} (t) \) is non-increasing and the space of monotone functions from \( \mathbb{R}_+ \to [0,1] \) is sequentially compact (by Helly’s selection theorem or Footnote 43), this sequence has a convergent subsequence \( \{v^{g_k}\} \) converging to some \( v \) on \( \mathbb{R}_+ \).

I claim that \( v = v^* \). Toward a contradiction, suppose not. Since \( v^* \) is the unique maxmin continuation payoff function in \( g^{ct} \), there exists \( \eta > 0 \) such that \( u_1^* > \lim_{k \to \infty} u_1^* (v^{g_k}) + \eta \).

By Lemma 6, \( \lim_{k \to \infty} u_1^{*,g_k} (v^{g_k}) = \lim_{k \to \infty} u_1^* (v^{g_k}) \). Finally, again by Lemma 6, there exists an alternative sequence of postures \( \{\gamma^{g_{k'}}\} \) such that \( \lim_{k \to \infty} u_1^{*,g_k} (\gamma^{g_{k'}}) \geq u_1^* \). Combining
these observations implies that there exists $K > 0$ such that, for all $k \geq K$,

$$u_1^{*g_k}(\gamma^{g_k}) > u_1^* - \eta/3 > u_1^*(u^{g_k}) + 2\eta/3 > u_1^{*g_k}(u^{g_k}) + \eta/3,$$

which contradicts the fact that $\lim_{k \to \infty} |u_1^{*g_k} - u_1^{*g_k}(\gamma^{g_k})| = 0$. Therefore, $v = v^*$. Since this argument applies to any convergent subsequence of $\{v^{g_n}\}$, and every subsequence of $\{v^{g_n}\}$ has a convergent sub-subsequence, this implies that $v^{g_n} \to v^*$ pointwise.

A similar contradiction argument shows that $\lim_{k \to \infty} u_1^{*g_k}(v^{g_k}) = u_1^*$, for any convergent subsequence $\{v^{g_k}\} \subseteq \{v^{g_n}\}$. Since $\lim_{k \to \infty} |u_1^{*g_k} - u_1^{*g_k}(\gamma^{g_k})| = 0$, it follows that $u_1^{*g_k} \to u_1^*$. And, since this argument applies to any convergent subsequence of $\{v^{g_n}\}$, this implies that $u_1^{*g_n} \to u_1^*$. ■
3 Cooperation with Network Monitoring

3.1 Introduction

The question of how groups can sustain cooperation and the related question of what kinds of groups can sustain cooperation best are fundamental in the social sciences (Olson, 1965; Ostrom, 1990; Coleman, 1990; Greif, 1993; Putnam, 2000). In economics, existing work on the theory of repeated games provides a framework for answering these questions when individuals can perfectly observe each other’s actions (e.g., Abreu, 1988), but has much less to say about the more realistic case where monitoring is imperfect. This weakness is particularly acute in settings where public signals are not very informative about each individual’s actions and high quality—but dispersed—private signals are the basis for cooperation. Consider the construction of a series of infrastructure projects in a small village (wells, schools, roads, etc.). The quality of each project is a poor signal of each villager’s contribution to its construction, but each villager may always know whether the other members of her household worked on the project. Similarly, the stock price of a Fortune 500 company is a poor signal of each employee’s effort, but each employee may observe her officemates’ effort; and price is a poor signal of each firm’s output in a large market, but each firm may observe the output of its local competitors. Thus, it is certainly plausible that local, private monitoring plays a larger role than public monitoring in sustaining cooperation in many interesting economic examples, and very little is known about how cooperation is best sustained under this sort of monitoring.

This paper studies cooperation in repeated games with network monitoring, where in every period a network is independently drawn from a (possibly degenerate) known distribution, and players perfectly observe the actions of their neighbors but observe nothing about any other player’s action. Each player’s action is simply her level of cooperation, in that higher actions are privately costly but benefit others. The main result is that, under some assumptions on the stage game, each player’s expected discounted level of cooperation is maximized (in sequential equilibrium) by grim trigger strategies, in which each player cooperates at a fixed level in every period unless she ever observes another player fail to cooperate at her prescribed level, in which case she stops cooperating forever. I then apply this result...
to derive comparative statics on the maximum (equilibrium level of) cooperation in two important special cases: equal monitoring, where in expectation players are monitored “equally well”; and fixed monitoring networks, where the monitoring network is constant over time. For example, I provide a simple characterization of when maximum cooperation is increasing in group size with equal monitoring, and show that more central players in fixed monitoring networks have greater maximum cooperation.

The fact that maximum cooperation is sustainable in grim trigger strategies follows from a novel kind of “strategic complementarity” that emerges in repeated cooperation games with network monitoring. The key observation is that the highest action that a player is willing to take at any on-path history is non-decreasing in the actions of every other player at every on-path history; this is because a player benefits when her opponents take higher actions and, with network monitoring and grim trigger strategies, deviating makes every on-path history weakly less likely. When players’ utility functions are separable and concave in actions, and players’ observe the realized monitoring network at the end of every period, this complementarity implies that maximum cooperation is sustainable in grim trigger strategies.49 In the leading case where every player’s maximum cooperation is below the first-best level, this result characterizes the equilibrium that maximizes utilitarian social welfare at any fixed discount factor, which is usually impossible in repeated games with imperfect monitoring.

The result that grim trigger strategies sustain maximum cooperation implies particularly sharp comparative statics in games with equal monitoring. The most striking result is that, when all players benefit equally from all other players’ actions (i.e., when a player’s level of cooperation is her contribution to a global public good), maximum cooperation is determined by the product of two terms: the marginal benefit a player receives from another player’s

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49 Interestingly, maximum cooperation may not be sustainable in grim trigger strategies when players do not observe the realized monitoring network at the end of every period (Example A2). Roughly speaking, this is because in more complicated strategy profiles differences in players’ beliefs about each other’s histories can be exploited to provide stronger incentives for cooperation than are possible in grim trigger strategy profiles.
action, and the *effective contagiousness* of the monitoring technology, defined as

\[ \sum_{t=0}^{\infty} \delta^t E_{[\text{number of players who learn about a deviation within } t \text{ periods}]} \].

In particular, one monitoring technology supports greater maximum cooperation than another if and only if its effective contagiousness is higher; all other properties of the monitoring technology (the variance of the number of players who learn about a deviation within \( t \) periods, the identity of these players, etc.) are irrelevant for supporting cooperation.

This characterization of comparative statics on maximum cooperation in games of global public good provision with equal monitoring leads to three more special comparative statics results. First, if the marginal benefit a player receives from another player’s action is independent of group size (the case of *pure* global public goods), then maximum cooperation is increasing in group size as long as monitoring does not degrade with group size so quickly that the expected number of players who learn about a deviation within \( t \) periods is decreasing in group size (for some \( t \)). On the other hand, if this marginal benefit is proportional to the reciprocal of group size (the case of *divisible* global public goods), then maximum cooperation is decreasing in group size whenever the expected fraction of players who learn about a deviation within \( t \) periods is decreasing in group size. For example, maximum cooperation is increasing in group size with pure global public goods but decreasing in group size with divisible global public goods when the monitoring technology is either random matching or monitoring on a fixed circle. Finally, holding group size fixed, making monitoring more uncertain in the second-order stochastic dominance sense reduces maximum cooperation.

The last part of the paper studies games with a fixed monitoring network (without equal monitoring). I introduce a new notion of network centrality and show that more central players have greater maximum cooperation, thus linking the graph-theoretical property of centrality with the game-theoretic property of maximum cooperation. I also provide simple graph-theoretic tools for determining which players are more central than others, and show that centrality can also be used to determine which of two networks supports greater maximum cooperation; for example, adding links to the monitoring network necessarily increases all players’ maximum cooperation, which formalizes the idea that individuals in better-
connected groups have greater capacities for cooperation. Finally, I show that adding links to the monitoring network is always stable when players only benefit from their neighbors' contributions (the case of local public goods), but that in general players may wish to sever links in the monitoring network in order to facilitate free-riding on others' cooperation.

The paper proceeds as follows: Section 3.2 relates this paper to the literature. Section 3.3 describes the model. Section 3.4 presents the main result that maximum cooperation can be sustained in grim trigger strategies. Section 3.5 derives comparative statics in games with equal monitoring. Section 3.6 studies games with fixed monitoring networks. Section 3.7 concludes and discusses directions for future research. Appendix A contains omitted examples, and Appendix B contains omitted proofs.

3.2 Related Literature

This paper lies at the intersection of the literature on repeated games with community enforcement and the literature on repeated public good provision. The study of repeated games with community enforcement was pioneered by Kandori (1992) and Ellison (1994), who show that cooperation is sustainable in a modified version of grim trigger strategies in the (two-action) prisoner's dilemma with random matching when the discount factor is sufficiently high. Two comments are in order. First, these papers, along with much of the subsequent literature (e.g., Deb, 2009; Takahashi, 2010) focus on the case of sufficiently high discount factors and do not characterize efficient equilibria at fixed discount factors, unlike my paper. Second, a key concern in these papers is ensuring that players do not prefer to cooperate off the equilibrium path. The issue is that grim trigger strategies may provide such a strong incentive to cooperate on-path that players prefer to cooperate even after observing a defection. Ellison resolves this problem by introducing a “relenting” version of grim trigger strategies tailored to make players indifferent between cooperating and defecting on-path, and then noting that cooperation is more appealing on-path than off-path (since off-path at least one opponent is already defecting). This issue does not arise in my analysis because, with continuous action spaces, players must be just indifferent between taking their prescribed actions and shirking on the equilibrium path, as otherwise they could be made to take slightly higher actions. By essentially the same argument as in Ellison, this implies
that players weakly prefer to shirk off-path. Hence, the key contribution of this paper is showing that grim trigger strategies provide the strongest possible incentives for cooperation on-path, not that they provide incentives for shirking off-path.

The maximum level of cooperation that can be sustained for given discount factors has been considered to some degree in the small existing literature on repeated public good provision. Bendor and Mookherjee (1987) study repeated provision of divisible public goods with a particular form of imperfect public monitoring, and present numerical evidence suggesting that in this context small groups can provide higher payoffs when only trigger strategies are considered; however, trigger strategies are not optimal in their model, and they do not characterize optimal equilibria. Bendor and Mookherjee (1990) ask when “multilateral” punishments, in which player $i$ may punish $j$ if $j$ cheats in her relationship with $k$, can improve on “unilateral” punishments, where this behavior is not present, in a repeated collective action game with perfect monitoring. They find an ambiguous relationship between group size and maximum cooperation. Pecorino (1999) studies repeated public good provision with perfect monitoring. He shows that public good provision is easier in larger groups with perfect monitoring, because shirking—and thus inducing everyone else to stop cooperating—is more costly in larger groups. Haag and Lagunoff (2007) study repeated collective action games with perfect monitoring when players have heterogeneous discount factors. Restricting attention to equilibria in which the same action is played every period, they show that maximum cooperation is increasing in group size, a result that in their model depends on heterogeneous discounting. In contrast, I show that equilibria in which the same action is played every period sustain maximum cooperation in my model, and I assume common discounting. Both Pecorino and Haag and Lagunoff suggest in their conclusions that imperfect monitoring might lead to less cooperation in large groups, but do not pursue this possibility in their papers. None of these papers characterize maximum cooperation in games with imperfect monitoring.

The paper most closely related to mine is Ali and Miller (2011). Ali and Miller study a network game in which links between players are recognized according to a Poisson process, and when a link is recognized the linked players play a prisoner’s dilemma with variable stakes, and can also make transfers to each other. In their model, equilibria exist in grim
trigger strategies with binding on-path incentive constraints, and these equilibria are optimal for symmetric networks; this result is similar to my main result, with the differences that Ali and Miller study staggered prisoner’s dilemmas with transfers rather than repeated cooperation games, the underlying network in their model is fixed over time, and they show that grim trigger strategies sustain maximum cooperation only for symmetric networks. Like my model, Ali and Miller’s features smooth actions and payoffs, so that, with grim trigger strategies, binding on-path incentive constraints imply slack off-path incentive constraints. Unlike my analysis, Ali and Miller’s does not emphasize strategic complementarity. Ali and Miller also discuss network formation and comparisons among networks, developing insights that are complementary to mine.

Finally, this paper is a contribution to the study of repeated games with private monitoring. By restricting attention to cooperation games with network monitoring, I am able to characterize efficient equilibria at fixed discount factors, whereas most papers in this literature study more general games and either prove folk theorems (Compte, 1998; Kandori and Matsushima, 1998; Matsushima, 2004; Hörner and Olszewski, 2006; Fong et al, 2007; Yamamoto, 2009; Sugaya, 2010) or study robustness to small deviations from public monitoring (Mailath and Morris, 2002, 2006; Sugaya and Takahashi, 2010). My approach is based on using the strategic complementarity discussed in the introduction to derive an upper bound on each player’s maximum cooperation, and then showing that this bound can be attained with grim trigger strategies. In particular, I make no attempt to characterize the entire set of sequential equilibria, or any large subset thereof. It would be interesting to see if similar indirect approaches, perhaps also based on strategic complementarity, can be useful in other classes of repeated games with private monitoring of applied interest.

3.3 Model

There are $N$ players; I also write $N$ for the set of players, abusively. In every period $t \in \mathbb{N} = \{0, 1, \ldots\}$, every player $i$ simultaneously chooses an action (“level of cooperation,” “contribution”) $x_i \in \mathbb{R}_+$. The players have common discount factor $\delta \in (0, 1)$. If the

\footnote{For folk theorems for repeated games with a fixed monitoring network, see Ben-Porath and Kahneman (1996) and Renault and Tomala (1998).}
players choose actions \( x = (x_1, \ldots, x_N) \) in period \( t \), player \( i \)'s expected period-\( t \) payoff is
\[
 u_i(x) = \left( \sum_{j \neq i} f_{i,j}(x_j) \right) - x_i,
\]
where the functions \( f_{i,j} : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfy
\begin{itemize}
  \item \( f_{i,j}(0) = 0 \), \( f_{i,j} \) is non-decreasing, and \( f_{i,j} \) is either strictly concave or identically 0;
  \item \( \lim_{x \to -\infty} \left( \sum_{j \neq i} f_{j,i}(x) \right) - x = -\infty \), and \( \lim_{x \to -\infty} \left( \sum_{j \neq i} f_{i,j}(x) \right) - x < 0 \).
\end{itemize}

The assumption that \( f_{i,j} \) is non-decreasing for all \( i \neq j \) is essential for interpreting \( x_j \) as player \( j \)'s level of cooperation. Note that the stage game is a prisoner's dilemma, in that playing \( x_i = 0 \) ("defecting," "shirking") is a dominant strategy for player \( i \) in the stage game. The second assumption states that the cost of cooperation becomes infinitely greater than the benefit for sufficiently high levels of cooperation. Concavity and the assumption that \( u_i(x) \) is separable in \((x_1, \ldots, x_N)\) play important roles in the analysis, and are discussed below.

Every period \( t \), a monitoring network \( L_t = \{l_{i,j,t}\}_{i,j \in N \times N}, l_{i,j,t} \in \{0, 1\} \), is drawn independently from a fixed probability distribution \( \mu \) on \((0,1)^{N^2}\). At the end of period \( t \), player \( i \) observes \( h_{i,t} = \{z_{i,1,t}, \ldots, z_{i,N,t}, L_t\} \), where \( z_{i,j,t} = x_{j,t} \) if \( l_{i,j,t} = 1 \), and \( z_{i,j,t} = \emptyset \) if \( l_{i,j,t} = 0 \). That is, at the end of period \( t \), player \( i \) observes the action of each of her out-neighbors and also observes the realized monitoring network \( L_t \). Assume that \( \Pr(l_{i,i} = 1) = 1 \) for all \( i \in N \); that is, there is perfect recall. A repeated game with such a monitoring structure has network monitoring, and the distribution \( \mu \) is the monitoring technology. Let \( h_i^t = (h_{i,0}, h_{i,1}, \ldots, h_{i,t-1}) \) be player \( i \)'s private history at time-\( t \), and denote the null history

\footnote{The assumption that each player observes the realized monitoring network was erroneously omitted from an earlier version of this paper but plays an important (but subtle) role in the analysis, as discussed in Section 3.4.}

\footnote{The model is agnostic as to whether players observe their realized stage-game payoffs. That is, \( f_{i,j}(x_j) \) is player \( i \)'s expected benefit from player \( j \)'s action, and player \( i \) may only benefit from player \( j \)'s action when \( l_{i,j,t} = 1 \). However, some combinations of assumptions on \( f_{i,j} \) and \( \mu \) are not consistent with this interpretation, such as monitoring on a fixed network with global public goods, where \( \Pr(l_{i,j,t} = 1) = 0 \) but \( f_{i,j} \neq 0 \) for some \( i,j \in N \). An alternative interpretation is required in these cases: for example, the infinite time horizon could be replaced with an uncertain finite horizon without discounting, with payoffs revealed at the end of the game and \( \delta \) viewed as the probability of the game's continuing. I thank a referee for helpful comments on this point.}
at the beginning of the game by \( h^0 = h^0_i \) for all \( i \). A (behavior) strategy of player \( i \)'s, \( \sigma_i \), specifies a probability distribution over period \( t \) actions as a function of \( h^t_i \).

Many important repeated games have network monitoring, including random matching (as in Kandori (1992) and Ellison (1994), with the modification that players observe the realized monitoring network) and monitoring on a fixed network. To fix ideas, note that a repeated game in which players observe the actions of their neighbors on a random graph that is determined in period 0 and then fixed for the duration of play does not have network monitoring, because the monitoring network is not drawn independently every period (e.g., player \( i \) observes player \( j \)'s action in period 1 with probability 1 if she observes it in period 0, but she does not observe player \( j \)'s action with probability 1 in period 0).

Throughout, I study sequential equilibria (SE) of this model with the property that, for every player \( i \), time \( t \), and monitoring network \( L_{t'} \), for \( t' < t \), the sum \( \sum_{t'=t}^\infty \delta^{t-t'} \mathbb{E} \left[ u_i \left( \sigma_j \left( (h^N_j)_{j=1}^N \right) \right) \ \bigg| \ L_{t'} \right] \) is well-defined; that is, \( \lim_{s \to \infty} \sum_{t'=t}^s \delta^{t-t'} \mathbb{E} \left[ u_i \left( \sigma_j \left( (h^N_j)_{j=1}^N \right) \right) \ \bigg| \ L_{t'} \right] \) exists.\(^{53}\) This technical restriction ensures that players’ continuation payoffs are well-defined, conditional on any past realized monitoring network. Let \( \Sigma_{SE} \) be the set of SE strategy profiles. The main object of interest is the SE that sustains each player’s maximum (equilibrium level of) cooperation, defined as follows.

**Definition 13** Player \( i \)'s maximum cooperation is

\[
x_i^* \equiv \sup_{\sigma \in \Sigma_{SE}} (1 - \delta) \sum_{t=0}^\infty \delta^t \mathbb{E} \left[ \sigma_i \left( (h^t_i) \right) \right].
\]

A strategy profile \( \sigma \) sustains player \( i \)'s maximum cooperation if \( \sigma \in \Sigma_{SE} \) and \( x_i^* = (1 - \delta) \sum_{t=0}^\infty \delta^t \mathbb{E} \left[ \sigma_i \left( (h^t_i) \right) \right] \).

My main result (Theorem 5) shows that there exists a grim trigger strategy profile that simultaneously sustains each player’s maximum cooperation, and the applied analysis in Sections 3.5 and 3.6 focuses on this equilibrium. This equilibrium is particularly important when it is also the SE that maximizes social welfare. This is the primary case of interest

\(^{53}\)It suffices for this paper to define a sequential equilibrium as a strategy profile in which, for every player \( i \) and private history \( h^t_i \), player \( i \)'s continuation strategy is optimal given beliefs about the vector of private histories \( (h^N_j)_{j=1}^N \) that are updated using Bayes’ rule whenever possible. In particular, sequential equilibrium could be replaced with “perfect Bayesian equilibrium.”
in the literature on public good provision, where the focus is on providing incentives for sufficient cooperation, rather than on avoiding providing incentives for excessive cooperation. For example, the grim trigger strategy profile that simultaneously sustains each player's maximum cooperation also maximizes utilitarian social welfare if \( x_i^* \) is below the first-best level for every \( i \in N \). Letting \( f'_{j,i} \) denote the left-derivative of \( f_{j,i} \) (which exists by concavity of \( f_{j,i} \)), this sufficient condition is

\[
\sum_{j \neq i} f'_{j,i}(x_i^*) \geq 1 \text{ for all } i \in N.
\]

This condition can be checked easily using the formula for \( (x_i^*)_i \) given by Theorem 5. In addition, a straightforward extension of Theorem 5 shows that if the grim trigger strategy profile that sustains each player's maximum cooperation also maximizes utilitarian social welfare, then it is effectively the only SE that does so, in that every such SE has the same path of play.

### 3.4 Characterization of Maximum Cooperation

This section presents the main theoretical result of the paper, which shows that all players' maximum cooperation can be sustained simultaneously in a grim trigger strategy profile.

I first define grim-trigger strategies.

**Definition 14** A strategy profile \( \sigma \) is a grim trigger strategy profile if there exist actions \( (x_i)_i \) such that \( \sigma_i(h_i) = x \) if \( z_{i,j,\tau} \in \{x_j, \emptyset\} \) for all \( z_{i,j,\tau} \in h_{i,\tau} \) and all \( \tau < t \), and \( \sigma_i(h_i) = 0 \) otherwise.

In a grim trigger strategy profile player \( i \)'s action at an off-path history \( h_i \) does not depend on the identity of the initial deviator. In particular, by perfect recall, player \( i \) plays \( x_i = 0 \) in every period following a deviation by player \( i \) herself. Also, if a grim trigger strategy profile \( \sigma \) sustains each player's maximum cooperation, then under \( \sigma \) each player \( i \) plays \( x_i^* \) at every on-path history.
Next, I introduce an important piece of notation: define \( D(\tau, t, i) \) recursively by

\[
D(\tau, t, i) = \emptyset \text{ if } \tau < t
\]

\[
D(t, t, i) = \{i\}
\]

\[
D(\tau + 1, t, i) = D(\tau, t, i) \cup \{j : z_{j,\tau} = x_{k,\tau} \text{ for some } k \in D(\tau, t, i)\} \text{ if } \tau \geq t.
\]

That is, \( D(\tau, t, i) \) is the set of players in period \( \tau \) who have observed a player who has observed... player \( i \) since time \( t \). The set is important because \( j \in D(\tau, t, i) \) is a necessary condition for player \( j \)'s time \( \tau \) history to vary with player \( i \)'s actions at times after \( t \). In particular, if players are using grim trigger strategies and player \( i \) defects at time \( t \), then \( D(\tau, t, i) \) is the set of players who defect at time \( \tau \). Note that the probability distribution of \( D(\tau, t, i) \) is the same as the probability distribution of \( D(\tau - t, i) \equiv D(\tau - t, 0, i) \), for all \( i \) and \( \tau \geq t \).

I now state the main result of the paper.

**Theorem 5** There is a grim trigger strategy profile \( \sigma^* \) that sustains each player’s maximum cooperation. Furthermore, \( (x^*_i)_{i=1}^N \) is the (component-wise) greatest vector \( (x_i)_{i=1}^N \) such that

\[
x_i = (1 - \delta) \sum_{i=0}^{\infty} \delta^i \sum_{j \neq i} \Pr(j \in D(t, i)) f_{i,j}(x_j) \text{ for all } i \in N. \tag{35}
\]

Theorem 5 is intuitive: one’s first thought might be that grim trigger strategies sustain each player’s maximum cooperation. In addition, given that grim trigger strategies sustain each player’s maximum cooperation, equation (35) is almost immediate: the left-hand side of (35) is the cost to player \( i \) from conforming to \( \sigma^* \); and the right-hand side of (35) is the benefit to player \( i \) from conforming to \( \sigma^* \), which is that, if player \( i \) deviated, she would lose her benefit from player \( j \)'s cooperation whenever \( j \in D(t, i) \). Thus, (35) states that the vector of maximum cooperations is the highest vector of cooperations that equalizes the cost and benefit of cooperation for each player. In addition, it is easy to compute the vector \( (x^*_i)_{i=1}^N \), as discussed in Footnote 64 in Appendix B.

However, Theorem 5 is not true without the assumptions that player \( i \)'s utility function is separable in \( (x_1, \ldots, x_N) \) and that each player observes the realized monitoring network
at the end of each period, as shown by Examples A1 and A2 in Appendix A. It is not very surprising that an assumption on utility functions like separability is required; for example, if the players' actions are substitutes, greater cooperation may be sustainable when players take turns cooperating (as in Example A1). In contrast, the role of the assumption that players observe the realized monitoring network is quite subtle and is discussed below.

The key idea behind Theorem 5 is that a player is willing to cooperate (weakly) more at any on-path history if any other player cooperates more at any on-path history, because the first player is more likely to benefit from this increased cooperation when she conforms than when she deviates. Thus, there is a kind of strategic complementarity between the actions of any two players at any two on-path histories. This suggests the following "proof" of Theorem 5: Define a function $\phi$ that maps the vector of all players’ on-path actions at every on-path history, $\tilde{x}$, to the vector of the highest actions that each player is willing to take at each on-path history when actions at all other on-path histories are as in $\tilde{x}$, and players shirk at off-path histories. Let $\tilde{X}$ be an action greater than any SE action, and let $\tilde{X}$ be the vector of on-path actions $\tilde{X}$. By complementarity among on-path actions, iterating $\phi$ on $\tilde{X}$ yields a sequence of vectors of (on-path) actions that are all constant across periods and weakly greater than the greatest fixed point of $\phi$, and this sequence converges monotonically to the greatest fixed point of $\phi$. Therefore, the greatest fixed point of $\phi$ is constant across periods, and it provides an upper bound on each player’s maximum cooperation. Finally, verify that the grim trigger strategy profile with on-path actions given by the greatest fixed point of $\phi$ is a SE.\footnote{For this last step, one might be concerned that grim trigger strategies do not satisfy off-path incentive constraints, as a player might want to cooperate off-path in order to slow the "contagion" of defecting, as in Kandori (1992) and Ellison (1994). As discussed in Section 3.2, this problem does not arise with continuous actions and payoffs.}

The problem with this "proof" is that, while the highest action that a player is willing to take at any on-path history is non-decreasing in every other player’s on-path actions, it is decreasing in her own future on-path actions. Hence, the function $\phi$ as defined in the previous paragraph is not isotone, and thus may not have a greatest fixed point. This problem may be addressed by working not with players’ stage-game actions $\sigma_i(h_i)$, but
rather with their continuation actions, \( X'_i \equiv (1 - \delta) \sum_{\tau \geq t} \delta^{\tau-t} \sigma_i (h'_i) \). Indeed, it can be shown that

\[
\mathbb{E} [X'_i | h'_i] \leq \sum_{\tau = t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} \Pr (j \in D (\tau, t, i) \setminus D (\tau - 1, t, i)) f_{i,j} \left( \mathbb{E} [X'_j | h'_i, j \in D (\tau, t, i) \setminus D (\tau - 1, t, i)] \right),
\]

for every player \( i \) and on-path history \( h'_i \); this intuition for this inequality is that, if player \( i \) defects at time \( t \), then player \( j \) starts shirking at time \( \tau \) with probability \( \Pr (j \in D (\tau, t, i) \setminus D (\tau - 1, t, i)) \), and this yields lost benefits of at least \( f_{i,j} \left( \mathbb{E} [X'_j | h'_i, j \in D (\tau, t, i) \setminus D (\tau - 1, t, i)] \right) \) to player \( i \).

This inequality yields an upper bound on player \( i \)'s expected continuation action, \( \mathbb{E} [X'_i | h'_i] \), in terms of her expectation of other players' continuation actions only. This raises the possibility that the function \( \phi \) could be isotone when defined in terms of continuation actions \( X'_i \), rather than stage-game actions. For an approach along these lines to work, however, one must be able to express \( \mathbb{E} [X'_j | j \in D (\tau, t, i) \setminus D (\tau - 1, t, i)] \) in terms of \( \mathbb{E} [X'_j | h'_j] \) for player \( j \)'s private histories \( h'_j \). By the assumption that past realizations of the monitoring network are observable,

\[
\mathbb{E} [X'_j | j \in D (\tau, t, i) \setminus D (\tau - 1, t, i)] = \mathbb{E} \left[ \mathbb{E} [X'_j | h'_j] | j \in D (\tau, t, i) \setminus D (\tau - 1, t, i) \right],
\]

so such an approach is possible. However, if past realizations of the monitoring network are unobservable, this may be impossible. The problem is that, if past realizations of the monitoring network are unobservable, disagreement between players \( i \) and \( j \) about player \( j \)'s continuation action in states of the world where \( j \in D (\tau, t, i) \) (but player \( j \) is not aware of this fact) may be leveraged to provide stronger incentives for player \( i \) to cooperate at time \( t \) without increasing player \( j \)'s expected continuation action at time \( \tau \) (from his perspective) by so much that he prefers to shirk. This is illustrated by Example A2 in Appendix A, which shows that a player's maximum cooperation may not be sustainable in grim trigger strategies when past realizations of the monitoring network are unobservable.
3.5 Equal Monitoring

This section imposes the assumption that all players’ actions are equally well-monitored in a sense that leads to sharp comparative statics results. In particular, assume throughout this section:

- **Parallel Benefit Functions:** There exists a function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) and scalars \( \alpha_{i,j} \in \mathbb{R}^+ \) such that \( f_{i,j} (x) = \alpha_{i,j} f(x) \) for all \( i, j \in N \) and all \( x \in \mathbb{R}^+ \).

- **Equal Monitoring:**
  \[
  \sum_{t=0}^{\infty} \delta^t \sum_{k \neq i} \Pr(k \in D(t,i)) \alpha_{i,k} = \sum_{t=0}^{\infty} \delta^t \sum_{k \neq j} \Pr(k \in D(t,j)) \alpha_{j,k}
  \]
  for all \( i, j \in N \).

Parallel benefit functions imply that the importance of player \( j \)'s cooperation to player \( i \) may be summarized by a real number \( \alpha_{i,j} \). With this assumption, equal monitoring states that the expected discounted number of players who may be influenced by player \( i \)'s action, weighted by the importance of their actions to player \( i \), equals the expected discounted number of players who may be influenced by player \( j \)'s action, weighted by the importance of their actions to player \( j \), for all \( i, j \in N \). To help interpret these assumptions, note that if \( \alpha_{i,j} \) is constant across players \( i \) and \( j \) then, for generic discount factors \( \delta \), equal monitoring holds if and only if \( \mathbb{E}[\#D(t,i)] = \mathbb{E}[\#D(t,j)] \) for all \( i, j \in N \) and \( t \in N \); that is, if and only if the expected number of players who find out about a defection by player \( i \) within \( t \) periods equals the expected number of players who find out about a defection by player \( j \) within \( t \) periods.

Section 3.5.1 derives a simple and general formula for comparative statics on maximum cooperation under equal monitoring. Sections 3.5.2 and 3.5.3 apply this formula to the leading special case of (global) public good provision, where \( \alpha_{i,j} = \alpha \) for all \( i \neq j \); that is, where all players value each other’s actions equally. Section 3.5.2 studies the effect of group size on public good provision, and Section 3.5.3 considers the effect of “uncertainty” in monitoring on public good provision.

3.5.1 Comparative Statics Under Equal Monitoring

The section derives a formula for comparative statics on maximum cooperation under equal monitoring. The first step is noting that each player’s maximum cooperation is the same
under equal monitoring.

**Corollary 5** With equal monitoring, \( x^*_i = x^*_j \) for all \( i, j \in N \).

Thus, under equal monitoring each player has the same maximum cooperation \( x^* \). I wish to characterize when \( x^* \) is higher in one game than another, when both games satisfy equal monitoring and have the same underlying benefit function \( f \). Formally, a *game with equal monitoring* \( \Gamma = \left( N, (\alpha_{i,j})_{(i,j) \in N \times N}, \delta, \mu \right) \) is a model satisfying the assumptions of Section 3.3 as well as equal monitoring. For any game with equal monitoring \( \Gamma \), let \( x^*(\Gamma) \) be the maximum cooperation in \( \Gamma \), and let

$$ B(\Gamma) \equiv (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{j \neq i} \Pr(j \in D(t, i)) \alpha_{i,j} $$

be player \( i \)'s benefit of cooperation (i.e., the right-hand side of (35)) when \( f(x_j) = 1 \) for all \( j \in N \), which is independent of the choice of \( i \in N \) by equal monitoring. The comparative statics result for games with equal monitoring is the following:

**Theorem 6** Let \( \Gamma' \) and \( \Gamma \) be two games with equal monitoring. Then \( x^*(\Gamma') \geq x^*(\Gamma) \) if \( B(\Gamma') \geq B(\Gamma) \), with strict inequality if \( B(\Gamma') > B(\Gamma) \) and \( x^*(\Gamma') > 0 \).

**Proof.** Since \( x^*_i = x^* \) for all \( i \in N \), (35) may be rewritten as

$$ x^* = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{j \neq i} \Pr(j \in D(t, i)) \alpha_{i,j} f(x^*) = B(\Gamma) f(x^*). $$

Hence, \( x^*(\Gamma) \) is the greatest zero of the concave function \( B(\Gamma) f(x) - x \). If \( B(\Gamma') \geq B(\Gamma) \), then \( B(\Gamma') f(x^*(\Gamma)) - x^*(\Gamma) \geq B(\Gamma) f(x^*(\Gamma)) - x^*(\Gamma) = 0 \), which implies that \( x^*(\Gamma') \geq x^*(\Gamma) \). If \( B(\Gamma') > B(\Gamma) \) and \( x^*(\Gamma') > 0 \), then either \( x^*(\Gamma) = 0 \) (in which case \( x^*(\Gamma') > x^*(\Gamma) \) trivially) or \( x^*(\Gamma) > 0 \), in which case \( B(\Gamma') f(x^*(\Gamma)) - x^*(\Gamma) > B(\Gamma) f(x^*(\Gamma)) - x^*(\Gamma) = 0 \), which implies that \( x^*(\Gamma') > x^*(\Gamma) \). 

Theorem 6 gives a complete characterization of when \( x^*(\Gamma) \) is greater or less than \( x^*(\Gamma') \), for any two games with equal monitoring \( \Gamma \) and \( \Gamma' \). In particular, maximum cooperation is higher when the expected discounted number of players who may be influenced by a player's
action, weighted by the importance of their actions to that player, is higher. For example, in the case of global public good provision (where all players value all other players’ actions equally), maximum cooperation is greater when the sets \( D(t, i) \) are likely to be larger; while if each player only values the actions of a subset of the other players (e.g., the other members of her household, office, or local market), then maximum cooperation is greater when, for each player \( i \), the intersection of the sets \( D(t, i) \) and the set of players whose actions she values is likely to be larger. Hence, Theorem 6 characterizes how different monitoring technologies sustain different kinds of cooperative behaviors.

### 3.5.2 The Effect of Group Size on Global Public Good Provision

This section uses Theorem 6 to analyze the effect of group size on maximum cooperation in the leading special case of global public good provision, where \( \alpha_{i,j} = \alpha \) for all \( i \neq j \). I first discuss general considerations and then present examples.

In the case of (global) public good provision,

\[
B(\Gamma) = \alpha \sum_{i=0}^{\infty} \delta^t \left( \mathbb{E}[\#D(t, i)] - 1 \right).
\]

Thus, for public goods, all the information needed to determine whether changing the game increases or decreases the maximum per capita level of public good provision is contained in the product of two terms: the marginal benefit to each player of public good provision, \( \alpha \), and (one less than) the effective contagiousness of the monitoring technology, \( \sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t, i)] \). Information such as group size, higher moments of the distribution of \( \#D(t, i) \), and which players are more likely to observe which other players are all irrelevant. In particular, the single number \( \sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t, i)] \)—the effective contagiousness—completely determines the effectiveness of a monitoring technology in supporting public good provision.

This finding that comparative statics on the per-capita level of public good provision are determined by the product of the marginal benefit of the public good to each player and the effective contagiousness of the monitoring technology yields useful intuitions about the effect of group size on the per capita level of public good provision. In particular, index
a game \( \Gamma \) by its group size, \( N \), and write \( \alpha (N) \) for the corresponding marginal benefit of contributions and \( \sum_{t=0}^{\infty} \delta^t \mathbb{E}[\# D(t, N)] \) \( \text{for the effective contagiousness (I use this simpler notation for the remainder of this section).} \) Normally, one would expect \( \alpha (N) \) to be decreasing in \( N \) (a larger population reduces player \( i \)'s benefit from player \( j \)'s contribution to the public good) and \( \sum_{t=0}^{\infty} \delta^t \mathbb{E}[\# D(t, N)] \) to be increasing in \( N \) (a larger population makes it more likely that player \( i \)'s action is observed by more individuals), yielding a tradeoff between the marginal benefit of contributions and the effective contagiousness. Consider again the example of constructing a local infrastructure project, like a well. In this case, \( \alpha (N) \) is likely to be decreasing and concave: since each individual uses the well only occasionally, there are few externalities among the first few individuals, but eventually it starts to become difficult to find times when the well is available, water shortages become a problem, etc.. Similarly, \( \sum_{t=0}^{\infty} \delta^t \mathbb{E}[\# D(t, N)] \) is likely to be increasing, and may be concave if there are “congestion” effects in monitoring. Thus, it seems likely that in typical applications \( \alpha (N) \sum_{t=0}^{\infty} \delta^t (\mathbb{E}[\# D(t, N)] - 1) \), and therefore per capita public good provision, is maximized at an intermediate value of \( N \).

Theorem 6 yields particularly simple comparative statics for the leading cases of pure public goods (\( \alpha (N) = 1 \)) and divisible public goods (\( \alpha (N) = 1/N \)), which are useful in examples below.

**Corollary 6** With pure public goods (\( \alpha (N) = 1 \)), if \( \mathbb{E}[\# D(t, N')] \geq \mathbb{E}[\# D(t, N)] \) for all \( t \) then \( x^*(N') \geq x^*(N) \), with strict inequality if \( \mathbb{E}[\# D(t, N')] > \mathbb{E}[\# D(t, N)] \) for some \( t \geq 1 \) and \( x^*(N') > 0 \).

With pure public goods, \( x^*(N) \) is increasing unless monitoring degrades so quickly as \( N \) increases that the expected number of players who find out about a deviation within \( t \) periods is decreasing in \( N \), for some \( t \). This suggests that \( x^*(N) \) is increasing in \( N \) in many applications.

**Corollary 7** With divisible public goods (\( \alpha (N) = 1/N \)), if \( \mathbb{E}[\# D(t, N')] / N' \geq \mathbb{E}[\# D(t, N)] / N \) for all \( t \) then \( x^*(N') \geq x^*(N) \), with strict inequality if \( \mathbb{E}[\# D(t, N')] / N' > \mathbb{E}[\# D(t, N)] / N \) for some \( t \geq 1 \) and \( x^*(N') > 0 \).
With divisible public goods, \( x^* (N) \) is increasing only if the expected fraction of players who find out about a deviation within \( t \) periods is non-decreasing in \( N \), for all \( t \). This suggests that, with divisible public goods, \( x^* (N) \) is decreasing in many applications.

The following two examples demonstrate the usefulness of Theorem 6 and Corollaries 6 and 7. An earlier version of this paper (available upon request) contains additional examples.

**Random Matching** Monitoring is *random matching* if in each period each player is linked with one other player at random, and \( l_{i,j,t} = l_{j,i,t} \) for all \( i, j \in N \) and all \( t \). This is possible only if \( N \) is even.

It can be shown that, with random matching, \( E[\#D(t,N)] \) is non-decreasing in \( N \) and is increasing in \( N \) for \( t = 2 \). Therefore, Corollary 6 implies that, with pure public goods, maximum cooperation is increasing in group size.

**Proposition 6** With random matching and pure public goods, if \( N' > N \) then \( x^* (N') \geq x^* (N) \), with strict inequality if \( x^* (N') > 0 \).

However, it can also be shown that \( \sum_{t=0}^{\infty} \delta^t E[\#D(t,N')] / N' < \sum_{t=0}^{\infty} \delta^t E[\#D(t,N)] / N \) whenever \( N' > N \), \( N' \) and \( N \) are sufficiently large, and \( \delta < 1/2 \). In this case, Theorem 6 implies that, with divisible public goods, maximum cooperation is decreasing in group size.

**Proposition 7** With random matching and divisible public goods, if \( \delta < \frac{1}{2} \) then, for any \( \gamma > 0 \), there exists \( \bar{N} \) such that \( x^* (N') \leq x^* (N) \) if \( N' > (1 + \gamma) N \geq \bar{N} \), with strict inequality if \( x^* (N') > 0 \).

**Monitoring on a Circle** Monitoring is *on a circle* if the players are arranged in a fixed circle and there exists an integer \( k \geq 1 \) such that \( l_{i,j,t} = 1 \) if and only if the distance between \( i \) and \( j \) is at most \( k \).

It is a straightforward consequence of Corollary 6 that maximum cooperation is increasing in group size with monitoring on a circle and pure public goods.

**Proposition 8** With monitoring on a circle and pure public goods, if \( N' > N \) then \( x^* (N') \geq x^* (N) \), with strict inequality if \( x^* (N') > 0 \).
Proof. $\mathbb{E}[\#D(t,N)] = \min \{1 + 2kt, N\}$, so $N' > N$ implies that $\mathbb{E}[\#D(t,N')] \geq \mathbb{E}[\#D(t,N)]$ for all $t$, and $\mathbb{E}[\#D(t,N')] > \mathbb{E}[\#D(t,N)]$ for $t = N'$. The result follows from Corollary 6. ■

Finally, Corollary 7 implies that maximum cooperation is decreasing in group size with monitoring on a circle and divisible public goods.

**Proposition 9** With monitoring on a circle and divisible public goods, if $N' > N$ then $x^*(N') \leq x^*(N)$, with strict inequality if $k < (N' - 1) / 2$ and $x^*(N') > 0$.

**Proof.** $\mathbb{E}[\#D(t,N)] / N = \min \{(1 + 2kt) / N, 1\}$, so $N' > N$ implies that $\mathbb{E}[\#D(t,N')] \leq \mathbb{E}[\#D(t,N)]$ for all $t$, and (if $k < (N' - 1) / 2$) $\mathbb{E}[\#D(t,N')] < \mathbb{E}[\#D(t,N)]$ for $t = 1$. The result follows from Corollary 7. ■

### 3.5.3 The Effect of Uncertain Monitoring on Global Public Good Provision

This section provides a result comparing monitoring technologies in terms of the maximum level of global public good provision they support, for a fixed group size. As discussed in the previous subsection, a monitoring technology supports greater maximum cooperation in global public good provision if and only if it has greater expected contagiousness, $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t)]$, where the parameter $N$ is omitted because it is held fixed in this subsection. I compare “less certain” monitoring, where it is likely that either a large or small fraction about the population finds out about a deviation, with “more certain” monitoring, where it is likely that an intermediate fraction of the population finds out about it, in the sense of second-order stochastic dominance. Under fairly broad conditions, more certain monitoring supports greater maximum cooperation.

The analysis of this subsection relies on the following assumption, which states that the distribution over $\#D(t + 1)$ depends only on $\#D(t)$.

- There exists a family of functions $\{g_k : \{0, \ldots, N\} \to [0,1]\}_{k=1}^{N}$ with such that, whenever $\#D(t) = k$, $\Pr(\#D(t + 1) = k') = g_k(k')$, for all $t$, $k$, and $k'$.

This assumption is satisfied by random matching, for example, but not by monitoring on a circle, because with monitoring on a circle the distribution of $\#D(t + 1)$ depends on the identities of the of the members of $D(t)$. 

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Given a probability mass function \( g_k \), define the corresponding distribution function \( G_k(k') \equiv \sum_{s=0}^{k'} g_k(s) \). Recall that a distribution \( \tilde{G}_k \) strictly second-order stochastically dominates \( G_k \) if \( \sum_{s=0}^N \eta(s) \tilde{g}_k(s) > \sum_{s=0}^N \eta(s) g_k(s) \) for all increasing and strictly concave functions \( \eta : \mathbb{R} \to \mathbb{R} \). The following result compares monitoring under \( \{ g_k \}_{k=1}^N \) and \( \{ \tilde{g}_k \}_{k=1}^N \).

**Theorem 7** Suppose that \( \tilde{G}_k(k') \) and \( G_k(k') \) are decreasing and strictly convex in \( k \) for \( k \in \{0, \ldots, k'\} \) and \( k' \in \{0, \ldots, N\} \), and that \( \tilde{G}_k \) strictly second-order stochastically dominates \( G_k \) for \( k \in \{1, \ldots, N-1\} \). Then maximum cooperation is strictly greater under a monitoring technology corresponding to \( \{ \tilde{g}_k(\cdot) \}_{k=1}^N \) than under a monitoring technology corresponding to \( \{ g_k(\cdot) \}_{k=1}^N \).

The condition that \( G_k(k') \) is decreasing and convex in \( k \) means that, as the number of defectors in period \( t \) (or, more generally, \( #D(t) \)) increases, the probability that there are fewer than \( k' \) defectors in period \( t+1 \) decreases at a decreasing rate. The condition that \( \tilde{G}_k \) strictly second-order stochastically dominates \( G_k \) means that, for any number of defectors \( k \) in period \( t \) (other than 0 or \( N \)), the distribution of the number of defectors in period \( t+1 \) under \( \tilde{G}_k \) strictly second-order stochastically dominates the number of defectors in period \( t+1 \) under \( G_k \).

The intuition for Theorem 7 is fairly simple: If \( \tilde{G}_k \) strictly second-order stochastically dominates \( G_k \) for all \( k \), then under \( \tilde{G}_k \) it is more likely that an intermediate number of players find out about an initial deviation each period. Since \( G_k(k') \) and \( \tilde{G}_k(k') \) are decreasing and convex in \( k \), the expected number of players who find out about the deviation within \( t \) periods increases in \( t \) more quickly when it is more likely that an intermediate number of players find out about the deviation each period. Hence, \( \sum_{t=0}^\infty \delta^t \mathbb{E} [ #D(t) ] \) is strictly higher under a monitoring technology corresponding to \( \{ \tilde{g}_k(\cdot) \}_{k=1}^N \) than under a monitoring technology corresponding to \( \{ g_k(\cdot) \}_{k=1}^N \), and the theorem then follows from Theorem 6.

### 3.6 Fixed Monitoring Networks

This section studies both global and local public good provision with network monitoring when the monitoring network is fixed over time. That is, throughout this section I make the following assumption on the (deterministic) monitoring technology.
• **Fixed Undirected Monitoring Network:** There exists a network \( L = (l_{i,j})_{(i,j) \in N \times N} \) such that \( l_{i,j,t} = l_{i,j} = l_{j,i} \) for all \( t \).

I also assume that the stage game satisfies one of the following two properties, where \( N(i) \) is the set of player \( i \)'s neighbors in \( L \).

- **Global Public Goods:** \( u_i(x) = \left( \sum_{j \neq i} f(x_j) \right) - x_i \).
- **Local Public Goods:** \( u_i(x) = \left( \sum_{j \in N(i)} f(x_j) \right) - x_i \).

The extensions of all of the results in this section to directed networks is straightforward. I discuss below where the assumption of global or local public goods can be relaxed.

Section 3.6.1 introduces a new definition of centrality in networks, and uses Theorem 5 to show that more central players have greater maximum cooperation. This result not only allows for comparisons among players in a given network but is also the key tool for comparing maximum cooperation across networks. In particular, Section 3.6.2 shows that if player \( i' \) in network \( L' \) is more central than player \( i \) in (connected) network \( L \), then for every player \( j \) in network \( L \) there exists a corresponding player \( j' \) in network \( L' \) such that player \( j' \)'s maximum cooperation is greater than player \( j \)'s; in this sense, network \( L' \) “dominates” network \( L \) in terms of supporting cooperation. Finally, Section 3.6.3 remarks on the stability of monitoring networks, emphasizing differences between the cases of global and local public goods.

### 3.6.1 Centrality and Maximum Cooperation

Theorem 5 provides a general characterization of players’ maximum cooperation as a function of the discount factor and benefit functions. Here, my goal is to provide a partial ordering (“centrality”) of players in terms of their network characteristics under which higher players have greater maximum cooperation for all discount factors and benefit functions. Intuitively, player \( i \) is “more central” than player \( j \) if \( i \) has more neighbors (within distance \( t \), for all \( t \in \mathbb{N} \)) than \( j \), \( i \)'s neighbors have more neighbors than \( j \)'s neighbors, \( i \)'s neighbors’ neighbors have more neighbors than \( j \)'s neighbors’ neighbors, and so on. Formally, let \( d(i,j) \) be the distance (shortest path length) between players \( i \) and \( j \), with \( d(i,j) \equiv \infty \) if there is no path between \( i \) and \( j \). The definition of centrality is the following.
Definition 15  Player $i$ is 1-more central than player $j$ if, for all $t \in \mathbb{N}$, $\# \{k \in N : d(i, k) \leq t\} \geq \# \{k \in N : d(j, k) \leq t\}$. Player $i$ is strictly 1-more central than player $j$ if in addition $\# \{k \in N : d(i, k) \leq t\} > \# \{k \in N : d(j, k) \leq t\}$ for some $t$.

For all integers $s \geq 2$, player $i$ is $s$-more central than player $j$ if, for all $t \in \mathbb{N}$, there exists a surjection $\psi : \{k \in N : d(i, k) \leq t\} \rightarrow \{k \in N : d(j, k) \leq t\}$ such that, for all $k$ with $d(j, k) \leq t$, there exists $k' \in \psi^{-1}(k)$ such that $k'$ is $s-1$-more central than $k$. Player $i$ is strictly $s$-more central than player $j$ if in addition $k'$ is strictly $s-1$-more central than $k$ for some $t$, $\psi$, $k$, and $k'$.

Player $i$ is more central than player $j$ if $i$ is $s$-more central than $j$ for all $s \in \mathbb{N}$. Player $i$ is strictly more central than player $j$ if in addition $i$ is strictly $s$-more central than $j$ for some $s \in \mathbb{N}$.

As a first example, consider five players arranged in a line (Figure 4). Player 3 is strictly more central than players 2 and 4, who are in turn strictly more central than players 1 and 5. To see this, note first that player 3 is strictly 1-more central than players 2 and 4, who are in turn each strictly 1-more central than players 1 and 5. For example, player 2 is strictly 1-more central than player 5 because player 2 has 3 neighbors within distance 1 (including player 2 herself), 4 neighbors within distance 2, and 5 neighbors within distance 3 or more; while player 5 has 2 neighbors within distance 1, 3 neighbors within distance 2, 4 neighbors within distance 3, and 5 neighbors within distance 4 or more. Next, suppose that player 3 is $s$-more central than players 2 and 4, and that players 2 and 4 are both $s$-more central than players 1 and 5. Then it is easy to check that player 3 is also $s+1$-more central than players 2 and 4, who in turn are both $s+1$-more central than players 1 and 5; for example, one surjection $\psi : \{k \in N : d(2, k) \leq 2\} \rightarrow \{k \in N : d(5, k) \leq 2\}$ that satisfies the terms of the definition is given by $\psi(1) = \psi(2) = 5$, $\psi(3) = 3$, $\psi(4) = 4$ (noting that a player is always more central than herself, because in this case $\psi$ can be taken to be the identity mapping). Thus, by induction on $s$, player 3 is strictly more central than player 2 and 4, who are in turn strictly more central than players 1 and 5.

The main result of this section states that, with either global or local public goods, more central players have greater maximum cooperation, regardless of the discount factor $\delta$ and benefit function $f$. The result can easily be generalized to allow for utility functions
intermediate between global and local public goods, where a player’s benefit from another player’s action is a decreasing function of the distance between them.\textsuperscript{56} The proof uses a monotonicity argument similar to that in the proof of Theorem 5, which shows that more central players cooperate more at every step of a sequence of vectors of cooperations that converges to the vector of maximum cooperations.

**Theorem 8** With either global or local public goods, if player \(i\) is more central than player \(j\), then \(x_i^* \geq x_j^*\). The inequality is strict if player \(i\) is strictly more central than player \(j\) and \(x_k^* > 0\) for all \(k \in N\).

The proof of the strict inequality in Theorem 8 uses the following important lemma.

**Lemma 7** Player \(i\) is more central than player \(j\) if and only if for all \(t \in \mathbb{N}\) there exists a surjection \(\psi : \{k \in N : d(i,k) \leq t\} \rightarrow \{k \in N : d(j,k) \leq t\}\) such that, for all \(k\) with \(d(j,k) \leq t\), there exists \(k' \in \psi^{-1}(k)\) such that \(k'\) is more central than \(k\).

**Proof of Theorem 8.** I prove the result for global public goods. The proof for local public goods is similar.

Let \(\phi : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N\), be defined as in the proof of Theorem 5; with a fixed monitoring network and global public goods, this simplifies to

\[
\phi_i\left((x_j)_{j=1}^N\right) = \sum_{j \neq i} \delta^{d(i,j)} f(x_j) \text{ for all } i.
\]

As in the proof of Theorem 5, define \(x_i^m\) recursively by \(x_1^1 = \bar{X}\) (a constant defined in Step 1a of the proof of Theorem 5) and \(x_i^{m+1} = \phi_i\left((x_j^m)_{j=1}^N\right)\). The proof of Theorem 5 shows that \(x_i^* = \lim_{m \rightarrow \infty} x_i^m\).

\textsuperscript{56}Formally, Theorem 8 holds whenever there exist a function \(f : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) and constants \(\alpha_d \in \mathbb{R}_+\) such that \(\alpha_d \geq \alpha_{d+1} \geq 0\) for all \(d \in \mathbb{N}\) and \(u_i(x) = \left(\sum_{j \neq i} \alpha_{d(i,j)} f(x_j)\right) - x_i\) for all \(i \in N\).
Suppose that player $i$ is more central than player $j$. I claim that $x^m_i \geq x^m_j$ for all $m \in \mathbb{N}$, which proves the weak inequality. Trivially, $x^1_i = \bar{X} \geq \bar{X} = x^1_j$. Now suppose that $x^m_i \geq x^m_k$ whenever player $k'$ is more central than player $k$, for some $m \in \mathbb{N}$. Since player $i$ is $m+1$-more central than player $j$, for any $t \in \mathbb{N}$ there exists a surjection $\psi : \{k \in N : d(i, k) \leq t\} \to \{k \in N : d(j, k) \leq t\}$ such that, for all $k$ with $d(j, k) \leq t$, there exists $k' \in \psi^{-1}(k)$ such that $k'$ is $m$-more central than $k$. Since $x^m_k \geq x^m_k$, this implies that $\sum_{k : d(i, k) \leq t} f(x^m_k) \geq \sum_{k : d(j, k) \leq t} f(x^m_k)$. This holds for all $t$, which implies that $x^{m+1}_i = (1-\delta) \sum_{t=0}^{\infty} \delta^t \sum_{k : d(i, k) \leq t} f(x^m_k) \geq (1-\delta) \sum_{t=0}^{\infty} \delta^t \sum_{k : d(j, k) \leq t} f(x^m_j) = x^{m+1}_j$. It follows by induction that $x^m_i \geq x^m_j$ for all $m \in \mathbb{N}$.

To prove the strict inequality, suppose that player $i$ is strictly more central than player $j$ and that $x^*_k > 0$ for all $k \in N$. Rewrite (35) as

$$x^*_i = \sum_{j \neq i} \delta^{d(i,j)} f(x^*_j). \quad (36)$$

Suppose that $i$ is more central than $j$ and strictly 1-more central than $j$, let $x^* \equiv \min_k x^*_k$ (which is positive by assumption), and let $\bar{d}$ be the diameter of $L$ (i.e., the maximum distance between any two path-connected nodes in $L$). Then, by Lemma 7 and (36), $x^*_i \geq x^*_j + \delta^{\bar{d}-1} \min \{\delta, 1-\delta\} f(x^*)$, as player $i$ has at least one more distance-$t$ neighbor than player $j$ for some $t \in \mathbb{N}$. Therefore, there exists $\varepsilon_1 > 0$ such that $x^*_i - x^*_j \geq \varepsilon_1 > 0$ whenever $i$ is more central than $j$ and strictly 1-more central than $j$. Now suppose that there exists $\varepsilon_s > 0$ such that $x^*_i - x^*_j \geq \varepsilon_s > 0$ whenever $i$ is more central than $j$ and strictly $s$-more central than $j$. Suppose that $i$ is more central than $j$ and strictly $s+1$-more central than $j$. Then $x^*_i \geq x^*_j + \delta^{\bar{d}-1} \max \{\delta, 1-\delta\} f(\varepsilon_s)$, by Lemma 7 and (36), which implies that there exists $\varepsilon_{s+1} > 0$ such that $x^*_i - x^*_j \geq \varepsilon_{s+1} > 0$. By induction on $s$, it follows that $x^*_i > x^*_j$ whenever $i$ is strictly more central than $j$.

Four remarks on Theorem 8 are in order. First, the conclusion of Theorem 8 would still hold for local public goods (but not global public goods) if the definition of centrality was weakened by specifying that player $i$ is 1-more central than player $j$ whenever $\#N(i) \geq \min \{\delta, 1-\delta\}$ term corresponds to the possibility that player $i$ may have one more distance-$\bar{d}$ neighbor than player $j$, or may have one more distance-$\bar{d}-1$ neighbor and the same number of distance-$\bar{d}$ neighbors.
Thus, players’ maximum cooperations can be ordered for more networks with local public goods than with global public goods. Second, the fixed point equation (36)—which is substantially simpler than the general fixed point equation (35)—orders players’ maximum cooperations for any fixed monitoring network.

Third, Theorem 8 provides a new perspective on the Olsonian idea of the “exploitation of the great by the small.” Olson (1965) notes that small players may free ride on large players if larger players have greater private incentives to contribute to public goods. Theorem 8 illustrates a reason why larger players might be expected to contribute disproportionately much to public goods even if they do not have greater private incentives to contribute: larger players may be more central, in which case they may be punished more effectively for shirking. While this “exploitation” implies that more central players receive lower payoffs than less central players with global public goods, Corollary 11 below implies that more central players receive higher payoffs than less central players with local public goods, because with local public goods the benefit of having more neighbors more than offsets the cost of contributing more.

Fourth, my definition of centrality is related on an intuitive level to existing centrality measures based on players’ neighbors’ characteristics, such as Bonacich centrality (Bonacich, 1987). My definition of centrality is a partial order, as it ranks players in a way that is invariant to the benefit function and discount factor, so a more direct comparison with Bonacich centrality results from comparing players’ maximum cooperation for a fixed benefit function, \( f \), and discount factor, \( \delta \); in this case, \( \delta \) is analogous to the decay factor in the definition of Bonacich centrality, \( \beta \). There are two differences between a player’s maximum cooperation and Bonacich centrality. First, a player’s maximum cooperation depends on other players’ maximum cooperations through the concave function \( f \), while this dependence in linear for Bonacich centrality. An important consequence is that the vector of players’ maximum cooperations is unique, while the vector of players’ Bonacich centralities is determined only up to multiplication by a constant. Second, player \( i \)’s maximum cooperation depends on player \( j \)’s maximum cooperation only through the distance between \( i \) and \( j \), not through the number of paths between \( i \) and \( j \) (as in Bonacich centrality). This reflects the

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58I thank a referee for pointing out this connection.
fact that only the time it takes for player \( j \) to learn about a deviation by player \( i \) is relevant for computing player \( i \)'s maximum cooperation.

For general monitoring networks, it may be difficult to verify that one player is more central than another, making it hard to apply Theorem 8. Sometimes, however, symmetries in the network can be exploited to determine which players are more central than others more easily. The remainder of this section shows how this can be done. Corollary 8 states that, if player \( i \) is closer to all players \( k \neq i, j \) than is player \( j \), then player \( i \) is more central than player \( j \). Corollary 9 shows that if players \( i \) and \( k \) are in “symmetric” positions in the monitoring network (in that there exists a graph automorphism \( \rho \) on \( L \) such that \( k = \rho (i) \)) and player \( k \) is more central than player \( j \), then player \( i \) is more central than player \( j \) as well.\(^5\)

**Corollary 8** If \( d(i, k) \leq d(j, k) \) for all \( k \neq i, j \), then player \( i \) is more central than player \( j \). Player \( i \) is strictly more central than player \( j \) if in addition the inequality is strict for some \( k \neq i, j \).

**Corollary 9** If there exists a graph automorphism \( \rho : N \to N \) such that \( \rho (i) \) is more central (resp., strictly more central) than \( j \), then \( i \) is more central (resp., strictly more central) than \( j \).

The example in Figure 5 illustrates the usefulness of Corollaries 8 and 9.\(^6\) First, Corollary 8 immediately implies that player 3 is more central than players 1 and 2, and that player 5 is more central than players 6 and 7. Second, observe that the following map \( \rho \) is an automorphism of \( L \): \( \rho (1) = 7, \rho (2) = 6, \rho (3) = 5, \rho (4) = 4, \rho (5) = 3, \rho (6) = 2, \) and \( \rho (7) = 1 \). Thus, Corollary 9 implies that each player in \( \{3, 5\} \) is more central than each player in \( \{1, 2, 6, 7\} \). Given this observation, it is not hard to show that player 4 is more central than each player in \( \{1, 2, 6, 7\} \). Finally, neither of players 3 and 4 are more central

\(^{5}\)A graph automorphism on \( L \) is a permutation \( \rho \) on \( N \) such that \( b_{i,j} = b_{\rho (i), \rho (j)} \) for all \( i, j \in N \). That is, a graph automorphism is a permutation of vertices that preserves links.

\(^{6}\)This example is the same as that in Figure 2.13 of Jackson (2008), which Jackson uses to illustrate various network-theoretic concepts of centrality. As shown in the text, my definition of centrality is similar to the concepts discussed by Jackson in that players 3, 4, and 5 are all more central than players 1, 2, 6, and 7. One important difference between my definition and those discussed by Jackson is that my definition gives a partial order on nodes, while all the definitions discussed by Jackson give total orders.
than the other, as player 3 has more immediate neighbors while player 4 has more neighbors within distance 2. Therefore, Theorem 8 does not say whether player 3 or player 4 has a higher maximum cooperation. This is reassuring, because one can easily construct examples in which $x_3^* > x_4^*$ and others in which the reverse inequality holds: for example, if $f(x) = \sqrt{x}$ (with global public goods), then $x_1^* \approx 2.638$, $x_3^* \approx 3.425$, and $x_4^* \approx 3.475$ if $\delta = .5$, whereas if $\delta = .4$ then $x_1^* \approx 1.378$, $x_3^* \approx 1.849$, and $x_4^* \approx 1.839$. Indeed, it is not surprising that player 3 contributes more relative to player 4 when $\delta$ is lower, as in this case the fact that player 3 has more immediate neighbors is more important, while player 4’s greater number of distance-2 neighbors matters more when $\delta$ is higher (since $\delta^2$ is low relative to $\delta$ when $\delta$ is low). However, there are networks in which a player $i$ is not more central than player $j$ but nonetheless $x_i^* \geq x_j^*$ for every concave benefit function $f$ and discount factor $\delta$, as shown by Example A3 in Appendix A.

### 3.6.2 Comparing Networks

This section shows that centrality is a key tool for comparing different networks in terms of their capacity to support cooperation, not just for comparing individuals within a fixed network. To see this, note that the “more central” relation can be immediately extended to pairs of players in different (connected) networks $L'$ and $L$ by specifying that player $i' \in L'$ is more central than player $i \in L$ if player $i'$ is more central than player $i$ in the network consisting of disjoint components $L'$ and $L$. With this definition, the result is the following.
Theorem 9 For any network $L'$ and connected network $L$, if there exists players $i' \in L'$ and $i \in L$ such that player $i'$ is more central than player $i$, then there exists a surjection $\psi : L' \to L$ such that, for all $j \in L$, there exists $j' \in \psi^{-1}(j)$ such that $x^*_j \geq x^*_{j'}$.

Proof. Let $\bar{d}$ be the diameter of $L$. Since player $i'$ is more central than player $i$, Lemma 7 implies that there exists a surjection $\psi : \{j \in L' : d(i', j) \leq \bar{d}\} \to \{\{j \in L : d(i, j) \leq \bar{d}\}\}$ such that, for all $j$ with $d(i, j) \leq \bar{d}$, there exists $j' \in \psi^{-1}(j)$ such that $j'$ is more central than $j$. By Theorem 8, $x^*_j \geq x^*_{j'}$ for any such $j'$ and $j$. Finally, $\{j \in L : d(i, j) \leq \bar{d}\} = L$, by definition of $\bar{d}$ and the assumption that $L$ is connected.

It is easy to see that Theorem 9 applies if $L' \supseteq L$, in which case any surjection $\psi : L' \to L$ such that $\psi(i) = i$ for all $i \in L$ satisfies the condition of the theorem. This implies the following corollary, which formalizes in a natural way the widespread idea that better-connected societies can provide more public goods.\textsuperscript{61}

Corollary 10 Adding links to a network weakly increases each player’s maximum cooperation.

However, Theorem 9 is much more general than this. For example, if $L'$ is a circle with $N'$ nodes and $L$ is a circle with $N$ nodes, then Theorem 9 applies whenever $N' \geq N$. Similarly, if $L'$ is a symmetric graph of degree $k'$ on $N$ nodes and $L$ is a symmetric graph of degree $k$ on $N$ nodes, then Theorem 9 applies whenever $k' \geq k$. Finally, the example in Figure 6 shows that Theorem 9 can even apply if $L'$ and $L$ have the same number of nodes and the same number of links (here, six and seven, respectively), because a simple application of Lemma 7 and Corollary 8 shows that players 1, 2, 5, and 6 are more central than players 7, 8, 11, and 12, and that players 3 and 4 are more central than players 9 and 10.

3.6.3 Network Stability

This section briefly considers the implications of allowing players to sever links in the monitoring network before the beginning of play. I assume that the resulting equilibrium involves

\textsuperscript{61}An earlier version of this paper proves that adding a link to a network strictly increases the maximum cooperation of every player in the same component as the added link, if the maximum cooperation of every such player is strictly positive.
each player making her maximum contribution with respect to the remaining monitoring network. I show that, with local public good provision, no player ever has an incentive to sever a link, but that this is not true with global public good provision. Given that adding any link to a monitoring network increases all players’ maximum cooperation (by Corollary 10), these results suggest that it may be easier to sustain monitoring networks that support large maximum cooperation with local public goods than with global public goods.

With local public goods, every player is made worse-off when any link in the monitoring network is severed. This implies that any monitoring network is stable, in that no individual can benefit from severing a link; if players can also add links, then only the complete network is stable. Note that a less restrictive definition of local public goods is needed for this result.

**Corollary 11** Suppose that $L'$ contains more links than $L$. If $u_i\left(\left(x_j\right)_{j=1}^{N}\right) = \left(\sum_{j \in N(i)} f_{i,j}(x_j)\right) - x_i$ for all $i \in N$, then every player $i$’s payoff when all players make their maximum contributions is weakly greater with monitoring network $L'$ than with monitoring network $L$.

**Proof.** Note that (35) simplifies to $x_i^* = \delta \sum_{j \in N(i)} f_{i,j}(x_j^*)$. Therefore,

$$u_i\left(\left(x_j^*\right)_{j=1}^{N}\right) = \left(\sum_{j \in N(i)} f_{i,j}(x_j^*)\right) - x_i^* = (1 - \delta) \sum_{j \in N(i)} f_{i,j}(x_j^*).$$  \hspace{1cm} (37)

The set $N(i)$ is weakly larger in $L'$ than in $L$ (in the set-inclusion sense), and by Corollary 10 every player’s maximum cooperation is weakly greater with monitoring network $L'$ than with monitoring network $L$. Hence, the result follows from (37).
Corollary 11 does not hold with global public goods. The key difference between global and local public goods is that with global public goods a player can benefit from another player’s cooperation even if she is not observed by the other player, and in this case her own maximum cooperation is lower. Formally, with global public goods (37) becomes

\[ u_i \left( \left( x_j^* \right)_{j=1}^N \right) = \sum_{j \neq i} \left( 1 - \delta^{d(i,j)} \right) f_{i,j}(x_j^*) . \]

This equation clarifies the tradeoff player \( i \) faces when deciding whether to sever a link with player \( j \): severing the link increases \( d(i,k) \) for some players \( k \in N \), which increases \( u_i \left( \left( x_j^* \right)_{j=1}^N \right) \) (by reducing player \( i \)'s maximum cooperation, \( x_i^* \)), but also decreases \( x_k^* \) for some players \( k \in N \), which decreases \( u_i \left( \left( x_j^* \right)_{j=1}^N \right) \). It is easy to construct examples where the first effect dominates (e.g., three players arranged in a line with \( f(x) = x^2 \) and \( \delta = .8 \)).

### 3.7 Conclusion

This paper studies repeated cooperation games with network monitoring and provides comparative statics on the maximum equilibrium level of cooperation with respect to group size and structure. The key theorem, which underlies all the results in the paper, is that maximum cooperation can be sustained in grim trigger strategies. This theorem is driven by the strategic complementarity, in repeated cooperation games with network monitoring, between any two player’s actions at any two on-path histories. With equal monitoring, maximum cooperation is typically increasing in group size with pure public goods and decreasing in group size with divisible public goods; in general, comparative statics on maximum cooperation depend on the product of the marginal benefit of cooperation and the effective contagiousness of the monitoring technology. Less uncertain monitoring, which in some cases may be interpreted as reliable local monitoring rather than unreliable public monitoring, sustains greater maximum cooperation. With a fixed monitoring network, more central players have greater maximum cooperation. In addition, all players have greater maximum cooperation when the network is better connected, though better connected networks are more likely to be stable with local public goods than with global public goods.

I conclude by discussing directions for future research. First, my analysis of optimal
equilibria with network monitoring may facilitate further investigations of the relationship between public and private monitoring as means of supporting cooperation. In many economic examples, it seems likely that only extremely weak incentives can be provided by public monitoring, but this intuition is not captured clearly by existing models of repeated games with imperfect public monitoring. A model in which players learn about each other’s actions through both network monitoring and imperfect public monitoring could clarify the extent to which large groups are able to avoid the problems associated with public monitoring by relying on local, private monitoring of the kind studied in this paper.

Second, grim trigger strategies are fragile in that one instance of shirking eventually leads to the complete breakdown of cooperation. This is especially problematic in (realistic) cases where the cost of cooperation is stochastic and is sometimes prohibitively high. Hence, extending the model to allow for stochastic costs of cooperation is important for deriving more robust predictions about which strategies best sustain cooperation, and this also seems to be an interesting and challenging problem from a theoretical perspective.

Finally, my analysis makes strong predictions about the effects of group size and structure on the level of public good provision, and on how these differ depending on whether the public good is pure or divisible and whether it is global or local. A natural next step would be to study these predictions empirically. This would be easiest to pursue experimentally, but some insight can be gained from recent field data. For example, Karlan et al. (2009) find that indirect network connections between individuals in Peruvian shantytowns support lending and borrowing, consistent with my finding that more central individuals have higher capacities for cooperation. Allcott et al. (2007) provide suggestive evidence that greater network closure may lead US high school students to help each other more both socially and academically; Allcott et al. interpret this evidence as indicating that network closure leads to provision of higher-value public goods, but it can also be interpreted as indicating that network closure leads to provision of more local public goods, which is consistent with my model (interpreting network closure as smaller distance to individuals whose actions one values; cf Theorem 6). Finally, Jackson, Rodriguez-Barraquer, and Tan (2010) find that favor-exchange networks in rural India exhibit high support, the property that linked players share at least one common neighbor. While it seems natural that support (which is the key
determinant of cooperation in Jackson, Rodriguez-Barraquer, and Tan’s model) should be correlated with my measure of maximum cooperation, it would be interesting to study the precise relationship between the two concepts, and in particular to determine which measure better predicts cooperation in which environments.

3.8 Appendix A: Omitted Examples

3.8.1 Example A1: Counterexample to Theorem 5 with Non-Separable Utility

This example shows that a player’s maximum cooperation may not be sustainable in grim trigger strategies if utility is not separable.\(^{62}\) The intuition is that, if players’ actions are substitutes, they may benefit from taking turns cooperating.

There are three players. Monitoring is perfect. Utility functions are

\[
\begin{align*}
    u_1 \left( (x_i)_{i=1}^3 \right) &= .01\sqrt{x_2} + .01\sqrt{x_3} + 1 - \min \{x_2 - 1, 0\} \min \{x_3 - 1, 0\} - x_1 \\
    u_2 \left( (x_i)_{i=1}^3 \right) &= .01\sqrt{x_1} + .01\sqrt{x_3} + 1 - \min \{x_1 - 1, 0\} \min \{x_3 - 1, 0\} - x_2 \\
    u_3 \left( (x_i)_{i=1}^3 \right) &= .01\sqrt{x_1} + .01\sqrt{x_2} + 1 - \min \{x_1 - 1, 0\} \min \{x_2 - 1, 0\} - x_3.
\end{align*}
\]

Note that every player’s utility function is increasing and strictly concave in every other player’s action and that \(u_i(0, 0, 0) = 0\) for all \(i \in N\). Let \(\delta = .8\).

First, consider equilibria in grim trigger strategies. By the same argument as in the case with separable utility, player \(i\)’s maximum cooperation in grim trigger strategies is the greatest value of \(x_i\) such that there exists a vector \((x_i)_{i=1}^3\) satisfying

\[
\begin{align*}
    x_1 &= .8 \left( .01\sqrt{x_2} + .01\sqrt{x_3} + 1 - \min \{x_2 - 1, 0\} \min \{x_3 - 1, 0\} \right) \\
    x_2 &= .8 \left( .01\sqrt{x_1} + .01\sqrt{x_3} + 1 - \min \{x_1 - 1, 0\} \min \{x_3 - 1, 0\} \right) \\
    x_3 &= .8 \left( .01\sqrt{x_1} + .01\sqrt{x_2} + 1 - \min \{x_1 - 1, 0\} \min \{x_2 - 1, 0\} \right).
\end{align*}
\]

It can be easily verified that the solution to this problem involves \(x_1 = x_2 = x_3 < 1\), and computation yields \(x_i \approx .7728\).

\(^{62}\) I thank Gabriel Carroll for help with this example.
Next, consider the following non-grim trigger strategy profile. On-path, player $i$ plays $x_i = 1.099$ in period $t$ if $t \neq i \mod 3$ and plays $x_i = 0$ in period $t$ if $t = i \mod 3$. Thus, each player takes every third period off from cooperating. Each player $i$ plays $x_i = 0$ in every subsequent period if she ever observes a deviation. To verify that this strategy profile is a SE, the key incentive constraint to check is that player $i$ does not prefer to deviate to $x_i = 0$ in a period $t = i + 1 \mod 3$. If she conforms in such a period, her payoff equals

$$\frac{1}{1 - .8^3} \left( .01 \sqrt{1.099} + 1 - 1.099 + .8 \left( .01 \sqrt{1.099} + 1 - 1.099 \right) + .8^2 \left( .01 \sqrt{1.099} + .01 \sqrt{1.099} + 1 \right) \right);$$

while if she deviates to $x_i = 0$, her payoff equals $.01 \sqrt{1.099} + 1$, which is strictly less. Finally, player 1’s equilibrium cooperation in this SE equals

$$\frac{1 - .8}{1 - .8^3} (1.099 + .8 (1.099)) \approx .8107,$$

which is greater than $.7728$, her maximum cooperation in grim trigger strategies.

3.8.2 Example A2: Counterexample to Theorem 5 with Unobserved Monitoring Network

This example shows that a player’s maximum cooperation may not be sustainable in grim trigger strategies if the realized monitoring network is unobserved.

There are three players. Player 1 is observed by player 2 with probability $1/2$ and is never observed by player 3. Players 2 and 3 always observe each other. Player 1 observes nothing. The realized monitoring network (drawn independently every period) is unobserved; in particular, Player 3 does not observe when player 2 observes player 1 and when he does not. For each player $i$, $u_i \left( (x_j)_{j=1}^3 \right) = \left( \sum_{j \neq i} \sqrt{x_j} \right) - x_i$, and $\delta = .5$. I will show that player 1’s maximum cooperation in grim trigger strategies equals $.25$, but that there exists a SE in which player 1’s equilibrium cooperation equals $0.2505$.

First, consider grim trigger strategies. By the same argument as in the case where the monitoring network is observable, player $i$’s maximum cooperation in grim trigger strategies
equals \( x_i \), where \( (x_i)_i^{3} \) is the greatest vector satisfying

\[
x_i = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{j \neq i} \Pr(j \in D(t,i)) \sqrt{x_j} \quad \text{for all } i.
\]

This may be rewritten as

\[
\begin{align*}
x_1 &= \frac{\delta/2}{1 - \delta/2} \sqrt{x_2} + \delta \left( \frac{\delta/2}{1 - \delta/2} \right) \sqrt{x_3} \\
x_2 &= \delta \sqrt{x_3} \\
x_3 &= \delta \sqrt{x_2}.
\end{align*}
\]

Solving this system of equations with \( \delta = .5 \) yields \( x_1 = x_2 = x_3 = .25 \). Hence, player 1’s maximum cooperation in grim trigger strategies equals .25.

Next, consider the following strategy profile. Player 1 plays \( x_1 = .2505 \) in every period. Players 2 and 3 each have two on-path actions, denoted \( x_2^L, x_2^H, x_3^L \), and \( x_3^H \), with \( x_2^L < x_2^H \) and \( x_3^L < x_3^H \); these numbers are defined below. Player 2 plays \( x_2 = x_2^H \) in period 0. At subsequent odd-numbered periods \( t \), player 2 plays \( x_2^H \) with probability 1 if he observed player 1’s period-\( t - 1 \) action, and otherwise plays each of \( x_2^H \) and \( x_2^L \) with probability .5. At subsequent even-numbered periods \( t \), player 2 plays \( x_2^H \) with probability 1 if he observed player 1’s period-\( t - 2 \) action, and otherwise plays each of \( x_2^H \) and \( x_2^L \) with probability .5. Thus, if player 2 observes player 1’s action in period \( t \), he then plays \( x_2^H \) with probability 1 in both periods \( t + 1 \) and \( t + 2 \). Finally, player 3 plays \( x_3 = x_3^H \) in period 0, and in every period \( t \geq 1 \) he plays \( x_3^H \) if player 2 played \( x_2^H \) in period \( t - 1 \), and plays \( x_2^L \) if player 2 played \( x_2^L \) in period \( t - 1 \). If any player \( i \) observes a deviation from this specification of on-path play (i.e., if any player deviates herself; if player 2 observes \( x_1 \neq .2505 \) or observes player 3 failing to take her prescribed action; or if player 3 observes \( x_2 \notin \{ x_2^L, x_2^H \} \)), she then plays \( x_i = 0 \) in all subsequent periods.

Before presenting the equations that define \( x_2^L, x_2^H, x_3^L, \) and \( x_3^H \), and verifying that the resulting strategy profile is a SE, I discuss why it is possible that a strategy profile of this form sustains a greater maximum cooperation of player 1’s than does any grim trigger strategy profile. The key is that the difference between player 1’s expectation of player 3’s
continuation action when player 1 cooperates and when player 1 shirks, conditional on the event that player 2 observes player 1 (which is the only event that matters for player 1’s incentives), is larger than in any equilibrium in grim trigger strategies. To understand this, consider what happens after period 2 sees player 1 take action .2505 in period \( t - 1 \), for \( t \) odd. Conditional on this event, player 1’s expectation of player 3’s action in both periods \( t + 1 \) and \( t + 2 \) equals \( x_2^H \); but player 3’s expectation of her own action in period \( t + 2 \) after seeing player 2 play \( x_2^H \) in period \( t \) is less than \( x_2^H \) (because he is not sure that player 2 observed player 1 in period \( t - 1 \)). Indeed, when player 3 sees player 2 play \( x_2^H \) in period \( t \), he would not be willing to play \( x_3^H \) if he knew that player 2 had observed player 1 in period \( t - 1 \) (as this would require him to play \( x_2^H \) in period \( t + 2 \) in addition to \( t + 1 \)). Thus, there is disagreement between player 1’s expectation of player 3’s continuation action, conditional on player 2 observing player 1, and player 3’s (unconditional) expectation of player 3’s continuation action, and this disagreement improves player 1’s incentive to cooperate without causing player 3 to shirk.

I now define \( x_2^L, x_2^H, x_3^L, \) and \( x_3^H \). For clarity of exposition, I begin by defining the first three numbers in terms of \( x_3^H \). First, let \( \sqrt{x_3^H} \equiv \sqrt{x_3^H} - .1 \). This gap between \( x_3^L \) and \( x_3^H \) differentiates the resulting strategy profile from a grim trigger strategy profile. Next, I want player 2 to be indifferent among contributing \( 0, x_2^L, \) and \( x_2^H \) at every on-path history, which is the case if

\[
\delta \sqrt{x_3^H} - x_2^H = \delta \sqrt{x_3^L} - x_2^L = 0.
\]

In order to satisfy this condition, let \( x_2^H \equiv .5 \sqrt{x_3^H} \) and \( x_2^L \equiv .5 \left( \sqrt{x_3^H} - .1 \right) \).

Given these definitions of \( x_2^L, x_2^H, \) and \( x_3^L \) in terms of \( x_3^H \), I define \( x_3^H \) to be the number that makes player 3 indifferent between actions \( x_3^H \) and \( 0 \) after he sees player 2 play \( x_2^H \) in an odd-numbered period \( t - 1 \); intuitively, this is the binding incentive constraint for player 3 because the fact that player 2 plays \( x_2^H \) in period \( t \) is evidence that he observed player 1 in period \( t - 1 \), in which case he plays \( x_2^H \) with probability 1 in period \( t + 1 \) and thus requires player 3 to play \( x_3^H \) in period \( t + 2 \) in addition to \( t + 1 \). To compute this number, note that player 1’s continuation action does not depend on player 3’s strategy, so player 3 is indifferent between contributing \( x_3^H \) and \( 0 \) if and only if \((1 - \delta) x_3^H \) equals the difference.
in player 3’s continuation value following actions \( x_3^{\text{H}} \) and 0, excluding player 1’s actions. Clearly, this continuation value equals 0 after action 0, as players 2 and 3 play 0 in every period after player 3 plays 0. To compute this continuation value after action \( x_3^{\text{H}} \), note that the probability that player 2 observed player 1’s action in period \( t - 2 \) conditional on his playing \( x_2^{\text{H}} \) in period \( t - 1 \) equals \( \frac{5}{5 + 5(5)} = 2/3 \). Therefore, player 3’s assessment of the probability that player 2 plays \( x_2^{\text{H}} \) in period \( t \) equals

\[
\frac{2}{3} (1) + \frac{1}{3} (0.5) = 5/6.
\]

In contrast, player 3’s assessment of the probability that player 2 plays \( x_2^{\text{H}} \) in every period \( \tau \geq t + 1 \) equals \(.5 (1) + .5 (.5) = 3/4 \). Hence, since player 3’s assessment of the probability that he himself plays \( x_3^{\text{H}} \) in period \( \tau + 1 \) equals his assessment of the probability that player 2 plays \( x_2^{\text{H}} \) in period \( \tau \), for all \( \tau \), his continuation value after playing \( x_3^{\text{H}} \) in \( t \) equals

\[
\delta \left( (1 - \delta) \left( \frac{5}{6} (x_3^{\text{H}}) + \frac{1}{6} (x_3^{L}) \right) + \frac{3}{4} \left( \sqrt{x_2^{\text{H}} - \delta x_3^{\text{H}}} \right) + \frac{1}{4} \left( \sqrt{x_2^{L} - \delta x_3^{L}} \right) \right) = 1/2 \left( \frac{3}{4} \left( \sqrt{x_3^{\text{H}}}/2 - \frac{1}{2} x_3^{H} \right) + \frac{1}{4} \left( \sqrt{\left( \sqrt{x_3^{H}} - .1 \right) / 2} - \frac{1}{2} \left( \sqrt{x_3^{H}} - .1 \right)^2 \right) \right). \tag{38}
\]

Define \( x_3^{\text{H}} \) to be the number such that \((.5) x_3^{\text{H}} \) equals (38). Computing this number yields \( x_3^{\text{H}} \approx .25384 \), and thus \( x_3^{L} \approx .16307 \), \( x_2^{L} \approx .25191 \), and \( x_2^{L} \approx .20191 \).

It remains to show that this strategy profile is a SE. The one-shot deviation principle applies, by standard arguments. Player 2 is indifferent among actions 0, \( x_2^{L} \), and \( x_2^{H} \) at every on-path history, and clearly weakly prefers to play 0 at every off-path history, so he has no profitable one-shot deviation (as any deviation yields a lower stage-game payoff and a weakly lower continuation payoff than does \( x_2 = 0 \)). It is also straightforward to verify that the fact that player 3 has no profitable deviation after seeing player 2 play \( x_2^{L} \) in an odd-numbered period implies that he has no profitable deviation at any history; in particular, all other one-shot incentive constraints of player 3’s are slack. Finally, player 1’s most profitable deviation at any on-path history is playing \( x_1 = 0 \). If player 1 conforms in period \( t \), for any
\[ t \geq 1, \text{ her expected payoff equals} \]
\[ \frac{3}{4} \left( \sqrt{x_2^H} + \sqrt{x_3^L} \right) + \frac{1}{4} \left( \sqrt{x_2^L} + \sqrt{x_3^L} \right) - .2505 \approx .71709. \]

If player 1 deviates to \( x_1 = 0 \) in an odd-numbered period, her expected payoff may be shown to equal
\[ (1 - \delta) \left( \frac{1}{4} \right) \left( \frac{1 + \delta}{2} \right) \left( \sqrt{x_2^H} - \sqrt{x_2^L} \right) + \left( 1 + \left( \delta \over 2 \right)^2 \right) \left( \sqrt{x_3^H} - \sqrt{x_3^L} \right) \]
\[ + \frac{1 - \delta}{1 - \delta^2} \left( \frac{1}{2} \sqrt{x_2^H} + \frac{1}{2} \sqrt{x_2^L} + \left( 1 + \delta \right) \left( \frac{1}{2} \sqrt{x_3^H} + \frac{1}{2} \sqrt{x_3^L} \right) \right) \approx .71676. \]

If player 1 deviates to \( x_1 = 0 \) in an even-numbered period \( t \geq 2 \), her expected payoff is strictly less than this; intuitively, this is because if player 1’s period-\( t \) deviation is unobserved, player 2 plays \( x_2^H \) in period \( t + 1 \) with probability 3/4 if \( t \) is odd but plays \( x_2^H \) with probability only 1/2 if \( t \) is even. In addition, it is clear that the difference between player 1’s expected payoff from conforming and from deviating to \( x_1 = 0 \) in period \( t = 0 \) is the same as the difference between her expected payoff from conforming and from deviating to \( x_1 = 0 \) in any other even-numbered period. Therefore, player 1 does not have a profitable deviation at any on-path history. Finally, it can be verified that deviating to \( x_1 = .2505 \) is not profitable for player 1 at any off-path history, and it is clear that no other off-path deviation is profitable.

### 3.8.3 Example A3: Centrality is not Necessary to Order Maximum Cooperations

The following example shows that player \( i \)'s maximum cooperation may be greater than player \( j \)'s for all discount factors \( \delta \) and benefit functions \( f \) even if player \( i \) is not more central than player \( j \).

The monitoring network consists of two components, as shown in Figure 7: Component 1 is four players on a line, and Component 2 is four players in a star. Let \( x_i^m \equiv x_i^m \) for either of the peripheral players in Component 1 \( (i \in \{1, 4\}) \), let \( y_i^m \equiv y_i^m \) for either of the central players in Component 1 \( (i \in \{2, 3\}) \), let \( x_i^m \equiv x_i^m \) for any of the peripheral players in Component 2 \( (i \in \{6, 7, 8\}) \), and let \( y_i^m \equiv y_i^m \) for the center player in Component 2 \( (i = 5) \).
Note that, for example, player 6 is not more central than player 1, and player 5 is not more central than player 2. However, I claim that, for any discount factor $\delta$, benefit function $f$, and integer $m \in \mathbb{N}$, $x_2^m \geq x_1^m$ and $y_2^m \geq y_1^m$, and therefore that $x_2^* \geq x_1^*$ and $y_2^* \geq y_1^*$. Hence, it is not always necessary that one player is more central than another to be able to order their maximum cooperations for all discount factors and benefit functions.

The proof that $x_2^m \geq x_1^m$ and $y_2^m \geq y_1^m$ for all $m \in \mathbb{N}$ is by induction. Specifically, I show that these inequality hold for $m \in \{1, 2\}$, and then show that they hold for $m + 1$ if they hold for $m$ and $m - 1$, for all $m \in \mathbb{N}$. Trivially, $x_1^1 = x_2^1 = y_1^1 = y_2^1 = \bar{X}$. Observe that

\[
x_1^{m+1} = (\delta + \delta^2) f (y_1^m) + \delta \delta f (x_1^m) \\
y_1^{m+1} = \delta f (y_1^m) + (\delta + \delta^2) f (x_1^m) \\
x_2^{m+1} = \delta f (y_2^m) + 2 \delta^2 f (x_2^m) \\
y_2^{m+1} = 3 \delta f (x_2^m).
\]

Therefore, $x_1^2 = (\delta + \delta^2 + \delta^3) f (\bar{X}) \leq (\delta + 2 \delta^2) f (\bar{X}) = x_2^2$, and $y_1^2 = (2 \delta + \delta^2) f (\bar{X}) \leq 3 \delta f (\bar{X}) = y_2^2$.

For the inductive step, suppose that $x_2^m \geq x_1^m$, $y_2^m \geq y_1^m$, $x_2^{m-1} \geq x_1^{m-1}$, and $y_2^{m} \geq y_1^{m}$. I show that $x_2^{m+1} \geq x_1^{m+1}$ and $y_2^{m+1} \geq y_1^{m+1}$. First, note that

\[
y_2^{m+1} - y_1^{m+1} = 3 \delta f (x_2^m) - \delta f (y_1^m) - \delta \delta f (x_1^m) \\
\quad = \delta f (x_2^m) - f (y_1^m) + (1 - \delta) f (x_2^m) + (1 + \delta) f (x_2^m) - f (x_1^m))
\]
\[ x_2^{m+1} - x_1^{m+1} = \delta f(y_2^m) + 2\delta^2 f(x_2^m) - (\delta + \delta^2) f(y_1^m) - \delta^3 f(x_1^m) \]

By hypothesis, \( x_2^m \geq x_1^m \) and \( y_2^m \geq y_1^m \), so to show that both of these expressions are non-negative it suffices to show that \( f(x_2^m) - f(y_1^m) + (1 - \delta) f(x_2^m) \) is non-negative. This is trivial if \( x_2^m > y_1^m \), so suppose that \( x_2^m \leq y_1^m \). By concavity of \( f \), this implies that

\[
f(x_2^m) - f(y_1^m) \geq -f'(x_2^m)(y_1^m - x_2^m) = -f'(x_2^m)(\delta f(y_1^m) + (\delta + \delta^2) f(x_1^m) - \delta f(y_2^m) - 2\delta^2 f(x_2^m)),
\]

where \( f' \) denotes the left-derivative of \( f \). In addition, \( f(0) = 0 \) and concavity of \( f \) imply that

\[
f(x_2^m) \geq f'(x_2^m) x_2^m = f'(x_2^m)(\delta f(y_2^m) + 2\delta^2 f(x_2^m)).
\]

Combining these inequalities yields

\[
f(x_2^m) - f(y_1^m) + (1 - \delta) f(x_2^m) \geq \delta f'(x_2^m) \left( -f(y_2^m) - (1 + \delta) f(x_1^m) + f(y_1^m) + 2\delta f(x_2^m) + (1 - \delta) f(y_2^m) + 2\delta (1 - \delta) f(x_2^m) \right) \geq 0,
\]

where the second inequality follows because \( f(y_2^m) \geq f(y_1^m) \), \( f(x_2^m) \geq f(x_1^m) \), and \( f(x_2^m) \geq 0 \); and the third inequality follows because the fact that \( y_2^m \geq x_2^m \geq x_1^m \) implies that \(-f(x_1^m) + \delta f(x_2^m) + (1 - \delta) f(y_1^m) \geq 0 \), and \( f' \) is non-negative.\(^{63}\)

\(^{63}\)The intuitive fact that \( y_2^m \geq x_2^m \) can be verified by a separate inductive argument.
3.9 Appendix B: Omitted Proofs

Proof of Theorem 5. There are three steps. Step 1 shows that there exists a (component-wise) greatest vector \((\hat{x}_i)^N_{i=1}\) satisfying (35), and also makes the technical point (used in Step 2d) that there exists an upper bound \(\bar{X} \in \mathbb{R}_+\) on any player’s expected action, conditional on any set of monitoring realizations, at any time in any \(SE\). Step 2 shows that \(\hat{x}_i\) is an upper bound on player \(i\)’s maximum cooperation, \(x^*_i\). Step 3 exhibits a \(SE\) in grim trigger strategies, \(\sigma^*\), such that \((1-\delta)\sum_{t=0}^{\infty} \delta^t \mathbb{E} [\sigma^*_i (h^*_j)] = \hat{x}_i\) for all \(i\), which proves that \(x^*_i = \hat{x}_i\) for all \(i\).

Step 1a: There exists a number \(\bar{X} \in \mathbb{R}_+\) such that for every \(\sigma \in SE\), player \(i\), time \(t\), and set of monitoring realizations up to time \(t\), \(F\), \(\mathbb{E} [\sigma_i (h^*_i) | F] \leq \bar{X}\) and \(\sum_{j \neq i} f_{i,j} (\bar{X}) = 0\).

Proof: Recall that \((1-\delta) \sum_{t=1}^{\infty} \delta^t [\sum_{i=1}^{N} \mathbb{E} \left[ u_i (\left(\sigma_j (h^*_j)\right)^N_{j=1}) | F \right]]\) is well-defined for all sets of monitoring realizations, \(F\), by assumption. The assumptions that \(f_{i,j}\) is concave for all \(i, j \in N\) and \(\lim_{x \to -\infty} \left( \sum_{j \neq i} f_{j,i} (x) \right) - x = -\infty\) for all \(i\) imply that there exists a number \(x^*_i \in \mathbb{R}_+\) that maximizes \(\left( \sum_{j \neq i} f_{j,i} (x) \right) - x\). For every player \(i\), let \(\bar{X}' \in \mathbb{R}_+\) be the number such that the sum of the players’ continuation payoffs from period \(t\) onward equals 0 when player \(i\) plays \(\bar{X}'\) in period \(t\), every player \(j \neq i\) plays \(x^*_j\) in period \(t\), and every player \(j\) (including player \(i\)) plays \(x^*_j\) in every subsequent period; that is, \(\bar{X}'\) is defined by

\[
(1 - \delta) \left( \left( \sum_{j \neq i} f_{i,j} (x^*_j) \right) - \bar{X}' \right) + \sum_{j \neq i} \left( \left( \sum_{k \neq \{i,j\}} f_{j,k} (x^*_k) \right) + f_{i,j} (\bar{X}') - x^*_j \right) + \delta \left( \sum_{j \in N} \left( \sum_{k \neq j} f_{j,k} (x^*_k) \right) - x^*_j \right) = 0.
\]

Let \(\bar{X}' = \max_{i \in N} \bar{X}_i\). Note that \((1 - \delta) \sum_{t=1}^{\infty} \delta^t [\sum_{i=1}^{N} \mathbb{E} \left[ u_i (\left(\sigma_k (h^*_k)\right)^N_{k=1}) | F \right]] < 0\) whenever \(\mathbb{E} [\sigma_i (h^*_i) | F] > \bar{X}'\) for some player \(i\). Since \(h^*_j\) determines the monitoring realization up to time \(t\), and thus whether \(F\) has occurred, \((1 - \delta) \sum_{t=1}^{\infty} \delta^t \mathbb{E} \left[ u_j \left( \left(\sigma_k (h^*_k)\right)^N_{k=1} \right) | F \right] = \mathbb{E} \left[ (1 - \delta) \sum_{t=1}^{\infty} \delta^t \mathbb{E} \left[ u_j \left( \left(\sigma_k (h^*_k)\right)^N_{k=1} \right) | h^*_j \right] \right].\) Therefore, there exists a player \(j\) such that, if \(\mathbb{E} [\sigma_i (h^*_i) | F] > \bar{X}'\), then \((1 - \delta) \sum_{t=1}^{\infty} \delta^t \mathbb{E} \left[ u_j \left( \left(\sigma_k (h^*_k)\right)^N_{k=1} \right) | h^*_j \right] < 0\) with positive probability. But \((1 - \delta) \sum_{t=1}^{\infty} \delta^t \mathbb{E} \left[ u_j \left( \left(\sigma_k (h^*_k)\right)^N_{k=1} \right) | h^*_j \right] \geq 0\) at any on-path history \(h^*_j\),
because \( \sigma \in \Sigma_{SE} \) and 0 is each player’s minmax value. Hence, \( \mathbb{E}[\sigma_i(h_t^i) | F] \leq \tilde{X} \) for every player \( i \).

Finally, for every player \( i \), the assumption that \( \lim_{x \to \infty} \left( \sum_{j \neq i} f_{i,j}(x) \right) - x < 0 \) (combined with concavity of \( f_{i,j} \)) implies there exists a number \( \tilde{X}_i \in \mathbb{R}_+ \) such that \( \left( \sum_{j \neq i} f_{i,j}(\tilde{X}_i) \right) - \tilde{X}_i = 0 \) and \( \left( \sum_{j \neq i} f_{i,j}(x) \right) - x < 0 \) for all \( x > \tilde{X}_i \). Taking \( \tilde{X} \equiv \max \{ \tilde{X}', \max_{i \in N} \tilde{X}_i \} + 1 \) completes the proof.

**Step 1b:** There exists a greatest vector \((\hat{x}_i)_{i=1}^N\) satisfying (35), and \( \hat{x}_i \leq \tilde{X} \) for all \( i \).

**Proof:** Define the function \( \phi : \mathbb{R}_+^N \to \mathbb{R}_+^N \) by

\[
\phi_i \left( (x_j)_{j=1}^N \right) \equiv (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{j \neq i} \Pr (j \in D(t,i)) f_{i,j} (x_j) \text{ for all } i.
\]

The fixed points of \( \phi \) are precisely those vectors satisfying (35). Observe that \( \phi \) is isotone. In addition, \( \left( (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{j \neq i} \Pr (j \in D(t,i)) f_{i,j} (\tilde{X}) \right) - \tilde{X} \leq 0 \) for every player \( i \), which implies that \( \phi \left( (\tilde{X})_{j=1}^N \right) \leq (\tilde{X})_{j=1}^N \). Hence, the image of the set \( [0, \tilde{X}]_N \) under \( \phi \) is contained in \( [0, \tilde{X}]^N \). Therefore, Tarski’s fixed point theorem implies that \( \phi \) has a greatest fixed point \((\hat{x}_i)_{i=1}^N\) in the set \([0, \tilde{X}]^N\). Finally, if \( x_i > \tilde{X} \) for some player \( i \), then there exists a player \( j \) (possibly equal to \( i \)) such that \( \left( (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{k \neq j} \Pr (k \in D(t,j)) f_{j,k} (x_k) \right) - x_j < 0 \). This implies that every fixed point of \( \phi \) must lie in the set \([0, \tilde{X}]_N\), and it follows that \((\hat{x}_i)_{i=1}^N\) is the greatest vector satisfying (35).

**Step 2a:** If \( \sigma \in \Sigma_{SE} \), then for every player \( i \) and every on-path history \( h_{t}^i \),

\[
(1 - \delta) \sum_{\tau=t}^{\infty} \delta^{\tau-t} \mathbb{E} \left[ \sigma_i(h_{\tau}^i) | h_{t}^i \right] \leq (1 - \delta) \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} \Pr (j \in D(\tau,t,i)) \mathbb{E} \left[ f_{i,j} \left( \sigma_j \left( h_{\tau}^j \right) \right) | h_{t}^i, j \in D(\tau,t,i) \right].
\]

(39)

**Proof:** Fix strategy profile \( \sigma \), player \( i \), and on-path history \( h_{t}^i \). For any player \( j \) and history \( h_{\tau}^j \), let \( \mathbb{E} \left[ f_{i,j} \left( \sigma_j \left( h_{\tau}^j \right) \right) | h_{t}^i, 0 \right] \) be the expectation of \( f_{i,j} \left( \sigma_j \left( h_{\tau}^j \right) \right) \) conditional on each

---

\( ^{64} \)As an aside, note that the vector \((\hat{x}_i)_{i=1}^N\) (which by Theorem 5 equals \((x^*_i)_{i=1}^N\)) may be easily computed by iterating \( \phi \) on \((\tilde{X})_{j=1}^N\). Thus, computing the vector of maximum cooperations is like computing the greatest equilibrium in a supermodular game (cf Milgrom and Roberts (1990)).
player k ≠ i following σ_k, player i following σ_i at every time τ < t, history h^i_τ being reached, and player i playing x_i = 0 at every time τ ≥ t. If σ ∈ Σ_SE, then player i’s expected payoff from conforming to σ from h^i_τ onward is weakly greater than her expected payoff from playing x_i = 0 at every time τ ≥ t. That is,

$$(1 - \delta) \sum_{\tau = t}^{\infty} \delta^{\tau-t} \mathbb{E} \left[ \left( \sum_{j \neq i} f_{i,j} \left( \sigma_j \left( h^j_\tau \right) \right) \right) - \sigma_i \left( h^i_\tau \right) \right] \geq (1 - \delta) \sum_{\tau = t}^{\infty} \delta^{\tau-t} \mathbb{E} \left[ \left( \sum_{j \neq i} f_{i,j} \left( \sigma_j \left( h^j_\tau \right) \right) \right) \right] | h^i_\tau, 0],
$$

or, equivalently,

$$(1 - \delta) \sum_{\tau = t}^{\infty} \delta^{\tau-t} \mathbb{E} \left[ \sigma_i \left( h^i_\tau \right) \right] \leq (1 - \delta) \sum_{\tau = t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} (\mathbb{E} \left[ f_{i,j} \left( \sigma_j \left( h^j_\tau \right) \right) \right] | h^i_\tau] - \mathbb{E} \left[ f_{i,j} \left( \sigma_j \left( h^j_\tau \right) \right) \right] | h^i_\tau, 0] \right).
$$

(40)

Observe that, conditional on the event j \notin D(\tau, t, i), the probability distribution over histories h^j_\tau does not depend on player i’s actions following history h^i_\tau. Therefore,

$$\mathbb{E} \left[ f_{i,j} \left( \sigma_j \left( h^j_\tau \right) \right) \right] | h^i_\tau, j \notin D(\tau, t, i) = \mathbb{E} \left[ f_{i,j} \left( \sigma_j \left( h^j_\tau \right) \right) \right] | h^i_\tau, 0, j \notin D(\tau, t, i) \right].
$$

Hence, the right-hand side of (40) equals

$$(1 - \delta) \sum_{\tau = t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} \Pr \left( j \in D(\tau, t, i) \right) \left( \mathbb{E} \left[ f_{i,j} \left( \sigma_j \left( h^j_\tau \right) \right) \right] | h^i_\tau, j \in D(\tau, t, i) \right) - \mathbb{E} \left[ f_{i,j} \left( \sigma_j \left( h^j_\tau \right) \right) \right] | h^i_\tau, 0, j \in D(\tau, t, i) \right]
$$

which is not more than the right-hand side of (39). Therefore, the fact that (40) holds for all players i and on-path histories h^i_\tau implies that (39) holds for all players i and on-path histories h^i_\tau.

**Step 2b:** For every player i, define the random variable X^i_0 by

$$X^i_0 = (1 - \delta) \sum_{\tau = t}^{\infty} \delta^{\tau-t} \sigma_i \left( h^i_\tau \right).$$
The right-hand side of (39) is not more than

$$\sum_{\tau=t}^{\infty} \delta^{\tau - t} \sum_{j \neq i} \Pr(j \in D(\tau, t, i) \setminus D(\tau - 1, t, i)) f_{i, j} \left( \mathbb{E} \left[ X_j^\tau | h_i^\tau, j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right] \right).$$

(41)

**Proof:** Fix a player \( j \). To simplify notation, define the random variable \( X_{i,j}^\tau \) by \( X_{i,j}^\tau \equiv (1 - \delta) \sum_{\tau=t}^{\infty} \delta^{\tau - t} f_{i, j} \left( \sigma_j (h_j^\tau) \right) \); this notation is used only in this step of the proof. Note that

\[
(1 - \delta) \mathbb{E} \left[ f_{i, j} \left( \sigma_j (h_j^\tau) \right) \right] | h_i^\tau, j \in D(\tau, t, i) = \mathbb{E} \left[ X_j^\tau | h_i^\tau, j \in D(\tau, t, i) \right] - \delta \mathbb{E} \left[ X_j^{\tau + 1} | h_i^\tau, j \in D(\tau, t, i) \right].
\]

Therefore,

\[
(1 - \delta) \sum_{\tau=t}^{\infty} \delta^{\tau - t} \Pr(j \in D(\tau, t, i)) \mathbb{E} \left[ f_{i, j} \left( \sigma_j (h_j^\tau) \right) \right] | h_i^\tau, j \in D(\tau, t, i) = \mathbb{E} \left[ X_j^\tau | h_i^\tau, j \in D(\tau, t, i) \right] - \delta \mathbb{E} \left[ X_j^{\tau + 1} | h_i^\tau, j \in D(\tau, t, i) \right]
\]

\[
= \sum_{\tau=t}^{\infty} \delta^{\tau - t} \left( \Pr(j \in D(\tau, t, i)) \mathbb{E} \left[ X_j^\tau | h_i^\tau, j \in D(\tau, t, i) \right] - \delta \mathbb{E} \left[ X_j^{\tau + 1} | h_i^\tau, j \in D(\tau, t, i) \right] \right)
\]

\[
= \sum_{\tau=t}^{\infty} \delta^{\tau - t} \left( \Pr(j \in D(\tau, t, i) \setminus D(\tau - 1, t, i)) \mathbb{E} \left[ X_j^\tau | h_i^\tau, j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right] + \Pr(j \in D(\tau - 1, t, i)) \mathbb{E} \left[ X_j^\tau | h_i^\tau, j \in D(\tau - 1, t, i) \right] \right)
\]

\[
= \sum_{\tau=t}^{\infty} \delta^{\tau - t} \Pr(j \in D(\tau, t, i) \setminus D(\tau - 1, t, i)) \mathbb{E} \left[ X_j^\tau | h_i^\tau, j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right]
\]

\[
= \sum_{\tau=t}^{\infty} \delta^{\tau - t} \Pr(j \in D(\tau, t, i) \setminus D(\tau - 1, t, i)) \mathbb{E} \left[ (1 - \delta) \sum_{s=\tau}^{\infty} \delta^{s - \tau} f_{i, j} \left( \sigma_j (h_j^s) \right) | h_i^\tau, j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right]
\]

\[
\leq \sum_{\tau=t}^{\infty} \delta^{\tau - t} \Pr(j \in D(\tau, t, i) \setminus D(\tau - 1, t, i)) f_{i, j} \left( \mathbb{E} \left[ X_j^\tau | h_i^\tau, j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right] \right)
\]

where the second equality uses the fact that \( \Pr(j \in D(t - 1, t, i)) = 0 \), the third equality uses the fact that \( D(\tau - 1, t, i) \subseteq D(\tau, t, i) \), and the inequality uses concavity of \( f_{i, j} \) and
Jensen’s inequality. Summing over $j \neq i$ completes the proof.

**Step 2c:** If $\sigma \in \Sigma_{SE}$, then for every player $i$, time $t$, and subset of monitoring realizations up to time $t$, $F$,

$$
\mathbb{E} \left[ X_i^t | F \right] \\
\leq \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} \Pr \left( j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right) f_{i,j} \left( \mathbb{E} \left[ X_j^\tau | j \in D(\tau, t, i) \setminus D(\tau - 1, t, i), F \right] \right)
$$

*Proof:* If $\sigma \in \Sigma_{SE}$, then (39) and Step 2b imply that, for every player $i$ and every on-path history $h_i^t$,

$$
\mathbb{E} \left[ X_i^t | h_i^t \right] \leq \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} \Pr \left( j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right) f_{i,j} \left( \mathbb{E} \left[ X_j^\tau | h_i^t, j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right] \right).
$$

Thus, by concavity of $f_{i,j}$, and Jensen’s inequality,

$$
\mathbb{E} \left[ \mathbb{E} \left[ X_i^t | h_i^t \right] | F \right] \\
\leq \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} \Pr \left( j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right) f_{i,j} \left( \mathbb{E} \left[ \mathbb{E} \left[ X_j^\tau | h_i^t, j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right] | F \right] \right).
$$

Finally, the assumption that $h_i^t$ (or $h_j^t$) determines the monitoring realization up to time $t$ (and thus whether the events $j \in D(\tau, t, i) \setminus D(\tau - 1, t, i)$ and $F$ have occurred) implies that

$$
\mathbb{E} \left[ \mathbb{E} \left[ X_i^t | h_i^t \right] | F \right] = \mathbb{E} \left[ X_i^t | F \right],
$$

and

$$
\mathbb{E} \left[ \mathbb{E} \left[ X_j^\tau | h_i^t, j \in D(\tau, t, i) \setminus D(\tau - 1, t, i) \right] | F \right] = \mathbb{E} \left[ X_j^\tau | j \in D(\tau, t, i) \setminus D(\tau - 1, t, i), F \right].
$$

**Step 2d:** If $\sigma \in \Sigma_{SE}$, then $\mathbb{E}[X_i^0] \leq \hat{x}_i$. In addition, if $\sigma \in \Sigma_{SE}$ and $\mathbb{E}[X_i^0] = \hat{x}_i$ for all $i$, then $\sigma_i(h_i^t) = \hat{x}_i$ for every player $i$ and on-path history $h_i^t$.

*Proof:* Define $x_i^m$ recursively, for all $m \in \mathbb{N}$, by letting $x_i^1 \equiv \bar{X}$ and letting $x_i^{m+1} \equiv \bar{X}$.
\[ \phi_i \left( \left( x_j^m \right)_{j=1}^N \right) \] for all \( i \). I first claim that \( \mathbb{E}[X_i^t | F] \leq x_i^m \) for every player \( i \), time \( t \), subset of monitoring realizations up to time \( t \), \( F \), and number \( m \in \mathbb{N} \). The proof is by induction on \( m \). For \( m = 1 \), the result follows because \( \mathbb{E}[\sigma_i(h_i^1) | F] \leq \hat{X} \) for all \( \tau \geq t \), by Step 1a, and therefore \( \mathbb{E}[X_i^t | F] = (1 - \delta) \sum_{\tau=t}^{\infty} \delta^{\tau-t} \mathbb{E}[\sigma_i(h_i^\tau) | F] \leq \hat{X} \). Suppose the result is proved for some \( m \in \mathbb{N} \). Then

\[
\begin{align*}
\mathbb{E}[X_i^t | F] &\leq \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} \Pr(j \in D(\tau, t, i) \setminus D(\tau - 1, t, i)) f_{i,j} \left( \mathbb{E}[X_j^\tau | j \in D(\tau, t, i) \setminus D(\tau - 1, t, i), F] \right) \\
&\leq \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} \Pr(j \in D(\tau, t, i) \setminus D(\tau - 1, t, i)) f_{i,j} \left( x_j^m \right) \\
&= (1 - \delta) \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{j \neq i} \Pr(j \in D(\tau, t, i)) f_{i,j} \left( x_j^m \right) \\
&= x_i^{m+1},
\end{align*}
\]

where the first inequality follows by Step 2c and the second inequality follows by the inductive hypothesis.

Since \( \phi \) is isotone, \( \hat{x}_i \leq x_i^m \) for all \( m \in \mathbb{N} \), and in addition \( \hat{x}_i \leq \phi_i(\lim_{m \to \infty} x_i^m) \). Also, \( \phi \) is continuous, which implies that \( \phi_i(\lim_{m \to \infty} x_i^m) = \lim_{m \to \infty} x_i^m \). The fact that \( \hat{x} \) is the greatest fixed point of \( \phi \) thus implies that \( \hat{x} = \lim_{m \to \infty} x^m \). Therefore, the fact that \( \mathbb{E}[X_i^t | F] \leq x_i^m \) for all \( m \in \mathbb{N} \) implies that \( \mathbb{E}[X_i^t | F] \leq \hat{x}_i \). Taking \( t = 0 \) and \( F = \emptyset \) yields \( \mathbb{E}[X_i^0] \leq \hat{x}_i \).

**Step 3:** Let \( \sigma^* \) be the strategy profile given by \( \sigma_i^*(h_i^t) = \hat{x}_i \) if \( z_{i,j,\tau} \in \{ \hat{x}_j, \emptyset \} \) for all \( z_{i,j,\tau} \in h_i^\tau \), and \( \sigma_i^*(h_i^t) = 0 \) otherwise, for all \( i \). Then \( \sigma^* \in \Sigma_{SE} \), and \( \mathbb{E}[X_i^0] = \hat{x}_i \) for all \( i \).

**Proof:** It is immediate that \( \mathbb{E}[X_i^0] = \hat{x}_i \) for all \( i \). To see that \( \sigma^* \in \Sigma_{SE} \), note that the one-shot deviation principle applies, by standard arguments. I first show that no player has a profitable one-shot deviation at any on-path history, and then show that no player has a profitable one-shot deviation at any off-path history.

Fix a player \( i \) and an on-path history \( h_i^t \). If \( \hat{x}_i = 0 \), then it is clear that player \( i \) does not have a profitable deviation at \( h_i^t \). So suppose that \( \hat{x}_i > 0 \). Player \( i \)'s continuation payoff
if she conforms to $\sigma^*$ equals $\sum_{j \neq i} f_{i,j}(\hat{x}_j) - \hat{x}_i$. The most profitable deviation from $\sigma^*$ is playing $x_i = 0$, as every other deviation yields the same continuation payoff and a lower stage-game payoff. I claim that player $i$’s continuation payoff (including period $t$) after such a deviation equals

$$(1 - \delta) \sum_{\tau = t}^{\infty} \delta^{\tau - t} \sum_{j \neq i} \Pr(j \notin D(\tau, t, i)) f_{i,j}(\hat{x}_j).$$

(42)

Given this claim, the difference between player $i$’s payoff from conforming to $\sigma^*$ and from playing her most profitable deviation equals

$$(1 - \delta) \sum_{\tau = t}^{\infty} \delta^{\tau - t} \sum_{j \neq i} \Pr(j \in D(\tau, t, i)) f_{i,j}(\hat{x}_j) - \hat{x}_i,$$

which equals 0 because the vector $(\hat{x}_i)_i^{N}$ satisfies (35). Therefore, to show that player $i$ has no profitable deviation, it suffices to prove that player $i$’s continuation payoff after playing $x_i = 0$ at on-path history $h^i$ equals (42).

If player $i$ deviates from $\sigma^*$ at on-path history $h^i$ and $j \notin D(\tau, t, i)$ for some player $j$ and time $\tau$, then $\sigma^*_i(h^i_\tau) = \hat{x}_j$. Hence, the claim that player $i$’s continuation payoff equals (42) is equivalent to the claim that $\sigma^*_i(h^i_\tau) = 0$ whenever $j \in D(\tau, t, i)$ and $\Pr(j \in D(\tau, t, i)) > 0$.

Thus, suppose that player $i$ plays $x_i = 0$ at on-path history $h^i$, that $\Pr(j \in D(\tau, t, i)) > 0$, and that the monitoring realization up to time $\tau$, $\omega^\tau$, is such that $j \in D(\tau, t, i)$ given $\omega^\tau$ and $\Pr((L_s)_s^{\tau+1} = \omega^\tau) > 0$. I claim that $\sigma^*_i(h^i_\tau) = 0$ given $\omega^\tau$. This claim is trivial if $\hat{x}_j = 0$, so assume that $\hat{x}_j > 0$. Proceed by induction on $\tau$: If $\tau = t + 1$, then $\hat{x}_j(t) = 0$ given $\omega^\tau$, so the fact that $0 \notin \{\hat{x}_i, \emptyset\}$ implies that $\sigma^*_j(h^i_\tau) = 0$. Suppose that the claim holds for all $\tau \leq \tau_0$, and consider the case where $\tau = \tau_0 + 1$. Since $j \in D(\tau_0 + 1, t, i)$, player $j$ observes the action of some player $k \in D(\tau_0, t, i)$ at time $\tau_0$ given $\omega^\tau$, and the fact that $\Pr((L_s)_s^{\tau_0} = \omega^\tau) > 0$ implies that $\Pr(j \in D(\tau_0 + 1, \tau_0, k)) > 0$. Since $\hat{x}_j > 0$, the fact that $\Pr(j \in D(\tau_0 + 1, \tau_0, k)) > 0$ implies that $\hat{x}_k > 0$, by the definition of $(\hat{x}_i)_i^{N}$. Therefore, by the inductive hypothesis, $\sigma^*_k(h^i_{\tau_0}) = 0$ given $\omega^\tau$, and $0 \notin \{\hat{x}_k, \emptyset\}$. Hence, $\sigma^*_j(h^i_\tau) = 0$, completing the proof of the claim.

It remains only to show that no player has a profitable deviation at any off-path history. Intuitively, given that each player $i$ is indifferent between playing $x_i = \hat{x}_i$ and $x_i = 0$ at every
on-path history $h_t^1$, this follows from Ellison’s (1994) observation that a player’s incentive to cooperate in a grim trigger strategy profile is reduced after a defection by another player. Formally, for any subset of players $S \subseteq N$, define $D(\tau, t, S)$ by

$$
D(\tau, t, S) = \emptyset \text{ if } \tau < t \\
D(t, t, S) = S \\
D(\tau + 1, t, S) = D(\tau, t, S) \cup \{ j : z_{j, k, \tau} = x_{k, \tau} \text{ for some } k \in D(\tau, t, S) \} \text{ if } \tau \geq t;
$$

note that this generalizes the definition of $D(\tau, t, i)$. Fix a player $i$ and an off-path history $h_t^1$. If player $i$ has a profitable deviation from $\sigma^*$ at $h_t^1$, it must be playing $x_i = \hat{x}_i$, as all other actions yield the same continuation payoff as $x_i = 0$ and a strictly lower stage game payoff. By a similar argument to that in the previous two paragraphs, if $\hat{D}(t)$ is set of players such that $z_{i,j, \tau} \notin \{ \hat{x}_j, \emptyset \}$ for some $z_{i,j, \tau} \in h_t^1$, then the difference between player $i$’s payoff from conforming to $\sigma^*$ and her payoff from deviating to $x_i = \hat{x}_i$ (and subsequently following $\sigma^*$) equals

$$
\hat{x}_i - \sum_{\tau=t}^{\infty} \delta^{t-\tau} \sum_{j \neq i} \Pr \left( j \in D(\tau, t, \hat{D}(t)) \setminus \left( D(\tau, t, \hat{D}(t) \setminus \{i\}) \cup D(\tau, t+1, i) \right) \right) f_{i,j}(\hat{x}_j)
$$

$$
= \hat{x}_i - \sum_{\tau=t}^{\infty} \delta^{t-\tau} \sum_{j \neq i} \Pr \left( j \in D(\tau, t, i) \setminus D(\tau, t+1, i) \right) f_{i,j}(\hat{x}_j)
$$

$$
\geq \hat{x}_i - \sum_{\tau=t}^{\infty} \delta^{t-\tau} \sum_{j \neq i} \Pr \left( j \in D(\tau, t, i) \setminus D(\tau, t+1, i) \right) f_{i,j}(\hat{x}_j)
$$

$$
= \hat{x}_i - \sum_{\tau=t}^{\infty} \delta^{t-\tau} \sum_{j \neq i} \left( \Pr( j \in D(\tau, t, i)) - \Pr( j \in D(\tau, t+1, i)) \right) f_{i,j}(\hat{x}_j)
$$

$$
= \hat{x}_i - \frac{1}{1- \delta} \hat{x}_i + \frac{\delta}{1- \delta} \hat{x}_i = 0,
$$

where the last equality follows because

$$
\sum_{\tau=t}^{\infty} \delta^{t-\tau} \sum_{j \neq i} \Pr \left( j \in D(\tau, t, i) \right) f_{i,j}(\hat{x}_j) = \hat{x}_i / (1 - \delta)
$$

and

$$
\sum_{\tau=t}^{\infty} \delta^{t-\tau} \sum_{j \neq i} \Pr \left( j \in D(\tau, t+1, i) \right) f_{i,j}(\hat{x}_j) = \delta \hat{x}_i / (1 - \delta).
$$

Hence, player $i$ does not have a profitable deviation at history $h_t^1$ for any set $\hat{D}(t)$, and therefore player $i$ does not have a profitable deviation at history $h_t^1$ for any belief about the vector of private histories $(h_j^1)_{j=1}^N$. □
Proof of Corollary 5. By Theorem 5, it suffices to show that \( \hat{x}_i = \hat{x}_j \) for all \( i, j \in N \). Let \( x_i^m \) be defined as in Step 2d of the proof of Theorem 5. I claim that \( x_i^m = x_j^m \) for all \( i, j \in N \) and \( m \in \mathbb{N} \). The proof is by induction. For \( m = 1 \), \( x_i^1 = x_j^1 = \bar{X} \). Suppose the result is proved for some \( m \in \mathbb{N} \); that is, that there exists \( x^m \in \mathbb{N} \) such that \( x_k^m = x^m \) for all \( k \in N \). Then

\[
x_i^{m+1} = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{k \neq i} \Pr(k \in D(t, i)) \alpha_{i,k} f(x_k^m) \quad \text{(by parallel benefit functions)}
\]

\[
= (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{k \neq i} \Pr(k \in D(t, i)) \alpha_{i,k} f(x^m)
\]

\[
= (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{k \neq j} \Pr(k \in D(t, j)) \alpha_{j,k} f(x_j^m) \quad \text{(by equal monitoring)}
\]

\[
= (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{k \neq j} \Pr(k \in D(t, j)) \alpha_{j,k} f(x_k^m)
\]

\[
= x_j^{m+1},
\]

proving the claim. Finally, \( \hat{x}_i = \lim_{m \to \infty} x_i^m = \lim_{m \to \infty} x_j^m = \hat{x}_j \). ■

Proof of Proposition 6. Since \( N \) and \( N' \) are even, it must be that \( N \geq 2 \) and \( N' \geq 4 \).

By Corollary 6, it suffices to show that \( \mathbb{E} \left[ \#D(t, N') \right] \geq \mathbb{E} \left[ \#D(t, N) \right] \) for all \( t \) and that \( \mathbb{E} \left[ \#D(2, N') \right] > \mathbb{E} \left[ \#D(2, N) \right] \). Since \( \#D(1, N') = \#D(1, N) = 2 \) with probability 1, it follows that \( \mathbb{E} \left[ \#D(2, N') \right] = \frac{N-2}{N-1} \cdot 4 + \frac{1}{N-1} \cdot 2 > \frac{N-2}{N-1} \cdot 4 + \frac{1}{N-1} \cdot 2 = \mathbb{E} \left[ \#D(2, N) \right] \).

To show that \( \mathbb{E} \left[ \#D(t, N') \right] \geq \mathbb{E} \left[ \#D(t, N) \right] \) for all \( t \), I show that the probability distribution over random matchings among \( N \) players may be parametrized so that \( \#D(t, N') \geq \#D(t, N) \) for all \( t \) and all realizations of the monitoring technology. The proof is by induction on \( t \). Suppose that \( \#D(t, N') \geq \#D(t, N) \). First, renumber the \( N' \) players so that \( D(t, N) \subseteq D(t, N') \), and let \( S = D(t, N) \) to simplify notation. Next, assume without loss of generality that in period \( t \) the random matching among \( N \) players is obtained by first randomly matching \( N' \) players and then randomly rematching those players in \( \{1, \ldots, N\} \) who were matched with players in \( \{N+1, \ldots, N'\} \) with other such players. Note that \( \#D(t+1, N') \geq \#S + \# \{ j \in N' \setminus S : l_{j,i,t} = 1 \text{ for some } i \in S \} \), while \( \#D(t+1, N) = \#S + \# \{ j \in N \setminus S : l_{j,i,t} = 1 \text{ for some } i \in S \} \). If a player \( i \in S \) matches with a player \( j \in N \setminus S \),
in the random matching among \(N\) players, then either \(i\) also matches with \(j\) in the random matching among \(N'\) players or \(i\) matches with a player \(k \in \{N + 1, \ldots, N'\} \setminus S\). Hence, \(\# \{j \in N' \setminus S : l_{j,i,t} = 1 \text{ for some } i \in S\} \geq \# \{j \in N \setminus S : l_{j,i,t} = 1 \text{ for some } i \in S\}\), and therefore \(#D(t + 1, N') \geq #D(t + 1, N)\).

**Proof of Proposition 7.** By Theorem 6, it suffices to show that for any \(\gamma > 0\) there exists \(\bar{N} > 0\) such that, if \(N' > (1 + \gamma)N \geq \bar{N}\), then \(\sum_{t=0}^{\infty} \delta^t (\mathbb{E}[#D(t, N')] - 1) / N' < \sum_{t=0}^{\infty} \delta^t (\mathbb{E}[#D(t, N)] - 1) / N\), or equivalently \(\sum_{t=0}^{\infty} \delta^t (N \mathbb{E}[#D(t, N')] - N' \mathbb{E}[#D(t, N)] + N' - N) < 0\). For any \(t\) and any \(\varepsilon > 0\), there exists \(\bar{N}'\) such that \(\mathbb{E}[#D(t, N')] - \mathbb{E}[#D(t, N)] < \varepsilon\) for any \(N', N \geq \bar{N}'\), as the probability that any two defectors match with each other within the first \(t\) periods when there are either \(N'\) or \(N\) players converges to 0 as \(N \to \infty\). Furthermore, \(\mathbb{E}[#D(t, N')] - \mathbb{E}[#D(t, N)] \leq 2\varepsilon\), since \(\mathbb{E}[#D(t, N')] \leq 2\varepsilon\) and \(\mathbb{E}[#D(t, N)] \geq 0\). Therefore, for any \(\varepsilon' > 0\), there exists \(\bar{N}'\) such that \(\sum_{t=0}^{\infty} \delta^t (\mathbb{E}[#D(t, N')] - \mathbb{E}[#D(t, N)]) < \varepsilon'\) for any \(N', N \geq \bar{N}'\), as each of the first \(T\) terms in the sum converges to 0 as \(\bar{N}' \to \infty\), for any \(T\), and the sum of the remaining terms is less than \(\sum_{t=T}^{\infty} \delta^t 2^{2t} = \frac{(2\delta)^T}{1 - 2\delta}\), which converges to 0 as \(T \to \infty\), under the assumption that \(\delta < \frac{1}{2}\). Let \(\varepsilon' = 2\delta\gamma\), let \(\bar{N}'(\gamma)\) be the corresponding \(\bar{N}'\), and let \(\bar{N} = (1 + \gamma) \bar{N}'(\gamma)\), which guarantees that \(N \geq \bar{N}'(\gamma)\) if \((1 + \gamma)N \geq \bar{N}\). Then,

\[
\sum_{t=0}^{\infty} \delta^t (N \mathbb{E}[#D(t, N')] - N' \mathbb{E}[#D(t, N)] + N' - N)
= N \left(\sum_{t=0}^{\infty} \delta^t (\mathbb{E}[#D(t, N')] - \mathbb{E}[#D(t, N)])\right) - (N' - N) \sum_{t=0}^{\infty} \delta^t (\mathbb{E}[#D(t, N)] - 1)
\leq N (2\delta\gamma) - (N' - N) \sum_{t=0}^{\infty} \delta^t (\mathbb{E}[#D(t, N)] - 1)
\leq N (2\delta\gamma) - (N' - N) (2\delta)
< 0,
\]

where the first inequality follows because \(N', N \geq \bar{N}'(\gamma)\), the second inequality follows because \(\sum_{t=0}^{\infty} \delta^t \mathbb{E}[#D(t, N)] \geq 1 + 2\delta\), and the third inequality follows because \(N' > (1 + \gamma)N\).

**Proof of Theorem 7.** Let \(g_k'(k') \equiv \Pr (#D(t) = k' | #D(0) = k)\), let \(G_k'\) be the corresponding distribution function, and let \(\mathbb{E}_{g_k'}[k'] \equiv \sum_{k'=0}^{N} k' g_k'(k')\). By Theorem 6, it suffices
to show that $\sum_{t=0}^{\infty} \delta^t E_{g_t'} [k'] > \sum_{t=0}^{\infty} \delta^t E_{g_t} [k']$.

I claim that $\tilde{G}_k^{t'}$ strictly second-order stochastically dominates $G_k^{t'}$ for all $t \geq 1$ and $k \in \{1, \ldots, N - 1\}$, which is equivalent to $\sum_{s=0}^{k'} \tilde{G}_k^{t'} (s) < \sum_{s=0}^{k'} G_k^{t'} (s)$ for all $t \geq 1$ and $k' \in \{k, \ldots, N - 1\}$.$^{65}$ The proof is by induction on $t$. The $t = 1$ case is the assumption that $\tilde{G}_k$ strictly second-order stochastically dominates $G_k$. Assume the result is proved for $t - 1$. Then

$$\sum_{s=0}^{k'} \tilde{G}_k^{t'} (s) = \sum_{s=0}^{k'} \sum_{r=0}^{s} \tilde{g}_k^{t'-1} (r) \tilde{G}_r (s) = \sum_{r=0}^{k'} \tilde{g}_k^{t'-1} (r) \sum_{s=r}^{k'} \tilde{G}_r (s) = \sum_{r=0}^{k'} \tilde{g}_k^{t'-1} (r) \sum_{s=0}^{k'} G_r (s) < \sum_{r=0}^{k'} \tilde{g}_k^{t'-1} (r) \sum_{s=0}^{k'} G_r (s) = \sum_{s=0}^{k'} G_k^{t'} (s),$$

where the first line follows because $\tilde{G}_k^{t'} (s) = \sum_{r=0}^{s} \tilde{g}_k^{t'-1} (r) \tilde{G}_r (s)$, the second line reverses the order of sums, the third line follows because $\tilde{G}_r (s) = 0$ if $s < r$, the fourth line follows because $\tilde{G}_r (s)$ strictly second-order stochastically dominates $G_r (s)$ for all $r \in \{1, \ldots, k'\}$, the fifth line follows because $\tilde{G}_k^{t'-1} (r)$ strictly second-order stochastically dominates $G_k^{t'-1} (r)$ (by the inductive hypothesis) and $\sum_{s=0}^{k'} G_r (s)$ is decreasing and strictly convex in $r$ for $r \in \{0, \ldots, k'\}$ (because $G_r (s)$ is decreasing and strictly convex in $r$ for $r = \{0, \ldots, s\}$ and $s \in \{0, \ldots, N\}$, so the sum of such functions is decreasing and strictly convex in $r$ for $r = \{0, \ldots, k'\}$), and the sixth line follows from undoing the rearrangement of the first two lines for $G_k^{t'} (s)$ rather than $\tilde{G}_k^{t'} (s)$. This proves the claim.

Trivially, $E_{g_t'} [k'] = E_{g_t} [k'] = 1$, and $E_{g_t'} [k'] \geq E_{g_t} [k']$ because $\tilde{G}_k$ second-order stochas-

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$^{65}$Note that this claim implies that $\sum_{t=0}^{\infty} \delta^t E_{g_t'} [k'] \geq \sum_{t=0}^{\infty} \delta^t E_{g_t} [k']$, but not necessarily that $\sum_{t=0}^{\infty} \delta^t E_{g_t'} [k'] > \sum_{t=0}^{\infty} \delta^t E_{g_t} [k']$, which is what needs to be shown.
tically dominates $G_k$. I now show that $\mathbb{E}_{g'_i} [k'] > \mathbb{E}_{g'_j} [k']$ for all $t \geq 2$. This follows because

\[
\mathbb{E}_{g'_i} [k'] = \sum_{s=0}^{k'} g'_{i}^{-1} (s) \mathbb{E}_{g'_i} [k'] \\
\geq \sum_{s=0}^{k'} g'_{i}^{-1} (s) \mathbb{E}_{g'_i} [k'] \\
> \sum_{s=0}^{k'} g'_{i}^{-1} (s) \mathbb{E}_{g'_i} [k'] \\
= \mathbb{E}_{g'_i} [k'],
\]

where the first line follows by the law of iterated expectation, the second line follows because $\tilde{G}_s^{-1} (k')$ second-order stochastically dominates $G_s^{-1} (k')$ if $t \geq 2$ (by the claim), the third line follows because $\mathbb{E}_{g'_i} [k']$ is increasing and strictly concave in $s$ for $s \in \{0, \ldots, N\}$ (since $G_k (k')$ is decreasing and strictly convex in $k$ for $k \in \{0, \ldots, k'\}$ and $k' \in \{0, \ldots, N\}$) and $\tilde{G}_s^{-1} (s)$ strictly second-order stochastically dominates $G_s^{-1} (s)$ (by the claim), and the fourth line follows from undoing the rearrangement of the first line. Summing over $t$ completes the proof. \( \blacksquare \)

**Proof of Lemma 7.** If such a surjection exists for all $t \in \mathbb{N}$, taking $t = 0$ implies that player $i$ is more central than player $j$.

For the converse, I first claim that if player $i$ is $s$-more central than player $j$, then player $i$ is $s-1$-more central than player $j$. The proof is by induction on $s$. If $i$ is 2-more central than $j$, then for all $t \in \mathbb{N}$ there exists a surjection $\psi : \{k \in \mathbb{N} : d (i, k) \leq t\} \to \{k \in \mathbb{N} : d (j, k) \leq t\}$, and therefore $\# \{k \in \mathbb{N} : d (i, k) \leq t\} \geq \# \{k \in \mathbb{N} : d (j, k) \leq t\}$, so $i$ is 1-more central than $j$. Suppose that if $i'$ is $s-1$-more central than $j'$ then $i'$ is $s-2$-more central than $j'$, for all $i', j' \in \mathbb{N}$, and suppose that $i$ is $s$-more central than $j$. Then for all $t \in \mathbb{N}$ there exists a surjection $\psi : \{k \in \mathbb{N} : d (i, k) \leq t\} \to \{k \in \mathbb{N} : d (j, k) \leq t\}$ such that, for all $k$ with $d (j, k) \leq t$, there exists a $k' \in \psi^{-1} (k)$ such that $k'$ is $s-1$-more central than $k$. By hypothesis, this implies that $k'$ is $s-2$-more central than $k$, which, by the definition of $s-1$-more central, implies that $i$ is $s-1$-more central than $j$. This establishes the claim.

The claim shows that, for any players $i$ and $k$, the set of players $k'$ such that $d (i, k') \leq t$
and $k'$ is $s$-more central than $k$ is weakly decreasing in $s$ (in the set-inclusion sense). Since the sets $\{k \in N : d(i,k) \leq t\}$ and $\{k \in N : d(j,k) \leq t\}$ are finite, this implies that there exists $\bar{s} \in \mathbb{N}$ such that, for all $k$ with $d(j,k) \leq t$, the set of players $k'$ such that $d(i,k') \leq t$ and $k'$ is $s$-more central than $k$ is the same for all $s \geq \bar{s}$. Hence, if player $i$ is more central than player $j$, there exists a surjection $\psi : \{k \in N : d(i,k) \leq t\} \rightarrow \{k \in N : d(j,k) \leq t\}$ such that, for all $k$ with $d(j,k) \leq t$, there exists a player $k' \in \psi^{-1}(k)$ such that $k'$ is $s$-more central than $k$ for all $s \geq \bar{s}$. By the claim, $k'$ is also $\bar{s} - m$-more central than $k$ for all $m$, so $k'$ is $s$-more central than $k$ for all $s \in \mathbb{N}$ and is therefore more central than $k$. $\blacksquare$

**Proof of Corollary 8.** It is immediate that player $i$ is 1-more central than player $j$, and that player $i$ is strictly 1-more central if at least one of the inequalities is strict. Suppose that player $i$ is $s$-more central. To see that player $i$ is $s + 1$-more central, for any $t$ let $\psi$ be any surjection such that $\psi(k) = k$ if $d(j,k) \leq t$ and $\psi(i) = j$. Then $\psi$ satisfies the conditions in the definition of centrality, because every player $k$ is $s$-more central than herself and player $i$ is $s$-more central than player $j$ by hypothesis, and player $i$ is strictly 1-more central than player $j$ if at least one of the inequalities is strict. $\blacksquare$

**Proof of Corollary 9.** Since $\# \{k : d(i,k) \leq t\} = \# \{k : d(\rho(i),k) \leq t\}$ for all $t$ (as $\rho$ preserves distance), it is clear that $i$ is 1-more central than $j$ whenever $\rho(i)$ is 1-more central than $j$. Suppose that $i$ is $s$-more central than $j$ whenever $\rho(i)$ is $s$-more central than $j$, for all $i,j \in N$. Let $\psi : \{k \in N : d(\rho(i),k) \leq t\} \rightarrow \{k \in N : d(j,k) \leq t\}$ be a surjection such that, for all $k$ with $d(j,k) \leq t$, there exists $k' \in \psi^{-1}(k)$ such that $k'$ is $s$-more central than $k$. I claim that $\psi \circ \rho$ is a surjection from $\{k \in N : d(i,k) \leq t\} \rightarrow \{k \in N : d(j,k) \leq t\}$ such that, for all $k$ with $d(j,k) \leq t$, there exists $k'' \in (\psi \circ \rho)^{-1}(k)$ such that $k''$ is $s$-more central than $k$. This follows because, if $d(\rho(i),k') \leq t$ then $d(i,\rho^{-1}(k')) \leq t$, and if $k'$ is $s$-more central than $k$, then $\rho^{-1}(k')$ is $s$-more central than $k$ as well, by hypothesis. It follows by induction that $i$ is more central than $j$. The argument for $i$ strictly more central than $j$ is similar. $\blacksquare$
4 Dynamic Monopoly with Relational Incentives

4.1 Introduction

The possibility of trade is often threatened by the possibility of opportunism. For example, a consumer who purchases a good from an online retailer must trust that the good will actually be delivered—as taking legal action in the case of nondelivery would be very costly—and must also believe that the retailer is not about to cut its price dramatically. Fortunately, long term incentives can mitigate the risk of opportunistic behavior: in the above example, the retailer may both deliver the good and keep prices high in order to preserve its standing with its consumers, even if it has no fear of the legal consequences of nondelivery. In particular, either failing to deliver the good or cutting prices may lead consumers to believe that the firm will not deliver the good in the future, as either of these actions could be interpreted as an indication that the firm is trying to maximize its short run profits and then quit the market. This reasoning suggests that a seller who is tempted to fail to deliver her product may still do quite well if the future is sufficiently important. This paper studies this idea in the context of both non-durable and durable goods monopoly, focusing primarily on the more involved durable goods case.

The above intuition contrasts starkly with the Coase conjecture (Coase, 1972) that a patient durable-goods seller that cannot commit to future prices earns little profit. As we will see, the Coase conjecture relies on the assumption that the seller is committed to delivering the good at her quoted price. In particular, the Coasian temptation to cut prices is absent when a price cut leads to a continuation equilibrium in which no consumers make purchases (expecting nondelivery) and the seller never delivers (expecting no future purchases). Thus, even if the seller cannot commit to a price path she can still earn high profits if she is not committed to delivering the good, either. This suggests that the Coasian temptation to cut prices is absent when a price cut leads to a continuation equilibrium in which no consumers make purchases (expecting nondelivery) and the seller never delivers (expecting no future purchases).

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66 An alternative story, discussed below, is that the retailer is contractually obligated to deliver something but that the quality of the good it delivers is unverifiable. In this case, it is natural to think that a price cut may suggest to consumers that the retailer intends to deliver a low quality good. For an example of an online market in which lower-priced goods seem to be of extremely low quality, see Ellison and Ellison (2009).

67 Of course, the seller now has an incentive to fail to deliver the good, so the result that the seller can earn high profits is not trivial.

68 This reasoning is similar to Bernheim and Whinston’s (1998) point that if some aspects of behavior are
conjecture may not apply to any institutional setting: If the seller can legally commit herself to both a price path and delivery of the good, she should do so. If she can legally commit herself to delivery, but not to a price path, she should not.\(^{69,70}\)

Throughout, I consider an infinitely-repeated interaction between a monopoly seller and a continuum of buyers, where in every period the seller first sets a price, consumers then choose whether or not to pay, and finally the seller chooses whether or not to deliver the good to each consumer. All actions are perfectly observable. If the good is non-durable and consumers are anonymous, I completely characterize the optimal perfect Bayesian equilibrium of this game for the seller: if the seller is sufficiently patient, she sets the static monopoly price each period and delivers the good to all consumers who purchase, while if she is less patient she charges a higher price in order to reduce the quantity demanded and thereby reduce her temptation to fail to deliver.\(^{71}\)

When the good is durable, the structure of any equilibrium in which the seller delivers the good is complicated: sales must continue forever, since the seller would never deliver the good to the last consumer, and the price path must fall slowly enough that consumers do not always wait for lower prices but quickly enough that sales do not occur so rapidly that the seller gives in to her temptation to fail to deliver. Indeed, with a general distribution of consumer valuations, it is very difficult to construct any equilibria in which the seller always delivers the good.\(^{72}\) I therefore take an indirect approach to analyzing this model by first considering an auxiliary model where the seller has the ability to set a maximum sales quantity each period in addition to the price, thereby rationing the good. The main result noncontractible it is often optimal to fail to contract on other aspects as well.

\(^{69}\)This is a slight oversimplification as there will be many equilibria in the model, not all of which yield high profits. For example, if consumers believe that the monopoly will never deliver the good unless it legally commits itself to do so, then of course so committing is the right move. On the other hand, the dynamic contracting literature often uses profit maximization as an equilibrium refinement and it does not seem more unreasonable than usual to do so here.

\(^{70}\)In some environments, the seller may be "automatically" committed to delivering the good, for example if nondelivery is viewed by courts as breaching an "implicit" contract. To address this issue, in Section 7 I show that my results extend to a setting where in each period the seller has an exogenous chance of being unable to deliver the good. In such a setting the issue that nondelivery may be viewed as breaching an implicit contract may not arise, since nondelivery always occurs occasionally.

\(^{71}\)The first part of this statement also holds when consumers are non-anonymous, in contrast with the results of Hart and Tirole (1988). See the discussion following Proposition 10.

\(^{72}\)As discussed below, it is much easier to construct equilibria in which the seller sometimes fails to deliver the good, but these equilibria may be unappealing for other reasons.
in this model with rationing, which I see as being of some independent interest, is that using rationing is never optimal for the seller. I then show that the seller’s optimal profit in the original model must exceed her profit in any equilibrium involving rationing.

This observation allows us to derive a lower bound on the seller’s profit in the original model—where constructing equilibria is very difficult—by constructing simple equilibria in the model with rationing. In particular, I construct equilibria in which price is constant over time but quantity sold every period is restricted via rationing. These quantity restrictions lead to positive residual demand, which gives the seller a reason to deliver the good. I show that a patient seller can approximate her static optimal profit level by setting price equal to the static monopoly price every period and selling to those consumers who are willing to buy at this price at a constant rate. Furthermore, for any discount factor $\delta$, the seller’s optimal profit is at least as high as the static monopoly profit of a seller with cost of delivering the good equal to $c/\delta$, where $c$ is the cost of delivering the good in the dynamic model, as this is precisely the profit level that can be attained by setting price equal to the static monopoly price of a seller with cost $c/\delta$ and then selling (at cost $c$) at the fastest rate at which the seller is willing to deliver in the dynamic model. I also use the relationship between my model and the model with rationing to show that the best equilibria for the seller in which she delivers the good to all consumers who purchase involve a strictly declining price path that asymptotes to a price no lower than $c/\delta$.

I proceed as follows: Section 4.2 relates this paper to the literatures on the Coase conjecture, strategic rationing, and relational contracting. Section 4.3 introduces the general model of both durable and non-durable goods monopoly with relational incentives. Section 4.4 analyzes the model in the simpler case of a non-durable goods monopoly. It is included both for completeness and because of connections between it and the subsequent analysis of the durable goods model. Section 4.5 introduces the model with a durable goods monopoly, as well as the model with rationing, and studies the connection between the two, ultimately showing that the best equilibrium without on-path non-delivery for the seller in the model without rationing yields profit at least as high as that in any equilibrium without on-path non-delivery in the model with rationing. Building off this insight, Section 4.6 presents my main results on the durable goods model: profits are bounded from below by those of a static...
monopoly with cost $c/\delta$, and the best equilibrium price path along which the seller always delivers strictly declines over time and asymptotes to at least $c/\delta$. Section 4.7 extends the analysis to a setting in which the seller is sometimes (exogenously) unable to deliver the good, where the assumption that the seller has the option of nondelivery seems particularly appropriate. Section 4.8 concludes and discusses some applications and empirical predictions of the model. Several proofs are deferred to Appendix A, and Appendix B discusses equilibria in which the seller does not always deliver the good along the equilibrium path.

4.2 Relation to the Literature

As indicated above, my results stand in stark contrast with the Coase conjecture (Coase, 1972), which was formalized and explored by Stokey (1982), Bulow (1982), Fudenberg, Levine, and Tirole (1985), and Gul, Sonnenschein, and Wilson (1986). My model would coincide with the standard "no-commitment" durable goods monopoly model if the seller, while still lacking commitment power over prices, was committed to delivering the good to all consumers who purchase. In this sense, my model has "less commitment" than this standard "no-commitment" case, though of course the reason the seller does better in my model is not that it has less commitment power but rather that committing to delivering the good to all consumers who purchase may not be wise, as after making such a commitment the seller is tempted to cut prices.

The literature on the Coase conjecture draws a sharp distinction between the "gap case"

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74In traditional Coase conjecture papers, like Fudenberg, Levine, and Tirole (1985); Gul, Sonnenschein, and Wilson (1986); and Ausubel and Deneckere (1989), the model may be interpreted as a monopoly selling to either a continuum of consumers with a known distribution of valuations or to a single consumer with unknown valuation. In the current paper, only the first interpretation is applicable, as in the single-buyer case the monopoly would never deliver the good after the buyer purchased, so there would be no equilibrium in which trade occurs.
in which the lowest consumer valuation is strictly greater than the seller’s marginal cost and the alternative "no-gap case." In the gap case, Fudenberg, Levine, and Tirole (1985) and Gul, Sonnenschein, and Wilson (1986) show that there is generically a unique perfect Bayesian equilibrium, which is Markovian and satisfies the Coase conjecture. In the no-gap case, a seminal paper by Ausubel and Deneckere (1989) constructs non-Markovian equilibria that yield static monopoly profits as the discount factor approaches one. The reason for the difference between the cases is that in the gap case the seller is always tempted to cut prices to the lowest consumer valuation, which allows the problem to be solved by backward induction, while in the no-gap case the possibility that price may fall to marginal cost very quickly if the seller deviates from a prescribed price path allows the seller to maintain high prices in equilibrium. This distinction between the gap and no-gap cases does not arise in my model, since in my model the off-path expectation that prevents the seller from cutting prices is that the seller will not deliver the good, not that the seller will rapidly cut prices. My analysis of durable goods monopoly does more than showing that the possibility of non-delivery allows Ausubel and Deneckere-style equilibria to be constructed in the gap case, however: as indicated above, I also provide a natural lower bound on seller payoffs for a fixed discount factor δ and prove that, for any δ, the best equilibrium for the seller in which there is no non-delivery has declining prices converging to a price no lower than \( c/\delta \). Results for fixed δ and characterizations of optimal equilibria are rare in the durable-goods monopoly literature. For example, for δ bounded away from one, none of the early papers on the Coase conjecture cited above contain results about optimal seller profits or the asymptotic behavior of the optimal price path.

Because my approach relies on comparing the model to an auxiliary model in which the seller is able to ration the good, the paper connects to the literature on strategic rationing. One lesson from this literature is that rationing in the absence of an efficient resale market, i.e., when the highest-valuation consumers do not always receive the good when there is a shortage, can help the seller both when it can commit to a price path (Van Cayseele, 1991) and when it cannot (Denicolò and Garella, 1999). Both Van Cayseele and Denicolò and Garella consider short finite horizons and state that rationing in the presence of an efficient resale market is never optimal. As part of the analysis of the durable-goods model, I show
that this result holds in an infinite-horizon setting. My focus is very different from that of Van Cayseele and Denicolò and Garella, as they are interested primarily in cases where allowing rationing can increase profits, while I am interested precisely in cases where allowing rationing cannot increase profits, so that I can use the model with rationing to derive results about the model without rationing.

Finally, my paper is related to the literature on relational contracting, particularly that part of the relational contracting literature that studies durable goods with hidden quality, which originated with the famous papers of Klein and Leffler (1981) and Shapiro (1982, 1983). While traditional models of durable goods monopoly can be thought of as "relational" in that they study the effect of dynamic incentives on a seller's decision to cut prices, I go further and assume that dynamic incentives also govern the seller's decision to deliver the good. Thus, the difference between my model and the existing literature on dynamic seller is that I move a decision—delivery—from formal to relational enforcement. Also, the equilibria I construct induce cooperation through the Nash threat of breaking off trade, as in many relational contracting models (e.g., Bull, 1987; Levin, 2003). Indeed, a key difference between my model and traditional models of dynamic monopoly is that my model admits a Nash equilibrium in which the seller receives her minmax value.

4.3 Model

Throughout, consider a seller who can provide a good at marginal cost $c > 0$ facing a continuum of consumers of mass 1 with valuations (per period in the case of non-durables, net present value in the case of durables) $v \sim F(v)$ with bounded support $[v, \bar{v}]$ with $v \geq 0$, $\bar{v} > c$, and $F$ continuously differentiable with strictly positive density $f$. There is a continuum of consumers with each valuation in $[v, \bar{v}]$, so that if a random fraction $x$ of consumers receive the good in some period then that fraction $x$ of consumers with every valuation receive the good. I do not make any assumptions as to whether $v$ is greater than or less than $c$, i.e., as to whether we are in the gap or no-gap case. Let $p^m$ be the static monopoly price of a

\footnote{The relevant result (Proposition 13) assumes that the seller has the option of failing to deliver the good, but the proof shows that the result continues to hold when the seller does not have this option.}

\footnote{For an up-to-date survey of this rapidly expanding literature, see Malcomson (2009). For a recent contribution with some similarities to the current paper, see Masten and Kosová (2009).}
seller facing consumers with valuations \( v \sim F(v) \) and marginal cost \( c \).

The traditional "no-commitment" model of dynamic monopoly is the following infinitely repeated game:

1. At time \( t \in \{0, 1, \ldots \} \), the seller chooses a menu of price-delivery probability pairs \( \{(p_{t,n}, x_{t,n})\} \).

2. Every consumer either selects a price-delivery probability pair \( (p_{t,n}, x_{t,n}) \in \{(p_{t,n}, x_{t,n})\} \) or rejects. Consumers who select \( (p_{t,n}, x_{t,n}) \) pay \( p_{t,n} \) and receive the good with probability \( x_{t,n} \). The seller gets payoff \( p_{t,n} - c \) from each consumer who pays \( p_{t,n} \) and receives the good, and gets \( p_{t,n} \) from each consumer who pays \( p_{t,n} \) and does not receive the good. A consumer with valuation \( v \) who pays \( p_{t,n} \) gets payoff \( v - p_{t,n} \) if she receives the good and gets payoff \( -p_{t,n} \) if she does not receive the good.

3. Repeat 1-2, discounting by (common) discount factor \( \delta \).

In the current model, the seller has the option of nondelivery. The game becomes:

1. At time \( t \in \{0, 1, \ldots \} \), the seller chooses a menu of prices \( \{p_{t,n}\} \).

2. Every consumer either selects a price \( p_{t,n} \in \{p_{t,n}\} \) or rejects. Consumers who select \( p_{t,n} \) pay \( p_{t,n} \). Let \( Q_{t,n} \) be the mass of consumers who pay \( p_{t,n} \).

3. For each \( p_{t,n} \), the seller chooses what fraction \( x_{t,n} \in [0, 1] \) of those \( Q_{t,n} \) consumers who pay \( p_{t,n} \) receive the good. Each consumer who pays \( p_{t,n} \) receives the good with probability \( x_{t,n} \). Payoffs are as above.

4. Repeat 1-3, discounting by \( \delta \).

I assume that players use strategies that depend on consumers’ decisions at time \( t \) only through \( Q_{t,n} \). This entails assuming that the seller does not condition her strategy on play by measure 0 sets of consumers, as is standard in the durable goods monopoly literature, as well as that consumers are anonymous.\(^77\) In particular, the seller cannot discriminate among consumers on the basis of their past play in either her pricing or delivery decisions.

\(^77\) See the discussion following Proposition 10 for more on this point.
Crucially, I assume that all decisions of the seller are publicly observed. Formally, let the history $h^t$ at the start of period $t$ be

$$\{\{p_{0,n}\}, \{Q_{0,n}\}, \{x_{0,n}\}, \ldots, \{p_{t-1,n}\}, \{Q_{t-1,n}\}, \{x_{t-1,n}\}\}.$$ 

Each of the seller’s (pure) strategies is a pair of maps from histories $h^t$ to $\{p_{t,n}\}$, where $p_{t,n} \in [0, \infty)$ for all $t, n$, and from histories $(h^t, \{p_{t,n}\}, \{Q_{t,n}\})$ to $x_{t,n} \in [0, 1]$ for all $Q_{t,n}$; while a consumer’s (pure) strategy is a map from histories $(h^t, \{p_{t,n}\} \rightarrow \{\{p_{t,n}\}, \emptyset\}$, corresponding to accepting a price $p_{t,n}$ or rejecting. Note that, for any strategy profile, changing the strategy of a single consumer does not affect the probability distribution over histories $h^t$ for any $t$; that is, a deviation by a single consumer does not affect the path of play.

Throughout, the solution concept is pure strategy Perfect Bayesian Equilibrium, which I simply abbreviate as PBE. Of course, the assumption that the seller uses a pure strategy does not imply that she chooses $x_{t,n} \in \{0, 1\}$, but rather that she does not randomize over different choices of $\{p_{t,n}\}$ or $\{x_{t,n}\}$. I have not explored whether mixed strategy equilibria can differ substantially from pure strategy equilibria; however, the main results that the seller can earn high profits in equilibrium can only be strengthened by considering mixed strategy equilibria.

I observe immediately that in either the non-durable or durable goods version of the model there is a Nash equilibrium in which consumers reject all price offers and the seller sets $x_{t,n} = 0$ for all $t, n$. The threat of reversion to this equilibrium following any deviation may induce the seller to conform to a prescribed price path as well as to deliver the good to those consumers who purchase. No such Nash equilibrium exists in the traditional no-commitment model.

I make frequent use of the following definition:

**Definition 16** A PBE is optimal if there is no other PBE that yields strictly higher payoff for the seller.

Finally, I briefly note an alternative interpretation of the model in terms of product quality. Suppose that the seller is (for whatever reason) contractually obligated to deliver...
at least a low-quality good (at cost normalized to zero) to any consumer who purchases and is able to deliver a high-quality good at additional cost $c$, and that quality is noncontractable. If every consumer has valuation zero for the low-quality good, the model is unchanged, with "low-quality delivery" substituted for "nondelivery." This interpretation depends on every consumer's having valuation zero for the low-quality good, and thus may be most attractive when quality is extremely difficult to verify. For example, the good may be a complicated, high-tech upgrade of an existing piece of hardware or software, which has no value at all for consumers if it is not superior to the original product, and outside observers are unable to verify whether the "upgrade" is in fact better than the original.\textsuperscript{78}

4.4 Non-Durable Goods Monopoly

In this section, each consumer demands one unit of the good each period, and $v$ is a consumer's per-period valuation. I also assume, for this section only, that $v - \frac{1 - F(v)}{f(v)}$ is weakly increasing, so that in the static monopoly allocation every consumer with positive virtual surplus receives the good.\textsuperscript{79}

The main result in this section is that, in the optimal equilibrium,\textsuperscript{80} the seller sets the (single) price equal to the static monopoly price if she is sufficiently patient, and otherwise sets the lowest price at which she is willing to deliver the good. The intuition is that the seller's incentive to fail to deliver the good is increasing in quantity; so if the seller is impatient she must restrict quantity in order to credibly commit to delivery; and the most profitable way to do this is to increase price. In particular, the seller sets $p = \max \{p^*, c/\delta\}$ every period. To see why $c/\delta$ is the lowest price at which the seller is willing to deliver the good, let $D(p) \equiv 1 - F(p)$ be demand at price $p$, and note that in every period the seller gains $cD(p)$ from failing to deliver and gains $\frac{\delta}{1 - \delta} (p - c) D(p)$ from delivering. The latter is

\textsuperscript{78}My results do not apply if consumers have positive valuations for the low-quality good, since in this case the model need not have a Nash equilibrium that yields zero profit. However, two recent papers illustrate interesting phenomena that may occur in such settings. Inderst (2008) shows that a durable goods monopoly that sells low- and high-quality goods may serve the entire market in the first period, selling the low-quality good to low-valuation consumers as a means of committing itself not to subsequently offer the high-quality good at a lower price. Hahn (2006) shows that this logic may provide an incentive for a durable goods monopoly to introduce a damaged version of its good and argues that this often has negative welfare consequences.

\textsuperscript{79}This assumption is for technical convenience only.

\textsuperscript{80}The proof of Proposition 10 shows existence and uniqueness of an optimal equilibrium.
weakly greater than the former if and only if \( p \geq c/\delta \). The idea of the proof is to first note that the seller can in effect commit to any price path, since deviations in price-setting may lead consumers to believe that the seller will not deliver the good and thus lead to zero sales; next observe that the best dynamic sales mechanism for the seller is stationary, as increasing one period’s profits also relaxes the seller’s incentive compatibility (willingness to deliver) constraints from earlier periods; and finally use standard static mechanism techniques to characterize the optimal stationary mechanism that is incentive compatible for the seller. The proof is deferred to Appendix A.

**Proposition 10** If \( \bar{v} \geq \frac{c}{\delta} \), the equilibrium path of the optimal PBE of the non-durable goods model is given by \( p_{t,n} = \max \{ p^m, \frac{c}{\delta} \} \) for all \( t, n \), buyers accept if and only if \( v \geq p_{t,n} \), and the seller delivers the good with probability 1 to all buyers who accept each period. That is, the seller offers only a posted price \( p \) in every period, \( p = p^m \) if \( \delta \geq \frac{c}{p^m} \), and \( p = \frac{c}{\delta} > p^m \) if \( \delta < \frac{c}{p^m} \). If \( \bar{v} < \frac{c}{\delta} \), there is no PBE in which the seller ever delivers the good or receives positive payments.

Recall that I have assumed that buyers are anonymous. Nonetheless, it is not hard to construct equilibria that yield static monopoly profits even if buyers are non-anonymous, provided that \( \delta \geq c/p^m \). For example, let the seller set \( p = p^m \) in every period and deliver the good if and only if she has both always delivered the good to all consumers who have purchased and set \( p = p^m \) in the past, and let each consumer purchase the good every period if and only if her valuation exceeds \( p^m \) and the seller has always delivered the good to all consumers who have purchased and has always set \( p = p^m \). In every period, the seller gains \( cD(p^m) \) from failing to deliver and gains \( \frac{\delta}{1-\delta} (p^m - c) D(p^m) \) from delivering, so the seller will deliver if \( \delta \geq c/p^m \). This result differs dramatically from the classic analysis of non-durable goods monopoly with non-anonymous consumers provided by Hart and Tirole (1988). Hart and Tirole show that, in a finite-horizon model with non-durable goods and non-anonymous consumers, equilibrium is governed by the ratchet effect: in every PBE, if \( v > c \), then \( p_t = v \) for all but the last few periods. Technically, the difference between my result and theirs comes from the fact that the stage game in my model has a bad Nash equilibrium ("reject any offer, never deliver"), which can be used as an off-equilibrium threat to prevent the seller
from using information revealed early on against high-valuation buyers.\textsuperscript{81} The key economic point is that the usual repeated game tradeoff between a short term gain from cheating and a long term gain from cooperation on the part of the seller is absent in the Hart-Tirole model: in their model, the seller is free to "cheat" by raising the price she charges to buyers that reveal themselves to have high valuations, but buyers cannot credibly retaliate by refusing to buy at the higher price. In my model, the option of the seller to fail to deliver the good lets the buyer credibly punish the seller for raising the price, allowing the seller to "commit" to keeping the price constant. On the other hand, I must now keep track of the seller’s incentive to deliver the good. If $\delta \geq c/p^m$, this incentive constraint is slack, so the seller can attain her full-commitment optimum.

4.5 Durable Goods Monopoly and Rationing

4.5.1 Preliminaries

For the remainder of the paper, each consumer demands only one unit of the (durable) good, and $v$ is a consumer’s net present value of receiving the good. In the traditional model of this situation (see Section 4.3), Fudenberg, Levine, and Tirole (1985) and Gul, Sonnenschein, and Wilson (1986) show that the Coase conjecture applies if the lowest valuation $v$ is greater than $c$: for generic parameters there is a unique PBE, and as $\delta$ goes to 1 the seller’s profit goes to $v - c$ and the price drops to $v$ very quickly.

The main result implies that the Coase conjecture does not apply to this model when the seller has the option of nondelivery (see Section 4.3), which I call the "relational contracting model," or $\Gamma$. Much of the analysis focuses on a particular class of PBE, which I call "full-delivery PBE." A full-delivery PBE is a PBE in which the seller sets $x_{t,n} = 1$ for all $n$ at all histories on the equilibrium path. It is important to note that the seller may set $x_{t,n} < 1$ off the equilibrium path in a full-delivery PBE. A full-delivery PBE is a best full-delivery PBE if there is no other full-delivery PBE that yields strictly higher payoff for the seller— I use the word "optimal" for the best PBE overall and "best" for the best full-delivery PBE to help

\textsuperscript{81}The infinite-horizon version of the Hart-Tirole model has equilibria that yield seller profits above $v - c$, though how much above $v - c$ has to our knowledge not been studied in the literature. Thus, it is possible that some of the difference in results is due to the difference in time horizons.
avoid confusion. Note that on the equilibrium path of a full-delivery PBE there is no reason for the seller to offer a menu of prices, as each consumer will either accept the lowest offered price or reject, so I simplify notation by writing $p_t$ for the lowest price offered by the seller at time $t$ on the equilibrium path. Furthermore, a consumer who pays $p_t$ always receives the good at time $t$; I say that a consumer who pays $p_t$ at time $t$ on the equilibrium path of a full-delivery PBE purchases the good at time $t$. Since I have restricted attention to pure strategy equilibria, every consumer purchases at exactly one time in every full-delivery PBE, with the convention that a consumer who never receives the good "purchases" at $t = \infty$.

Clearly, an optimal PBE of the relational contracting model can yield no higher payoff to the seller than an optimal PBE of the "full-commitment" model in which the requirement that the seller’s strategy is sequentially rational is relaxed, and it follows from standard results that an optimal PBE of this full-commitment model yields profits equal to optimal static monopoly profits. The main result is the following, which implies that the Coase conjecture does not hold in this game regardless of the relationship between $v$ and $c$ and also provides a lower bound on the seller’s profit for any fixed $\delta$:

**Theorem 10** In the relational contracting model:

1. An optimal PBE exists.
3. As $\delta$ approaches 1, profit in a best full-delivery PBE approaches static monopoly profit.
4. If $\bar{v} > \frac{c}{\delta}$ and cost equals $c$, there exists a full-delivery PBE in which profit is strictly greater than static monopoly profit when cost equals $\frac{c}{\delta}$.
5. If $\bar{v} > \frac{c}{\delta}$, any best full-delivery PBE has a strictly decreasing price path and involves positive sales in every period.
6. If $\bar{v} > \frac{c}{\delta}$, $p_t \geq \max \{ v, \frac{c}{\delta} \}$ for all $t$ in any best full-delivery PBE.
7. If $\bar{v} \leq \frac{c}{\delta}$, there is no PBE in which the seller ever delivers the good or receives positive payments.
Sections 5 and 6 devoted to establishing Theorem 10: parts 1 and 2 are proved in this section (in Propositions 11 and 14) and parts 3 through 7 are proved in Section 4.6 (in Propositions 15 through 18). I therefore take a moment to motivate devoting so much attention to full-delivery PBE. Full-delivery PBE are those equilibria in which on-path delivery is as in both the full-commitment model (in which the seller commits to both a price path \((p)_t\) and a delivery path \((x)_t\)) and in the traditional no-commitment model described in Section 4.3, which makes them a natural class of equilibria to study. Indeed, on-path non-delivery—the equivalent of the seller selling "lottery tickets" that entitle consumers to receive the good with some probability less than 1—may be unappealing in some settings, for example if consumers can tell whether the seller has failed to deliver the good to anyone but not whether the seller has delivered to some exact fraction of consumers. Furthermore, Theorem 10 implies that the profit lost by the seller in a best full-delivery PBE as opposed to an optimal PBE is bounded from above by the difference between static monopoly profit when cost equals \(c\) and when cost equals \(c/\delta\), which is small for \(\delta\) close to 1. Nonetheless, I conjecture that in general the optimal PBE is not full-delivery, for reasons discussed in Appendix B. Appendix B proves the analogs of parts 3 and 4 of Theorem 10 for non-full-delivery equilibria directly, i.e., without relying on the connection between the relational contracting model and the related model with rationing introduced below. The approach of Appendix B also has the advantage of explicitly constructing equilibria in the relational contracting model, while the approach taken in the body of the paper in nonconstructive. Thus, there are at least two very different kinds of PBE that yield high seller profits: full-delivery PBE with declining price paths, whose existence is proven nonconstructively in the text; and non-full-delivery equilibria with constant price paths, which are constructed in Appendix B.

I adopt a novel approach to proving Theorem 10. I first introduce the following variant of the relational contracting model, in which the seller can artificially restrict the quantity of the good supplied each period:

1. The seller chooses a price \(p_t\) and a maximum quantity to supply \(q_t \in [0, 1]\).\(^{82}\)

\(^{82}\)For the remainder of the paper, \(q_t\) refers to the quantity cap in period \(t\) and \(Q_t\) refers to the number of consumers who pay in period \(t\) (i.e., the period \(t\) quantity). By construction of the model with rationing,
2. Every consumer chooses whether or not to accept $p_t$. If less than $q_t$ consumers accept, all consumers who accept pay $p_t$. Otherwise, the $q_t$ consumer with the highest valuations among those who accept pay $p_t$. Formally, a consumer with valuation $v$ who accepts pays if and only if the mass of consumers with valuation strictly greater than $v$ who accept is strictly less than $q_t$.

3. If measure $Q_t$ of consumers pay $p_t$ (which I call the period $t$ quantity), the seller chooses what fraction $x_t \in [0, 1]$ of these consumers receive the good. Each consumer who pays $p_t$ receives the good with probability $x_t$.

4. Repeat 1-3, discounting by $\delta$.

I have not allowed the seller to offer menus of prices as this would only complicate notation, since I restrict attention to full-delivery PBE in what follows.

I call this game the "relational contracting model with rationing," or simply the "model with rationing," or $\Gamma_R$. Optimal, full-delivery, and best full-delivery PBE in $\Gamma_R$ are defined as in $\Gamma$. The main reason I introduce $\Gamma_R$ is that full-delivery equilibria in $\Gamma_R$ may have flat price paths, while every full-delivery equilibrium in $\Gamma$ must involve price cuts, as otherwise there would be no way to delay sales and thereby induce delivery. Full-delivery equilibria with flat price paths are easy to analyze, as consumers' incentives in such equilibria are trivial: if the price is fixed at $p$ in a full-delivery equilibrium, a consumer with valuation $v \geq p$ wants to purchase as soon as possible, while a consumer with $v < p$ will never purchase. I will show that full-delivery equilibria with flat price paths exist in $\Gamma_R$ that approximate static monopoly profits for high $\delta$. Furthermore, I will show that a price-quantity path $(p, Q)_t$ is a best full-delivery PBE price-quantity path in $\Gamma$ if and only if it is a best full-delivery

\[ Q_t \leq q_t. \]

In defining $\Gamma_R$ I have made two assumptions on the rationing technology: that types "on the boundary" between receiving the good and not do not receive the good, and that any rationing that occurs is "efficient," in that the highest-valuation consumers are eligible to receive the good. The first assumption is only for technical convenience and simplifies the proof of Lemma 10. The second assumption is substantive, as Van Cayseele (1991) shows that under full-commitment a monopoly can achieve profits above static monopoly profits by using "inefficient" rationing. The second assumption is descriptive in the presence of a frictionless resale market. Alternatively, one could view the model with rationing entirely as a technical aid in analyzing the model without rationing.

The results about $\Gamma_R$, especially Proposition 13, may also be of independent value to readers interested in strategic rationing.
PBE price-quantity path in $\Gamma_R$ (Corollary 12, in Section 4.5.4). Therefore, the best full-delivery PBE profit attainable by the seller is the same in $\Gamma$ and $\Gamma_R$, so the above observation that simple full-delivery PBE exist in $\Gamma_R$ in which profits approximate static optimal profits immediately yields part 3 of Theorem 10, even though no such simple full-delivery PBE exist in $\Gamma$. The proofs of parts 2 and 4 through 7 of Theorem 10 also rely on Corollary 12, as we will see; thus, Corollary 12 is the key to my approach to proving Theorem 10.

To summarize the above roadmap, Sections 4.5 and 4.6 establish the following chain of inequalities:

\[
\text{Optimal PBE Profit in } \Gamma \geq \text{Best Full-Delivery PBE Profit in } \Gamma \text{ (by definition)} = \text{Best Full-Delivery PBE Profit in } \Gamma^R \text{ (by Corollary 12)} > \text{Best Full-Delivery, Constant-Price PBE Profit in } \Gamma^R \text{ (by Proposition 16)} = \text{Static Monopoly Profit with Cost } c/\delta \text{ (by Corollary 13)}.\]

Before beginning the analysis of $\Gamma_R$, I first prove part 1 of Theorem 10 directly. The proof proceeds by first showing that the seller’s profit is continuous in price-delivery paths $(p, x)_t$ and then showing that any price-delivery path can be supported in PBE by endowing consumers with the belief that the seller will never deliver the good if she ever deviates from her prescribed price-delivery path. The details are deferred to Appendix A.

**Proposition 11 (Theorem 10.1)** *An optimal PBE exists in $\Gamma$.*

### 4.5.2 Existence of Best Full-Delivery PBE in the Model with Rationing

I now begin the analysis of the full-delivery PBE of $\Gamma$ and $\Gamma_R$ and the relationship between them. The goal of this subsection is to show that a best full-delivery PBE exists in $\Gamma_R$. Start with a definition:

**Definition 17** *Given a price path $(p)_t$, a valuation $v$ is generic with respect to $(p)_t$ if\[
\delta^t(v - p_t) \neq \delta^t'(v - p_{t'})\]
for all \( t \neq t' \). If not, \( v \) is nongeneric with respect to \((p)_t\).

That is, a valuation \( v \) is generic with respect to \((p)_t\) if a consumer with valuation \( v \) is not indifferent between purchasing at any two times \( t \) and \( t' \) when prices are given by \((p)_t\). For any price path \((p)_t\), there are only countably many valuations which are nongeneric with respect to \((p)_t\), so the assumption that \( F \) admits a strictly positive density immediately yields the following observation:

**Lemma 8** For any price path \((p)_t\), the set of valuations \( v \in [\underline{v}, \bar{v}] \) that are generic with respect to \((p)_t\) has measure 1.

I now present a series of lemmas that are needed to prove existence of a best full-delivery PBE in \( \Gamma_R \). The longer proofs are deferred to Appendix A.

Lemma 9 simply states that any two consumers with the same valuation receive the same payoff in any PBE, and consumers with higher valuations receive higher payoffs:

**Lemma 9** In any PBE of \( \Gamma \) or \( \Gamma_R \), any two consumers with the same valuation, \( v \), receive the same PBE payoff, \( V_v \). If \( v \geq v' \), then \( V_v \geq V_{v'} \).

**Proof.** The first part follows because at any PBE a consumer with valuation \( v \) can deviate to the strategy of another consumer with valuation \( v \) and receive the same payoff as him, because the actions of a single consumer do not affect the path of play (in either \( \Gamma \) or \( \Gamma_R \)). The second part follows because at any PBE a consumer with valuation \( v \geq v' \) can deviate to the strategy of a consumer with valuation \( v' \) and receive a weakly higher payoff than him (in \( \Gamma_R \), this relies on the fact that a consumer with higher valuation can purchase whenever a consumer with lower valuation can do so), again because the actions of a single consumer do not affect the path of play. ■

The next two lemmas show that, across all full-delivery PBE, the price-rationing path \((p, q)_t\) uniquely determines the quantity path \((Q)_t\). Lemma 10 is not trivial because the set of times at which a consumer is able to purchase under price-rationing path \((p, q)_t\) depends on the times at which higher-valuation consumers are purchasing. The intuition for the result is that if a consumer with valuation \( v \) cannot purchase at the same set of times under two PBE,
then there must be a nontrivial mass of higher-valuation consumers who cannot purchase at
the same set of times under the two PBE, either, as otherwise almost all higher-valuation
consumers would purchase at the same times under both PBE and the original consumer
would not have been "rationed out" of purchasing at his preferred time. Therefore, there
can be no valuation \( v \) that is "approximately" the highest valuation that gets "rationed out,"
which implies that no valuation can be "rationed out."

**Lemma 10** Given a price-rationing path \((p, q)_t\) in \( \Gamma_R \) and a valuation \( v \) that is generic with
respect to \((p)_t\), there exists a time \( \tau_v \) such that every consumer with valuation \( v \) purchases
at \( \tau_v \) in any full-delivery PBE in \( \Gamma_R \) with price-rationing path \((p, q)_t\).

Combining Lemma 8 and Lemma 10 immediately yields the following:

**Lemma 11** Given price-rationing path \((p, q)_t\), every full-delivery PBE in \( \Gamma_R \) with price-rationing path \((p, q)_t\) has the same quantity path \((Q)_t\).

In fact, this quantity path \((Q)_t\) can be viewed as a continuous function of the price-rationing path \((p, q)_t\):

**Lemma 12** The unique quantity path \((Q)_t\) that may occur in a full-delivery PBE in \( \Gamma_R \) with
price-rationing path \((p, q)_t\) is continuous in \((p, q)_t\) in the product topology.

I now show that a best full-delivery PBE exists in the model with rationing (Proposition
12). This holds because the set of full-delivery PBE price-rationing-quantity paths can
be shown to be compact in the product topology, and the seller’s profit is continuous in
price-rationing-quantity paths. It is straightforward to show that the set of full-delivery
PBE price-rationing paths is compact: the seller can be induced to set any price-rationing
path if consumers believe that she will never deliver the good if she sets the wrong path,
and the seller is willing to deliver \( Q \) units of the good if she is willing to deliver \( Q - \varepsilon \) for
all small \( \varepsilon \). The difficulty is showing that small changes in the price-rationing path induce
small changes in the quantity path. This is taken care of by Lemmas 11 and 12, which are
both proved in Appendix A.

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85 Technically, this holds for price paths with \( p_t \in [\underline{p}, \overline{p}] \) for all \( t \), which one can restrict attention to without loss of generality.
Proposition 12 A best full-delivery PBE exists in $\Gamma_R$.

Proof. Let $\mathcal{F}$ be the set of full-delivery PBE price-rationing-quantity paths $(p, q, Q)_t$ in $\Gamma_R$ satisfying $p_t \in [v, \bar{v}]$ for all $t$. Note that if a PBE is best in the set of PBE with price-rationing paths in $\mathcal{F}$, then it is best overall, as any PBE with $p_t > \bar{v}$ for some $t$ yields no more profit than a PBE with an identical price-rationing path but with $p_t = \bar{v}$ for all such $t$ instead, and similarly for $p_t < v$. Given a price-rationing-quantity path $(p, q, Q)_t$, the associated profit for the seller is

$$\sum_{t=0}^{\infty} \delta^t (p_t - c) \min \{q_t, Q_t\},$$

which is obviously continuous in $(p, q, Q)_t$ in the product topology. I will show that $\mathcal{F}$ is compact in the product topology, and then apply Weierstrass's Theorem to complete the proof.

Observe that $\mathcal{F} \subseteq \prod_{t=0}^{\infty} ([v, \bar{v}], [0,1], [0,1])_t$, which is compact by Tychonoff's Theorem. Therefore, to show that $\mathcal{F}$ is compact in the product topology it suffices to show that $\mathcal{F}$ is closed in the product topology. To see that it is, consider a sequence of paths $\{(p, q, Q)_t\}_n \in \mathcal{F}$ converging pointwise to $(p^*, q^*, Q^*)_t$. I must show that there exists a full-delivery PBE with price-rationing-quantity path $(p^*, q^*, Q^*)_t$. Consider the following strategy profile:

1. The seller sets price-rationing path $(p^*, q^*)_t$ and $x_t = 1$ as long as she has conformed to this strategy in the past. Otherwise, she sets $p_t = \bar{v}, q_t = 1, x_t = 0$ for all future periods. In particular, the seller sets $x_t = 0$ in any period in which has set $p_t \neq p_t^*$.

2. A consumer with valuation $v$ who has not yet received the good at $t$ accepts at $t$ if and only if the seller has never deviated from her prescribed strategy and $\delta^\tau (v - p_t^*) \geq \delta^\tau (v - p_t^*)$ for all $\tau \geq t$.

To establish that this profile is a PBE, first observe that if the seller ever sets $p_\tau \neq p_\tau^*$, she receives zero continuation payoff. Since this is her minmax value, she cannot receive continuation payoff strictly less than this in any PBE, so in particular her on-path continuation value after $\tau$ along $(p, q, Q)_{t,n}$ is weakly positive for every $n$, so by continuity of profits in
we see that her on-path continuation value after $\tau$ along $(p^*, q^*, Q^*)_t$ is also weakly positive. This implies that setting $p_\tau \neq p^*_\tau$ on-path is not a profitable deviation. Similarly, the fact that setting $x_t = 1$ is optimal on-path along $(p, q, Q)_t$, for all $n$ implies that setting $x_t = 1$ is optimal on-path in this strategy profile, because the cost of delivery and on-path continuation values are continuous in $(p, q, Q)_t$, while the payoff of zero that results from deviating from the equilibrium path in this profile is at least as bad as the payoff from deviating in any PBE. Also, the seller’s off-path play is optimal because off-path price-setting does not affect her payoffs and off-path delivery imposes a positive cost at no benefit.

I next check that each consumer’s play is optimal. It is again obvious that his off-path play is optimal, as paying is costly and yields no benefit when the seller sets $x_t = 0$. To see that his on-path play is optimal given $(p^*, q^*, Q^*)_t$, note that accepting at $t$ yields $\delta^t (v - p^*_t)$ if he pays (i.e., if he is allowed to purchase the good) and his continuation payoff otherwise, while rejecting always yields his continuation payoff, and $\delta^t (v - p^*_t)$ is weakly greater than his continuation payoff if $\delta^t (v - p^*_t) \geq \delta^\tau (v - p^*_\tau)$ for all $\tau \geq t$.

Finally, one must check that the prescribed consumer behavior actually induces quantity path $(Q^*)_t$. By Lemma 11, for any price-rationing path $(p, q)_t$ there is a unique quantity path $(Q)_t$ that occurs in a full-delivery PBE with price-rationing path $(p, q)_t$, and $(Q)_t$ is continuous in $(p, q)_t$ by Lemma 12. Therefore, the fact that $(p, q)_t$ converges to $(p^*, q^*)_t$ implies that $(Q)_t$ converges to $(Q^*)_t$. Thus, there there exists a full-delivery PBE with price-rationing-quantity path $(p^*, q^*, Q^*)_t$.

I have shown that $\mathcal{F}$ is closed, and therefore compact, in the product topology. Weierstrass’s Theorem now implies that there is a point in $\mathcal{F}$ that maximizes profits, which completes the proof.

4.5.3 Nonoptimality of Rationing in the Model with Rationing

I now show that any best full-delivery PBE in $\Gamma_R$ involves no rationing on the equilibrium path. This is the central step in showing equivalence of best full-delivery PBE in $\Gamma$ and $\Gamma_R$ (Corollary 12), which is in turn the main tool in proving Theorem 10.

By Lemma 10, the path of play of a full-delivery PBE is given by a price-rationing path $(p, q)_t$, up to differences in the play of the measure-0 set of consumers with nongeneric
valuations with respect to \((p)_t\). Let us write \(D_\tau((p,q)_t)\) for the quantity demanded at time \(\tau\) given price-rationing path \((p,q)_t\), i.e., the measure of consumers who would prefer to receive the good at time \(\tau\) at price \(p_\tau\) than to receive their PBE payoff.\(^{86}\) Similarly, say that a consumer demands the good at \(\tau\) if she prefers receiving the good at time \(\tau\) at price \(p_\tau\) to receiving her PBE payoff. Finally, say that rationing occurs along a price-quantity-rationing path \((p,q)_t\) if there exists a time \(\tau\) such that \(D_\tau((p,q)_t) > q_\tau > 0.\(^{87}\) Note that in a full-delivery PBE in which \(D_\tau((p,q)_t) \leq q_\tau\), a consumer with nongeneric valuation who demands the good at \(\tau\) must purchase at \(\tau.\(^{88}\)

I show that every best full-delivery PBE in \(\Gamma_R\) involves no rationing by arguing that any full-delivery PBE involving rationing can be strictly improved upon by another full-delivery PBE. The basic idea is that if rationing occurs at time \(t^*\), modifying the equilibrium by slightly increasing price at \(t^*\), such that quantity sold at \(t^*\) remains constant, and using additional rationing to ensure that quantity sold in every other period does not increase, leads the timing of all sales to remain constant and therefore yields an increase in profits. However, the proof is complicated by the fact that, without first ruling out rationing, I cannot ensure that the price path is decreasing and cannot establish the usual skimming property that higher-valuation consumers purchase earlier. The heart of the proof involves showing that slightly increasing price at \(t^*\) and using additional rationing to ensure that sales do not increase elsewhere cannot lead to a decrease in sales at some other time \(\tau\). If it did, then those consumers who used to purchase at \(\tau\) must now purchase at some other time that is better for them than \(\tau\), as they still have the option of earning surplus by purchasing at \(\tau\). And the fact that they have this new opportunity means that some other, higher-valuation consumers must also be purchasing at a different time. Since higher-valuation consumers must purchase at some point rather than never purchasing if lower-valuation consumers do so, following this "trail" of consumers who purchase at different times ultimately shows that every consumer (with generic valuation) who purchased before the price increase still

\(^{86}\)Throughout the paper, \(D(p) = 1 - F(p)\) is the static demand at price \(p\), while \(D_\tau((p,q)_t)\) is the time-\(\tau\) demand in the dynamic model under price-rationing path \((p,q)_t\).

\(^{87}\)If \(q_t = 0\), it is irrelevant whether one considers the monopoly to be rationing at \(t\) or to be setting price equal to infinity. I do not refer to this case as rationing for technical convenience.

\(^{88}\)If \(D_\tau((p,q)_t) = q_\tau\), this may fail for a measure-zero set of consumers who demand the good at \(\tau\) but are unable to purchase at \(\tau\) due to rationing. Since measure-zero sets of consumers are irrelevant for the analysis, I ignore this case in the discussion.
purchases after the price increase. The details of the proof are deferred to Appendix A.

**Proposition 13** In $\Gamma_R$, no rationing occurs along a best full-delivery PBE price-quantity-rationing path.

### 4.5.4 Equivalence of Best Full-Delivery PBE in the Model with and without Rationing

We are finally ready to prove Corollary 12, which establishes a very close relationship between best full-delivery PBE in the relational contracting model with and without rationing. The intuition for Corollary 12 is simple: by Proposition 13, no rationing occurs on the equilibrium path in a best full-delivery PBE of $\Gamma_R$, and the worst possible off-path punishment (breaking off trade) does not require rationing, so a best full-delivery PBE of $\Gamma_R$ can be no better than a best full-delivery PBE of $\Gamma$. The details of the proof, which involves constructing a PBE in $\Gamma$ corresponding to a given price-quantity path in $\Gamma_R$, and vice versa, is deferred to the appendix. The constructed PBE have the same grim-trigger structure as the PBE described in the proof of Proposition 12 and in Section 4.6.1.

**Corollary 12** A price-quantity path $(p, Q)_t$ is a best full-delivery PBE price-quantity path in $\Gamma_R$ if and only if it is a best full-delivery PBE price-quantity path in $\Gamma$.

Corollary 12 combined with Proposition 12 immediately yields part 2 of Theorem 10:

**Proposition 14 (Theorem 10.2)** A best full-delivery PBE exists in $\Gamma$.

### 4.6 Properties of Best Full-Delivery Equilibria

#### 4.6.1 High Profits and Super-Monopoly Pricing

In this subsection, I use the facts about $\Gamma_R$ and its relationship to $\Gamma$ established in Section to prove parts 3 and 4 of Theorem 10.

I first show that profits in a best full-delivery PBE in $\Gamma_R$ (which exists, by Proposition 12) converge to the static monopoly profit as $\delta$ approaches 1, which is not difficult. Corollary 12 then implies that the same is true in $\Gamma$. To see why payoffs in the best full-delivery
PBE in $\Gamma_R$ converge to static monopoly profits as $\delta$ approaches 1, let $D(p) \equiv 1 - F(p)$—the static demand at price $p$—and consider the following strategy profile, where $\gamma$ is a constant in $(0, \frac{p^m - c}{p^m})$:

1. The seller sets price-rationing-delivery path $p_t = p^m$, $q_t = \gamma (1 - \gamma)^t D(p^m)$, $x_t = 1$ as long as she has conformed to this strategy in the past. Otherwise, she sets $p_t = \bar{v}$, $q_t = 1$, $x_t = 0$ for all future periods. In particular, the seller sets $x_t = 0$ in any period in which has set $p_t \neq p^m$.

2. A consumer with valuation $v$ who has not yet received the good accepts if and only if the seller has never deviated from her prescribed strategy and $v > p^m$.

That is, the seller keeps price fixed at the static monopoly price, $p^m$, and sells to fraction $\gamma$ of those consumers who demand the good each period, while consumers accept if and only if $v \geq p^m$ and the seller has never deviated. It is clear that consumers’ play is optimal, and that the seller can never benefit from setting a different value of $p_t$ or $q_t$, so checking that this profile is an equilibrium reduces to checking that the seller prefers to deliver the good. The proof of Proposition 15 shows that the seller does in fact prefer to deliver the good if $\gamma \leq \frac{\delta p^m - c}{\delta p^m}$, and if $\delta$ is close to 1 then this strategy profile yields approximately static monopoly profits, as the cost of delay involved in selling to only fraction $\gamma$ of the consumers who demand the good each period is small. Therefore, profits in a best full-delivery PBE in $\Gamma_R$ must approximate static monopoly profits for $\delta$ close to 1 as well.

**Proposition 15 (Theorem 10.3)** For both $\Gamma$ and $\Gamma_R$, for all $\varepsilon > 0$, there exists $\overline{\delta} < 1$ such that, for all $\delta > \overline{\delta}$, there exists a full-delivery PBE under which the seller’s payoff is within $\varepsilon$ of the static monopoly payoff.

**Proof.** I prove the result for $\Gamma_R$ below. Proposition 12 then implies that, for every $\delta > \overline{\delta}$, there exists a best full-delivery PBE in $\Gamma_R$ under which the seller’s payoff is within $\varepsilon$ of the static monopoly payoff. Corollary 12 in turn implies that the same is true in $\Gamma$.

Recall that $p^m$ is the static monopoly price, so the static monopoly payoff is $(p^m - c) D(p^m)$. Suppose that $p^m > c$, i.e., that positive profits are possible—the case where this fails is trivial.
Consider the strategy profile described above, for \( \gamma \) some constant in \((0, \frac{p^m - c}{p^m})\). It is clear that each consumer’s strategy is a best-reply. Note also that \( q_t = Q_t \) for all \( t \) along the equilibrium path. To check that this profile describes a PBE, one must only check that the seller has an incentive to deliver the good along the equilibrium path, since any other deviation yields continuation payoff zero against positive continuation payoff from conforming. This condition is

\[
\sum_{t=1}^{\infty} \delta^t q_{t+\tau}(p_t+c) \geq q_t c \text{ for all } t \geq 0.
\]

For any \( t \), this can be rewritten as

\[
\gamma (1 - \gamma)^t \left( \frac{\delta (1 - \gamma)}{1 - \delta (1 - \gamma)} \right) D(p^m)(p^m - c) \geq \gamma (1 - \gamma)^t D(p^m)c,
\]

or

\[
\left( \frac{\delta (1 - \gamma)}{1 - \delta (1 - \gamma)} \right) (p^m - c) \geq c.
\]

Rearranging this inequality gives

\[
\gamma \leq \frac{\delta p^m - c}{\delta p^m}.
\]

Thus, the strategy profile above is a PBE for any \( \gamma \) satisfying (43). Since \( p^m > c \), there exists \( \gamma > 0 \) such that the strategy profile above is a PBE for high enough \( \delta \), in particular for \( \delta > \frac{c}{p^m} \).

Suppose that \( \delta > \frac{c}{p^m} \) and fix any positive \( \gamma \) satisfying (43). Note that this strategy profile yields profit

\[
\left( \frac{\gamma}{1 - \delta (1 - \gamma)} \right) D(p^m)(p^m - c)
\]

for the seller. As \( \delta \) approaches 1, this converges to \( D(p^m)(p^m - c) \), completing the proof.

The intuition for this result is that, for \( \delta \) high enough (\( \delta > c/p^m \)), the seller can credibly deliver the good to those consumers willing to pay the monopoly price at a fixed positive rate \( \gamma \), and taking \( \delta \) to 1 means that the loss from delay involved in this strategy is insignificant. Observe that, while the proof of Proposition 15 shows that, in \( \Gamma_R \), there exists a single strategy profile which is a PBE for all sufficiently high \( \delta \) and which yields profits converging
to static monopoly profits as \( \delta \) converges to 1, such a strategy profile need not exist in \( \Gamma \).

Note that the strategy profile described in the proof of Proposition 15, with \( p = p^m \), is not a best full-delivery PBE in \( \Gamma_R \) for fixed \( \delta < 1 \). Indeed, there exist full-delivery PBE in \( \Gamma_R \) with constant price paths (i.e., \( p_t = p_{t'} \) for all \( t, t' \)) that yield higher profits. To see this, consider the strategy profile in the proof of Proposition 15 with \( p^m \) replaced by some price \( p \). Let

\[
\gamma^*(p) = \frac{\delta p - c}{\delta p}.
\]

The argument in the proof of Proposition 15 that led to equation (43) shows that \( \gamma^*(p) \) is the fastest rate at which the seller can sell in a full-delivery PBE in which price is fixed at \( p \). This implies that the seller’s profit in the best full-delivery PBE with a constant price path at \( p \) and a constant sales rate \( \gamma \) is

\[
\left( \frac{\gamma^*(p)}{1 - \delta(1 - \gamma^*(p))} \right) D(p)(p - c),
\]

which equals

\[
\left( \frac{p - \frac{c}{\delta}}{p - c} \right) D(p)(p - c), \tag{44}
\]

Note that the first term of (44) represents the cost of the delay in sales required to induce the seller to deliver, while the second term is simply the static profit at price \( p \). Raising \( p \) above \( p^m \) yields a first-order increase in the first term in (44) and a second-order decrease in the product of the second and third terms, so the seller does better to sell at price above \( p^m \).

The intuition is similar to that of Section 4.4: raising price reduces quantity, which reduces the seller’s temptation to fail to deliver, and, with durable goods, this allows the seller to sell at a faster rate. More specifically, the required delay in sales forces a seller who would receive \( p - c \) per unit sold under full commitment to receive only \( p - \frac{c}{\delta} \) per unit sold, so, with a constant price path, a seller with cost \( c \) can do no better than imitating the pricing of a static monopoly with cost \( c/\delta \). That is, (44) equals

\[
\left( p - \frac{c}{\delta} \right) D(p),
\]

from which it is clear that the best full-delivery, fixed-price PBE in which the seller sells at
a constant rate is given by price $p^m \left( \frac{c}{\delta} \right)$, the monopoly price when cost equals $c/\delta$, and sales rate $\gamma = \gamma^* \left( p^m \left( \frac{c}{\delta} \right) \right)$. In fact, it is not hard to show that this is the best full-delivery, fixed-price PBE overall: all that remains to show this is to establish that selling at the constant rate $\gamma^* (p)$ is optimal given that prices are fixed at any given $p$, which follows from a standard dynamic programming argument.⁸⁹

**Corollary 13** If $\bar{v} > \frac{c}{\delta}$, the best full-delivery, constant-price PBE in $\Gamma_R$ is given by $p_t = p^m \left( \frac{c}{\delta} \right)$ and $q_t = \gamma^* \left( p^m \left( \frac{c}{\delta} \right) \right) \left(1 - \gamma^* \left( p^m \left( \frac{c}{\delta} \right) \right) \right) D \left( p^m \left( \frac{c}{\delta} \right) \right)$. Furthermore, $D \left( p^m \left( \frac{c}{\delta} \right) \right) D \left( p^m \left( \frac{c}{\delta} \right) \right)$ is a lower bound on the best full-delivery PBE profit in both $\Gamma$ and $\Gamma_R$.

**Proof.** Given the first part of the result, the second part follows immediately from Corollary 12.

Suppose $p_t = p$ for all $t$. Let $Q$ be the static demand for price $p$. The problem of finding the best full-delivery PBE with a constant price $p$ in $\Gamma^R$ reduces to finding the best number of consumers to sell to in every period while maintaining the seller’s incentive to deliver the good; i.e., to solving the following functional equation:

$$V(Q) = \max_{q \leq Q \text{ such that } \delta V(Q-q) \geq qc} (p - c) q + \delta V(Q - q).$$

(45)

Standard dynamic programming results imply that there is at most one solution to this equation with a non-trivial set satisfying the constraints. Conjecture that $V(Q) \leq \frac{\delta p - c}{\delta} Q$. The right-hand side of (45) then becomes

$$\max_{q \leq \left( \frac{\delta p - c}{\delta} \right) Q} (p - c) q + (\delta p - c) (Q - q)$$

$$= (p - c) \left( \frac{\delta p - c}{\delta p} \right) Q + (\delta p - c) \left( \frac{c}{\delta p} \right) Q$$

$$= \left( p - \frac{c}{\delta} \right) Q,$$

where the constraint set is non-trivial if $p > c/\delta$. Therefore, $\left( p - \frac{c}{\delta} \right) Q$ is the highest profit attainable by a price path fixed at $p > c/\delta$ when there are $Q$ remaining consumers with

⁸⁹Corollary 13 applies only to the case $\bar{v} > \frac{c}{\delta}$. Proposition 18 shows that, if $\bar{v} \leq \frac{c}{\delta}$, there is no full-delivery PBE in $\Gamma$ or $\Gamma_R$ in which the seller ever delivers the good or receives positive payments.
valuations greater than $p$, and $0$ is the highest such profit if $p \leq c/\delta$ (as the solution to (45) must be nonincreasing in $p$). Setting $Q = D(p)$ and maximizing over $p$ completes the proof.

Finally, I note that (non-constant price) full-delivery PBE of $\Gamma_R$ exist that yield profits strictly above static monopoly profits with cost equal to $c/\delta$, if $\bar{v} > c/\delta$. For example, consider modifying the best full-delivery, constant price path by increasing $p_0$ from $p^m(\xi) - \varepsilon$, for $\varepsilon$ small. I claim that, for small $\varepsilon$, $q_0$ consumers will still pay $p_0$. This follows because a consumer with valuation $v$ demands the good at time 0 and price $p_0$ if $v - p^m(\xi) - \varepsilon \geq \delta (v - p^m(\xi))$, or $\varepsilon \leq (1 - \delta) (v - p^m(\xi))$. This holds for all consumers with $v > p^m(\xi)$ in the limit as $\varepsilon$ goes to 0, and $q_0 = \left(\frac{\delta p^m(\xi) - c}{\delta p^m(\xi)}\right) D\left(p^m(\xi)\right)$, which is strictly less than $1 - F\left(p^m(\xi)\right)$. Therefore, there exists $\varepsilon > 0$ such that more than $q_0$ consumers demand the good at time 0 when $p_0 = p^m(\xi) + \varepsilon$. And the continuation path of play from $t = 1$ onward is the same under the modified strategy profile as under the best constant price PBE, so the modified profile yields strictly higher profits overall. This yields part 4 of Theorem 10:

**Proposition 16 (Theorem 10.4)** If $\bar{v} > c/\delta$, there exists a full-delivery PBE of $\Gamma_R$ (when cost equals $c/\delta$) yielding profits strictly greater than static monopoly profits when cost equals $c/\delta$. By Corollary 12, the same is true of full-delivery PBE of $\Gamma$.

Before leaving this subsection, note that Corollary 13 suggests that the best full-delivery PBE of the relational contracting model may involve pricing above the static monopoly level. I demonstrate this here in a simple, two-type example.\footnote{This example does not exactly fit the model as I have assumed a continuous distribution of valuations. However, the example can be slightly perturbed to yield a distribution that satisfies my assumptions, and, noting that every best full-delivery PBE price path is decreasing (by Proposition 17), I conjecture that the best full-delivery PBE in the perturbed example will have $p_0 > p^m$.}

**Example 1** Suppose that half the consumers have valuation 2.36 while the other half have valuation 2.12. Let $c = .38$ and $\delta = .4$. Note that the static monopoly price is 2.12, as this yields profit 1.74 while setting price equal to 2.36 yields profit .99. In the dynamic model, the discussion preceding Corollary 13 implies that the best full-delivery PBE with price fixed at 2.36 yields profit $\left(2.36 - \frac{.38}{4}\right) .5 = .71$ while the best PBE with price fixed

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at 2.12 yields profit \((2.12 - \frac{38}{4}) \cdot 1 = 1.17\). On the other hand, one can check that setting \(p_0 = 2.26\) and \(p_t = 2.12\) for all \(t \geq 1\) and selling to all high-valuation consumers in period zero and then selling to the low-valuation consumers at the fastest possible rate yields profit \((2.26 - .38) \cdot .5 + .4 \cdot (2.12 - \frac{38}{4}) \cdot .5 = 1.174\). Furthermore, this is a PBE price-quantity path, as high-valuation consumers receive \(2.36 - 2.26 = .1\) from purchasing in period zero and at most \(.4 \cdot (2.36 - 2.12) = .096\) from purchasing at a later date; while the seller gains \(.4 \cdot (2.12 - \frac{38}{4}) \cdot .5 = .234\) from delivering the good at time zero and gains \(.38 \cdot .5 = .19\) from failing to deliver. Since this full-delivery PBE yields higher profit than the best PBE that fixes price at the monopoly price of 2.12, which is clearly the best PBE in which all prices are weakly below the monopoly price, the best full-delivery PBE in this example must have \(p_t > p^m\) for some time \(t\).

### 4.6.2 Declining Prices

Finally, I establish three additional important properties of best full-delivery PBE of \(\Gamma\) and \(\Gamma_R\), which hold for any fixed discount factor (parts 5 through 7 of Theorem 10). I first use the possibility of rationing to ensure that best full-delivery PBE involve strictly decreasing price paths and positive sales each period. The idea is that delaying sales is wasteful and rationing can be used to ensure that speeding up sales does not violate the seller’s incentive compatibility constraint, which might otherwise be a concern.

**Proposition 17 (Theorem 10.5)** If \(\bar{v} > \frac{\bar{v}}{\bar{v}}\), any best full-delivery PBE of \(\Gamma\) or \(\Gamma_R\) has a strictly decreasing price path and strictly positive sales each period.

**Proof.** I prove the result for \(\Gamma_R\), whence the result for \(\Gamma\) follows by Corollary 12. If \(\bar{v} > \frac{\bar{v}}{\bar{v}}\), full-delivery PBE exist in which the seller makes positive profits (by Proposition 16), so any best full-delivery PBE of \(\Gamma_R\) yields positive profits. Suppose that \((p, q)_t\) is such a best full-delivery PBE price path (which exists by 12). By Proposition 13, \(D_\tau((p, q)_t) \leq q_\tau\) for all \(\tau\), so \(Q_\tau = D_\tau((p, q)_t)\) for all \(\tau\). Suppose that there exists some time \(\tau\) such that \(D_\tau((p, q)_t) = 0\). Let \(t^\tau\) be the first such time. If \(t^\tau = 0\), then define a new path by

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91 Corresponding off-path play may be taken to be as in the strategy profile in the proof of Proposition 15, for example.

92 See Proposition 18 for why this is not true if \(\bar{v} < \frac{\bar{v}}{\bar{v}}\).
letting $p'_t = p_{t+1}, q'_t = q_{t+1}$, i.e., shifting the original price-rationing path forward one period, which implies that $Q'_t = Q_{t+1}$, so profits under the new path are $\frac{1}{\delta}$ times profits under the original path, contradicting the optimality of the original path. If $t^* > 0$, let $v_{t^*-1}$ be the lowest valuation such that a consumer with valuation $v_{t^*-1}$ demands the good at $t^* - 1$, which is well-defined because a positive measure of consumers demand the good at $t^* - 1$, by definition of $t^*$. I first claim that $v_{t^*-1} > p_{t^*-1}$. To see this, first note that a consumer with valuation $v_{t^*-1}$ can demand the good at $t^* - 1$ only if $v_{t^*-1} \geq p_{t^*-1}$. If $v_{t^*-1} = p_{t^*-1}$, then it must be true that $p_{\tau} = p_{t^*-1}$ for all $\tau > t^* - 1$, since the price path is weakly decreasing by assumption; and if the price ever falls strictly below $p_{t^*-1}$ then all consumers with valuations sufficiently close to $p_{t^*-1}$ prefer to wait until this time to purchase, and all but at most a set of measure 0 of these consumers have the option of doing so since $D_\tau ((p, q)_t) \leq q_\tau$ for all $\tau$. The fact that $D_\tau ((p, q)_t) \leq q_\tau$ for all $\tau$ then implies that $Q_\tau = 0$ for all $\tau > t^* - 1$, as all consumers prefer to purchase at $t^* - 1$ than at any later time. Therefore, continuation profits from time $t^* - 1$ onward equal 0, which implies that the seller does not deliver at $t^* - 1$. This in turn implies that no consumers pay at $t^* - 1$, so that continuation profits from time $t^* - 2$ onward equal 0 as well. By induction, continuation profits from time 0 onward are 0, contradicting the fact that any best full-delivery PBE yields positive profits if $\bar{v} > \frac{e}{\delta}$.

Now consider modifying $(p, q)_t$ by changing $p_{t^*}$ to $p_{t^*}^{\star+1} = \frac{p_{t^*} - (1-\delta)v_{t^*}-1}{\delta^{t^*+1}}$. Since $v_{t^*-1} > p_{t^*-1}$, it follows that have $p_{t^*} < p_{t^*-1}$, and it is easy to check that all consumers with valuation weakly greater than $v_{t^*-1}$ continue to demand the good at $t^* - 1$. By the skimming property (which is easily seen to hold due to declining prices and no rationing), the seller can sell a positive quantity at date $t^* + \tau$ only if $p_{t^*+\tau} < p_{t^*}^{\star+1} = \frac{p_{t^*} - (1-\delta)v_{t^*}-1}{\delta^{t^*+1}}$, so the seller strictly prefers selling to some mass of consumers at $t^*$ at the new price to selling to them at any point in the future. Next, observe that under the new price there is strictly positive demand at $t^*$, since at the new price a consumer with valuation $v_{t^*}$ strictly prefers to purchase at $t^* - 1$ than to purchase at any other time except $t^*$, and is indifferent between purchasing at $t^* - 1$ and purchasing at $t^*$, so a consumer with valuation slightly below $v_{t^*-1}$ strictly prefers purchasing at $t^*$ to purchasing at any other time. Furthermore, the total sales at all future dates to consumers who do not buy at $t^*$ is left unchanged, so total profits are strictly
higher under the new path. Finally, the potential complication that the seller’s incentive compatibility constraint may be violated at $t^*$ can be addressed by rationing at $t^*$, since the necessity of positive continuation profits from $t^*$ on implies that the seller can credibly sell a strictly positive quantity at $t^*$. So the modified path (possibly with rationing at $t^*$) strictly improves on the original path, contradicting the assumption that $D_\tau ((p,q)_t) = 0$ for some $\tau$.

I have shown that every best full-delivery PBE induces strictly positive sales at every date. Since every best full-delivery PBE involves no rationing, this is possible only if every best full-delivery PBE has a strictly declining price path. ■

We are now ready to complete the proof of Theorem 10 by proving parts 6 and 7, which show that every best full-delivery PBE of $\Gamma$ (or $\Gamma_R$) has an equilibrium price path $(p)_t$ that asymptotes to a price at least as high as $\max \{v, c/\delta\}$ as $t$ goes to infinity. The intuition is that a best full-delivery PBE has a declining price path, by Proposition 17; there is no reason to price below $v$; and prices must be at least $c/\delta$ in any full-delivery PBE with a declining price path in which the seller ever delivers, in analogy with Proposition 10. The following Lemma formalizes the last part of this intuition:

**Lemma 13** In any full-delivery PBE of $\Gamma$ or $\Gamma_R$ with price-quantity path $(p,q)_t$ in which $p_t \geq p_{t+1}$ for all $t$ and a strictly positive quantity of the good is delivered along the equilibrium path, $p_t > \frac{c}{\delta}$ for all $t$.

**Proof.** Consider $\Gamma_R$ first. Suppose that $Q$ consumers have not yet received the good at time $t^*$. First note that the seller’s continuation profit from time $t^*$ onward is bounded from above by her continuation profit from time $t^*$ onward in a best full-delivery PBE of the modified continuation game where she is constrained to price weakly below $p_{t^*}$ and all remaining consumers’ valuations are set to $p_{t^*}$. This follows because in the modified game the seller can set the original continuation price path $(p)_{t \geq t^*}$ and use rationing in order to sell according to the original price-quantity path.

The seller’s continuation value at $t^*$ in a full-delivery PBE of the modified game is therefore bounded from above by the solution to equation (45) with $p = p_{t^*}$. As shown in the proof of Corollary 13, equation (45) has a solution with $V(Q) > 0$ if and only if $\delta p_{t^*} > c$. 

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So if \( p_{t^*} \leq \frac{v}{\delta} \), the seller’s continuation value at \( t^* \) equals 0 in any full-delivery PBE in the modified game, and therefore equals 0 in any full-delivery PBE of the unmodified game as well. This implies that the seller delivers 0 units of the good at time \( t^* \), which then implies that no buyers pay anything to the seller at time \( t^* \), so that the seller’s continuation value at \( t^* - 1 \) equals 0 as well. By induction, the seller’s continuation value equals 0 at all periods, and the seller never delivers a positive quantity of the good.

By Proposition 12 and Corollary 12, the above argument shows that in any full-delivery PBE of \( \Gamma \) with a declining price, the seller’s continuation value starting from any \( t^* \) satisfying \( p_{t^*} \leq \frac{v}{\delta} \) is 0. As above, this implies that the seller never delivers any positive quantity of the good.

**Proposition 18 (Theorem 10.6 and 10.7)** Any best full-delivery PBE of \( \Gamma \) or \( \Gamma_R \) has \( p_t > \frac{v}{\delta} \) and \( p_t \geq v \) for all \( t \) if \( \bar{v} > \frac{v}{\delta} \). If \( \bar{v} \leq \frac{v}{\delta} \), there is no PBE in \( \Gamma \) or \( \Gamma_R \) in which the seller ever delivers the good or receives positive payments.

**Proof.** If \( \bar{v} > \frac{v}{\delta} \), the price path of any best full-delivery PBE of \( \Gamma \) or \( \Gamma_R \) is declining, by Proposition 17; and any full-delivery PBE with a declining price path has \( p_t > \frac{v}{\delta} \) for all \( t \), by Lemma 13. Finally, modifying any declining price path in \( \Gamma_R \) by replacing all \( p_t < v \) with \( v \) and using rationing to ensure delivery yields a strict increase in profits if \( p_t < v \) for any \( t \) (as sales occur in every period in a best full-delivery PBE, by Proposition 17), so the result for \( \bar{v} > \frac{v}{\delta} \) holds for \( \Gamma_R \). Corollary 12 then implies that it also holds for \( \Gamma \).

Suppose that \( \bar{v} \leq \frac{v}{\delta} \) and that mass \( Q \) consumers have not yet received the good at some time \( t \) in \( \Gamma \) or \( \Gamma_R \). If the seller delivers \( q \) units of the good at time \( t \), she cannot receive total payments of more than \( \bar{v}q \) and must of course be willing to deliver the \( q \) units. Therefore, her continuation payoff from time \( t \) onward is bounded from above by the solution to equation (45) with \( p = \bar{v} \). As we have seen, the only solution to equation (45) when \( \bar{v} \leq \frac{v}{\delta} \) is \( V(Q) = 0 \) for all \( Q \). So no PBE in \( \Gamma \) or \( \Gamma_R \) yields positive profits if \( \bar{v} \leq \frac{v}{\delta} \), which, as in the proof of Lemma 13, implies that no PBE involves delivery or positive payments.
4.7 An Extension: Exogenous Chance of Nondelivery

The analysis is based on the assumption that the seller has the option of failing to deliver the good after receiving payment. We have argued that the presence of equilibria that yield high profits for the seller under this assumption suggests that sellers may try to avoid committing themselves to delivering the good. However, in some environments sellers may be "automatically" committed to delivery; for example, taking payment for a good and then failing to provide it may be viewed by courts as breaching an "implicit" contract, particularly if the seller has always provided the good to paying customers in the past (as is the case in full-delivery PBE). In this section, I show that the model can easily be extended to an environment in which this concern that the seller may be involuntarily committed to delivery does not apply. In particular, I assume that in every period there is an exogenous, independent probability \( q > 0 \) that the seller privately learns that she is unable to deliver the good after receiving payment.\(^\text{93}\) For example, the seller may require certain specialized inputs in order to produce the final good, and these inputs may not always be available (and consumers and courts may be unable to observe whether the inputs are available). In this model, the seller periodically fails to deliver the good even if she wishes to deliver in every period, and since courts cannot tell whether failure to deliver results from lack of inputs or opportunistic behavior by the seller there is no possibility that the seller can be involuntarily committed to trying to deliver the good in every period.

The equilibria I have constructed for both the non-durable and durable goods models can easily be adapted to this environment by specifying that no purchases or delivery occur after any nondelivery by the seller (so that trade eventually breaks down on the equilibrium path), and that prior to the breakdown of trade consumers take into account that they receive the good only with probability \( 1 - \eta \) even if they pay (since consumers are risk-neutral, this implies that the mass of consumers who wish to purchase at price \( p \) is now \( D\left(\frac{p}{1-\eta}\right) \) rather than \( D(p) \)). That is, the results are "continuous" in \( \eta \). Rather than formally stating this rather natural finding, I instead focus on characterizing the best full-delivery, constant-price PBE in \( \Gamma_R \), in analogy to Corollary 13, which provides an intuitive lower bound on the best full-delivery PBE profit in both \( \Gamma \) and \( \Gamma_R \). It turns out that the analysis of Section 4.6.1

\(^{93}\)I thank Edward Green for suggesting I pursue this analysis.
carries through with the sole modification that \( D(p) \) is replaced by \( D\left(\frac{p}{1-\eta}\right) \): the intuition for this result is that, in the best full-delivery PBE, the seller is indifferent between delivering the good and breaking off trade, so she is not made worse off by the possibility that trade may break off exogenously (except insofar as this causes consumers with valuations \( v \in \left[p, \frac{p}{1-\eta}\right] \) to reject price her price offer). Finally, note that the original definition of a full-delivery PBE does not allow for the possibility that trade breaks down in equilibrium, which leads us to use the following, somewhat ad hoc, definition in the statement of the result:

**Definition 18** A modified full-delivery PBE is a PBE in which the seller sets \( x_t = 1 \) at all on-path histories at which \( Q_t > 0 \) and sets \( x_t = 0 \) at all on-path histories at which \( Q_t = 0 \).

The earlier results pertaining to full-delivery PBE (in particular, Corollary 12) also apply to modified full-delivery PBE.

**Proposition 19** If \( \bar{v} > \frac{\zeta}{\delta} \), the best modified full-delivery, constant-price PBE in \( \Gamma_R \) is given by \( p_t = \arg\max_p \left( p - \frac{\zeta}{\delta} \right) D\left(\frac{p}{1-\eta}\right) \equiv p^*(\eta) \) and \( q_t = \gamma^*(p^*(\eta)) \left(1 - \gamma^*(p^*(\eta))\right)^t D\left(\frac{p^*(\eta)}{1-\eta}\right) \), where \( \gamma^*(\mu) = \frac{\delta \nu - \zeta}{\delta p} \) as in Section 4.6.1.

**Proof.** A consumer who demands the good at price \( p \) receives it with probability at most \( 1 - \eta \), so at most \( D\left(\frac{p}{1-\eta}\right) \) consumers ever purchase in a full-delivery PBE with constant price \( p \). The argument in the proof of Corollary 13 shows that, if the seller faces this demand curve and can freely choose what quantity to deliver in every period, her best (modified) full-delivery PBE profit with constant price \( p \) equals \( (p - \frac{\zeta}{\delta}) D\left(\frac{p}{1-\eta}\right) \). Therefore, \( (p^*(\eta) - \frac{\zeta}{\delta}) D\left(\frac{p^*(\eta)}{1-\eta}\right) \) is an upper bound on the seller’s best modified full-delivery, constant-price PBE profit when in each period she may be unable to deliver the good with probability \( \eta \).

I claim that the following strategy profile attains this upper bound: the seller sets \((p_t, q_t)\) as in the statement of the proposition and sets \( x_t = 1 \) until the first time that delivery is impossible and subsequently sets \( x_t = 0 \); and a consumer with valuation \( v \) demands the good if and only if \( v \geq \frac{p_t^*}{1-\eta} \) and the seller has always set \( p_t = p^*(\eta) \) and delivered the good in the past. The only nontrivial part of verifying that this profile is a PBE is checking that it is optimal for the seller to deliver the good when prescribed. Nondelivery leads to
continuation payoff 0, and in every period prior to the first non-delivery the seller fails to deliver with probability \( \eta \). Therefore, the condition that it is optimal for the seller to deliver the good when prescribed at time \( t \) is

\[
\sum_{\tau=1}^{\infty} (1 - \eta)^{\tau-1} \delta^{\tau} q_{t+\tau} (p_{t+\tau} - c (1 - \eta)) \geq q_{t} c.
\]

Substituting in the specified \((p_{t}, q_{t})\) yields

\[
\gamma^* (p^* (\eta)) (1 - \gamma^* (p^* (\eta)))^t \left( \frac{\delta (1 - \gamma^* (p^* (\eta)))}{1 - \delta (1 - \gamma^* (p^* (\eta))) (1 - \eta)} \right) D \left( \frac{p^* (\eta)}{1 - \eta} \right) (p^* (\eta) - c (1 - \eta))
\]

\[
\geq \gamma^* (p^* (\eta)) (1 - \gamma^* (p^* (\eta)))^t D \left( \frac{p^* (\eta)}{1 - \eta} \right) c,
\]

or

\[
\left( \frac{\delta (1 - \gamma^* (p^* (\eta)))}{1 - \delta (1 - \gamma^* (p^* (\eta))) (1 - \eta)} \right) (p^* (\eta) - c (1 - \eta)) \geq c.
\]

This can be rewritten as

\[
\gamma^* (p^* (\eta)) \leq \frac{\delta p^* (\eta) - c}{\delta p^* (\eta)},
\]

exactly as in (43), which holds by definition of \( \gamma^* (p^* (\eta)) \). This verifies that the above strategy profile is a modified full-delivery, constant-price PBE, and it is straightforward to check that it yields expected profit \((p^* (\eta) - \frac{c}{\delta}) D \left( \frac{p^* (\eta)}{1 - \eta} \right)\).

Thus, Proposition 19 shows that the lower bound on optimal monopoly profits derived in Section 4.6.1 extends naturally to environments with an exogenous change of non-delivery, where it may be more realistic to view the seller as having the option of non-delivery.

### 4.8 Conclusion

The main insight of this paper is that the optimal pricing strategy of a dynamic monopoly may be very different from that in traditional models when the relationship between the seller and consumers is regulated by relational incentives. Unlike in Hart and Tirole (1988), a non-durable goods monopoly in my model can earn high profits even if consumers are non-anonymous, provided the discount factor is sufficiently high. And unlike in Coase (1972), a durable goods monopoly can earn approximately static monopoly profits in the limit as the
discount factor approaches one, even if the lowest consumer valuation is above the marginal cost of production. A durable goods monopoly can also earn high profits when the discount factor is bounded away from one.

While the model has many equilibria, restricting attention to the best equilibria for the seller brings out some novel economic intuitions and empirical predictions. First, for both non-durable and durable goods monopolies, the temptation to fail to deliver provides an incentive for pricing above the static monopoly level. The intuition is the same in both cases: the larger the quantity of the good a monopoly is supposed to deliver, the greater is its incentive to renege. So the monopoly benefits from restricting quantity, and the most profitable way for it to restrict quantity is to raise price. Second, in the durable goods case, the monopoly has an incentive to gradually cut prices over time, using high prices rather than rationing to restrict sales early on. These new effects have potentially interesting applications for regulation: in traditional models, observing a monopoly cutting its price is a sign that consumers are doing better than they would be if the monopoly had full commitment power, since they are paying lower prices and (if the discount factor is high) are not facing costly delays in purchasing. In my model, however, consumers may be better off when the monopoly has full commitment power, for two reasons: they may face lower prices (since without commitment the monopoly may price above the static monopoly price), and they may receive the good significantly faster. This also points to an important empirical prediction of my model: in contrast to the standard full-commitment and "no-commitment" models of durable good monopoly, my model predicts that a monopoly will cut prices over time, but will do so slowly enough that the costs from delay are significant.

I also introduce two methodological innovations. First, I use an augmented "model with rationing" to help analyze the durable-goods seller problem. This greatly simplifies the analysis by allowing us to construct simple equilibria with flat price paths in the model with rationing and then use the relationship between the model with and without rationing to draw conclusions about best full-delivery equilibria in the model without rationing. Second, and more generally, I use relational incentives to replace the temptation to deviate at the

This possibility that dynamic monopolies may price higher than static monopolies is a prediction of the model which differs from standard models of dynamic monopoly pricing.
contract offer stage (price offers in the model) with the temptation to deviate at the contract
execution stage (delivery of the good in the model), which may have applications to other
areas where studying dynamics in the presence of adverse selection has proved difficult. For
example, recall that in the model of non-durable goods and non-anonymous consumers, the
"dynamic enforcement" constraint that the seller delivers the good replaced the ratchet effect
in price setting. Perhaps further insights may be gained from applying this idea to dynamic
principal-agent problems with adverse selection, where characterizing dynamics in models
with "no commitment" is difficult due to the ratchet effect (see, e.g., Laffont and Tirole,
1988).

4.9 Appendix A: Omitted Proofs

Proof of Proposition 1. First observe that the problem of finding the best PBE for the
seller is equivalent to finding the best PBE for the seller when she can fully commitment
to her sequence of prices \(\{p_{t,n}\}_{n=1}^N\). To see this, note that one can specify off-path beliefs
for buyers such that each buyer expects the seller to never deliver the good following any
deviation in price-setting by the seller. Given these beliefs, no buyer will ever accept a
strictly positive price in any period following a deviation in price-setting by the seller, so the
seller always receives continuation payoff zero, equal to her minmax payoff, after any such
deviation.

Using this observation and applying the revelation principle to each period, one can write
the problem in a standard mechanism design notation, writing \(T\) for transfers:

\[
\max_{\{T_t\}, x_t} \sum_{t=0}^\infty \delta^t \int_0^\bar{v} (T_t(v) - c_x(v)) f(v) dv
\]

subject to

\[
v x_t(v) - T_t(v) \in \arg\max_{v'} v x_t(v') - T_t(v') \quad \text{for all } v \text{ and } t \quad \text{(IC)}
\]

\[
v x_t(v) - T_t(v) \geq 0 \quad \text{for all } v \text{ and } t \quad \text{(IR)}
\]

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and
\[ \sum_{r=1}^{\infty} \delta^r \int_\mathcal{U} (T_{r+T}(v) - c x_{r+T}(v)) f(v)dv \geq c \int_\mathcal{U} x_t(v)f(v)dv \text{ for all } t. \] (DE)

Note that the third constraint is the seller’s incentive compatibility constraint, which I also refer to as the dynamic enforcement or DE constraint. Substituting for \( T_t(v) \) using the IR and IC constraints in the usual way and temporarily ignoring the resulting monotonicity constraint lets us rewrite the problem as

\[ \max_{\{x_t(v)\}_t} \sum_{t=0}^{\infty} \delta^t \int_\mathcal{U} ((v - c) f(v) - (1 - F(v))) x_t(v)dv \]

subject to the DE constraint

\[ \sum_{r=1}^{\infty} \delta^r \int_\mathcal{U} ((v - c) f(v) - (1 - F(v))) x_{r+T}(v)dv \geq c \int_\mathcal{U} x_t(v)f(v)dv \text{ for all } t. \]

Let \( \{x^*_t(v)\}_t \) be a solution to this problem. Note that, for all \( t \), \( x^*_t(v) \) must solve

\[ \max_{x_t(\cdot)} \int_\mathcal{U} ((v - c) f(v) - (1 - F(v))) x_t(v)dv \]

subject to

\[ \sum_{r=1}^{\infty} \delta^r \int_\mathcal{U} ((v - c) f(v) - (1 - F(v))) x^*_{r+T}(v)dv \geq c \int_\mathcal{U} x_t(v)f(v)dv, \]

since the solution to this program maximizes both the original objective and the left-hand side of each original constraint over all \( x_t(\cdot) \) that satisfy the original time \( t \) constraint.

This implies that, for all \( t, t' \), if \( (v - c) f(v) - (1 - F(v)) > 0 \), then \( x^*_t(v) > x^*_{t'}(v) \) if \( \sum_{r=1}^{\infty} \delta^r \int_\mathcal{U} x^*_{r+T}(v)dv > \sum_{r=1}^{\infty} \delta^r \int_\mathcal{U} x^*_{r+T}(v)dv \); while if \( (v - c) f(v) - (1 - F(v)) < 0 \), then \( x^*_t(v) = x^*_{t'}(v) = 0 \). Since \( \sum_{r=1}^{\infty} \delta^{r+T} \int_\mathcal{U} x^*_{r+T}(v)dv \) is bounded from above, there exists a finite \( x^*(\cdot) \) such that \( x^*(v) = \sup_t x^*_t(v) \) if \( (v - c) f(v) - (1 - F(v)) \geq 0 \) and \( x^*(v) = 0 \) otherwise.

I claim that \( x_t(v) = x^*(v) \) for all \( t \) and \( v \) in any solution to this problem. Clearly, the profit corresponding to this allocation is an upper bound on the profit in any solution.
Furthermore,

\[
\sum_{\tau=1}^{\infty} \delta^\tau \int_{\mathbb{R}} \left[ (v - c) f(v) - (1 - F(v)) \right] x^*(v) dv = \sum_{\tau=1}^{\infty} \delta^\tau \int_{\mathbb{R}} \left[ (v - c) f(v) - (1 - F(v)) \right] \sup_t x^*_t(v) dv
\]

\[
\geq \sup_t \sum_{\tau=1}^{\infty} \delta^\tau \int_{\mathbb{R}} \left[ (v - c) f(v) - (1 - F(v)) \right] x^*_t(v) dv
\]

\[
\geq \sup_t c \int_{\mathbb{R}} x^*_t(v) f(v) dv
\]

\[
= c \int_{\mathbb{R}} x^*(v) f(v) dv,
\]

where the first line is by the definition of \( x^*(v) \), the second is immediate, the third follows because \( \{x^*_t(v)\}_t \) satisfies the DE constraint for all \( t \), and the fourth follows because \( x^*_t(v) \geq x^*_{t'}(v) \) if and only if \( x^*_t(v') \geq x^*_{t'}(v') \) for any \( t, t', v, \) and \( v' \), so the sup may be moved inside the integral. The above chain of inequalities implies that repeating \( x^*(v) \) satisfies the seller’s incentive compatibility constraint. Finally, if there exists \( t \) such that \( x^*_t(v) \neq x^*(v) \), then the allocation \( \{x^*_t(v)\}_t \) yields strictly lower profit than repeating \( x^*(v) \) in period \( t \) and yields weakly lower profit in all other periods, so every solution to the original problem has the same allocation rule in every period.

I have shown that the optimal allocation rule is stationary, so the problem becomes

\[
\max_{x(\cdot)} \int_{\mathbb{R}} \left[ (v - c) f(v) - (1 - F(v)) \right] x(v) dv
\]

subject to the DE constraint

\[
\sum_{\tau=1}^{\infty} \delta^\tau \int_{\mathbb{R}} \left[ (v - c) f(v) - (1 - F(v)) \right] x(v) dv \geq c \int_{\mathbb{R}} x(v) f(v) dv \text{ for all } t.
\]

The DE constraint may be rewritten as

\[
\int_{\mathbb{R}} (vf(v) - (1 - F(v))) x(v) dv \geq \left( \frac{c}{\delta} \right) \int_{\mathbb{R}} x(v) f(v) dv.
\]

If the constraint is slack, we have standard monopoly pricing. If the constraint is binding, noting that the assumptions on \( F(v) \) imply that \( x(v) \) continues to take a cutoff form whereby
If \( \bar{v} \leq \frac{v^*}{\delta} \), then \( v^* > \bar{v} \), so \( x(v) = 0 \) for all \( v \), which implies that the seller never delivers the good or receives positive payments in any optimal PBE. Since the seller’s minmax payoff is zero, every PBE is optimal if \( \bar{v} < \frac{v^*}{\delta} \), which proves the result in the \( \bar{v} < \frac{v^*}{\delta} \) case. ■

**Proof of Proposition 11.** The proof is similar to the proof of Proposition 12, so I omit some details. Let \( \mathcal{F} \) be the set of PBE price-quantity-delivery paths \((p, Q, x)_t\) satisfying \( p_t \in [\underline{v}, \bar{v}] \) for all \( t \). If a PBE is optimal in the set of PBE with price-demand-delivery paths in \( \mathcal{F} \), then it is optimal overall. Furthermore, it is clear that the seller’s PBE payoff is continuous in price-quantity-delivery paths \((p, Q, x)_t\) in the product topology.

Next, note that the continuation value of a consumer with valuation \( v \) facing price-quantity-delivery path \((p, Q, x)_t\) at time \( t \) is continuous in \((p, Q, x)_t\) in the product topology.\(^{95}\) To see this, observe that the maximum gain in continuation value over a \( \varepsilon \)-ball about \((p, Q, x)_t \in \mathcal{F}\) is no more than \( \frac{(1+\delta)^t - \varepsilon}{1-\delta} \), corresponding to receiving the good, valued at \( \bar{v} \), with additional probability \( \varepsilon \) in each period, and paying \( \varepsilon \) less in each period. This converges to 0 as \( \varepsilon \) does.

I now show that \( \mathcal{F} \) is compact in the product topology. Observe that \( \mathcal{F} \subseteq \prod_{t=0}^{\infty} ([\underline{v}, \bar{v}], [0, 1], [0, 1])_t \), which is compact by Tychonoff’s Theorem. Therefore, it suffices to show that \( \mathcal{F} \) is closed in the product topology. To see that it is, consider a sequence of paths \( \{(p, Q, x)_t\}_n \in \mathcal{F} \) converging pointwise to \((p^*, Q^*, x^*)_t \). I must show that there exists a PBE with price-demand-delivery path \((p^*, Q^*, x^*)_t \). Consider the following strategy profile:

1. The seller sets price-delivery path \((p^*, x^*)_t \) as long as she has conformed to this strategy in the past. Otherwise, she sets \( p_t = \bar{v} \), \( x_t = 0 \) for all future periods. In particular, the seller sets \( x_t = 0 \) in any period in which she has set \( p_t \neq p_t^* \).

2. A consumer with valuation \( v \) who has not yet received the good at \( t \) pays at \( t \) if and only

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\(^{95}\)This continuation value is well-defined here by standard dynamic programming arguments, because, unlike in the model with rationing, each consumer faces the same optimization problem regardless of the behavior of the other consumers.
if the seller has never deviated from her prescribed strategy and \( x_t^*v - p_t^* \geq \delta x_t^*V_{t+1}^v \),
where \( V_{t+1}^v \) is the continuation value of such a consumer facing \((p^*, Q^*, x^*)_t\).

The proof that the seller's play is optimal is as in the proof of Proposition 12. To see that each consumer's play is optimal, first note that it is obvious that her off-path play is optimal, as paying is costly and yields no benefit when the seller sets \( x_t = 0 \). To see that her on-path play is optimal given \((p^*, Q^*, x^*)_t\), note that paying at \( t \) gives expected payoff \( x_t^*v - p_t^* + \delta (1 - x_t^*) V_{t+1}^v \), while not paying gives \( \delta V_{t+1}^v \), so paying is optimal if and only if \( x_t^*v - p_t^* \geq \delta x_t^*V_{t+1}^v \).

That the prescribed consumer behavior induces quantity path \((Q^*)_t\) follows from the observation that each consumer's payoff is continuous in \((p, Q, x)_t\), and that each consumer plays a best response to each \((p, Q, x)_t\) in equilibrium. This completes the argument that \( \mathcal{F} \) is closed, and therefore compact, in the product topology. Weierstrass's Theorem then implies that there is a point in \( \mathcal{F} \) that maximizes profits, completing the proof.

**Proof of Lemma 10.** Fix a price-rationing path \((p, q)_t\) and two full-delivery PBE \( \sigma \) and \( \sigma' \). Let \( \mathcal{V} \) be the set of generic valuations \( v \) such that there exists a consumer with valuation \( v \) who purchases at different times under \( \sigma \) and \( \sigma' \). Suppose, towards a contradiction, that \( \mathcal{V} \) is nonempty. Then \( \mathcal{V} \) has a supremum, which I denote by \( v^* \). Let \( V_{v^*} \) be the payoff of a consumer with valuation \( v^* \) under \( \sigma \), let \( V_{v'}^v \) be the payoff of a consumer with valuation \( v^* \) under \( \sigma' \), and without loss of generality assume that \( \sigma' \) maximizes profits, completing the proof.

I first claim that \( V_{v^*} = V_{v'}^{v^*} \). To see this, suppose that there exists a consumer with valuation \( v^* \) who purchases at time \( \tau_{v^*} \) under \( \sigma \) and purchases at time \( \tau'_{v^*} \) under \( \sigma' \), with \( \delta_{\tau_{v^*}} (v^* - p_{\tau_{v^*}}) > \delta_{\tau'_{v^*}} (v^* - p_{\tau'_{v^*}}) \), so that the consumer receives a higher payoff under \( \sigma \). This is possible only if the consumer is unable to purchase at time \( \tau_{v^*} \) under \( \sigma' \), which in turn is possible only if strictly more than \( q_{\tau_{v^*}} \) consumers accept price \( p_{\tau_{v^*}} \) at time \( \tau_{v^*} \) under \( \sigma' \). Since the consumer is able to purchase at time \( \tau_{v^*} \) under \( \sigma \), which is possible only if no more than \( q_{\tau_{v^*}} \) consumers accept price \( p_{\tau_{v^*}} \) at time \( \tau_{v^*} \) under \( \sigma \), this implies that there is a positive measure \( \mu \) of consumers with valuations greater than \( v^* \) who purchase at \( \tau_{v^*} \) under \( \sigma \) but not under \( \sigma' \). By Lemma 8, this implies that there exists a consumer with valuation \( v' > v^* \) and \( v' \) generic with respect to \((p)_t\) who purchases at different times under \( \sigma \) and \( \sigma' \), which contradicts the fact that \( v^* = \sup \{ v : v \in \mathcal{V} \} \). This implies that \( V_{v^*} = V_{v'}^{v^*} \).
which also implies that \( v^* \notin \mathcal{V} \), as if \( V_{\sigma^*} = V'_{\sigma} \) then either every consumer with valuation \( v^* \) purchases at the same time under \( \sigma \) and \( \sigma' \) or \( v^* \) is nongeneric with respect to \((p)\).

If \( V_{\sigma^*} = V'_{\sigma} = 0 \), then is no time \( t \) at which \( v^* > p_t \) and a consumer with valuation \( v^* \) is able to purchase under either \( \sigma \) or \( \sigma' \). This implies that there is no time \( t \) at which \( v \geq p_t \) and a consumer with valuation \( v \) is able to purchase under either \( \sigma \) or \( \sigma' \), for any \( v \in \mathcal{V} \), as \( v < v^* \) for all \( v \in \mathcal{V} \) and a consumer with a lower valuation is able to purchase at a weakly smaller set of times. Therefore, a consumer with valuation \( v \) never purchases under either \( \sigma \) or \( \sigma' \), for all \( v \in \mathcal{V} \), which implies that \( \mathcal{V} \) is empty, a contradiction.

If \( V_{\sigma^*} = V'_{\sigma} > 0 \), then for any \( \eta \in (0, V_{\sigma^*}) \) there exist at most finitely many times \( t \) such that there exists \( v \in [\bar{v}, \hat{v}] \) such that \( \delta' (v - p_t) \geq V_{\sigma^*} - \eta \) and \( q_t \geq Q_t \), where \( Q_t \) is the measure of consumers who purchase at time \( t \) under \( \sigma \) and have valuations greater than \( v^* \) (as \( p_t \geq 0 \) for all \( t \)); call the set of such times \( T \). Let \( \varepsilon_t = q_t - Q_t \), and let \( \varepsilon = \min \{ \varepsilon_t : t \in T \} /2 > 0 \).

Since every consumer with generic valuation greater than \( v^* \) purchases at the same time under \( \sigma \) and \( \sigma' \), by definition of \( \mathcal{V} \), and the set of consumers with nongeneric valuations is of measure 0, by Lemma 8, the measure of consumers with valuations greater than \( v^* - \varepsilon \) who purchase at any \( t \) under \( \sigma' \) is less than \( q_t - \varepsilon \). By definition of \( \varepsilon \), this implies that the measure of consumers with valuations greater than \( v^* - \varepsilon \) who purchase at any \( t \) under \( \sigma' \) is less than \( Q_t + \varepsilon \). So any consumer with valuation \( v > v^* - \varepsilon \) can purchase at any time \( t \) with \( \delta' (v - p_t) \geq V_{\sigma^*} - \eta \) under \( \sigma' \) at which she can purchase under \( \sigma \). By the same argument, there exists \( \varepsilon' > 0 \) such that a consumer with valuation \( v > v^* - \varepsilon' \) can purchase at any time \( t \) with \( \delta' (v - p_t) \geq V_{\sigma^*} - \eta \) under \( \sigma \) at which she can purchase under \( \sigma' \). Therefore, letting \( \varepsilon'' = \min \{ \varepsilon, \varepsilon' \} \), we see that a consumer with valuation \( v > v^* - \varepsilon'' \) can purchase at the same set of times \( t \) with \( \delta' (v - p_t) \geq V_{\sigma^*} - \eta \) under both \( \sigma \) and \( \sigma' \). Furthermore, a consumer with valuation close enough to \( v^* \) can purchase at any time at which a consumer with valuation \( v^* \) can purchase, by the specification of rationing, so there exists \( \varepsilon^* \) such that a consumer with valuation \( v > v^* - \varepsilon^* \) receives a payoff of at least \( V_{\sigma^*} - \eta \) under both \( \sigma \) and \( \sigma' \). Finally, by definition of \( v^* \), there exists \( v \in \mathcal{V} \) such that \( v > v^* - \min \{ \varepsilon'', \varepsilon^* \} \). A consumer with valuation \( v \) receives a payoff of at least \( V_{\sigma^*} - \eta \) under both \( \sigma \) and \( \sigma' \), which implies that he purchases at a time \( t \) with \( \delta' (v - p_t) \geq V_{\sigma^*} - \eta \) under both \( \sigma \) and \( \sigma' \). The set of such times at which the consumer can purchase is the same under both \( \sigma \) and \( \sigma' \). Since \( v \)
is generic with respect to \((p)_t\), the consumer has a strict preference ordering over purchase times, which implies that he purchases at the same time under \(\sigma\) and \(\sigma'\), which contradicts the assumption that \(v \in V\). ■

**Proof of Lemma 12.** Consider the problem of maximizing \(Q_t\) over price-rationing paths \((p', q')_t\) in an \(\varepsilon\)-ball about \((p, q)_t\). As \(\varepsilon \to 0\), the measure of consumers who have different preference orderings over purchase times (i.e., over the \(\{\delta^i(v - p_t)\}_t\) under \((p', q')_t\) and \((p, q)_t\) converges to 0. Furthermore, the maximum difference between \(Q_t\) and a \(Q'_t\) corresponding to \((p, q)_t\) in an \(\varepsilon\)-ball about \((p, q)_t\) (holding \((p)_t\) fixed) is no more than \(\sum_{t=0}^{\infty} \max \{\varepsilon, Q_t\}\), the maximum measure of consumers whose purchasing times can be affected decreasing \(q_t\) by \(\varepsilon\) for all \(t\), holding other consumers’ purchasing times fixed; this follows because if rationing prevents measure \(\mu\) consumers from purchasing at some time \(t\), each of these consumers cannot alter his play in a way that leads more than one total consumer to purchase at time \(\tau\) (i.e., he can purchase at time \(\tau\) himself, or he can displace one other consumer through rationing at some other time). Thus, the maximum variation in \(Q_t\) over an \(\varepsilon\)-ball about \((p, q)_t\) converges to \(\lim_{\varepsilon \to 0} \sum_{t=0}^{\infty} \max \{\varepsilon, Q_t\}\) as \(\varepsilon \to 0\), so the following technical lemma completes the proof:

**Lemma 14** Given any quantity path \((Q)_t\), \(\lim_{\varepsilon \to 0} \sum_{t=0}^{\infty} \max \{\varepsilon, Q_t\} = 0\).

**Proof.** First, note that

\[
\lim_{\varepsilon \to 0} \sum_{t=0}^{\infty} \max \{\varepsilon, Q_t\} = \left(\lim_{\varepsilon \to 0} \# \{t : Q_t > \varepsilon\}\right) + \left(\lim_{\varepsilon \to 0} \sum_{t : Q_t < \varepsilon} Q_t\right).
\]

Let \(N_\varepsilon \equiv \# \{t : Q_t > \varepsilon\}\) to simplify notation. Assume, towards a contradiction, that the lemma is false, i.e., that there exists \(\delta > 0\) such that for all \(\varepsilon > 0\) there exists \(\varepsilon < \bar{\varepsilon}\) satisfying \(\varepsilon N_\varepsilon > \delta\). Fix such a \(\delta > 0\), and let \(\varepsilon_0 > 0\) satisfy \(\varepsilon_0 N_0 > \delta\). Now for all \(n \geq 1\), let \(\bar{\varepsilon}_n = \frac{\varepsilon_{n-1}}{2^n}\), and let \(\varepsilon_n\) be a strictly positive number strictly less than \(\bar{\varepsilon}_n\) satisfying \(\varepsilon_n N_{\varepsilon_n} > \delta\). Note that

\[
\frac{\varepsilon_n}{\varepsilon_{n-1}} < \frac{1}{2^n}.
\]

I omit the measure-theoretic details of this argument, which are similar to those in the proof of Proposition 13.

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Observe that, for any \( n, N_{\varepsilon_n} < \frac{1}{\varepsilon_n} \), for otherwise the total quantity of sales made in the \( N_{\varepsilon_n} \) periods in which \( Q_t > \varepsilon_n \) would exceed 1. Since \( N_{\varepsilon_n} < \frac{1}{\varepsilon_n} \), and \( \varepsilon_{n+1} N_{\varepsilon_{n+1}} > \delta \), it follows that \( N_{\varepsilon_{n+1}} - N_{\varepsilon_n} > \frac{\delta}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \). Now \( N_{\varepsilon_{n+1}} - N_{\varepsilon_n} \) is the number of periods in which \( Q_t \) is between \( \varepsilon_{n+1} \) and \( \varepsilon_n \), so total sales made in all periods is at least

\[
\sum_{n \geq 0} (N_{\varepsilon_{n+1}} - N_{\varepsilon_n}) \varepsilon_{n+1} > \sum_{n \geq 0} \left( \frac{\delta}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right) > \sum_{n \geq 0} \left( \frac{1}{2n+1} \right) = \infty.
\]

This contradicts the assumption that the population of consumers is of measure 1. ■ ■

**Proof of Proposition 13.** Suppose that rationing occurs at time \( t^* \) along a full-delivery PBE path \((p, q)_t\). I show that \((p, q)_t\) cannot be a best full-delivery PBE path.

First, consider the path \((p', q')_t\) given by \( p'_t = p_t \) for all \( t \) and \( q'_t = Q_t \) for all \( t \), where \((Q)_t\) is the unique (by Lemma 11) quantity path corresponding to \((p, q)_t\). All consumers are best-responding if they purchase at the same time under \((p', q')_t\) as they did under \((p, q)_t\), and by Lemma 10 this purchasing schedule is unique up to the measure-0 set of consumers who are indifferent between purchasing at different times, so the seller’s profit is the same in any full-delivery PBE corresponding to \((p', q')_t\) and in any full-delivery PBE corresponding to \((p, q)_t\). Furthermore, \( D_t((p', q')_t) > q'_t \). Since \( F \) admits a strictly positive density, there is a small enough strict increase in \( p_t^* \), \( \Delta_p \), such that demand at \( t^* \) still exceeds \( q_t^* \) when price at \( t^* \) is increased by \( \Delta_p \). So consider the path \((p^*, q^*)_t\) given by \( p_t^* = p_t + \Delta_p, p_t^* = p_t \) for all \( t \neq t^* \), and \( q_t^* = Q_t \) for all \( t \). I claim that \( Q_t^* = Q_t \) for all \( t \), which then implies that profit is higher under \((p^*, q^*)_t\) than under \((p', q')_t\) (and therefore \((p, q)_t\)), since \( Q_t^* > 0 \) (by the definition of rationing occurring at \( t^* \)).

Since \( q_t^* = Q_t \) for all \( t \), it follows that \( Q_t^* \leq Q_t \) for all \( t \), so since \( Q_t^* = Q_t^* \) by definition of \( \Delta_p \) it suffices to show that \( \sum_{t \neq t^*} Q_t^* \geq \sum_{t \neq t^*} Q_t \). Suppose, towards a contradiction, that \( \sum_{t \neq t^*} Q_t - \sum_{t \neq t^*} Q_t^* \equiv \mu > 0 \). For any \( \tau \neq t^* \), if \( Q_\tau - Q_\tau^* \equiv \mu_\tau > 0 \), then \( D_\tau((p^*, q^*)_t) = q_t^* - \mu_\tau \). Since the price at \( \tau \) is the same under \((p^*, q^*)_t\) and \((p, q)_t\), this is possible only if there are measure \( \mu_\tau \) consumers who demanded the good at \( \tau \) under \((p, q)_t\) and have higher PBE
payoffs under \((p^*, q^*)_t\). Since prices are weakly higher in each period under \((p^*, q^*)_t\), this implies that at least \(\mu_t\) consumers who purchase at \(\tau\) under \((p, q)_t\) must purchase at times under \((p^*, q^*)_t\) at which they could not purchase under \((p, q)_t\). This argument applies to all \(\tau\) such that \(\mu_t > 0\), so at least \(\mu = \sum_t \mu_t\) consumer purchase at times under \((p^*, q^*)_t\) at which they could not purchase under \((p, q)_t\), and receive higher payoffs under \((p^*, q^*)_t\). Let \(\mathcal{D}\) be the set of consumers who purchase at times under \((p^*, q^*)_t\) at which they could not purchase under \((p, q)_t\) and receive higher payoffs under \((p^*, q^*)_t\). Now measure \(\mu\) of consumers can purchase at times under \((p^*, q^*)_t\) at which none of them can purchase under \((p, q)_t\) only if there exists a measure-preserving injection \(\psi: \mathcal{D} \rightarrow [v, \tilde{v}]\) (mapping consumers who do better under \((p^*, q^*)_t\) to consumers they "displace") from these consumers to a another set of consumers of mass \(\mu\) satisfying

1. \(\psi(v) > v\) for all \(v \in \mathcal{D}\)

2. If a consumer with generic (with respect to \((p^*)_t\)) valuation \(v\) purchases at time \(t\) under \((p^*, q^*)_t\), then every consumer with valuation \(\psi(v)\) purchases at time \(t\) under \((p, q)_t\), and (since \(\psi\) is measure-preserving) for every \(t\) the measure of consumers in the preimage who purchase at time \(t\) under \((p^*, q^*)_t\) equals the measure of consumers in the image who purchase at time \(t\) under \((p, q)_t\).

3. A consumer in the image of \(\psi\) who purchases at time \(t\) under \((p, q)_t\) purchases at some time \(t' \neq t\) under \((p^*, q^*)_t\)

Note that each of the consumers in the image of \(\psi\) retains under \((p^*, q^*)_t\) the option of purchasing at the same time at which she purchased under \((p, q)_t\), because her valuation is higher than that of the corresponding consumer in the preimage, so since she does not do so it must either be that she purchases at a time \(t'\) at which she could not purchase under \((p^*, q^*)_t\) and receives a higher payoff under \((p^*, q^*)_t\) or that \(t = t^*\), in which case purchasing at \(t\) has become less attractive. That is, if a consumer is in the image of \(\psi\), then either he is also in \(\mathcal{D}\) (the preimage of \(\psi\)) or he purchases at \(t^*\) under \((p, q)_t\) but not under \((p^* q^*)_t\). Iterating the procedure of constructing such a measure-preserving injection from consumers who purchase at different times under \((p^*, q^*)_t\) and \((p, q)_t\) and receive higher payoffs under
(p*, q*) to the consumers they "displace" implies that there are μ consumer who did not purchase at t* under (p, q) who do purchase at t* under (p*, q*)t, that all of them receive higher payoffs under (p*, q*)t than under (p, q)t, and that a measure-preserving bijection satisfying 1 through 3 exists between the set of consumers who receive a higher payoff under (p*, q*)t than under (p, q)t and the set of consumers who purchase at t* under (p, q)t who do not purchase at t* under (p*, q*)t.

By the preceding paragraph, the measure of consumers who purchase at t* under (p, q)t who do not purchase at t* under (p*, q*)t is at least μ. Since all consumers who purchase at t* under (p*, q*)t but not under (p, q)t receive a higher payoff under (p*, q*)t, it follows that every consumer who purchases at t* under (p, q)t has a higher valuation than any of these consumers, and therefore has a higher valuation than any consumer who receives a higher payoff under (p*, q*)t than under (p, q)t. Therefore, every consumer who purchases at t* under (p, q)t but not under (p*, q*)t prefers to purchase at any τ satisfying μτ > 0 to never purchasing. Furthermore, suppose that mass ε of such consumers, with valuations with infimum v, purchase at time τ satisfying μτ = 0 under (p*, q*)t. Then there must exist mass ε of consumers each with valuation strictly less than v who purchase at τ under (p, q)t but not under (p*, q*)t. Consider such a consumer with valuation v' < v, fix any τ satisfying μτ > 0, and suppose towards a contradiction that v' < pτ. We have that v > pτ, a consumer with valuation v prefers purchasing at time τ and price pτ to purchasing at time τ and price pτ (by revealed preference at (p*, q*)t, since there is no rationing at τ under (p*, q*)t), and a consumer with valuation v' also prefers purchasing at time τ and price pτ to purchasing at time τ and price pτ (since v' > pτ by revealed preference at (p, q)t and v' < pτ by assumption). Now there also exists a consumer who purchases at time τ and price pτ under (p, q)t and obtains a higher payoff under (p*, q*)t, since μτ > 0. Such a consumer must have valuation v'' ∈ [pτ, v), so v'' > v', which implies that such a consumer has the option of purchasing at τ under (p, q)t. Therefore, such a consumer must prefer purchasing at time τ and price pτ to purchasing at time τ and price pτ. Thus, the assumption that v' < pτ ≤ v'' < v yields a violation of single-crossing. Therefore, each of the μ consumers who purchases at t* under (p, q)t but not under (p*, q*)t either purchases at a τ such that μτ > 0 under (p*, q*)t or else displaces another consumer who prefers to purchase at any
such \( \tau \) to never purchasing. So the measure of consumers who purchase at some (finite) time under \((p^*, q^*)_t\) must weakly exceed the measure of consumers who purchase at some time under \((p, q)_t\). Since \(Q^*_t = Q_t\), this implies that \(\sum_{t \neq t^*} Q^*_t \geq \sum_{t \neq t} Q_t\), completing the proof that profit is higher under \((p^*, q^*)_t\) than under \((p, q)_t\).

It remains only to check that there exists a full-delivery PBE with price-rationing path \((p^*, q^*)_t\). This follows from the fact that there exists a full-delivery PBE with price-rationing path \((p, q)_t\), because, since \(Q^*_t = Q_t\) for all \(t\) and \(p^*_t \geq p_t\) for all \(t\), the seller's gain from nondelivery is the same in every period under \((p^*, q^*)_t\) as under \((p, q)_t\), and her gain from delivery is weakly higher in every period under \((p^*, q^*)_t\), in a strategy profile in which consumers expect the seller to never deliver in the future if she does not deliver in the current period.

\[\text{Proof of Corollary 1.}\] Suppose that \((p^*, q^*)_t\) is a best full-delivery PBE price-rationing path in \(\Gamma_R\). Consider the following strategy profile in \(\Gamma\), which I denote by \(\sigma\):

1. The seller sets price path \((p^*)_t\) and \(x_t = 1\) as long as she has conformed to this strategy in the past. Otherwise, she sets \(p_t = \bar{v}, x_t = 0\) for all future periods. In particular, the seller sets \(x_t = 0\) in any period in which she has set \(p_t \neq p^*_t\).

2. A consumer with valuation \(v\) who has not yet received the good at \(\tau\) accepts at \(\tau\) if and only if the seller has never deviated from her prescribed strategy and \(\tau \in \arg \max_{t} \delta^t (v - p^*_t)^{97}\).

To establish that \(\sigma\) is a PBE, first observe that a consumer with valuation \(v\) receives the same payoff \(V_v\) under \(\sigma\) as under any full-delivery PBE with price-rationing path \((p^*, q^*)_t\) in \(\Gamma_R\). This follows because, since no rationing occurs along \((p^*, q^*)_t\) in \(\Gamma_R\) (by Proposition 13) and the path of play does not depend on an individual consumer's actions, a consumer with generic valuation \(v\) facing \((p^*, q^*)_t\) in \(\Gamma_R\) purchases at time \(\tau\) if and only if \(\tau \in \arg \max_{t} \delta^t (v - p^*_t)\) in any full-delivery PBE. Furthermore, if valuation \(v\) is generic with respect to \((p^*)_t\), then the payoff of a consumer with valuation \(v\) uniquely determines her

\[^{97}\text{The case where there are multiple maximizers is irrelevant, as this occurs for a set of measure zero consumers.}\]
purchase time. Therefore, $(Q)_t$ is the same under any full-delivery PBE with price-rationing path $(p^*, q^*)_t$ in $\Gamma_R$ as under $\sigma$.

Next, note that if the seller ever sets $p_t \neq p^*_t$, she receives zero continuation payoff. Since this is her minmax value in $\Gamma_R$, she cannot receive continuation payoff strictly less than this in the continuation game from $\tau + 1$ onward in $\Gamma_R$ under a full-delivery PBE with price-rationing path $(p^*, q^*)_t$. Now we have seen that $(Q)_t$ is the same in any full-delivery PBE with price-rationing path $(p^*, q^*)_t$ in $\Gamma_R$ as in $\sigma$, and by construction $(p)_t$ is the same as well, so the seller's on-path continuation payoff from $\tau + 1$ onward must be the same, too, so in particular this continuation payoff must be nonnegative. This implies that setting $p_t \neq p^*_t$ on-path is not a profitable deviation. Similarly, the fact that setting $q_t = q^*_t$ is optimal on-path along $(p^*, q^*)_t$ implies that setting $q_t = q^*_t$ is optimal on-path in $\sigma$, because the cost of delivery and on-path continuation values are identical, while the payoff of zero that results from deviating from the equilibrium path in $\sigma$ is at least as bad as the payoff from deviating in any PBE of $\Gamma_R$. Also, the seller's off-path play is optimal because off-path price-setting does not affect her payoffs and off-path delivery imposes a positive cost at no benefit.

I next check that each consumer's play is optimal. It is again obvious that his off-path play is optimal, as paying is costly and yields no benefit when the seller sets $q_t = 0$. That his on-path play is optimal follows from the fact that the seller's strategy is full-delivery. So $\sigma$ is a full-delivery PBE of $\Gamma$.

The above argument shows that if a price-quantity path $(p, Q)_t$ is a best full-delivery PBE price-quantity path in $\Gamma_R$, then it is also a best full-delivery PBE price-quantity path in $\Gamma$. For the converse, suppose that $(p^*, Q^*)_t$ is a full-delivery PBE price-quantity path in $\Gamma$. Consider the following strategy profile in $\Gamma_R$:

1. The seller sets price path $(p^*)_t$ and $q_t = 1$, $x_t = 1$, as long as she has conformed to this strategy in the past. Otherwise, she sets $p_t = \bar{v}$, $q_t = 1$, and $x_t = 0$ for all future periods. In particular, the seller sets $x_t = 0$ in any period in which has set $p_t \neq p^*_t$.

2. A consumer with valuation $v$ who has not yet received the good at $\tau$ pays at $\tau$ if and only if the seller has never deviated from her prescribed strategy and $\tau \in$
arg max_t \delta^t (v - p^*_t).

It is easy to check that this is a PBE in \Gamma_R. Furthermore, since no other players condition play on an individual consumer’s actions, a consumer with generic valuation \( v \) purchases at time \( \tau \) under this strategy profile if and only if a consumer with this valuation purchases at \( \tau \) in any full-delivery PBE in \( \Gamma \) with price-quantity path \( (p^*, Q^*)_\tau \). This implies that the mass of consumers who purchase at each period under this profile is the same as the mass of consumers who purchase at each period in any full-delivery PBE in \( \Gamma \) with price-quantity path \( (p^*, Q^*)_\tau \), which then implies that the seller’s profit under this strategy profile is the same as under any full-delivery PBE in \( \Gamma \) with price-quantity path \( (p^*, Q^*)_\tau \). This completes the proof. ■

4.10 Appendix B: Non-Full-Delivery Equilibria

This appendix considers non-full-delivery PBE of the relational contracting model of Section 4.5. I conjecture that optimal PBE of \( \Gamma \) are not fully-delivery PBE, though the difference in payoff between an optimal PBE and a best full-delivery PBE must converge to 0 as \( \delta \) converges to 1, as argued in the text. This is because setting \( x < 1 \) allows the seller to sell to some lower-valuation consumers before higher-valuation consumers. This may be useful for the seller, as selling to low-valuation consumers before high-valuation consumers may be a way of increasing continuation payoffs without increasing quantity sold today, allowing the seller to sell more quickly.

While a complete analysis of optimal (non-full-delivery) PBE is outside the scope of the paper, I show here that analogues of parts 3 and 4 of Theorem 10 for non-full-delivery PBE can be established without reference to the model with rationing. I view these results as complementary to those in the text, because full-delivery PBE are of particular interest for reasons discussed in the text. The results in this Appendix do not establish that full-delivery equilibria exist that yield profits close to static monopoly profits; I do not know how to establish this result without using the connection to the model with rationing developed in Section 4.5.

Intuitively, one can prove analogues of parts 3 and 4 of Theorem 10 directly for non-full-
delivery PBE because one can use non-delivery to substitute for rationing. That is, instead of using rationing to ensure that only fraction $\gamma$ of those consumers who demand the good at price $p_t$ at time $t$ are allowed to purchase at $t$, the seller can charge $\gamma p_t$ to each of these consumers in exchange for delivering the good to each of them with probability $\gamma$. With this idea in hand, the proof of parts 3 and 4 of Theorem 10 follows easily from the proof of Proposition 15 in Section 4.6:

**Proposition 20** There exists a strategy profile in $\Gamma$ that is a non-full-delivery PBE for high enough $\delta$ under which the seller’s payoff converges to her static monopoly payoff as $\delta \to 1$.

**Proof.** Consider the following strategy profile:

1. The seller sets $p_t = \gamma p^m$, $x_t = \gamma$ for all $t$, for $\gamma$ an arbitrary positive constant less than 1, as long as she has conformed to this strategy in the past. Otherwise, she sets $p_t = \bar{v}$, $x_t = 0$ for all future periods, and in particular sets $x_t = 0$ in any period in which has set $p_t \neq \gamma p^m$.

2. A consumer with valuation $v$ who has not yet received the good pays if and only if $v \geq p^m$ and the seller has never deviated from her prescribed strategy.

At any period $t$ along the equilibrium path, a consumer with valuation $v < p^m$ has continuation value 0, while a consumer with valuation $v \geq p^m$ who has not yet received the good has continuation value $\frac{\delta \gamma}{1-\delta(1-\gamma)} (v- p^m) < \frac{\gamma}{1-\delta(1-\gamma)} (v- p^m)$, so every consumer’s play is optimal by the one-shot deviation principle. It is clear that the seller’s off-path play and on-path price setting is optimal. It remains only check that the seller has an incentive to deliver the good along the equilibrium path. This condition is

$$
\sum_{\tau=1}^{\infty} \delta^\tau ((1-\gamma)^{i+\tau} p_{t+\tau} - \gamma (1-\gamma)^{i+\tau} c) \geq \gamma (1-\gamma)^{i} c \text{ for all } t \geq 0.
$$

For any $t$, this can be rewritten as inequality (43). Now if $\delta > c/p^m$ then there exists a positive $\gamma$ that satisfies (43). The above strategy profile then yields profit $\left(\frac{\gamma}{1-\delta(1-\gamma)}\right) D(p^m) (p^m - c)$ for the seller, which converges to $D(p^m) (p^m - c)$ as $\delta$ converges to 1. 
For the analogue of part 4 of Theorem 10 for non-full-delivery PBE, I argue as in the discussion following Proposition 15. Consider the strategy profile where the seller fixes the price of a $\gamma$ chance of receiving the good at some given $\gamma p$. Recall that

$$\gamma^*(p) \equiv \frac{\delta p - c}{\delta}.$$

By the same argument that led to (43), $\gamma^*(p)$ is the greatest probability of receiving the good that the seller can credibly offer at price $\gamma^*(p)p$ in a PBE with fixed price and delivery probability. The best PBE profit for the seller with a constant price path at $\gamma p$ and a constant sales rate $\gamma$ is therefore

$$\left( \frac{\gamma^*(p)}{1 - \delta(1 - \gamma^*(p))} \right) D(p)(p - c),$$

which can be rewritten as

$$\left( p - \frac{c}{\delta} \right) D(p).$$

Therefore, if the seller sets $p_t = \gamma p^m \left( \frac{c}{\delta} \right)$ and $x_t = \gamma^* \left( p^m \left( \frac{c}{\delta} \right) \right) = \frac{\delta p^m \left( \frac{c}{\delta} \right) - c}{\delta p^m \left( \frac{c}{\delta} \right)}$ for all $t$ on the equilibrium path, and off-path play is given as in the strategy profile in the proof of Proposition 20, the seller’s profit is equal to the static monopoly profit when cost equals $c/\delta$. Finally, the seller can achieve a strictly higher payoff than this by slightly raising price and delivery probability early on while keeping quantity delivered constant in every period, in analogy with the discussion preceding Proposition 16.

References


