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ON THE EFFICIENCY OF KEYNESIAN EQUILIBRIUM

by

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Introduction

Theorems characterizing equilibrium in economics that fail to satisfy some of the strictures of the Arrow-Debreu model have recently abounded. In particular, papers by Grossman [7], Grossman and Hart [8], and Hahn [10] have studied the efficiency properties of equilibrium with incomplete market structures and have established analogues of the two principal theorems of welfare economics. In this paper we undertake a study of the efficiency of a model with a different kind of imperfection, fixed prices. More precisely, we start by showing (Proposition 2) that Grandmont's [5] notion of K-equilibrium in fixed-price models, which embraces both the Dreze [3] and Benassy[1] [1] equilibrium concepts, is equivalent to a kind of Social Nash Optimum[2], in which optimization is incompletely coordinated across markets and where the control variables are quantity constraints. Viewing K-equilibria as Social Nash Optima, we believe, permits a better understanding of the structure of the set of equilibria (see Proposition 3).

K-equilibria possess two important properties: order (the requirement that at most one side of the market can be quantity-constrained) and voluntary exchange (no one trades more of any good than he wants to). After studying K-equilibria, we examine order and voluntary exchange and their connection with the two principal concepts of optimality in a fixed-price economy: constrained Pareto optimality (optimality relative to trades that are feasible at the fixed prices) and implementable Pareto optimality (optimality relative to feasible trades that satisfy voluntary exchange). We show first (Proposition 4) that the usual definition of order (c.f., Hahn [9] and Grandmont [5]), in fact,

implies that exchange is voluntary (assuming that preferences are convex and differentiable). We, therefore, introduce a less demanding notion of order, weak order, which is distinct from voluntary exchange. We prove (Corollary 6) that, with convexity and differentiability, order is equivalent to the conjunction of implementability (voluntary exchange) and weak order. We then demonstrate (Proposition 7) that constrained Pareto optima, although weakly orderly, are, except by accident, non-implementable and, hence, non-orderly. Furthermore, (Proposition 8) implementable Pareto optima need not even be weakly orderly (although in an interesting special case--the absence of "spillover" effects--they will be).

In section 1 we introduce the notation and definitions. Section 2 treats the existence and characterization of K-equilibria. Section 3 studies the relationships among order, weak order and voluntary exchange. We consider optimality in section 4. Finally, in section 5, we compute the dimension of the various sets of optima.

1. Notation and Definitions

Consider an economy of $m + 1$ goods indexed by $h(h = 0,1,\ldots,m)$, whose price vector p is fixed ($p_0 = 1$), and n traders indexed by $i(i = 1,\ldots,n)$ where trader i has a feasible net trade set $X^i \subseteq R^{m+1}$. We assume that X^i is convex and contains the origin (so that trading nothing is possible) and that trader i's preferences (denoted by $\succsim_i$) are continuous and strictly convex on this set. We will at times require preferences to be differentiable as well. Following Grandmont [5], we define an equilibrium for such an economy as follows:

Definition 1: A K-equilibrium is a vector of net trades $(t^1,\dots,t^n)$ associated with the vector of quantity constraints $((\underline{Z}^1, \overline{Z}^1),\dots,(\underline{Z}^n, \overline{Z}^n))$ (with $\underline{Z}^i \le 0$, $\overline{Z}^i \ge 0$, $\underline{Z}^i_0 = -\infty$, and $\overline{Z}^i_0 = +\infty$) such that, for all i,

(a) t^i is feasible at prices p: $t^i \varepsilon \tilde{X}^i = X^i \cap \{t^i \mid p \cdot t^i = 0\}$

(b) quantity constraints are observed : $\underline{Z}^i \le t^i \le \overline{Z}^i$

(c) exchange is voluntary: t^i is the $\succsim_i$ - maximal element among net trades satisfying (a) and (b)

(d) exchange is orderly: if, for some commodity h, $(\tilde{t}^i, \tilde{t}^j) \varepsilon \gamma^i_h(\underline{Z},\overline{Z}) \times \gamma^j_h(\underline{Z},\overline{Z})$ $\tilde{t}^i \succ_i t^i$ and $\tilde{t}^j \succ t^j$, then $(t^i_h - \overline{Z}^i_h)(t^j_h - \underline{Z}^j_h) \ge 0$, where

$$\gamma^i_h(\underline{Z},\overline{Z}) = \{\tilde{t}^i \varepsilon \tilde{X}^i \mid \underline{Z}^i_k \le t^i_k \le \overline{Z}^i_k \quad \forall k \ne 0,h\}.$$

(e) aggregate feasibility: $\sum_i t^i = 0$.

For any trade t^i by trader i, we may confine our attention to the "canonical" rations $\underline{Z}(t^i)$ and $\overline{Z}(t^i)$, associated with that trade: for $h \ne 0$, if $t^i_h \ge 0$, then $\overline{Z}^i_h(t^i) = t^i_h$ and $\underline{Z}^i_h(t^i) = 0$; if $t^i_h \le 0$, then $\overline{Z}^i(t^i) = 0$ and $\underline{Z}^i_h(t^i) = t^i_h$.

Voluntary exchange implies that agents are not forced to trade more of any good than they want to. An allocation characterized by voluntary exchange is said to be implementable. Formally, we have

Definition 2: An implementable allocation is a vector of net trades $(t^1,\dots,t^n)$ satisfying conditions (a), (b), (c), and (e) for the canonical rations associated with these trades.

A market is orderly if buyers and sellers are not both constrained on that market. The next two definitions represent alternative attempts to capture the idea of order. First we introduce property (d'), which is equivalent to (see Proposition 1) but somewhat easier to work with than (d).

(d'): A vector of net trades $(t^1,\ldots,t^n)$ satisfies property (d') if, for all markets h, there exists no alternative vector $(\tilde{t}^1,\ldots,\tilde{t}^n)\varepsilon \prod_i \gamma_h^i(\underline{Z}(t^i),\overline{Z}(t^i))$ such that $\tilde{t}^i \succsim_i t^i$ (with at least one strict preference) and $\sum_i \tilde{t}_h^i = 0$.[3]

The following definition is standard:

Definition 3: An orderly allocation is a vector of net trades $(t^1,\ldots,t^n)$ satisfying (a), (b), (d'), and (e) for the canonical rations associated with those trades.

The problem with the above definition of an orderly allocation, if one is attempting to distinguish between the notions of order and voluntary exchange, is that it itself embodies elements of voluntary exchange. Indeed, we will show below (Proposition 4) that, with differentiability, the above concept of order implies voluntary exchange. Heuristically, this is because, under the definition of order, the trade $\tilde{t}^i$ in $\gamma_h^i(\underline{Z}(t^i), \overline{Z}(t^i))$ could be preferred to t^i simply because t^i involves forced trading on a market $k \neq h$ and not because $\tilde{t}^i$ relaxes a constraint on market h. Therefore, we propose a new notion of order that is free from the taint of voluntary exchange.

We first define property (d").

(d"): A vector of net trades $(t^1,\ldots,t^n)$ satisfies property (d") if, for all markets h, there exists no alternative vector $(\tilde{t}^1,\ldots,\tilde{t}^n)\ \varepsilon \prod_{i=1}^{n} \overline{\gamma}_h^i\ (t^i)$ such that, for each i, $\tilde{t}^i \succsim_i t^i$ (with at least one strict preference) and $\sum_i \tilde{t}_h^i = 0$, where $\overline{\gamma}_h^i(t^i)=\{\overline{t}^i\ \varepsilon\ \tilde{X}^i \mid \overline{t}_k^i = t_k^i,\ k \neq 0,\ h\}$.

Notice that properties (d') and (d") are identical except that the latter requires that alternative net trade vectors be identical to the original trades in all markets other than h and 0.

Definition 4: A weakly orderly allocation is a vector of net trades $(t^1,\ldots,t^n)$ satisfying property (a), (b), (d"), and (e) for the canonical rations associated with the trades.

An orderly allocation is obviously weakly orderly.

Below we shall be interested in the Pareto-maximal elements in the sets of K-equilibria, implementable allocations, orderly allocations and weakly orderly allocations, which will be called K-Pareto optima (KPO), implementable Pareto optima (IPO), orderly Pareto optima (OPO), and weakly orderly Pareto optima (WPO), respectively. A still stronger notion of optimality, selecting Pareto-maximal elements in the set of all feasible allocations, is constrained Pareto optimality:

Definition 5: A constrained Pareto optimum (CPO) is a Pareto optimum of the economy for feasible consumption sets $\tilde{X}^i = X^i \cap \{t^i \mid p\cdot t^i = 0\}$. I.e., it solves the program

$$(*)\ \max \sum_{i=1}^{n} \lambda^i u^i(t^i) \text{ subject to } t^i \varepsilon \tilde{X}^i \text{ and } \Sigma t^i = 0,$$

for some choice of non-negative λ^i's, where the u^i's are utility functions representing preferences over net trades.

2. Characterization and Existence of K-equilibrium

We first check the consistency of the definitions:

Proposition 1: {K-equilibrium allocations} = {Implementable allocations} $\cap$ {Orderly allocations}.

Proof: We need just check that (d) and (d') are equivalent.[4] If (d') is not satisfied for $(t^1,\ldots,t^n)$, there exist h and $(\tilde{t}^i,\ldots,\tilde{t}^n) \varepsilon \prod_{i=1}^{n} \gamma_h^i(\underline{Z}(t^i),\overline{Z}(t^i))$ such that $(\tilde{t}^1,\ldots,\tilde{t}^n)$ Pareto-dominates $(t^1,\ldots,t^n)$.

Because $(\tilde{t}^1,\ldots,\tilde{t}^n)$ maintains equilibrium on market h, we can infer that at least one agent is demand-constrained and one supply-constrained in $(t^1,\ldots,t^n)$, which contradicts (d). If, on the other hand, (d) is not satisfied by $(t^1,\ldots,t^n)$, there exist h, i, j, and

$$(\tilde{t}^i, \tilde{t}^j) \in \gamma_h^i(\underline{Z}(t^i),\overline{Z}(t^i)) \times \gamma_h^j(\underline{Z}(t^j), \overline{Z}(t^j))$$

such that $(\tilde{t}^i, \tilde{t}^j)$ Pareto-dominates (t^i, t^j) and $(\tilde{t}_h^i - t_h^i)(\tilde{t}_h^j - t_h^j) < 0$. Therefore, if the constraints on market h are relaxed by the amount $\min\{|\tilde{t}_h^i - t_h^i|, |\tilde{t}_h^j - t_h^j|\}$, property (d') is contradicted. Q.E.D.

We now turn to the characterization of K-equilibria in terms of social Nash optima. As indicated above, the idea behind a social Nash optimum is to consider m uncoordinated planners, one for each market h ($h \neq 0$), who choose quantity constraints to maximize a weighted sum of consumers' utilities, subject to keeping equilibrium on their own market and given the rations chosen by the other planners. A social Nash optimum is then defined as a Nash equilibrium of that "game." Formally:

Definition 6: For each i, let $\overset{\circ}{t}{}^i(p, \underline{Z}^i, \overline{Z}^i)$ solve trader i's preference maximization problem, given prices p and rations $\underline{Z}^i$ and $\overline{Z}^i$. Define the indirect utility function $V^i(p, \underline{Z}^i, \overline{Z}^i) = u^i(\overset{\circ}{t}{}^i(p,\underline{Z}^i,\overline{Z}^i))$, where u^i is a utility function representing i's preferences. Suppose that, for fixed positive weights $\{\lambda_h^i\}$, the manager of market h chooses $\underline{Z}_h^i$ and $\overline{Z}_h^i$ for each i so as to maximize $\sum_{i=1}^n \lambda_h^i V^i(p, \underline{Z}^i, \overline{Z}^i)$ subject only to the constraint $\sum_{i=1}^n \overset{\circ}{t}{}_h^i(p, \underline{Z}^i, \overline{Z}^i) = 0$ and taking as given the rations $\underline{Z}_k^i$ and $\overline{Z}_k^i$ in each market $k \neq h$, 0. The allocation corresponding to the equilibrium of such a "game" is a social Nash optimum.

By definition, a social Nash optimum is an implementable allocation. From the equivalence between (d) and (d'), it is also orderly. Conversely,

a K-equilibrium is a social Nash optimum. We have thus characterized the set of K-equilibria:

Proposition 2: {K-equilibria} = {Social Nash optima}

The set of weakly orderly and orderly allocations can be characterized similarly. In particular, a weakly orderly allocation is equivalent to a social Nash optimum where the instruments of planner h are the trades $\{t_h^i\}$ on his own market. The characterization of orderly allocations, although straightforward, is less natural because of the hybrid nature of these allocations: an orderly allocation is a social Nash optimum where each planner chooses trades on his own market given the canonical rations associated with the allocations chosen by the other planners. In other words, each planner assumes that the others have the power only to choose rations, whereas, in fact, they choose the actual trades. Under differentiability, Corollary 6 below guarantees that this kind of social Nash optimum is identical to that of Definition 6.

As a by-product of the characterizaton of K-equilibria as social Nash optima, we obtain a straightforward proof of the existence of a K-equilibrium at prices p based on the Social Equilibrium Existence Theorem of Debreu [2]:

Proposition 3: Under the above assumptions, for any vector of positive prices p and any choice of positive weights $\{\lambda_h^i\}$, there exists a social Nash optimum and, hence, a K-equilibrium, associated with those prices and weights.

Proof: We must show that each planner faces a concave, continuous objective function and that his feasible strategy space is a convex- and compact-valued correspondence of the strategies of the other planners.

To see the concavity of the objective function, consider two alternative choices of constraints, $((\underline{Z}^1, \overline{Z}^1),\ldots,(\underline{Z}^n,\overline{Z}^n))$ and $((\tilde{\underline{Z}}^1, \tilde{\overline{Z}}^1),\ldots,(\tilde{\underline{Z}}^n, \tilde{\overline{Z}}^n))$. For any α, $0 \leq \alpha \leq 1$, the trade $\alpha \overset{\circ}{t}{}^i(p, \underline{Z}^i, \overline{Z}^i) + (1-\alpha)\overset{\circ}{t}{}^i(p,\tilde{\underline{Z}}, \tilde{\overline{Z}}^i)$ is feasible for the constraints $(\alpha\underline{Z}^i + (1-\alpha)\, \tilde{\underline{Z}}^i, \alpha\overline{Z}^i + (1-\alpha)\, \tilde{\overline{Z}}^i)$. The concavity of the utility functions then implies the concavity of the planner's objective function. Continuity follows immediately from the continuity of preferences. The set of feasible strategies for planner h is defined by

$$\{((\underline{Z}_h^1, \overline{Z}_h^1),\ldots,(\underline{Z}_h^n, \overline{Z}_h^n)) \mid \sum_{i=1}^{n} \overset{\circ}{t}{}_h^i(p, \underline{Z}_h^i, \underline{Z}^i_{)h(}, \overline{Z}_h^i, \overline{Z}^i_{)h(}) = 0\}$$ [5]

and is denoted by $\Gamma_h(\underline{Z}_{)h(}, \overline{Z}_{)h(})$. Without loss of generality, we can restrict $\Gamma_h(\underline{Z}_{)h(}, \overline{Z}_{)h(})$ to <u>canonical</u> quantity constraints. $\Gamma_h(\underline{Z}_{)h(}, \overline{Z}_{)h(})$ is not empty because it contains (0,0). It is bounded because if t^i is the preferred vector in $\gamma_h^i(\underline{Z}_{)h(}, \overline{Z}_{)h(})$, $\min\{t_h^i, 0\} \leq \underline{Z}_h^i \leq 0$ and $0 \leq \overline{Z}_h^i \leq \max\{t_h^i, 0\}$. It is closed because of the continuity of the $\overset{\circ}{t}_i$'s. To see that $\Gamma_h(\underline{Z}_{)h(}, \overline{Z}_{)h(})$ is convex, choose $(\underline{Z}_h, \overline{Z}_h) = ((\underline{Z}_h^1, \overline{Z}_h^1),\ldots,(\underline{Z}_h^n, \overline{Z}_h^n))$ and $(\tilde{\underline{Z}}_h, \tilde{\overline{Z}}_h)$ in $\Gamma_h(\underline{Z}_{)h(}, \overline{Z}_{)h(})$ and consider $\alpha(\underline{Z}_h, \overline{Z}_h) + (1-\alpha)\,(\tilde{\underline{Z}}_h, \tilde{\overline{Z}}_h)$ for $0 \leq \alpha \leq 1$. If, for example, $\overset{\circ}{t}{}_h^i(p, \underline{Z}_h^i, \underline{Z}^i_{)h(}, \overline{Z}_h^i, \overline{Z}^i_{)h(}) = \underline{Z}_h^i$, then, because constraints are canonical, $\overset{\circ}{t}{}_h^i(p, \tilde{\underline{Z}}_h^i, \underline{Z}^i_{)h(}, \tilde{\overline{Z}}{}^i_h, Z^i_{)h(}) = \tilde{\underline{Z}}_h^i$ and $\overset{\circ}{t}{}_h^i(p, \alpha\underline{Z}_h^i + (1-\alpha)\,\tilde{\underline{Z}}_h^i, \underline{Z}^i_{)h(}, \alpha\overline{Z}_h^i + (1-\alpha)\,\tilde{\overline{Z}}_h^i, \overline{Z}^i_{)h(}) = \alpha\underline{Z}_h^i + (1-\alpha)\,\tilde{\underline{Z}}_h^i$. Similarly, for the upper constraints. That the correspondence Γ_h is upper hemi-continuous follows from the continuity and convexity of preferences. It is also immediate that Γ_h is lower hemi-continuous. Finally, we can restrict the domain of Γ_h to only those rations $(\underline{Z}_{)h(}, \overline{Z}_{)h(})$ which could ever be canonical constraints. This domain is obviously closed, bounded and convex. We can thus apply the Social Equilibrium Existence Theorem to conclude that a social Nash optimum exists. Q.E.D.

3. Order and Voluntary Exchange

We can now demonstrate that if preferences are differentiable and there are at least three markets, order implies voluntary exchange. For convenience, we shall, from now on, delete the component of trade vectors corresponding to good zero. Thus t^i corresponds to trades $t^i_1, \ldots, t^i_m$ on markets 1 to m, with an implicit trade $t^i_0 = -\sum_{h=1}^{m} p_h t^i_h$ on market h.

Proposition 4: If preferences are differentiable and $m \geq 2$, an orderly allocation is implementable.

Proof: Consider an orderly allocation $(\bar{t}^1, \ldots, \bar{t}^n)$. If this allocation is not implementable, then, in particular, if $(t^{*1}, \ldots, t^{*n})$ is the vector of feasible, preference-maximizing net trades associated with the canonical rations for $(\bar{t}^1, \ldots, \bar{t}^n)$, $t^{*i} \neq \bar{t}^i$ for some i. For this i, there exist h and $\tilde{t}^i_h$ such that $\tilde{t}^i = (\tilde{t}^i_h, \bar{t}^i_{)h(})$ is feasible and $\tilde{t}^i \succ_i \bar{t}^i$ (otherwise, by differentiability and convexity, $\bar{t}^i$ maximizes i's preferences among all net trades satisfying $p \cdot t^i = 0$). If $\tilde{t}^i$ satisfies the canonical quantity constraints associated with $\bar{t}^i$, choose $h' \neq h$. Then $\tilde{t}^i \,\varepsilon \gamma^i_{h'}\, (\underline{Z}(\bar{t}^i), \bar{Z}(\bar{t}^i))$, and since $\sum_{j=1}^{n} \bar{t}^j_{h'} = 0$ and $\tilde{t}^i_{h'} = t^i_{h'}$, $\tilde{t}^i$ contradicts the order of $(\bar{t}^1, \ldots \bar{t}^n)$.

Therefore, $\tilde{t}^i$ cannot satisfy the canonical constraints associated with $\bar{t}^i$. This, in turn, implies that either (i) $|\tilde{t}^i_h| > |\bar{t}^i_h|$ and $\tilde{t}^i_h \cdot \bar{t}^i_h \geq 0$ or (ii) $\tilde{t}^i_h \cdot \bar{t}^i_h < 0$. If (i) holds, then, since $|\bar{t}^i_h| \geq |t^{*i}_h|$, there exists scalar $\lambda \varepsilon [0,1]$ such that $\lambda \tilde{t}^i_h + (1-\lambda)\, t^{*i}_h = \bar{t}^i_h$. Let $t^i(\lambda) = \lambda\, \tilde{t}^i + (1-\lambda) t^{*i}$. Then $t^i_h(\lambda) + \sum_{j \neq i} \bar{t}^j_h = 0$ By convexity, $t^i(\lambda) \succ_i \bar{t}^i$. Furthermore, $t^i(\lambda)$ satisfies the canonical constraints associated with $\bar{t}^i$. Therefore $(\bar{t}^1, \ldots, \bar{t}^n)$ is not orderly, afterall. If (ii) holds, then 0 is a convex combination of $\bar{t}^i_h$ and $\tilde{t}^i_h$. If $\bar{t}^i_h = 0$, then $\bar{t}^i_h = t^{*i}_h$, which violates our assumption above. If $|\bar{t}^i_h| > 0$ then, by strict convexity, $(0, \bar{t}^i_{)h(}) \succ_i \bar{t}^i$. Because $(0, \bar{t}^i_{)h(})$ satisfies the canonical constraints associated with $\bar{t}^i$, we again conclude that $(\bar{t}^i, \ldots, \bar{t}^n)$ is not orderly. Thus,

in all cases, if $(\bar{t}^1,...,\bar{t}^n)$ is not implementable, we conclude that it is not orderly, a contradiction. Thus order implies implementability. Q.E.D.

That there be at least three markets and that preferences be differentiable are hypotheses essential for the validity of the preceding proposition. Consider, for example, a two-market economy as represented in the Edgeworth box in Figure 1. Point A represents the initial endowment;

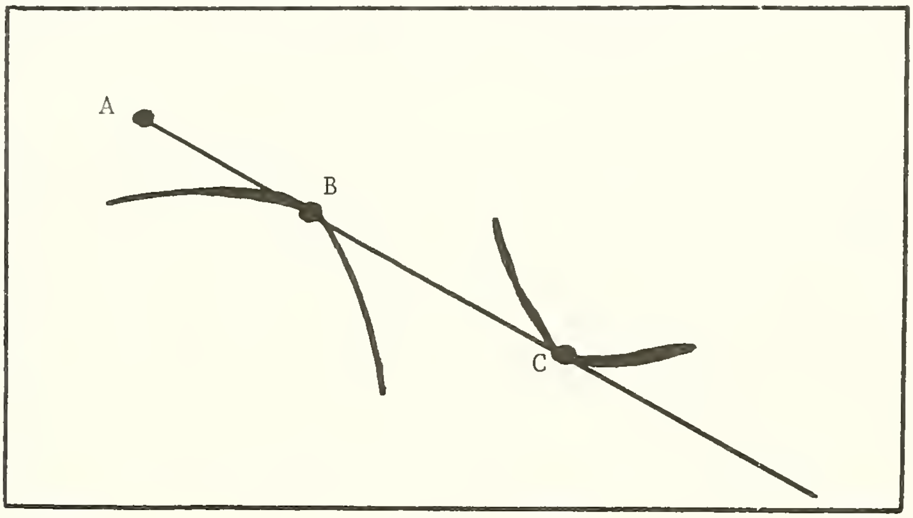

Figure 1

the line through A, prices; and the curves tangent to the line, indifference curves. Any allocation between B and C is clearly orderly, but not implementable since it involves forced trading by the agent whose indifference curve is tangent at B. To see that differentiability is crucial, consider a two-person three-good economy where agents have preferences of the form log min $\{x_1, x_2\}$ + log x_0. Given these preferences, we can treat goods 1 and 2 together as a composite commodity, since traders will always hold goods 1 and 2 in equal amounts. Thus the economy is, in effect, reduced to two goods, and so Figure 1 again becomes applicable.

We next show that voluntary exchange and weak order together imply order. To do so, we make use of the following:

Definition 6: Agent i is said to be constrained on market h for feasible trade t^i if there exists $\tilde{t}^i_h$ with $|\tilde{t}^i_h| > |t^i_h|$ such that $(\tilde{t}^i_h, t^i_{)h(})$ is feasible and $(\tilde{t}^i_h, t^i_{)h(}) \succ_i t^i$.

Proposition 5: If preferences are differentiable, {Implementable allocations}$\cap$ {Weakly orderly allocations}$\subseteq$ {Orderly allocations}.

Proof: Suppose that $(t^1,\ldots,t^n)$ is a weakly orderly and implementable vector of net trades. If it is not orderly, there exists market h and feasible net trades $(\tilde{t}^1,\ldots,\tilde{t}^n)$ that Pareto dominate $(t^1,\ldots,t^n)$ such that $\sum_{i=1}^{n} \tilde{t}^i_h = 0$ and, for all i, $\tilde{t}^i \in \gamma^i_h(\underline{Z}(t^i), \bar{Z}(t^i))$. There are two cases, depending on whether all agents i are constrained on market h for trade t^i.

Case I: Every agent i constrained on market h for t^i.

Either everyone is constrained from buying or all are constrained from selling - a mixture is not possible because of weak order. Suppose, without loss of generality, that everyone is constrained from buying. Then $t^i_h = 0$ for all i, and there exists j such that $\tilde{t}^j_h < 0$ (if $\tilde{t}^i_h = 0$ for all i then $(\tilde{t}^1,\ldots,\tilde{t}^n)$ satisfies the canonical constraints associated with $(t^1,\ldots,t^n)$, a contradiction of implementability). Because j is constrained from buying on market h, there exists $\bar{t}^j_h > 0$ such that $(\bar{t}^j_h, t^j_{)h(})$ is feasible and $(\bar{t}^j_h, t^j_{)h(}) \succ_j t^j$. Choose $\lambda\varepsilon(0,1)$ such that $\lambda\bar{t}^j_h + (1-\lambda)\tilde{t}^j_h = 0$.

Let $t^j(\lambda) = \lambda(\bar{t}^j_h, t^j_{)h(}) + (1-\lambda)\tilde{t}^j$. By strict convexity $t^j(\lambda) \;{}_j\, t^j$, yet $t^j(\lambda)$ satisfies the canonical constraints associated with t^j, a violation of $(t^1,\ldots,t^n)$'s implementability. Therefore, in this case, we conclude that $(t^1,\ldots,t^n)$ is orderly.

<u>Case II</u>: Not all agents are constrained on market h.

Suppose, in fact, that the sellers are not constrained. Let S be the set of all sellers. From strict convexity and implementability, we have $|\tilde{t}^i_h| \geq |t^i_h|$ for all i, with strict inequality if and only if $\tilde{t}^i \succ_i t^i$. Thus if $\tilde{t}^i_h = t^i_h$ for all $i \varepsilon S$, there exists $j \notin S$ such that $\tilde{t}^j_h > t^j_h$ and therefore $\Sigma \tilde{t}^i_h > 0$, a contradiction. Therefore, there exists $i \;\varepsilon\; S$ such that $\tilde{t}^i_h < t^i_h \leq 0$. Define $H = \{k \mid i$ is constrained on market k for t^i, and $|\tilde{t}^i_k| < |t^i_k|\}$. Assume, for the moment, that $H \neq \phi$. For each $k \;\varepsilon\; H$, choose $\tilde{\tilde{t}}^i_k$ so that $|\tilde{\tilde{t}}^i_k| > |t^i_k|$ and, if $\tilde{\tilde{t}}^i(k) = (\tilde{\tilde{t}}^i_k, t^i_{)k(})$, $\tilde{\tilde{t}}^i(k) \succ_i t^i$ (such a choice is possible since i is t^i-constrained on market k). By convexity, we can choose the $\tilde{\tilde{t}}^i_k$'s so that there exists $\lambda \varepsilon (0,1)$ for which

$$\lambda(\tilde{\tilde{t}}^i_k - t^i_k) = (1-\lambda)\ |H|\ (t^i_k - \tilde{t}^i_k)$$ for all $k \varepsilon H$, where $|H|$ is the number of elements in H. Define

$$\bar{t}^i = \begin{cases} \dfrac{\lambda \sum\limits_{k \varepsilon H} \tilde{\tilde{t}}^i(k)}{|H|} + (1-\lambda)\tilde{t}^i, & \text{if } |H| > 0 \\[2ex] \tilde{t}^i, & \text{if } |H| = 0 \end{cases}$$

By convexity, $\bar{t}^i$ is strictly preferred to t^i, and, by construction $\bar{t}^i_k = t^i_k$ for all $k \varepsilon H$. But by differentiability, convexity, and implementability,

t^i is the best trade that i can make if forced to trade t_k^i on each market $k \varepsilon H$, a contradiction. Thus, in this case too we conclude that $(t^1,...,t^n)$ is orderly. Q.E.D.

Corollary 6: If $m \geq 2$ and preferences are differentiable then {Implementable allocations} $\cap$ {Weakly orderly allocations} = {Orderly allocations} = {K-equilibrium allocations}

4. Optimality

We next turn to constrained Pareto optimality. We show that although a constrained Pareto optimal allocation is weakly orderly, it is ordinarily neither implementable nor orderly, at least when preferences are differentiable.

Proposition 7: A constrained Pareto optimum (CPO) is (i) weakly orderly (which implies that {Constrained Pareto optimal allocations} = {weakly orderly Pareto optimal allocations} and (ii) with differentiable preferences, neither implementable nor (when $m \geq 2$) orderly, if it is not a Walrasian equilibrium allocation, if each trader is assigned a strictly positive weight in the program (*), and if there is some (i.e., non-zero) trade on every market.

Proof: Let $(t^1,...,t^n)$ be a CPO.

If it were not weakly orderly, then trades could be altered on some market h, leaving trades on other markets undisturbed, in a Pareto-improving way, a contradiction of optimality. Therefore, (i) is established.

Suppose that $(t^1,\ldots,t^n)$ is not a Walrasian equilibrium allocation, that preferences are differentiable, that there is non-zero trade on every market, and that all traders have positive weight. We will establish that $(t^1,\ldots,t^n)$ is not implementable. Because it is not Walrasian, there exists at least one market h and one agent i who would prefer a trade different from t^i_h, given his trades on other markets $k \neq 0$, h. If, say, trader i is a net buyer of h, either he would like to buy more or to buy less of good h. If less, the non-implementability of $(t^1,\ldots,t^n)$ follows immediately. Assume, therefore, that he would like to buy more. Because, by assumption, there is non-zero trade on market h, there are traders who sell positive quantities of good h. If among these traders, there exists an agent j who would like to sell less of good h, the proof is, again, complete. If j would like to sell more than $-t^j_h$ units of good h (given his trades on markets other than 0 and h), i and j can arrange a mutually beneficial trade at prices p, contradicting constrained Pareto optimality. Therefore, assume that all sellers on market h are unconstrained. From differentiability, forcing them to sell a bit more of good h does not change their utility to the first order but does increase i's utility. Therefore, if the allocation assigns positive weight to i in (*), it involves forced trading. Thus $(t^1,\ldots,t^n)$ is not implementable. If $m \geq 2$, Proposition 4 implies it is not orderly. Q.E.D.

Remark: This proposition demonstrates that Hahn's Proposition 2.2 (see [9]), which asserts the existence of a K-equilibrium that is also constrained Pareto optimal, is false.

By the absence of spillovers, we mean that a change in a constraint on a market does not alter net trades in any of the other markets, except the unconstrained market. A sufficient condition to obtain no spillovers is that traders' utility functions take the form $U^i = t_0^i + \Sigma\, \Phi_h^i\, (t_h^i)$. In the no spillover case, the only change in Figure 3 is that the IPO set shrinks to coincide with the KPO and OPO sets. We have

Proposition 9: In the case of no spillovers, {IPO} = {KPO} = {OPO}.

Proof: An IPO must be orderly. Otherwise, slightly relaxing the constraints in market h for one demand-constrained and one supply-constrained agent would be implementable (since it would not disturb the other markets) and Pareto improving. Q.E.D.

The hypothesis of differentiability in Proposition 7 is, as in previous results, essential. Crucial too is the assumption that all traders have positive weight in the program (*). To see this, refer again to Figure 1. Point B is both constrained Pareto optimal and implementable. However, the trader whose indifference curve is tangent to C has zero weight. (Note, incidentally, that all the other CPO's - the line segment between B and C - are non-implementable.) Finally, the hypothesis of non-zero trade on each market is necessary. Refer, for example, to the Edgeworth box economy of Figure 2. Initial endowments

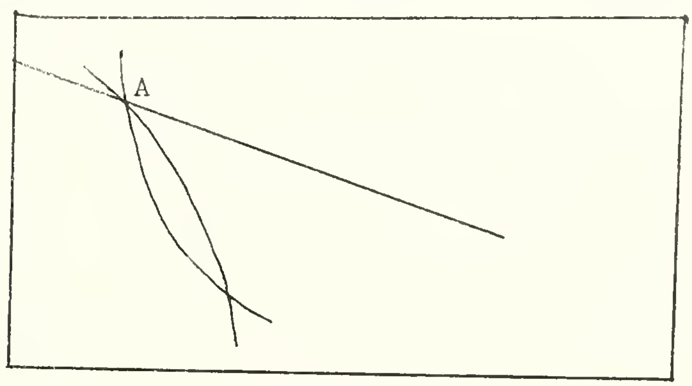

Figure 2

are given by A, which is also a constrained Pareto optimum relative to the price line drawn. Although A does not involve forced trading, it does not violate the Proposition, as it involves no trade at all.

Although differentiability is a restrictive assumption, the non-zero weight and trade assumptions rule out only negligibly many CPO's. On the basis of Proposition 7, we may conclude that, with differentiability, CPO's are generically non-implementable and non-orderly.

We now consider the set of Pareto optima among implementable allocations: the Implementable Pareto optima. As opposed to a Social Nash Optimum where an uncoordinated planners choose the rations on their own market according to their own weights, an IPO is a Pareto optimum for a unique planner choosing all the rations. Obvious questions are whether IPO's are necessarily orderly or even weakly orderly. The following proposition demonstrates that this is not the case.

Proposition 8: Implementable Pareto optima need not be weakly orderly. (nor, *a fortiori*, orderly).

Proof: The proof takes the form of an example. Consider a two-trader, three-good economy in which trader A derives utility only from good 0 and has an endowment of one unit each of goods 1 and 2. Trader B has a utility function of the form

$$U(x_0,x_1,x_2) = \frac{15}{8} x_1 + \frac{3}{2} x_2 - 3x_1x_2 - 3x_1^2 - x_2^2 + x_0,$$

where x_i is consumption of good i, and an endowment of one unit of good 0. All prices are fixed at 1. It can verified that trader B's unconstrained demands for goods 1 and 2 at these prices are $\frac{1}{12}$ and $\frac{1}{8}$, respectively. This is an IPO in which all the weight is assigned to trader B. In this IPO, trader A is constrained on both markets. Now suppose that trader B is constrained from buying more than $\frac{1}{24}$ units of good 1. It can be verified that his corresponding demand for good 2 is $\frac{3}{16}$. Because $\frac{1}{24} + \frac{3}{16} > \frac{1}{12} + \frac{1}{8}$, trader A obtains more of

of good 3 when B is constrained on market 1 than when unconstrained. Therefore the allocation in which A is constrained from selling more than the vector $(\frac{1}{24}, \frac{3}{16})$ and B from buying more than $\frac{1}{24}$ units of good 1 is an IPO. Furthermore, it is not weakly orderly, because given a purchase of 3/16 units of good 2, trader B would like to buy $\frac{5}{96}$ units of good A, and $5/96 > \frac{1}{24}$. Q.E.D.

We can summarize the results (with differentiability) to this point in a schematic diagram (Figure 3).

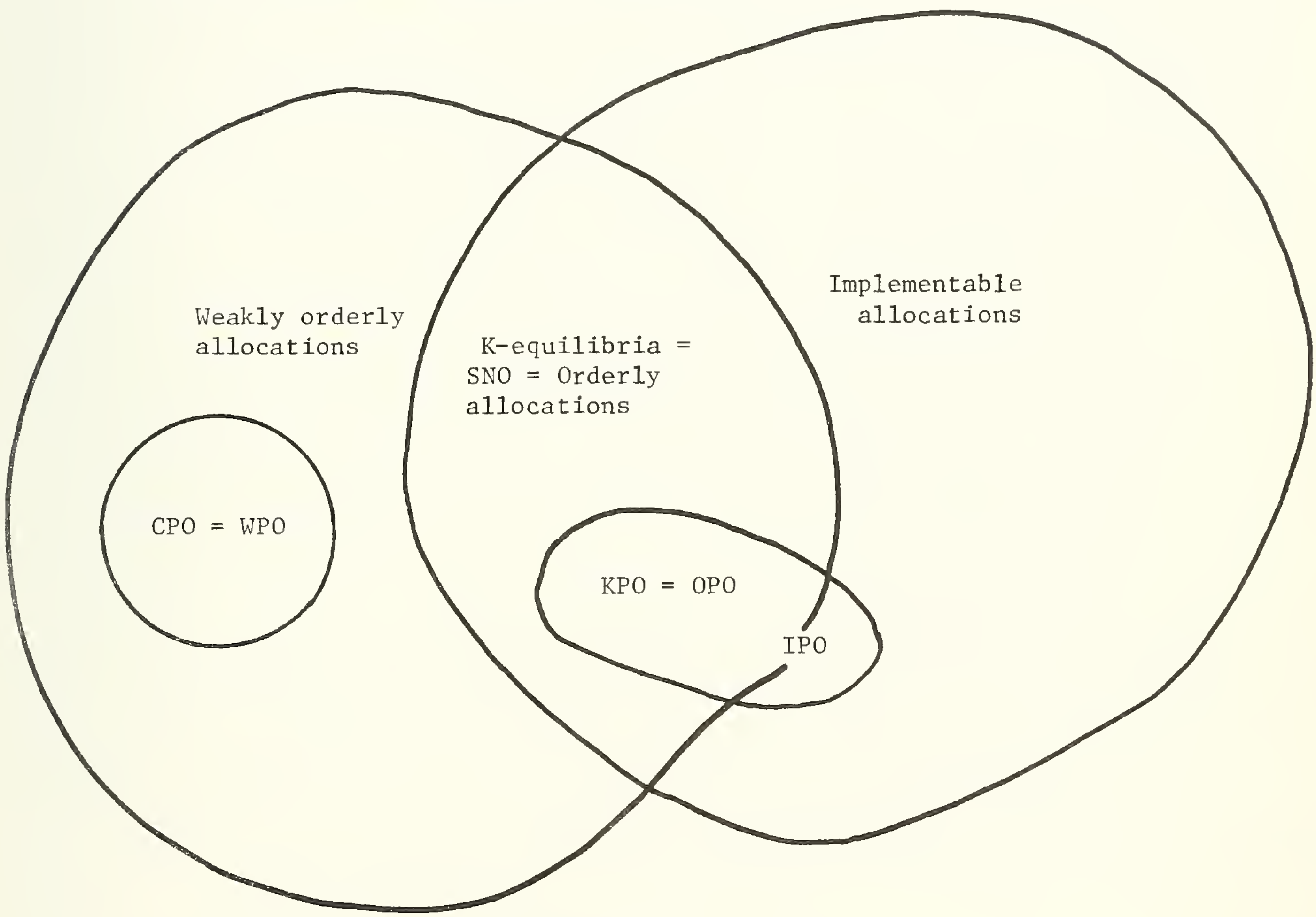

Figure 3

One "unappealing" feature of Figure 3 is the fact that the set of IPO's is neither completely within nor without the set of weakly orderly allocations. With an additional hypothesis often made in the disequilibrim literature - the absence of spillover effects - this unaesthetic property disappears.

5. Structure of K-Equilibria, Constrained Pareto Optima and Keynesian Pareto Optima Sets.

In this section we undertake a study of the local dimension of the different sets.

5.1 K-Equilibria. Assume that at a K-equilibrium the m markets ($h = 1,\ldots,m$) comprise at least one binding constraint (a constraint is binding if the derivative of the indirect utility function with respect to the constraint is different from zero). Let us show that, on market h, the number of degrees of freedom is equal to the number of binding constraints (b^h), minus one; for that let us remember that a K-equilibrium is a Social Nash Optimum. The first order condition for a Social Nash Optimum with a binding ration $\bar{Z}_h^i$ yields

$\lambda_h^i \dfrac{\partial V^i}{\partial \bar{Z}_h^i} = \mu_h$, where μ_h is the multiplier associated with market h's equilibrium constraint. Let $\tilde{\lambda}_h^i = \dfrac{\lambda_h^i}{\mu_h}$ and $F = \begin{bmatrix} \tilde{\lambda}_h^i \dfrac{\partial V^i}{\partial \bar{Z}_h^i} - 1 \\ \sum_i Z_h^i\,(p,\underline{Z}^i,\bar{Z}^i) \end{bmatrix}$.

It is easy to see that the jacobian of F is of rank $[b^h + 1]$. Because F is a function of ($2b^h$) variables, the inverse image $F^{-1}(0)$ is a manifold of dimension ($b^h - 1$). The local dimension of the set of K-equilibria is thus $\sum_h (b^h - 1) = b - m$, where b is the total number of binding constraints.

5.2 Constrained Pareto Optima.

The local dimension of this set is (n-1) since a Constrained Pareto Optimum is a Pareto Optimum of the economy with consumption sets $\tilde{X}^i$ and induced preferences.

<u>5.3 Keynesian Pareto Optima</u>. The only (direct) way to change the weight between two traders is to change their rations for a market on which they are both constrained. Call $T = \{(i,j) \mid i$ and j are both constrained on at least one market$\}$. $\tilde{T}$ is obtained from T by eliminating the redundant pairs; more precisely, in $\tilde{T}$, starting from i there can be at most one sequence of pairs: $(i,j), (j,k), \ldots, (\ell,i)$ leading back to i. The local dimension of the set of Keynesian Pareto Optima is then: $\text{Min}\,[\,|\tilde{T}|\,, n - 1]$.

FOOTNOTES

[1]Benassy's equilibrium is a K-equilibrium if preferences are convex.

[2]The term is due to Grossman [7]. Grossman's SNO, however, is related only in spirit to our own.

[3]Grandmont, Laroque, and Younes [6] call property (d') market-by-market efficiency.

[4]This equivalence is demonstrated by Grandmont, Laroque, and Younes [6].

[5]If x is an m-dimensional vector, the notation $x_{)h(}$ denotes the vector $(x_1,\ldots,x_{h-1}, x_{h+1},\ldots,x_m)$, i.e., the vector obtained by deleting component h.

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