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OPTIMAL TAXATION AND PUBLIC PRODUCTION

BY

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Theories of optimal production in a planned economy have usually assumed that the tax system can allow the government to achieve any desired redistribution of property.[1/] On the other hand, some recent discussions of public investment criteria in a mixed economy have tended to ignore taxation as a complementary method of controlling the economy.[2/] Although lump-sum transfers of the kind required for full optimality[3/] are not feasible today, commodity and income taxes can certainly be used to increase welfare.[4/] We shall therefore examine the maximization of social welfare using both taxes and public production as control variables. In doing so, we intend to bring together the theories of taxation, public investment, and welfare economics.

On the tax side, this analysis can be viewed as a series of extensions of the work of Ramsey (10) and Samuelson (11) , who have discussed the optimal commodity tax structure for raising a given revenue from a single consumer (or a community within which marginal utilities of income are equalized). The addition of public production to the set of control variables does not alter the nature of the optimal tax structure. Using the tax structure to improve income distribution, as well as to collect revenue, leads to a different optimal tax structure, but does not alter the nature of the analysis. Other complications, including the addition of an income tax, will also be considered.

On the expenditure side, the deliberate use of tax variables alters, and simplifies, public investment criteria. Many problems that apparently justify the use of public production rules different from private production rules are better treated by variations in the tax rates. Thus, in the face of various complications, the presence of the optimal tax structure implies the desirability of aggregate production efficiency. This will only be possible when marginal rates of transformation are the same in publicly and privately controlled production. The result can be viewed as a dominance of taxes, which affect both public and private production, over public production changes, which operate on a smaller "base."

Alternatively, our results can be viewed as an extension of the fundamental theorem of welfare economics. The latter can be interpreted as saying that any Pareto optimum can be achieved in a decentralized economy by employing lump-sum taxes to achieve the correct distribution of income. For a timeless economy, various ways of redistributing income in a lump-sum do not seem unattainable. But, in a multiperiod economy, any tax that varies with economic position will be noticed and will affect decisions at the margin. Only poll taxes (and subsidies) seem to be feasible lump-sum taxes. Assuming that no lump-sum taxes are employed, the only Pareto optimum that can be achieved is the competitive equilibrium (or equilibria) arising from the initial distribution of income. With many social welfare functions, including those that respect individual tastes, social welfare may be improved by moving to a competitive equilibrium with distorting taxes, which is thus not a Pareto optimum. We shall show, however, that the welfare maximum will usually require aggregate production efficiency. Thus, the optimum can be attained by decentralization employing two price vectors, one for consumers and a second for producers.

The first two sections of the paper contain analysis of a one consumer economy. Geometrical and calculus analyses are presented successively, showing the desirability of production efficiency and the calculation of the optimal tax structure. In the third section, the assumption of differentiability is not employed and the general case is considered using the methods of general equilibrium theory. In the fourth section, we examine the optimal tax structure in an economy with many households and a progressive income tax. In the fifth section, a number of complications and extensions are briefly considered.

I ONE CONSUMER ECONOMY - GEOMETRIC ANALYSIS

To present the basic structure of our argument clearly, let us begin by considering an economy with a single, price-taking consumer and just two commodities. We shall assume that all production possibilities are controlled by the government. While there is clearly no scope for redistribution of income in this economy, the government might need to raise revenue to cover losses if there are increasing returns to scale or if there are fixed expenditures (such as defense) and constant returns to scale. Alternatively the technology might exhibit decreasing returns to scale, facing the government with the problem of disposing of a surplus, if all transactions are carried out at market prices. The optimal solution to either raising or disposing of revenue is well known. A poll tax or subsidy, as the case may be, will permit the hiring of the needed resources and permit the economy to achieve a Pareto optimum, which, in a one consumer economy, is equivalent to the maximization of the consumer's utility. While this is a reasonable possibility in a one consumer economy, lump-sum taxes varying from individual to individual do not seem feasible in a much

larger economy. Thus we shall consider the use of commodity taxes when lump-sum taxes are not permitted to the planner, not for the intrinsic interest of this question in a one consumer economy but as an introduction to the many consumer case.

In an economy with free disposal, the technological constraint on the planner is that the government supply be on or under the production frontier. Such a constraint is shown by the shaded area in Diagram 1. Let us measure on the axes the quantities supplied to the consumer. Thus, the output being produced is measured positively, while the input is measured negatively. The case drawn is the familiar one of decreasing returns to scale. If the government needed a fixed bundle of resources, for national defense say, then the production possibility frontier (describing the potential transactions with the consumer) would not pass through the origin. With constant returns to scale this might appear as in Diagram 2, where a units of good one are needed for defense. (It is perhaps convenient to think of good one as labor and good two as a consumption good.)

With a totally planned economy, where the consumer is given a fixed consumption bundle (including labor to be supplied), the planner would have no further constraint and could choose any point that was technologically feasible. Again, this is not unreasonable for the planner in a one consumer economy, but becomes so as the number of households grows. A more realistic assumption, then, is to assume that the planner can only deal with consumer through the market place, hiring labor and selling the consumer good. (We shall assume further that the planner is constrained to charge uniform prices.) The planner must now set the price of the consumer good relative to the wage (or inversely the real wage), and is constrained to transactions which the consumer is

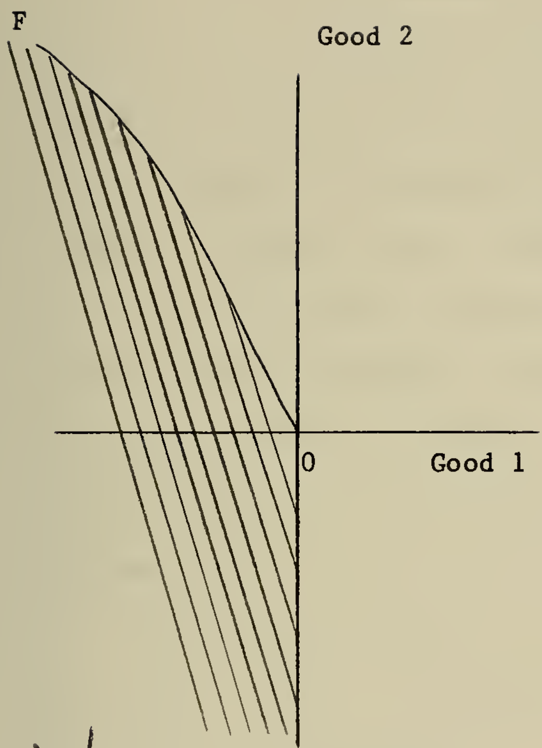


DIAGRAM 1

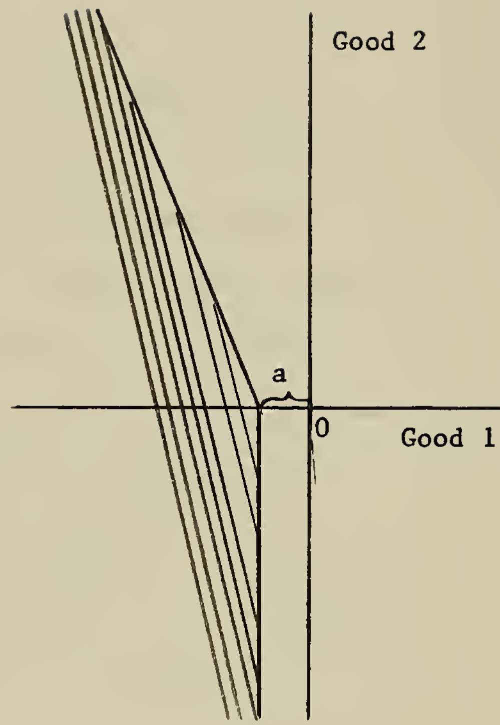


DIAGRAM 2

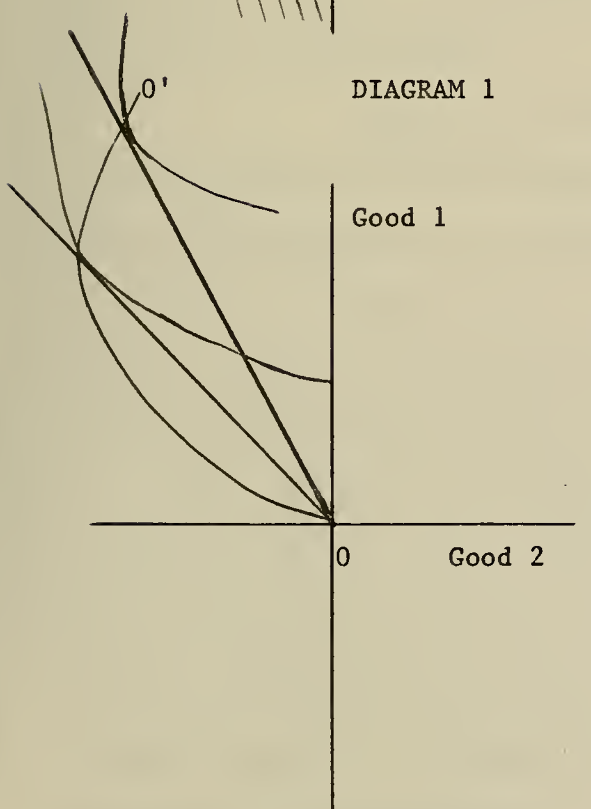


DIAGRAM 3

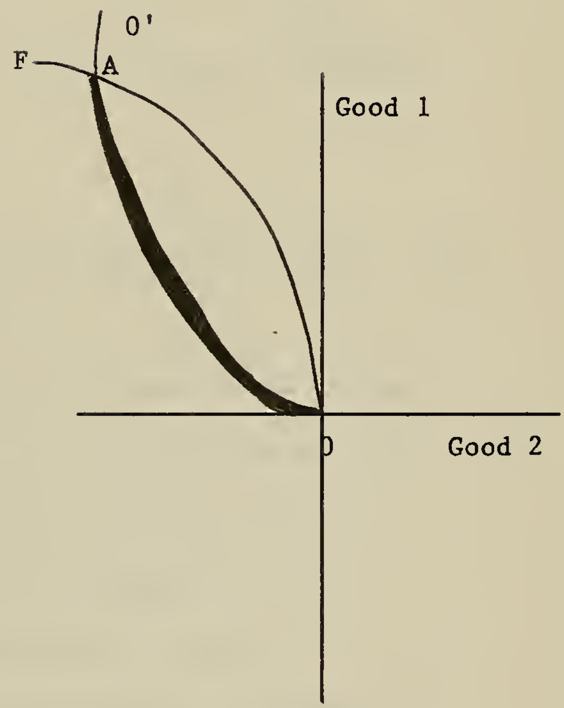


DIAGRAM 4

willing to undertake at some relative price. The locus of consumption bundles to which the consumer is willing to trade from the origin is the offer curve or price-consumption locus. It represents the bundles of goods that the consumer would purchase at different possible price ratios. Diagram 3 contains an example of an offer curve, with several hypothetical budget lines, and the corresponding indifference curves drawn in. The planner thus has two constraints: he must choose a point which is both technologically feasible and an equilibrium bundle from the point of view of the consumer. Combining these two constraints, the range of consumption bundles which are both feasible and potential consumer equilibria is shown as the heavy line in Diagram 4.

We can state these two constraints algebraically. Let us denote by $z = (z_1,\ldots,z_n)$ the vector of government supply. The production constraint is then written

1. $G(z) \leqq 0$, or equivalently, $z_1 \leqq g(z_2,z_3,\ldots,z_n)$.

The constraint that the government supply equal the consumer demand for some price can be written in vector notation

2. $x(q) = z,$

where $x = (x_1,\ldots,x_n)$ is the vector of consumer demands and $q = (q_1,\ldots,q_n)$ is the vector of prices faced by the consumer.

We can now add an objective function to our constraints to seek a welfare maximum. Let us consider the case where the planner seeks to maximize the same function of consumption as the consumer's utility function. The welfare function is said to be individualistic, or to respect individual preferences, since welfare can be written as a function of individual utility. Returning to Diagram 3 we see that the consumer moves to higher indifference curves as he proceeds along the offer curve away from the origin. Thus, in Diagram 4 we wish to move as

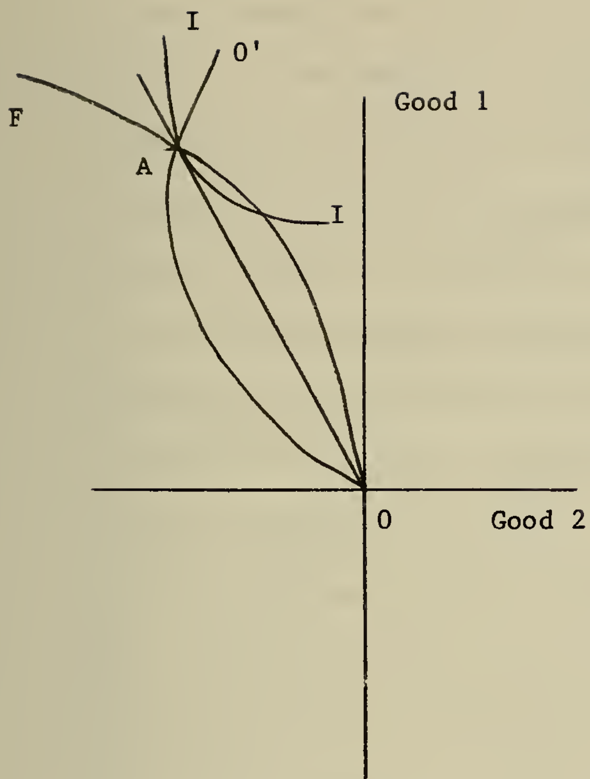


DIAGRAM 5

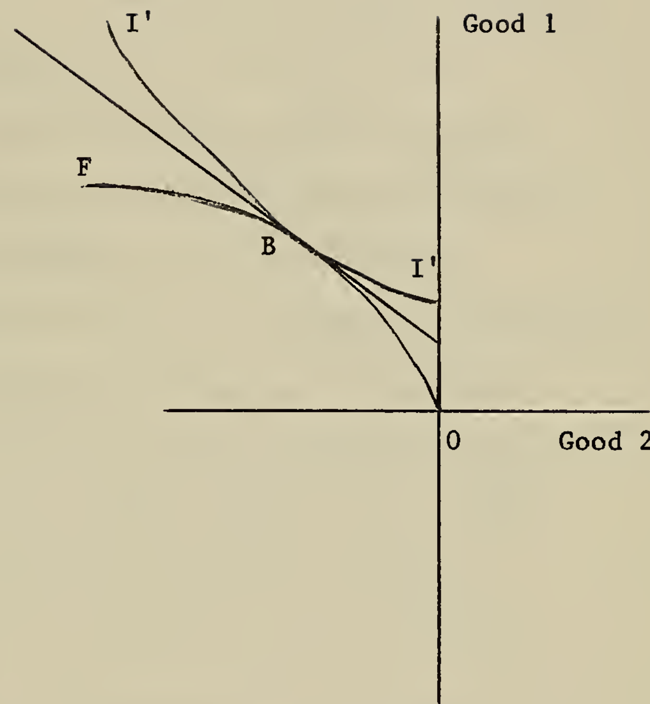


DIAGRAM 6

far along 00' as possible subject to the constraint of being on or under 0F. The optimal point will thus be A, where the offer curve and the production frontier intersect. The prices which will induce the consumer to purchase the optimal consumption bundle are defined by the budget line 0A. In Diagram 5 we show the optimal point and the implied budget line, and indifference curve II. All the points between II and 0F are Pareto superior to A and technologically feasible, but not attainable by market transactions without lump-sum transfers. For contrast, in Diagram 6, we show the Pareto optimal point, B, and the implied budget line, and indifference curve I'I', which will permit decentralization. In the case drawn, the consumer's budget line does not pass through the origin, representing his receipt of a lump-sum transfer from the profits of government production.

We can now see one property of the optimal configuration of the economy, namely that the optimal point is on the production possibility frontier of the economy, not inside it. This important property of the optimum point can easily be seen to carry over to the case of many commodities, but still one consumer. With many commodities, the offer curve is a union of loci, each of which is obtained by holding the prices of all but one commodity constant and varying the price of that one commodity. Doing this for each commodity and for all possible configurations of prices for the other commodities generates all the loci. The offer curve is the union of such loci. On each locus the point which is also on the production frontier is better than the other points on the locus. Thus, any point which is not on the production frontier is dominated by some point which is on the frontier. Therefore, the optimal point is one of the points on the frontier. The implications of this result will be seen more clearly below, when we consider both public and private production. For this result to carry over to the case of many consumers requires one further, mild, assumption which will be discussed in the third section. Before proceeding to the many consumer case, let us consider the one consumer economy algebraically, with both public and private production, to show by calculus the desirability of aggregate production efficiency and the relationship between consumer prices and the slope of the production possibilities. This relationship defines the optimal tax structure.

II ONE CONSUMER ECONOMY - ALGEBRAIC ANALYSIS

It is valuable to restate the problem of welfare maximization to incorporate private production explicitly and to state clearly the constraint that the government is selecting among the equilibrium positions of the economy. It is natural to begin with the statement of

the welfare function. The most general way to state the objective function in an equilibrium setting is to assume that social welfare is a function of prices to the consumer and the distribution of all lump-sum transfers in the economy. Transfers can come from three sources. First, consumers might give resources to other consumers (e.g. bequests). We shall rule out this possibility. Second, if firms earn profits these will be distributed to the owners of the firms. We shall assume constant returns to scale throughout the paper. This implies that in equilibrium there will be no profits to be distributed. (There is a brief discussion of this assumption in Section 5.) For this section we assume that the only tax variables at the command of the government are commodity taxes.[5/] This set of assumptions implies that there is zero lump-sum income. Thus, we can write the welfare function as a function of prices faced by consumers $v(q)$.

In the special case where social preferences coincide with those of the single consumer in the economy, the indirect welfare function in terms of prices is equal to the consumer's utility function, evaluated at the demand functions, which in turn, are functions solely of price. Algebraically,

3. $$v(q) = u(x(q)).$$

We shall not use this special form for $v(q)$ in the analysis below until we come to evaluate the tax structure explicitly. Until that point, the analysis applies to welfare functions that are not individualistic.

For future reference, it is convenient to evaluate the derivatives of the welfare function. A subscript on a function will refer to a partial derivative, that is $v_k = \partial v/\partial q_k$. Using (3) it can be shown that

4. $$v_k = \sum u_i \frac{\partial x_i}{\partial q_k} = - \alpha x_k,$$

where α is a positive constant (i.e., independent of k), representing

the marginal utility of income. Since the demand curves were derived from utility maximization, they satisfy the budget constraint,

$$5. \qquad \sum q_i x_i = 0.$$

Differentiating (5) with respect to q_k,

$$6. \qquad x_k + \sum q_i \frac{\partial x_i}{\partial q_k} = 0.$$

Thus, the relationships $u_i = \alpha q_i$, which are necessary for utility maximization, allow us to deduce (4).

Production

We assume constant return to scale in privately controlled production; and we shall further assume that private entrepreneurs are price takers. This implies that the supplies of privately produced goods are functions solely of the prices that producers face. Let us denote by $p = (p_1, \ldots, p_n)$ the vector of prices faced by private producers. These differ from the prices faced by consumers by the tax structure, $q_i = p_i + t_i$ $(i = 1, \ldots, n)$. Let us denote by $y = (y_1, \ldots, y_n)$ the vector of commodities privately supplied[6/] (i.e., factor demands are negative supplies). By the assumption of constant return to scale, we know that maximized profits are zero in an equilibrium:

$$7. \qquad \sum p_i y_i = 0.$$

We further assume that private supplies (and all other functions) are continuously differentiable. So that we may conveniently employ calculus, we shall assume that the government production constraint, equation (1), is satisfied with an equality rather than an inequality. Thus we do not give the government the option of inefficient government production. Rather, we shift our attention to aggregate production efficiency. Efficiency will be present if marginal rates of transformation are the same in publicly and privately controlled production.

Walras Law

We have chosen an objective function and expressed the government's production constraint above. To complete the formulation of the maximization problem, it remains to add the requirement that the economy be in equilibrium. The conditions that all markets clear can be stated

$$8. \qquad x(q) = y(p) + z.$$

The reader may be puzzled that at no place in this formulation has a budget constraint been introduced for the government. (Other readers may be puzzled by our failure to include only n-1 markets in our market clearance equations. These are aspects of the same phenomenon.) Walras' Law implies that if all economic agents satisfy their budget constraints and all markets but one are in equilibrium, then the last market is also in equilibrium. It also implies that when all markets clear and all economic agents but one are on their budget constraints, then the last economic agent is on his budget constraint. In setting up our problem, we have assumed that the household and the private firms are on their budget constraints. Thus, if we assume that all markets clear, this will imply that the government is satisfying its budget constraint,[7/] which we can express as

$$9. \qquad \sum(q_i - p_i)\, x_i + \sum p_i z_i = 0 = \sum t_i x_i + \sum p_i z_i .$$

Alternatively, if we consider the government budget balance as one of the constraints, then it is only necessary to impose market clearance in n-1 of the markets.

There is a further choice in setting up the maximization problem. We can consider both sets of prices, q and p, as under government control with the full set of market equations as constraints. But instead we can assume - and this at first seems more natural - that it is taxes which

are under government control[8/] and that the producer and consumer prices are related to the control variables by means of the market clearance equations. In this formulation these equations are not constraints of the maximization, but rather define the prices in terms of the control variables.

In this model we can make two price normalizations, one for each price structure. Since both consumer demand and firm supply are homogeneous of degree zero in respective prices, changing either price level without altering relative prices leaves the equilibrium unchanged. As normalizations let us assume,

10. $p_1 = 1,\ q_1 = 1,\ t_1 = 0.$

It may seem surprising that it does not matter whether the government can tax good one. But the reader should remember the budget balance of the consumer. Since, there are no lump-sum transfers to the consumer, net consumer expenditures are zero. Thus, levying a tax at a fixed proportional rate on all consumer transactions results in no revenue. (It should be noticed that a positive tax rate applied to a good supplied by the consumer is in effect a subsidy and results in a loss of revenue to the government.)

Welfare Maximization

We can now state the maximization problem. Rather than calculating the first order conditions from the formulation we have spelled out above, we shall consider various changes to simplify the calculations. First let us restate the basic problem. We have to choose

11. $q_2,\ldots,q_n,\ p_2,\ldots,p_n,\ z_1,\ldots,z_n,$

to maximize $v(q)$

subject to $x_i(q) - y_i(p) - z_i = 0 \qquad i = 1,2,\ldots,n$

$G(z_1,\ldots,z_n) = 0.$

Since the producer prices will be determined by market clearance for any given choice of the other variables, we can remove n-1 control variables and change n-1 constraints to definitional equations for the price variables. We can also eliminate z_1 by rewriting the production constraint as in equation (1).

12. Choose $z_2,\ldots,z_n$, $q_2,\ldots,q_n$,

$$\text{to maximize } v(q)$$

$$\text{subject to } x_1(q) - y_1(p) - g(z_2,\ldots,z_n) = 0$$

$$\text{where } x_i(q) - y_i(p) - z_i = 0 \qquad i = 2,3,\ldots,n.$$

To simplify further we can use the private production constraint,

13. $$F(y) = 0 \quad \text{or } y_1 = f(y_2,\ldots,y_n)$$

to replace y_1 in the constraint. The remaining private supplies can be eliminated by market clearance giving us a simple form for the maximization:

14. Choose $z_2,\ldots,z_n$, $q_2,\ldots,q_n$,

$$\text{to maximize } v(q)$$

$$\text{subject to } x_1(q) - f(x_2(q) - z_2,\ldots,x_n(q) - z_n) - g(z_2,\ldots,z_n) = 0.$$

Having solved this maximization the producer prices can be determined from the market clearance equations, or more simply from private profit maximization first order conditions

15. $$p_i = - f_i(y) = - f_i(x - z).$$

Forming a Lagrangian expression from (14), with multiplier λ,

16. $$L = v(q) - \lambda[x_1(q) - f(x_2 - z_2,\ldots,x_n - z_n) - g(z_2,\ldots,z_n)]$$

we can differentiate with respect to q_k:

17. $$v_k - \lambda\left(\frac{\partial x_1}{\partial q_k} - \sum_{i=2}^{n} f_i \frac{\partial x_i}{\partial q_k}\right) = 0. \qquad k = 2,3,\ldots,n.$$

Making use of the equations for producer prices this can be written

$$18. \qquad v_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_k} = 0. \qquad k = 2,3,\ldots,n.$$

Differentiating L with respect to z_k we have

$$19. \qquad \lambda(f_k - g_k) = 0. \qquad k = 2,3,\ldots,n.$$

Provided that λ is unequal to zero (provided that there is a social cost to a marginal need for additional resources), equation (19) implies equal marginal rates of transformation in public and private production and thus aggregate production efficiency as was argued above. The relations given by (18) determine the optimal tax structure, which we shall now examine.

Optimal Tax Structure

There is a striking asymmetry in the way demand and supply curves appear in the description of the optimum. The optimum taxes depend upon demand elasticities but not on supply elasticities. One can see how this asymmetry arises from the different ways in which production and consumption enter the constraints. The production constraint simply states that the equilibrium quantities be feasible, a statement about quantities alone. The consumption constraint, on the other hand, requires that the equilibrium lie on the price-consumption locus, or, in other words, that the supporting budget line pass through the origin, a statement about quantities and the slope of the indifference curve. Assuming the offer curve and production frontier are not tangent, a small change in the slope of the production frontier at the equilibrium point A in Diagram 5 leaves equilibrium quantities unchanged. But a change in the slope of the indifference curve through A makes that point no longer attainable and shifts the optimum. All this contrasts sharply with an economy in which lump-sum taxes are used, for in that case a change in the slope of either the indifference curve or the production frontier shifts the optimal point.

Since x_i is a function of p+t, $\frac{\partial x_i}{\partial q_k} = \frac{\partial x_i}{\partial t_k}$. Consequently, the optimal tax structure, (18), can be rewritten:

$$20. \qquad v_k = \lambda \frac{\partial}{\partial t_k} (\sum p_i x_i)$$
$$= - \lambda \frac{\partial}{\partial t_k} (\sum t_i x_i),$$

where the derivatives are defined at constant producer prices, and use is made of the consumer budget constraint, $\sum q_i x_i = 0$.

This last set of equations asserts the proportionality of the marginal utility of a change in the price of a commodity to the change in tax revenue resulting from a change in the corresponding tax rate, calculated at constant producer prices constant. Like the first order conditions for the optimum in elementary welfare economics, our first order conditions are expressions in constant prices. The tax administrator, like the production planner, need not be concerned with the response of prices to government action when looking at the first order conditions.

The first order conditions, as expressed in equations (20), show quite conveniently what information is needed to discover whether we are at the optimum. They do not directly indicate the size of the tax rates required, nor the impact upon demand that the optimal tax rates would have. In his pioneering study of optimal tax structure, Ramsey manipulated the first order conditions so as to shed light on the latter question. He employed the concept of demand curves calculated at a constant marginal utility of income. Samuelson reformulated this using the more familiar demand curves calculated at a constant level of utility. We reproduce the Samuelson analysis here.

We shall eliminate the derivatives of welfare from (18), assuming an individualistic welfare function, by employing equations (4).

Further we shall use the Slutsky equation,

$$\text{21.} \qquad \frac{\partial x_i}{\partial q_k} = s_{ik} - x_k \frac{\partial x_i}{\partial I} ,$$

where s_{ik} is the derivative of the compensated demand curve for i with respect to q_k, and $\frac{\partial x_i}{\partial I}$ is the derivative of the uncompensated demand with respect to income. We shall make use of the well-known result that $s_{ik} = s_{ki}$.

Substituting from (4) and (21) into the first order conditions, in the form (20), we have:

$$\text{22.} \qquad - \alpha x_k = - \lambda x_k - \lambda \sum t_i s_{ik} + \lambda x_k \sum t_i \frac{\partial x_i}{\partial I} . \quad k = 2,\ldots,n.$$

Rearranging terms, we can write this in the form:

$$\text{23.} \qquad \frac{\sum\limits_i t_i s_{ik}}{x_k} = \frac{\alpha + \lambda - \lambda \sum t_i \frac{\partial x_i}{\partial I}}{\lambda} .$$

The point to be noticed is that the right-hand side of this equation is independent of k. Finally, using the symmetry of the Slutsky matrix, we write the first order conditions as:

$$\text{24.} \qquad \frac{\sum\limits_i s_{ki} t_i}{x_k} = \text{constant}.$$

The left-hand side of this expression is the percentage change in the demand for good k that would result from the tax change if the consumer were compensated so as to stay on the same indifference curve and if the derivatives of the compensated demand curves were constant at the same level as at the optimum point. Formally:

$$\text{25.} \qquad \Delta x_k = \sum_i \int_o^{t_i} \frac{\partial x_k}{\partial t_i} dt_i = \sum_i \int_o^{t_i} s_{ki} dt_i$$

$$= \sum_i s_{ki} \int_o^{t_i} dt_i = \sum_i s_{ki} t_i .$$

Thus the optimal tax structure implies an equal percentage change in compensated demand at constant producer prices. We can also calculate the actual changes in demand arising from the tax structure (assuming price derivatives of demand and production prices are constant) by resubstituting from the Slutsky equation in equation (24). Denote the right-hand side of (24) by θ. Then, upon substitution, we have:

26. $$\sum \frac{\partial x_k}{\partial q_i} t_i + \frac{\partial x_k}{\partial I} \sum t_i x_i = \theta x_k;$$

or

27 $$\frac{\sum \frac{\partial x_k}{\partial q_i} t_i}{x_k} = \theta - x_k^{-1} \frac{\partial x_k}{\partial I} \sum t_i x_i.$$

The actual changes in demand induced by the tax structure differ from proportionality, with a larger than average percentage fall in demand for goods with a large income derivative. Given the equal percentage change in compensated demands, it is not surprising that uncompensated demands differ in their percentage changes depending on differences in income derivatives.

It should be noted that although equations (24) (and (27)) were derived from the first order conditions and thus have been shown to hold only for goods 2 through n, equations (24) and (27) also hold for good one. These equations were obtained by substituting the utility maximization conditions (4) and the Slutsky relations (21) into the first order conditions (18). (4) and (21) certainly hold for $k = 1$. To see that (18) holds for $k = 1$, multiply (18) by $p_k + t_k$ and sum over $k = 2,\ldots,n$. Since everything is homogeneous of degree zero in prices, and $p_1 + t_1 = 1$, we get (18) for $k = 1$.

The above argument is only an approximation, useful when the revenue collected by taxation is sufficiently small. Indeed it is not possible for all the demand derivatives to be constant simultaneously over a range.

Examples

The implications of the above model are very diverse, depending upon the nature of the demand functions. A simple example will show how the theory can be used. If we define demand elasticities by the usual formula

28. $$\varepsilon_{ik} = q_k x_i^{-1} \frac{\partial x_i}{\partial q_k},$$

we can rewrite the optimal taxation formula in the form

29. $$v_k = q_k^{-1} \lambda \sum p_i x_i \varepsilon_{ik} ;$$

which becomes, when the welfare function is individualistic and (4) applies,

30. $$- \alpha q_k x_k = \lambda \sum p_i x_i \varepsilon_{ik} \quad \text{or} \quad q_k p_k^{-1} = - \frac{\lambda}{\alpha} \sum \frac{p_i x_i}{p_k x_k} \varepsilon_{ik} :$$

If we have a good the price of which does not affect other demands, (implying a unitary own price elasticity), equation (30) simplifies to yeild the optimal tax of that good:

31. If $\varepsilon_{ik} = 0 \quad (i \neq k)$ and $\varepsilon_{kk} = - 1$,
then $q_k p_k^{-1} = \lambda \alpha^{-1}$,

where $q_k p_k^{-1}$ equals one plus the percentage tax rate. Recalling that α is the marginal utility of income while λ reflects the change in welfare from allowing a government deficit, their ratio gives a cost (in terms of the numeraire good) of raising revenue. Thus the optimal tax rate on such a good gives the cost to society of raising the marginal dollar of tax.

An example of a utility function exhibiting such demand curves is the Cobb-Douglas, where only labor is supplied. As an example consider:

32. $$u(x) = \beta_1 \log (x_1+\omega_1) + \sum_{i=2}^{n} \beta_i \log x_i.$$

If we choose labor as the untaxed numeraire, all other goods satisfy (31) and we see that the optimal tax structure is a proportional tax structure.

It is easy to exhibit examples where the optimal tax structure is not proportional. Consider the example:

33. $$u(x) = \sum \beta_i \log (x_i+\omega_i), \quad \sum \beta_i = 1.$$

The demands arising from these preferences are:

34. $$x_i = q_i^{-1} \beta_i \sum q_j \omega_j - \omega_i.$$

Therefore the demand elasticities are:

35. $$\varepsilon_{ik} = \beta_i \omega_k x_i^{-1} \frac{q_k}{q_i} \qquad (k \neq i),$$

$$\varepsilon_{kk} = - \beta_k x_k^{-1} \sum_{j \neq k} \omega_j q_j / q_k.$$

Substituting in the formula for the optimal taxes,

36. $$- \alpha q_k x_k = \lambda [\sum_{j \neq k} \beta_j \frac{p_j}{q_j} \omega_k q_k - \beta_k \frac{p_k}{q_k} \sum_{j \neq k} \omega_j q_j]$$

$$= \lambda \sum_j [\beta_j \omega_k \frac{p_j q_k}{q_j} - \beta_k \omega_j \frac{p_k q_j}{q_k}].$$

Since the assumption $\sum \beta_j = 1$ allows us to write the demand functions (34) in the form:

37. $$q_k x_k = \sum_j [\beta_k \omega_j q_j - \beta_j \omega_k q_k],$$

we can deduce from (36) that

$$38. \qquad \sum_j [\beta_j \omega_k q_k (\frac{p_j}{q_j} + \frac{\alpha}{\lambda}) - \beta_k \omega_j q_j (\frac{p_k}{q_k} + \frac{\alpha}{\lambda})] = 0.$$

These equations allow us to calculate p for any given q, and in that way give the optimal taxation rules. In general, taxes cannot be proportional; for if $\frac{p_j}{q_j}$ were the same for all j, we should have

$$39. \qquad \sum_j [\beta_j \omega_k q_k - \beta_k \omega_j q_j] = 0;$$

which is in general impossible, and in any case holds only when optimal producer prices are in a special relationship to one another.

III THE GENERAL CASE

In an economy with many households, taxes and subsidies are imposed both to finance public expenditures and to redistribute income. For the present we continue to suppose that only proportional taxes on goods and services are possible. Even taxes of this kind can improve the distribution of income, although the degree of improvement depends on the diversity of tastes. It might be supposed that suitable departures from efficient production could also improve the distribution of income: but, if taxes are optimal, this is true only in exceptional cases. We have to assume, however, that <u>all</u> goods and services can be taxed (or subsidized) to <u>any</u> extent we choose.

In some ways, it is easier to see what is going on if we think of the economy as a planned economy. Production is controlled by the government, and so are the prices that consumers pay or receive for goods and services. Households determine their activities freely, subject to these prices, and their own initial wealth. We shall show that, in general, production should take place on the production frontier. Further, if production possibilities form a convex set, there will exist producers'

prices at which optimal production will maximize profit. The difference between optimal consumer prices and optimal producer prices can be regarded as the optimal taxes and subsidies if production is to be run in an entirely decentralized manner, or by private, but perfectly competitive, firms.

We therefore consider the following problem. Given a welfare function depending upon the consumption of households; given that households make their consumption choices constrained by prices set by government; and given that aggregate household demands for goods must be capable of being satisfied; we seek consumer prices that will maximize the welfare function. (We assume there are no externalities between consumers, or between consumers and producers.)

Suppose that our problem has an answer: q^* is a vector of optimum consumer prices. If one of the commodities has a positive price, and is purchased by all households, and at least one of the households would be better off if it had more of it, a reduction in the price of that commodity would increase social welfare, if the latter reflects individual preferences. Therefore there is a sequence of price vectors tending to q^*, none of which leads to household demands that the economy can satisfy - for otherwise q^* would not be optimal. If the demand functions are continuous functions of prices, the production vectors corresponding to this sequence of price vectors - none of which are in the set of feasible production vectors - likewise form a sequence converging to the production-vector that corresponds to q^*, which we can denote by x^*. This simple argument shows that x^* must actually be <u>on the frontier</u> of the production set - i.e., on the production frontier.[9/]

This brief sketch of the argument is not rigorous, but it does assume that the government can arrange for consumer prices to take on any value. The reader may suspect that the same argument does go through if other

taxes - e.g., progressive taxes - are allowed. This is indeed the case, as we shall see in the next section.

To establish our propositions rigorously, and elucidate the assumptions that are required, we now proceed more formally.[10]

Assumptions

There are H households in the economy. For h = 1,2,...,H, household h seeks the most preferred consumption vector in his consumption set X^h, subject to his budget constraint,

40. $$q.x \leqq q.x_o^h ,$$

where x_o^h is the initial endowment of the h^{th} household. It should be made clear that the vector x has, in general, both positive and negative components corresponding to purchases and sales by the household.

There are several different assumptions on household preferences which will be employed:

(a.1) X^h is closed and convex,

(a.2) there exists a vector a^h such that $a^h \leqq x$ for all x in X^h,

(a.3) there exists a vector $\hat{x}^h$ in X^h such that $\hat{x}^h << x_o^h$,

(a.4) for every x' in X^h, the sets $\{x \text{ in } X^h \mid x \underset{h}{\succsim} x'\}$ and $\{x \text{ in } X^h \mid x' \underset{h}{\succsim} x\}$ are closed,

(a.5)[11] if x^1 and x^2 are two points in X^h and if t is a real number in]0,1[then $x^2 \underset{h}{\succ} x^1$ implies $tx^2 + (1-t)x^1 \underset{h}{\succ} x^1$,

(a.6) there is no satiation consumption in X^h.

Assumptions (a.1) and (a.4) guarantee the existence of a continuous utility function which we shall write u^h (Debreu 4.6). Under assumptions (a.1) to (a.6) demand is a continuous function of prices for positive prices.[12] (If we further restrict the consumption set by an upper bound, demand is a continuous function of prices for all prices.) We can write individual demand as

41. $$D^h(q) = x^h - x_o^h.$$

Aggregate demand will be written as

42. $$D(q) = \sum_h D^h(q).$$

We shall denote the set of feasible production vectors by F. Assumptions on production will be selected from the following list:

(b.1) $-\Omega$ is in F, (where Ω is the non-negative orthant),

(b.2) 0 is in F,

(b.3) F is closed,

(b.4) F is convex,

(b.5)[13/] there exists a vector a such that $x \leqq a$ for all x in $F \cap \Omega$.

The welfare function will be denoted by $U(x^1,\ldots,x^H)$. It is said to respect household preferences if it can be written

43. $$U(x^1,\ldots,x^H) = W(u^1(x^1),\ldots,u^H(x^H)),$$

with W strictly increasing in each argument. Since demand is a function of prices we can write the indirect welfare function

44. $$V(q) = U(D^1(q),\ldots,D^H(q)).$$

The assumption which will be employed on preferences is

(c.1) U is a continuous function of $(x^1,\ldots,x^H)$.

Existence of an Optimum

We can now state the government's maximization problem as

45. Maximize $V(q)$ subject to $D(q)$ being in F.

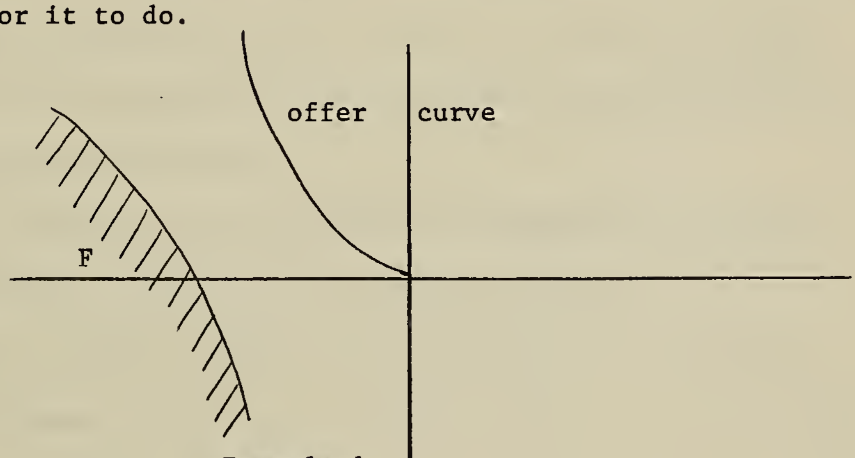
A commodity vector will be called attainable if it is feasible and if there exist prices such that aggregate demand equals this vector. The set of attainable vectors will be called the attainable set. We seek the best attainable vector. The approach to this problem will be via six theorems. The first two give conditions for the attainable set to be non-empty and bounded, respectively. Since the attainable set may have these properties without satisfying the hypotheses of these theorems, the third theorem gives conditions which, together with the boundedness and

non-emptiness of the attainable set, imply the existence of an optimum. The fourth theorem presents conditions for the optimal production vector to be on the production frontier. The fifth and sixth theorems refer to the optimal tax structure.

Theorem 1. If assumptions (a.1) - (a.6), and (b.1) hold, then there exists an x in F and a $q \geq 0$ such that $x = D(q)$.

Proof: Consider an economy with these consumers, where the only production possibilities are those of free disposal. Then, for this exchange economy, there exists an equilibrium (Debreu 5.7). Let q^* be the equilibrium prices. Then $D(q^*)$ is in $-\Omega$ which is contained in F.

Let us note that assumption (b.1) seems excessively strong in that we may wish to consider economies which need to provide for public defense, or perhaps pay tribute to a foreign power. For such a country doing nothing may not be a feasible production vector. It is thus more appealing to assume that the attainable set is non-empty in the analysis below. One can construct examples of economies not satisfying (b.1) for which the attainable set is empty (and so no optimum exists). Consider an economy with a single consumer, as depicted in the diagram. Here the government wants to do more than it is possible for it to do.



Example 1.

Theorem 2. If assumptions (a.2) and (b.2) - (b.5) hold, then the attainable set is bounded.

Proof: Suppose the attainable set, A, is not bounded. Then there exists a sequence of vectors x_n, x_n in A, such that $||x_n||$ is an unbounded increasing sequence of real numbers. There exists an n' such that $||x_{n'}||>||a||$, where a is a vector employed in (b.5). Consider the sequence of vectors $(||x_{n'}||/||x_n||)x_n$ for $n \geqq n'$. Each vector is in the feasible set (being a convex combination of the origin and x_n). Further the sequence is bounded. Thus there is a limit point, z, which is in F and satisfies $||z||>||a||$. Let $b = \sum_h a_h$, where a_h is the vector employed in (a.2). Then $x_n = \sum_h x_n^h \geqq \sum_h a_h = b$. Further $(||x_{n'}||/||x_n||)x_n \geqq (||x_{n'}||/||x_n||)b$. But the latter sequence of vectors converges to zero. Thus $z \geqq 0$. This is a contradiction.

As with theorem 1, we may very well have the attainable set bounded without satisfying the hypotheses of the theorem. For example, F or X^h might be bounded above.

Theorem 3. If assumptions (a.1) - (a.6), (b.3), and (c.1) hold, and if the attainable set, A, is non-empty and bounded, then there exists an optimum.

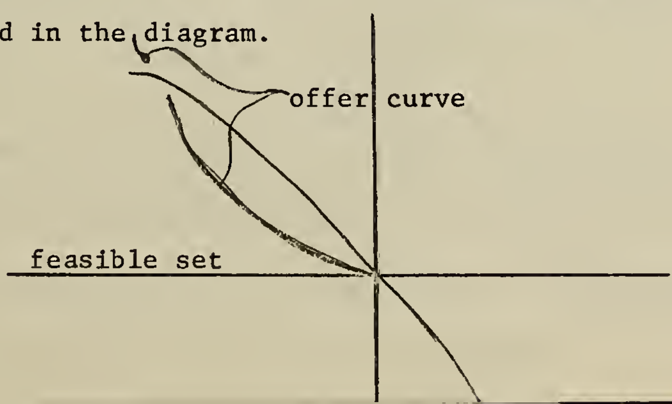
Proof: Consider an economy where all consumption sets are bounded by a larger bound than the bound on A. For this economy (denoted by $\sim$) the demand function $\tilde{D}^h$ are continuous for all price vectors unequal to zero. Further, $A = \tilde{A}$ and $D^h = \tilde{D}^h$ for any q corresponding to an attainable vector. Thus an optimum for this economy is an optimum

for the original economy and without loss of generality we can assume that demands are continuous for $q \neq 0$. Since consumers are not satiated, $q = 0$ does not give an attainable demand. Thus we can restrict analysis to price vectors satisfying $\sum q_i = 1$.

We shall now demonstrate that the set $\{q|D(Q) \text{ in } F\}$ is closed. Let q_n be a sequence of price vectors converging to q' and satisfying $D(q_n)$ in F for all n. Let x' be a limit point of $\{D(q_n)\}$. Since F is closed, x' is in F. Since D is continuous $x' = D(q')$. Thus q' is in $\{q|D(q) \text{ in } F\}$. Since A is not empty, $\{q|D(q) \text{ in } F\}$ is closed, bounded, and non-empty. Since $V(q)$ is continuous, it assumes its maximum.

The conditions leading to the existence of an optimum are not extremely strong, and no doubt, optima can exist with weaker conditions. (The appendix shows that strong convexity can be weakened to convexity.) However, it is useful to remember that there can exist cases where an optimum does not exist. One such example was given above. As further examples, consider:

Example 2: an economy with a single consumer. Assume further that the consumer has lexicographic preferences defined on what would otherwise be his indifference curves. If his "indifference curve" has a linear portion, demand curves will not be continuous and there may not exist an optimum. Such a case is depicted in the diagram.



Example 3: Demand curves may also be discontinuous when some prices equal zero if the initial endowments are not strictly positive. This can prevent the existence of an optimum. In the example presented here, it is desirable to raise the price of the zero[th] good without limit. With $\sum q_i = 1$, this means lowering other prices toward zero. The existence problem here can be solved by giving the government the power to outlaw production and consumption of the zero[th] good.

Assume there is a single consumer, whose demands[14/] are

46. $D^i(q) = \alpha_i q_i^{-1} \sum_i q_i x_i^o \qquad i = 0,1,\ldots,n.$

Let the social welfare function be

47. $$V = \sum_{i=1}^{n} \alpha_i \log x_i .$$

Thus, individuals desire the zero[th] good to which society is neutral. Let production possibilities be

48. $$\sum_{i=0}^{n} p_i (x_i - x_i^o) = 0.$$

If an optimum exists, it will satisfy the first order conditions for the constrained maximization:

49. $$\underset{q_i}{\text{Maximize}} \sum_{i=1}^{n} \alpha_i \log (\alpha_i q_i^{-1} \sum_j q_j x_j^o)$$

$$\text{subject to} \sum_{i=0}^{n} p_i \alpha_i q_i^{-1} \sum_j q_j x_j^o = \sum p_i x_i^o .$$

Differentiating with respect to q_0 (assuming $x_0^0 = 0$) we have

50. $$- \lambda q_o^{-2} p_0 \alpha_0 \sum_j q_j x_j^o = 0.$$

Since λ is unequal [15/] to zero, this condition cannot be satisfied. As suggested above, the optimum has $q_0 = +\infty$, for society does not want to waste resources producing good zero. Simply banning the good solves the problem.

Efficiency

Even if an optimum exists, we shall have to impose further conditions before we can deduce that optimal production is on the frontier. We first present a general, but strangely formulated Lemma giving sufficient conditions. Theorem 4 then presents the conditions when the welfare function respects individual preferences.

Lemma 1. Assume an optimum exists. If aggregate demand functions and the indirect welfare function are continuous in the neighborhood of the optimal prices and if either

(1) for some i, V is a strictly increasing function of q_i in the neighborhood of q^*; or

(2) for some i with $q_i^* > 0$, V is a strictly decreasing function of q_i in the neighborhood of q^*;

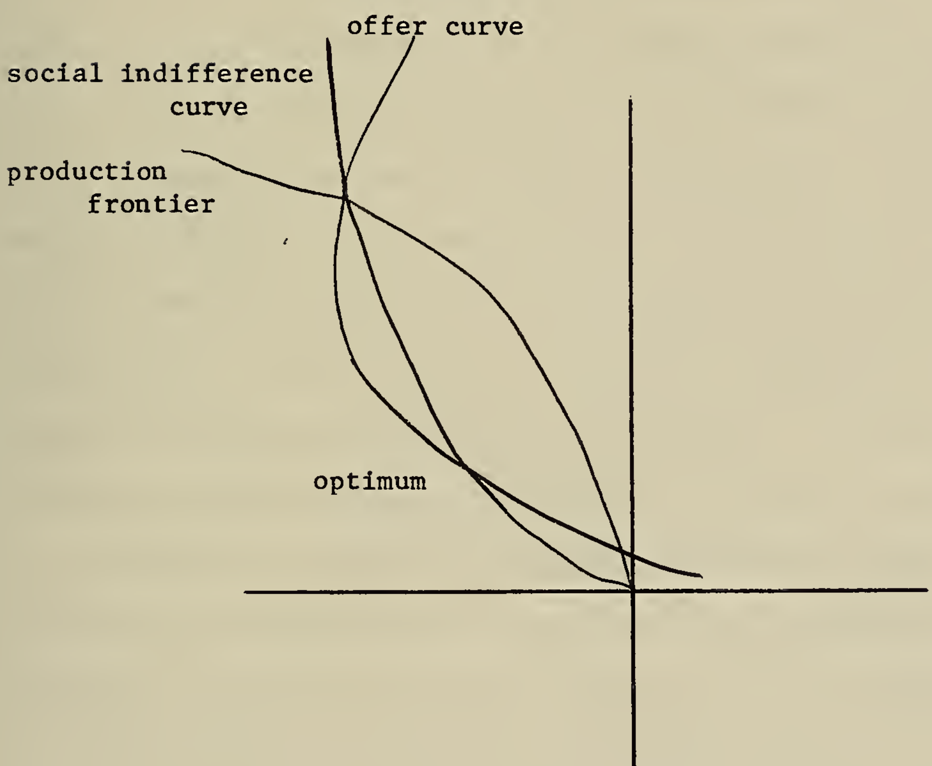
then production at the optimum occurs on the frontier of the feasible production set.

Proof: Let ℓ_i be the vector with all zero components except the $i^{\underline{th}}$, which is one. In case 1, for ε sufficiently small $V(q^* + \varepsilon\ell_i) > V(q^*)$. Hence $D(q^* + \varepsilon\ell_i)$ is not in F. Letting ε decrease to zero, the continuity of D shows that $D(q^*)$ is a limit of points not in F, and therefore belongs to the boundary of F. In case 2, a similar argument can be made using $V(q^* - \varepsilon\ell_i)$.

This lemma will be the basis for showing that optimal production is on the frontier when social welfare respects individual preferences. When preferences are not respected, the conditions of the lemma may still be satisfied, although alternatively they may not be and optimal production need not be on the frontier as is shown in

Example 4: Consider an economy with a single consumer. Then, social

indifference curves can be drawn which differ from individual curves. It can then occur that a social indifference curve is tangent to the offer curve at a point below the frontier and this point is a welfare maximum. (This cannot happen with strict convexity if the welfare function respects individual tastes.)



Theorem 4. If (a.1) - (a.6) and (c.1) hold, if social welfare respects individual preferences and if either

(1) for some i, $x_i^h - x_{oi}^h \leq 0$ for all h, and $x_i^{h'} - x_{oi}^{h'} < 0$ for some h'; or

(2) for some i with $q_i > 0$, $x_i^h - x_{oi}^h \geq 0$ for all h and $x_i^{h'} - x_{oi}^{h'} > 0$ for some h';

then if an optimum exists, production for the optimum is on the frontier of the feasible set.

Proof: Individual demand functions are continuous in the neighbor-

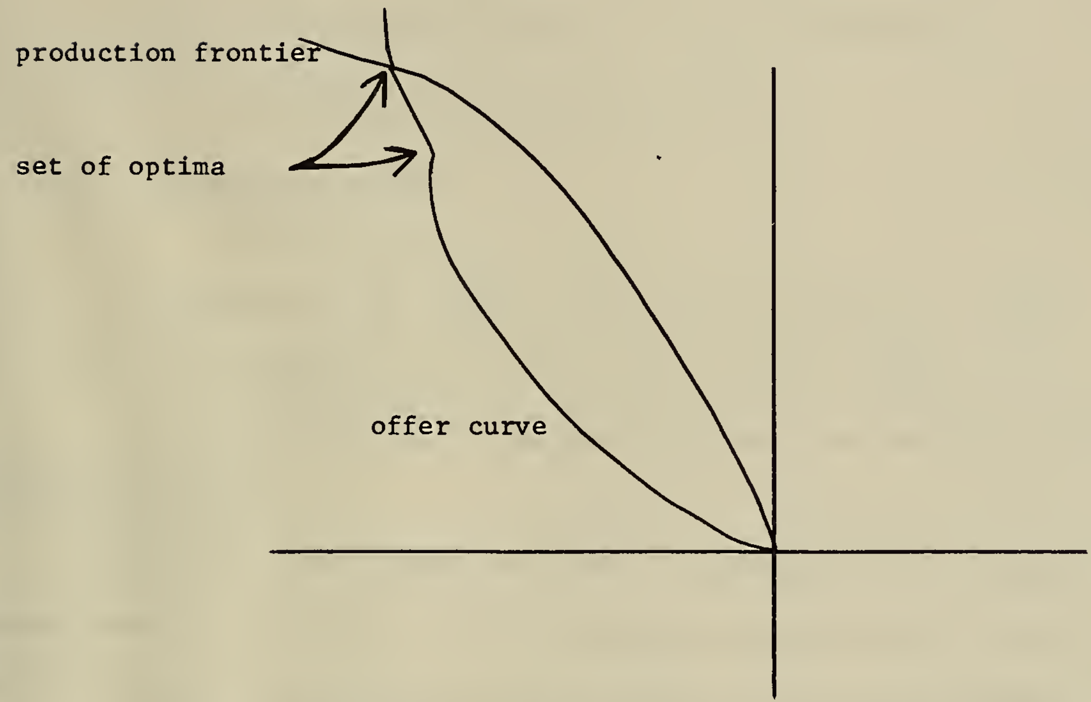
hood of the optimum (see Lemma in the appendix) and thus aggregate demands and the indirect welfare function are continuous. Since social welfare respects preferences, indirect social welfare can be written as an increasing function of indirect utilities. In case 1 indirect utilities are a non-decreasing function of q_i in the neighborhood of q^* for all h while the indirect utility function of h' is strictly increasing in q_i. Thus V increases with q_i. Case 2 follows similarly.

Assuming that demand curves are continuous, if there is an optimum which is internal to the production set, then any small change in prices from the optimal prices still leaves aggregate demands which are in the feasible set. Thus we will have an internal optimum only if no price change at the optimum increases social welfare. If we have conditions guaranteeing the monotonicity of welfare in at least one price, then we can eliminate the possibility of having an internal optimum. Theorem 4 contains such an assumption for the case of an individualistic welfare function; namely that there exist a good for which consumers are found on only one side of the market. If consumers sell but do not buy a good, we can raise its price making someone better off and no one worse off. This does not seem to be a stringent condition. This will be satisfied if there is a particular kind of labor supplied only to producers, and not to other consumers, or if there is a non-durable manufactured good of which consumers have no stocks.

It is in Theorem 4 that the assumption of strict convexity of preferences is needed to obtain the result as stated. This can be seen by considering Example 5:

Example 5: Consider an economy with one consumer whose indifference

curves have a linear section. Then, the offer curve may coincide with the linear part of an indifference curve, giving a set of optima, only one of which is on the production frontier.



Further the assumption that all consumers coincide in demanding or supplying one good is indeed required for this result, as indicated by:

Example 6: In this example there are two commodities and two households. One has utility function x^2y, the other utility function xy^2; each has consumption set $\{(x,y): x \geqq 0,\ y \geqq 0\}$. The first has three units of the first commodity initially, the second one unit of the second commodity. The welfare function is

$$-\frac{1}{x_1^2 y_1} - \frac{1}{x_2 y_2^2}.$$

The two commodities can be transformed into one another according to the production relation

$$x + 10y \leqq 0.$$

Let the prices of the commodities be p,q. Then the first household's net demands are

- 1 of the first commodity,

$^p/_q$ if the second commodity.

The second household has net demands

$$\frac{1}{3}\,{}^{p}/_{q} \text{ and } -\frac{1}{3}.$$

Thus, the net market demand for the commodities is

$$x = \frac{1}{3}\,{}^{p}/_{q} - 1 \text{ and } y = {}^{p}/_{q} - \frac{1}{3}.$$

These must satisfy

$$x + 10y = 0.$$

Welfare is $-\frac{q}{4p} - \frac{27p}{4q}$, which is maximum when ${}^{p}/_{q} = 3\sqrt{3}$: the corresponding production vector $\sqrt{3} - 1,\ \frac{1}{3}(\frac{1}{\sqrt{3}} - 1)$ is actually interior to the production set, not on the frontier.

Optimal Tax Structure

In Section 2, we derived the optimal tax structure for the one consumer economy. No use was made in obtaining equations (18) of the particular form of the indirect welfare function. Thus we can expect the same equations to continue to describe the first order conditions for the optimum in a many consumer economy. In the next section we will use those equations along with particular forms for the welfare and utility functions to calculate examples of the optimal tax structure. First we wish to derive these equations more rigorously, allowing for the possibility that there may be free goods, implying some inequality first order conditions rather than equalities. In the process, we hope to shed some light on the asymmetry in the roles of demand and supply in the optimal tax conditions. To this end, we shall generalize the solution of a maximization problem of Kuhn and Tucker (8) to allow for the particular form of constraint for our problem.

The general problem is to maximize V(q) subject to the constraint that D(q) lie in F. (The appearance of F rather than the non-negative orthant in the constraint requires an extension of the basic maximization

theorem.) We shall make several specific assumptions for this subsection:

(d1) F is convex - this assumption will be weakened in the corollary to theorem 5 - closed and contains at least two points.

(d2) There exists an optimum, with consumer prices $q^* \neq 0$.

(d3) Both V and D are continuously differentiable.

(d4) There is no $q \geqq 0$ such that $V'(q) \leqq 0$. (Since $V'(q) \cdot q = 0$, this means that V has no local maximum in the non-negative orthant.)

(d5) $(D(\cdot), F)$ satisfies the Constraint Qualification at q^* which we will define below.

It seems that two awkward possibilities may prevent the existence of a producer price vector which will give us the first order conditions derived above. The set of consumer prices that lead to feasible demands might have a cusp on its frontier; or the tangent to the attainable set at the optimum production point might be the unique tangent to the production frontier there. These are surely rare contingencies. To rule them out, we make a constraint qualification which parallels that employed by Kuhn and Tucker. We think that the qualification does not impose serious restrictions and can, in any event, be checked fairly easily in any particular case.

Constraint Qualification

If there exists a differentiable arc $[z(\theta) : 0 \leqq \theta \leqq 1]$ such that for some $a \geqq 0$,

$z(0) = D(q), \quad z'(0) = D'(q) \cdot a, \quad z(\theta) \in F \quad (0 \leqq \theta \leqq 1);$

then there exists a differentiable arc $[q(\theta) : 0 \leqq \theta \leqq 1]$ such that

$q(0) = q, \quad q'(0) = a, \quad D(q(\theta)) \in F \quad (0 \leqq \theta \leqq 1).$

When this is satisfied, we say that $(D(\cdot), F)$ satisfies the constraint qualification at q. The assumption states that when a small price change alters demands in a direction such that smooth changes of

production initially in that direction can keep production within the production set, then suitable smooth changes of production initially in that direction can keep production within the attainable set. Kuhn and Tuckers' constraint qualification is the particular case of this one for F the non-negative orthant.

We are now in a position to state and prove a theorem giving the first order conditions assuming that an optimum exists. We shall then go on to consider certain stronger assumptions which equate the optimality of a point with the satisfaction of the first order conditions.

Theorem 5. Assume that (d.1) to (d.5) are satisfied. Then there exists a non-zero vector, p, such that

(i) $D(q^*)$ maximizes $p \cdot x$ for x in F;

(ii) $V'(q^*) - p \cdot D'(q^*) \leq 0$.

Before proceeding to the proof, let us note that condition (ii) implies the first order conditions (18) for each consumer price, q_k, that is positive. Since V and D are both homogeneous of degree zero in consumer prices, $(V' - p \cdot D') \cdot q$ is identically equal to zero. In particular, this is true for the optimal prices, q^*. Since these prices are non-negative, each term in the sum must be zero, i.e.,

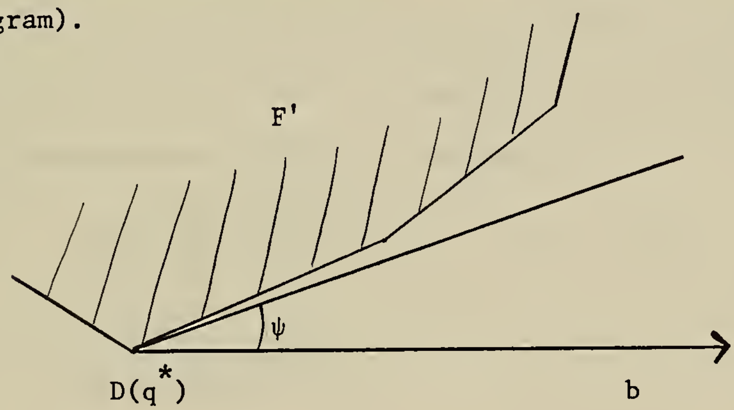
$$\left(\frac{\partial V}{\partial q_k} - \sum p_i \frac{\partial D_i}{\partial q_k}\right) q_k^* = 0, \qquad (k = 1,\ldots,n).$$

Proof of Theorem 5: Let P be the set of p such that $p \cdot x \leq p \cdot D(q^*)$ for all x in F, i.e., the set of tangent hyperplanes to F at $D(q^*)$ plus the zero vector. P is a closed convex cone. Let C be the closed convex cone $P \cdot D'(q^*)$ consisting of vectors $p \cdot D'(q^*)$ for p in P.

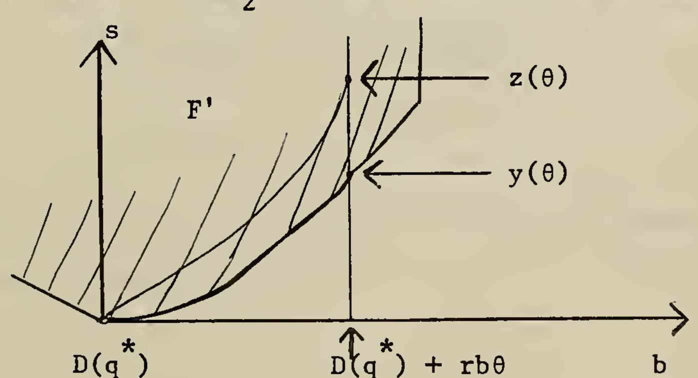
Lemma. If $p \cdot b \leq 0$ for all p in P, there exists a differentiable arc $[z(\theta)]$ lying in F such that $z(0) = D(q^*)$ and $z'(0) = rb$ for some $r > 0$.

Proof: (i) If $p \cdot b < 0$ for all $p \neq 0$ in P, $D(q^*) + rb$ is in F for some $r > 0$. Otherwise, we could separate the half-line $[D(q^*) + \lambda b : \lambda > 0]$ from F by a hyperplane containing $D(q^*)$, so that $q \cdot b \geq 0$ for some q in P. The arc $z(\theta) = D(q^*) +$ rb has the desired properties.

(ii) Suppose that $p \cdot b = 0$ for some $p \neq 0$ in P, and $p' \cdot b \leq 0$ for all other p' in P. Then one of the planes containing $D(q^*)$ and $D(q^*) + b$ intersects F in a set F' whose frontier makes a minimum angle θ with b (as in the diagram).



Suppose $\frac{\pi}{2} > \psi > 0$. Then the convex set[16/]
$[x : (x - D(q^*)) \cdot b > \{||x - D(q^*)|| \; ||b||\} \cos \psi]$ does not intersect F. Separating it from F, we have a non-zero vector p' such that $p' \cdot (D(q^*) + b) > q \cdot x$ for all x in F, i.e. $p' \cdot b > 0$. If $\psi \geq \frac{\pi}{2}$ the above argument is still valid replacing $\cos \psi$ by $\cos \frac{\pi}{2}$. This contradiction shows that $\psi = 0$.



Consider the situation in a plane where the frontier of F' is tangent to the direction b at $D(q^*)$, as shown. s is the direction perpendicular to b on the same side as F'. Let $y(\theta)$ be the point of the frontier of F' lying vertically above $D(q^*) + r\theta b$, if there is such a point. Choose $r > 0$ so small that there is such a point for all θ in $[0,2]$. Let $\lambda(\theta)$ be the height of this point above the b-direction, so that

51. $y(\theta) = D(q^*) + \theta rb + \lambda(\theta)s$

$\lambda(\theta)$ is a convex function of θ, with $\lambda(0) = 0$ and right derivative zero at $\theta = 0$. It may not be differentiable in a neighborhood of 0, but μ defined by

52. $$\mu(\theta) = \frac{1}{2\theta}\int_0^{2\theta} \lambda(\phi)\, d\phi \qquad 0 < \theta \leqq 1$$

$$\mu(0) = 0$$

is: it has derivative

53. $$\frac{1}{\theta}\,\lambda(2\theta) - \frac{1}{2\theta^2}\int_0^{2\theta} \lambda(\phi)\, d\phi$$

if $0 < \theta \leqq 1$. When $\theta = 0$, we have

54. $$\mu'(0) = \lim_{\theta\to 0}\left[\frac{1}{2\theta^2}\int_0^{2\theta}\lambda(\phi)d\phi - \frac{1}{\theta}\,\lambda(0)\right]$$

$$\leqq \lim_{\theta\to 0}\left[\frac{1}{2\theta^2}\int_0^{2\theta}\left\{\left(\frac{\phi}{2\theta}\right)\lambda\,(0) + \left(1 - \frac{\phi}{2\theta}\right)\lambda(2\theta)\right\} d\phi - \frac{1}{\theta}\,\lambda(0)\right]$$

$$= \lim_{\theta\to 0}\left[\frac{1}{2\theta}\{\lambda(2\theta) - \lambda(0)\}\right]$$

$$= \lambda'(0) = 0;$$

and since $\mu \geqq 0$, $\frac{1}{\theta}[\mu(\theta) - \mu(0)] \geqq 0$ $(\theta > 0)$. Hence $\mu'(0)$ exists and is equal to 0.

Define

55. $z(\theta) = D(q^*) + r\theta b + \mu(\theta)s.$

From convexity of $\lambda(\theta)$

56. $\lambda(\theta) \leqq \mu(\theta) \leqq \frac{\theta}{2} \lambda(0) + \frac{2-\theta}{2} \lambda(2)$.

Thus $z(\theta)$ can be expressed as a convex combination of two points in F, $y(\theta)$ and $\frac{\theta}{2} y(0) + (\frac{2-\theta}{2}) y(2)$. Hence $z(\theta) \varepsilon F (0 \leqq \theta \leqq 1)$; it is a differentiable arc, $z(0) = D^*(q)$ and $z'(0) = b$. Hence it has all the desired properties. This completes the proof of the lemma.

Put $b = D'(q^*) \cdot a$ for $a \geqq 0$. Then, by the constraint qualification, there exists a differentiable arc, $q(\theta)$, such that $q(0) = q^*$, $q'(0) = ra$, and $D(q(\theta)) \varepsilon F$. The last statement implies that $V(q(\theta)) \leqq V(q^*)$. Hence

57. $0 \geqq \frac{d}{d\theta} V(q(\theta)) = V'(q^*) \cdot q'(0) = rV'(q^*) \cdot a$.

We have shown, then, that when $a \geqq 0$, and $p \cdot D'(q^*) \cdot a \leqq 0$ for all p in P, $V'(q^*) \cdot a \geqq 0$. In other words

58. $c \cdot a \leqq 0$ (all c in C) and $a \geqq 0$

imply $V'(q^*) \cdot a \leqq 0$.

By the duality theorem for closed convex cones (Karlin ( ), Theorem B.3.1,(I)), we can deduce that $V'(q^*)$ lies in the cone spanned by C and the non-positive orthant. That is, for some p,

59. $V'(q^*) \leqq p \cdot D'(q^*)$, and

$p \cdot x \leqq p \cdot D(q^*)$ for all x in F.

By (d.4), p cannot be zero.

We have thus obtained the first order conditions (18) for the optimum, with an inequality if a price is zero and an equality for any positive price. Of the assumptions employed to reach this result, we wish to examine (d1), that F is convex. One of the standard reasons for government control of production is non-convexity of the production set. We are all familiar with the complications this implies for controlling a sector with increasing returns to scale. However, it is usually felt that the price system permits decentralization for those sectors which do not have increasing returns to scale. To show that this analysis carries over to this case we can consider a weakening of (d1) and exhibit two corollaries which show the possibility of decentralization where production possibilities are convex.

(d.1') F = G + H where G is convex.

Corollary 1: Assume that (d.1'), (d.2) - (d.5) hold. Then there exists a non-zero vector, p, such that

(i) $D(q^*) - h^*$ maximizes $p \cdot x$ for x in G

(ii) $V'(q^*) - pD'(q^*) \leqq 0$,

where h^* is the optimal production vector in H. The proof follows exactly as that of Theorem 5, replacing F by G and D by $D-h^*$.

With suitable differentiability assumptions, we can, of course, say rather more, that h^* occurs where H is tangent to a hyperplane defined by the producer prices p. Differentiability is employed to ensure that the concept of tangency is meaningful. We now assume

(d.1'') F = G + H where G is convex, and H has a differential manifold as frontier.

Corollary 2: Assume (d.1''), (d.2) - (d.5). Then there exists a non-zero vector, p, such that

(i) p is supporting to G at g^* and tangent to H at h^*

(ii) $V'(q^*) - p \cdot D'(q^*) \leq 0$.

A proof would follow that given for Theorem 5 with minor changes, for the properties of the cone of tangent hyperplanes used in that proof hold also for the more general kind of set here considered.

There are two uniqueness problems which may arise in the consideration of the application of the first order conditions to achieve an optimum. One problem is that there may be more than one pair of price vectors, (p,q), that satisfy the first order conditions and clear markets. This is similar to the problem that arises in seeking the full optimum in the presence of a non-convex production set - there may be two or more points satisfying the first order conditions implying the necessity to have recourse to global considerations to choose among them. This situation can arise here even in the absence of non-convexity. Sometimes, however, there may be just one such point. Theorem 6, and its corollary, give examples of assumptions which ensure that all points satisfying the first order conditions provide the maximum of welfare.

With the fundamental theorem of welfare economics, it may occur that when the optimizing distribution of income is brought about by lump sum redistribution, there are be several competitive equilibria that can arise with that distribution of income. The same problem can arise here if we employ the taxes rather than the consumer prices as the government control variables.[17/] When the consumer prices are the control variables, the demand functions give us a unique equilibrium position in the presence of strict convexity of preferences. Now let us turn to Theorem 6.

Theorem 6. Assume that F is convex; V is a concave function; and that there exists $\bar{q} \geq 0$ for which $D(\bar{q})$ is in the interior of F. Suppose that either (i) D is linear,

or (ii) D is convex and F includes free disposal.

Then q^* maximizes $V(q)$ for $D(q)$ in F if and only if there exists p such that $D(q^*)$ maximizes $p \cdot x$ over F and q^* maximizes $V(q) - p \cdot D(q)$ for $q \geq 0$.

Proof: The sufficiency of the conditions is trivial, and the result does not depend upon the particular assumptions made about V, D, and F: but, in general, no such p would exist. Assume not.

For some q', $V(q') > V(q^*)$, but $D(q') \leq p \cdot D(q^*)$

and $V(q') - p \cdot D(q') \leq V(q^*) - p \cdot D(q^*)$,

this is a contradiction.

To prove necessity, we separate the convex hull of

$$A = [(u_o, u) : u_o \leq V(q),\ u = D(q),\ q \geq 0],$$

and the set

$$B = [(v_o, v) : v_o > V(q^*),\ v \text{ in the interior of } F].$$

The sets are both convex, the latter open, and neither empty. In case (i), A itself is convex, and it is clear that A and B do not intersect. In case (ii), if B intersected the convex hull of A, we should have

$$D(\lambda q^1 + (1-\lambda)q^2) \leq \lambda D(q^1) + (1-\lambda)D(q^2) \varepsilon F,$$

for some λ, q^1, q^2. Using free disposal, we see that

$$V(q^*) \geq V(\lambda q^1 + (1-\lambda)q^2) \geq \lambda V(q^1) + (1-\lambda)V(q^2).$$

Thus $\lambda(V(q^1), D(q^1)) + (1-\lambda)(V(q^2), D(q^2))$ cannot be in B, which there fore does not intersect the convex hull of A.

Separating the two sets, by the separating hyperplane theorem, we obtain the existence of p_o and p such that

60. $p_o u_o + p \cdot D(q) < p_o v_o + p \cdot v$,

if $u_o \leq V(q)$, $v_o > V(q^*)$, and v is in the interior of F. If p_o were less than or equal to zero, and we put $q = \bar{q}$ and $v = D(\bar{q})$, we should have $p_o(V(\bar{q}) - v_o) < 0$, which would be impossible, since $V(\bar{q}) \leq V(q^*)$. Therefore we can put $p_o = 1$.

Now let v_o tend to $V(q^*)$, and v tend to an arbitrary point, x, in F. Then (60) implies

61. $V(q) - p \cdot D(q) \leq V(q^*) - p \cdot x \quad (q \geq 0,\ x \in F)$.

Putting $q = q^*$, we obtain the profit maximizing property; and putting $x = D(q^*)$, we find that q^* maximizes $V - p \cdot D$ for $q \neq 0$. This completes the proof.

Note that p = 0 only if q^* maximizes V over the whole non-negative orthant - a rather uninteresting case.

Corollary: Assume that $q = h(s)$, where s is an m vector; that $h(s) \geqq 0$ when $s \geqq 0$; and that $D(h(s))$ takes all the values $D(q)$ takes for $q \geqq 0$ as s varies in the non-negative orthant of R^m. Assume that there exists $\bar{q}$ such that $D(\bar{q})$ is in the interior of F. If $V(h(s))$ is concave in s; and $D(h(s))$ is linear in s, or F includes free disposal and $D(h(s))$ is convex in s, the result of the theorem still holds.

Proof: Apply the theorem as though s were the consumer prices.

The corollary is the useful result. It applies, for instance, to the example considered in the next section, where $s = (\frac{1}{q_2}, \frac{1}{q_3}, \ldots, \frac{1}{q_n})$. One would not expect to be able to apply the theorem directly. In the case of one consumer, the simplest case, V is a quasi-convex function, and therefore concave only if linear.

When Theorem 6, or its corollary, applies, and V and D are differentiable, any price vectors satisfying the first order conditions and clearing markets

maximize welfare. The function V - p'D is concave in the two cases covered by the theorem. Therefore it is maximized when its derivatives are all less than are equal to zero at $q^* \geq 0$. If V is strictly concave, a unique state of the economy is defined by these relations.

APPENDIX TO SECTION III

We shall now re-examine the theorems in this section, replacing the strict convexity assumptions, (a.5) by

(a.5') If x^1 and x^2 are two points in X^h and if t is a real number of $]0,1[$, then $x^2 \succsim_h x^1$ implies $tx^2 + (1-t)x^1 \succ_h x^1$.

Weakening (a.5) to (a.5') implies replacing the continuous demand functions employed in the text by upper semi-continuous demand correspondences. Since a consumer is indifferent between two demand vectors in his demand correspondence for any given price, if the social welfare function respects individual preferences and is continuous, the indirect social welfare function is well defined and continuous. Examining the proofs of theorems 1 and 2, we see that weakening (a.5) to (a.5') does not alter either proof. Thus we have

Theorems 1' and 2': Assumptions (a.5) can be replaced by assumption (a.5') in theorems 1 and 2.

Since theorem 3 employed the continuity of the indirect welfare function, we must strengthen the other assumptions to carry through the same argument.

Theorem 3': If the social welfare function respects individual preferences, assumption (a.5) can be replaced by (a.5') in theorem 3.

Proof: It is necessary to replace the demand functions (D) by demand correspondences, D. It is also necessary to replace the set $\{q|D(q) \text{ in } F\}$ by $\{q|D(q) \cap F \neq \emptyset\}$. To demonstrate the closedness of the latter consider a sequence q_n in this set, converging to q'. For each n, select x_n in $D(q_n) \cap F$. Let x' be a limit point of $\{x_n\}$. One exists since A is bounded. Then x' is in $D(q') \cap F$. Thus the proof carries through.

As was shown by Example 5, without strict convexity, there may exist an optimum which is not on the production frontier. We shall show in theorem 4' that with convexity, if there exists an optimum, then there exists an optimum which is on the frontier (although there may exist other socially indifferent points which are not on the frontier).

Lemma 1': Assume an optimum exists. If aggregate demand functions are upper semi-continuous, and the indirect welfare function continuous in the neighborhood of the optimal prices, if F is closed, and if either

(1) for some i, V is a strictly increasing function of q_i in the neighborhood of q^*, or

(2) for some i with $q_i^* > 0$, V is a strictly decreasing function of q_i in the neighborhood of q^*,

then there exists an optimum with production on the frontier of the feasible set.

We shall consider case 1.

Proof: For ε sufficiently small, $V(q^*+\varepsilon\ell_i) > V(q^*)$. Thus $D(q^*+\varepsilon\ell_i) \cap F = \emptyset$. Let z be a limit point of $\{D(q^*+\varepsilon\ell_i)\}$ as ε goes to zero. Then z is in $D(q^*)$. If z is feasible, we have an optimum with production on the frontier. If z is not feasible, the line $[z,x^*]$ is in $D(q^*)$ and there exists a point on the line on the boundary of F. This point is a production point for an optimum.

Theorem 4': Replacing (a.5) by (a.5') and adding (b.3) to the hypothesis of theorem 4 implies that if an optimum exists, there exists an optimum with production on the production frontier.

It remains to show that assumptions (a.1) – (a.6) lead to continuous demand functions.

Lemma 2a: If assumptions (a.1) - (a.6) hold then demand functions are continuous at positive prices.

Lemma 2b: If assumptions (a.1) - (a.6) hold and X^h is compact, then demand functions are continuous at non-negative prices.

Lemma 2a': If assumptions (a.1) - (a.4), (a.5'), and (a.6) hold then demand correspondences are upper semi-continuous at positive prices.

Lemma 2b': If assumptions (a.1) - (a.4), (a.5') and (a.6) hold and X^h is compact, then demand correspondences are upper semi-continuous at non-negative prices.

Proof: With strict convexity demand correspondences are necessarily functions, so 2a' and 2b' imply 2a and 2b.

Let $\gamma(q) = \{x \text{ in } X | q \cdot x = q \cdot x^o\}$. For strictly positive prices, $\gamma(q)$ is bounded since X is bounded below. Thus we can replace X by a compact subset strictly containing $\gamma(q')$ for all q' in a sufficiently small neighborhood of q. Therefore 2b' implies 2a'.

By assumption (a.3), $q \cdot x^o$ is always greater than Min. $q \cdot x$ for x in X. Thus result (1) of 4.8 (Debreu) holds and γ is a continuous correspondence for non-negative prices. The upper semi-continuity of demand (and continuity of the indirect utility function) then follows from 4.10 (Debreu).

IV. OPTIMAL TAX STRUCTURE

Reviewing the derivation of the optimal tax structure, 18, and of the desirability of efficiency, 19, in Section 2, it is seen that no use was made of the relation between the form of the welfare function and the structure of demand. Thus, having many consumers in the economy does not alter either result. However, when we come to examine the tax structure, the derivatives of the welfare function with respect to any price will reflect which particular consumers damend that good. This can be easily brought out by an example. This example satisfies the conditions for the corollary to Theorem 6 (with $s = (q_1^{-1}, q_2^{-1}, \ldots, q_n^{-1})$). Thus we know that the solution to the first order conditions will give us a welfare maximum.

Example

We will assume that each consumer has a Cobb-Douglas utility function,

62. $$u^h = \alpha_0^h \log(x_0^h + \omega^h) + \sum_1^n \alpha_i^h \log x_i, \quad \sum_0^n \alpha_i^h = 1.$$

Choosing good 0 as numeraire, we saw above that with a one consumer economy, taxation would be proportional. This will not, in general, be true in a many consumer economy where each consumer has this utility function. The individual demand curves coming from this utility function are:

63. $$x_i^h = q_i^{-1} \alpha_i^h q_0 \omega^h, \qquad i = 1, 2, \ldots, n,$$

and

$$x_0^h = -(1 - \alpha_0^h)\omega^h.$$

We shall assume that the welfare function respects individual tastes.

64. $$U(x^1, \ldots, x^H) = W(u^1(x^1), \ldots, u^H(x^H))$$

or

$$V(q) = W(v^1(q), \ldots, v^H(q)).$$

Differentiating this expression we have

65. $$V_k = \sum_h \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial q_k} = - \sum_h \frac{\partial W}{\partial v^h} \alpha^h x_k^h = - \sum \beta_h x_k^h,$$

where α^h is the marginal utility of income for the $h^{\underline{th}}$ individual while β_h is the social marginal utility of income for the $h^{\underline{th}}$ individual. Substituting in the first order conditions for the optimal tax structure, we have (where $X = \sum_h x^h$),

66. $$V_k = - \lambda (X_k + \sum_i t_i \frac{\partial X_i}{\partial q_k})$$

or

67. $$- \sum_h \beta_h x_k^h = - \lambda (\sum_h x_k^h - t_k \sum_h q_k^{-2} \alpha_k^h q_0 \omega^h) \qquad (k = 1,\ldots,n)$$

or

68. $$\sum_h \beta_h x_k^h = \lambda (\sum_h x_k^h - t_k q_k^{-1} \sum_h x_k^h) \qquad (k = 1,\ldots,n)$$

or

69. $$\sum_h \beta_h x_k^h = \lambda \; p_k q_k^{-1} \sum_h x_k^h . \qquad (k = 1,\ldots,n)$$

This implies the following formula:

70. $$\frac{q_k}{p_k} = \lambda \frac{\sum x_k^h}{\sum \beta_h x_k^h} = \lambda \frac{\sum \alpha_k^h \omega_k^h}{\sum \beta_h \alpha_k^h \omega_k^h} \qquad (k = 1,\ldots,n)$$

To complete the determination of the optimal taxes, we must find the relationship between λ, p_o and q_o. This is obtained from the Walras identity. The value of net consumer demand in producer prices is equal to minus the profit in production. (Alternatively, we could determine λ so that the government budget is balanced.) That is

71. $$- p_o \sum_h (1 - \alpha_o^h) \omega^h + \sum_{i,h} p_i q_i^{-1} \alpha_i^h q_o \omega^h = \gamma,$$

where γ is the maximized profit of production ($= \sum p_i z_i$). Substituting from (70) and rearranging, we obtain

$$72. \qquad \frac{q_o}{p_o} = -\lambda \frac{\sum(1 - \alpha_o^h)\omega^h - \gamma p_o^{-1}}{\sum \beta_h \alpha_i^h \omega^h}$$

$$= -\lambda \frac{\sum(1 - \alpha_o^h)\omega^h - \gamma p_o^{-1}}{\sum \beta_h (1 - \alpha_o^h)\, \omega^h} .$$

The number γp_o^{-1} is determined by the technology and the government expenditure decision.

Equations (70) and (72) determine the optimal tax rates. If the social marginal utilities, β_h, are independent of taxation, the optimal tax rates can be read off at once. This is true if W has the special form $\sum_h v^h$; for in that case $\beta_h = 1/\omega^h$. It should be noticed that, although each household's social marginal utility of income is unaffected by taxation, it is desirable to have taxation in general. If households with relatively low social marginal utility of income predominate among the purchasers of a commodity, that commodity should be relatively highly taxed. Although such taxation does nothing to bring social marginal utilities of income closer together, it does increase total welfare.

In general, taxation does affect social marginal utilities of income. The β_h depend upon the tax rates, and equations (68) do not, therefore, give explicit formulae for the optimum taxes. In case $W = -\mu^{-1} \sum_h e^{-\mu v^h}$ - so that there is a stronger bias toward equality than in the additive case - it can be verified quite easily that the optimum taxes have to satisfy

$$73. \qquad \frac{q_k}{p_k} \sum_h \alpha_k^h (\omega_h)^{-\mu} \prod_i (\alpha_i^h)^{-\mu\alpha_i^h} q_i^{\mu\alpha_i^h} = \lambda \sum_h \alpha_k^h \omega^h . \quad (k = 1,\ldots,n)$$

In this case, marginal utilities of income are brought closer together.[18/] It is not immediately obvious from the equations (70) that the q are determined given the p. However, the method of deriving the equations,

by maximizing a quasi-concave function subject to a linear constraint shows that they must have a unique solution. In fact the relations (70) (along with (71)) would, if followed by government, certainly lead to maximum welfare if production were perfectly competitive, since any state of the economy satisfying these conditions maximizes welfare, and the maximum is unique for the welfare function considered. We have suggested in Section 3 that this convenient property is not general.

From equation (69) we can identify two cases where optimal taxation is proportional. If the social margin utility of income is the same for everyone ($\beta_h = \beta$, for all h), then equation (69) reduces to $q_k p_k^{-1} = \lambda/\beta$. In this case there is no welfare gain to be achieved by redistributing income, and so no need to tax differently (on average) the expenditures of different individuals. Thus the optimal tax formula has the same form as the one consumer case. When the β_h do differ, taxes are greater on commodities purchased more heavily by individuals with a low social marginal utility of income. If, for example, the welfare function treats all individuals symmetrically and if there is diminishing social marginal utility with income, then there is greater taxation on goods purchased more heavily by the rich.

The second case leading to proportional taxation occurs when demand vectors are proportional for all individuals, $x^h = \rho_h x$, and thus $\alpha_k^h = \alpha_k$ for all h. With all individuals demanding goods in the same proportions, it is impossible to redistribute income by commodity taxation, implying that the tax structure again assumes the form it has in a one consumer economy.

Optimal Tax Formulae

The description in Section 2 of some possible interpretations of the optimal tax formula carries over to the many consumer case. Thus, as was true there, demand elasticities but not supply elasticities enter the

equations, and at the optimum the social marginal utility of a price change is proportional to the marginal change in tax revenue from raising that tax, calculated at constant producer prices. Analysis of the change in demand can also be carried out, but is naturally more complicated. Assuming an individualistic welfare function, the first order conditions can be written[19/]

$$74. \qquad \sum \beta_h x_k^h = \lambda \sum_h \sum_i t_i \frac{\partial x_i^h}{\partial q_k} + \lambda \sum_h x_k^h.$$

From the Slutsky equation, we know that

$$75. \qquad \frac{\partial x_i}{\partial q_k} = s_{ik} - x_k \frac{\partial x_i}{\partial I} = s_{ki} - x_k \frac{\partial x_i}{\partial I} = \frac{\partial x_k}{\partial q_i} - x_k \frac{\partial x_i}{\partial I} + x_i \frac{\partial x_k}{\partial I} .$$

Substituting from (75) in (74) we can write the optimal tax formula as:

$$76. \qquad \sum \beta_h x_k^h = \lambda \sum_h \sum_i t_i \frac{\partial x_k^h}{\partial q_i} + \lambda \sum_h \sum_i t_i \left(x_i^h \frac{\partial x_k^h}{\partial I} - x_k^h \frac{\partial x_i^h}{\partial I}\right) + \lambda \sum_h x_k^h.$$

Rearranging terms we can write

$$77. \qquad \frac{\sum_h \sum_i t_i \frac{\partial x_k^h}{\partial q_i}}{\sum_h x_k^h} = \lambda^{-1} \frac{\sum \beta_h x_k^h}{\sum x_k^h} - 1 + \frac{\sum_h \left(\sum_i t_i \frac{\partial x_i^h}{\partial I}\right) x_k^h}{\sum_h x_k^h} - \frac{\sum_h \left(\sum_i t_i x_i^h\right) \frac{\partial x_k^h}{\partial I}}{\sum_h x_k^h} .$$

With constant producer prices equation (77) gives the change in demand as a result of taxation for a good with constant price derivatives of the demand function. Considering two such goods, we see that the percentage decrease in demand is greater for the good the demand for which is concentrated among:

(1) individuals with low social marginal utility of income,

(2) individuals with small decreases in taxes paid with a decrease in income,

(3) individuals for whom the product of the income elasticity of demand for good k and the fraction of income paid in tax is large.

Other Taxes

Thus far we have examined the combined use of public production and commodity taxation as control variables. It is natural to examine the changes brought about by adding additional tax variables to those controlled by the government. In particular, in the next subsection we will briefly consider income taxation. But first, let us examine a general class of taxes such that the consumer budget constraint depends on consumer prices and on tax variables. We shall replace the budget constraint $\sum q_i x_i = 0$ by the more general constraint $\phi(x,q,\zeta) = 0$, where ζ represents a shift parameter to reflect the choice among different systems of additional taxation (for example, the degree of progression in the income tax). Let us note that this formulation continues to assume that all taxes are levied on consumers and that there are no profits in the economy.

The key assumption to permit an extension of the analysis above is an independence of the two constraints. We need to assume that the choice of tax variables does not affect the production possibilities, and further that the choice of a production point does not affect the possible demand configurations (ignoring the need for market clearance). In particular, the formulation implies that producer prices do not affect consumer budget constraints. Thus the income tax, to fit this formulation, needs to be levied on the wages that consumers receive, not on the cost of wages to the firm. Similarly it is assumed that there are no sales tax deductions from the income tax base.

With such an enlarged tax system, the case for production efficiency is similar to the one given above, when just commodity taxation was considered. We shall make this argument in two ways. First a direct calculus argument paralleling that of Section 2. Then a non-rigorous discussion of the carryover of the arguments of Section 3 to this case.

First let us restate the basic maximization problem. We wish to maximize an indirect welfare function, which is now a function of the consumer prices and the other tax variables, $V(q,\zeta)$. The constraint is that aggregate demand, $D(q,\zeta)$ result in an equilibrium. Repeating the structure of the problem given in equation (14) we can state the problem as

78. Choose $q_2,\ldots,q_n,\ z_2,\ldots,z_n,\ \zeta$

so as to maximize $V(q,\zeta)$

subject to $X_1(q,\zeta) - f(X_2(q,\zeta) - z_2,\ldots,X_n(q,\zeta) - z_n)$

$$- g(z_2,\ldots,z_n) = 0.$$

Forming a Lagrangian expression from (78) and differentiating with respect to q_k, we obtain a first order condition similar in form to that obtained above

79. $$V_k - \lambda\left(\frac{\partial X_1}{\partial q_k} - \sum f_i \frac{\partial X_i}{\partial q_k}\right) = 0.$$

Now differentiating with respect to z_k we obtain

80. $$-\lambda(f_k - g_k) = 0.$$

Thus, provided that λ is unequal to zero, we again obtain the condition for aggregate production efficiency. For use in the next section, let us differentiate the Lagrangian with respect to the shift parameter for our other taxes

81. $$V_\zeta - \lambda\left(\frac{\partial X_1}{\partial \zeta} - \sum f_i \frac{\partial X_i}{\partial \zeta}\right) = 0.$$

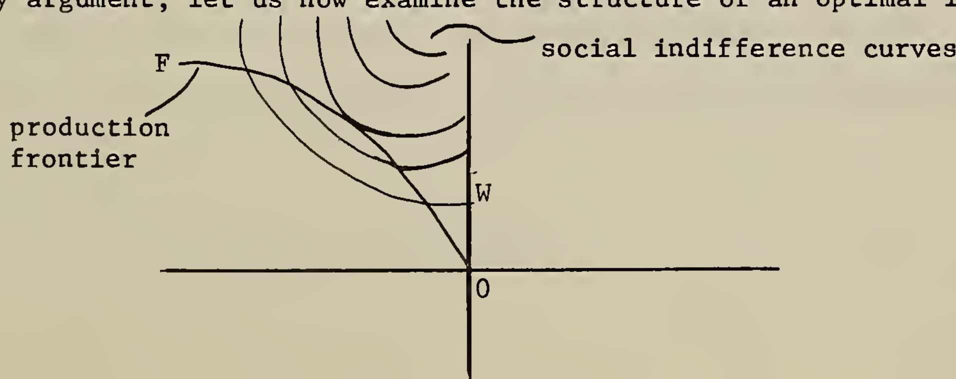
Paralleling the argument of the previous section, we will find production at the optimum occuring on the production frontier if we can find a sequence of consumer price and tax variables (q^i,ζ^i) each resulting in a higher level of welfare than the optimum and converging to the optimal level (q^*,ζ^*). If demands are continuous, this argument implies production on the frontier. The conditions leading to this above should continue to result in this conclusion provided that the set of

other taxes is large enough. Put differently, assuming an optimum exists, we can choose the level of other taxes, ζ^*, and now repeat the argument above that for choice of q^* results in production on the frontier provided that the assumptions above continue to hold in the presence of the tax ζ^*. We would not expect to be able to follow this line of argument if for some reason we were not able to levy commodity taxation on all commodities.

It might be useful at this point to restate the efficiency argument in terms of the familiar concept of social indifference curves. For a point, X, in commodity space let us assign the maximal level of welfare that can be achieved by setting the tax variables resulting in aggregate demand equal to this vector. We can define the welfare function by

82. $$W(X) \equiv \text{Max } V(q,\zeta) \text{ subject to } D(q,\zeta) = X.$$

From the argument given for the one consumer economy, it may be true that the welfare function is not defined for some values of X. Under suitable conditions, the indifference curves of this welfare function will be well behaved and will increase as we move to higher aggregate quantity levels. (This is not equivalent to having national income as the welfare criterion, it does assume that the tax variables are strong enough to be able to improve welfare by appropriate distribution of a larger aggregate output. As we saw in example 6, in Section 3, this need not always be true.) When these conditions hold, we would expect to find the optimum on the frontier of the production set as indicated in the diagram. Having retraced the efficiency argument, let us now examine the structure of an optimal income tax.



Income Taxation

The analysis of the use of commodity taxation to redistribute income and collect revenue is not meant to suggest that commodity taxation is necessarily superior to income taxation. The analysis above simply asked how to employ the commodity tax tool to maximize social welfare. It is natural to go on to ask how one employs both commodity taxation and income taxation. The formulation of income taxation raises a problem. If the planners are free to select any income tax structure and if there are a finite number of tax payers, the tax structure can be selected so that the marginal tax rate is zero for each taxpayer at his equilibrium income (although this does not necessarily bring the economy to the full welfare maximum). This eliminates much of our problem, but like lump sum taxation, seems to be beyond the planning tools available in a large economy. The natural treatment of this problem, avoiding this dilemma, would be a continuum of tax payers. However, we shall take the alternate route from this problem by assuming a limited set of alternatives for the income tax structure.

Formally, let $\psi(x^h,p)$ be the taxes paid by a household for purchasing the vector x^h. We shall restrict ψ to lie in a family of functions $\psi(x^h,p,\zeta)$ where ψ is continuously differentiable in all variables. The important restriction on policy tools is that the tax function ψ is independent of h. For example, with just commodity taxation,

83. $$\psi = \sum_i t_i x_i .$$

To add income taxation to the tax structure we can select a subset of commodities, T, e.g., labor services, and tax the value of transactions on this subset,

84. $$\psi = \sum_i t_i x_i + \tau\left(\sum_{i \text{ in } T} q_i x_i \right).$$

With a tax on services (x_i negative) we would expect τ to be decreasing in its tax base, with a derivative between zero and minus one.

In terms of the notation employed above, we can define the budget constraint $\phi(x^h,q,\zeta)$ by

$$85. \qquad \phi(x^h,q,\zeta) = \sum p_i x_i + \psi(x^h,q,t,\zeta)$$

$$= \sum q_i x_i + \tau(\sum_{i \text{ in } T} q_i x_i).$$

Thus the consumer's budget constraint can be expressed in a form depending on consumer prices and independent of producer prices.

Optimal Income Taxation

We can now employ the first order conditions calculated above to examine the optimal income tax. Let us define δ_i to be 1 or 0 as i is or is not in T. Then, the first order conditions for individual utility maximization are

$$86. \qquad u_i^h = -\lambda(p_i + t_i)(1 + \delta_i \frac{\partial\tau}{\partial I}),$$

where $\frac{\partial\tau}{\partial I}$ is the derivative of the income tax with respect to taxable income. From (86) and the budget constraint we can obtain the derivatives of the indirect utility function in terms of consumer prices and income tax parameters,

$$87. \qquad \frac{\partial v^h}{\partial q_k} = \lambda^h x_k^h (1 + \delta_k \frac{\partial\tau}{\partial I})$$

$$\frac{\partial v^h}{\partial\zeta} = \lambda^h \frac{\partial\tau}{\partial\zeta}.$$

From the first order conditions (79) and (81) we have

$$88. \qquad \frac{\partial V}{\partial t_k} + \lambda(\sum_h x_k^h + \sum_h \sum_i t_i \frac{\partial x_i^h}{\partial q_k} + \sum_h \frac{\partial\tau}{\partial I}(\delta_k x_k^h + \sum_i (p_i+t_i)\delta_i \frac{\partial x_i^h}{\partial q_k})) = 0.$$

$$89. \qquad \frac{\partial V}{\partial\zeta} + \lambda(\sum_h \sum_i t_i \frac{\partial x_i^h}{\partial\zeta} + \sum_h (\frac{\partial\tau}{\partial\zeta} + \frac{\partial\tau}{\partial I}\sum_i (p_i + t_i)\delta_i \frac{\partial x_i^h}{\partial\zeta})) = 0.$$

Thus it remains true that the social marginal utility of a tax variable change is proportional to the marginal change in tax revenue calculated at constant producer prices. With an individualistic welfare function, the social marginal utilities satisfy

$$90. \qquad \frac{\partial V}{\partial t_k} = \sum_h \beta_h x_k^h \left(1 + \delta_k \frac{\partial \tau(I^h)}{\partial I}\right)$$

$$\frac{\partial V}{\partial \zeta} = \sum_h \beta_h \frac{\partial \tau(I^h)}{\partial \zeta} ,$$

where β_h is the social marginal utility of income for individual h.

From the equation (81) we can write the first order condition for an optimal income tax as

$$91. \qquad \sum \beta_h \frac{\partial \tau(I^h)}{\partial \zeta} = \lambda \sum_h \sum_i p_i \frac{\partial x_i^h}{\partial \zeta} .$$

Thus, at the optimum, for two different income tax structure changes, the social marginal utility weighted change in taxation (producer prices and taxable consumer quantities held constant) is proportional to the change in total tax revenue (income and excise tax revenue, calculated at fixed producer prices).

V. EXTENSIONS

Certain further complications leave our conclusions about production efficiency almost unchanged. The essential condition is that changes in production should not affect the distribution of income in ways that taxation cannot mimic. In particular the assumption of constant returns to scale in the private sector is necessary for our conclusions, since there is then no need to keep track of the recipients of economic profits, i.e., pure rents, arising from the ownership of special production opportunities. (Insofar as the rents can be treated as payments for special inputs, and taxed accordingly,

the problem disappears.) In this section, we consider, briefly, the taxation of transactions between producers, and between consumers; subsistence agriculture; international trade; migration; capital market imperfections; and consumption externalities.

Intermediate Good Taxation

If taxes were imposed on transactions between firms, there would, in general, be inefficiency of production: production would take place in the interior of the production set. Consequently such transactions should not be taxed. In particular, sales by the public sector to the private production sector should not be taxed, nor should they be subsidised. There is a straightforward interpretation of this result, which helps to explain the desirability of production efficiency. In the absence of profits, taxation of intermediate goods must be reflected in changes in final good prices. Therefore, the revenue could have been collected by final good taxation, causing no greater change in final good prices and avoiding production inefficiency. This interpretation highlights the necessity of our assumption of constant returns to scale in privately controlled production.

However, it may well be desirable to tax transactions between consumers or to charge different taxes on producer sales to different consumers. There are two ways in which we can consider doing this: the country might be geographically partitioned with different consumer prices in different regions. Ignoring migration and consumers making purchases in neighboring regions, the analysis above can be applied to determine taxes region by region. In general the tax structure will vary over the country.

Alternatively, we might consider taxation on all consumer - consumer transactions. Here, too, we would expect to be able to increase

social welfare by having these additional tax controls. Neither of these additions to the available tax structure alters the desirability of production efficiency.

The statements above can be interpreted in the following way. Consider dividing the economy into many sectors, some of the sectors containing consumers and other sectors containing producers, and giving to the government the power to tax any transaction between individuals in different sectors at a rate depending on the particular sectors involved. We can distinguish consumers from producers by whether their transactions directly affect social welfare. Then, the desirability of aggregate production efficiency (and thus the undesirability of taxation of transactions between firms) can be viewed as a lumping together of the sectors containing just producers. Partitioning of this sector adds nothing to the government's ability to affect social welfare. Put alternatively, we would say that we want the same tax rate on any transaction between an individual not in one of the sectors lumped together and an individual in any of the sectors which we are combining. This indifference to the ability to distinguish sectors does not hold when we consider partitioning the set of consumers. Any partitioning would, in general, permit an increase in social welfare.

Another problem can be analysed in terms of these ideas. What if there are several sectors which cannot be distinguished for tax purposes? This is a formulation of the problem that was considered by Boiteux. (3). He considered a public enterprise which had a given budget constraint (and no tax powers). This is similar to an inability to separate, for tax purposes, the consumer sectors from the private production sectors. (He also assumed that the private sector was equivalent to a single consumer because of lump sum transfers.) The

optimal production rule for the public sector can then be deduced from the optimal tax rules for this economy. (We can consider shadow taxes as levied on all sales by the public sector and an equating of marginal rates of transformation to the net of tax prices.) The first order conditions, $V' = \lambda\ pD'$, carry over to this case: the demand derivatives are obtained from the demand functions faced by the public enterprise. Usually, we would expect the government to increase social welfare by being flexible in the deficit allowed a public enterprise and by making use of the ability to distinguish sectors more fully.

Another complication that can be analysed by the same means is that of subsistence agriculture. It can be argued that in an under-developed economy where much of agricultural output is consumed on the farm it is impossible to tax agricultural output to a large extent. Tax evasion of any of the taxes mentioned above is not an element which has been included in the model and would presumably affect the analysis given. However, if it is possible to tax the transactions between subsistence farmers and the industrialized sector (sales of seed and fertilizer for example) then while it is not possible to achieve as good a welfare position as when agriculture is taxable, it is still true that it is desirable to preserve production efficiency in the industrialized sector.

Let us consider this second proposition first. Combining the small farm with the household running it we have a new household with continuous demand curves. Treating this household as part of the consumption sector rather than part of the production sector (thus changing the sign convention for measuring purchases and sales), we have a problem equivalent to our original problem, and thus reach the conclusion that we want to preserve efficiency within the production sector.

The argument above, that with two production sectors, it is best to tax them identically shows that a better welfare point could be achieved if these farms were labeled firms rather than households. Thus, this form of tax evasion lowers social welfare and requires a change in the structure of optimal taxes but does not change the advantage of the taxes as a policy tool over investment rules implying inefficiency in the advanced sector.

International Trade

So long as we are completely indifferent to the welfare of the rest of the world, international trade merely provides us with an additional production sector. We would want to equate marginal rates of transformation between producing and importing. If there is a monopoly position to be exploited, then this is called for. If international prices are unaffected by this country's demand, intermediate goods should not be subject to a tariff, but final good sales directly to consumers should be subject to a tariff equal to the tax on the same sale by a domestic producer.

Consideration of the rest of the world's welfare should seldom be negligible. We have another set of consumers who can trade with us at prices different from consumers in our own country. The case is similar to the possibility of having different consumer prices in different regions of the economy. In general, it will not be optimal either to trade with the rest of the world at domestic producer prices, p, or at domestic consumer prices, q. But it will still be desirable to produce on the production frontier.

Migration of Population

One of the most persuasive-seeming arguments advanced to justify inequality of incomes in countries other than the United States is the "brain drain." Our discussion would be seriously incomplete if we did

not allow for the influence on the size and composition of the equilibrium population within the country of the taxes that are imposed.

The form of our problem - the maximization of V(q) for D(q) in F - remains unaltered. We have to consider the properties of V and D anew. With a finite number of consumers, D becomes discontinuous, since each consumer has a particular set of q that will lead him to live in our economy, and will move out (or in) as soon as q crosses the frontier of this set. But this is certainly not a consideration of any importance, single consumers being a small fraction of the total populace. We must, therefore, allow for a continuum of consumers, and suppose - as will be reasonable in general - that demand function are continuous.

If we are prepared to assume that consumers have perfect information about their prospects in other countries, we may suppose that each remains in our country so long as his utility is greater than a certain level, u_o^h, but leaves and enjoys that level of utility elsewhere if he cannot attain this level in this economy. Respect for individual tastes should make the social welfare function depend upon the utility levels of all potential consumers.

Without entering into a rigorous treatment, we can see what must happen to the function V. A small change in consumer prices will change the utility level of those who do not leave or enter the economy, but leave unchangedthe utility of the others who remain outside, enter, or leave. Therefore, the conditions given earlier for V to be a strictly increasing function of prices when they change in some direction will also hold in the present case. If, for example, everyone is a net supplier of labor-effort, optimal production will still have to take place on the production frontier.

The possibility of migration does not, in general, then, change the derivatives of the indirect welfare function, V_k. It does change the

derivatives of the demand functions D (supposing they are differentiable). We shall not attempt to examine here to what extent, and in what direction, these considerations would be likely to affect the tax structure. They would surely play an important part in any thorough study of the structure of optimal income taxation.

A similar analysis could be applied to internal migration if there is geographic price discrimination. However, this analysis may not be appropriate for population changes due to births and deaths. It is parents, not babies, who control the number of births and the difference in contribution to social welfare from being born may not be zero for someone whose parents are on the margin of deciding on an additional child. In any case it is unclear what formulation of a social welfare function would be an adequate treatment of not yet born generations.

Capital Market Imperfections

One element that frequently appears in the discussion of public investment is imperfection in the capital market which limits consumers' ability to borrow. (A second element, not treated here, is a similar limitation for firms.) If the degree of limitation is not affected by the production decision, then this element, while altering the optimal tax structure is not a reason for aggregate production inefficiency. This can be seen most clearly by assuming that consumers can lend but cannot borrow. This still leaves well-defined, continuous, consumer demand functions in terms solely of prices. Thus the argument as made above goes through without further complication. The optimal tax structure is now changed because the derivative of an indirect utility function with respect to a price depends both on the quantity of the good purchased and the budget out of which the good is financed,

$$v_k = - \alpha_t \, x_k,$$

assuming good k is a good purchased in year t. For a consumer who is saving between years t and t+1 we would have $\alpha_t = \alpha_{t+1}$, but if he were not saving and would like to borrow, then $\alpha_t > \alpha_{t+1}$. Thus the inability of consumers to borrow would tend to lead to taxation that favored present over future consumption relative to a case where borrowing was possible.

Public Consumption and Consumer Externalities

The presence of externalities between consumers need not interfere with the continuity of demand functions, nor does it present a case where the production decision directly affects demands or utility (although existence of an optimum may be a more serious problem). Consequently, there is no justification for production inefficiency, although the tax structure will change to favor goods that give rise to positive externalities, if there are no private side payments.

We have treated public consumption as a fixed vector of requirements to be met out of production. This is unsatisfactory. We ought to think of government expenditures being undertaken because they affect total welfare, either by affecting the utility of certain individuals, or because the State insists upon valuing certain outputs itself. It is not always natural to quantify the output of public goods, but we shall proceed as though one could do so.

Let z be the vector of "public consumption," i.e., a vector of outputs that do not enter into trade but do affect welfare. We can write the indirect welfare function as $V(q,z)$. The problem is to maximize

(92) $\quad V(q,z)$ subject to $D(q,z) + z$ being in F.

Formally, this is a problem of the same kind as we have been studying. For the same reasons, production efficiency will, in general, be desirable. It follows that the government should seek to minimize the cost of its expenditure programmes, measuring the cost in terms of producer prices.

The form of the optimal tax relations

$$(93) \qquad V_q = \lambda p \cdot D_q$$

will remain unaltered. Finally, there are additional relations to determine the level of public consumption:

$$(94) \qquad V_z = \lambda \, [p \cdot D_z + p].$$

In other words, the shadow price of a public good should be equal to its aggregate social marginal utility, divided by the social marginal utility of government revenue, λ, plus the additional tax revenue generated, $t \cdot D_z = (q - p) \cdot D_z = - p \cdot D_z$.[20/]

This formulation of the public consumption problem, like the general formulation above, has ignored the distinction between publicly and privately controlled production. Many of the considerations that divide production possibilities between the public and private sectors are not easily captured by the type of model we have been analyzing. This raises the possibility that there may be additional political constraints which need to be added to the problem. One can construct examples where these constraints imply the desirability of aggregate inefficiency in goods being both given away publicly (presumably in limited amounts) and sold privately - although we will still want efficiency among the goods that are either just sold or just given away, but not both.

As an example consider an investment opportunity which is subject to the constraint that its output must be given away. Then comparing the marginal unit of output from this investment with that of the same good from an investment not subject to this constraint, we see two differences. First, the good being given away is allocated to a particular set of consumers and represents a lump sum income transfer

to them. Second, and the other side of the same transaction, the good given away does not directly contribute to government revenue as does the good sold. Thus we have different first order conditions for optimal production of these two investments and therefore, a lack of aggregate efficiency. We would still expect the same marginal rate of substitution between inputs for the two investments, for the above considerations do not apply to purchased inputs. As examples of goods both appearing in the market and transacted at non-market prices we have police protection (publicly given away and privately sold), and labor (publicly drafted and privately hired).

VI CONCLUSION

Welfare economics has usually been concerned with characterizing the best of attainable worlds, accepting only the basic technological constraints. As everyone has been aware, the omitted constraints on communication, calculation, and administration of an economy (not to mention political constraints) limit the direct applicability of the implications of this theory to policy problems, although great insight into these problems has certainly been acquired. We have not attempted to come directly to grips with the problem of incorporating these complications into economic theory. Instead, we have explored the implications of viewing these constraints as limits on the set of policy tools that can be applied. There are many sets of policy tools which might be examined in this way. Specifically, we have assumed that the policy tools available to the government include commodity taxation (and subsidization) to any extent. For these tools we have derived the rules for optimal tax policy and have shown the desirability of aggregate production efficiency, in the presence of optimal taxation. We have also considered expansion of the set of policy tools in a way which does not

violate the condition that production decisions do not change the class of possible budget constraints. For example, this condition is still preserved with the inclusion of poll taxes, progressive income taxation, regional differences in taxation, taxation on transactions between consumers, and most kinds of rationing. This type of expansion of the set of policy tools does not alter the desirability of production efficiency nor does it alter the conditions for the optimal commodity tax structure, although the tax rates themselves will change in general. We have, however, ignored the cost of administering taxes. Since there are indivisibilities in setting up taxes, we would expect to find that some of the available taxes are not used in general. It may then also be of interest to consider sets of policy variables that do not include commodity taxation. While the independence condition stated above may still hold, we would not expect this to be true with great generality. When this is violated we can no longer expect efficiency to be desirable. Similarly, we can examine the introduction of political, legal, or constitutional constraints into the model. If the constraints still permit sufficient flexibility in the choice of policy tools to maintain the independence condition, we can expect the analysis to parallel that above. Again, this condition may be violated.

Let us briefly consider the type of policy implications that are raised by our analysis. In the context of a planned economy our analysis implies the desirability of using a single price vector in all production decisions, although these prices will, in general, differ from the prices at which commodities are sold to consumers.

As an application of this analysis to a mixed economy, let us briefly examine the discussion of a proper criterion for public investment decisions. As has been widely noted, there are considerable differences in western economies between the intertemporal marginal rates of trans-

formation and substitution. This has been the basis of analyses leading to investment criteria which would imply aggregate production inefficiency because they employ an interest rate for determining the margins of production which differs from the private marginal rate of transformation. One argument used against these criteria is that the government, recognizing the divergence between rates of transformation and substitution, should use its power to achieve the full Pareto optimum, bringing these rates into equality. When this is done, the single interest rate then existing will be the appropriate rate to use in public investment decisions. We begin by presuming that the government does not have the power to achieve any Pareto optimum that it chooses. Then, from the maximization of a social welfare function, we argued that the government will, in general, prefer one of the non-Pareto optima to the Pareto optima, if any, that can be achieved. At the constrained optimum, which is the social welfare function maximizing position of the economy for the available policy tools, we saw that the economy will still be characterized by a divergence between marginal rates of substitution and transformation, not just intertemporally, but also elsewhere, e.g., in the choice between leisure and goods. However, we concluded that in this situation we desired aggregate production efficiency. This implies the use of interest rates for public investment decisions which equate public and private marginal rates of transformation.

We have obtained the first order conditions for public production, but we have not considered the correct method of evaluating indivisible investments. This is one problem that deserves examination. In examining the optimal tax structure, we have briefly considered the tax rates implied by particular utility functions. This analysis should be extended to more general and more interesting sets of consumers,

particularly in the context of a continuum of households. We have not considered the problem of calculation of the optimal tax structure. This raises questions on the econometric problems of obtaining the required information and the calculation problems of extending to this type of economy, where the indirect welfare function and the demand functions are not concave, the recent work on gradient methods and similar techniques of successive approximation.

Several of our assumptions also seem ripe candidates for further research. For a choice of the tax structure it is necessary to consider the costs of administering the tax and the implications of partial tax evasion. The assumption of constant returns to scale and competitive behavior in privately controlled production is also worth examining. In the presence of profits from either decreasing returns to scale or non-competitive market behavior, it is presumably desirable to add a profits tax to the policy tools. The problems in directly extending our analysis to this case can be seen by considering the fact that the choice of a production point implies a set of producer prices and thus a level of profits. Depending on the pattern of firm ownership in the economy, this then has an impact on consumer demands and thus on the indirect welfare function. It would also be useful to know whether, as one would suppose, it is possible to get close to the optimum with efficient aggregate production if pure profits are small.

We hope at any rate to have shown the possibility of analysis of a realistically "second-best" situation and the need to reconsider some policy recommendations.

FOOTNOTES

* The authors are at M.I.T. and Trinity College, Cambridge respectively. During some of the work the authors were at Churchill College, Cambridge and M.I.T. respectively. They wish to thank D. K. Foley, P. A. Samuelson, and K. Shell for helpful discussions on this subject. Diamond was supported in part by the National Science Foundation under grant GS 1585. The authors bear sole responsibility for opinions and errors.

1. For a discussion of this literature see A. Bergson, [1], [2].

2. For a survey of this literature see A. Prest and R. Turvey, [9]. For analysis quite close in spi to that employed here see M. Boiteux, [3], or the discussion of his work by J. Drèze, [6]. Boiteux's work will be briefly discussed in Section 5.

3. We wish to distinguish here between lump-sum taxes, which may vary from individual to individual, while being unaffected by the individual's behavior, and poll taxes which are the same for all individuals, or perhaps for all individuals within several large groups, distinguished perhaps by age, sex, or region.

4. For another study of the general equilibrium impact of taxation, which does not explore the optimality question, see G. Debreu, [4].

5. In Section 4 we consider the extension of these results to economies containing additional taxes, such as progressive income taxes. The restriction to commodity taxes is made for simplicity.

6. The reader is no doubt aware that with constant returns to scale, relative quantities are determined by prices but there is still the problem of determining the level of production. It is usual in equilibrium analysis to assume that firms produce at the appropriate level for equilibrium. We shall make the same assumption here.

7. In an intertemporal interpretation of this model, the government budget is in balance over the horizon of the model, not year by year.

8. This ignores the problem of the uniqueness of equilibrium for a given set of taxes.

9. x^* need not be productively efficient (in the usual sense, that $x \geqq x^*$ and x feasibly imply $x = x^*$). Even on the assumption of free disposal, it might be possible to do without some of the inputs if production of one commodity were at its maximum level. But it is being on the frontier that is relevant for the existence of prices, not efficiency.

10. Notational conventions, assumptions, and arguments are freely borrowed from Debreu, [5].

11. This strict convexity assumption can be weakened without affecting Theorems 1, 2, or 3, but causing a weakening of Theorem 5. These results are presented in the appendix to this section.

12. A proof is given in the appendix, Lemma 2.

13. This assumption is similar to the assumption that inputs are required to obtain outputs, but permits the government to own a vector of inputs.

14. These demands are derived from $u = \sum_{i=0}^{n} \alpha_i \log x_i$ with an initial endowment x^o.

15. Assuming $x_1^o = 0$, differentiation with respect to q_1 gives

$$- \alpha_1 q_1^{-1} - \lambda p_1 \alpha_1 q_1^{-2} \sum_j q_j x_j^o = 0. \quad \text{Since } q_1 \neq 0, \lambda \neq 0.$$

16. The norm $||\cdot||$ is the Euclidean norm.

17. For a discussion of multiple equilibria in a related problem see E. Foster and H. Sonnenschein [7].

18. If $\mu < 0$, utilities and marginal utilities are moved further apart.

19. We neglect the possibility of a free good when the first order condition would be an inequality.

20. We can contrast this with the first order conditions in the presence lump-sum taxation as presented by P. Samuelson, [12].

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