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6.061 / 6.690 Introduction to Electric Power Systems
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Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.061 Introduction to Power Systems
Class Notes
Chapter 1: Review of Network Theory*

J.L. Kirtley Jr.

1 Introduction

This note is a review of some of the most salient points of electric network theory. In it we do not prove any of the assertions that are made. We deal only with passive, linear network elements.

2 Network Primitives

Electric network theory deals with two primitive quantities, which we will refer to as:

1. Potential (or voltage), and
2. Current.

Current is the actual flow of charged carriers, while difference in potential is the force that causes that flow. As we will see, potential is a single-valued function that may be uniquely defined over the *nodes* of a network. Current, on the other hand, flows through the *branches* of the network. Figure 1 shows the basic notion of a *branch*, in which a *voltage* is defined across the branch and a *current* is defined to flow through the branch. A *network* is a collection of such elements, connected together by *wires*.

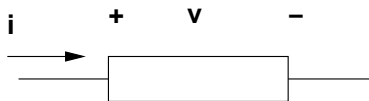


Figure 1: Basic Circuit Element

Network *topology* is the interconnection of its elements. That, plus the constraints on voltage and current imposed by the elements themselves, determines the performance of the network, described by the distribution of voltages and currents throughout the network.

Two important concepts must be described initially. These are of “loop” and “node”.

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1. A *loop* in the network is any *closed* path through two or more elements of the network. Any non-trivial network will have at least one such loop.

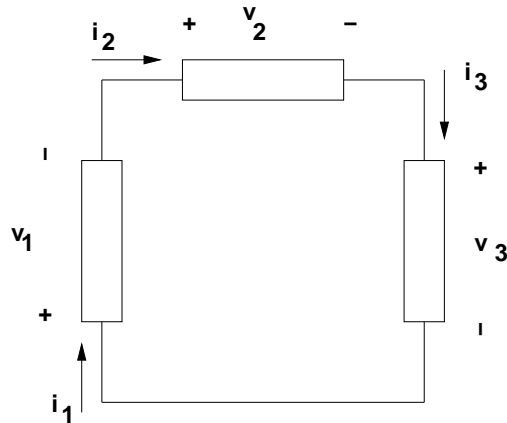


Figure 2: This is a loop

2. a *node* is a point at which two or more elements are interconnected.

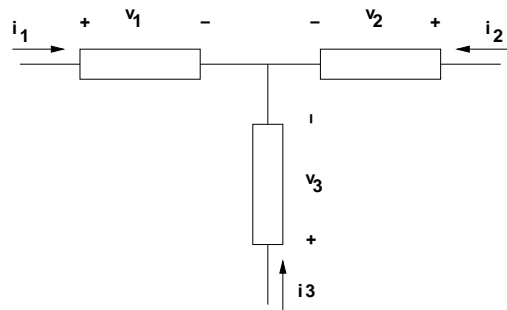


Figure 3: This is a node

The two fundamental laws of network theory are known as *Kirchoff's Voltage Law (KVL)*, and *Kirchoff's Current Law (KCL)*. These laws describe the topology of the network, and arise directly from the fundamental laws of electromagnetics. They are simply stated as:

- Kirchoff's Voltage Law states that, around any *loop* of a network, the sum of all voltages, taken in the same direction, is zero:

$$\sum_{loop} v_k = 0 \quad (1)$$

- Kirchoff's Current Law states that, at any *node* of a network, the sum of all currents entering the node is zero:

$$\sum_{node} i_k = 0 \quad (2)$$

Note that KVL is a discrete version of Faraday's Law, valid to the extent that no time-varying flux links the loop. KCL is just conservation of current, allowing for no accumulation of charge at the node.

Network elements affect voltages and currents in one of three ways:

1. *Voltage* sources constrain the potential difference across their terminals to be of some fixed value (the value of the source).
2. *Current* sources constrain the current through the branch to be of some fixed value.
3. All other elements impose some sort of relationship, either linear or nonlinear, between voltage across and current through the branch.

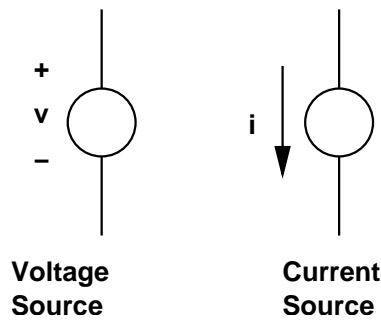


Figure 4: Notation for voltage and current sources

Voltage and current sources can be either *independent* or *dependent*. Independent sources have values which are, as the name implies, independent of other variables in a circuit. Dependent sources have values which depend on some other variable in a circuit. A common example of a dependent source is the equivalent current source used for modeling the collector junction in a transistor. Typically, this is modeled as a *current dependent current source*, in which collector current is taken to be directly dependent on emitter current. Such dependent sources must be handled with some care, for certain tricks we will be discussing below do not work with them.

For the present time, we will consider, in addition to voltage and current sources, only *impedance* elements, which impose a linear relationship between voltage and current. The most common of these is the *resistance*, which imposes the relationship which is often referred to as *Ohm's law*:

$$v_r = Ri_r \tag{3}$$

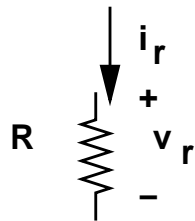


Figure 5: Resistance Circuit Element

A bit later on in this note, we will extend this notion of *impedance* to other elements, but for the moment the resistance will serve our purposes.

3 Examples: Voltage and Current Dividers

Figure 6 may be used as an example to show how we use all of this. See that it has one *loop* and three *nodes*. Around the loop, KVL is:

$$V_s - v_1 - v_2 = 0$$

At the upper right- hand node, we have, by KCL:

$$i_1 - i_2 = 0$$

The *constitutive relations* imposed by the resistances are:

$$v_1 = R_1 i_1$$

$$v_2 = R_2 i_2$$

Combining these, we find that:

$$V_s = (R_1 + R_2) i_1$$

We may solve for the voltage across, say, R_2 , to obtain the so-called *voltage divider* relationship:

$$v_2 = V_s \frac{R_2}{R_1 + R_2} \quad (4)$$

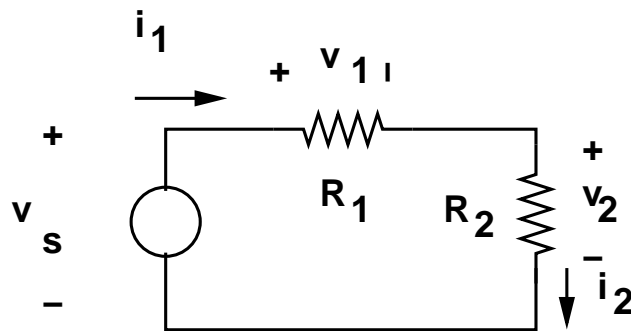


Figure 6: Voltage Divider

A second example is illustrated by Figure 7. Here, KCL at the top node yields:

$$I_s - i_1 - i_2 = 0$$

And KVL, written around the loop that has the two resistances, is:

$$R_1 i_1 - R_2 i_2 = 0$$

Combining these together, we have the *current divider* relationship:

$$i_2 = I_s \frac{R_1}{R_1 + R_2} \quad (5)$$

Once we have derived the voltage and current divider relationships, we can use them as part of our “intellectual toolkit”, because they will always be true.

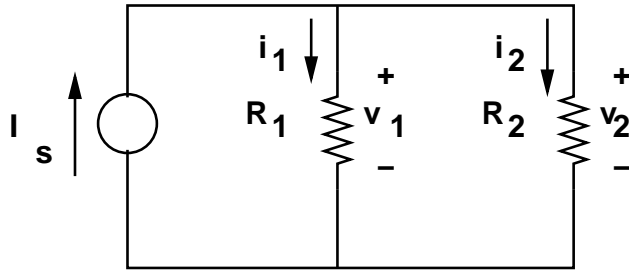


Figure 7: Current Divider

4 Node Voltages and Reference

One of the consequences of KVL is that every node in a network will have a potential which is uniquely specified with respect to some other node. Thus, if we take one of the nodes in the network to be a reference, or *datum*, each of the other nodes will have a unique potential. The voltage across any network branch is then the difference between the potentials at the nodes to which the element is connected. The potential of a node is the *sum* of voltages encountered when traversing some path between that node and the *datum* node. Note that any path will do. If KVL is satisfied, all paths between each pair of nodes will yield the same potential.

A commonly used electric circuit is the Wheatstone Bridge, shown in its simplest form in Figure 8. The output voltage is found simply from the input voltage as just the difference between two voltage dividers:

$$v_o = v_s \left(\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right)$$

This circuit is used in situations in which one or more resistors varies with, say temperature or mechanical strain. The bridge can be *balanced* so that the output voltage is zero by adjusting one of the other resistors. Then relatively small variations in the sensing element can result in relatively big differences in the output voltage. If, for example R_2 is the sensing element, R_4 can be adjusted to balance the bridge.

5 Serial and Parallel Combinations

There are a number of techniques for handling network problems, and we will not be able to investigate each of them in depth. We will, however, look into a few techniques for analysis which involve progressive simplification of the network. To start, we consider how one might handle series and parallel combinations of elements. A pair of elements is in *series* if the same current flows through both of them. If these elements are resistors and if the detail of voltage division between them is not required, it is possible to lump the two together as a single resistance. This is illustrated in Figure 9. The voltage across the current source is:

$$v_s = v_1 + v_2 = i_s R_1 + i_s R_2 = i_s (R_1 + R_2)$$

The equivalent resistance for the series combination is then:

$$R_{series} = R_1 + R_2 \tag{6}$$

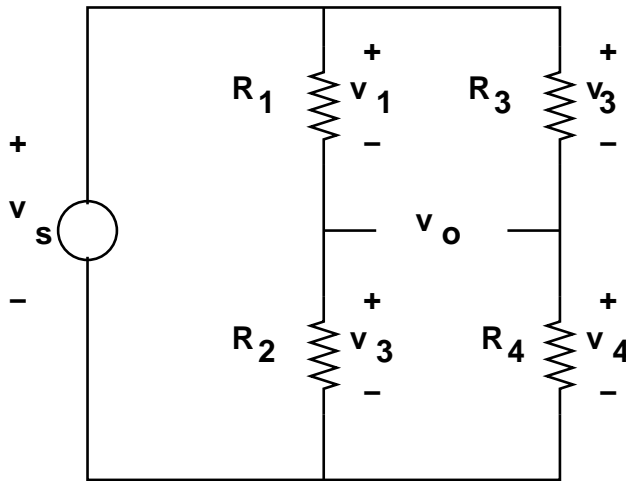


Figure 8: Wheatstone Bridge

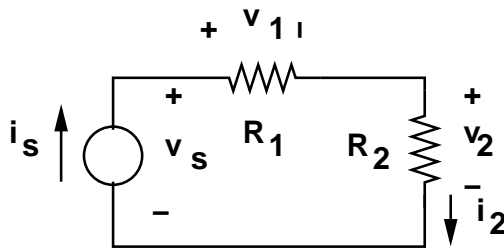


Figure 9: Series Resistance Combination

Similarly, resistance elements connected in *parallel* can be lumped if it is not necessary to know the details of division of current between them. Figure 10 shows this combination.

Here, current i is simply:

$$i = \frac{v}{R_1} + \frac{v}{R_2} = v \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

The equivalent resistance for the parallel combination is then:

$$R_{par} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2} \quad (7)$$

Because of the importance of parallel connection of resistances (and of other impedances), a special symbolic form is used for parallel construction. This is:

$$R_1 || R_2 = \frac{R_1 R_2}{R_1 + R_2} \quad (8)$$

As an example, consider the circuit shown in Figure 11, part (a). Here, we have four resistors arranged in an odd way to form a two-terminal network. To find the equivalent resistance of this thing, we can do a series of series-parallel combinations.

The two resistors on the right can be combined as a series combination to form a single, two ohm resistor as shown in part (b). Then the equivalent resistor, which is in parallel with one of the

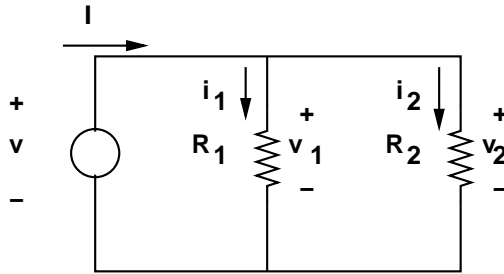


Figure 10: Parallel Resistance Combination

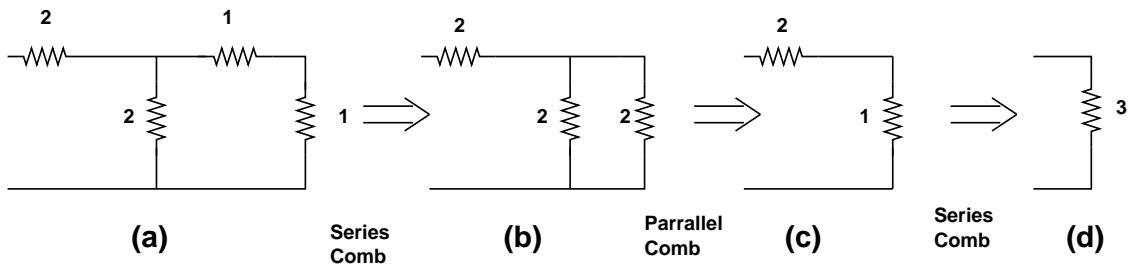


Figure 11: Series-Parallel Reduction

two ohm resistors can be combined to form a single combination part(c). That is in series with the remaining resistor, leaving us with an equivalent input resistance of $R = 3\Omega$.

6 Loop and Node Equations

There are two well- developed formal ways of solving for the potentials and currents in networks, often referred to as *loop* and *node* equation methods. They are closely related, using KCL and KVL together with element constraints to build sets of equations which may then be solved for potentials and currents.

- In the *node equation method*, KCL is written at each node of the network, with currents expressed in terms of the node potentials. KVL is satisfied because the node potentials are unique.
- In the *loop equation method*, KVL is written about a collection of closed paths in the network. “Loop currents” are defined, and made to satisfy KCL, and the branch voltages are expressed in terms of them.

The two methods are equivalent and a choice between them is usually a matter of personal preference. The node equation method is probably more widely used, and lends itself well to computer analysis.

To illustrate how these methods work, consider the network of Figure 12.

This network has three nodes. We are going to write KCL for each of the nodes, but note that only two explicit equations are required. If KCL is satisfied at two of the nodes, it is automatically satisfied at the third. Usually the *datum* node is the one for which we do not write the expression.

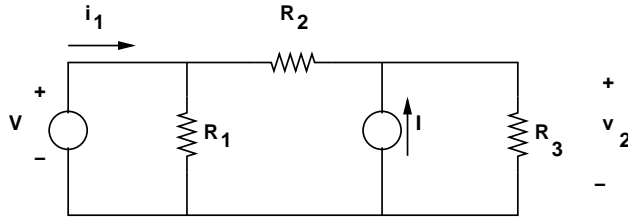


Figure 12: Sample Network

KCL written for the two upper nodes of the network is:

$$-i_1 + \frac{V}{R_1} + \frac{V - v_2}{R_2} = 0 \quad (9)$$

$$-I + \frac{v_2 - V}{R_2} + \frac{v_2}{R_3} = 0 \quad (10)$$

These two expressions are easily solved for the two unknowns, i_1 and v_2 :

$$v_2 = \frac{R_3}{R_2 + R_3}V + \frac{R_2 R_3}{R_2 + R_3}I$$

$$i_1 = \frac{R_1 + R_2 + R_3}{R_1(R_2 + R_3)}V - \frac{R_3}{R_2 + R_3}I$$

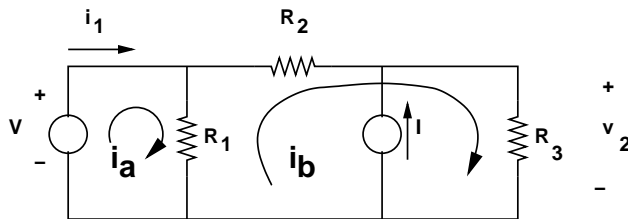


Figure 13: Sample Network Showing Loops

The *loop equation* method is similar. We need the same number of independent expressions (two), so we need to take two independent loops. For this, take as the loops as is shown in Figure 13:

1. The loop that includes the voltage source and R_1 .
2. The loop that includes R_1 , R_2 , and R_3 .

It is also necessary to define *loop currents*, which we will denote as i_a and i_b . These are the currents circulating around the two loops. Note that where the loops intersect, the actual *branch* current will be the sum of or difference between loop currents. For this example, assume the loop currents are defined to be circulating counter-clockwise in the two loops. The two loop equations are:

$$V + R_1(i_a - i_b) = 0 \quad (11)$$

$$R_1(i_b - i_a) + R_2 i_b + R_3(i_b - I) = 0 \quad (12)$$

These are equally easily solved for the two unknowns, in this case the two loop currents i_a and i_b .

7 Linearity and Superposition

An extraordinarily powerful notion of network theory is *linearity*. This property has two essential elements, stated as follows:

1. For any *single* input x yielding output y , the response to an input kx is ky for *any* value of k .
2. If, in a multi-input network the input x_1 by itself yields output y_1 and a second input x_2 by itself yields y_2 , then the combination of inputs x_1 and x_2 yields the output $y = y_1 + y_2$.

This is important to us at the present moment for two reasons:

1. It tells us that the solution to certain problems involving networks with multiple inputs is actually easier than we might expect: if a network is linear, we may solve for the output with each separate input, then add the outputs. This is called *superposition*.
2. It also tells us that, for networks that are linear, it is not necessary to actually consider the *value* of the inputs in calculating response. What is important is a *system function*, or a ratio of output to input.

Superposition is an important principle when dealing with linear networks, and can be used to make analysis easier. If a network has multiple independent sources, it is possible to find the response to each source separately, then add up all of the responses to find total response. Note that this can only be done with *independent* sources!

Consider, for example, the example circuit shown in Figure 12. If we are only interested in the output voltage v_2 , we may find the response to the voltage source first, then the response to the current source, then the total response is the sum of the two. To find the response to the voltage source, we must “turn off” the current source. This is done by assuming that it is not there. (After all, a current source with zero current is just an open circuit!). The resulting network is as in Figure 14.

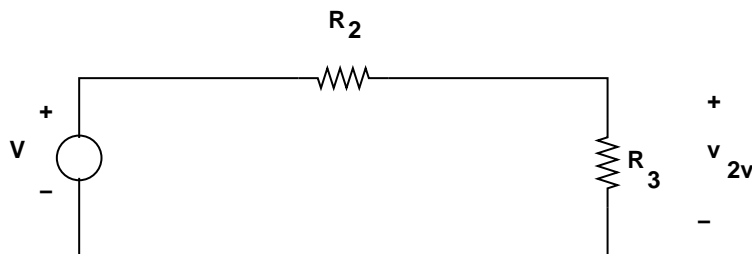


Figure 14: Superposition Fragment: Voltage Source

Note that the resistance R_1 does not appear here. This is because a resistance in parallel with a voltage source is just a voltage source, unless one is interested in current in the resistance. The output voltage is just:

$$v_{2v} = V \frac{R_3}{R_2 + R_3}$$

Next, we “turn off” the voltage source and “turn on” the current source. Note that a voltage source that has been turned off is a *short* circuit, because that implies zero voltage. The network is as shown in Figure 15

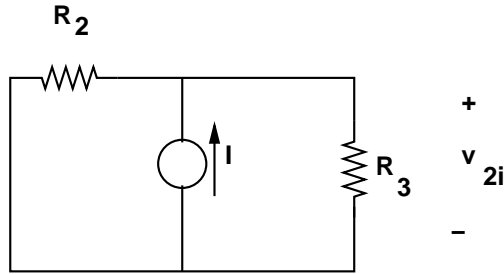


Figure 15: Superposition Fragment: Current Source

The response to this is:

$$v_{2i} = IR_2 || R_3$$

The total response is then just:

$$v_2 = v_{2v} + v_{2i} = V \frac{R_3}{R_2 + R_3} + I \frac{R_2 R_3}{R_2 + R_3}$$

8 Thevenin and Norton Equivalent Circuits

A particularly important ramification of the property of linearity is expressed in the notion of *equivalent circuits*. To wit: if we are considering the response of a network at any given *terminal pair*, that is a pair of nodes that have been brought out to the outside world, it follows from the properties of linearity that, if the network is linear, the output at a single terminal pair (either voltage or current) is the sum of two components:

1. The response that would exist if the *excitation* at the terminal pair were zero and
2. The response forced at the terminal pair by the exciting voltage or current.

This notion may be expressed with either *voltage* or *current* as the response. These yield the *Thevenin* and *Norton* equivalent networks, which are exactly equivalent. At *any* terminal pair, the properties of a *linear* network may be expressed in terms of either Thevenin or Norton equivalents. The *Thevenin* equivalent circuit is shown in Figure 16, while the *Norton* equivalent circuit is shown in Figure 17.

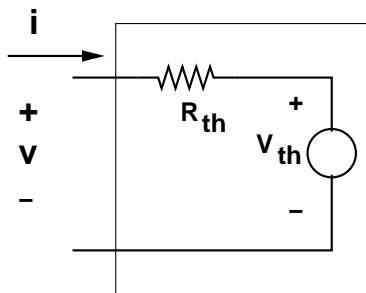


Figure 16: Thevenin Equivalent Network

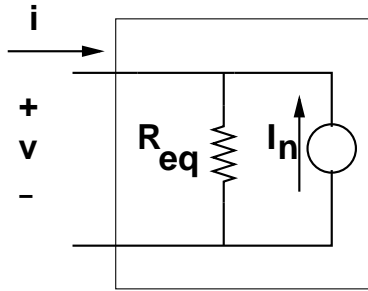


Figure 17: Norton Equivalent Network

The Thevenin and Norton equivalent networks have the *same* impedance. Further, the equivalent sources are related by the simple relationship:

$$V_{Th} = R_{eq}I_N \quad (13)$$

The *Thevenin Equivalent Voltage*, the source internal to the Thevenin equivalent network, is the same as the *open circuit* voltage, which is the voltage that would appear at the terminals of the equivalent circuit were it to be open circuited. Similarly, the *Norton Equivalent Current* is the same as minus the *short circuit* current.

To consider how we might use these equivalent networks, consider what would happen if the Wheatstone bridge were connected by some resistance across its output, as shown in Figure 18

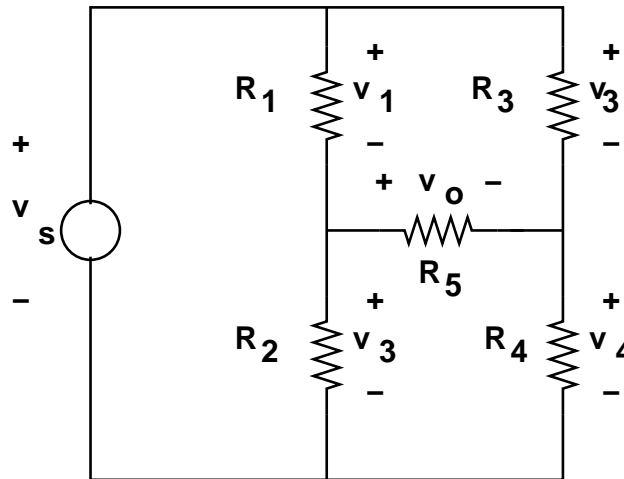


Figure 18: Wheatstone Bridge With Output Resistance

The analysis of this situation is simplified substantially if one recognizes that *each* side of the bridge can be expressed as either a Thevenin or Norton equivalent network. We may proceed to solve the problem by finding the equivalent networks for each side, then paste them together to form the whole solution. So: consider the equivalent network for the left-hand side of the network, formed by the elements V , R_1 and R_2 . This is shown in Figure 19.

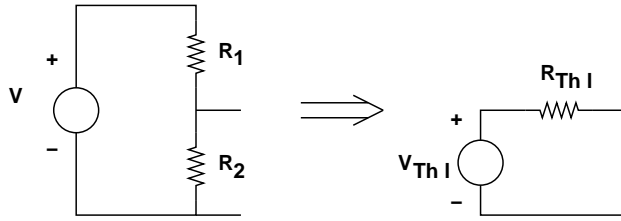


Figure 19: Construction of Equivalent Circuit

Where, here, the components of the equivalent circuit are:

$$v_{Thl} = V \frac{R_2}{R_1 + R_2}$$

$$R_{eql} = R_1 || R_2$$

Similarly, the *right* side of the network is found to have an equivalent source and resistance:

$$v_{Thr} = V \frac{R_4}{R_3 + R_4}$$

$$R_{eqr} = R_3 || R_4$$

And the whole thing behaves as the equivalent circuit shown in Figure 20

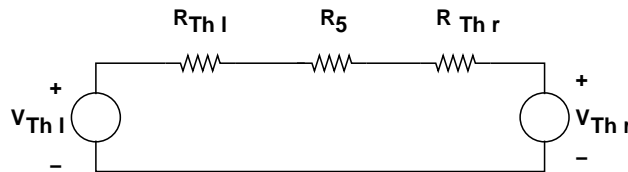


Figure 20: Equivalent Circuit

This is, of course, easily solved for the current through, and hence the voltage across, the resistance R_5 , which was desired in the first place:

$$v_5 = (v_{Thl} - v_{Thr}) \frac{R_5}{R_5 + r_{eql} + r_{eqr}} = V \left(\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right) \frac{R_5}{R_5 + R_1 || R_2 + R_3 || R_4}$$

9 Two Port Networks

So far, we have dealt with a number of networks which may be said to be *one port* or single-terminal-pair circuits. That is, the important action occurs at a single terminal pair, and is characterized by an *impedance* and by either a *open circuit voltage* or a *short circuit current*, thus forming either a Thevenin or Norton equivalent circuit. A second, and for us very important, class of electrical network has two (or sometimes more) terminal pairs. We will consider formally here the *two port* network, illustrated schematically in Figure 21.

There are a number of ways of characterizing this type of network. For the time being, consider that it is *passive*, so that there is no output without some input and there are no dependent sources.

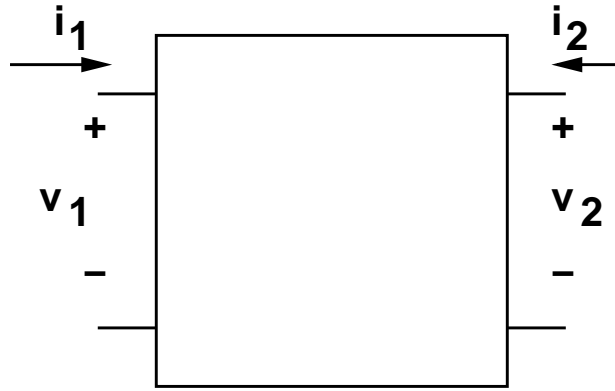


Figure 21: Two-Port Network

Then we may characterize the network in terms of the *currents* at its terminals in terms of the *voltages*, or, conversely, we may describe the *voltages* in terms of the *currents* at the terminals. These two ways of describing the network are said to be the *admittance* or *impedance* parameters. These may be written in the following way:

The *impedance parameter* point of view would yield, for a resistive network, the following relationship between voltages and currents:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad (14)$$

Similarly, the *admittance parameter* point of view would yield a similar relationship:

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (15)$$

These two relationships are, of course, the inverses of each other. That is:

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}^{-1} \quad (16)$$

If the networks are *linear* and *passive* (i.e. there are no *dependent* sources inside), they also exhibit the property of *reciprocity*. In a reciprocal network, the *transfer impedance* or *transfer admittance* is the same in both directions. That is:

$$\begin{aligned} R_{12} &= R_{21} \\ G_{12} &= G_{21} \end{aligned} \quad (17)$$

It is often useful to express two-port networks in terms of \mathbf{T} or $\mathbf{\Pi}$ networks, shown in Figures 22 and 23.

Sometimes it is useful to cascade two-port networks, as is shown in Figure 24. The resulting combination is itself a two-port. Suppose we have a pair of networks characterized by impedance parameters:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

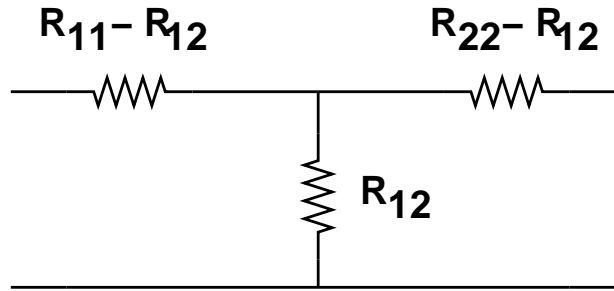


Figure 22: T- Equivalent Network

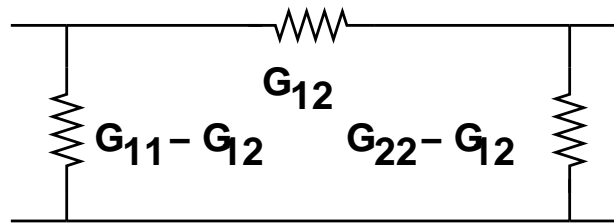


Figure 23: Pi-Equivalent Network

$$\begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} R_{33} & R_{34} \\ R_{34} & R_{44} \end{bmatrix} \begin{bmatrix} i_3 \\ i_4 \end{bmatrix}$$

By noting that $v_2 = v_3$ and $i_3 = -i_2$, it is possible to show, with a little manipulation, that:

$$\begin{bmatrix} v_1 \\ v_4 \end{bmatrix} = \begin{bmatrix} R'_{11} & R_{14} \\ R_{14} & R'_{44} \end{bmatrix} \begin{bmatrix} i_1 \\ i_4 \end{bmatrix}$$

where

$$R'_{11} = R_{11} - \frac{R_{12}^2}{R_{22} + R_{33}}$$

$$R'_{44} = R_{44} - \frac{R_{34}^2}{R_{22} + R_{33}}$$

$$R_{14} = \frac{R_{12}R_{34}}{R_{22} + R_{33}}$$

10 Inductive and Capacitive Circuit Elements

So far, we have dealt with circuit elements which have no memory and which, therefore, are characterized by instantaneous behavior. The expressions which are used to calculate what these elements are doing are algebraic (and for most elements are linear too). As it turns out, much of the circuitry we will be studying can be so characterized, with complex parameters.

However, we take a quick diversion to discuss briefly the transient behavior of circuits containing capacitors and inductors.

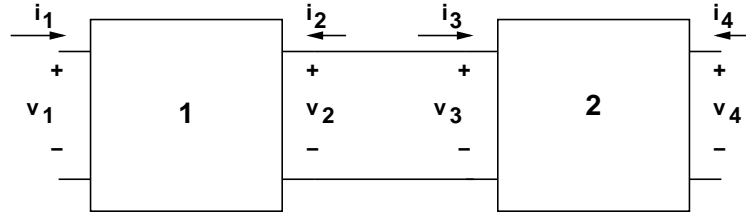


Figure 24: Cascade of Two-Port Networks

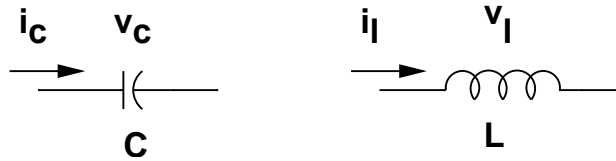


Figure 25: Capacitance and Inductance

Symbols for capacitive and inductive circuit elements are shown in Figure 25. They are characterized by the relationships between voltage and current:

$$i_c = C \frac{dv_c}{dt} \qquad v_\ell = L \frac{di_\ell}{dt} \qquad (18)$$

Note that, while these elements are *linear*, since time derivatives are involved in their characterization, expressions describing their behavior in networks will become ordinary differential equations.

10.1 Simple Case: R-C

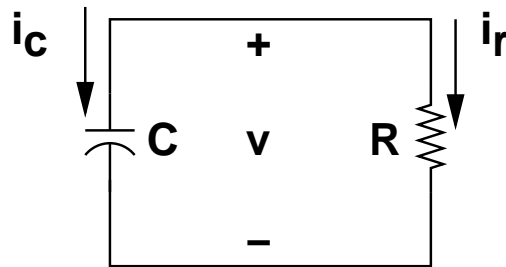


Figure 26: Simple Case: R-C

Figure 26 shows a simple connection of a resistance and a capacitance. This circuit has only two nodes, so there is a single voltage v across both elements. The two elements produce the constraints:

$$\begin{aligned} i_r &= \frac{v}{R} \\ i_c &= \frac{dv}{dt} \end{aligned}$$

and, since $i_r = -i_c$,

$$\frac{dv}{dt} + \frac{1}{RC}v = 0$$

Now, we know that this sort of first-order, linear equation is solved by:

$$v \sim e^{-\frac{t}{RC}}$$

(To confirm this, just substitute the exponential into the differential equation.) Then, if we have some *initial condition*, say $v(t=0) = V_0$, then

$$v = V_0 e^{-\frac{t}{RC}}$$

This was a trivial case, since we don't describe *how* that initial condition might have taken place. But consider a closely related problem, illustrated in Figure 27.

10.2 Simple Case with Drive

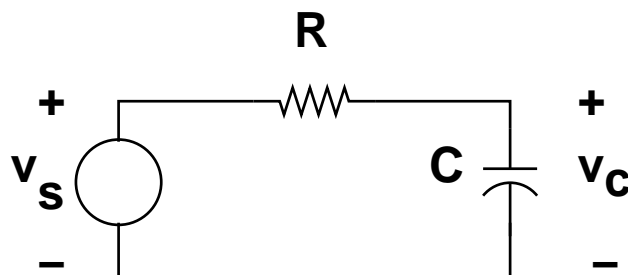


Figure 27: RC Circuit with Drive

The analysis of this circuit is accomplished by noting that it contains a single loop, and adding up the voltages around the loop we find:

$$RC \frac{dv_c}{dt} + v_c = v_s$$

Now, assume that the voltage source is a *step*:

$$v_s = V_s u_{-1}(t)$$

We should define the step function with some care, since it is of quite a lot of use. The step is one of a hierarchy of *singularity functions*. It is defined as:

$$u_{-1}(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad (19)$$

Now, remembering that differential equations have *particular* and *homogeneous* solutions, and that for $t > 0$ a particular solution which solves the differential equation is:

$$v_{cp} = V$$

Of course this does not satisfy the initial condition which is that the capacitance be uncharged: $v_c(t = 0+) = 0$. Again, remember that the whole solution is the sum of the particular and a homogeneous solution, and that the homogeneous solution is the un-driven case. To satisfy the initial condition, the homogeneous solution must be:

$$c_{ch} = -V e^{-\frac{t}{RC}}$$

So that the total solution is simply:

$$v_c = V \left(1 - e^{-\frac{t}{RC}} \right)$$

Next, suppose $v_s = u_{-1}(t)V \cos \omega t$. We know the homogeneous solution must be of the same form, but the particular solution is a bit more complicated. In later chapters we will learn how to make the process of extracting the particular solution easier, but for the time being, let's assume that with a sinusoidal drive we will get a sinusoidal response of the same frequency. Thus we will guess

$$v_{cp} = V_{cp} \cos(\omega t - \phi)$$

The time derivative is

$$\frac{dv_{cp}}{dt} = \omega V_{cp} \sin(\omega t - \phi)$$

so that we can find an algebraic equation for the particular solution:

$$V \cos \omega t = V_{cp} (\cos(\omega t - \phi) + \omega RC \sin(\omega t - \phi))$$

Note the trigonometric identities:

$$\begin{aligned} \cos(\omega t - \phi) &= \cos \phi \cos \omega t + \sin \phi \sin \omega t \\ \sin(\omega t - \phi) &= -\sin \phi \cos \omega t + \cos \phi \sin \omega t \end{aligned}$$

Since the sine and cosine terms are orthogonal, we can equate coefficients of sine and cosine to get:

$$\begin{aligned} V &= V_{cp} [\cos \phi + \omega RC \sin \phi] \\ 0 &= V_{cp} [\sin \phi + \omega RC \cos \phi] \end{aligned}$$

The second of these can be solved for the phase angle:

$$\phi = \tan^{-1} \omega RC$$

and squaring both equations and adding:

$$V^2 = V_{cp}^2 (1 + (\omega RC)^2)$$

so that the *particular* solution is:

$$v_{cp} = \frac{V}{\sqrt{1 + (\omega RC)^2}} \cos(\omega t - \phi)$$

Finally, if the capacitor is initially uncharged ($v_c(t = 0+) = 0$), we can add in the homogeneous solution (we already know the form of this), and find the total solution to be:

$$v_{cp} = \frac{V}{\sqrt{1 + (\omega RC)^2}} \left[\cos(\omega t - \phi) - \cos \phi e^{-\frac{t}{RC}} \right]$$

This is shown in Figure 28

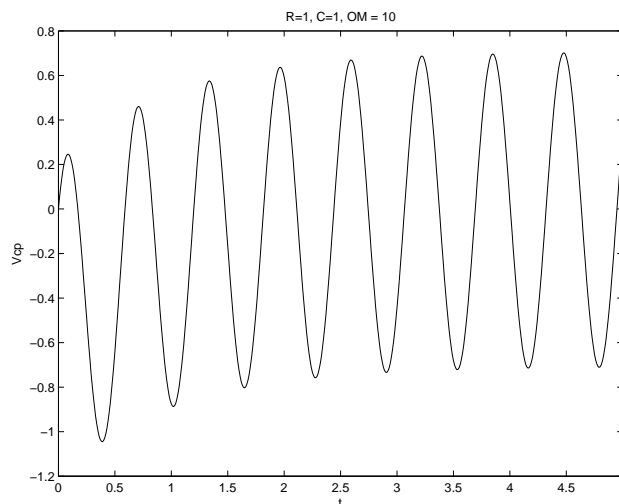


Figure 28: Output Voltage for RC Example

10.3 Second-Order System Example

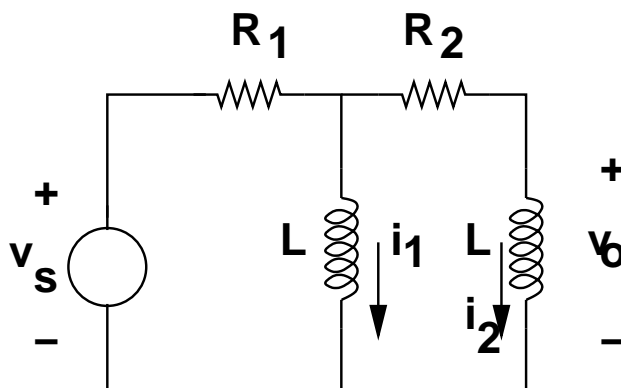


Figure 29: Two-Inductor Circuit

Figure 29 shows a network with two inductances and two resistances. Assume that this is driven by a voltage step: $v_s = V_s u_{-1}(t)$. Note that, with two inductances, we will require two initial conditions to complete the solution.

The steady state (particular) solution is $v_o = 0$. There will, of course, be current flowing in each of the inductances, but if excitation is constant there will be no time derivative of current so that voltage across each of the inductances will eventually fall to zero.

The initial conditions may be found by inspection. Right *after* $t = 0$ (note we use $t = 0+$ for this), output voltage must be:

$$v_o(t = 0+) = V_s$$

This must be so since current cannot be made to flow instantaneously in either inductance, so that there is no current in either resistance.

The second initial condition is the rate of change of voltage right after the instant of the voltage step. To find this, note that output voltage is equal to the source voltage minus the voltage drops across each of the two resistances.

$$v_o = v_s - R_2 i_2 - R_1 (i_1 + i_2)$$

If we differentiate this with respect to time and note that the time derivative of a constant (after the step the input voltage is constant) is zero:

$$\frac{dv_o}{dt}(t = 0+) = -(R_1 + R_2) \frac{di_2}{dt} - R_1 \frac{di_1}{dt}$$

Now, since right after the instant of the step both inductances have the source voltage V_s across them:

$$\left. \frac{di_1}{dt} \right|_{t=0+} = \left. \frac{di_2}{dt} \right|_{t=0+} = \frac{V_s}{L}$$

the rate of change of voltage at $t = 0+$ is:

$$\left. \frac{dv_o}{dt} \right|_{t=0+} = -\frac{2R_1 + R_2}{L} V_s$$

Now, we can find the homogeneous solution using the loop method. Setting the source to zero, assume a current i_a in the left-hand loop and i_b in the right-hand loop. KVL around these two loops yields:

$$\begin{aligned} R_1 i_a + L \frac{d}{dt} (i_a - i_b) &= 0 \\ R_2 i_b + 2L \frac{di_b}{dt} - L \frac{di_a}{dt} &= 0 \end{aligned}$$

With a little manipulation, these become:

$$\begin{aligned} L \frac{di_a}{dt} + 2R_1 i_a + R_2 i_b &= 0 \\ L \frac{di_b}{dt} + R_1 i_a + R_2 i_b &= 0 \end{aligned}$$

Assume that solutions are of the form Ie^{st} , and this set of simultaneous equations becomes:

$$\begin{bmatrix} (sL + 2R_1) & R_2 \\ R_1 & (sL + R_2) \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We need to solve this for s (to find values of s for which this set is true, and that is simply the solution of the “characteristic equation”

$$(sL + 2R_1)(sL + R_2) - R_1 R_2 = 0$$

which is the same as:

$$s^2 + \frac{2R_1 + R_2}{L} s + \frac{R_1}{L} \frac{R_2}{L} = 0$$

Now, for the sake of “nice numbers”, assume that $R_1 = 2R$, $R_2 = 3R$. The characteristic equation is:

$$s^2 + 7\frac{R}{L}s + 6\left(\frac{R}{L}\right)^2 = 0$$

which factors nicely into $(s + \frac{R}{L})(s + 6\frac{R}{L}) = 0$, or the two values of s are $s = -\frac{R}{L}$ and $s = -6\frac{R}{L}$.

Since the particular solution to this one is zero, we have a total solution which is:

$$v_o = Ae^{-\frac{R}{L}t} + Be^{-6\frac{R}{L}t}$$

The initial conditions are:

$$\begin{aligned} v_o|_{t=0+} &= A + B = V_s \\ \frac{dv_o}{dt}|_{t=0+} &= -\frac{R}{L}(A + 6B) = -7\frac{R}{L}V_s \end{aligned}$$

The solution to that pair of expressions is:

$$A = -\frac{V_s}{5} \qquad B = \frac{6V_s}{5}$$

and this is shown in Figure 30.

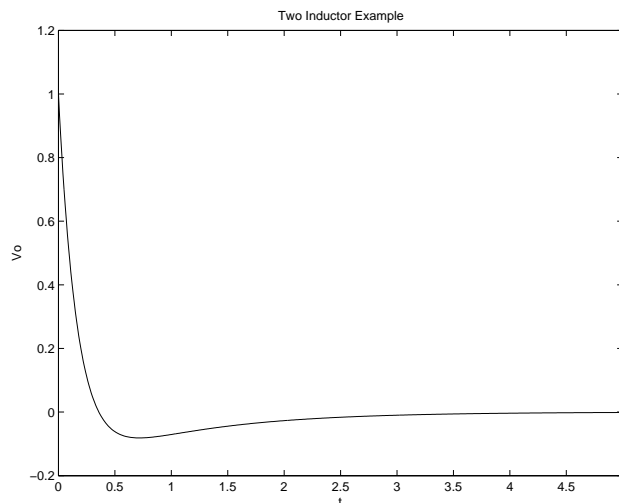


Figure 30: Output Voltage for Two Inductor Example