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PRIVATE CONSTANT RETURNS AND PUBLIC SHADOW PRICES

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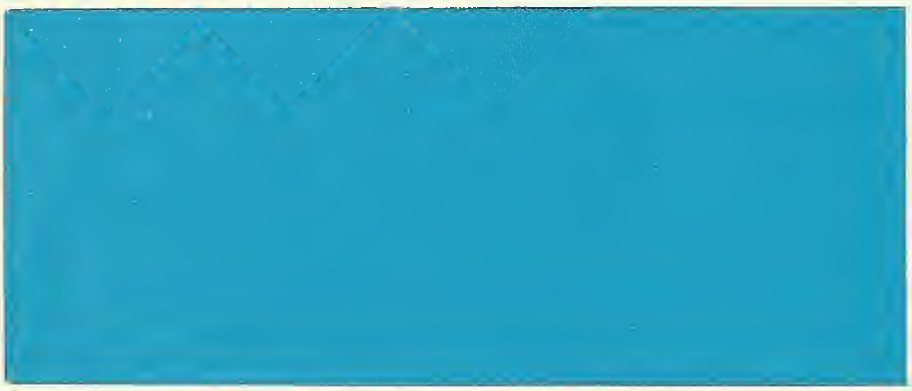

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1. Introduction

Although there has been little analysis of the extent of its validity, a widely (but sometimes only implicitly) accepted proposition is that governments should generally select production plans which are on the frontiers of their production possibility sets. This implies the existence of a set of shadow prices which can be used for government production decisions.[3/] If private production is characterized by constant returns to scale and controlled by price taking profit maximizers and if the economy has optimal commodity taxes and no further distortions, then government production plans should generally be such that aggregate production (government plus private production) is on the frontier of the aggregate production possibility set [5].[4/] It follows that the shadow prices associated with optimal government production are the market prices faced by private producers. In this note we shall explore the situation where part of the economy exhibits constant returns to scale and is controlled by price taking profit maximizers. The behavior of the rest of the private economy is assumed to be determinable from equilibrium prices and aggregate quantities. We shall not assume that taxes are necessarily optimal. Rather it is assumed that government policy is such that all markets can clear, and that in setting taxes it shows no concern for the division of its revenue between taxes and the profits of public production, or the division between public and private

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3/ With a convex production possibility set, government production would maximize profits evaluated at shadow prices.

4/ A similar analysis follows without constant returns with the presence, also, of optimal profits taxes [2], [8].

production as such. In this setting, assuming that government production should be efficient, and that government transactions with the private economy take place at market prices, it is shown that the shadow prices for government production must be such that the shadow profits of any constant returns to scale industry (i.e. profits calculated using shadow prices rather than market prices) equal zero.[5/]

This rule can be used to check a set of alternatively derived shadow prices, or as an estimation system for particular prices given estimates of other prices. It implies that the government should not employ all inputs at lower marginal productivities than their private use in the constant returns part of the economy. For example, in a two input case, a shadow wage below market wage implies a discount rate above the market rate.

As a special case, consider a constant returns industry with one input and one output. The rule says that relative shadow prices should be the same as relative producer prices. If, for some commodities, the economy trades on the world market at fixed prices, any two of these commodities can be viewed as a constant returns industry. Thus relative shadow prices should be the same as world prices for these commodities.[6/] We get another interesting special case when the set of constant returns industries spans the commodity set. Under these circumstances, the optimal shadow prices equal the market prices.

We begin with a discussion of the intuitive source of this result. We then present both calculus and noncalculus proofs, since they each contribute

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5/ In the context of special models this result has been derived by the authors previously, one model with constant returns and several commodities necessarily taxed at the same rate, [4], and the other, unpublished, with decreasing returns. This note explores the generality of this condition of the optimum.

6/ For previous derivations of this result, see [3], [7].

to possible insight into the nature of the result. The basic model in which the result is derived can readily be generalized in many directions without altering the relations obtained. Some of these extensions are discussed in the concluding section.

2. Basic Argument

Consider a government production plan and an implied equilibrium position of the economy. If the plan is optimal, there would be no change in social welfare from small transfers of inputs between public and private production. Consider transferring to the public sector a small proportion of all the inputs of a single constant returns industry (which produces a single output). The assumption of constant returns implies that these firms remain in equilibrium at the prevailing prices. Prices then do not change, provided that the government output from the transferred inputs precisely equals the decline in private production resulting from the loss of these inputs. If this condition holds, prices and aggregate quantities do not change, and consequently behavior elsewhere in the economy is unchanged. The loss in private output is the sum of the lost inputs weighted by marginal products. Since marginal products equal price ratios, we can write this as $\theta\Sigma p_i y_i^1$, where $\{p_i\}$ are the producer prices (with output as numeraire), $\{y_i^1\}$ are total inputs to industry 1 and θ is the proportion of inputs transferred to the public sector. The increase in public sector output is the sum of the new inputs weighted by their marginal products in the public sector. But these marginal products are proportional to the shadow prices characterizing the optimum, $\{s_i\}$. Thus we have the relationship

$$\sum_{\text{inputs}} p_i y_i^1 = \sum_{\text{inputs}} s_i y_i^1 \qquad (1)$$

Since a constant returns industry has zero profits evaluated in market prices, the value of output equals the value of inputs in market prices

which in turn equals the value of inputs in shadow prices. Thus we reach our conclusion that profits measured in shadow prices are zero

$$s \cdot y^1 = 0 \qquad (2)$$

provided that industry one has constant returns to scale. In the next section, we provide a formal derivation of this result, so as to make clear the role played by the various assumptions.

3. Calculus Proof

In deriving the result by calculus methods we ignore nonnegativity constraints, since these are handled in the rigorous proof to follow. We use the following notation:

- q — the vector of consumer prices
- p — the vector of producer prices
- W — a measure of welfare depending only on the consumption of individuals
- x — the vector of net consumer demands
- y^1 — the vector of net supplies by producer 1
- y^2 — the vector of net supplies of all other producers
- z — the vector of net supplies of the public sector

We assume that producer 1 is a profit-maximizing price-taker, with a constant returns technology described by

$$f(y^1) = 0 . \qquad (1)$$

f is twice continuously differentiable. For profit maximization by producer 1, provided that the optimum does not imply zero production, we have the first order conditions

$$f_y(y^1) = \lambda p \qquad (4)$$

where λ is a factor of proportionality. By taking these equations as constraints in social welfare maximization, we can consider the vector of supplies, y^1, as control variables.

For all other private producers in the economy we assume that the vector of net supplies is uniquely determined by the vector of producer prices,

$$y^2 = y^2(p) \tag{5}$$

(It would not alter the result to have some constant returns firms in this sector.) The profits generated by these producers, net of any profits taxes, are distributed to the consumers who own the firms. In this way the vector of net consumer demands depends on consumer prices, q, producer prices, p, and the variables describing the level of profit taxation, τ:

$$x = x(q,p,\tau) \tag{6}$$

Granted this description of individual demands, and the assumption that welfare depends only on individual consumption bundles, we can write welfare as a function of the same three variables

$$W = V(q,p,\tau) \tag{7}$$

To complete the description of individual agents in the economy we need to consider the constraints on public production and the tax setting policy of the government. The vector of net supplies is assumed to be constrained by a smooth production function

$$g(z) = 0 \tag{8}$$

The fact that g(0) may not be zero allows this formulation to include a fixed vector of government expenditure needs. To construct a model of tax setting let us first consider the government budget constraint, that tax revenue equal the net loss on public production

$$(q-p)\cdot x(q,p,\tau) + T(p,\tau) = -p\cdot z \tag{9}$$

where T is the revenue from profit taxation as a function of the tax parameters τ. Write the profits from public production as

$$\Pi = p\cdot z \tag{10}$$

Given the beliefs about demands and supplies held by public tax setters, it is reasonable to argue that consumer prices and profit tax parameters can be described as functions of producer prices and the governments net profits from expenditures

$$q = \phi(p,\Pi) = \phi(p,p\cdot z)$$
$$\tau = \psi(p,\Pi) = \psi(p,p\cdot z) \tag{11}$$

Since demands depend on q and τ as well as p this formulation represents a reduced form of tax setting depending on demand quantities as well as prices and loss. Since partial optimization of taxes would often fit this general framework, the taxes need not necessarily be nonoptimal. The formulation (11) does imply that the division of net consumer demand among different suppliers has no effect on tax setting policy or on tax revenue. This ignores variations in administrative costs of tax enforcement as well as political pressures.

Given these models of the behavior of each agent we complete the description of the economy by adding the market clearance conditions,

$$x(q,p,\tau) = y^1 + y^2(p) + z \tag{12}$$

We can form a Lagrangian expression to describe the maximization of social welfare subject to this description of the economy

$$L = V(\phi(p,p\cdot z),p,\psi(p,p\cdot z))$$
$$+ \alpha[x(\phi(p,p\cdot z),p,\psi(p,p\cdot z) - y^1 - y^2(p) - z]$$
$$+ \beta g(z)$$
$$+ \gamma f(y^1)$$
$$+ \xi[f_y(y^1) - \lambda p] \tag{13}$$

where α and ξ are vectors of multipliers while β and γ are scalars. Fortunately, for our purposes we need only consider the derivatives of L with respect to z and y^1. Differentiation with respect to y^1 yields

$$\alpha = \gamma f_y + \xi f_{yy} \tag{14}$$

Differentiation with respect to z gives

$$[V_q\phi_\Pi + V_\tau\psi_\Pi + \alpha x_q\phi_\Pi + \alpha x_\tau\psi_\Pi]\ p - \alpha + \beta g_z = 0 \tag{15}$$

Writing the expression in brackets as σ, which is a scalar, this becomes

$$\alpha = \beta g_z + \sigma p \tag{16}$$

Combining (14) and (16) we have

$$\beta g_z = \gamma f_y + \xi f_{yy} - \sigma p$$
$$= (\gamma\lambda - \sigma)\ p + \xi f_{yy} \tag{17}$$

where use has been made of (4). In the two commodity case, f is linear and $f_{yy} = 0$. In the many commodity nonlinear case, equation (17) gives us no reason in general for the shadow prices, g_z, to be proportional to p, as

there is no reason in the model for ξf_{yy} to be proportional to f_y. Since f is homogeneous of degree one and f_y therefore homogeneous of degree zero we have

$$\lambda p \cdot y^1 = f_y \cdot y^1 = f = 0 \tag{18}$$

and

$$f_{yy} \cdot y^1 = 0 \tag{19}$$

Therefore multiplying (17) by the vector y^1 yields the implication

$$\beta g_z \cdot y^1 = 0 \tag{20}$$

Provided β is not zero (i.e. provided the public production constraint "bites") we have the desired result

$$s \cdot y^1 = 0 \tag{21}$$

where s is the vector of shadow prices characterizing public production at the optimum. It is also true that

$$p \cdot y^1 = 0 \ , \tag{22}$$

but this does not imply that p and s are proportional, except in the special case of two commodities. Clearly the same analysis could be done identifying any constant returns producer as firm 1, provided we recognized that supply by the rest of the economy $y^2(p)$ would then be a correspondence rather than a function. This would not create difficulty, since p and y^2 are kept fixed while we derive the first-order conditions we need.

4. Noncalculus Proof

In the analysis above we ignored any complications arising from nonnegativity constraints. We now give a rigorous statement and proof of the theorem which does not have this shortcoming. We denote the net supply vectors of individual firms by y^j, $j = 1,2,...,J$. We will call the set of firms that are price-taking profit-maximizers, and have constant returns to scale, the C-sector. The set is denoted by C, and the equilibrium supply vectors of the firms are denoted by y^{*j}. We call the set of remaining producers the R-sector. The production set available to the public sector is denoted by Z.

Theorem: Let Z be convex. Let consumer demand and firm supply depend (in reduced form) on producer prices and public sector profits. Assume that the public sector transacts at market prices. Let welfare depend on consumer demand, and supply by producers in the R-sector. Assume that optimal public production must be on the relative frontier of the public production possibility set (z^* on the relative frontier of Z). Then there exists a nonzero vector of shadow prices s such that optimal public production maximizes $s \cdot z$ on Z and

$$s \cdot y^{*j} = 0$$

for all j in C.

Proof: If z^* and $\{y^{*j} | j \in C\}$ are equilibrium supplies at the optimal level of public production, then $z^* + \Sigma_j \lambda_j y^{*j}$ and $\{(1-\lambda_j)\, y^{*j} | j \in C\}$ are also net supplies which, we shall show, result in the same equilibrium levels of consumer and R-sector behavior. Net supply being unchanged, producer prices

are unchanged if public sector profits are unchanged. Since $p \cdot y^{*j} = 0$ for all j in C where p is the producer price vector (competitive constant returns firms make zero profits), public sector profits are unchanged.

The supply $(1-\lambda_j)\, y^{*j}$ is feasible for producer j provided $\lambda_j < 1$. Thus the vector $z^* + \sum_j \lambda_j y^{*j}$ is an optimal public production plan, and by hypothesis cannot lie in the relative interior of Z. Therefore there exists a hyperplane, containing all the vectors $z^* + \Sigma\lambda_j y^{*j}, \lambda_j < 1$, which does not intersect the relative interior of Z. This hyperplane defines the desired shadow prices s and satisfies

$$s \cdot y^{*j} = 0$$

for j in C. QED.

This proof captures the essence of the matter. There is no social loss or gain from shifting production between the public and C-sectors in exact proportion to the activity of one of the C-producers. Therefore the value of the shift, in shadow price terms, must be zero. Undoubtedly the theorem is evasive in postulating, rather than deducing, that optimal public production must be efficient. Yet, on consideration, there are considerable difficulties in formulating an adequate general criterion for efficiency. One would try to prove it by showing that more public production of <u>some</u> good must raise welfare. But the general equilibrium effects of changing public production are very complicated, and might easily produce no net benefit by helping some consumers while hurting others.[7/] Even with a one

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7/ For an example of desired inefficiency in the public sector in a two consumer economy with optimal taxation see Example b in [5], page 18. (The welfare function in that example is misprinted and should be $-\frac{1}{x_1 y_1^2} - \frac{1}{x_2 y_2^2}$.)

consumer economy and optimal taxes it may be true that the optimum need not lie on the frontier where there is a set of optima.[8/] These complications are present at given tax rates. A further problem, depending on tax setting policy, is that tax rates might change in a way which offset the gain from greater production. With the extensions to be considered below, like consumption externalities, the possible ways of constructing examples of desirable inefficiency expand. From the perspective of foreign aid however, this limitation is not important since a public sector producing inside its production possibility set can make no use of foreign aid (at least in small quantities).

An obvious corollary of this theorem is that the presence of enough linearly independent producers in the C-sector implies that producer prices p are shadow prices for optimal public production. Specifically, if there are n commodities and n-1 C-sector firms whose equilibrium bundles y^{*j} are linearly independent, then the conditions $s \cdot y^{*j} = 0$ and $p \cdot y^{*j} = 0$ imply that $p = s$. The corollary applies for example if there is a single nonproduced input, no joint production, and at least one competitive constant returns firm producing each produced commodity. This case is obviously extremely special, but it is interesting that it exists, without an assumption of optimal taxation.

8/ See Example d in [5], page 23, where the indifference curve has a linear segment all points of which are optimal but only one point of which is on the frontier.

5. Extensions

The explicit model contained in Section 3 had consumers and producers making decisions based solely on prices. This is more restrictive than is needed to carry through the argument. What matters is that no decisions depend explicitly on z and y^j separately. Thus individual consumer demands could depend on the demands of other consumers or on supply aggregates. In this way the model could be extended to cover consumption externalities. Individual firms in the R-sector need not be competitive. If they exert monopoly power, it is a natural assumption that their supplies depend on aggregate quantities as well as prices.[9/] With this formulation, and suitable restrictions to ensure equilibrium, the results again carry through.

The models considered above were equilibrium models where all markets cleared. The extension to models with some types of unemployment, such as that associated with rural-urban migration [10/] ought to be possible, provided the migration decisions depend only on aggregate job opportunities. (That is, would-be workers do not prefer public to private employment or vice versa.) However, in such a model it is generally desirable for the public sector to pay wages which are not market wages.[11/] If this is possible, optimal behavior violates one of the conditions of the theorem.

We have considered economies with only two price vectors, one for consumers and one for producers. The presence of multiple consumer price

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9/ For a model of equilibrium with monopoly, see [1].

10/ For an example, see [6].

11/ See [9] or [10].

vectors (as with regionally varying prices) clearly causes no complication for the analysis. The analysis of multiple producer price vectors (brought about by intermediate good taxes or tariffs on imports by firms) is slightly more complicated and calls for a somewhat different formulation, which still yields the conclusion. Let us denote by x^{+i}, y^{+j}, z^{+}, vectors of demands (with zeros for quantities supplied) and by x^{-i}, y^{-j}, z^{-}, vectors of supplies. Equilibrium is now written as

$$\sum_i x^{+i} + \sum_j y^{+j} + z^{+} = \sum_i x^{-i} + \sum_j y^{-j} + z^{-} \qquad (23)$$

Assume that consumer i pays a tax t_i' on his purchases and a tax t_i'' on his supplies while firm j pays a tax θ_j' on his purchases and one of θ_j'' on his supplies. Denoting market prices by p, demands of the i^{th} consumer depend on (p, t_i', t_i'', Π_i) where Π_i are profits received by consumer i. Demands of the j^{th} producer depend on $(p, \theta_j', \theta_j'')$. Profits therefore depend on p, all the taxes $\{\theta_j'\}$, $\{\theta_j''\}$ and whatever profit taxes exist.

The theorem still holds, provided that the government makes its transactions at the price vector p, since equilibrium is not upset by the transfer of λy^{*j} to the public sector. The equality of market and shadow prices when the supply vectors of the firms in the C-sector span the commodity space does not hold in this case because prices to producers may vary across firms.

References

1. K. Arrow and F. Hahn, General Competitive Analysis (Oliver and Boyd, Edinburgh, and Holden Day, San Francisco), 1971

2. P. Dasgupta and J. Stiglitz, "On Optimal Taxation and Public Production", Review of Economic Studies, 39 (1), Jan. 1972

3. __________ and __________, "Benefit-Cost Analysis and Trade Policies", Journal of Political Economy, 82 (1), Jan.-Feb. 1974

4. P. Diamond, "Taxation and Public Production in a Growth Setting" in Mirrlees and Stern (eds.) Models of Economic Growth (MacMillan, 1973)

5. __________ and J. Mirrlees, "Optimal Taxation and Public Production", American Economic Review, 61 (1 and 3), March and June 1971

6. J. Harris and M. Todaro, "Migration, Unemployment, and Development: A Two-Sector Analysis", American Economic Review, 60 (1), March 1970

7. I.M.D. Little and J. Mirrlees, Project Appraisal and Planning for Developing Countries (Heinemann, London, and Basic Books, New York, 1974, p. 370)

8. J. Mirrlees, "On Producer Taxation", Review of Economic Studies, 39 (11), Jan. 1972

9. __________, "A Pure Theory of Underdeveloped Economies" in L. Reynolds (ed.) Agriculture and Development Theory (Yale University Press, 1975)

10. J. Stiglitz, "Unemployment in LDS: II. The Efficiency Wage Case", Cowles Foundation Discussion Paper, March 1973