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A Dual Approach to Regularity In Production Economies

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Abstract

Using smooth profit functions to characterize production possibilities we are able to extend the concepts of regularity and fixed point index to economies with very general technologies, involving both constant and decreasing returns. To prove the genericity of regular economies we rely on an approach taken by Mas-Colell, which utilizes in an elegant and insightful manner the topological concept of transversality. We also generalize the index theorem given by Kehoe. Our results shed new light on the question of when an economy has a unique equilibrium.

A Dual Approach to Regularity in Production Economics

by

Timothy J. Kehoe*

1. INTRODUCTION

Differential topology has, over the past decade, provided economists with a unified framework for studying both the local and the global properties of solutions to general equilibrium models. Debreu (1970) initiated this line of research with his introduction of the concept of a regular economy, a model whose equilibria are locally unique and vary continuously with the underlying economic parameters. Dierker (1972) pointed out the close connection of this concept with that of the fixed point index, a concept ideally suited to the study of existence and uniqueness of equilibria. Both of these studies focused attention on pure exchange economies that allow no production. More recently a number of different researchers, among them Fuchs (1974, 1977), Mas-Colell (1975, 1976, 1978), Smale (1976), and Kehoe (1979a, 1979b, 1980a), have extended these concepts to models with production. In all of these studies the concepts of genericity and transversality have played an important role in ruling out degenerate situations.

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*Many people have influenced the ideas presented in this paper. Most of my understanding of the concept of regularity has developed as a result of conversations and correspondence that I have had with Andreu Mas-Colell. Sidney Winter taught me the importance of the concept of duality in characterizing production technologies. Franklin Fisher and David Levine provided helpful suggestions. Above all, I am grateful to Herbert Scarf, who introduced me to index theorems and encouraged me to apply them to production economies.

The approach taken in this paper is in the spirit of Mas-Colell (1978) and Kehoe (1980a), who emphasize the development of a formula for computing the index of an equilibrium and the connection between this formula and theorems dealing with the uniqueness of equilibrium. Both of these writers model the production side of an economy as an activity analysis technology. Unfortunately, the results obtained by Mas-Colell and Kehoe are not immediately applicable to economies with more general production technologies. It is true, of course, that any constant-returns technology can be aproximated in a continuous manner by an activity analysis technology. Furthermore, any decreasing-returns technology can be represented as a constant-returns technology with certain non-marketed factors of production. As we shall see, however, the differentiable nature of our approach makes an activity analysis approxiation to a smooth production technology unsuitable. An essential local property of an equilibrium price vector is the curvature of the dual cone at that point. An activity analysis approximation misses this property.

Using smooth profit functions to characterize production possibilities, we are able to extend the concepts of regularity and fixed point index to economies with very general technologies, involving both constant and decreasing returns. To prove the genericity of regular economies we rely on an approach taken by Mas-Colell (1978), which utilizes in an elegant and insightful manner the topological concept of transversality. We also generalize the index theorem given by Kehoe (1980a). Our results shed new light on the question of when an economy has a unique equilibrium. Providing a satisfactory answer to this question is crucial to the applicability of general equilibrium models in comparative statics excercises.

2. ECONOMIES WITH SMOOTH PROFIT FUNCTIONS

We initially deal with constant-returns production technologies. We later treat decreasing returns as a special case. The model that we employ is identical to the one used by Kehoe (1980a) except for its description of the production technology. The consumption side of the model is completely described by an aggregate excess demand function. In an economy with n commodities the excess demand function ξ is assumed to be completely arbitrary except for the following assumptions:

ASSUMPTION 1 (Differentiability): $\xi : R^n_+ \setminus \{0\} \to R^n$ is a continuously differentiable function, in other words, is C^1. Here $R^n_+ \setminus \{0\}$ is the set of all non-negative prices except the origin.

ASSUMPTION 2 (Homogeneity): ξ is homogeneous of degree zero; $\xi(t\pi) \equiv \xi(\pi)$ for all $t > 0$.

ASSUMPTION 3 (Walras's law): ξ obeys Walras's law; $\pi'\xi(\pi) \equiv 0$.

Kehoe (1979b) generalizes Assumption 1 to one that allows the norm of excess demand to become unbounded as some prices approach zero. For the sake of simplicity, however, we assume here that ξ is defined and continuous over all non-negative prices except the origin.

We use a dual approach to characterize the production side of the model. The technology is specified by m C^2 profit functions

$a_j : R^n_+ \backslash \{0\} \to R$, which can be regarded as a mapping from $R^n_+ \backslash \{0\}$ into R^m, $a(\pi) = (a_1(\pi), \ldots, a_m(\pi))$. To motivate this approach, let us consider

the problem of maximizing profits when production possibilities are specified directly by a production function, Suppose that a vector of feasible net-output combinations is one that satisfies the constraints

$$f(\pi) = 0$$

$$x_i \geq 0, \ i = 1, \ldots, h$$

$$x_i \geq 0, \ i = h+1, \ldots, n.$$

Here $f : R^n \to R$ is a constant-returns production function that produces the first h commodities as outputs employing the final n-h commodities as inputs. We assume that f is homogeneous of degree one and concave. Suppose that we attempt to find the vector that maximizes $\pi'x$ subject to the feasibility constraints where π is a fixed vector of non-negative prices. The problem that immediately arises is that, given the assumption of constant returns, profit is unbounded if there is some feasible vector x for which $\pi'x > 0$. There are several ways to get around this difficulty. For example, if h = 1, that is, if f produces a single output, we can impose the additional constraint $x_1 = 1$. Another, more general, solution to this problem is to impose the constraint $\| x \| = 1$.

Consider, for example, the Cobb-Douglas production function

$f(x_1, x_2, x_3) = (-x_2)^{\alpha} (-x_3)^{1-\alpha} - x_1$, $0 < \alpha < 1$. The problem to be solved is

$$\max \quad \pi_1 x_1 + \pi_2 x_2 + \pi_3 x_3$$

$$\text{s.t.} \quad (-x_2)^{\alpha} (-x_3)^{1-\alpha} - x_1 = 0$$

$$x_2, x_3 \leq 0$$

$$x_1 = 1.$$

At any price vector π the optimal net-output vector is easily calculated to be

$$x_1(\pi) = 1 \ , \ x_2(\pi) = -\left(\frac{\alpha\pi_3}{\pi_2 - \alpha\pi_2}\right)^{1-\alpha} \ , \quad x_3(\pi) = -\left(\frac{\pi_2 - \alpha\pi_2}{\alpha\pi_3}\right)^{\alpha} .$$

The profit function is defined by the rule $a(\pi) \equiv \pi' x(\pi)$; in this case

$$a(\pi) = \pi_1 - \left(\frac{\pi_2}{\alpha}\right)^{\alpha} \left(\frac{\pi_3}{1-\alpha}\right)^{1-\alpha} .$$

It is well known that $a(\pi)$ is homogeneous of degree one, convex, and continuous as long as the feasible set is non-empty, even when the optimal net-output vector is not single-valued. When a is differentiable Hotelling's lemma tells us how to recover the profit maximizing net-output vector for any vector of prices: We merely find the gradient vector Da_{π} and check that it equals $x(\pi)$ (see, for example, Diewert (1974). Given the constant-returns nature of the production technology we can consider this gradient vector as an activity analysis vector: Any non-negative scalar multiple of it is a feasible input-output combination.

Let us now consider again the general case $a : R^n_+ \backslash \{0\} \to R^m$.

The Jacobian matrix Da_π maps R^n into R^m. Define the mapping $A : R_+^n \setminus \{0\} \to R^{n \times m}$ by the rule $A(\pi) \equiv (Da_\pi)'$. $A(\pi)$ is a generalization of the concept of an activity analysis matrix. Indeed, in the situation where each a_j is the linear function $\Sigma_{i=1}^n a_{ij}\pi_i$ $A(\pi)$ is a matrix of constants. The set of feasible net-output vectors corresponding to $a(\pi)$ is the production cone $Y_a = \{x \in R^n \mid x = A(\pi)y$ for some $\pi \in R_+^n \setminus \{0\}$, $y \in R_+^m\}$. Observe that Y_a contains the origin, is convex, and is closed if a is C^1. We specify the production side of our model by imposing restrictions on the mapping a.

ASSUMPTION 4 (Differentiability): $a : R_+^n \setminus \{0\} \to R^m$ has continuous first and second order partial derivatives, in other words, is C^2.

ASSUMPTION 5 (Homogeneity): a is homogeneous of degree one; $a(t\pi) = ta(\pi)$ for any $t > 0$.

ASSUMPTION 6 (Convexity): Each function a_j is convex; $a_j(t\pi^1 + (1-t)\pi^2) \leq ta_j(\pi^1) + (1-t)a_j(\pi^2)$ for any $0 < t < 1$.

ASSUMPTION 7 (Free disposal): $A(\pi)$ always includes n free disposal activities, one for each commodity. Letting these activities be the first $n \leq m$, we set $a_j(\pi) = -\pi_j$, $j = 1, \ldots, n$.

ASSUMPTION 8 (Boundedness): There exists some $\pi > 0$ such that $a(\pi) \leq 0$.

The convexity of a implies that $a(\pi^1) - a(\pi^2) \geq Da_{\pi^2}(\pi^1 - \pi^2)$ for all $\pi^1, \pi^2 \in R^n_+ \setminus \{0\}$. Combining this observation with the homogeneity of a yields

$$a(\pi^1) - a(\pi^2) \geq Da_{\pi^2}\pi^1 - a(\pi^2)$$

$$Da_{\pi^1}\pi^1 \geq Da_{\pi^2}\pi^1$$

$$\pi^{1\prime}A(\pi^1) \geq \pi^{1\prime}A(\pi^2)$$

for all $\pi^1, \pi^2 \in R^n_+ \setminus \{0\}$. It is easy to use this result to demonstrate that Assumption 8 is equivalent to the assumption that there is no output possible without any inputs in the sense that $Y_a \cap R^n_+ = \{0\}$.

Notice that the activity analysis specification used by Mas-Colell and Kehoe is a special case of this type of technology. Assumptions 5-8 are quite natural; it is the differentiability part of Assumption 4 that is restrictive. It would be possible to impose conditions on production functions that would give rise to such smoothness in the corresponding profit functions and net-output functions. These conditions would be similar to those on a consumer's utility function that imply smoothness in the corresponding indirect utility function and individual demand function. Since we have chosen to specify the production side of the economy using profit functions rather than production functions, we shall not pursue this issue. A further restriction embodied in Assumption 4 is that we require the net-output functions to be continuous even on the boundary of R^n_+. Notice that in the example that we worked out this condition does not hold. We would have avoided this problem if we had imposed the constraint $\| x \| = 1$ rather than $x_1 = 1$. This is not an important conceptual issue, however. We shall ignore it.

An economy is specified as a pair (ξ, a) that satisfies Assumptions 1-3.

DEFINITION: An equilibrium of an economy (ξ, a) is a price vector $\hat{\pi}$ that satisfies the following conditions:

a. $a(\hat{\pi}) \leq 0$

b. There exists $\hat{y} \geq 0$ such that $\xi(\hat{\pi}) = A(\hat{\pi})\hat{y}$.

c. $\hat{\pi}'e = 1$ where $e = (1, \ldots, 1)$.

The condition $a(\hat{\pi}) \leq 0$ implies that at π no excess profits can be made. The second condition, when combined with Walras's law and the homogeneity of a, implies that $\hat{\pi}'\xi(\hat{\pi}) = \hat{\pi}'A(\hat{\pi})\hat{y} = a(\hat{\pi})'\hat{y} = 0$. Thus any activity actually in use at equilibrium earns zero profit. Since $\hat{\pi}'A(\pi) \leq \hat{\pi}'A(\hat{\pi})$ for all $\pi \in R^n_+ \backslash \{0\}$ this implies that the production plan $A(\hat{\pi})\hat{y}$ is profit maximizing at prices $\hat{\pi}$. We can justify the final condition as follows: Since a is homogeneous of degree one, A is homogeneous of degree zero. Consequently the homogeneity assumptions on ξ and a imply that, if $\hat{\pi}$ satisfies the first two equilibrium conditions, $t\hat{\pi}$ also does for any $t > 0$. Therefore, when examining equilibrium positions, we possess a degree of freedom that we use to impose the restriction $\hat{\pi}'e = 1$. The free disposal assumption allows us to restrict our attention even further to the unit simplex $S = \{\pi \in R^n \mid \pi \geq 0, \quad \pi'e = 1\}$.

When discussing the space of economies that satisfy Assumptions 1-8, we shall need to utilize some sort of topological structure. We now give the space of economies the structure of a metric space. Let $\mathcal{D}$ be the space of excess demand functions that satisfy Assumptions 1-3. We endow $\mathcal{D}$ with the topology of uniform C^1 convergence by defining the metric

$$d(\xi^1, \xi^2) = \sup_{i,\pi\in S} \left| \xi_i^1(\pi) - \xi_i^2(\pi) \right| + \sup_{i,j,\pi\in S} \left| \frac{\partial \xi_i^1}{\partial \pi_j}(\pi) - \frac{\partial \xi_i^2}{\partial \pi_j}(\pi) \right|$$

for any $\xi^1, \xi^2 \in \mathcal{D}$. Let $\mathcal{A}$ be the space of profit maps that satisfy Assumptions 4-8. Notice that the first n components of any such map are fixed by the free disposal assumption. We endow $\mathcal{A}$ with the topology of uniform C^2 convergence by defining the metric

$$d(a^1, a^2) = \sup_{i,\pi\in S} \left| a_i^1(\pi) - a_i^2(\pi) \right| + \sup_{i,j,\pi\in S} \left| \frac{\partial a_i^1}{\partial \pi_j}(\pi) - \frac{\partial a_i^2}{\partial \pi_j}(\pi) \right|$$

$$+ \sup_{i,j,k,\pi\in S} \left| \frac{\partial^2 a_i^1}{\partial \pi_j \partial \pi_k}(\pi) - \frac{\partial^2 a_i^2}{\partial \pi_j \partial \pi_k}(\pi) \right|$$

for any $a^1, a^2 \in \mathcal{A}$. The space of economics $\mathcal{E} = \mathcal{D} x \mathcal{A}$ receives the product topology induced by the metric $d[(\xi^1, a^1), (\xi^2, a^2)] = d(\xi^1, \xi^2) + d(a^1, a^2)$.

The proof of the existence of an equilibrium follows the same lines as that in Kehoe (1980a). We begin with a few preliminary definitions.

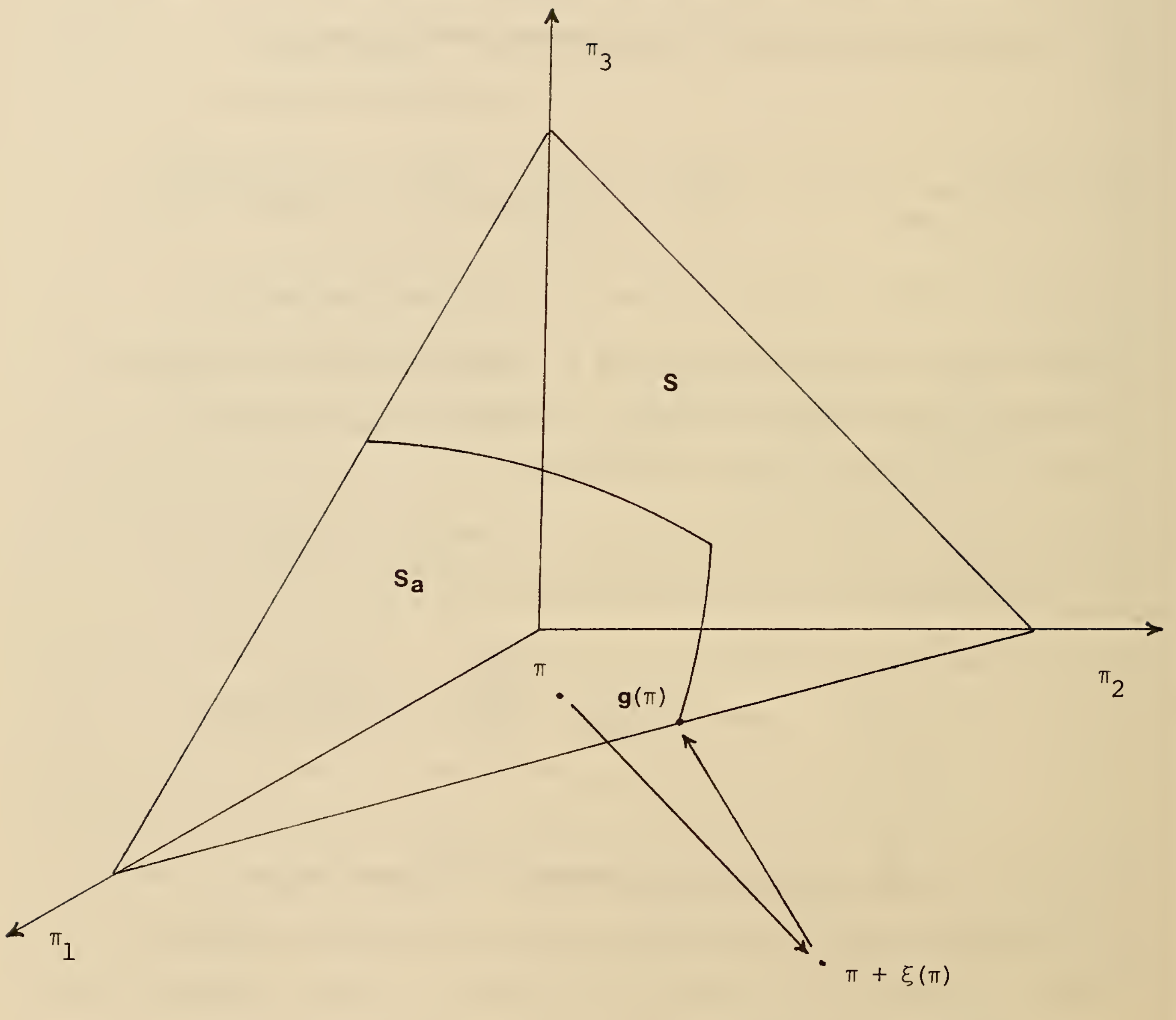

Figure 1

Letting N be any non-empty, closed, convex subset of R^n, we define the continuous projection $p^N : R^n \rightarrow N$ by the rule that associates any point $q \in R^n$ with the point $p^N(q)$ that is closest to q in terms of Euclidean distance. We also define the set $S_a = \{\pi \in R^n \mid a(\pi) \leq 0, \pi'e = 1\}$. The boundedness of production implies that S_a is non-empty. The convexity of a implies that it is convex . It is compact by definition and the continuity of a. We define the map $g : S \rightarrow S$ by the rule $g(\pi) = p^{S_a}(\pi + \xi(\pi))$.

Figure 1

THEOREM 1: Fixed points $\hat{\pi} = g(\hat{\pi})$ of the map g and equilibria of (ξ, a) are equivalent.

PROOF: At any point $\pi \in S$, $g = g(\pi)$ can be computed by solving the quadratic programming problem

$$\min \ 1/2 \ (g - \pi - \xi(\pi))'(g - \pi - \xi(\pi))$$

$$\text{s.t.} \quad a(g) \leq 0$$

$$g'e = 1 \ .$$

The Kuhn-Tucker theorem implies that the vector g solves this problem if and only if there exists $y \in R^m_+$ and $\lambda \in R$ such that

$$g - \pi - \xi(\pi) + A(g)y + \lambda e = 0.$$

and $a(g)'y = 0$. Consequently, $\hat{\pi} = g(\hat{\pi})$ is a fixed point if and only if $-\xi(\hat{\pi}) + A(\hat{\pi})\hat{y} + \hat{\lambda}e = 0$. Walras's law implies that this relationship holds if and only if $\hat{\lambda} = 0$. Therefore, $\xi(\hat{\pi}) = A(\hat{\pi})\hat{y}$ and $\hat{\pi} \in S_a$ is equivalent to $\hat{\pi} = g(\hat{\pi})$. □

Since S is compact and convex and g continuous, Brouwer's fixed point theorem implies the existence of an equilibrium of (ξ, a).

3. REGULAR PRODUCTION ECONOMIES

We have just resolved a fundamental issue concerning the applicablity of our model in comparative statics exercises. We now know that the model is internally consistent in the sense that an equilibrium always exists. Even when an equilibrium exists, however, there are circumstances in which the comparative statics method makes little sense. A relevant concern is whether there is a unique equilibrium consistent with a given vector of parameters. If a solution is not unique, the question becomes whether it is locally unique. Finally, when comparing the equilibria associated with different parameter values, the question that naturally arises is whether the equilibrium varies continuously with the parameters.

The concept of regularity provides us with answers to the questions of local uniqueness and continuity of equilibria. In the subsequent discussion we focus our attention on the partial derivatives of g at its fixed points. For proofs and more detailed discussion of many of the results presented here we refer to Kehoe (1980a). To make matters simple, we define X as a smooth (that is, C^1) n dimensional manifold with boundary that is a compact, convex subset of R^n chosen so that it contains S in its interior and does not contain the origin. It is easy to smoothly extend the domain of ξ to X. We are justifed, therefore, in viewing X as the domain of g.

Unfortunately, g is not everywhere differentiable except in very special cases. All we really need is that if is differentiable at its fixed points. To ensure that this requirement holds we need to impose two additional restrictions on (ξ, a). Consider the mapping $b : R^n_+ \setminus \{0\} \to R^k$, $0 \leq k \leq m$, made up of k of the profit functions $(a_1(\pi), \ldots, a_m(\pi))$. Let $B : R^n_+ \setminus \{0\} \to R^{n \times k}$ be the corresponding matrix function whose columns are the gradients of the individual profit functions b_j, $j = 1, \ldots, k$. In the case where $k = 0$ the images of both b and B are empty.

ASSUMPTION 9: At any point $\pi \in S$ the profit functions b that satisfy $b(\pi) = 0$ are such that the columns of $B(\pi)$ are linearly independent.

ASSUMPTION 10: Suppose that b is the vector of profit functions that earn zero profit at some equilibrium $\hat{\pi}$. Then the vector $\hat{y} \in R^k_+$ is strictly positive in the equation $\xi(\hat{\pi}) = B(\hat{\pi})\hat{y}$.

We later justify these assumptions on the grounds that they hold for any open dense subset of economies in $\mathcal{E}$. Actually, Assumption 9 is stronger than needed. What we require for g to be differentiable at its fixed points is that the matrix of activities in use at every equilibrium has linearly independent columns. Assumption 9 implies that this condition holds but has the advantage of being easier to deal with in genericity arguments. Assumption 10 rules out the possibility of an activity earning zero profit but not being used at equilibrium.

Suppose that $\hat{\pi}$ is an equilibrium of (ξ, a). Let $C = [B(\hat{\pi})\ e]$ where $B(\hat{\pi})$ is the matrix of activities that are used at equilibrium. Further let H be the nxn matrix formed by taking the Hessian matrices of the k profit functions b evaluated at π, multiplying them by the corresponding activity levels, then adding them together. In the next section we prove the following theorem.

THEOREM 2: If an economy $(\xi, a) \in \mathcal{E}$ satisfies Assumptions 9 and 10, then is differentiable in some open neighborhood of every fixed point $\hat{\pi}$. Moreover, $Dg_{\hat{\pi}} = (I + (I-C(C'C)^{-1}C')H)^{-1}(I-C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}})$.

Notice that in the activity analysis case every element of H is zero and, consequently, $Dg_{\hat{\pi}} = (I-C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}})$.

Let us consider a subset of economics that satisfy Assumptions 9 and 10 and the further restriction that 0 is a regular value of $(g - I)$: $X \to R^n$. Here, of course, I is the identity mapping. Recall that a point $x \in M$ is a regular point of a C^1 map $f: M \to N$ from a smooth manifold of dimension m to a smooth manifold of dimension n if $Df_x : T_x(M) \to T_{f(x)}(N)$ has rank n; in other words, is onto. A point

$y \varepsilon N$ is a regular value if every point x for which $f(x) = y$ is a regular point. By convention, any point y for which the set $f^{-1}(y)$ is empty is a regular value. Points in M that are not regular points are critical points. Points in N that are not regular values are critical values.

DEFINITION: An economy $(\xi, a) \varepsilon \mathcal{E}$ that satisfies Assumptions 9 and 10 and is such that $Dg_{\pi} - I$ is non-singular at every equilibrium is a <u>regular economy</u>. The set of regular economics is denoted $\mathcal{R}$.

Regular economies possess many desirable properties. For example, the inverse function theorem applied to g - I at every equilibrium $\hat{\pi}$ implies that the equilibria of a regular economy are isolated. Since the set of equilibria lie in the compact set S , this implies that a regular economy has a finite number of equilibria. Consider the equilibrium price correspondence $\Pi: \mathcal{E} \rightarrow S$ that associates any economy with the set of its equilibria. The topology on $\mathcal{E}$ is fine enough to imply that Π is an upper-semi-continuous correspondence. On $\mathcal{R}$, moreover, Π is continuous and the number of equilibria is locally constant.

4. PROOF OF THEOREM 2

Demonstrating that the two non-degeneracy assumptions imply that g is continuously differentiable in some open neighborhood of every fixed point is a straightforward, if somewhat laborious, application of the implicit function theorem. It is, nevertheless, a worthwhile exercise to work through because the expression that we derive for Dg_{π} plays a central role in the subsequent analysis. Readers willing to accept Theorem 2 on faith can skip this section.

The function g is the composition of the functions p^{S_a} and $I + \xi$. It is p^{S_a} that we must prove is differentiable. At any $q \in R^n$ $p^{S_a}(q)$ is calculated by solving the problem

$$\min\ 1/2\ (p - q)'(p - q)$$

$$\text{s.t.} \quad a(p) \leq 0$$

$$p'e = 1 \, .$$

Assumption 10 implies that if $\hat{\pi}$ is an equilibrium, then in some open neighborhood of $\hat{q} = \hat{\pi} + \xi(\hat{\pi})$, $p^{S_a}(\hat{q})$ can be calculated by solving the same problem where the m constraints are replaced by a subset of constraints, $b(p) \leq 0$, that all hold with equality. Let $c : R^n_+ \setminus \{0\} \rightarrow R^{k+1}$ be defined by the rule $c(p) = \begin{bmatrix} b(p) \\ p'e \end{bmatrix}$. Notice that c is homogeneous of degree one and C^2. Furthermore, each of its components is a convex function. Let $d \in R^{k+1}$ be a vector each of whose first k elements is zero and whose last is unity. Also let $C(p) = [B(p)\ e]$. The Kuhn-Tucker theorem implies that, in some open neighborhood of $\hat{q}$, $p = p^{S_a}(q)$ satisfies

$$p - q + C(p)\lambda = 0$$

for some $\lambda \in R^{k+1}$ whose first k elements are positive. Assumption 9 and the continuity of p^{S_a} imply that this neighborhood can be chosen so that the columns of C(p) are linearly independent at every $p = p^{S_a}(q)$. We can use this observation and the homogeneity of c, which implies that $C'(p)p \equiv c(p)$, to solve for λ.

$$C'(p)p - C'(p)q + C'(p)C(p)\lambda = 0$$

$$d - C'(p)q + C'(p)\ C(p)\lambda = 0$$

$$\lambda = (C'(p)C(p))^{-1}(C'(p)q - d).$$

The conditions that determine p^{S_a} can therefore be rewritten as

$$p - q + C(p)(C'(p)C(p))^{-1}(C'(p)q-d) = 0.$$

If matrix of partial derivatives with respect to p of the left-hand side of this expression is non-singular then the implicit function theorem implies that p^{S_a} is locally C^1.

Differentiating the matrix function C(p) with respect to the vector p is a complex procedure. We can simplify it by differentiating with respect to the scalars p_j, j=1,...,n, and then stacking the results. See any advanced econometrics text for the matrix differentiation techniques that are used.

$$D[C(C'C)^{-1}(C'q-d)]_{p_j} = DC_{p_j}(C'C)^{-1}(C'q-d)$$

$$- C(C'C)^{-1}(DC'_{p_j}C+C'DC_{p_j})(C'C)^{-1}(C'q-d)$$

$$+ C(C'C)^{-1}DC'_{p_j}q$$

$$= (I-C(C'C)^{-1}C')\ DC_{p_j}(C'C)^{-1}(C'q-d)$$

$$+ C(C'C)^{-1}DC'_{p_j}(q-C(C'C)^{-1}(C'q-d)).$$

notice that $(C'C)^{-1}(C'q - d) = \lambda$ and $q - C(C'C)^{-1}(C'q - d) = p$. Our results can therefore be rewritten as

$$D[C(C'C)^{-1}(C'q - d)]_{p_j} = (I - C(C'C)^{-1}C')DC_{p_j}\lambda + C(C'C)^{-1}DC'_{p_j}p.$$

If we carefully differentiate the identity $C'(p)p \equiv c(p)$ with respect to p_j, we find that $DC'_{p_j}p \equiv 0$. Let us now stack the columns

$D[C(C'C)^{-1}(C'q - d)]_{p_j}$, $j = 1,\ldots,n$, to obtain the expression

$$D[C(C'C)^{-1}(C'q - d)]_p = (I - C(C'C)^{-1}C')[DC_{p_1}\lambda \ \ldots \ DC_{p_n}\lambda].$$

Let the nxn matrix H be formed by taking the Hessian matrices of each of the k+1 component functions of $c(p)$ evaluated at $p^{S_a}(q)$, multiplying them by the corresponding Lagrange multipliers λ_i, $i = 1,\ldots,k+1$, and adding them together. Observe that, since the Hessian matrix of $p'e$ is identically zero and the first k multipliers λ_i are strictly positive, the convexity of each $c_i(p)$ implies that H is a symmetric positive semi-definite matrix. We demonstrated in the proof of Theorem 1 that, if $p^{S_a}(q)$ is an equilibrium, then these Lagrange multipliers are the activty levels that correspond to activities $B(\hat{\pi})$. The above expression can be rewritten as

$$D[C(C'C)^{-1}(C'q - d)]_p = (I - C(C'C)^{-1}C')H.$$

Remember that what we want to do is to demonstrate that $D[p - q + C(C'C)^{-1}(C'q - d)]_p = I + (I - C(C'C)^{-1}C')H$ is non-singular.

LEMMA 1: Let C be any nx(k+1) matrix of full column rank and H be any nxn positive semi-definite matrix. Then $\det[I + (I - C(C'C)^{-1}C')H] > 0$.

PROOF: Consider the matrix $I + (I - C(C'C)^{-1}C')H(I - C(C'C)^{-1}C')$. Since it is the sum of a positive definite matrix and a positive semi-definite matrix, it is positive definite. Hence it has a positive determinant. We prove our contention by using elementary row and column operations to reduce the determinant $I + (I - C(C'C)^{-1}C')H$ to that of the (n+k+1)x(n+k+1) matrix

$$\begin{bmatrix} I + (I - C(C'C)^{-1}C')H(I - C(C'C)^{-1}C') & (I - C(C'C)^{-1}C')HC \\ 0 & I \end{bmatrix}$$

Adding the second column of this matrix post-multiplied by $(C'C)^{-1}C'$ to the first, we obtain

$$\begin{bmatrix} I + (I - C(C'C)^{-1}C')H & (I - C(C'C)^{-1}C')HC \\ (C'C)^{-1}C' & I \end{bmatrix}.$$

Now adding the first row of this matrix pre-multiplied by $(C'C)^{-1}C$ to the second, we obtain

$$\begin{bmatrix} I + (I - C(C'C)^{-1}C')H & (I - C(C'C)^{-1}C')HC \\ 0 & I \end{bmatrix},$$

which has the same determinant as $I + (I - C(C'C)^{-1}C')H$. □

This lemma and the implicit function theorem imply that p^{S_a} is continuously differentiable in some open neighborhood of $\hat{\pi} + \xi(\hat{\pi})$ if $\hat{\pi}$ is an equilibrium. $I + \xi$ is, of course continuously differentiable. We use the chain rule to establish that

$$Dg_{\pi} = (I + (I - C(C'C)^{-1}C')H)^{-1}(I - C(C'C)^{-1}C')(I + D\xi_{\pi})$$

in some open neighborhood of every fixed point.

5. THE INDEX THEOREM

The concept of regularity tells us when an equilibrium is locally unique and varies continuously with the parameters of the economy. We are still left with the question of when an economy has a unique equilibrium. The version of the Lefschetz fixed point theorem given by Saigal and Simon (1973) provides us with a tool for counting the equilibria of (ξ, a). If (ξ, a) is a regular economy, then the local Lefschetz number of any fixed point of g can be calculated as $L_{\hat{\pi}}(g) = \text{sgn}(\det[Dg_{\hat{\pi}} - I])$. Saigal and Simon prove that $\sum_{\pi = g(\pi)} L_{\pi}(g) = (-1)^n$. A regular economy therefore has an odd number of equilibria. Furthermore, a necessary and sufficient condition for a regular economy to have a unique equilibrium is that $L_{\hat{\pi}}(g) = (-1)^n$ at every equilibrium.

To make much economic sense of this result we need to develop alternative expressions for $\text{sgn}(\det[Dg_{\hat{\pi}} - I])$. According to Theorem 2,

$$Dg_{\hat{\pi}} - I = (I + (I - C(C'C)^{-1}C')H)^{-1}(I - C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}}) - I.$$

Lemma 1 implies that we can pre-multiply this matrix by $(I + (I - C(C'C)^{-1}C')H)$ without changing the sign of its determinant.

$$\begin{aligned} \text{sgn}(\det[Dg_{\hat{\pi}} - I]) &= \text{sgn}(\det[(I - C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}}) \\ &\qquad - (I + (I - C(C'C)^{-1}C')H)]) \\ &= \text{sgn}(\det[(I - C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}} - H) - I]). \end{aligned}$$

Using elementary row operations, we can easily transform this expression into

$$\text{sgn}(\det[Dg_{\hat{\pi}} - I]) = \text{sgn}\left(\det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} - H(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix}\right).$$

(See Lemma 4 of Kehoe (1980a).)

DEFINITION: If $\hat{\pi}$ is an equilibrium of a regular economy (ξ, a), then index($\hat{\pi}$) is defined as

$$(-1)^n \operatorname{sgn}(\det[Dg_{\hat{\pi}} - I]) = (-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} - H(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix}\right) .$$

The following theorem is an immediate consequence of the Lefschetz fixed point theorem and the definition of index$(\hat{\pi})$.

THEOREM 3: Suppose that (ξ, a) is a regular economy. Then

$\sum_{\pi \in \Pi(\xi,a)} \text{index}(\pi) = +1.$

Notice that the existence of an equilibrium for (ξ, a) follows directly from this theorem.

In applications of this theorem other expressions for index(π) are useful. They can be derived using simple matrix manipulations. Let the matrix $\bar{J}$ be formed by deleting from $D\xi_{\hat{\pi}}$ all rows and columns corresponding to commodities with zero prices at equilibrium $\hat{\pi}$ and then deleting one more row and corresponding column. Let $\bar{H}$ be formed similarly by deleting the same rows and columns from $H(\hat{\pi})$. Let $\bar{B}$ be formed by deleting the same rows from $B(\hat{\pi})$ as well as all columns corresponding to disposal activities. It is easy to verify that

$$\text{index}(\hat{\pi}) = \operatorname{sgn}\left(\det\begin{bmatrix} -\bar{J} + \bar{H} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix}\right) .$$

If only one commodity has positive price at $\hat{\pi}$, then index$(\hat{\pi}) = +1$.

Another formula for index$(\hat{\pi})$ can be expressed as follows: Choose an $n \times (n-k)$ matrix V whose columns span the null space of the columns of $B(\hat{\pi})$.

Let E be the nxn matrix whose every element is unity. Then it is possible to demonstrate that

$$\text{index}(\hat{\pi}) = \text{sgn}(\det[V'(E + H(\hat{\pi}) - D\xi_{\hat{\pi}})V]).$$

For the derivations of these, and other, formulas for index($\hat{\pi}$) see Kehoe (1979a).

6. A TRANSVERSALITY APPROACH TO REGULARITY

Mas-Colell (1978) has developed an interpretation of the definition of regular economy based on the concept of transversality. Using his interpretation, we can reduce the demonstration that regularity is a generic property of the space of economics to an argument dealing with transversal intersections of manifolds. In this section we translate the regularity conditions into transversality conditions. In the next section we prove that $\mathcal{R}$ is an open dense subset of $\mathcal{E}$. References for the technical concepts employed here are the books on differential topology by Guillemin and Pollack (1974) and Hirsch (1976).

DEFINITION: Let $f: W \to Y$ be a C^1 map of smooth manifolds, and let Z be a smooth submanifold of Y. The map f is <u>transversal</u> to Z, denoted $f \pitchfork Z$ if, for every point $x \in f^{-1}(Z)$, $\text{Image}(Df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$.

An important special case of this concept is where W is a submanifold of Y and f is the inclusion map. The derivative map

$Df_x : T_x(W) \to T_x(Y)$ is then the inclusion map of $T_x(W)$ into $T_x(Y)$. In this case, if f is transversal to Z, we say W and Z are transversal, $W \pitchfork Z$, which means $T_x(W) + T_x(Z) = T_x(Y)$ for all $x \in W \cap Z$.

Figure 2

Consider, for example, the situations depicted in Figure 2. Here $Y = R^2$. The manifolds W and Z intersect transversally in (a); they do not in (b). Notice, however, that if $Y = R^3$ then W and Z are not transversal even in (a). In fact, if the dimensions of W and Z do not add up to at least the dimension of Y, then $W \pitchfork Z$ only if $W \cap Z = \emptyset$. The intuitive appeal of the concept of transversality is that transversal intersections are stable against small perturbations while non-transversal intersections are not.

Our goal is to rephrase the definition of regularity in terms of transversality. A few preliminaries are necessary before we start. Recall that, if $f : R^n \to R^k$ is a C^1 map and $y \in R^k$ is a regular value of f, then $f^{-1}(y)$ is a smooth n-k dimensional manifold. Let Y denote this manifold, $\{x \in R^n \mid f(x) = y\}$; notice that it can be non-empty only if $k \leq n$. Suppose that $x \in Y$ and that $\phi : U \to V$ is a local parametrization of some relatively open subset $V \subset Y$ containing x; here U is an open subset of R^{n-k}. Then $f(\phi(u)) \equiv y$ for all $u \in U$.

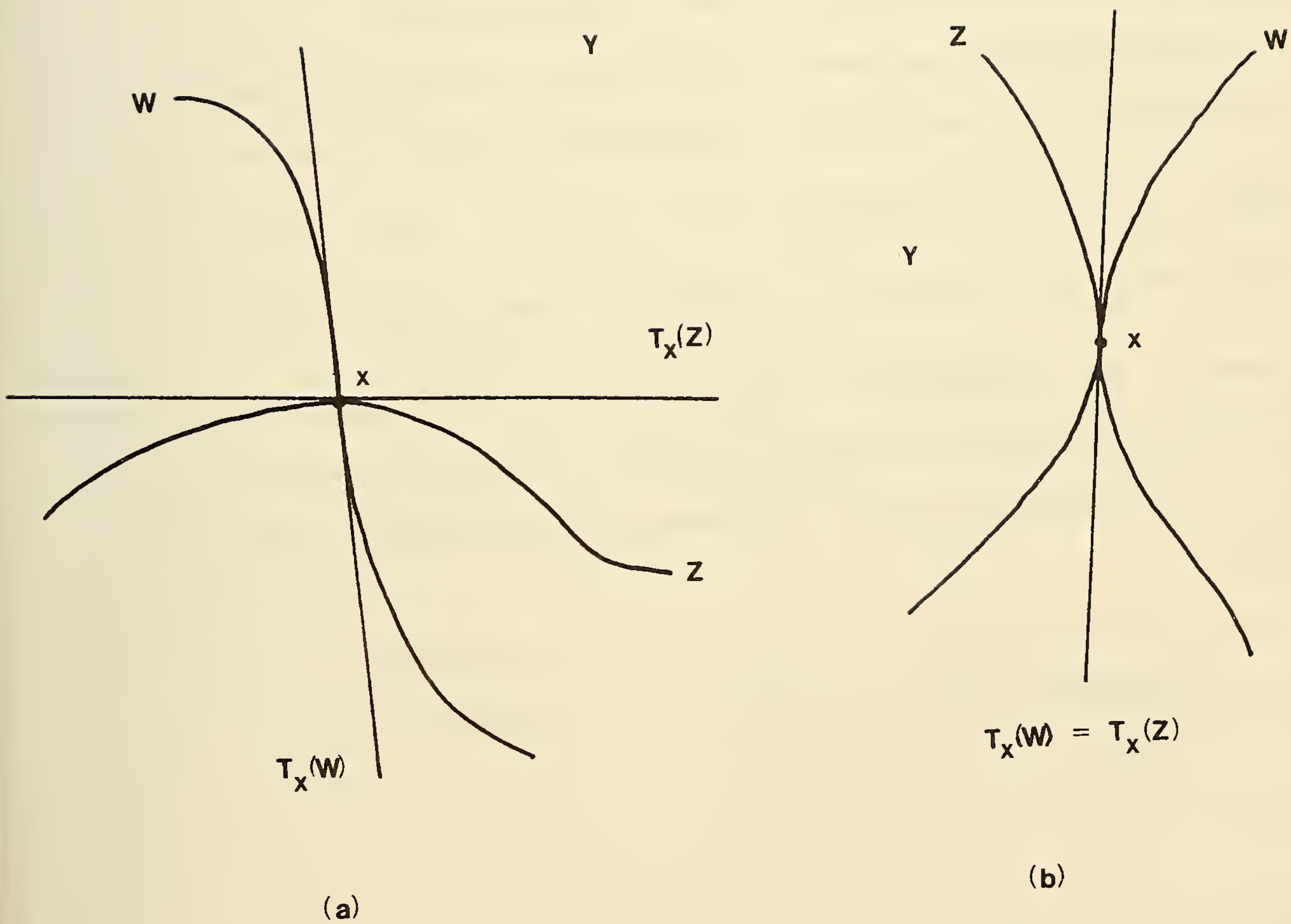

Figure 2

Differentiating this identity, we obtain $Df_x D\phi_u \equiv 0$. Recall that the tangent space $T_x(Y)$ is just the image of the linear map $D\phi_u : R^{n-k} \to R^n$; in other words, it is the n-k dimensional subspace of R^n $\{z \in R^n \mid z = D\phi_u v,\ v \in R^{n-k}\}$. Consider the set $\{z \in R^n \mid Df_x z = 0\}$; our arguments imply that this set contains $T_x(Y)$. In fact it is equal to $T_x(Y)$ since Df_x has rank k if x is a regular value. We shall use this observation to calculate explicit representations for tangent spaces that do not depend on local parametrizations.

In the subsequent discussion we shall find it convenient to change our normalization rule for prices. By Assumptions 2 and 5 the vector $\hat{p} = \frac{1}{\|\hat{\pi}\|}\hat{\pi}$ satisfies the conditions $a(\hat{p}) \leq 0$ and $\xi(\hat{p}) = A(\hat{p})\hat{y}$ for some $\hat{y} \in R^m_+$ if and only if the vector $\hat{\pi} = \frac{1}{e'\hat{p}}\hat{p}$ is an equilibrium of (ξ, a). Furthermore, $D\xi_{\hat{p}} = \|\hat{\pi}\| D\xi_{\hat{\pi}}$, $A(\hat{p}) = A(\hat{\pi})$, and $H(\hat{p}) = \|\hat{\pi}\| H(\hat{\pi})$. In this section and the next we normalize our equilibrium price vectors so that $\|\hat{p}\| = 1$. The motivation for this change will become obvious as we proceed.

Let $\sum$ be the unit sphere $\{p \in R^n \mid \|p\| = 1\}$. We define the set X so that it now contains $\sum \cap R^n$ in its interior, but otherwise has the

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Figure 3

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same properties as before. We shall do most of out work on the insersection of $\sum$ and the interior of X, intX. We denote this set P; it

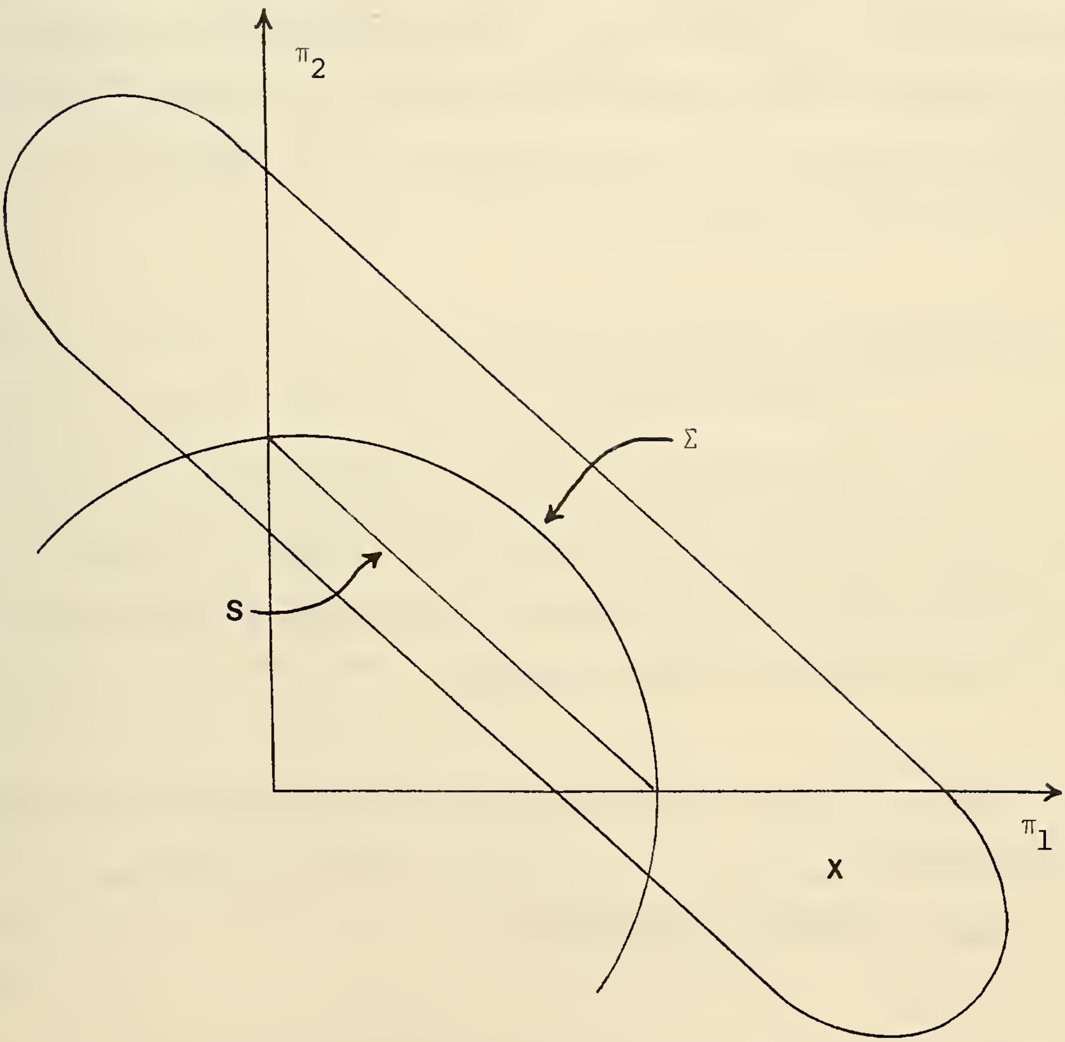

Figure 3

is an n-1 dimensional smooth manifold with boundary. To avoid problems with the boundary of R^n, we extend both ξ and A to C^1 maps on X. See Kehoe (1980a) for details. Define $a(p) = A'(p)p$ at points not in R^n_+.

The requirement that (ξ, a) is a regular economy involves three conditions: It must satisfy Assumption 9; it must satisfy Assumption 10; and it must satisfy the condition that a certain determinant is non-zero at every equilibrium. Let us first turn our attention to Assumption 9. Define M to be the power set of the integers $\{1,\ldots,m\}$. For every $M_j \in M$, $j = 1,\ldots,2^m$, define the set

$$Q(M_j) = \{x \in R^m \mid x_i = 0 \text{ if } i \in M_j\}.$$

In other words, $Q(M_j)$ is just the coordinate subspace of R^m on which the coordinates whose indices are elements of M_j are zero. If M_j has k elements, then $Q(M_j)$ is an m-k dimensional linear subspace and, quite naturally, an m-k dimensional smooth manifold.

LEMMA 2: Consider a vector of profit functions $a \in \mathcal{A}$ extended to a C^1 map from intX into R^m. Suppose that, for every $M_j \in M$, a is transversal to $Q(M_j)$. Then a satisfies Assumption 9.

PROOF: The proof is an immediate application of the definition of transversality. Suppose that $x \in Q(M_j)$. Then, if $\pi \in a^{-1}(x)$,

$$\text{Image}(Da_\pi) + T_x(Q(M_j)) = T_x(R^m).$$

Notice that $T_x(Q(M_j)) = Q(M_j)$ and $T_x(R^m) = R^m$. Therefore, since $a \pitchfork Q(M_j)$, Image(Da_π) must make up for a k dimensional linear subspace of R^m left over by $Q(M_j)$. Consequently, if $b: \text{int}X \to R^k$ is made up of the coordinate functions of a corresponding to elements of M_j, Image(Db_π) must span R^k. This is precisely the requirement that $B(\pi)$ has full column rank. □

Observe that, since Db_π is kxn and $Db_\pi \pi = 0$ if $b(\pi) = 0$, the rank of Db_π can be no greater than n-1. If the number of elements in M_j is greater than n-1, then the only way that it is possible for $a \pitchfork Q(M_j)$ is for $a^{-1}(Q(M_j))$ to be empty. In other words, there can never be more than n-1 profit functions that earn zero profit at some price vector if the transversality condition is met. We are justified, therefore, in considering only cases where $k \leq n-1$.

An alternative way of looking at Assumption 9 is that it is the requirement that 0 is a regular value of $b : \text{int}X \to R^k$. If Assumption 9 is satisfied, then $b^{-1}(0)$ is a smooth manifold of dimension n-k. Let us define

$$K_B = \{p \in \text{int}X \mid p'B(p) = 0,\ p'p = 1\}$$

By Assumptions 5 and 9 K_B is a smooth n-k-1 dimensional manifold. At any point $p \in K_B$

$$T_p(K_B) = \{v \in R^n | v'B(p) = 0,\ p'v = 0\}$$

Figure 4

Let B be an nxk matrix of net-output functions and let B* be an nxk*, $k \leq k^* \leq n-1$, matrix that includes B as a submatrix. Define the sets

$$T(P) = \{(p, x) \in P \times R^n | p'x = 0\}$$

$$\text{graph}(\xi) = \{(p, x) \in P \times R^n | x = \xi(p)\}$$

$$L(B^*, B) = \{(p, x) \in P \times R^n | p'B^*(p) = 0,\ x = B(p)y \text{ for some } y \in R^k\}.$$

T(P) is a smooth 2(n-1) dimensional manifold called the tangent bundle of P. Walras's law implies that graph(ξ) is a smooth n-1 dimensional submanifold of T(P). At any (p, x) $\in$ graph(ξ)

$$T_{(p,x)}(\text{graph}(\xi)) = \{(v, u) \in R^n \times R^n | p'v = 0,\ u = D\xi_p v\}.$$

If a $\in \mathcal{A}$ satisfies Assumption 9, then L(B*, B) is a smooth (n - 1) - (k* - k) dimensional manifold for any submatrices B and B* of the matrix of net-output functions A. It too is a submanifold of T(P). Notice that, if $\hat{p}$ is an equilibrium of (ξ, a), then ($\hat{p}$, $\xi(\hat{p})$) $\in$ graph(ξ)$\cap$L(B*, B) where B* is the matrix of activities that earn zero profit at $\hat{p}$ and B is the matrix of activities that are actually in use.

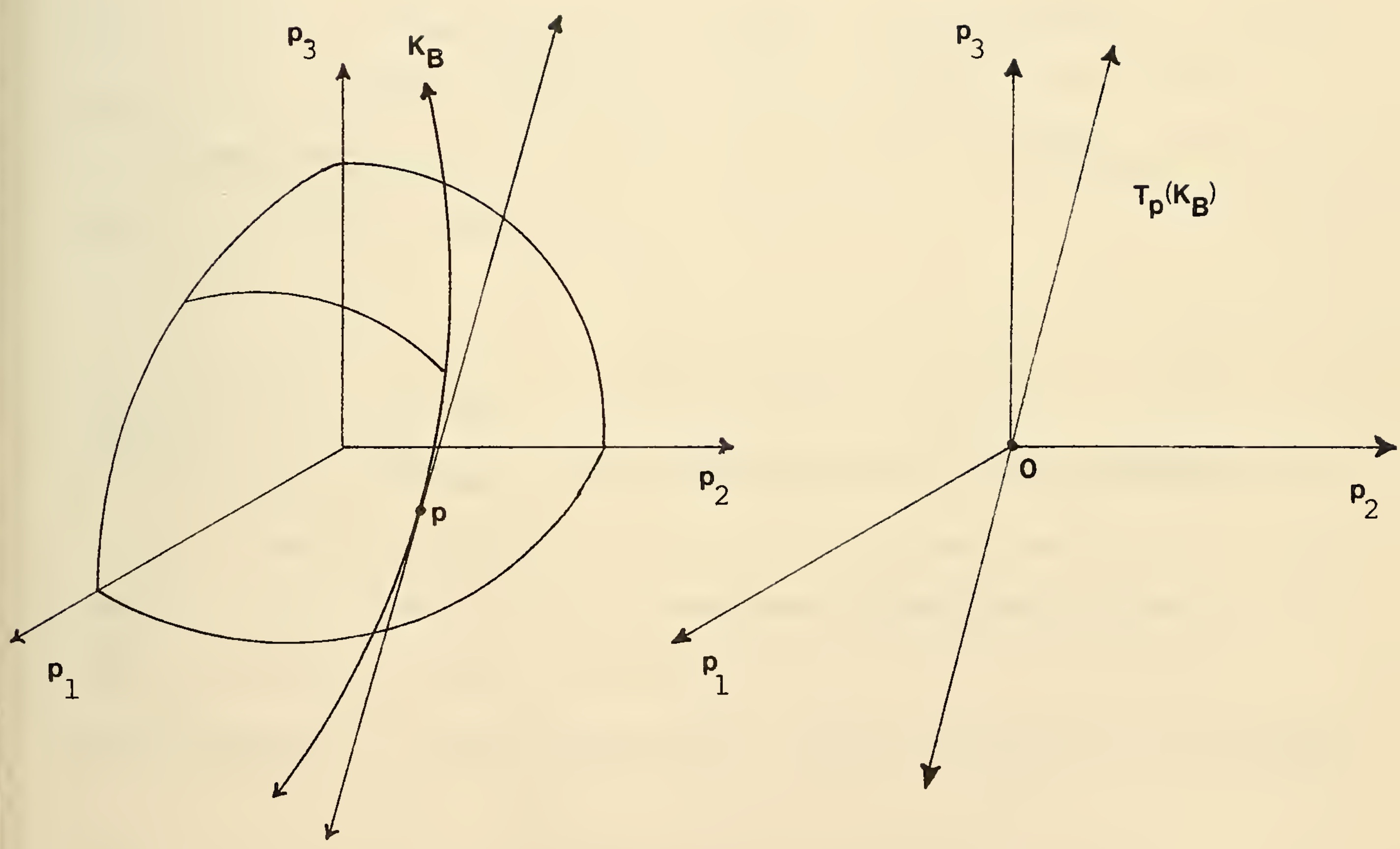

Figure 4

We shall need to have an expression for the tangent space to $L(B^*, B)$ at an equilibrium pair $(\hat{p}, \xi(\hat{p}))$. Any vector $(v, u) \varepsilon T_{(p,x)}(L(B^*, B))$ must clearly satisfy the restriction that $v \varepsilon T_p(K_{B^*})$. The restrictions on u are less obvious. The requirement that $x = B(p)y$ for some $y \varepsilon R^k$ can be rewritten

$$(I - B(p)(B'(p)B(p))^{-1}B'(p))x = 0$$

where $y = (B'(p)B(p))^{-1}B'(p)x$. Define the function $h(p, x) = (I - B(p)(B'(p)B(p))^{-1}B'(p))x$. We have argued that any vector $(u, v) \varepsilon T_{(p,x)}(L(B^*, B))$ must satisfy $Dh_p(p, x)v + Dh_x(p, x)u = 0$. Notice that

$$Dh_x(p, x) = (I - B(p)(B'(p)B(p))^{-1}B'(p)).$$

To calculate $Dh_p(p, x)$ we can differentiate with respect to the scalars p_j, $j = 1,\ldots,n$, and then stack the results. Actually, we only shall need to know what $Dh_{\hat{p}}(\hat{p}, \xi(\hat{p}))$ looks like.

$$Dh_{\hat{p}_j}(\hat{p}, \xi(\hat{p})) = -DB_{\hat{p}_j}(B'B)^{-1}B'\xi(\hat{p}) + B(B'B)^{-1}(DB'_{\hat{p}_j}B+B'DB_{p_j})(B'B)^{-1}B'\xi(\hat{p})$$

$$-B(B'B)^{-1}DB'_{\hat{p}_j}\xi(\hat{p})$$

$$= -(I + B(B'B)^{-1}B')DB_{\hat{p}_j}(B'B)^{-1}B'\xi(\hat{p})$$

$$-B(B'B)DB'_{\hat{p}}\ (I-B(B'B)B')\xi(\hat{p})$$

(Here, to simplify notation, we use B to refer to both the function and its value at $\hat{p}$.) Notice that $(I - B(B'B)^{-1}B')\xi(\hat{p}) = 0$ since $\xi(\hat{p}) = B(\hat{p})\hat{y}$. Consequently,

$$Dh_{\hat{p}}(\hat{p},\ \xi(\hat{p})) = -(I - B(\hat{p})(B'(\hat{p})B(\hat{p}))^{-1}B'(\hat{p}))H(\hat{p})$$

where, as before, the matrix $H(\hat{p})$ is constructed by multiplying the Hessian matrices of the k components of b by the k weights $\hat{y} = (B'(\hat{p})B(\hat{p}))^{-1}B'(\hat{p})\xi(\hat{p})$ and then summing them up. These arguments and a simple counting of dimensions imply that

$$T_{(\hat{p},\xi(\hat{p}))}(L(B^*,\ B)) = \Big\{(v,\ u)\in R \times R \ \Big|\ v'B^*(\hat{p}) = 0,\ v'\hat{p} = 0,$$
$$(I - B(\hat{p})(B'(\hat{p})B(\hat{p}))^{-1}B'(\hat{p}))(u - H(\hat{p})v)=0\Big\}$$

LEMMA 3: Suppose that $(\xi,\ a)$ satisfies Assumption 9 and that, for all possible combinations B and B*, $\text{graph}(\xi) \pitchfork L(B^*,\ B)$. Then $(\xi,\ a)$ satisfies Assumption 10.

PROOF: If $(\xi,\ a)$ violates Assumption 10, then the matrix of activities that earn zero profit at some equilibrium $\hat{p}$, $B^*(\hat{p})$, has more columns than the matrix in use, $B(\hat{p})$. For this particular pair of matrix functions B* and B

$$\dim L(B^*, B) = n - 1 - (k^* - k) < n - 1.$$

If, however, $L(B^*, B)$ and graph(ξ) are transversal, then $\dim L(B^*, B)$ + $\dim$ graph(ξ) $\geq \dim T(P)$, which implies $\dim L(B^*, B) \geq n - 1$.

□

Let B be an nxk matrix of net-output functions. Consider the function $f^B : \text{int}X \to R^n$ defined by the rule

$$f^B(p) = (I - B(p)(B'(p)B(p))^{-1} B'(p))\xi(p).$$

The advantage of our normalization $\|p\| = 1$ is that f^B is a tangent vector field on K_B since $p'f^B(p) = 0$ for all $p \in K_B$. In other words, $f^B(p) \in T_p(K_B)$ for all $p \in K_B$. Another advantage is that, if $\hat{p}$ is an equilibrium of (ξ, a), then $f^B(\hat{p}) = 0$. We can differentiate the condition $p'f(p) \equiv 0$ at $\hat{p}$ to obtain

$$\hat{p}'Df^B_{\hat{p}} + (f^B(\hat{p}))' = \hat{p}'Df^B_{\hat{p}} = 0.$$

Similarly differentiating the condition $B'(p)f^B(p) \equiv 0$, we establish that $Df^B_{\hat{p}}$ maps $T_{\hat{p}}(K_B)$ into itself. In fact, differentiating f^B at $\hat{p}$ yields

$$Df^B_{\hat{p}} = (I - B(\hat{p})(B'(\hat{p})B(\hat{p}))^{-1}B'(\hat{p}))(D\xi_{\hat{p}} - H(\hat{p})).$$

LEMMA 4: Suppose that (ξ, a) satisfies Assumption 9 and that, for all possible combinations B and B*, $\text{graph}(\xi) \pitchfork L(B^*, B)$. Suppose further that $\hat{p}$ is an equilibrium of (ξ, a) and that $B(\hat{p})$ is the associated matrix of activities in use. Then the matrix

$$\begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{p}} - H(\hat{p}) & B(\hat{p}) \\ 0 & B'(\hat{p}) & 0 \end{bmatrix}$$

is non-singular.

PROOF: Lemma 3 implies that we need only concern ourselves with the case $B = B^*$. We begin by arguing that $Df^B_{\hat{p}} : T_{\hat{p}}(K_B) \to T_{\hat{p}}(K_B)$ is onto, that is, has rank n-k-1, if $\text{graph}(\xi) \pitchfork L(B, B)$. Suppose that it does not. Then there exists $v \in T_{\hat{p}}(K_B)$, not equal to zero, such that

$$Df^B_{\hat{p}} v = (I - B(B'B)^{-1}B')(D\xi_{\hat{p}} - H(\hat{p}))v = 0.$$

This implies that the tangent spaces $T_{(\hat{p}, \xi(\hat{p}))}(\text{graph}(\xi))$ and $T_{(\hat{p}, \xi(\hat{p}))}(L(B, B))$ overlap; both include the non-zero vector $(v, D\xi_{\hat{p}}v)$. Since both tangent spaces have dimension n-1, it is therefore impossible for their sum to be a space of dimension 2(n-1). As a result, graph(ξ) and L(B, B) cannot be transversal, which is the desired contradiction.

The next step of the proof is to demonstrate that rank $[(I - B(B'B)^{-1}B')(D\xi_{\hat{p}} - H(\hat{p}))] = n - k - 1$ implies our contention. This is a matter of simple, but tedious, algebraic arguments similar to those used by Kehoe (1980a) in the proofs of his Lemma 4 and Theorem 6. We omit it here.

□

Suppose that we renormalize prices $\hat{\pi} = \frac{1}{e'\hat{p}}\hat{p}$. Observe that

$$\det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{p}} - H(\hat{p}) & B(\hat{p}) \\ 0 & B'(\hat{p}) & 0 \end{bmatrix} = \|\hat{\pi}\|^{n-k-1} \det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} - H(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix}.$$

Consequently, we can combine Lemmas 3 and 4 to obtain the following theorem.

THEOREM 4. Suppose that (ξ, a) satisfies Assumption 9 and is such that $\text{graph}(\xi) \pitchfork L(B^*, B)$ for all possible combinations B^* and B, then (ξ, a) is a regular economy.

7. GENERICITY OF REGULAR ECONOMIES

The arguments of the previous section suggest that economies that are not regular are somehow pathological because they correspond to non-transveral intersections of certain manifolds. If we are able to perturb these manifolds in a sufficient number of directions, then the smallest perturbation results in the manifolds becoming transversal. We would therefore expect most economies to be regular. The following theorem formalizes this intuition (see Guillemin and Pollack (1974, pp. 67-69)).

TRANSVERSALITY THEOREM: Let M, V, and N be smooth manifolds where dim $M = m$, dim $N = n$, and $m \leq n$, and let Z be a smooth submanifold of N. Suppose that $F : M \times V \to N$ is a C^1 map transversal to Z. For any $v \in V$ let $f_v : M \to N$ be defined by the rule $f_v(x) = F(x,v)$. Then the set $U \subset V$ for which $f_v \pitchfork Z$, $v \in U$, has full Lesbesque measure.

This theorem says that almost all maps are transversal to a given submanifold in the target space if the maps come from a rich enough family.

A problem that we face in using this theorem to demonstrate that almost all economies are regular is that we must translate the measure-theoretic concept of genericity involved in statement of the theorem into a topological concept. For the infinite dimensional space of economicss a natural concept of a generic property is one that holds for an open dense set. We shall actually use the transversality theorem only to prove the density of regular economics. Openness, as we shall see, follows directly from definitions. It should be stressed, however, that, if we are willing to restrict ourselves to some appropriately definite finite dimensional subset of $\mathcal{E}$, we could prove that the set of regular economies has full Lebesque measure. In fact, it is by doing just this that we shall prove the density of regular economies.

We first prove that the set of profit maps that satisfy Assumption 9 is an open dense subset of $\mathcal{A}$. We then consider a fixed profit map $a \in \mathcal{A}$ that satisfies Assumption 9. Any such vector of profit functions is associated with a finite number of matrices of net-output functions B* and B. We prove that for any fixed combination B* and B the set of excess demand functions for which graph(ξ)$\pitchfork$L(B*, B) is an open dense subset of $\mathcal{D}$. Since the intersection of a finite number of open dense sets is open dense, this implies the genericity of regular economies.

LEMMA 5: The set of profit maps that satisfy Assumption 9 is an open dense subset of $\mathcal{A}$.

PROOF: Lemma 2 implies that $a : (\text{int}X) \to R^m$ satisfies Assumption 9 if it is transversal to a finite number of submanifolds, $Q(M_j) \subset R^m$, $j=1,\ldots, 2^m$. Standard arguments imply that the set of maps that satisfy this property is an open dense subset of all maps from intX into R^m endowed with the uniform C^2 topology. See, for example, the version of the transversality theorem given by Hirsch (1976, pp. 74-77). This immediately implies that the set of maps that satisfy Assumption 9 is open in $\mathcal{A}$.

We need to prove that this set is dense in $\mathcal{A}$. Consider the subset $\mathcal{B}$ of profit maps that satisfy the restriction that there exists some $\pi > 0$ such that $a(\pi) < 0$. That $\mathcal{B}$ is an open dense subset of $\mathcal{A}$ follows immediately from Assumption 8 and the joint continuity of $a(\pi)$ in a and π. If we can prove that the set of profit maps that satisfy Assumption 9 is dense in $\mathcal{B}$, we will have demonstrated our contention.

Choose an $(m-n)x(m-n)$ matrix G that is non-singular. For any $v \in R^{(m-n)}$ define the function δ: $(\text{int}X) \times R^{(m-n)} \to R^m$ by the rule

$$\delta(p,v) = p'e \begin{bmatrix} 0 \\ G \end{bmatrix} v$$

where $\begin{bmatrix} 0 \\ G \end{bmatrix}$ is $mx(m-n)$ and e is again the nx1 vector whose every element is unity. For any fixed $v \in R^{(n-m)}$ and $a \in \mathcal{A}$ define $a_v(p) = a(p) + \delta(p, v)$. It is easy to check that $a_v \in \mathcal{A}$ for all v in some open set $V \subset R^{(m-n)}$ that contains the origin. Define the C^1 map F: $(\text{int}X) \times V \to R^m$ by the rule

$F(p,v) = a(p) + \delta(p,v)$. The transversality theorem implies that, if F is transversal to some submanifold of R^m, then, for all v in some set $U \subset V$ of full Lebesque measure, a_v is also transversal to that submanifold. Clearly, however, F is transversal to any submanifold of R^m because

$$DF_{(p,v)} = \begin{bmatrix} -I & 0 \\ * & G \end{bmatrix}$$

is non-singular and hence includes all R^m in its image. (Here the elements denoted *, the partial derivatives of the final m-n components of F with respect to p, are of no consequence.)

It is now a straightforward matter to argue that for any $\varepsilon > 0$ there exists some $\varepsilon' > 0$ such that $d(a, a_v) < \varepsilon$ if $\|v\| < \varepsilon'$. Recall that the intersection of a finite number of sets with full Lebesque measure also has full Lebesque measure and that a set with full Lebesque measure is dense. Consequently, since the set V contains the origin, the set of profit maps that satisfy Assumption 9 is dense in $\mathcal{A}$. □

These same arguments could be used to prove genericity of Assumption 9 in the subset of $\mathcal{A}$ made up of linear profit functions. Notice that the perturbation function $\delta(\pi, v)$ can be regarded as the vector of linear profit functions associated with the m-n activities ev'G'. Similar arguments could be used to prove the genericity of Assumption 9 in other subsets of $\mathcal{A}$: the sets of profit maps associated with Cobb-Douglas production functions, with C.E.S. production functions, with trans-log production functions, to give a few examples.

We can now choose a fixed profit map $a \in \mathcal{A}$ that satisfies Assumption 9 and let perturbations of the excess demand function ξ do all the work.

LEMMA 6: Suppose that B* and B are matrices of net-output functions, where B is a submatrix of B*, associated with a profit map $a \in \mathcal{A}$ that satisfies Assumption 9. Then the set of excess demand functions for which graph(ξ) $\pitchfork$ L(B*, B) is an open dense subset of $\mathcal{D}$.

PROOF: Once again openness follows directly from the transversality condition. We need to find a finite dimensional family of perturbations that allows us both to remain in $\mathcal{D}$ and to satisfy the requirements of the transversality theorem. Re-using some of the notation from the proof of Lemma 5, we define the function $\delta : (\text{int}X) \times R^n \to R^n$ by the rule

$$\delta(p,v) = \frac{p'v}{p'e} - v.$$

Notice that, for any fixed $v \in R^n$, δ satisfies Assumptions 1-3. Consequently, if we define ξ_v by the rule $\xi_v(p) = \xi(p) + \delta(p, v)$, it is an element of   if ξ is. In addition, δ satisfies the condition that $D\delta_v(p, v)$ has rank n - 1 for any $(p, v) \in (\text{int}X) \times R^n$ since it can easily be verified that the only vectors $x \in R^n$ for which $x'D\delta_v(p,v) = 0$ are scalar multiples of p.

Define the C^1 map $F : P \times R^n \to T(P)$ by the rule

$$F(p,v) = \begin{bmatrix} p \\ \xi(p) + \delta(p,v) \end{bmatrix}.$$

For a fixed $v \in R^n$ the image of F is, obviously, graph(ξ_v). We want to prove that $F \pitchfork L(B^*, B)$. Differentiating F, we obtain

$$DF_{(p,v)} = \begin{bmatrix} I & 0 \\ D\xi_p + D\delta_p & D\delta_v \end{bmatrix}.$$

Notice that the image of the linear map $DF_{(p,v)} : T_p(P) \times R^n \to T_p(P) \times T_p(P)$ has dimension $2(n-1)$ since $T_p(P)$ has dimension $n - 1$ and $D\delta_v$ has rank $n - 1$. Consequently, this image must fill up the tangent space to $T(P)$ at any point $(p,x) \in T_p(P) \times T_p(P)$, since it too has dimension $2(n-1)$. The transversality theorem therefore implies that graph(ξ_v) is transversal to any submanifold of $T(P)$ for all v in some set $U \subset R^n$ of full Lebesque measure. It is now easy to prove the density of the transversality condition in $\mathcal{D}$.

□

The genericity of regular economies follows directly from Lemmas 2-6.

THEOREM 5: The set of regular economies $\mathcal{R}$ is open and dense in $\mathcal{E}$.

One problem with our demonstration of this result is that it relies on perturbations of the excess demand function that may not be appropriate in economies where production plays an important role. In such economies there are likely to primary commodities, which are inelastically supplied as inputs to the production process, and intermediate commodities, which are only produced in order to produce other commodities. Obviously, if we perturb the excess demand function of such an economy, we may destroy the primary and intermediate characteristics of these goods. It is possible to extend our argument to such situations, but there are several minor technical problems. First, we must explicitly confront the possibility of excess demand being unbounded at some, but not all, prices on the boundary of R^n to deal with primary commodities. Second, we must slightly alter our definition of regularity to deal with the possibility of prices of intermediate commodities being undefined at equilibria where no production takes place. Kehoe (1979b) resolves these problems for the special case of economies with activity analysis production technologies. His analysis can easily be extended to the more general model that we are using here.

8. DECREASING RETURNS

In this section we sketch a procedure for extending our results to economies with decreasing-returns production technologies. The production side of such an economy is again specified by a vector of C^2 profit functions. The first n profit functions correspond to disposal activities. In addition there are k profit functions,

$r_j: R^n_+ \backslash \{0\} \rightarrow R$, $j=1,\ldots,k$, that correspond to production functions that exhibit strictly decreasing returns. We assume that these functions satisfy Assumptions 4-6. Implicit is the assumption that these functions have been smoothly bounded away from infinity. These functions differ from the ones that we have defined for constant returns production technologies in that they are always non-negative. Although Hotelling's lemma still holds, it is no longer true that every non-negative scalar multiple of the profit function is a feasible net-output combination.

The positive profits that are earned by activities that are used in equilibrium must somehow be distributed to consumers. The easiest way to specify this process is to assign each consumer a fixed share of each profit function, which may be thought of as a firm. Consumer excess demand then depends on profits made on the production side of the model.

There are two ways to think about this type of model. The first is to specify an excess demand function that has both a consumption and a production component. We set

$$z(\pi) = \xi(\pi, r(\pi)) - B(\pi)e \ .$$

Here B is the matrix function whose columns are the gradients of the k non-disposal profit functions. The excess demand function z naturally satisfies Assumptions 1-3. We focus our attention on the pure exchange economy specified by excess demand ξ and free disposal. An equilibrium of such an economy is a price vector $\hat{\pi}$ such that $z(\hat{\pi}) \leq 0$. Differentiating z, we obtain

$$Dz_{\hat{\pi}} = D\xi_{\hat{\pi}} + D\xi_{\hat{r}}B'(\hat{\pi}) - H(\hat{\pi})$$

where H is defined as before and all activity levels are unity. H can now be thought of as the Jacobian matrix of the aggregate supply function $B(\hat{\pi})e$. If there are no zero prices at equilibrium $\hat{\pi}$, then we can calculate the index as

$$\text{index }(\hat{\pi}) = (-1)^n \operatorname{sgn}\left(\det \begin{bmatrix} 0 & e' \\ e & D\xi_{\hat{\pi}} + D\xi_{\hat{r}}B'(\hat{\pi}) - H(\hat{\pi}) \end{bmatrix}\right).$$

The second way to think about a model with decreasing-returns production involves defining an additional good to represent the non-marketed factors of production peculiar to each firm (see, for example, McKenzie (1959)). There are then n+k goods in the model. The first n profit functions, which allow free disposal of the first n goods, stay fixed. An additional k profit functions are defined by the rule

$$a_{n+j}(\pi_1,\ldots,\pi_{n+k}) = r_j(\pi_1,\ldots,\pi_n) - \pi_{n+j} \quad , \; j=1,\ldots,k.$$

The vector of profit functions $a: R_+^{n+k}\setminus\{0\} \to R^{n+k}$ would satisfy Assumptions 4-8 except that it lacks components that correspond to free disposed activities for the final k goods. The assumption of strictly decreasing returns implies, however, that good n+j has a zero price only if the corresponding firm j does not operate. In this case the profit function $a_{n+j}(\pi_1,\ldots,\pi_{n+k}) = -\pi_{n+j}$ if $r_j(\pi_1,\ldots,\pi_n) = 0$.

The restrictiveness of the assumption that r_j is C^2 is clear in this context. We are, in fact, assuming that a firm's optimal net-output function is C^1 even at prices where it just becomes optimal to shut down. The problem is similar to the one that is encountered in smoothing a consumer's excess demand function when there are corner solutions to the utility maximization problem. We have chosen to ignore both of these minor technical problems. It would be possible to deal with them, however, by demonstrating that, even if such non-differentiabilities existed in r and ξ, they would not occur at equilibria of almost all economies.

Let us specify the consumption side of the model by endowing each consumer with an initial endowment of good n+j equal to his share of the profits of firm j. Since the sum of profit shares for each firm is unity, the aggregate initial endowment of each good n+1,...,n+k is also unity. Each of these goods is considered a primary good in the sense that $\xi_{n+j}(\pi) \equiv -1$, $j=1,\ldots,k$.

To simplify the comparison of the calculation of the index for this formulation to the previous one, let us abuse notation a bit by partitioning the vector $\pi \in R^{n+k}$ into $\begin{bmatrix} \pi \\ r \end{bmatrix}$ where $\pi \in R^n$ and $r \in R^k$. Let us similarly partition the vector $\xi(\pi,r)$ into $\begin{bmatrix} \xi(\pi,r) \\ -e \end{bmatrix}$ and the matrix $B(\pi)$ into $\begin{bmatrix} B(\pi) \\ -I \end{bmatrix}$. Again assuming that there are no zero prices at equilibrium $(\hat{\pi},\hat{r})$ so that $B(\hat{\pi})$ is made up of the gradients of the final k profit functions, we can write

$$\text{index}\ (\hat{\pi}) = (-1)^{n+k}\ \text{sgn}\left(\det\begin{bmatrix} 0 & e' & e' & 0 \\ e & D\xi_{\hat{\pi}} - H(\hat{\pi}) & D\xi_{\hat{r}} & B(\hat{\pi}) \\ e & 0 & 0 & -I \\ 0 & B'(\hat{\pi}) & -I & 0 \end{bmatrix}\right) .$$

where e has the appropriate dimension. We can add the final row of this matrix pre-multiplied by $D\xi_{\hat{r}}$ to the second row without changing its determinant. We obtain

$$\text{index}\ (\hat{\pi}) = (-1)^{n+k}\ \text{sgn}\left(\det\begin{bmatrix} 0 & e' & e' & 0 \\ e & D\xi_{\hat{\pi}} + D\xi_{\hat{r}}B'(\hat{\pi}) - H(\hat{\pi}) & 0 & B(\hat{\pi}) \\ e & 0 & 0 & -I \\ 0 & B'(\hat{\pi}) & -I & 0 \end{bmatrix}\right) .$$

Pre-multiplying the third row by $B(\hat{\pi})$ and adding it to the second row, as well as post-multiplying the third column by $B'(\hat{\pi})$ and adding it to the second column, produce

$$\text{index}\ (\hat{\pi}) = (-1)^{n+k}\ \text{sgn}\left(\det\begin{bmatrix} 0 & e'+e'B'(\hat{\pi}) & e' & 0 \\ e+B(\hat{\pi})e & D\xi_{\hat{\pi}}+D\xi_{\hat{r}}B'(\hat{\pi})-H(\hat{\pi}) & 0 & 0 \\ e & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{bmatrix}\right)$$

$$= (-1)^{n}\ \text{sgn}\left(\det\begin{bmatrix} 0 & e'+e'B'(\hat{\pi}) \\ e+B(\hat{\pi})e & D\xi_{\hat{\pi}} + D\xi_{\hat{r}}B'(\hat{\pi}) - H(\hat{\pi}) \end{bmatrix}\right) .$$

This is, of course, the same as the expression that we derived previously. The only difference is that now we have rescaled so that

$e'\hat{\pi}+e'\hat{r} = e'\hat{\pi}+e'B'(\hat{\pi})\hat{\pi} = 1$ rather than $e'\hat{\pi} = 1$.

Equilibria that have zero prices have the same index in either of the above formulations. If some price is zero, then Assumption 10 implies that the corresponding disposal activity is used in equilibrium. By expanding the determinantal expression for the index along both the column and the row that contain this activity, we can easily show that the index is the same as that for the economy where the free good does not appear.

9. UNIQUENESS OF EQUILIBRIUM

The most significant consequence of our results is that they permit us to establish conditions sufficient for uniqueness of equilibria. If the parameters of a regular economy (ξ, a) are such that $\text{index}(\pi) = +1$ at every equilibrium $\pi \in \Pi(\xi, a)$, then the set of equilibrium prices consists of a single point. A partial converse to this observation is also valid. If an economy (ξ, a) has a unique equilibrium $\hat{\pi}$, then it cannot be the case that $\text{index}(\hat{\pi}) = -1$. The condition that $\text{index}(\hat{\pi}) = +1$ at every equilibrium is, therefore, necessary as well as sufficient for uniqueness in almost all cases.

Kehoe (1980b) has studied the implications of the index theorem for uniqueness of equilibrium in economies with activity analysis production technologies. His two principal results are that economy has a unique equilibrium if its excess demand function satisfies the weak axiom of revealed preference or if there are $n-1$ activities in use at every equilibrium. An economic interpretation of the first condition is that the aggregate excess demand function behaves like that of a single consumer. An interpretation of the second condition is that the economy is an input-output system; that is, there is no joint production, and consumers hold initial endowments of a single good, which cannot be produced.

Both of these conditions imply uniqueness of equilibrium in the more general model that we are considering here. The weak axiom of revealed perference, for example, implies that at any equilibrium $\hat{\pi}$ the Jacobian matrix $D\xi_{\hat{\pi}}$ is negative semi-definite (not necessarily symmetric) on the null space of the nxk matrix of activities in use, $B(\hat{\pi})$. Recall that

$$\text{index}(\hat{\pi}) = \text{sgn}(\det[V'(E + H(\hat{\pi}) - D\xi_{\hat{\pi}})V])$$

where V is any nx(n-k) matrix whose columns span the null space of the columns of $B(\hat{\pi})$. The matrices E and $H(\hat{\pi})$ are both positive semi-definite. If ξ satisfies the weak axiom of revealed preference, then $-V'D\xi_{\hat{\pi}}V$ and $V'(E + H(\hat{\pi}) - D\xi_{\hat{\pi}})V$ are also positive semi-definite. Consequently, if (ξ,a) is a regular economy, then index $(\hat{\pi}) = +1$ at every equilibrium.

An alternative expression for the index is

$$\text{index }(\hat{\pi}) = \text{sgn}\left(\det \begin{bmatrix} -\bar{J} + \bar{H} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix}\right).$$

If there are always n-1 activities in use at equilibrium, then $\bar{B}$ is an (n-1) x (n-1) square matrix. This implies that

$$\text{index } (\hat{\pi}) = \text{sgn}\left(\det\left[\bar{B}'\bar{B}\right]\right) = +1.$$

Therefore, an economy with n-1 activities in use at every equilibrium has a unique equilibrium.

Unfortunately, it seems that these two sets of conditions, which are extremely restrictive, are the only conditions that imply uniqueness of equilibrium in economies with production. For example, if an excess demand function $\xi \in \mathcal{D}$ does not satisfy the weak axiom, then it is possible to choose a vector of profit functions $a \in \mathcal{A}$ so that the economy (ξ, a) has a multiple equilbria. On the production side of the economy the situation is even worse. If a profit map $a \in \mathcal{A}$ satisfies the condition $a(\pi) \leq 0$ for more than $\pi \in S$, then it is easy to find an excess demand function $\xi \in \mathcal{D}$ such that the economy (ξ, a) has multiple equilibria. Obviously, general conditions that imply uniqueness of equilibrium would have to combine restrictions on the demand side with restrictions on the production side. An example of such a combination is the input-output condition mentioned earlier.

One direction to look in would seem to be combinations of restrictions on ξ and a that imply $-D\xi_{\hat{\pi}} + H(\hat{\pi})$ is positive semi-definite on the null space of $B(\hat{\pi})$ at every equilibrium $\hat{\pi}$. We already know that $H(\hat{\pi})$ satisfies this condition. What we want is that $H(\hat{\pi})$ somehow dominates $-D\xi_{\hat{\pi}}$ so that their sum is positive semi-definite. $H(\hat{\pi})$ measures the responsiveness of production techniques to price changes. $D\xi_{\hat{\pi}}$ measures the responsiveness of demand to price changes.

To get some idea of the relationship between these two, consider an economy with three goods and one profit function, $a: R^3_+ \setminus \{0\} \to R$, besides the three free disposal profit functions. If all prices are strictly positive at equilibrium, then $H(\hat{\pi})$ is just the 3x3 matrix of second partial derivatives of a weighted by a scalar activity level. Conditions on $D^2 a_\pi$ are conditions curvature of the boundary of the intersection of the dual cone and the simplex, S_a, at π. Given an excess demand function ξ and a price vector $\hat{\pi} \varepsilon$ int$S$ that satisfies $\xi(\hat{\pi}) \neq 0$, we can easily choose a so that $\hat{\pi}$ is an equilibrium with index $(\hat{\pi}) = +1$. We set $Da_{\hat{\pi}} = \xi(\hat{\pi})$ and twist the boundary of S_a until $D^2 a_{\hat{\pi}}$ is large enough so that index $(\hat{\pi}) = +1$. The condition that index $(\hat{\pi}) = +1$ is equivalent in this case to $\hat{\pi}$ being a sink of the vector field g-I. In Figure 5 $\hat{\pi}$

Figure 5

goes from being a saddle point in (a), to a degenerate equilibrium in (b) to a sink in (c) and (d). In (a) index $(\hat{\pi}) = -1$, in (c) and (d) index $(\hat{\pi}) = +1$, while $\hat{\pi}$ is a critical point of g-I in (b). Notice that (d) is the limiting case where the curvature of S_a is infinite at $\hat{\pi}$. Here there are actually 2 = n-1 activities in use at equilibrium; we already know that index $(\hat{\pi}) = +1$ in such cases.

To make statements about uniqueness of equilibrium we would want to impose global restrictions on $D^2 a_\pi$ on S. To make $D^2 a_\pi$ a martix of constants, for example, we would choose a $\varepsilon \mathcal{A}$ so that S_a is a sphere. To increase the curvature of the boundary of S_a at every point we would have to shrink the size of this sphere. The limiting case, of course, is

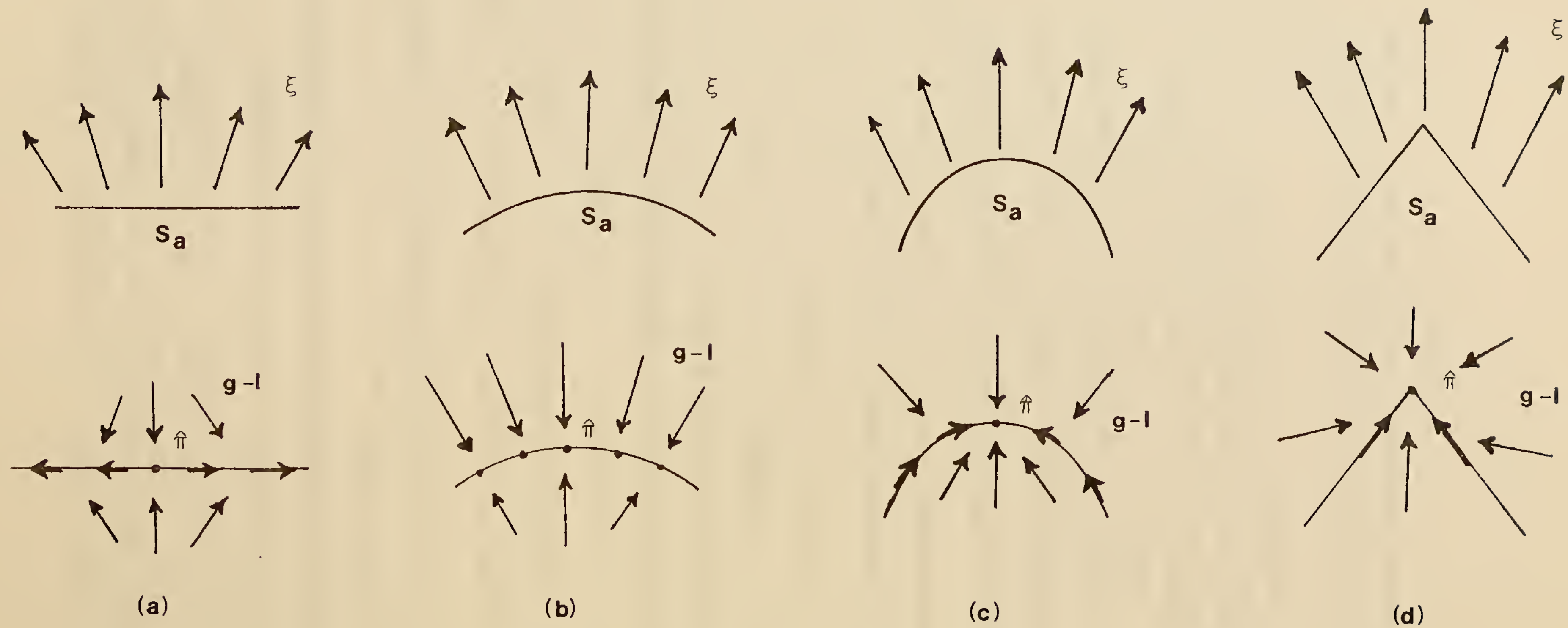

Figure 5

where S_a shrinks to a single point. This situation is one where the production set has a hyperplane as its upper boundary and there is complete reversibility of production in every direction. For a

Figure 6

given ξ it may be necessary to choose a $\varepsilon \mathcal{A}$ so that S_a consists of a single point in order to ensure uniqueness of equilibrium. Consider $\xi(\pi) \equiv 0$ for example. Nevertheless, it is possible to prove that for almost all $\xi \varepsilon \mathcal{D}$ we can find an a $\varepsilon \mathcal{A}$ so that (ξ, a) has a unique equilibrium and that S_a has an interior. We just keep shrinking S_a. Such a result does not seem to be of much interest, however, because it involves almost complete reversibility of production, a very unpalatale assumption.

Figure 7

An insight into the nature of an activity analysis approximation to a smooth production technology is provided by considering a situation where

$$\det \begin{bmatrix} -\bar{J} + H & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix} > 0$$

at some equilibrium π, but

$$\det \begin{bmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix} < 0.$$

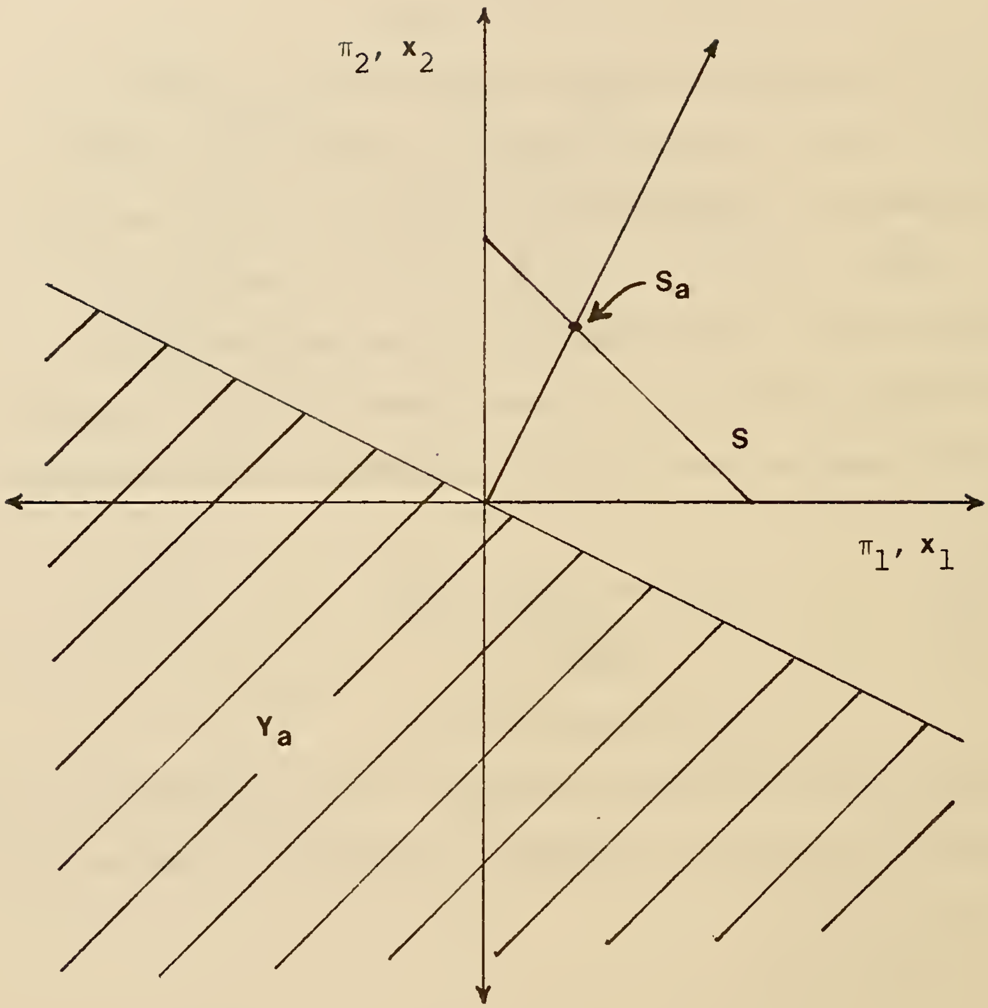

Figure 6

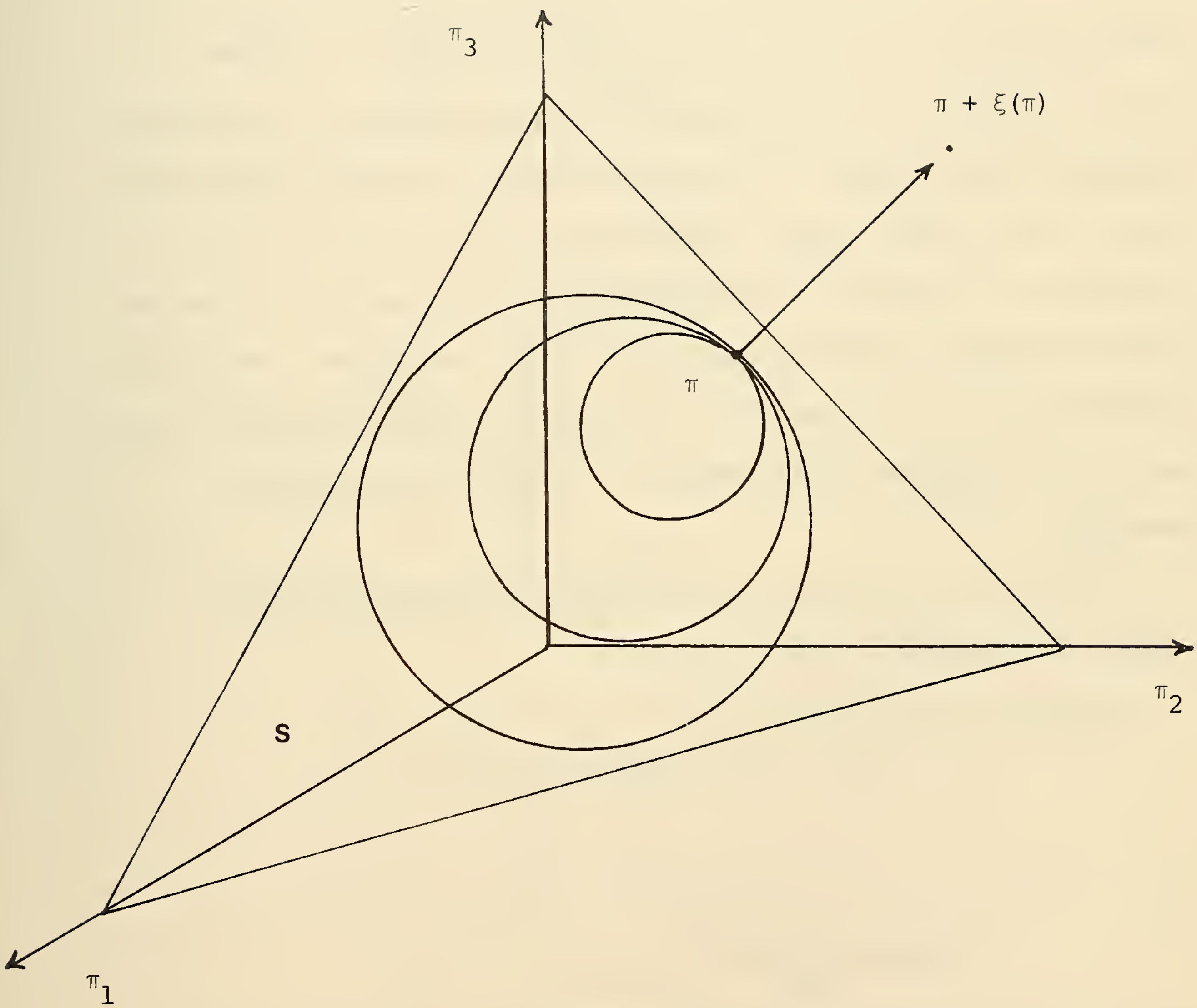

Figure 7

Such a situation is depicted in Figure 8. There are three equilibria in (a) that coalesce into a single equilibrium as the approximation to the smooth production set becomes more accurate. Let $a:R^n_+ \setminus \{0\} \to R$ be a profit

Figure 8

function and let $\pi^1,\ldots,\pi^k$ be a finite number of points in S. Then $b(\pi) = \max[\pi' a(\pi^1),\ldots,\pi' a(\pi^k)]$ provides an approximation to a that can be made arbitrarily accurate in the uniform C^0 metric by proper choice of the set $\pi^1,\ldots,\pi^k$. Unfortunately, such an approximation is not accurate enough in the C^2 topology we have defined on $\mathcal{A}$. The curvature of the dual core is an essential local characteristic of any equilibrium. The unique equilibrium in (c) can be mistaken for isolated, multiple equilibria if we use a activity analysis approximation to the underlying production technology.

Let us turn our attention to economies with decreasing returns production technologies. Here

$$\text{index}(\hat{\pi}) = (-1)^n \text{sgn}\left(\det\begin{bmatrix} 0 & e \\ e & D\xi_{\hat{\pi}} + D\xi_{\hat{r}} B'(\hat{\pi}) - H(\hat{\pi}) \end{bmatrix}\right)$$

$$= \text{sgn}\left(\det[-\bar{J}_1 - \bar{J}_2 \bar{B}' + \bar{H}]\right)$$

where $\bar{J}_1$ is $D\xi_{\hat{\pi}}$ with one row and column deleted and $\bar{J}_2$ is $D\xi_{\hat{r}}$ with the same row deleted. It is easy to use Walras's law and homogeneity prove that, if $D\xi_{\hat{\pi}} + D\xi_{\hat{r}} B'(\hat{\pi})$ has all its off diagonal elements positive and on

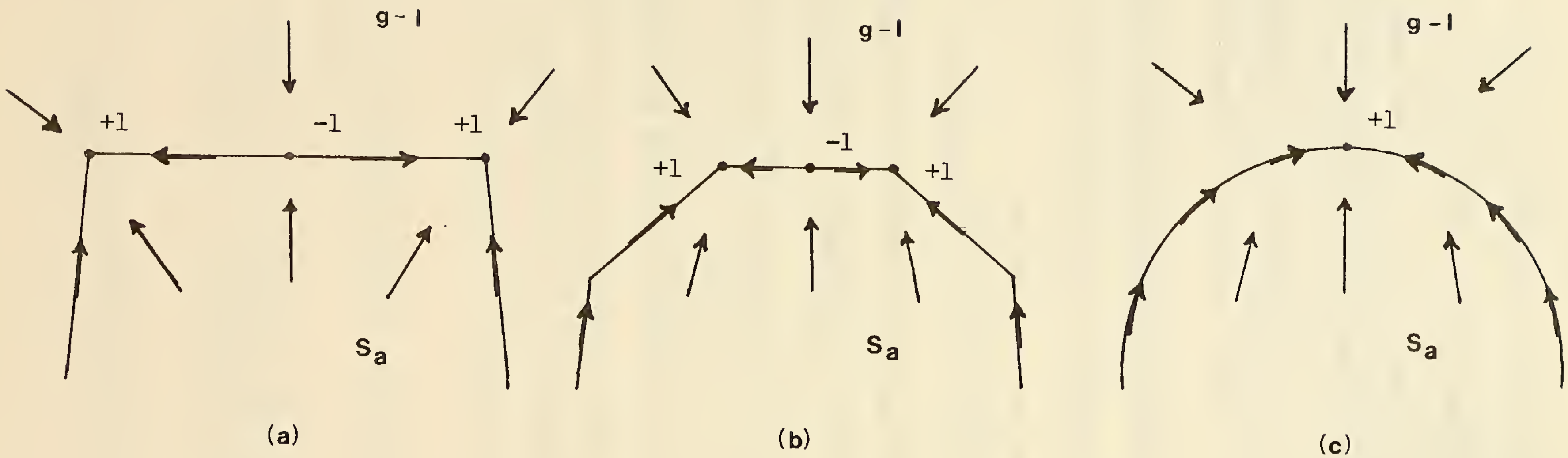

Figure 8

diagonal elements negative, then index $(\hat{\pi}) = +1$. This result was originally discovered by Rader (1972), who did not use an index theorem. The interpretation that he gave it was that gross substitutability in demand implies uniqueness of equilibrium regardless of what the production technology looks like. The problem with this interpretation is that the term $D\xi_{\hat{r}}B'(\hat{\pi})$ involves a complex interaction between income effects from the demand side of the model and activities from the production side. It seems impossible to develop easily checked conditions to guarantee that $D\xi_{\hat{\pi}}+D\xi_{\hat{r}}B'(\hat{\pi})$ has the required sign pattern.

Our results shed light on the applicability of the comparative statics method to general equilibrium models. The assumptions that the equilibria of an economy are locally unique and vary continuously with its parameters are not at all restrictive. Almost all economies satisfy these conditions. Unfortunately, uniqueness of equilibrium is a more elusive property. The conditions that imply uniqueness seem to be too restrictive for most applications. There is obviously a need for more discussion the relationship between comparative statics and uniqueness of equilibrium.

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