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by

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Number 20 -- April, 1968

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A DUAL MAXIMUM PRINCIPLE FOR DISCRETE-TIME LINEAR SYSTEMS WITH ECONOMIC APPLICATIONS

C. Duncan MacRae

Abstract - A discrete-time linear optimal control problem with given initial and terminal times, state-control constraints, and variable end points is set forth. Corresponding to this primal control problem, or maximization problem, is a dual linear control problem, or minimization problem. The fundamental properties of duality are stated. A dual maximum principle is proved with the aid of the duality theory of linear programming, where the dual of the Hamiltonian of the primal control problem is the Hamiltonian of the dual control problem. A discrete-time analogue of the Hamilton-Jacobi equation is derived; and economic applications are discussed.

INTRODUCTION

One of the major developments in modern control theory is the maximum principle. Although it was developed by Pontryagin [1] for continuous-time problems, it has been extended to discrete-time problems by Halkin [2,3], Holtzman [4,5] Jordan and Polak [6] through a geometric approach, and by Pearson and Sridhar [7], Mangasarian and Fromovitz [8], and Arimoto [9] with mathematical programming theory.

Another development of importance is the concept of duality in optimal control problems. The notion of duality was first discussed in modern control theory by Bellman [10] in the analysis of a continuous-time bottleneck process. He formulated a dual process both to verify a proposed solution to the allocation problem and to analyze the nature of the optimal process. His results have been extended by Tyndall [11] and Levinson [12] for a class of continuous linear programming problems. Kalman [13,14] observed duality in a different context. He recognized that the linear filtering problem is the dual of the linear regulator problem. Observability and controllability are dual concepts. His analysis has been generalized by Pearson [15] for continuous-time systems. Pearson [16], Hanson [17], and Ringlee [18] have exploited the idea of duality to derive a lower bound for a convex variational programming problem. Following the formulation of necessary conditions for the solution of the problem by Berkovitz [19], they derive a dual problem which yields the lower bound given a feasible solution to the primal problem. The

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knowledge of the lower bound is helpful in the computation of the solution. Pearson [20] and Kreindler [21] have analyzed reciprocity and duality in control programming problems. The reciprocal or dual of a primal control problem is obtained by interchanging the role of the state and co-state variables along the extremal of the primal. The statement of the reciprocal problem contains both state and co-state variables. Pearson distinguishes the dual problem from the reciprocal problem by the condition that the extremal of the primal problem is an extremal of the dual problem. Kreindler refers to the dual problem only as a reciprocal problem, since the reciprocal of the reciprocal is not the primal. He shows that along an extremal the Hamiltonian of the primal problem with inequality constraints is equal to the Hamiltonian of a reciprocal problem without inequality constraints. Pearson [22], Pallu de la Barriere [23], Mond and Hanson [24] have further discussed duality in continuous-time systems. With the exception of Kalman's work the discussion of duality in modern control theory has been limited to continuous-time systems. Moreover, with the exception of Kreindler's result there has been no examination of the relationship between the Hamiltonian of the primal control problem and the Hamiltonian of the reciprocal or dual control problem.

The primary contribution of this paper is a dual maximum principle for discrete-time linear systems. This theorem expresses the relationship between duality and the maximum principle in linear control problems and provides necessary and sufficient conditions for an optimal solution. Moreover, it has an immediate economic interpretation in terms of allocation and valuation of resources over time. The proof of the theorem follows from the complementary slackness theorem and the duality theory of linear programming.[1]

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[1] The application of linear programming to optimal control problems is not new. See Zadeh [25], Whalen [26], Dantzig [27], and Rosen [28].

DUALITY THEORY

In this section we examine duality in linear control problems. Consider a discrete-time linear control problem with given initial and terminal times, state-control constraints, and variable end points.

Primal Linear Control Problem

Choose $s_0 \geq 0$ and $u(k) \geq 0$ for $k = 0, 1, \ldots, N-1$, so as to maximize

$$J = f_0' s_0 + \sum_{k=0}^{N-1} [a'(k)x(k) + b'(k)u(k)] + f_N' x(N) \tag{1}$$

subject to

$$G_0 s_0 + x(0) = h_0, \tag{2}$$

$$D(k)u(k) - F(k)x(k) \leq d(k), \quad k = 0, 1, \ldots, N-1 \tag{3}$$

$$\Delta x(k) = A(k)x(k) + B(k)u(k) + c(k+1), \quad k = 0, 1, \ldots, N-1 \tag{4}$$

and

$$G_N x(N) \leq h_N, \tag{5}$$

where s_0 is a vector of primal pre-control variables, $x(k)$ is a vector of primal state variables, $u(k)$ is a vector of primal control variables, f_0, f_N, $a(k)$, $b(k)$, $c(k+1)$, $d(k)$, h_0, and h_N are given vectors, $A(k)$, $B(k)$, $D(k)$, $F(k)$, G_0, and G_N are given matrices, $\Delta x(k) = x(k+1) - x(k)$, 0 is the given primal initial time, N is the given primal terminal time, and $x(k)$ is taken as given in (3).

The general formulation of the discrete-time linear control problem given in (1) - (5) allows for fixed or free end points, equalities in the constraints, and pure control or state constraints. Although $x(k)$ is taken as given in (3), we can allow for primal state constraints of the form

$$M(k)x(k) \leq n(k) \qquad k = 1, 2, \ldots, N-1 \tag{6}$$

by substituting (4) into (6) to obtain

$$\{M(k+1)B(k)\}u(k) - \{-M(k+1)[I + A(k)]\}x(k) \leq \{n(k+1) - M(k+1)c(k+1)\}$$

$$k = 0, 1, \ldots, N - 2 \qquad (7)$$

which is of the form given by (3). Note that state constraints from period 1 to period N - 1 imply constraints on the control variables only up through period N - 2. If the constraints on the initial primal state are of the form (6) rather than (2), they can be represented by beginning the primal problem at $k = -1$ with a free state.

A feasible solution to the primal linear control problem is a sequence $\{x(k), u(k); s_0\}$ satisfying (2) - (5) with $u(k)$ and $s_0 \geq 0$. Let $J\{x(k), u(k); s_0\}$ be the value of J for the feasible solution $\{x(k), u(k); s_0\}$. An optimal solution to the primal control problem is a feasible solution $\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}$ such that

$$J\{\hat{x}(k), \hat{u}(k); \hat{s}_0\} \geq J\{x(k), u(k); s_0\} \qquad (8)$$

for all feasible solutions $\{x(k), u(k); s_0\}$.

The primal linear control problem can be reformulated as a linear programming problem.

<u>Primal Linear Programming Problem</u>

Find $s_0 \geq 0$, $x(k)$ unrestricted for $k = 0, 1, \ldots, N$, and $u(k) \geq 0$ for $k = 0, 1, \ldots, N - 1$ which maximize

$$J = f_0' s_0 + \sum_{k=0}^{N-1} [a'(k)x(k) + b'(k)u(k)] + f_N' x(N) \qquad (9)$$

subject to

$$G_0 s_0 + x(0) = h_0, \qquad (10)$$

$$-F(k)x(k) + D(k)u(k) \leq d(k) \qquad k = 0, 1, \ldots, N-1, \tag{11}$$

$$-[I + A(k)]x(k) - B(k)u(k) + x(k+1) = c(k+1) \qquad k = 0, 1, \ldots, N-1, \tag{12}$$

and

$$G_N x(N) \leq h_N. \tag{13}$$

Tucker Diagram

The structure of the primal linear programming problem can be seen more clearly with the aid of the detached coefficient array developed by A. W. Tucker [29]. The primal variables are listed across the top of the diagram along with their sign restrictions. Each row of the matrix gives the coefficients of one of the constraints, while the rightmost column gives the constants of the right-hand side of each constraint. The bottom row gives the coefficients of the primal variables in the objective function.

Corresponding to the primal linear programming problem is a dual linear programming problem. The dual variables are listed in the left-most column and the dual constraints are read down the columns of the matrix. The right-most column gives the coefficients of the dual objective function.

Dual Linear Programming Problem

Find $\gamma_N \geq 0$, $\Psi(k)$ unrestricted for $k = N, N-1, \ldots, 0$, and $\lambda(k) \geq 0$ for $k = N-1, N-2, \ldots, 0$, which minimize

$$\Phi = h_N' \gamma_N + \sum_{k=N-1}^{0} [d'(k)\lambda(k) + c'(k+1)\Psi(k+1)] + h_0' \Psi(0) \tag{14}$$

subject to

$$G_N' \gamma_N + \Psi(N) = f_N, \tag{15}$$

$$D'(k)\lambda(k) - B'(k)\Psi(k+1) \geq b(k) \qquad k = N-1, N-2, \ldots, 0, \tag{16}$$

Dual \ Primal	$s_0 \geq 0$	$x(0) \gtrless 0$	$u(0) \geq 0$	$x(1) \gtrless 0$	...	$x(N-1) \gtrless 0$	$u(N-1) \geq 0$	$x(N) \gtrless 0$		
$\Psi(0) \gtrless 0$	G_0	I							=	h_0
$\lambda(0) \geq 0$		$-F(0)$	$D(0)$						$\leq$	$d(0)$
$\Psi(1) \gtrless 0$		$-[I+A(0)]$	$-B(0)$	I					=	$c(1)$
...				...	...				...	...
$\Psi(N-1) \gtrless 0$					...	I			=	$c(N-1)$
$\lambda(N-1) \geq 0$						$-F(N-1)$	$D(N-1)$		$\leq$	$d(N-1)$
$\Psi(N) \gtrless 0$						$-[I+A(N-1)]$	$-B(N-1)$	I	=	$c(N)$
$\gamma_N \geq 0$								G_N	$\leq$	h_N
	$\geq$	=	$\geq$	=	...	=	$\geq$	=		
	f_0	$a(0)$	$b(0)$	$a(1)$	...	$a(N-1)$	$b(N-1)$	f_N		

FIGURE 1

$$\Psi(k) - F'(k)\lambda(k) - [I + A'(k)]\Psi(k+1) = a(k) \quad k = N-1, N-2, \ldots, 0, \tag{17}$$

and

$$G_0'\Psi(0) \geq f_0. \tag{18}$$

The dual linear programming problem can be reformulated as a dual linear control problem with given initial and terminal times, state-control constraints, and variable end points.

Dual Linear Control Problem

Choose $\gamma_N \geq 0$ and $\lambda(k) \geq 0$ for $k = N-1, N-2, \ldots, 0$ so as to minimize

$$\Phi = h_N'\gamma_N + \sum_{k=N-1}^{0} [c'(k+1)\Psi(k+1) + d'(k)\lambda(k)] + h_0'\Psi(0) \tag{19}$$

subject to

$$G_N'\gamma_N + \Psi(N) = f_N \tag{20}$$

$$D'(k)\lambda(k) - B'(k)\Psi(k+1) \geq b(k) \quad k = N-1, N-2, \ldots, 0, \tag{21}$$

$$\Delta\Psi(k+1) = A'(k)\Psi(k+1) + F'(k)\lambda(k) + a(k) \quad k = N-1, N-2, \ldots, 0, \tag{22}$$

and

$$G_0'\Psi(0) \geq f_0, \tag{23}$$

where $\Psi(k+1)$ is a vector of dual state variables, $\lambda(k)$ is a vector of dual control variables, γ_N is the vector of dual pre-control variables, $\Delta\Psi(k+1) = \Psi(k) - \Psi(k+1)$, N is the dual initial time, 0 is the dual terminal time, and $\Psi(k+1)$ is taken as given in (22).[2]

[2] Note that the statement of the dual linear control problem contains only dual variables. In general, this will not be true for nonlinear control problems. See Pearson [20] and Kreindler [21].

A feasible solution to the dual linear control problem is a sequence $\{\Psi(k+1),\ \lambda(k);\ \gamma_N\}$ satisfying (20) - (23). Let $\Phi\{\Psi(k+1),\ \lambda(k);\ \gamma_N\}$ be the value of Φ for the feasible solution $\{\Psi(k+1),\ \lambda(k);\ \gamma_N\}$. An optimal solution to the dual linear control problem is a feasible solution $\{\hat{\Psi}(k+1),\ \hat{\lambda}(k);\ \hat{\gamma}_N\}$ such that

$$\Phi\{\hat{\Psi}(k+1),\ \hat{\lambda}(k);\ \hat{\gamma}_N\} \le \Phi\{\Psi(k+1),\ \lambda(k);\ \gamma_N\} \tag{24}$$

for all feasible solutions $\{\Psi(k+1),\ \lambda(k);\ \gamma_N\}$.

Fundamental Properties of Duality

The primal (dual) problem is a maximization (minimization) problem. The primal (dual) system moves forward (backward) in time; and the primal initial (terminal) time is the dual terminal (initial) time.[3] Moreover, the dual of the dual control problem is the primal control problem.

Theorem 1 (Boundedness Theorem). If $\{x(k),\ u(k);\ s_0\}$ and $\{\Psi(k+1),\ \lambda(k);\ \gamma_N\}$ are feasible, then

$$J\{x(k),\ u(k);\ s_0\} \le \Phi\{\Psi(k+1),\ \lambda(k);\ \gamma_N\}. \tag{25}$$

Theorem 2 (Unboundedness Theorem). If the primal (dual) linear control problem is feasible, but the dual (primal) linear control problem is not feasible, J (Φ) is unbounded from above (below).

Theorem 3 (Existence Theorem). The primal (dual) linear control problem has an optimal solution if and only if both control problems have feasible solutions.

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[3]This property of duality was first observed by R. Bellman [10] in the analysis of a continuous-time bottleneck process. R. E. Kalman [13,14] also perceived this property in his study of the linear filtering problem.

Theorem 4 (Duality Theorem). A primal (dual) feasible solution $\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}$ ($\{\hat{\Psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\}$) is an optimal solution if and only if there exists a dual (primal) feasible solution $\{\hat{\Psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\}$ ($\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}$) with

$$J\{\hat{x}(k), \hat{u}(k); \hat{s}_0\} = \Phi\{\hat{\Psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\}. \tag{26}$$

Theorem 5 (Complementary Slackness Theorem). A primal (dual) feasible solution $\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}$ ($\{\hat{\Psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\}$) is an optimal solution if and only if there exists a dual (primal) feasible solution $\{\hat{\Psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\}$ ($\{\hat{x}(k), \hat{\lambda}(k); \hat{s}_0\}$) with

$$\hat{\lambda}'(k)[D(k)\hat{u}(k) - F(k)\hat{x}(k) - d(k)] = 0 \quad k = 0, 1, \ldots, N-1, \tag{27}$$

$$\hat{u}'(k)[D'(k)\hat{\lambda}(k) - B'(k)\hat{\Psi}(k+1) - b(k)] = 0 \quad k = N-1, N-2, \ldots, 0, \tag{28}$$

and the transversality conditions

$$\hat{s}_0'\hat{\gamma}_0 = 0, \tag{29}$$

and

$$\hat{\gamma}_N'\hat{s}_N = 0, \tag{30}$$

where

$$\hat{\gamma}_0 = G_0'\hat{\Psi}(0) - f_0 \text{ and } \hat{s}_N = h_N - G_N\hat{x}(N). \tag{31}$$

Theorems 1-5 are derived from the corresponding theorems of linear programming [29].

Theorem 5 is a statement of the Kuhn-Tucker [30] necessary and sufficient conditions for an optimum solution. Note that the initial (terminal) transversality condition for the primal control problem, (29) ((30)), is the terminal (initial) transversality condition for the dual control problem.

Theorem 6 (Sensitivity Theorem). If $\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}$ and $\{\hat{\Psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\}$ are optimal,

$$\nabla_{x(k)} J\{\hat{x}(k)\} = \hat{\Psi}(k) \quad k = 0, 1, \ldots, N, \tag{32}$$

and

$$\nabla_{\Psi(k)} \Phi\{\hat{\Psi}(k)\} = \hat{x}(k) \quad k = N, N-1, \ldots, 0 \tag{33}$$

where $\nabla_{x(k)} J\{\hat{x}(k)\}$ is the gradient of J with respect to x(k) evaluated on $\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}$ if the gradient exists, etc.

The proof of (32) and (33) follows immediately from the sensitivity theory of linear programming [31]. In (32) and (33) $\hat{x}(k)$ and $\hat{\Psi}(k)$ are treated as parameters. Note that the derivatives in (32) and (33) may be one-sided.

Although the sensitivity analysis is in terms of derivatives, we can be sure for finite changes in the parameters that (32) and (33) describe upper bounds on the change in J and lower bounds on the change in Φ since J is concave in the parameters and Φ is convex.

HAMILTONIAN THEORY

In this section we investigate the relationship between duality in linear control problems and the corresponding Hamiltonian programming problems. Consider a sequence of primal linear programming problems for k = 0, 1,...,N-1.

<u>Primal Hamiltonian Programming Problem</u>

Choose $u(k) \geq 0$ so as to maximize

$$\mathcal{H}(k) = a'(k)x(k) + b'(k)u(k)$$

$$+ \Psi'(k+1)[A(k)x(k) + B(k)u(k) + c(k+1)] \tag{34}$$

subject to

$$D(k)u(k) - F(k)x(k) \leq d(k), \tag{35}$$

where $x(k)$ and $\Psi(k+1)$ are given, $\mathcal{H}(k)$ is the Hamiltonian of the primal linear control problem, and (35) describes the state-control constraints of the primal control problem.

A feasible solution to the primal Hamiltonian programming problem is a vector $u(k) \geq 0$ satisfying (35). Let $\mathcal{H}[u(k); x(k), \Psi(k+1)]$ be the value of $\mathcal{H}$ for the feasible solution $u(k)$, given $x(k)$ and $\Psi(k+1)$. An optimal solution to the primal Hamiltonian program is a feasible solution $\hat{u}(k)$ such that

$$\mathcal{H}[\hat{u}(k); x(k), \Psi(k+1)] \geq \mathcal{H}[u(k); x(k), \Psi(k+1)] \tag{36}$$

for all feasible solutions $u(k)$, given $x(k)$ and $\Psi(k+1)$.

Corresponding to this sequence of primal Hamiltonian programming problems is a sequence of dual linear programming problems for $k = N-1, N-2, \ldots, 0$.

Dual Hamiltonian Programming Problem

Choose $\lambda(k) \geq 0$ so as to minimize

$$\begin{aligned} \mathcal{K}(k) = {} & c'(k+1)\Psi(k+1) + d'(k)\lambda(k) \\ & + x'(k)[A'(k)\Psi(k+1) + F'(k)\lambda(k) + a(k)] \end{aligned} \tag{37}$$

subject to

$$D'(k)\lambda(k) - B'(k)\Psi(k+1) \geq b(k), \tag{38}$$

where $\Psi(k+1)$ and $x(k)$ are given, $\mathcal{K}(k)$ is the Hamiltonian of the dual linear control problem, and (38) describes the state-control constraints of the dual control problem.

A feasible solution to the dual Hamiltonian programming problem is a vector $\lambda(k) \geq 0$ satisfying (38). Let $\mathcal{K}[\lambda(k); \Psi(k+1), x(k)]$ be the value of $\mathcal{K}(k)$ for the feasible solution $\lambda(k)$, given $\Psi(k+1)$ and $x(k)$. An optimal solution to the dual Hamiltonian programming problem is a feasible solution $\hat{\lambda}(k)$ such that

$$\mathcal{K}[\hat{\lambda}(k);\Psi(k+1),\ x(k)] \leq \mathcal{K}[\lambda(k);\Psi(k+1),\ x(k)] \quad (39)$$

for all feasible solutions $\lambda(k)$, given $\Psi(k+1)$ and $x(k)$.

Dual Maximum Principle

Lemma 1. A feasible solution $\hat{u}(k)$ ($\hat{\lambda}(k)$) to the primal (dual) Hamiltonian programming problem is an optimal solution if and only if there exists a feasible solution $\hat{\lambda}(k)$ ($\hat{u}(k)$) to the dual (primal) Hamiltonian problem with

$$\hat{\lambda}'(k)[D(k)\hat{u}(k) - F(k)x(k) - d(k)] = 0 \quad (40)$$

and

$$\hat{u}'(k)[D'(k)\hat{\lambda}(k) - B'(k)\Psi(k+1) - b(k)] = 0. \quad (41)$$

Corollary 1.

$$\mathcal{H}[\hat{u}(k);x(k),\ \Psi(k+1)] = \mathcal{K}[\hat{\lambda}(k);\Psi(k+1),\ x(k)]. \quad (42)$$

Lemma 1 and Corollary 1 are direct applications of the complementary slackness and duality theorems of linear programming [29].

The main result of this paper can now be stated.

Theorem 7 (Dual Maximum Principle). A feasible solution $\{\hat{x}(k),\ \hat{u}(k);\ \hat{s}_0\}$ ($\{\hat{\Psi}(k+1),\ \hat{\lambda}(k);\ \hat{\gamma}_N\}$) to the primal (dual) linear control problem is an optimal solution if and only if there exists a feasible solution $\{\hat{\Psi}(k+1),\ \hat{\lambda}(k);\ \hat{\gamma}_N\}$ ($\{\hat{x}(k),\ \hat{u}(k);\ \hat{s}_0\}$) to the dual (primal) control problem with

$$\mathcal{H}[\hat{u}(k);\ \hat{x}(k),\ \hat{\Psi}(k+1)] = \mathcal{K}[\hat{\lambda}(k);\ \hat{\Psi}(k+1),\ \hat{x}(k)] \quad (43)$$

and satisfying the primal and dual transversality conditions, (29) and (30),

where

$$\Delta\hat{x}(k) = \nabla_{\Psi(k+1)}\mathcal{H}[\hat{u}(k);\hat{x}(k),\ \hat{\Psi}(k+1)] \quad (44)$$

and $$\Delta\hat{\Psi}(k+1) = \nabla_{x(k)}\mathcal{K}[\hat{\lambda}(k);\hat{\Psi}(k+1),\ \hat{x}(k)]. \tag{45}$$

The proof follows from Theorem 5, Lemma 1, and Corollary 1.

Hamilton-Jacobi Inequality

We can use the dual maximum principle to derive a discrete-time analogue of the Hamilton-Jacobi partial differential equation for continuous-time optimal systems.

Theorem 8 (Hamilton-Jacobi Inequality). If $\{\hat{x}(k),\ \hat{u}(k);\ \hat{s}_0\}$ and $\{\hat{\Psi}(k+1),\ \hat{\lambda}(k);\ \hat{\gamma}_N\}$ are optimal solutions to the primal and dual linear control problems,

$$J_k\{\hat{x}(k)\} - J_{k+1}\{\hat{x}(k)\} \geq \mathcal{H}[\hat{u}(k);\ \hat{x}(k),\ \hat{\Psi}(k+1)]$$

$$= \mathcal{K}[\hat{\lambda}(k);\ \hat{\Psi}(k+1),\ \hat{x}(k)] \geq \Phi_{k+1}\{\hat{\Psi}(k+1)\} - \Phi_k\{\hat{\Psi}(k+1)\} \tag{46}$$

where $J_k\{\hat{x}(k)\}$ is the optimal value of J at the beginning of period k with initial primal state $\hat{x}(k)$, etc., and $\Phi_{k+1}\{\hat{\Psi}(k+1)\}$ is the optimal value of Φ at the end of period k with an initial dual state $\hat{\Psi}(k+1)$, etc.

The proof of the theorem employs the principle of optimality [10], the mean value theorem, and Theorem 7. Applying the principle of optimality to the primal and dual linear control problems we obtain

$$J_k\{\hat{x}(k)\} = a'(k)\hat{x}(k) + b'(k)\hat{u}(k) + J_{k+1}\{\hat{x}(k+1)\} \tag{47}$$

or $$J_k\{\hat{x}(k)\} - J_{k+1}\{\hat{x}(k)\}$$

$$= a'(k)\hat{x}(k) + b'(k)\hat{u}(k) + J_{k+1}\{\hat{x}(k+1)\} - J_{k+1}\{\hat{x}(k)\}, \tag{48}$$

and $$\Phi_{k+1}\{\hat{\Psi}(k+1)\} = c'(k+1)\hat{\Psi}(k+1) + d'(k)\hat{\lambda}(k) + \Phi_k\{\hat{\Psi}(k)\} \tag{49}$$

or
$$\Phi_{k+1}\{\hat{\Psi}(k+1)\} - \Phi_k\{\hat{\Psi}(k+1)\}$$

$$= c'(k+1)\hat{\Psi}(k+1) + d'(k)\hat{\lambda}(k) + \Phi_k\{\hat{\Psi}(k)\} - \Phi_k\{\hat{\Psi}(k+1)\}. \qquad (50)$$

From the mean value theorem we know there exists a $\theta(k)$ and a $\sigma(k)$ subject to $0 \leq \theta(k) \leq 1$ and $0 \leq \sigma(k) \leq 1$ such that

$$\nabla_{x(k+1)} J\{\hat{x}(k+1) - [1 - \theta(k)]\Delta\hat{x}(k)\}'\Delta\hat{x}(k)$$

$$= J_{k+1}\{\hat{x}(k+1)\} - J_{k+1}\{\hat{x}(k)\} \qquad (51)$$

and
$$\nabla_{\Psi(k)} \Phi\{\hat{\Psi}(k) - [1 - \sigma(k)]\Delta\hat{\Psi}(k+1)\}'\Delta\hat{\Psi}(k+1)$$

$$= \Phi_k\{\hat{\Psi}(k)\} - \Phi_k\{\hat{\Psi}(k+1)\}, \qquad (52)$$

or
$$\nabla_{x(k+1)} J\{\hat{x}(k)\}'\Delta\hat{x}(k) \leq J_{k+1}\{\hat{x}(k+1)\} - J_{k+1}\{\hat{x}(k)\} \qquad (53)$$

and
$$\nabla_{\Psi(k)} \Phi\{\hat{\Psi}(k+1)\}'\Delta\hat{\Psi}(k+1) \geq \Phi_k\{\hat{\Psi}(k)\} - \Phi_k\{\hat{\Psi}(k+1)\} \qquad (54)$$

since J is concave in x and Φ is convex in Ψ, where $\nabla_{x(k+1)} J\{\hat{x}(k+1) - [1-\theta(k)]\Delta\hat{x}(k)\}$ is the gradient of J with respect to x(k+1) evaluated at $\hat{x}(k+1) - [1-\theta(k)]\Delta\hat{x}(k)$ if it exists, etc.

Substituting (32), (34), and (53) into (48), and (33), (37), and (54) into (50), we obtain (46) from (42), (48), and (50). Since time is a discrete variable, (46) is an inequality rather than an equality.

ECONOMIC APPLICATIONS

The problem of allocating goods and services over time can be formulated as a primal linear control problem with the aid of linear activity analysis [32,33]. The flow of commodities, y(k), is assumed to be a linear function of the activity levels, u(k):

$$y(k) = C(k)u(k), \tag{55}$$

where C(k) is a given matrix describing the technology of the system. The primal state variables are the stocks of commodities, x(k); and the primal control variables are the activity levels, u(k). The primal control problem is to choose the activity levels over time so as to maximize net present value, (1), subject to the initial and terminal conditions on the stocks of commodities, (2) and (5), the capacity constraints, (3), and the transformation of stocks, (4).

Corresponding to the allocation problem is a dual problem or valuation problem. The dual state variables are the imputed prices of the stocks of commodities, $\Psi(k)$; and the dual control variables are the imputed prices of capacity, $\lambda(k)$. The dual control problem is to choose the capacity prices over time so as to minimize the imputed cost of commodities and capacities, (19), subject to the initial and terminal constraints on commodity prices, (20) and (23), the no unimputed revenue constraints, (21), and the transformation of prices, (22).

The economic interpretation of the dual maximum principle, therefore, is that the long run allocation and valuation problems can be decomposed into a sequence of short run allocation and valuation problems, or primal and dual Hamiltonian programming problems. The primal programming problem is to choose activity levels so as to maximize net income in terms of revenue, (34), subject

to the capacity constraints, (35), given commodity stocks; and the dual programming problem is to choose capacity prices so as to minimize net income in terms of imputed cost, (37), subject to the no unimputed revenue constraints, (38), given commodity prices. The Hamilton-Jacobi inequality, then, shows the way the allocation sequence moves forward in time describing the impact of pricing decisions tomorrow on imputed cost today and the valuation sequence moves backward in time describing the impact of allocation decisions today on revenue tomorrow.

There is an intimate relation between linear control and linear programming. Any problem which can be formulated as a discrete-time linear control problem with a finite horizon also can be formulated as a multi-stage linear programming problem and vice-versa. Applications include defense [32], production, transportation and investment [34], regional and national economic planning [35], education and manpower planning [36], and water resource development [37].

There are, however, a number of advantages to formulating a problem involving allocating resources over time as a problem in linear control rather than as a problem in linear programming. The formulation of dynamic allocation problems in forms of state and control variables is natural since the variables have immediate physical interpretation in terms of stocks of commodities and levels of activities. Moreover, the problem need only be formulated for the representative period using a standard format. Alternatives and generalizations, even to include non-linearities, then can be made more readily. Also the boundary conditions on the initial and terminal stocks of commodities, which are implicit or explicit in every allocation problem are more apparent in the linear control formulation.

Once the allocation problem is formulated as a linear control problem, there are a number of ways the dual maximum principle can be applied to the solution of the problem. The linear control problem can be reformulated as a multi-stage

linear programming problem. The simplex algorithm and its variants [38] can, then, be considered as applications of the dual maximum principle. The disadvantage of this approach is that the resulting linear programming problem may become so large that it is intractable. The problem, however, does have the special "staircase" structure, illustrated in Figure 1, to which we can apply the decomposition principle [38]. The maximum principle for linear systems may even be considered to be a consequence of the decomposition principle of linear programming [27]. The usual approach, however, is to use the maximum principle to express the optimal control as a function of the state and co-state vectors. The disadvantage of this approach to the solution of linear control problems is that the solution to the Hamiltonian programming problem may not be unique.

The main limitation to the application of linear control theory is the assumption of convexity. This assumption excludes economies of mass production and discrete changes in the state and control variables [38]. There are situations in which the absence of convexity is too blatent to be ignored. There are many situations, however, in which the assumption of convexity is not unreasonable.

CONCLUSIONS

The main result of this paper is a dual maximum principle for discrete-time linear systems. A feasible solution to the primal (dual) linear control problem is an optimal solution if and only if there exists a feasible solution to the dual (primal) control problem with the primal and dual Hamiltonians equal and satisfying the primal and dual transversality conditions, where the dual of the Hamiltonian programming problem associated with the primal linear control problem is the dual of the Hamiltonian problem associated with the dual control problem. The main application of the maximum principle in modern control theory has been in the solution of optimal control problems. The dual maximum principle can be helpful, however, not only in the solution of linear control problems, but also in their formulation, interpretation, and analysis. This is particularly true for problems involving allocation and valuation.

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