6.061 / 6.690 Introduction to Electric Power Systems
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Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.061 Introduction to Power Systems
Class Notes Chapter 2
AC Power Flow in Linear Networks *

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1 Introduction

Electric power systems usually involve sinusoidally varying (or nearly so) voltages and currents. That is, voltage and current are functions of time that are nearly pure sine waves at fixed frequency. In North America, most ships at sea and eastern Japan that frequency is 60 Hz. In most of the rest of the world it is 50 Hz. Normal power system operation is at this fixed frequency, which is why we study how systems operate in this mode. We will deal with transients later.

This note deals with alternating voltages and currents and with associated energy flows. The focus is on sinusoidal steady state conditions, in which virtually all quantities of interest may be represented by single, complex numbers.

Accordingly, this section opens with a review of complex numbers and with representation of voltage and current as complex amplitudes with complex exponential time dependence. The discussion proceeds, through impedance, to describe a pictorial representation of complex amplitudes, called *phasors*. Power is then defined and, in sinusoidal steady state, reduced to complex form. Finally, flow of power through impedances and a conservation law are discussed.

Secondarily, this section of the notes deals with transmission lines that have interesting behavior, both in the time and frequency domails.

2 Complex Exponential Notation

Start by recognizing a geometric interpretation for a complex number. If we plot the real part on the horizontal (x) axis and the imaginary part on the vertical (y) axis, then the complex number $\underline{z} = x + jy$ (where $j = \sqrt{-1}$) represents a vector as shown in Figure 1. Note that this vector may be represented not only by its real and imaginary components, but also by a magnitude and a *phase*

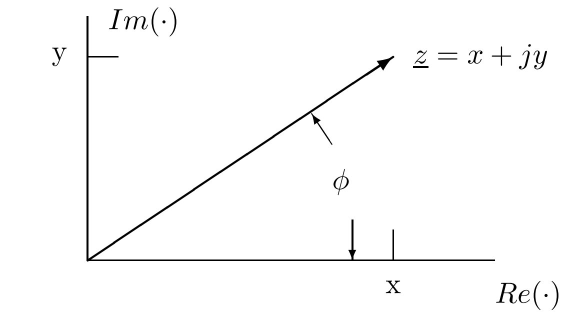

Figure 1: Representation of the complex number $\underline{z} = x + jy$

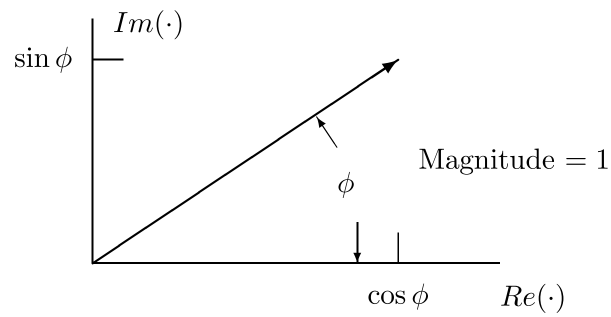

Figure 2: Representation of $e^{j\phi}$

angle:

$$|\underline{z}| = \sqrt{x^2 + y^2} \quad (1)$$

$$\phi = \arctan(\frac{y}{x}) \quad (2)$$

The basis for complex exponential notation is the celebrated *Euler Relation*:

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \quad (3)$$

which has a representation as shown in Figure 2.

Now, a comparison of Figures 1 and 2 makes it clear that, with definitions (1) and (2),

$$\underline{z} = x + jy = |\underline{z}|e^{j\phi} \quad (4)$$

It is straightforward, using (3) to show that:

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2} \quad (5)$$

$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j} \quad (6)$$

$$(7)$$

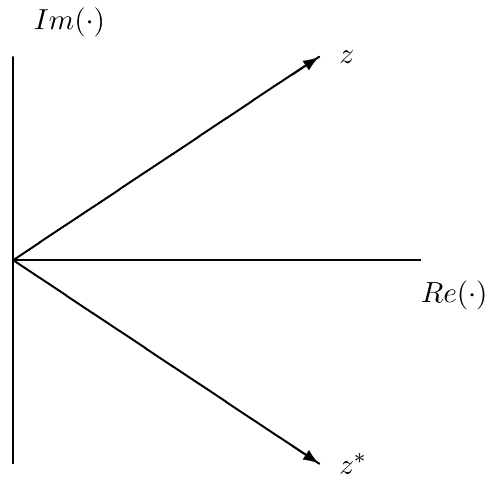

Figure 3: Representation Of A Complex Number And Its Conjugate

The complex exponential is a tremendously useful type of function. Note that the *product* of two numbers expressed as exponenentials is the same as the exponential of the *sums* of the two exponents:

$$e^a e^b = e^{a+b} \tag{8}$$

Note that it is also true that the *reciprocal* of a number in exponential notation is just the exponential of the *negative* of the exponent:

$$\frac{1}{e^a} = e^{-a} \tag{9}$$

Then, if we have two numbers $\underline{z}_1 = |\underline{z}_1| e^{j\phi_1}$ and $\underline{z}_2 = |\underline{z}_2| e^{j\phi_2}$, then the *product* of the two numbers is:

$$\underline{z}_1 \underline{z}_2 = |\underline{z}_1||\underline{z}_2| e^{j(\phi_1+\phi_2)} \tag{10}$$

and the *ratio* of the two numbers is:

$$\frac{\underline{z}_1}{\underline{z}_2} = \frac{|\underline{z}_1|}{|\underline{z}_2|} e^{j(\phi_1-\phi_2)} \tag{11}$$

The complex conjugate of a number $\underline{z} = x + jy$ is given by:

$$z^* = x - jy \tag{12}$$

The *sum* of a complex number and its conjugate is real:

$$\underline{z} + \underline{z}^* = 2Re(\underline{z}) = 2x \tag{13}$$

while the *difference* is imaginary:

$$\underline{z} - \underline{z}^* = 2jIm(\underline{z}) = 2jy \tag{14}$$

where we have used the two symbols $Re(\cdot)$ and $Im(\cdot)$ to represent the operators which extract the *real* and *imaginary* parts of the complex number.

The complex conjugate of a complex number $\underline{z} = |\underline{z}| e^{j\phi}$ may *also* be written as:

$$\underline{z}^* = |\underline{z}| e^{-j\phi} \tag{15}$$

so that the *product* of a complex number and its conjugate is *real*:

$$\underline{z}\underline{z}^* = |\underline{z}| e^{j\phi} |\underline{z}| e^{-j\phi} = |\underline{z}|^2 \tag{16}$$

3 Sinusoidal Time Functions

A sinusoidal function of time might be written in at least two ways:

$$f(t) = A\cos(\omega t + \phi) \tag{17}$$
$$f(t) = B\cos(\omega t) + C\sin(\omega t) \tag{18}$$

A third way of writing this time function is as the sum of two complex exponentials:

$$f(t) = \underline{X}e^{j\omega t} + \underline{X}^* e^{-j\omega t} \tag{19}$$

Note that the *form* of equation 19, in which complex conjugates are added together, guarantees that the resulting function is *real*.

Now, to relate equation 19 with the other forms of the sinusoidal function, equations 17 and 18, see that $\underline{X}$ may be expressed as:

$$\underline{X} = |\underline{X}|e^{j\psi} \tag{20}$$

Then equation 19 becomes:

$$\begin{aligned} f(t) &= |\underline{X}|e^{j\psi}e^{j\omega t} + |\underline{X}|^* e^{-j\psi}e^{-j\omega t} &(21)\\ &= |\underline{X}|e^{j(\psi+\omega t)} + |\underline{X}|^* e^{-j(\psi+\omega t)} &(22)\\ &= 2|\underline{X}|\cos(\omega t + \psi) &(23)\end{aligned}$$

Then, the coefficients in equation 17 are related to those of equation 19 by:

$$|\underline{X}| = \frac{A}{2} \tag{24}$$
$$\psi = \phi \tag{25}$$

Alternatively, we could write

$$\underline{X} = x + jy \tag{26}$$

in which the *real* and *imaginary* parts of $\underline{X}$ are:

$$x = |\underline{X}|\cos(\psi) \tag{27}$$
$$y = |\underline{X}|\sin(\psi) \tag{28}$$

Then the time function is written:

$$\begin{aligned} f(t) &= x(e^{j\omega t} + e^{-j\omega t}) + jy(e^{j\omega t} - e^{-j\omega t}) &(29)\\ &= 2x\cos(\omega t) - 2y\sin(\omega t) &(30)\end{aligned}$$

Thus:

$$A = 2x \tag{31}$$
$$B = -2y \tag{32}$$
$$X = \frac{A}{2} - j\frac{B}{2} \tag{33}$$

It is also possible to write equation 19 in the form:

$$f(t) = Re(2\underline{X}e^{j\omega t}) \tag{34}$$

While both expressions (19 and 34) are equivalent, it is advantageous to use one or the other of them, according to circumstances. The first notation (equation 19) is the full representation of that sinusoidal signal and may be used under any circumstances. It is, however, cumbersome, so that the somewhat more compact version(equation 34) is usually used. Chiefly when nonlinear products such as power are involved, it is necessary to be somewhat cautions in its use, however, as we will see later on.

4 Impedance

Because it is so easy to differentiate a complex exponential time signal, such a way of representing time signals has real advantages in electric circuits with all kinds of linear elements. In Section 1 of these notes, we introduced the linear *resistance* element, in which voltage and current are linearly related. We must now consider two other elements, inductances and capacitances. The *inductance*

Figure 4: Inductance and Capacitance Elements

produces a relationship between voltage and current which is:

$$v_L = L\frac{di_L}{dt} \tag{35}$$

If voltage and current are sinusoidal functions of time:

$$\begin{aligned} v &= \underline{V}e^{j\omega t} + \underline{V}^* e^{-j\omega t} \\ i &= \underline{I}e^{j\omega t} + \underline{I}^* e^{-j\omega t} \end{aligned}$$

Then the relationship between voltage and current is given simply by:

$$\underline{V} = j\omega L \underline{I} \tag{36}$$

This is a particularly simple form, and as can be seen is directly analogous to *resistance*. We can generalize our view of resistance to *complex impedance* (or simply impedance), in which inductances have impedance which is:

$$\underline{Z}_L = j\omega L \tag{37}$$

The *capacitance* element is similarly defined. A capacitance has a voltage-current relationship:

$$i = C\frac{dv_C}{dt} \tag{38}$$

Thus the *impedance* of a capacitance is:

$$\underline{Z}_C = \frac{1}{j\omega C} \tag{39}$$

The extension to resistive network behavior is now obvious. For problems in *sinusoidal steady state*, in which all excitations are sinusoidal, we may use all of the tricks of linear, resistive network analysis. However, we use *complex* impedance in place of resistance.

The inverse of *impedance* is *admittance*:

$$\underline{Y} = \frac{1}{\underline{Z}}$$

Series and parallel combinations of admittances and impedances are, of course, just like those of conductances and resistances. For two elements in series or in parallel:
Series:

$$\underline{Z} = \underline{Z}_1 + \underline{Z}_2 \tag{40}$$

$$\underline{Y} = \frac{\underline{Y}_1 \underline{Y}_2}{\underline{Y}_1 + \underline{Y}_2} \tag{41}$$

Parallel:

$$\underline{Z} = \frac{\underline{Z}_1 \underline{Z}_2}{\underline{Z}_1 + \underline{Z}_2} \tag{42}$$

$$\underline{Y} = \underline{Y}_1 + \underline{Y}_2 \tag{43}$$

4.1 Example

Suppose we are to find the voltage $v(t)$ in the network of Figure 5, in which $i(t) = I\cos(\omega t)$. The

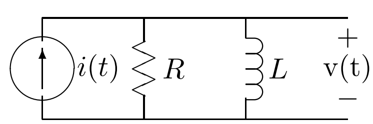

Figure 5: Complex Impedance Network

excitation may be written as:

$$i(t) = \frac{I}{2}e^{j\omega t} + \frac{I}{2}e^{-j\omega t} = Re\left(Ie^{j\omega t}\right)$$

Now, the *complex impedance* of the parallel combination of R and L is:

$$R||j\omega L = \frac{Rj\omega L}{R + j\omega L}$$

So that, if $v(t)$ is represented by:

$$\begin{aligned} v(t) &= \frac{\underline{V}}{2} e^{j\omega t} + \frac{\underline{V}}{2} e^{-j\omega t} \\ &= Re\left(\underline{V} e^{j\omega t}\right) \end{aligned}$$

Then

$$\underline{V} = \frac{Rj\omega L}{R + j\omega L} I$$

Now: the impedance $\underline{Z}$ may be represented by a magnitude and phase angle:

$$\begin{aligned} \underline{Z} &= |\underline{Z}| e^{j\phi} \\ |\underline{Z}| &= \frac{\omega L R}{\sqrt{(\omega L)^2 + R^2}} \\ \phi &= \frac{\pi}{2} - \arctan \frac{\omega L}{R} \end{aligned}$$

Then, using relations developed here, $v(t)$ may be written as:

$$v(t) = \frac{\omega L I}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} \cos(\omega t + \phi)$$

Note that this expression represents only the *sinusoidal steady state* solution, and therefore does not represent any starting transients.

5 System Functions and Frequency Response

If we are interested in the behavior of a linear system such as the circuits we have been discussing, we often speak of the *system function*. This is the (usually complex) ratio between *output* and *input* of the system. System functions can express *driving point* behavior (impedance or its reciprocal, admittance) or *transfer* behavior. We speak of voltage or current transfer ratios and of transfer impedance (output voltage related to input current) and transfer admittance (output current related to input voltage).

The system function may be expressed in a number of ways, often as a Laplace Transform. Such is beyond the scope of this subject. However, it is important to understand one way of expressing linear system behavior, in the form of *frequency response*. The frequency response of a system is the complex number that relates output of the system to input as a function of frequency. Usually it is expressed as a pair of numbers, magnitude and phase angle. Thus

$$H(j\omega) = |H(j\omega)| e^{j\phi(j\omega)}$$

Subjects in Signals and Systems or Network Theory often spend some time on how to obtain and plot the frequency response of a network in ways which are both useful and easy. For our purposes, a straightforward, perhaps even "brute force" approach will do. Consider, for example, the circuit shown in Figure 6.

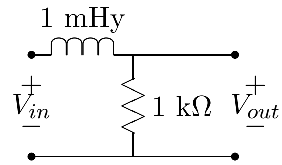

Figure 6: Example Circuit for Frequency Response

This is just a voltage divider between an inductance and a resistance. We seek to find, and then plot, the transfer ratio V_{out}/V_{in} of this network. A *very* little analysis yields an expression for the transfer function, which is:

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R}}$$

The magnitude and angle of this function can be extracted in a number of ways. For the purpose of these notes, we have done the mathematics using MATLAB. The specific instructions for producing the frequency response plot are shown in Figure 7. Funamentally what is done is to compute the system function for a number of frequencies (note that we use a way of computing specific frequencies which produces a uniform spacing on a logarithmic scale, and then plotting the magnitude (also on a logarithmic scale) and angle of that system function against frequency.

6 Phasors

Phasors are *not* weapons. They are a handy geometric trick which help us understand the nature of sinusoidal steady state signals and systems. To start, consider the basis for complex exponential time notation, the function $e^{j\omega t}$. At any instant of time, this is a complex number: at time $t = 0$ it is equal to 1, at time $\omega t = \frac{\pi}{2}$ it is equal to j, and so forth. We may describe this function as a *vector*, of length unity, rotating about the origin of the complex number plane, with angular velocity ω. It has, of course, both *real* and *imaginary* parts, which are just the projections of the vector onto the *real* and *imaginary* axes.

Now consider a sinusoidally varying signal $x(t)$, which may be represented by:

$$x(t) = \frac{\underline{X}}{2} e^{j\omega t} + \frac{\underline{X}^*}{2} e^{-j\omega t}$$

This is the sum of two numbers, complex conjugates, which are, as functions of time, rotating in *opposite directions* in the complex plane. The *sum* of the two is, of course, real. This is the same time function as:

$$x(t) = Re\left(\underline{X} e^{j\omega t}\right) \tag{44}$$

where the real part operator $Re(\cdot)$ simply takes the *projection* of the function on the real axis.

It might be helpful at this point to remember one of the features of complex arithmetic. Multiplication of two complex numbers results in a third complex number which has:

```
L=1e-3;                             % Set Parameter Values
R=1000;
e=3:.05:7;                          % This is a way of producing evenly
f=10 .^ e;                          % spaced points on a logarithmic chart
om=2*pi .* f;                       % Frequency in radians per second
H = 1 ./ (1 + j*L/R .* om);         % This is the frequency response
subplot(211);
loglog(f, abs(H))                   % Plot of magnitude
xlabel('Frequency, Hz');
ylabel('Magnitude');
grid
subplot(212);
semilogx(f, angle(H))               % Plot of angle
xlabel('Frequency, Hz')
ylabel('Angle')
grid
title('Frequency Response of L-R')
print('freq.ps')
```

Figure 7: MATLAB Program `freq.m`

1. a *magnitude* which is the *product* of the magnitudes of the two numbers begin multiplied and,
2. an *angle* which is the *sum* of the angles of the two numbers being multiplied.

Thus, multiplying a number by $e^{j\omega t}$, which has a *magnitude* of unity and an *angle* which is increasing with time at the rate ω, simply has the effect of setting that number spinning around the origin of the complex plane.

It is therefore relatively easy to represent sinusoidally varying signals with just their complex amplitudes, understanding that they also include $e^{j\omega t}$, which provides time variation. The *complex amplitude* includes not only the *magnitude* of the signal, but also a *phase angle*. Usually the phase angle by itself is of little use, and must be related to some time reference. That is, as we will see, it is the *difference* between phase angles that is important in most cases.

Impedances and admittances are also complex numbers, so that *phasors* can be used to visualize the relationship between voltages and currents in a network. The key here is that multiplication and division of complex numbers is the same as multiplication or division of *magnitudes* and addition or subtraction of *angles*.

6.1 Example

Consider the simple network shown in Figure 9, and suppose that the current source is sinusoidal:

$$i = Re\left(\underline{I}e^{j\omega t}\right)$$

The *impedance* of the R-L combination is a complex number:

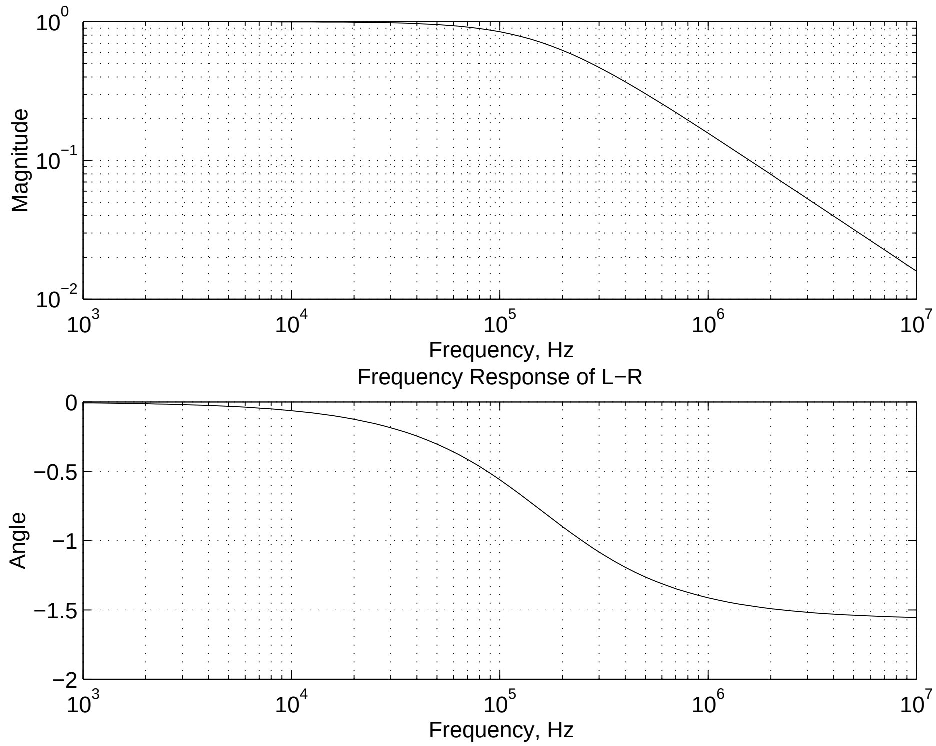

Figure 8: Frequency Response

$$\underline{Z} = R + j\omega L = 1 + j2$$

Now: the *impedance* may be represented in the complex plane as shown in Figure 10.

Voltage v is given by:

$$v = Re\left(\underline{V}e^{j\omega t}\right)$$

where:

$$\underline{V} = \underline{ZI}$$

Then the relationship between voltage and current is as shown in Figure 11. Note that the phase angle between voltage and current is the same as the phase angle of the impedance.

Note that KVL may be represented graphically in the fashion of Figure 12.

7 Energy and Power

For any terminal pair with voltage and current defined as shown in Figure 13, *power* flow *into* the element is:

$$p = vi \tag{45}$$

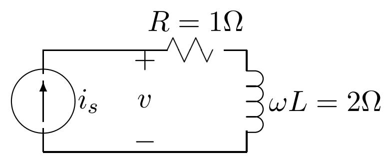

Figure 9: Example Circuit

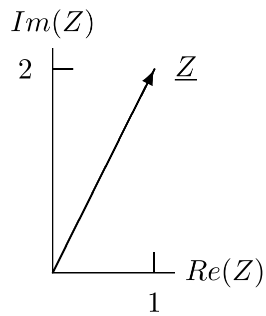

Figure 10: Complex Impedance

Power is expressed in *Watts* (**W**), and one *Watt* is the product of one *Volt* and one *Ampere*. Energy transferred over an interval of time t_0 to t_1 is the integral of power:

$$w = \int_{t_0}^{t_1} v(t)i(t)dt \tag{46}$$

Energy is expressed in *Joules*, and one *Joule* is one *Watt- Second*. A Joule is also a Newton-Meter (force times distance), and therefore a Watt is a Newton-Meter per Second.

Consider the behavior of the three types of linear, passive elements we have encountered:

- Resistance: $v = Ri$, Instantaneous power is:

$$p = Ri^2 = \frac{v^2}{R} \tag{47}$$

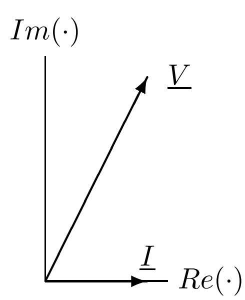

Figure 11: Voltage and Current

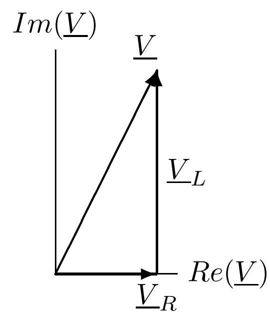

Figure 12: Components of Voltage

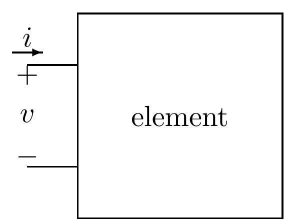

Figure 13: Definition for Power

- Inductance: $v = L\frac{di}{dt}$, Instantaneous power is:

$$p = iL\frac{di}{dt} = \frac{1}{2}L\frac{di^2}{dt} \tag{48}$$

  The quantity $w_L = \frac{1}{2}Li^2$ may be interpreted as energy stored in the inductance, so that $p = \frac{dw_L}{dt}$. We will need to refine this definition later, when we consider electromechanical interactions or nonlinear elements, but it will do for now.

- Capacitance: $i = C\frac{dv}{dt}$, Instantaneous power is:

$$p = vC\frac{dv}{dt} = \frac{1}{2}C\frac{dv^2}{dt} \tag{49}$$

  The quantity $w_C = \frac{1}{2}Cv^2$ may similarly be interpreted as energy stored in the capacitance.

Next, consider the power input to each of these three elements under sinusoidal steady state conditions:

- Resistance: if $i = I\cos(\omega t + \theta)$, then

$$\begin{aligned} p &= RI^2\cos^2(\omega t + \theta) \\ &= \frac{RI^2}{2}\left[1 + \cos 2(\omega t + \theta)\right] \end{aligned} \tag{50}$$

Thus, *average* power into the resistance is:

$$P = \frac{1}{2} R I^2 \tag{51}$$

- Inductance: if $i = I\cos(\omega t + \theta)$, then voltage is $v = -\omega L I \sin(\omega t + \theta)$, and power is:

$$\begin{aligned} p &= -\omega L I^2 \cos(\omega t + \theta)\sin(\omega t + \theta) \\ &= -\frac{\omega L I^2}{2} \sin 2(\omega t + \theta) \end{aligned} \tag{52}$$

  Average power into the inductance is zero. Instantaneous energy stored in the inductance is

$$w_L = \frac{1}{2} L I^2 \cos^2(\omega t + \theta)$$

  and *that* has an average value:

$$< w_L >= \frac{1}{4} L I^2 \tag{53}$$

- Capacitance: if $v = V\cos(\omega t + \phi)$, then $i = -\omega C V \sin(\omega t + \phi)$, and power is:

$$p = -\frac{\omega C V^2}{2} \sin 2(\omega t + \phi) \tag{54}$$

  which has zero time average. Energy stored in the capacitance is:

$$w_C = \frac{1}{2} C V^2 \cos^2(\omega t + \phi)$$

  which has time average:

$$< w_C >= \frac{1}{4} C V^2 \tag{55}$$

Now, consider power flow into a set of terminals in a situation in which both voltage and current are sinusoidal and have the same frequency, but possibly different phase angles:

$$\begin{aligned} v(t) &= V\cos(\omega t + \phi) \\ i(t) &= I\sin(\omega t + \theta) \end{aligned}$$

It is necessary to revert to the original form of complex notation, as in equation 19, to compute power.

$$v(t) = \frac{1}{2}\left[\underline{V} e^{j\omega t} + \underline{V}^* e^{-j\omega t}\right] \tag{56}$$

$$i(t) = \frac{1}{2}\left[\underline{I} e^{j\omega t} + \underline{I}^* e^{-j\omega t}\right] \tag{57}$$

Instantaneous power is the product of voltage and current:

$$p = \frac{1}{4}\left[\underline{V}\,\underline{I}^* + \underline{V}^*\underline{I} + \underline{V}\,\underline{I} e^{j2\omega t} + \underline{V}^*\underline{I}^* e^{-j2\omega t}\right] \tag{58}$$

This is directly equivalent to:

$$p = \frac{1}{2} Re \left[\underline{V} \underline{I}^* + \underline{V} \underline{I} e^{j2\omega t} \right] \tag{59}$$

This is, in turn, expressible as:

$$p = \frac{1}{2} |\underline{V}||\underline{I}| \left[\cos(\phi - \theta) + \cos(2\omega t + \phi + \theta) \right] \tag{60}$$

From this, we extract "real power", or time- average power:

$$P = \frac{1}{2} Re \left[\underline{V} \underline{I}^* \right] = \frac{1}{2} |\underline{V}||\underline{I}| \cos(\phi - \theta) \tag{61}$$

The ratio between *real* power and *apparent power* $P_a = \frac{1}{2}|\underline{V}||\underline{I}|$ is called the *power factor*, and is simply:

$$\text{power factor} = \cos\psi = \cos(\phi - \theta) \tag{62}$$

The *power factor angle* $\psi = \phi - \theta$ is the *relative* phase shift between voltage and current.

This expression for time- average power suggests a definition for something we might call *complex power*:

$$P + jQ = \frac{1}{2} \underline{V} \underline{I}^* \tag{63}$$

in which average power P is the real part. The magnitude of this complex quantity is the *apparent power*. The *imaginary* part is called *reactive power*. It has importance which will be discussed later.

Different *units* are used for real, reactive and apparent power, in order to gain some distinction between quantities. Usually we will express *real power* in *watts* (**W**) (or kW, MW,...). *Apparent power* is expressed in *volt-amperes* (**VA**), and *reactive power* is expressed in *volt-amperes-reactive* (**VAR's**).

To obtain some more feeling for reactive power, expand the time- varying part of the expression for instantaneous power:

$$p_{varying} = \frac{1}{2} |\underline{V}||\underline{I}| \cos(2\omega t + \phi + \theta)$$

Now, using the trig identity $\cos(x + y) = \cos x \cos y - \sin x \sin y$, and assigning $x = 2\omega t + 2\phi$ and $y = -\psi = \theta - \phi$, we have:

$$p_{varying} = \frac{1}{2} |\underline{V}||\underline{I}| \left[\cos 2(\omega t + \phi) + \sin\psi \sin 2(\omega t + \phi) \right]$$

Thus, total instantaneous power is:

$$p = \frac{1}{2} |\underline{V}||\underline{I}| \cos\psi \left[1 + \cos 2(\omega t + \phi) \right] + \frac{1}{2} |\underline{V}||\underline{I}| \sin\psi \sin 2(\omega t + \phi) \tag{64}$$

Now, if we note expressions for P and Q, we can re-write this as:

$$p = P \left[1 + \cos 2(\omega t + \phi) \right] + Q \sin 2(\omega t + \phi) \tag{65}$$

Thus, *real power* P represents not only time average power but also the pulsations that go with time average power. *Reactive power* Q represents energy exchange with zero average value.

7.1 RMS Amplitude

Note that, in all of the expressions for power used so far, a factor of $\frac{1}{2}$ appears. This is, of course, because the *average* value of the product of two sinusoids of the same frequency has a value of half of the products of their *peak* amplitudes multiplied by the cosine of the relative phase angle. It has become common to use a different measure of voltage amplitude, which is called *root-mean-square* or simply RMS. The proper definition for the RMS value of a waveform is somewhat complex, but boils down to that value which, if it were DC, would dissipate the same power in a resistor. It is possible to define RMS for *any* periodic waveform. However, since we will be dealing with sinusoids, the definition is even easier. Clearly, since power dissipated in a resistor is, in terms of *peak* amplitudes:

$$P = \frac{1}{2}\frac{|\underline{V}|^2}{R}$$

then the *RMS amplitude* must be:

$$V_{RMS} = \frac{|\underline{V}|}{\sqrt{2}} \tag{66}$$

Then,

$$P = \frac{V_{RMS}^2}{R}$$

As we will see, RMS amplitudes are the default for most situations: when a circuit is described as "120 Volts AC", the designation virtually always means 120 Volts, RMS. The peak amplitude of this is $|\underline{V}| = \sqrt{2} \cdot 120 \approx 170$ volts. Often you will see sinusoidal waveforms expressed in the form:

$$v = \sqrt{2} V_{RMS} \cos(\omega t)$$

in which V_{RMS} is obviously the RMS amplitude.

7.2 Example

Consider the simple network of Figure 14. We will calculate the *instantaneous* power flow into that network in terms we have been discussing. Assume that the voltage source has RMS amplitude

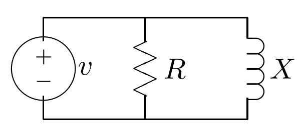

Figure 14: Example Circuit

of 120 volts and R and X are both 100 Ω. Then:

$$v(t) = 170 \cos \omega t$$

The admittance of this network is:

$$Y = \frac{1}{100} - \frac{j}{100}$$

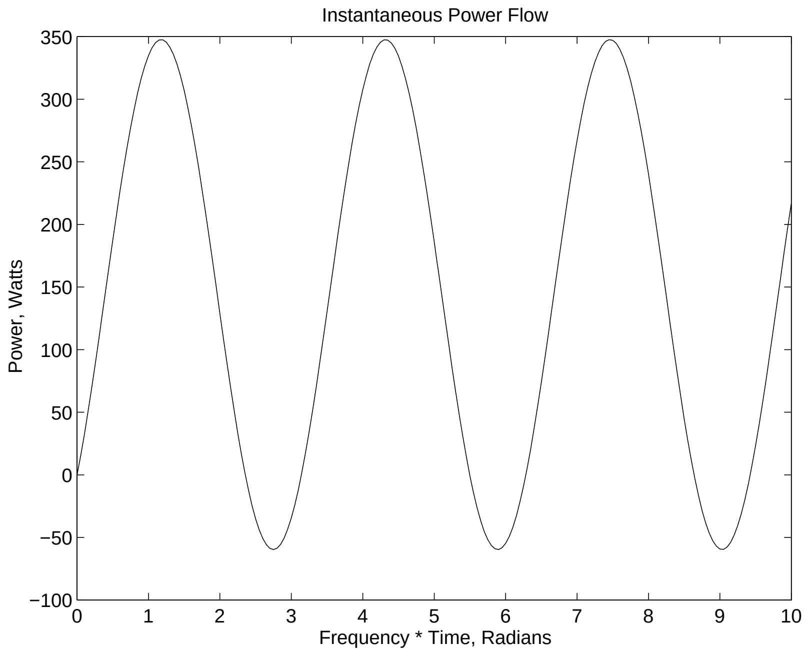

Figure 15: Power Flow For Example Circuit

so that the complex amplitude of current is:

$$I = 1.7 - j1.7$$

And then *complex* power is:

$$P + jQ = \frac{1}{2} 170(1.7 + j1.7)$$

Real and reactive power are, respectively: $P = 144$ W, $Q = 144$ VAR. This gives a *power factor angle* of $\psi = \arctan(1) = 45^\circ$. Then, instantaneous power is:

$$p = 144 \left[1 + \cos 2(\omega t - 45^\circ)\right] + 144 \sin 2(\omega t - 45^\circ)$$

This is illustrated in Figure 15.

8 A Conservation Law

It is possible to show that complex power is conserved in the same way as we expect time average power to be conserved. Consider a network with a collection of *terminals* and with a collection of internal *branches*. Instantaneous power flow *into* the network is:

$$p_{in} = \sum_{terminals} vi$$

Note that this expression holds for voltage and current expressed over *any* complete set of terminals. That is, if it is possible to delineate the terminals of the network into a set of *pairs*, the voltages might correspond to voltages across the pair, while currents would flow between the terminals of each pair. Alternatively, the voltages might correspond to single node-to-*datum* voltage, while currents would then be single input node currents. Since power can go *only* into network elements, it follows that the sum of internal branch powers must be equal to the sum of terminal powers:

$$\sum_{terminals} vi = \sum_{branches} vi \tag{67}$$

If this is true for *instantaneous* power, it is also true for *complex* power:

$$\sum_{terminals} \underline{V}\,\underline{I} = \sum_{branches} \underline{V}\,\underline{I} \tag{68}$$

Now, if the network is made up of resistances, capacitances and inductances,

$$\sum_{terminals} \underline{V}\,\underline{I} = \sum_{resistances} \underline{V}\,\underline{I} + \sum_{inductances} \underline{V}\,\underline{I} + \sum_{capacitances} \underline{V}\,\underline{I} \tag{69}$$

For these individual elements:

- Resistances: $\underline{V}\,\underline{I}^* = R|\underline{I}|^2$
- Inductances: $\underline{V}\,\underline{I}^* = j\omega L|\underline{I}|^2$
- Capacitances: $\underline{V}\,\underline{I}^* = -j\omega C|\underline{V}|^2$

Then equation 69 becomes:

$$\sum_{terminals} \underline{V}\,\underline{I} = \sum_{resistances} R|\underline{I}|^2 + j \sum_{inductances} \omega L|\underline{I}|^2 - j \sum_{capacitances} \omega C|\underline{V}|^2 \tag{70}$$

Then, identifying individual terms:

$$\begin{array}{rcll} \displaystyle\sum_{terminals} \underline{V}\,\underline{I} &=& 2(P+jQ) & \text{Total Complex Power into Network} \\ \displaystyle\sum_{resistances} R|\underline{I}|^2 &=& 2\sum < p_r > & \text{Power Dissipated in Resistors} \\ \displaystyle j\sum_{inductances} \omega L|\underline{I}|^2 &=& 4\omega\sum < w_L > & \text{Energy Stored in Inductances} \\ \displaystyle j\sum_{capacitances} \omega C|\underline{V}|^2 &=& 4\omega\sum < w_C > & \text{Energy Stored in Capacitances} \end{array}$$

So, for any RLC network:

$$P + jQ = \sum_{resistors} < p_r > +2j\omega \left[\sum_{inductors} < w_L > - \sum_{capacitors} < w_C > \right] \tag{71}$$

9 Power Flow Through An Impedance

Consider the situation shown in Figure 16. This actually represents a number of important situations in power systems, where the impedance $\underline{Z}$ might represent a transmission line, transformer or motor winding. Of interest to us is the flow of power through the impedance. Current is given

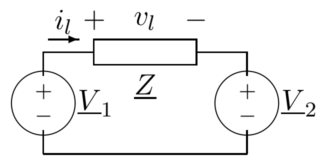

Figure 16: Power Flow Example

by:

$$\underline{i_l} = \frac{\underline{V}_1 - \underline{V}_2}{\underline{Z}} \tag{72}$$

Then, complex power flow out of the left- hand voltage source is:

$$P + jQ = \frac{1}{2}\underline{V}_1 \left(\frac{\underline{V}_1^* - \underline{V}_2^*}{\underline{Z}^*} \right) \tag{73}$$

Now, the complex amplitudes may be expressed as:

$$\underline{V}_1 = |\underline{V}_1| e^{j\theta} \tag{74}$$

$$\underline{V}_2 = |\underline{V}_2| e^{j\theta + \delta} \tag{75}$$

where δ is the *relative* phase angle between the two voltage sources. Complex power at the terminals of the voltage source $\underline{V}_1$ is now given by:

$$P + jQ = \frac{|\underline{V}_1|^2}{2\underline{Z}^*} - \frac{|\underline{V}_1||\underline{V}_2| e^{-j\delta}}{2\underline{Z}^*} \tag{76}$$

This is describable as a circle in the complex plane, with its origin at

$$\frac{|\underline{V}_1|^2}{2\underline{Z}^*}$$

and its radius equal to:

$$\frac{|\underline{V}_1||\underline{V}_2|}{2|\underline{Z}|}$$

Now suppose the impedance through which we are passing power is describable as a simple inductance as shown in Figure 17. This is perhaps the simplest of transmission line models which represents only the inductive impedance of the line. Line inductance arises because currents in the line produce magnetic fields, and this is a fair model for most lines which are fairly 'short'. More on that in the next section. This line has the impedance

$$Z = j\omega L = jX_L$$

Now, switching to RMS amplitudes, so that $\underline{V}_s = \sqrt{2}\underline{V}_1$ and $\underline{V}_r = \sqrt{2}\underline{V}_2$, Then real and reactive power flow are:

$$P_s + jQ_s = \underline{V}_s \underline{I}^* = j\frac{|V_s|^2 - \underline{V}_s \underline{V}_r^*}{X_l}$$

$$P_r + jQ_r = -\underline{V}_r \underline{I}^* = j\frac{|V_r|^2 - \underline{V}_s^* \underline{V}_r}{X_l}$$

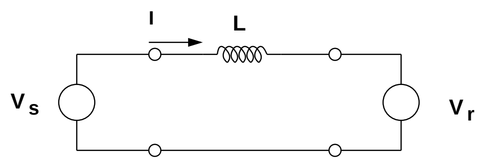

Figure 17: Simplest Transmission Line Model

Now if we assume that the voltages are of the form:

$$\underline{V}_s = V_s e^{j\phi}$$
$$\underline{V}_r = V_r e^{j\theta}$$

and that the relative phase angle between them is $\phi - \theta = \delta$ and doing a little trig:

$$P_s = \frac{V_s V_r \sin\delta}{X_L}$$
$$Q_s = \frac{V_s^2 - V_s V_r \cos\delta}{X_L}$$
$$P_r = -\frac{V_s V_r \sin\delta}{X_L}$$
$$Q_r = \frac{V_r^2 - V_s V_r \cos\delta}{X_L}$$

A picture of this locus is referred to as a *power circle diagram*, because of its shape. Figure 18 shows the construction of a sending end power circle diagram for equal sending-end and receiving-end voltages and a purely reactive impedance.

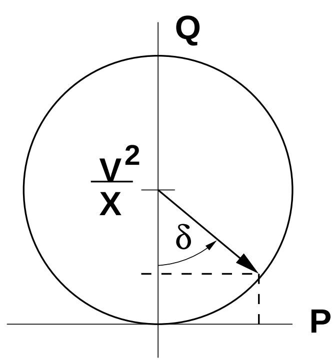

Figure 18: Power Circle, Equal Voltages

As a check, consider the reactive power consumed by the line itself: we expect that $Q_s + Q_r = Q_L$, and so:

$$Q_s + Q_r = \frac{V_s^2 + V_r^2 - 2V_s V_r \cos\delta}{X_L}$$

Note that the voltage across the line element itself is found using the law of cosines (see Figure 19:

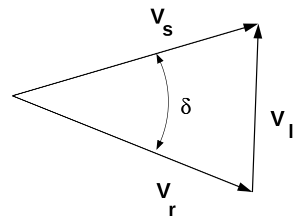

Figure 19: Illustration of the Law of Cosines

$$V_L^2 = V_s^2 + V_r^2 - 2V_sV_r\cos\delta$$

and, indeed,

$$Q_L = \frac{V_L^2}{X_L}$$

10 Compensated Line

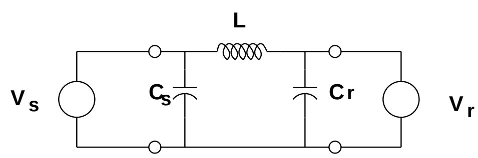

Figure 20: Transmission Line Model

Perhaps a more commonly used model for a transmission line is as shown in Figure 20. This represents not only the fact that most transmission lines have, in addition to series inductance, parallel capacitance but also the fact that many transmission lines are shunt compensated. This may be represented as a two-port network with the admittance parameters, using $X_L = j\omega L$ and $X_C = \frac{-j}{\omega C}$, :

$$\begin{aligned}
\underline{Y}_{ss} &= \frac{1}{jX_L} - \frac{1}{jX_{C1}} \\
\underline{Y}_{sr} = \underline{Y}_{rs} &= \frac{1}{jX_L} \\
\underline{Y}_{rr} &= \frac{1}{jX_L} - \frac{1}{jX_{C2}}
\end{aligned}$$

It is fairly clear that, for voltage sources at both ends, real and reactive power flow are:

$$P_s = \frac{V_sV_r\sin\delta}{X_L}$$

$$
\begin{aligned}
Q_s &= V_s^2\left(\frac{1}{X_L}-\frac{1}{X_{C1}}\right)-\frac{V_sV_r\cos\delta}{X_L} \\
P_r &= -\frac{V_sV_r\sin\delta}{X_L} \\
Q_r &= V_s^2\left(\frac{1}{X_L}-\frac{1}{X_{C2}}\right)-\frac{V_sV_r\cos\delta}{X_L}
\end{aligned}
$$

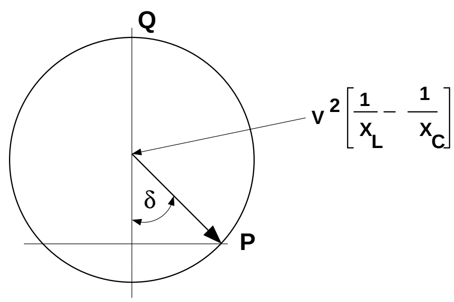

Figure 21: Power Circle, Equal Voltages, Compensation Offset

The power circle for this sort of line is similar to that of the simpler model, but the center is offset to smaller reactive component, as shown in Figure 21.

An interesting feature of transmission lines is illustrated by what might happen were the receiving line to be open: in that case:

$$\underline{V}_r = \underline{V}_s\frac{1}{1-\omega^2LC}$$

Depending on the values of frequency, inductance and capacitance this could be arbitrarily large, and this is a potential problem, particularly for longer lines, as we will discuss in the next section.

11 Transmission Lines

A transmission line is really a long, continuous thing. It has inductance which is really inductance per unit length multiplied by the line length, but it also has a continuous capacitance. We might attempt to represent a long transmission line as a series of relatively 'short' sections each represented by an inductance and a capacitance. These 'lumped parameter' models for lines are actually used in many system studies, particularly in physical analog models called 'Transmission System Simulators'. (We built one of these at MIT in the 1970's). After the next section you might contemplate the definition of 'short' for our purposes here, but generally each lumped parameter capacitance and resistance pair would represent a few to a few tens of miles.

11.1 Telegrapher's Equations

Peering at the model presented in Figure 22, one might divine that a proper representation of voltage and current, both of which are functions of time and distance along the line, might be:

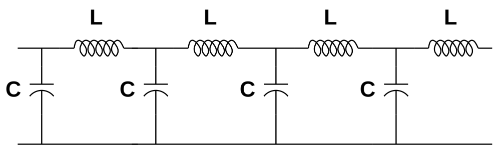

Figure 22: Transmission Line Lumped Parameter Model

$$\begin{aligned}\frac{\partial v}{\partial x} &= -L_l \frac{\partial i}{\partial t} \\ \frac{\partial i}{\partial x} &= -C_l \frac{\partial v}{\partial t}\end{aligned}$$

These are known as the "Telegrapher's Equations" and represent the fact that inductance presents voltage drop along the line in proportion to rate of change of current and that capacitance presents a change in current along the line in proportion to rate of change of voltage.

It is not difficult to eliminate either voltage or current from these to produce a wave equation. For example, take the cross-derivatives and substitute the second of these equations into the first to get:

$$\frac{\partial^2 v}{\partial x^2} = L_l C_l \frac{\partial^2 v}{\partial t^2}$$

Now: this equation is solved by arbitrary functions which are of the form:

$$v(x,t) = v(x \pm ut)$$

where the wave velocity is:

$$u = \frac{1}{\sqrt{L_l C_l}}$$

So now we can see that the voltage on the line is the sum of some waveform going in the 'positive' direction and something else going in the 'negative' direction:

$$v(x,t) = v_+(x - ut) + v_-(x + ut)$$

The same will be true of current, and substituting back into either of the telegrapher's equations yields:

$$i(x,t) = \frac{1}{L_l u} \left(v_+(x - ut) - v_-(x + ut) \right)$$

the product of inductance times wave velocity has the units of impedance:

$$L_l u = \sqrt{\frac{L_l}{C_l}} = Z_0$$

This is often referred to as the 'characteristic impedance' of the transmission line. This is also a commonly used term: transmission cables are often referred to by their characteristic impedances. For coaxial wires 50 to 72 ohms are common values. For high tension transmission lines somewhat higher values are to be expected.

11.2 Surges on Transmission Lines

Consider the situation shown in Figure 23. Here the left-hand end of the line is driven by a current source with a pulse (illustrated is a square pulse). This is actually not too far from the situation that transmission lines experience with lightning, which is usually representable as a current source, typically of magnitude between 20 and 100 kA and duration of about $1\mu S$. (Actually, it is not a square pulse but that is not important here).

What will happen, if the pulse is short enough, is that it will launch a traveling wave in which $v_+ = Z_0 i_+$ and i_+ is the current that was imposed. When this pulse reaches the far, or load end of the line, we have the situation in which at that point:

$$\begin{aligned} v(t) &= v_+ + v_- \\ i(t) &= \frac{v_+}{Z_0} - \frac{V_-}{Z_0} \end{aligned}$$

and, of coures, $v = Ri$.

The 'reflected', or negative going wave will have magnitude:

$$v_- = v_+ \frac{\frac{R}{Z_0} - 1}{\frac{R}{Z_0} + 1}$$

In the extreme case of an open circuit, the magnitude of the voltage pulse at the end of the transmission line is exactly twice that of the propagating pulse. In the case of a short circuit, of course, the magnitude of the voltage is zero, the current in the short is double the current of the pulse itself, and the pulse is reflected, but going in the reverse direction with a polarity the opposite of the forward-going pulse. This is illustrated in cartoon form in Figure 23.

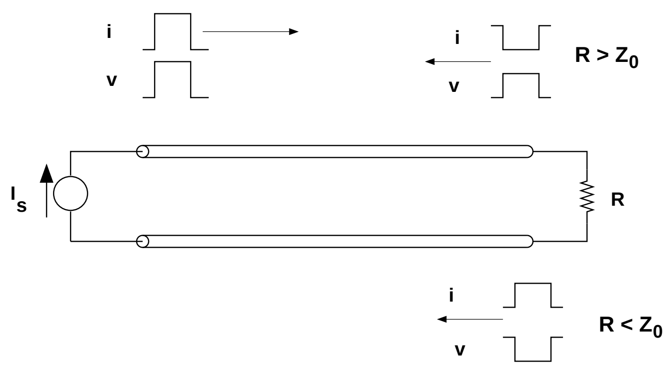

Figure 23: Pulse Propagation on a Transmission Line

11.3 Sinusoidal Steady State

Now, consider a transmission line operating in the sinusoidal steady state. As suggested by Figure 24, it is driven by a voltage source at one end and is loaded by a resistive load at the other. Consistent with the voltage and currents that we know can exist on such a line, we know they will be of this form:

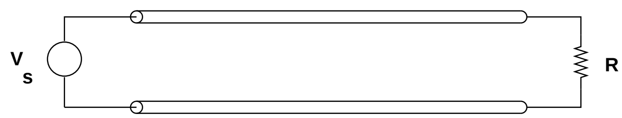

Figure 24: Transmission Line in Simple Configuration

$$v(x,t) = \text{Re}\left\{\underline{V}_+ e^{j(\omega t - kx)} + \underline{V}_- e^{j(\omega t + kx)}\right\}$$
$$i(x,t) = \text{Re}\left\{\frac{\underline{V}_+}{Z_0} e^{j(\omega t - kx)} - \frac{\underline{V}_-}{Z_0} e^{j(\omega t + kx)}\right\}$$

Where the phase velocity is $u = \frac{\omega}{k} = \frac{1}{\sqrt{L_l C_l}}$.
At the termination end of the line, at $x = \ell$

$$R = \frac{\underline{V}}{\underline{I}} = Z_0 \frac{\underline{V}_+ e^{-jk\ell} + \underline{V}_- e^{jk\ell}}{\underline{V}_+ e^{-jk\ell} - \underline{V}_- e^{jk\ell}}$$

This may be solved for the ratio of 'reverse' to 'forward' amplitude:

$$\underline{V}_- = \underline{V}_+ e^{-2jk\ell} \frac{\frac{R}{Z_0} - 1}{\frac{R}{Z_0} + 1}$$

Since at the 'sending' end:

$$V_s = \underline{V}_+ + \underline{V}_-$$

With a little manipulation it can be determined that

$$\underline{V}_r = V_s \frac{e^{-jk\ell}\left[\left(\frac{R}{Z_0} + 1\right) + \left(\frac{R}{Z_0} - 1\right)\right]}{\left(\frac{R}{Z_0} + 1\right) + e^{-2jk\ell}\left(\frac{R}{Z_0} - 1\right)}$$

Further manipulation yields:

$$\underline{V}_r = V_s \frac{\frac{R}{Z_0}}{\frac{R}{Z_0}\cos k\ell + j \sin k\ell}$$

This might be made a bit more comprehensible when turned into a magnitude:

$$\left|\frac{V_r}{V_s}\right| = \frac{\frac{R}{Z_0}}{\sqrt{\left(\frac{R}{Z_0}\cos k\ell\right)^2 + (\sin k\ell)^2}}$$

If the line is loaded with a resistance equivalent to the 'surge impedance' (so-called 'surge impedance loading', the receiving end voltage is the same as the sending end voltage. If it is more heavily loaded, the receiving end voltage is less than the sending end and if it is less heavily loaded the receiving end voltage is greater.