6.061 / 6.690 Introduction to Electric Power Systems
Spring 2007

Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.061/6.690 Introduction to Power Systems

Problem Set 4 Solutions February 28, 2007

Problem 1: Current in the line is:

$$I = \frac{V_s - V_r}{R + jX} = V\frac{1 - e^{-j\delta}}{R + jX}$$

since complex power is $P + jQ = VI^*$, real and reactive power at the sending and receiving ends are:

$$P_s + jQ_s = V^2\frac{1 - e^{j\delta}}{R - jX}$$

$$P_r + jQ_r = -V^2\frac{1 - e^{-j\delta}}{R - jX}$$

Note I have defined receiving end power as coming *out* of the line. The loci of these two complex quantities are circles around the two origins:

$$\text{Sending} \quad \frac{V^2}{R - jX}$$

$$\text{Receiving} \quad -\frac{V^2}{R - jX}$$

To construct the sending and receiving end circles, we rotate a vector of the same length as the distance to the origin: counter clockwise for sending end power and clockwise for receiving end power. The construction is shown in Figure 1.

The problem of finding the angle for a defined power flow involves solving a transcendental function. We could do this in a variety of ways, but MATLAB gives us lazy people a clever way of doing it. The function fzero(FOO(X), X0) returns a value of X that makes FOO(X) equal to zero. See the script attached for details. The answers are:

```
Required Power Angle =    0.546 radians =    31.30 degrees
Sending End Circle Center =     9901 + j   99010
Circle Radius =    99504 Watts
Receiving End Power =    50000
Sending End Power =    52882
Line Dissipation =     2882
Receiving End Reactive Power =   -19556
Sending End Reactive Power=     9268
Line Reactive Power=    28825
```

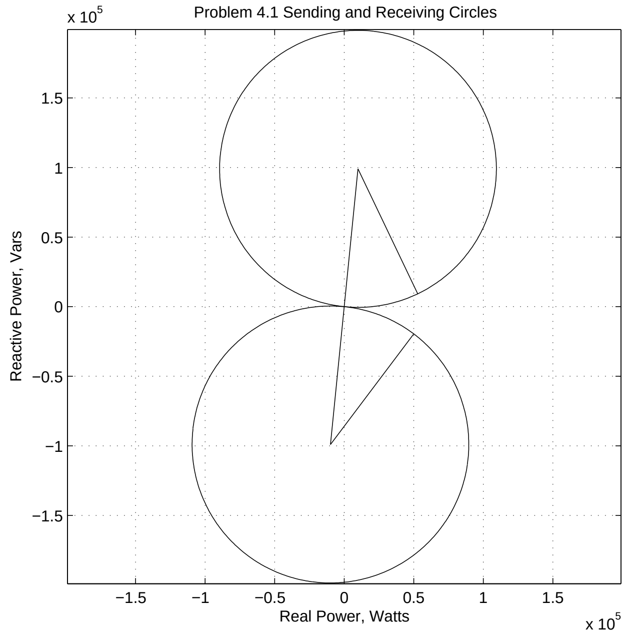

Figure 1: Sending and Receiving End Circles

Vectors from the centers of the power circle to the sending and receiving complex power points are plotted in Figure 1. Note that both real and reactive power are conserved.

Problem 2: The three phase voltages are:

$$\begin{aligned} v_a &= \sqrt{2} \cdot 120 \cos(\omega t) \\ v_b &= \sqrt{2} \cdot 120 \cos(\omega t - \frac{2\pi}{3}) \\ v_c &= \sqrt{2} \cdot 120 \cos(\omega t + \frac{2\pi}{3}) \end{aligned}$$

and the center point of this source is grounded.

B We take this one out of order as it is the easiest. The voltages across each of the resistances is defined by the matching source, so that:

$$\begin{aligned} i_a &= \sqrt{2} \cdot \cos(\omega t) \\ i_b &= \sqrt{2} \cdot \cos(\omega t - \frac{2\pi}{3}) \\ i_c &= \sqrt{2} \cdot \cos(\omega t + \frac{2\pi}{3}) \end{aligned}$$

A Noting that in part B, the sum of the three currents is zero, the neutral point at the junction of the three resistors can (and in fact will be) at zero potential and so the currents are exactly the same.

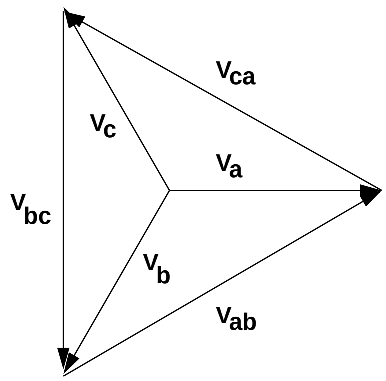

Figure 2: Three-Phase Voltages

C Refer to Figure 2. The voltage across the resistor is:

$$v_{ab} = v_a - v_b = \sqrt{2} \times 120 \left(\cos \omega t - \cos(\omega t - \frac{2\pi}{3}) = \sqrt{2} \times \sqrt{3} \times 277 \cos(\omega t + \frac{\pi}{6}) \right)$$

and of course $\sqrt{3} \times 120 \approx 208$. So

$$i_a = -i_b = \sqrt{2} \times \frac{208}{360} \left(\cos \omega t + \frac{\pi}{6} \right) = \sqrt{2} \times .578 \cos \left(\omega t + \frac{\pi}{6} \right) \approx .817 \times \cos \left(\omega t + \frac{\pi}{6} \right)$$

D Since $3 \times 120 = 360$ this load is equivalent to that of Part A and so the currents are the same. If you want you can do this the hard way by following the recipe for Part E and adding the two resistor currents at each node.

E This one involves computing the two resistor voltages:

$$\begin{aligned} v_{ab} &= v_a - v_b = & \sqrt{2} \times 120 \left(\cos \omega t - \cos(\omega t - \frac{2\pi}{3}) \right) = \sqrt{2} \times \sqrt{3} \times 120 \cos(\omega t + \frac{\pi}{6}) \\ v_{ca} &= v_c - v_a = & \sqrt{2} \times 120 \left(\cos(\omega t + \frac{4\pi}{3}) - \cos \omega t \right) = \sqrt{2} \times \sqrt{3} \times 120 \cos(\omega t + \frac{5\pi}{6}) \end{aligned}$$

Noting that $\cos \left(\omega t + \frac{5\pi}{6} \right) = - \cos \left(\omega t - \frac{\pi}{6} \right)$, we may use the identity:

$$\cos(\omega t - \frac{\pi}{6}) + \cos(\omega t + \frac{\pi}{6}) = 2 \cos \omega t \cos \frac{\pi}{6} = \sqrt{3} \cos \omega t$$

and, using the results obtained in Part C,

$$\begin{aligned} i_a &= \sqrt{2} \cos \omega t \\ i_b &= -\sqrt{2} \times .578 \cos(\omega t + \frac{\pi}{6}) \\ i_c &= -\sqrt{2} \times .578 \cos(\omega t - \frac{\pi}{6}) \end{aligned}$$

F This is just like case B, except for phase C is not connected:

$$\begin{aligned} i_a &= \sqrt{2} \times \cos \omega t \\ i_b &= \sqrt{2} \times \cos \left(\omega t - \frac{2\pi}{3} \right) \\ i_c &= 0 \end{aligned}$$

Problem 3: This one is best done graphically. Note that the current through the ground resistor is just the sum of the three phase currents. Shown in Figure 3 is the same figure that established the currents, but with this summation shown.

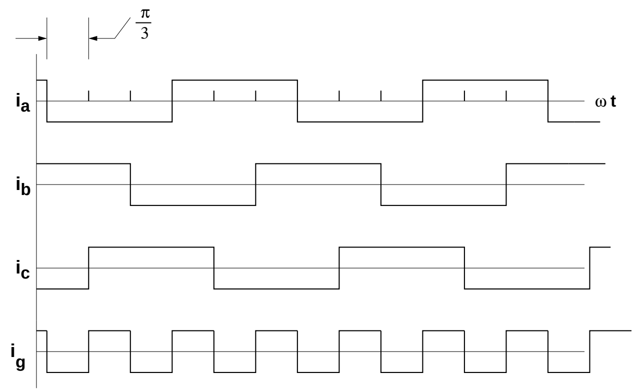

Figure 3: Currents

Now, the voltages in the individual resistors will be just the current sources times the one ohm resistance. The voltage across the ground resistance will, similarly, be just the bottom trace times five ohms.

Problem 4: For 6.690 If we are to write expressions for the currents, they would be:

$$
\begin{aligned}
i_a &= \frac{v_a - v_n}{R} \\
i_b &= \frac{v_b - v_n}{R} \\
i_c &= \frac{v_c - v_n}{R}
\end{aligned}
$$

where v_n is the voltage of the 'star point' with respect to ground. Now we may note that, since the sum of the three currents must be zero the star point voltage must be the average of the three phase voltages:

$$v_n = \frac{v_a + v_b + v_c}{3}$$

which may be estimated graphically: see Figure 4

Now we may use this to find the voltage across each of the resistances, by graphically subtracting the neutral, or star point voltage from the phase voltage. This is shown for Phase A in Figure 5. Current in that lead will have the same shape. The same procedure follows for the other two phases: the answers are identical but shifted by 120 degrees.

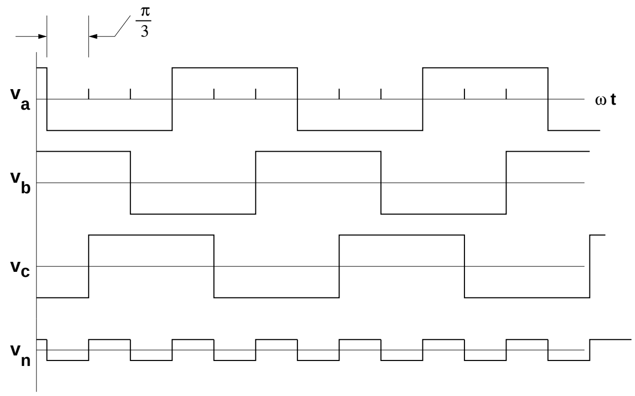

Figure 4: Voltage Source Waveforms

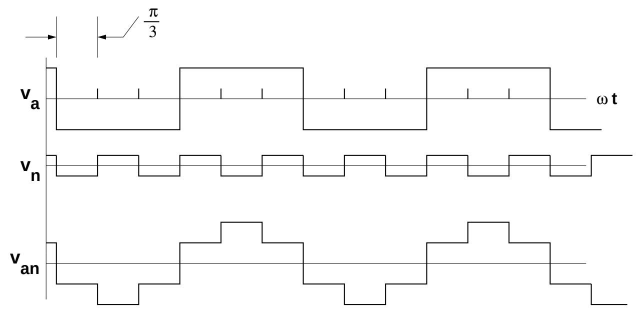

Figure 5: Reconstructed Phase A to Neutral Voltage

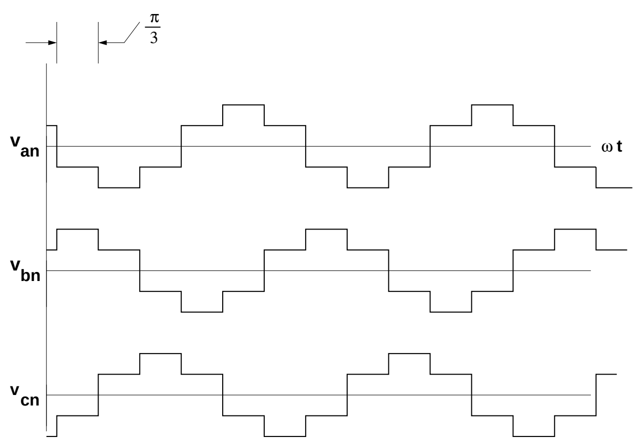

Figure 6: Three Phase Voltages with Neutral Open

Appendix: MATLAB script for Problem 1:

```
% 6.061/6.690 problem set 4, problem 1
% transmission line example
clear
global V  Y P_R                          % do this for a procedure to follow
% basic parameters
V = 1000;                                %Single Phase Voltage
R = 1;                                   %Series line resistance
X = 10;                                  %Series line reactance
P_R = 50000;                             %Target Power
Z = R+j*X;                               %Impedance
Y = 1/Z;                                 %Admittance
delt = 0:pi/200:2*pi;                    %Going full circle
A = V^2*conj(Y) .* (1 - exp(j .* delt)); % Complex sending end power
Ps = real(A);                            % sending end circle
Qs = imag(A);

B = -V^2*conj(Y) .* (1 - exp(-j .* delt)); % complex receiving end power
Pr = real(B);
Qr = imag(B);

Pz = V^2*R/X^2;                          % this establishes circle centers
Qz = V^2/X;
P3 = [0 Pz];                             % sending end
Q3 = [0 Qz];
P4 = - P3;                               % receiving end
Q4 = - Q3;

deltz = fzero('pbal', .1);               % angle for the target case
Az = V^2*conj(Y) * (1 - exp(j * deltz)); % Complex sending end power
Psz = real(Az);                          % real part of that
Qsz = imag(Az);                          % reactive part

Bz = -V^2*conj(Y) * (1 - exp(-j * deltz)); % receiving end power
Prz = real(Bz);                            % this had better be P_R
Qrz = imag(Bz);

P3z = [Pz Psz];                           % these are vectors to sending
Q3z = [Qz Qsz];                           % and receiving complex power
P4z = [-Pz Prz];
Q4z = [-Qz Qrz];

I = abs(V * (1 - exp(-j*deltz)) * conj(Y)); % as a check: here is current
P_d = I^2 * R;                              % dissipation
```

```
Q_L = I^2 * X;                                % line absorbed VARs

fprintf('Required Power Angle = %8.0f radians = %8.2f degrees\n', deltz, 180*deltz/pi)
fprintf('Receiving End Power = %8.0f\n', Prz)
fprintf('Sending End Power = %8.0f\n', Psz)
fprintf('Line Dissipation = %8.0f\n', P_d)
fprintf('Receiving End Reactive Power = %8.0f\n', Qrz)
fprintf('Sending End Reactive Power= %8.0f\n', Qsz)
fprintf('Line Reactive Power= %8.0f \n', Q_L)

%
Pscale = 2*V^2*abs(Y);                       % so it plots nicely
figure(1)
plot(Ps, Qs, Pr, Qr, P3, Q3, 'k', P4, Q4, 'k', P3z, Q3z, 'k', P4z, Q4z, 'k')
title('Problem 4.1 Sending and Receiving Circles')
ylabel('Reactive Power, Vars')
xlabel('Real Power, Watts')
axis([-Pscale Pscale -Pscale Pscale])
axis square
grid on
-----------------------------------------------
function Pbal = pbal(d)    % used in fzero() to find power angle
global V  Y P_R
B = -V^2*conj(Y) * (1 - exp(-j * d));
Pr = real(B);
Qr = imag(B);
Pbal = P_R - Pr;
```