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ON NON-WALRASIAN EQUILIBRIA

by

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On Non-Walrasian Equilibria

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There has recently been a resurgence of interest in specifying more completely the relationship between Walrasian microeconomic models of economic behavior and Keynesian macroeconomic models. On the face of it, these two approaches to economic reality seem very different. The Walrasian model assumes agents engage in maximizing behavior taking as given a common perception of relative prices. The relative prices then adjust to equilibrate the system. The Keynesian model specifies that agents' behavior obeys certain ad hoc rules relating quantity variables of the system. These quantities then adjust to equilibrate the system.

Keynesian analysis is often thought to concern itself primarily with a case where price signals are fixed or adjust very slowly; Walrasian analysis examines cases where realized and expected quantity signals do not affect agents' behavior. Microeconomics concerns itself with long run equilibrium phenomena, while macroeconomics concerns itself with short run equilibria and dynamic phenomena.

In simple dynamic Walrasian models where the rate of change of prices depends on excess demands, the only equilibria of the system are those where excess demand is zero; i.e., the Walrasian equilibria. A fundamental feature of such equilibria is the fact that the assumption of maximizing behavior implies that they must be pareto efficient.

On the other hand, macroeconomic models often allow for extended periods of underemployment of resources. The question of whether such

non-Walrasian equilibria can persist in models with flexible prices is very important. In this paper I present some simple dynamic models with flexible prices that exhibit non-Walrasian equilibria. A fundamental feature of these models is that the economic agents do not behave in the standard Walrasian manner. Walrasian firms maximize potential profits calculated only on the basis of observed prices; here we require firms to maximize profits given a constraint on expected sales. Walrasian consumers maximize utility on a potential budget set; here we require consumers to maximize utility on their actual budget set, as constrained by realized sales. Thus the economic agents' behavior depends on quantity signals as well as on price signals. It is this dependence that allows for the existence of the non-Walrasian equilibria.

Similar concepts of non-Walrasian behavior have been discussed by Leijonhufvud (L1), (L2), Benassy (B), Barro and Grossman (BG), Patinkin, Chapter 13 (P), and Varian (V1). The most direct antecedent of the equilibrium concept presented here is Benassy's concept of the "monopoly line" (B), p. 49.

Money plays two very important roles in these models: (1) it allows the markets for goods and for labor to be separated; and (2) it allows for a gap between savings and investment. Despite these important roles, money is not directly referred to in the formal structure of the models. To incorporate money into such a formal structure, one would need to specify a more complete theory of why economic agents hold money; such a theory must invoke many considerations which are not directly relevant to the main points of this paper. Hence, I have tried to avoid an explicit introduction of monetary phenomena into the models.

A Graphical Example

We will consider first a very simple example which displays in a concrete way the phenomena I wish to discuss. We imagine a flow economy with two goods: a perishable consumption good, (c), and labor, (q). Associated with each good is its price; the price of the consumption good will be denoted by p, the nominal wage rate will be denoted by r, and the real wage rate will be denoted by $w = r/p$. There are two types of agents in the economy, producers and consumers.

We imagine that the technology available to producers can be described by a production function, $f(q)$, which we will assume to exhibit decreasing returns to scale. Given a real wage, a classical Walrasian profit maximizing firm would choose an amount of labor that maximizes profits; such behavior gives rise to the Walrasian demand for labor function, which we will denote by $Q_d(w)$.

This behavioral hypothesis is very restrictive. It requires firms to base their behavior only on relative prices and to ignore any signals they receive concerning other economic conditions such as the effective demand for their product, the probability of actually completing desired transactions, and so on. If we allow firms to take account of such other signals, their "optimal" behavior may be quite different from the above.

Let us imagine that firms have some point expectations about the demand for their product. At any point in time, this expected demand will be denoted by y. On the basis of this expected demand, firms choose a production plan that maximizes profits given the constraint that output be less than or equal to y. In the case of the production function defined

above, such behavior will give rise to a constrained demand for labor, which we denote by $Q_d(w,y)$.

We turn now to the behavior of consumers. Faced with a real wage rate w, consumers determine a utility maximizing supply of labor and planned demand for consumption. The supply function of labor is denoted by $q_s(w)$; the Walrasian demand for consumption is simply $wq_s(w) + P(w,y)$, where $P(w,y)$ is the amount of real profits paid out to consumers. (The real level of profits is the money level divided by p.) If consumers cannot sell all of the labor they wish to - that is, if $Q_d(w,y) < q_s(w)$ - their Walrasian demand for consumption will be constrained. This constrained demand for consumption - which we call effective demand - will be the demand that is actually presented to the market. Let $Q(w,y) = \min\ (q_s(w), Q_d(w,y))$ be the actual amount of labor sold; the effective demand for consumption will then depend on real realized income $Y = wQ(w,y) + P(w,y)$. In the simple case we are considering, with the marginal propensity to consume equal to one, the effective demand for consumption is simply Y, the amount of real realized income.

We now turn to a specification of the dynamics of the economy just described.

If the firm's desired demand for labor is not equal to the supply of labor, the nominal wage rate will adjust. I assume this can be described by the following differential equation:

$$\dot{r} = Q_d(w,y) - q_s(w) \qquad \text{(WAGES)}$$

Similarly, if the expected demand for output is not equal to the actual effective demand for output, firms adjust their expectations; I

assume this process can be described by the following differential equation:

$$\dot{y} = Y(w,y) - y \qquad \text{(EXPECTATIONS)}$$

Thus, if actual demand is greater than expected demand, firms raise their expectations; if actual demand is less than expected demand, firms lower their expectations.

If actual demand is less than effective supply of output, prices will also change. I assume that this adjustment can be described by an equation of the form:

$$\dot{p} = Y(w,y) - f(Q(w,y)) \qquad \text{(PRICES)}$$

The complete dynamical system is now described by the three equations (WAGES), (PRICES), and (EXPECTATIONS). We are interested in the equilibria of this system; these are pairs of (y,w) such that (1) the expected demand equals actual demand so expectations are unchanging, (2) actual demand for consumption equals the actual supply of consumption so the price level is unchanging, and (3) the actual supply of labor equals the conditional demand for labor so that the wage rate is unchanging.

In Figure 1 I have drawn the production possibilities set as a function of leisure (i.e., 1 - Q), and the indifference curves of the consumers between consumption and leisure. At the Walras equilibrium (w^*,y^*) we have the familiar tangency condition that the marginal rate of transformation equals the marginal rate of substitution, so the economy is pareto efficient.

In general, we must expect a positive level of Walrasian profits, here denoted by $P(w^*,y^*) = P^*$. Let us imagine the following experiment: require

the firms to pay out profits P*, regardless of what their actual profits are, and vary the real wage. This variation of the real wage will sweep out the offer curve of the consumer, which is depicted in Figure 1. At w^*, the offer curve passes through the Walras equilibrium point; at $w = 0$, the consumer remains at his endowment point. The offer curve intersects the production frontier in two points; one is the Walrasian equilibrium, (w^*,y^*), the other is a non-Walrasian equilibrium, here denoted by $(\hat{w},\hat{y})$. At $(\hat{w},\hat{y})$ the <u>effective</u> demand for labor equals the supply of labor, the demand for consumption equals the effective supply of consumption, expectations are being satisfied, and actual profits are equal to P*. This is an equilibrium of the effective demand system.

What is happening at this non-Walrasian equilibrium? Firms have pessimistic expectations for the demand for their product; thus they demand little labor. The low real wage induces consumers to supply little labor. Hence demand equals supply and the system is in equilibrium. <u>There are no signals to tell any agent to change his behavior</u>.

Actually, if we allow the profit level to vary, there are many non-Walrasian equilibria of the above system. Figure 2 shows how to construct such equilibria. Pick any point the production frontier to the right of the Walrasian equilibrium. Draw the tangent line to the indifference curve at this point; find the level of profits that makes this the budget line of the consumer. If firms then choose to pay out such a level of profits, the entire system will be in equilibrium at the expected level of output. The set of non-Walrasian equilibrium will be the set of all (w,y) such that $Q_d(w,y) = q_s(w)$; as long as all profits are paid out, the consumption market will automatically clear.

A natural question to ask about such equilibria is the question of stability. If firms are required to pay out all earned profits, the stability issue is rather simple. When expectations are perturbed downward, firms hire less labor but pay out more profits, and the wage rate adjusts to clear the labor market. A more interesting case occurs when we allow some freedom in the profit behavior. Let us specify a profit function, $P(w,y)$. We can regard this in three ways: (1) as real profits payed out by the firm in state (w,y); (2) as real profits expected by consumers; or (3) as the contribution to consumption demand from profits. We do not require $P(w,y)$ to be equal to actual profits; hence, we have allowed a source of savings behavior in the model: in interpretation (1), firms can save or dissave, in (2) consumers can save or dissave when their expectations are wrong, and in (3) consumers determine voluntary savings and disavings. (I will later consider a more general model where savings can occur in labor income as well.) For the moment, let us just consider the case where $P(w,y) = P^*$, the Walrasian level of profits.

We start at the Walrasian equilibrium (w^*,y^*) and consider a perturbation to a $y < y^*$. If expectations are perturbed downwards, firms will lower both their demand for labor and their supply of output. Since price = p = marginal cost = $r(dq/dy)$ at the Walrasian equilibrium we have (heuristically) that $p\ dy = r\ dq$ so that the decrease in labor's income is exactly equal to the decrease in value of output. Thus the nominal price level remains unchanged. But there is now excess supply in the labor market, so the nominal wage rate falls implying the real wage falls. This contributes a term of the form $dr\ q$. The total effect is that aggregate income falls further than the value of aggregate supply. Thus effective demand will be

less than expected demand and firms will adjust their expectations downward even further.

Consider the situation at the non-Walrasian equilibrium, $(\hat{w},\hat{y})$. Suppose we perturb expectations upward to a $y > \hat{y}$. At $(\hat{w},\hat{y})$ price is greater than marginal cost; therefore $p\ dy > r\ dq$ so that the incremental value of consumption supply is greater than the increment to income. Thus the gap between actual demand and expected demand is negative which implies that firms revise their expectations downward, returning to $(\hat{w},\hat{y})$.[1]

If we allow profit payments to vary with the state of the economy, the same heuristic analysis applies, with the proviso that $D_y P(w,y)$ is positive: that is, more profits are saved (by firms or by agents) when expectations are perturbed downwards. This does not seem unrealistic.

The effective demand system described above has been shown (nonrigorously) to have some extremely unpleasant properties. Not only does there exist a non-Walrasian equilibrium, but it is a stable equilibrium of the system while the Walrasian equilibrium is unstable. In this effective demand models, this type of inefficient "self-fulfilling expectations" is the rule, not the exception. The Walrasian equilibrium is <u>always</u> unstable, and there will always exist non-Walrasian equilibria, at least under some quite plausible economic assumptions. The precise statement of the general model and a demonstration of this assertion are the content of the next section.

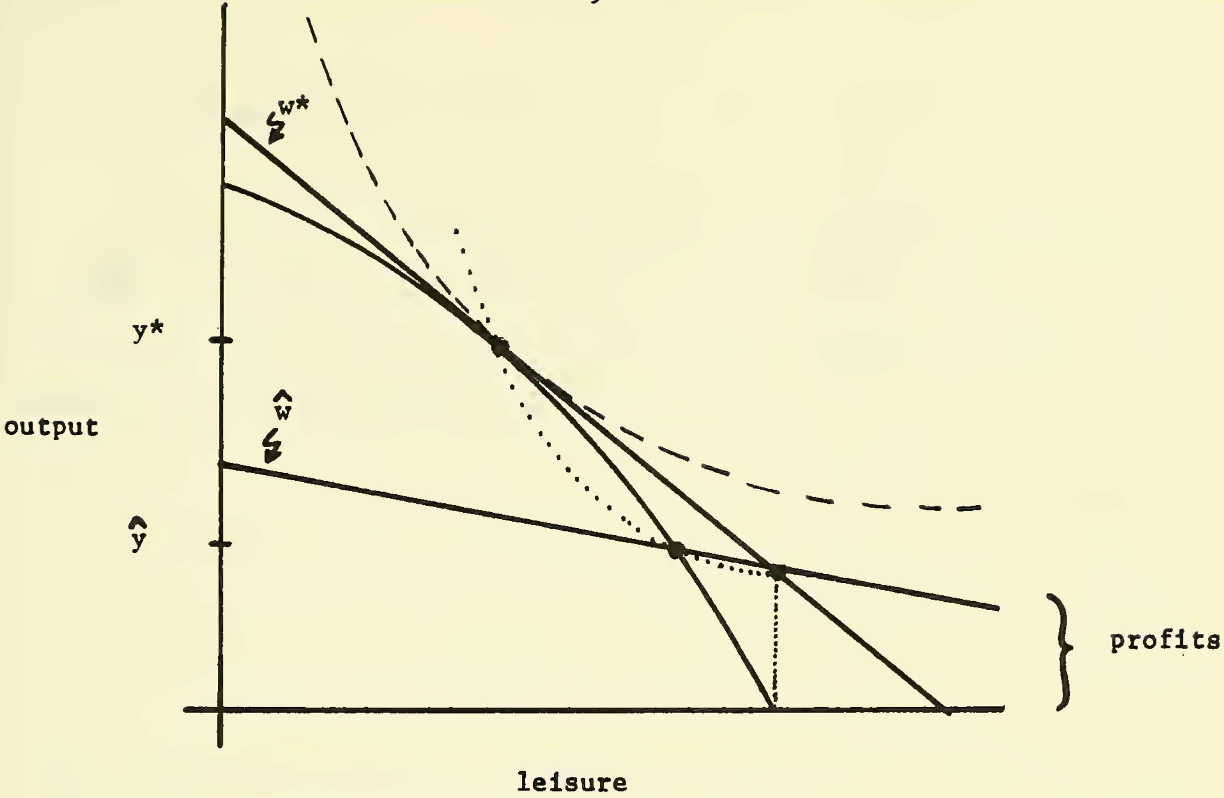


Figure 1 -- The Non-Walrasian and the Walrasian Equilibria

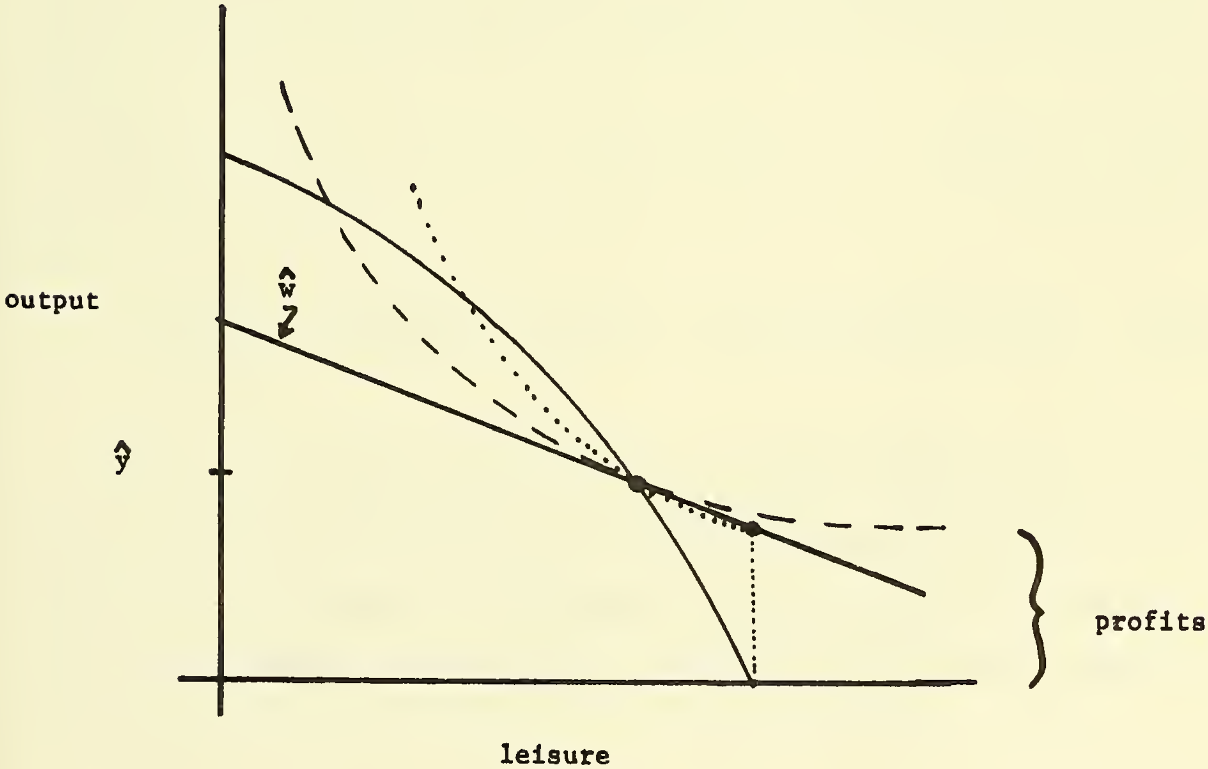


Figure 2 -- Construction of a Non-Walrasian Equilibrium

A Model with One Factor of Production

In this section I examine a more general and abstract model which includes the example discussed in the previous section. We now allow for different speeds of adjustments of expectations, prices, and wages. The dynamical system is therefore represented by:

$$\dot{y} = G_0(Y - y) \qquad \text{(EXPECTATIONS)}$$

$$\dot{p} = G_1(Y - f(Q(w,y))) \qquad \text{(PRICES)}$$

$$\dot{r} = G_2(Q_d(w,y) - q_s(w)) \qquad \text{(WAGES)}$$

The functions G_i, i=0,1,2 are assumed to be smooth sign preserving functions of their arguments.

There are three main points that need to be discussed: (1) the dynamics of the real wage as opposed to the nominal wage; (2) the description of the state space and global dynamics of the system; and (3) the distribution of profits.

The first point is the most straightforward. By the quotient rule for derivatives,

$$\dot{w} = \frac{p\dot{r} - r\dot{p}}{p^2} = \frac{G_2(Q_d(w,y) - q_s(w);\, w)}{p} - \frac{wG_1(Y - f(Q(w,y));\, w)}{p} \qquad \text{(REAL WAGE)}$$

We can now normalize p to be equal to 1, which implies that the dynamical system described by (EXPECTATIONS) and (REAL WAGE) gives us a complete description of the dynamics of the economy in terms of the state variables y and w. This dynamical system will be referred to as the "real system". It is given by:

$$\dot{y} = G_0(Y - y) \qquad \text{(EXPECTATIONS)}$$

RS:

$$\dot{w} = G_2(Q_d(w,y) - q_s(w)) - wG_1(Y - f(Q(w,y))) \qquad \text{(REAL WAGE)}$$

We now turn to the specification of the state space of the economy. The natural structure for this state space is that of a rectangle in R^2_+, $YW = \{(y,w) \text{ in } R^2_+ : \underline{y} \leq y \leq \bar{y} \text{ and } \underline{w} \leq w \leq \bar{w}\}$. The parameters $\underline{y}$ and $\underline{w}$ are lower bounds on expectations and wages - these are presumably zero - while $\bar{y}$ and $\bar{w}$ are upper bounds on the same variables. These bounds will be given explicitly below. For reasons that will become apparent, we will actually want to think of the state space as being a smooth manifold-with-boundary; to ensure this, we need to round off the corners of the above rectangle. This smoothed rectangle will be denoted by M.

We now turn to the problem of analyzing the existence and stability of the Walrasian and non-Walrasian equilibria. The problem is complicated by the fact that the dynamical system under consideration is nondifferentiable. To get around this, we resort to a kind of n-dimensional "one-sided" derivative. Our strategy will be as follows: we will define a new, "virtual" dynamical system on M which coincides with the "real" dynamical system at all equilibria. If we can show that the virtual dynamical system has non-Walrasian equilibria, then we will have shown that the real dynamical system has non-Walrasian equilibria. If we can show the Walrasian equilibrium is unstable for the virtual dynamical system, then it will in general be unstable for the real dynamical system.

We will assume that there exists an interior Walrasian equilibrium to the real dynamical system which we will denote by (w^*,y^*). The assump-

tions required to establish this existence are well known by now; they primarily involve the continuity of the Walrasian excess demands, and some monotonicity assumptions. In general we must allow for the possibility of nonzero profits at (w^*, y^*). We will denote this level of real profits by P^*.

Let $e(w,y) = wq_d(w,y)$ denote the conditional cost function of the firm; by this I mean the costs incurred by a firm required to produce an output equal to y. The conditional cost function satisfies two familiar derivative properties, namely that price equals marginal cost at an (unconstrained) maximum profit point and that the derivative of $e(w,y)$ with respect to w gives the conditional demand for labor, $q_d(w,y)$. I will assume without further comment that $q_d(w,y)$ and $q_s(w)$ are smooth functions on the domain M.

If we assume that the marginal cost curves of the firms are monotonically declining to the left of the Walras equilibrium level of output, then cost minimization will be equivalent to constrained profit maximization for y less than y^*. For no constrained profit maximizer would operate at a point where price was less than marginal cost since he could make a higher profit by cutting output. Likewise, if price is greater than marginal cost, a constrained maximizer will produce the maximum amount he thinks he can sell, namely, y.

Consider the following dynamical system on M, which we will call the <u>virtual system</u>:

$$\dot{y} = G_0(e(w,y) + P^* - y) \qquad \text{(V. EXPECTATIONS)}$$

VS:

$$\dot{w} = G_2(q_d(w,y) - q_s(w)) \qquad \text{(V. REAL WAGE)}$$

This virtual system differs from the real system in three ways: (1) it assumes realized labor income is $e(w,y) = wq_d(w,y)$ rather than $wQ(w,y)$; (2) it assumes the level of real profits distributed is always P^*; (3) it ignores the effect of the consumption market on the real wage.

For now we will simply fix profit payments to be P^*; the subsequent analysis applies equally well for variable profit behavior, as long as $D_y P(w,y)$ is positive and $D_w P(w,y)$ is small, and certain boundary assumptions are met. Formally, this behavior can be subsumed in the more general case of a consumption function which will be discussed in more detail below.

Given such profit behavior, the virtual system is a well-defined dynamical system on the state space M. We know that it has at least one zero, the Walras equilibrium (w^*,y^*); we want to know if it has any other equilibria. Suppose that we can find one of these other equilibria $(\hat{w},\hat{y})$ where price is greater than marginal cost. Then this must be an equilibrium of the real system. For if price is greater than marginal cost, we have already argued that cost minimizing behavior is equivalent to constrained profit maximizing behavior so that $e(w,y)$ represents actual factor payments and $q_d(w,y)$ is the same as $Q_d(w,y)$. Furthermore, since $q_d(w,y) = q_s(w)$, we have $y = f(q_s(w))$ so that the desired amount of output is actually produced and thus the consumption market clears. Hence, $(\hat{w},\hat{y})$ is also an equilibrium of the real system.

We have now reduced the problem of the existence of a non-Walrasian equilibrium of the real dynamical system to the study of the existence of a non-Walrasian equilibrium of the virtual dynamical system. In order to ensure the existence of such an equilibrium, we need to restrict the behavior of VS near the boundary of M.

We now introduce a concept from differential topology. Consider an arbitrary smooth dynamical system on M defined by $\dot{x} = f(x)$. Then the Gauss map on boundary M is defined by $G(x) = f(x)/||f(x)||$ where x is restricted to lie on the boundary of M and $|| \ ||$ is the ordinary Euclidean norm. Thus the Gauss map is a smooth map from the boundary of M to the unit sphere. If this map is not onto the unit sphere, we will say the Gauss map is nullhomotopic. The Gauss map will be nullhomotopic if there is some direction such that $\dot{x}$ never points in that direction along the boundary of M.

Consider the following economic assumptions; for sake of generality they are stated for n factors of production:

B1. $P^* \geqq 0$; Walrasian profits are nonnegative.

B2. There is a maximum level of output; i.e., there is at least one indispensible factor of production in limited supply. Choose $\bar{y}$ to be greater than this maximum level of output.

B3. There are some constants $k_i > 0$ such that $f(q_1,\cdots,q_n) > 0$ iff $q_i > k_i$ for $i=1,\cdots,n$. That is, all factors are indispensible. Choose $\bar{w}_i$ such that $\bar{w}_i k_i - \bar{y} + P^* > 0$.

B4. For any w, $e(w,0) = 0$. For any w such that $w_i = 0$, $q_s^i(w) = 0$.

It is shown in the appendix that these natural boundary assumptions imply that the Gauss map of VS is nullhomotopic on the boundary of M. This fact allows us to invoke the following elementary consequence of the Poincaré-Hopf Theorem:

Theorem: Let $v: M \to TM$ be a smooth vector field on the disk with the Gauss map nullhomotopic the boundary of M and that has a finite number of isolated

zeroes $p_1 \ldots p_n$ with $\det(Dv(p_i)) \neq 0$ for $p_i = 1,\ldots,n$. Then one can define the index of each zero as being:

$$\text{index}(p_i) = \begin{cases} +1 & \text{if } \det(-Dv(p_i)) > 0 \\ -1 & \text{if } \det(-Dv(p_i)) < 0 \end{cases}$$

and furthermore,

$$\sum_{i=1}^{n} \text{index}(p_i) = 0.$$

The proof of this theorem is an easy modification of the Hopf lemma on page 36 of Milnor (M). (See also the excellent discussion in (GP).) It is clear from this theorem that if we can show that the Walrasian equilibria must all have index -1, then we can conclude that there must exist other equilibria to the virtual system.

We consider the negative of the system VS and compute its Jacobian at a Walrasian equilibrium (w*,y*). This will be:

$$\det V = \det \begin{bmatrix} -D_y\dot{y}(w^*,y^*) & -D_w\dot{y}(w^*,y^*) \\ -D_y\dot{w}(w^*,y^*) & -D_w\dot{w}(w^*,y^*) \end{bmatrix}$$

where

$$-D_y\dot{y}(w^*,y^*) = -DG_0(0,y^*)[D_y e(w^*,y^*) - 1]$$

$$-D_y\dot{w}(w^*,y^*) = -DG_2(0,w^*)[D_y q_d(w^*,y^*)]$$

$$-D_w\dot{y}(w^*,y^*) = -DG_0(0,y^*)[q_d(w^*,y^*)]$$

$$-D_w\dot{w}(w^*,y^*) = -DG_2(0,y^*)[D_w z(w^*,y^*)]$$

Here I have denoted $q_d(w,y) - q_s(w)$ by $z(w,y)$ and made repeated use of the fact that the derivative of the conditional cost function is the conditional factor demand, $q_d(w,y)$.

Let us check the signs of each of these terms. We first recall that since (w^*,y^*) is a Walras equilibrium, firm are maximizing profits, and therefore price equals marginal cost. The nominal price level has been normalized to be 1 which implies $D_y e(w^*,y^*) = 1$. Thus $-D_y\dot{y}(w^*,y^*) = 0$. Since the derivative of the conditional demand for labor with respect to output is presumably positive, $-D_y\dot{w}(w^*,y^*)$ must be negative.

The conditional demand for labor is certainly positive so $-D_w\dot{y}(w^*,y^*)$ must be negative. Finally we assume that the excess demand for labor is globally downward sloping which implies that $-D_w\dot{w}(w^*,y^*)$ is positive. Putting these all together, we find that the Jacobian matrix V has sign pattern

$$\text{sign } V = \begin{bmatrix} 0 & - \\ - & + \end{bmatrix}$$

This clearly has a negative determinant and therefore every Walrasian equilibrium must have index -1. By the Theorem, there must therefore exist other, non-Walrasian, equilibria of index +1. In fact, if there are k Walrasian equilibria, there must be at least k non-Walrasian equilibria. The only sign that can change in the above Jacobian is the sign of $-D_y\dot{y}$. In order for the determinant of V to be positive, we need to have $-D_y\dot{y}$ positive, which means that price must be greater than marginal cost. Hence this virtual equilibrium is also a real equilibrium and the proof is done.

There is one further remark concerning this method of proof. We have assumed that the speed of adjustment functions, the $G_i(\cdot)$ functions, are globally sign preserving. We could just as well have assumed some more interesting behavior; for example that at large unemployment levels real wage adjustment becomes sticky and the real wage refuses to fall. With such an adjustment process it is very possible to get true "unemployment equilibria". The techniques presented in this section can easily be modified to demonstrate the existence of such equilibria.

We turn now to the question of the stability of the equilibria. Let $U = \{(w,y) \text{ in } M: q_d(w,y) \leqq q_s(w) \text{ and } 1 \leqq D_y e(w,y)\}$. This is the set of states of the economy where labor demand is no greater than labor supply and price is no greater than marginal cost. Consider the following virtual dynamical system on M:

$$\text{VS}': \quad \begin{aligned} \dot{y} &= G_0(e(w,y) + P^* - y) \\ \dot{w} &= G_2(q_d(w,y) - q_s(w)) - wG_1(e(w,y) + P^* - y) \end{aligned}$$

According to the previous discussion, VS' coincides with RS on the subspace U. We will discuss the stability of VS' at (w^*,y^*) and $(\hat{w},\hat{y})$ which will give us information about the "one-sided stability" of the real system at the two equilibria. It is easy to check that the additional term involving the consumption market does not affect the sign pattern of the matrix V.

To analyze stability, we need to find the eigenvalues of the matrix V. Notice that we can multiply the second row of V by some positive constant

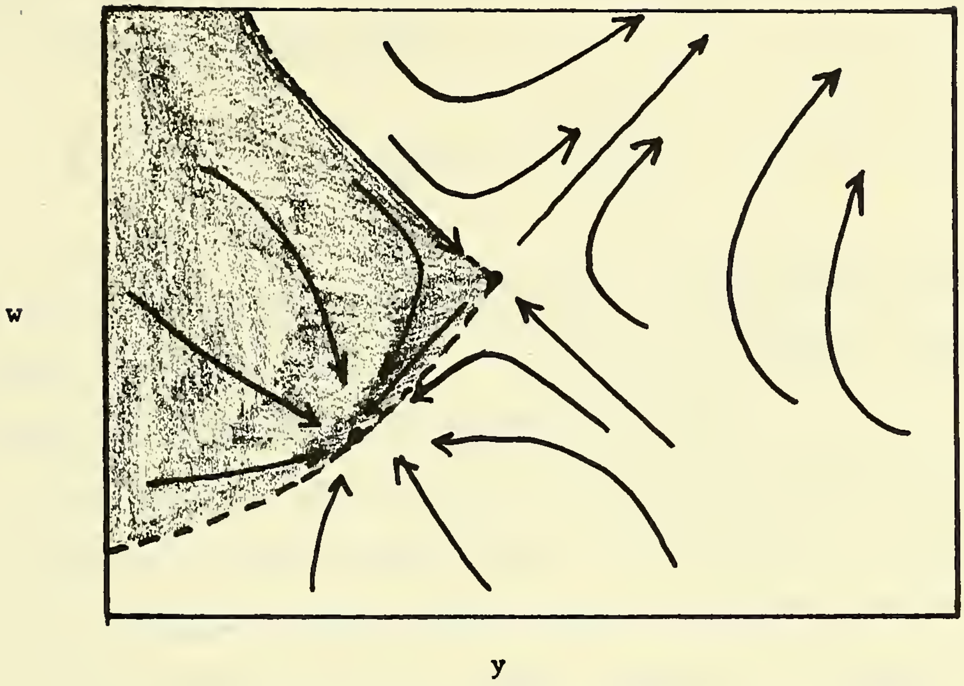


Figure 3 (a) -- The Virtual System

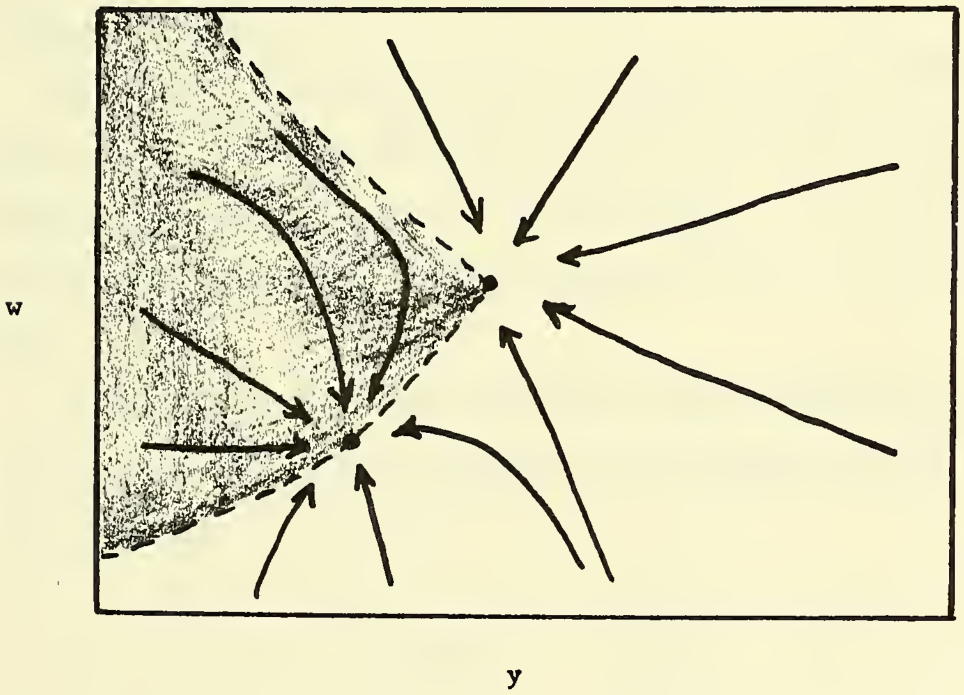


Figure 3 (b) -- The Real System

and add it to the first row in such a way as to transform the matrix V into a diagonal matrix V_1 with

$$\text{sign } V_1 = \begin{bmatrix} - & 0 \\ - & + \end{bmatrix}$$

The virtual dynamical system is therefore a saddle point at (w^*,y^*): one eigenvector points inward, and the other points outward. Such an equilibrium is depicted in Figure 3. It is clearly unstable.

What about the behavior of the real dynamical system? We know that it coincides with the virtual system on the subspace U, and that (w^*,y^*) is in U. As long as U contains points other than points along the one stable ray, there will be points near (w^*,y^*) that have no tendency to return to (w^*,y^*) under the influence of the real dynamical system. This statement is geometrically obvious, but is formally proved in the appendix. Hence we can conclude that even the real dynamical system is unstable at any Walrasian equilibrium.[2]

What about the stability of the non-Walrasian equilibrium? Here the situation is a bit more complicated; to show the Walrasian equilibrium is unstable, we only needed to show it was unstable on one side. To show the non-Walrasian equilibrium is stable we have to show it is stable on both sides.

We already know that at least some of the non-Walrasian equilibria have index +1. Thus the sign pattern of the Jacobian matrix at $(\hat{w},\hat{y})$ must be of the form:

$$\text{sign } V = \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

Diagonalize this matrix in the same way as before, giving a matrix of sign pattern:

$$\text{sign } V = \begin{bmatrix} ? & 0 \\ - & + \end{bmatrix}$$

In order for this matrix to have positive determinant, the entry indicated by the question mark must be positive. Hence the Jacobian of VS has two real negative eigenvalues and the system is stable at $(\hat{w},\hat{y})$ against perturbations into U.

To analyze perturbation on the other side, we must consider the following system, which coincides with the real system on $M \setminus U$:

$$\text{VS}'': \quad \begin{aligned} \dot{y} &= G_0(wq_s(w) - y) \\ \dot{w} &= G_2(q_d(w,y) - q_s(w)) - wG_1(wq_s(w) - f(q_s(w))) \end{aligned}$$

The conditions for stability are not especially informative, and are rather messy so I will not examine them here. However, it is perfectly reasonable to assume they are met at both equilibria. Hence we now have a clear picture of the dynamics around both equilibria: one sided instability at the Walras equilibrium, and stability at the non-Walrasian equilibrium.

An n-Factor Model

It is possible to extend the above model to a case with more than one factor of production. The restriction that there be only one kind of output is more fundamental. Of course, if all output behaves similarly with respect to factor price movements, the previous analysis is still possible. This is really the case where the Hicksian aggregation conditions are satisfied so that output can be treated as a composite good with respect to factors. These conditions are restrictive but not unreasonably so.

The virtual dynamical system in the n-factor case will be the natural generalization of the one-factor system:

$$\text{VS:}\quad \begin{aligned} \dot{y} &= G_0(Y - y) \\ \dot{w}_1 &= G_1(q_d^1(w,y) - q_s^1(w)) \\ &\vdots \\ \dot{w}_n &= G_n(q_d^n(w,y) - q_s^n(w)) \end{aligned}$$

Now realized income is just the constrained cost function e(w,y) plus some given level of profits P*. As before, $-D_y\dot{y}(w^*,y^*) = 0$, $-D_{w_i}\dot{y}(w^*,y^*)$ is negative, $D_y\dot{w}_i(w^*,y^*)$ is negative and if all excess demand curves are downward sloping, $-D_{w_i}\dot{w}_i(w^*,y^*)$ will be positive. We will also assume that cross price effects are small; that is, that $-D_{w_i}\dot{w}_j(w^*,y^*) \approx 0$. This is a strong assumption, but is not unreasonable in the present context. It simply assumes that there is little short run substitution possibilities between factors. Thus when expectations decline and relative prices shift, <u>all</u> factor payments will decline, so the economic analysis can proceed as before.

In order for the formal analysis to proceed, we need to ensure that the Jacobian of the negative of the virtual dynamical system just described has index -1. By the above assumptions on signs of the individual terms, this matrix will have sign pattern:

$$\text{sign } V = \begin{bmatrix} 0 & - & - & \cdots & - \\ - & + & & & \\ - & & + & & \\ \vdots & & & \ddots & \\ - & & & & + \end{bmatrix}$$

The omitted off diagonal terms are all zeroes. Such a matrix must always have a negative determinant. My original proof of this was by induction and involved an expansion by cofactors on the last column. Subsequently Bob Solow discovered the following simple proof: notice that we can multiply each row of the matrix by a positive constant and add it to the first row in a way that cancels out each of the negative elements to the right of the left hand corner element. The resulting matrix has sign pattern:

$$\text{sign } V_2 = \begin{bmatrix} - & 0 & 0 & \cdots & 0 \\ - & + & & & \\ - & & + & & \\ \vdots & & & \ddots & \\ - & & & & + \end{bmatrix}$$

which obviously has a negative determinant. This argument allows one to see clearly the geometrical structure of the dynamical system around (y^*,w^*): all of the eigenvectors point inward except for one which points outward. The Walrasian equilibrium is a saddle point of the system, with index $(y^*,w^*) = -1$. As before if we assume the appropriate boundary behavior, we can apply the Poincaré-Hopf theorem and argue for the existence of a non-Walrasian equilibrium.

A Model with Savings and Investment

As I have remarked earlier, the previous model has an interesting Keynesian interpretation. If we start at a Walras equilibrium and perturb expectations downward, the supply of output will exceed demand. The leftover output can be regarded as an excess of involuntary savings. What is needed in the model is a source of investment to create sufficient demand to consume this leftover product.

Let us therefore postulate an investment function, $I(w,y)$ which represents the value of desired investment when firms face a real wage w and expect the demand for their product to be y. As before, we will do a "one-sided" analysis and consider only the case where firms can realize their desires: where they can hire enough labor to satisfy the demand for current output and for investment purposes. In such circumstances, labor's realized income will be $Y = e(w,y) + P(w,y) + I(w,y)$, composed respectively of income from the production of consumption, profits, and income from investment. We will now regard firms as having no savings or dissavings behavior; all profits must be paid out. Since profits are by definition equal to $y - e(w,y)$, the above equation reduces to $Y = y + I(w,y)$.

We now postulate a consumption function $C(Y) = C(y + I(w,y))$. At a demand limited state of the economy, the following seems to be an appropriate expectations adjustment equation:

$$\dot{y} = G_0(C(y + I(w,y)) - y)$$

Exactly as before, firms revise their expectations about the demand for consumption upwards when demand exceeds their expectations, and revise their expectations downward when the reverse is the case. The adjustment equation for wages is the same as the previous virtual adjustment equation:

$$\dot{w} = G_2(q_d(w,y) - q_s(w))$$

We now allow $q_d(w,y)$ to include both demand for consumption and investment producing labor. Suppose that (w^*,y^*) is an equilibrium of the above system; then realized income $Y^* = y^* + I(w^*,y^*) = C(Y^*) + I(w^*,y^*)$. Hence, savings must equal investment, which is a familiar macroeconomic equilibrium condition.

Let us suppose that (w^*,y^*) is an efficient equilibrium in some appropriate sense, and ask ourselves what factors influence its index. Evaluating the various derivatives we find:

$$-D_y\dot{y} = -g_0(D_Y C(Y^*)(1 + D_y I(w^*,y^*)) - 1)$$

$$-D_w\dot{y} = -g_0\ D_Y C(Y^*)\ D_w I(w^*,y^*)$$

$$-D_y\dot{w} = -g_2\ D_y q_d(w^*,y^*)$$

$$-D_w\dot{w} = -g_2\ D_w z(w^*,y^*)$$

Here $g_i = DG_i(0)$ for $i=0,2$ and $z(w,y) = q_d(w,y) - q_s(w)$.

It is not obvious what sign to attach to $D_w I(w^*,y^*)$; it seems reasonable to assume it is positive since an increase in the real wage should both stimulate investment in an attempt to substitute capital for labor and, at the same time, make investment more expensive. If this is the case, all the signs of the Jacobian matrix will be the same as before with the exception of the term in the upper left hand corner. This term will be negative when the marginal propensity to consume times one plus the marginal propensity to invest is greater than one. This is exactly the macroeconomic condition for instability in the case where investment depends on consumption instead of total national income. Thus macroeconomic instability is a necessary,

but not sufficient condition for (w^*,y^*) to have index -1. A sufficient condition can be found by tracing through the elementary row operations used earlier; it is:

$$[D_Y C(Y^*)(1 + D_y I(w^*,y^*)) - 1] - \frac{D_y q_d(w^*,y^*)\ D_Y C(Y^*)\ D_w I(w^*,y^*)}{D_w z(w^*,y^*)} > 0$$

Even when the economy is macroeconomically stable, the positive effect of the last term may be sufficient to allow unstable behavior to reveal itself through the income feedback loop. In this case we can put together a stable Keynesian system with a stable Walrasian system and get an unstable system out! (This possibility has been noticed by Leijonhufvud (L1).)

If the efficient equilibrium (w^*,y^*) does have index -1 and the appropriate boundary conditions are met, we know that there must be another virtual equilibrium of the system with index +1. Such an equilibrium must be macroeconomically stable. However, it is difficult to assert with certainty that this virtual equilibrium must be a real equilibrium. For a proof of existence, it seems that one must analyze the real dynamical system in a more explicit manner. This analysis should be possible through use of some new techniques developed by Carl Futia.

Appendix:

Proposition Under assumptions B1-B4, the Gauss map is nullhomotopic on the boundary of M.

Proof. We will show the degree of the Gauss map is zero on the boundary of YW; a simple continuity argument gives the desired result. To show the degree of the Gauss map is zero, it suffices to show that there is some vector in the unit ball that is not in the image of the Gauss map.

We will show that there is no point (w,y) on boundary YW where $\dot{y} < 0$ and $\dot{w}_i < 0$, $i=1,\cdots,n$, which will establish the proposition.

Consider the possible cases:

(1) Can $w_i = 0$ for any i? No, for by B4 we would have demand no smaller than supply in some market.

(2) Can $y = 0$? No, for B1 and B4 imply $\dot{y}$ would then be positive.

(3) Can $y = \bar{y}$? No, for B2 implies there would be excess demand for some factor.

(4) Can $w_i = \bar{w}_i$? No, for B3 implies that $\Sigma\, \bar{w}_i q_d^i(\bar{w},y) + P^* - y \geqq \bar{w}_i k_i + P^* - \bar{y} > 0$. □

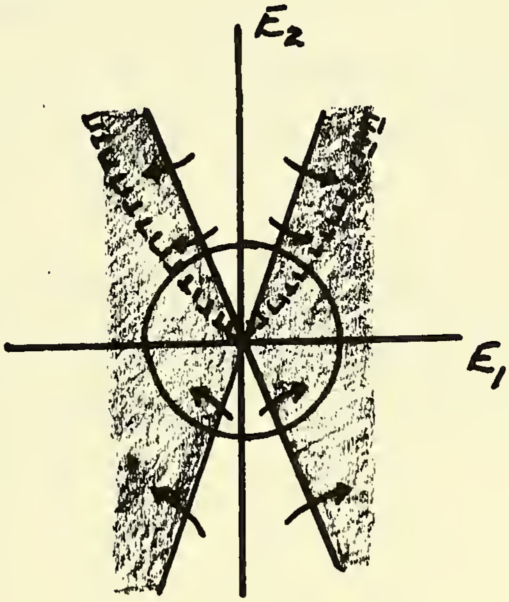
<u>Proposition</u> Let W be an open subset of R^n and let $f\colon W \to E \subset R^n$ be continuously differentiable. Consider the dynamical system $\dot{x} = f(x)$. Suppose that $f(\bar{x}) = 0$ and that some eigenvalue of $Df(\bar{x})$ has positive real part. Let $\bar{x}$ be adherent to the nonempty interior of some set V. Let U be a neighborhood of $\bar{x}$. Then for every neighborhood U_1 of $\bar{x}$ in U there is some solution to the differential equation $x(t)$ which starts at $x(0)$ in $U_1 \cap V$ which does not lie entirely in $U \cap V$.

<u>Proof</u>. The proof consists in a slight modification of the instability theorem on page 187 of (HS) and follows their notation. At the bottom of page 187, Hirsch and Smale choose a real number $a > 0$ such that every eigenvalue of A has real part greater than a. Then there is a Euclidean norm on E_1 such that $\langle Ax,x\rangle \geqq a|x|^2$ for all x in E_1. We will choose a slightly different Euclidean norm.

Let k be such that under the original choice of norms, $k|x| \geqq |y|$ for some (x,y) in $U_1 \cap V$. Then redefine $|x| = k|x|$ and $a = a/k$. This will ensure that the cone C defined on page 188 has a nonempty inter-

section with $U_1 \cap V$. Smale and Hirsch show that there are solutions starting arbitrarily close to zero in $C \cap U$ which leave U, which establishes the above proposition. The reader may find it helpful to compare Figure 5 to Figure E on page 188 of (HS) □

Figure 5 - The set V is cross-hatched, the cone C is shaded.



Footnotes

1. Since price is greater than marginal cost why don't firms cut their price? Many heuristic reasons can be suggested: (1) at this equilibrium demand equals supply so the market gives them no signal that sufficient demand will be forthcoming; (2) a price cut brings an immediate cut in profits with only a hope of more eventual sales; (3) if all prices are cut, the real wage changes; if the non-Walrasian equilibrium is stable with respect to such perturbations equilibrium will be re-established through the labor market.

2. This statement should be qualified slightly. We have shown that paths emanating from points in U near (w^*, y^*) have no tendency to move towards (w^*, y^*); however, when these paths leave U, they may begin to approach the Walrasian equilibrium.

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