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# On the Marginal Distribution of the Eigenvalues of Wishart Matrices

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***Abstract*—Random matrices play a crucial role in the design and analysis of multiple-input multiple-output (MIMO) systems. In particular, performance of MIMO systems depends on the statistical properties of a subclass of random matrices known as Wishart when the propagation environment is characterized by Rayleigh or Rician fading. This paper focuses on the stochastic analysis of this class of matrices and proposes a general methodology to evaluate some multiple nested integrals of interest. With this methodology we obtain a closed-form expression for the joint probability density function of $k$ consecutive ordered eigenvalues and, as a special case, the PDF of the $\ell^{\underline{\text{th}}}$ ordered eigenvalue of Wishart matrices. The distribution of the largest eigenvalue can be used to analyze the performance of MIMO maximal ratio combining systems. The PDF of the smallest eigenvalue can be used for MIMO antenna selection techniques. Finally, the PDF the $k^{\underline{\text{th}}}$ largest eigenvalue finds applications in the performance analysis of MIMO singular value decomposition systems.**

***Index Terms*—Multiple-input multiple-output (MIMO), Wishart matrices, eigenvalue distribution, marginal distribution**

## I. Introduction

THE increasing demand for higher capacity has generated interest in multiple antenna systems [1], [2] and, more recently, in multiple-input multiple-output (MIMO) systems [3]–[10]. Such systems can provide high spectral efficiency in rich and quasi-static scattering environments for which the elements of the channel gain matrix $\mathbf{H}$ are random variables[1] [3]–[10]. In particular, performance of MIMO systems depends on the distribution of the eigenvalues of Hermitian matrices of the form $\mathbf{H}\mathbf{H}^\dagger$, where the superscript $^\dagger$ denotes conjugation and transposition. In general the distribution of the eigenvalues is known, or can be expressed in a tractable form, only for some special cases. Fortunately, in several practical situations, the elements of the channel matrix can be modelled as complex Gaussian random variables; this is the case, for example, when the propagation environment is characterized by Rayleigh or Rician fading. Under these conditions, $\mathbf{H}\mathbf{H}^\dagger$ represents a particular case of random matrix, known as Wishart [11]–[13], whose joint probability density function (PDF) of the eigenvalues can be written in terms of hypergeometric functions [14]. The knowledge of the joint PDF of the eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ has been used extensively to analyze the performance of MIMO systems in terms of capacity [7], [8], [15] and symbol error probability [16].

The ergodic capacity of MIMO systems can be expressed in terms of the joint PDF of the generic (unordered) eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ [7]. Therefore, the knowledge of this PDF for a given propagation environment enables the evaluation of the expected value of the MIMO capacity [7], [8]. However, further analysis of this joint PDF is necessary to investigate the performance of some MIMO systems. For example, in MIMO maximal ratio combining (MIMO-MRC), the instantaneous (with respect to fading) signal-to-noise ratio (SNR) at the output of the combiner is proportional to the largest eigenvalue of $\mathbf{H}\mathbf{H}^\dagger$ [17], [18].[2] The cumulative density function (CDF) of this eigenvalue has been known for nearly four decades [19], [20] and has been recently applied to performance analysis of MIMO-MRC systems [18]. These examples reveal that the distribution of the eigenvalues of Wishart matrices has been investigated in the literature for a few special cases, specifically for the joint PDF of all the eigenvalues, or for the PDF of the largest eigenvalue [8], [18]–[20]. Results on the joint PDF of the eigenvalues for the case of fully correlated Wishart (with correlation among both rows and columns) are given in [21]. Although the knowledge of the joint PDF allows, in principle, the derivation of any marginal distribution, analysis of Wishart matrices can be challenging.

In the paper, we propose a general methodology to evaluate some multiple nested integrals with an integrand expressed as the product of two determinants. Since the expression for the joint PDF of the eigenvalues of a Wishart matrix can be written as a product of two determinants, we obtain closed-form expressions for the joint PDF of $k$ consecutive eigenvalues, as well as for the $\ell^{\underline{\text{th}}}$ largest eigenvalue in the cases of uncorrelated (both central and noncentral) and correlated (central) Wishart matrices.[3] These distributions enable the investigation of MIMO systems in the presence of Rayleigh (central Wishart) and Rician (noncentral Wishart) fading.

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[1]The matrix $\mathbf{H}$ is a $N_\text{R} \times N_\text{T}$ matrix, where $N_\text{R}$ and $N_\text{T}$ indicate the number of receive and transmit antennas, respectively.

[2]MIMO-MRC is equivalent to maximum ratio transmission.

[3]Note that the CDF of the $\ell^{\underline{\text{th}}}$ largest eigenvalue was also studied in [19], but the final expression given there still requires the evaluation of an infinite sum.

TABLE I
CONSTANTS AND MATRICES IN (1) FOR UNCORRELATED CENTRAL, UNCORRELATED NONCENTRAL AND CORRELATED CENTRAL WISHART

| | $K$ | $\mathbf{\Phi}(\mathbf{x})$ | $\mathbf{\Psi}(\mathbf{x})$ | $\xi(x)$ |
|---|---|---|---|---|
| uncorrelated central | $K_{\text{uc}} = \left[\prod_{i=1}^{q}(p-i)!\prod_{j=1}^{q}(q-j)!\right]^{-1}$ | $\mathbf{V}_1(\mathbf{x})$ | $\mathbf{V}_1(\mathbf{x})$ | $x^{p-q}e^{-x}$ |
| uncorrelated noncentral | $K_{\text{un}} = \frac{\prod_{i=1}^{q} e^{-\mu_i}}{[(p-q)!]^q \, \lvert\mathbf{V}_1(\boldsymbol{\mu})\rvert}$ | $\mathbf{V}_1(\mathbf{x})$ | $\mathbf{F}(\mathbf{x},\boldsymbol{\mu})$ | $x^{p-q}e^{-x}$ |
| correlated central | $K_{\text{cc}} = K_{\text{uc}}\prod_{i=1}^{q}(i-1)!\frac{\lvert\mathbf{\Sigma}\rvert^{-p}}{\lvert\mathbf{V}_2(\boldsymbol{\sigma})\rvert}$ | $\mathbf{V}_1(\mathbf{x})$ | $\mathbf{E}(\mathbf{x},\boldsymbol{\sigma})$ | $x^{p-q}$ |

The main contributions of the paper are as follows:

- derivation of the exact expression for the joint PDF of $k$ consecutive ordered eigenvalues;
- derivation of the PDF of the $\ell^{\underline{\text{th}}}$ ordered eigenvalue;
- a concise representation for the PDF of the largest and smallest eigenvalue.

These results, which extend the continuous analog of Cauchy-Binet formulas [8], can be applied to arbitrary Wishart matrices (uncorrelated central, correlated central, uncorrelated noncentral). Note that these results are expressed in closed-form in the case of central Wishart (both uncorrelated and correlated) and as an infinite series expansion in the case of uncorrelated noncentral Wishart. As discussed previously, the distribution of the largest eigenvalue can be used to analyze the performance of MIMO-MRC systems [17], [18], [22]–[25]. The PDF of the smallest eigenvalue can be used for MIMO antenna selection techniques [26]. Finally, the PDF the $\ell^{\underline{\text{th}}}$ largest eigenvalue finds applications in the performance analysis of MIMO systems with singular value decomposition (MIMO-SVD) [7], [24].

The paper is organized as follows: in Sec. II we provide a brief review of the joint PDF of the eigenvalues of a Wishart matrix and derive the CDF of the extreme eigenvalues. In Sec. III we obtain theorems that can be used to evaluate multiple nested integrals. The results of Sec. III are used in Sec. IV to obtain a concise representation for the joint PDF of consecutive eigenvalues of Wishart matrices. Some numerical examples are shown in Sec. V, and conclusions are given in Sec. VI.

## II. PRELIMINARIES

Throughout the paper, vectors and matrices are indicated by bold, $|\mathbf{A}|$ and $\text{tr}\{\mathbf{A}\}$ denote the determinant and the trace of a matrix $\mathbf{A}$, respectively. Let us define the $(q \times p)$, with $q \leq p$, complex matrix $\mathbf{A}$, with a common covariance matrix $\mathbf{\Sigma} = \mathbb{E}\left\{\mathbf{a}_j \, \mathbf{a}_j^\dagger\right\} \forall j$, where $\mathbf{a}_j$ is the $j^{\underline{\text{th}}}$ column vector of $\mathbf{A}$. The elements of two columns $\mathbf{a}_i$ and $\mathbf{a}_j$ are considered to be mutually independent. If the elements of $\mathbf{A}$, $a_{ij}$, are complex valued with real and imaginary part each belonging to a normal distribution $\mathcal{N}(0, 1/2)$ so that $\mathbb{E}\{\mathbf{A}\} = \mathbf{0}$, then the Hermitian matrix $\mathcal{W}_q(p, \mathbf{\Sigma}) = \mathbf{A}\mathbf{A}^\dagger$ is called central Wishart [27]. When $\mathbb{E}\{\mathbf{A}\} = \mathbf{M} \neq \mathbf{0}$, the matrix is called noncentral Wishart. We will denote the cases $\mathbf{\Sigma} = \mathbf{I}$ and $\mathbf{\Sigma} \neq \mathbf{I}$ as uncorrelated and correlated Wishart, respectively. It has been known (for more than four decades [14]) that the joint PDF of the eigenvalues of $\mathcal{W}_q(p, \mathbf{\Sigma})$ can be expressed in terms of hypergeometric functions of Hermitian matrices. More recently, a simpler form of this joint PDF was derived in terms of the product of two determinants [8].

Specifically, the joint PDF of the ordered eigenvalues for the cases of uncorrelated (both central and noncentral) and correlated (central) Wishart matrices can be written in the form

$$f_{\boldsymbol{\lambda}}(\mathbf{x}) = K|\mathbf{\Phi}(\mathbf{x})| \cdot |\mathbf{\Psi}(\mathbf{x})| \prod_{l=1}^{q} \xi(x_l) \tag{1}$$

where $\mathbf{x} = [x_1, \ldots, x_q]^T$ and $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_q]^T$ is the vector of the ordered ($\lambda_1 \geq \cdots \geq \lambda_q$) eigenvalues. The values of the normalizing constant $K$, $\mathbf{\Phi}(\mathbf{x})$, $\mathbf{\Psi}(\mathbf{x})$, and $\xi(x)$ for uncorrelated central, correlated central, and uncorrelated noncentral are due to [14], [8], and [18], respectively, and are summarized in Table I. In this Table, $\mathbf{V}_1(\mathbf{x})$ denotes a Vandermonde matrix [28, pp. 29] whose $(i,j)^{\underline{\text{th}}}$ element is $x_j^{i-1}$; $\mu_1 > \mu_2 > \cdots > \mu_q$ are the eigenvalues of $\mathbf{M}^\dagger\mathbf{M}$ with $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_q]^T$, and $\sigma_1 > \sigma_2 > \cdots > \sigma_q$ are the eigenvalues of $\mathbf{\Sigma}$ with $\boldsymbol{\sigma} = [\sigma_1, \ldots, \sigma_q]^T$. The $(i,j)^{\underline{\text{th}}}$ elements of the matrices $\mathbf{V}_2(\boldsymbol{\sigma})$, $\mathbf{F}(\mathbf{x}, \boldsymbol{\mu})$, and $\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})$ are $-\sigma_j^{1-i}$, ${}_0\mathcal{F}_1(p-q+1; x_i\mu_j)$, and $e^{-x_j/\sigma_i}$, respectively, where ${}_0\mathcal{F}_1$ is the Confluent hypergeometric function. Now, let us discuss some special cases.

### A. Pseudo Wishart matrices

When the correlation is among the elements of the rows of $\mathbf{A}$ instead of the columns, the matrix is usually referred to non full rank Wishart or pseudo Wishart. In that case, the distribution of the eigenvalues can still be written in the form of (1), but now either $\mathbf{\Phi}(\mathbf{x})$ or $\mathbf{\Psi}(\mathbf{x})$ are $(p \times p)$ matrices. For instance, if $\mathbf{\Phi}(\mathbf{x})$ is a $(p \times p)$ matrix (similarly for $\mathbf{\Psi}(\mathbf{x})$), then the $(i,j)^{\underline{\text{th}}}$ element is given by [9]

$$\{\mathbf{\Phi}(\mathbf{x})\}_{i,j} = \begin{cases} \phi_{i,j} & j = 1, \ldots p-q \\ \phi_i(x_j) & j = p-q+1, \ldots, p. \end{cases} \tag{2}$$

Although in this paper we focus on the distribution of the eigenvalues of full rank Wishart matrices, the results of Sec. III can be easily extended to matrices having the form of (2), and all the results of this paper can be extended to the pseudo Wishart case.

### B. Covariance matrix $\mathbf{\Sigma}$ with coincident eigenvalues

In the case of correlated central Wishart, the joint PDF of the eigenvalues takes the form of (1) with $K$, $\mathbf{\Phi}(\mathbf{x})$, $\mathbf{\Psi}(\mathbf{x})$ and $\xi(x)$ replaced by $K_{\text{cc}}$, $\mathbf{V}_1(\mathbf{x})$, $\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})$, and $x^{p-q}$. If the covariance matrix $\mathbf{\Sigma}$ presents some coincident eigenvalues (say $\sigma_1 = \sigma_2 = \cdots = \sigma_\ell$), we need to calculate the following limit

$$\lim_{\sigma_2 \cdots \sigma_\ell \to \sigma_1} \frac{|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})|}{|\mathbf{V}_2(\boldsymbol{\sigma})|} \tag{3}$$

which can be evaluated by means of Lemma 2 of [29], [30]. Note that (3) has an impact only on the constant $K_{\text{cc}}$ and on $\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})$. In particular, the $(i,j)^{\underline{\text{th}}}$ element of $\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})$ becomes

$$\{\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\}_{i,j} = \begin{cases} e^{-x_j/\sigma_i} & i = 1, \ldots, q-\ell \\ x_j^{q-i} & i = q-\ell+1, \ldots, q. \end{cases} \tag{4}$$

It is straightforward to observe that the $(i,j)^{\underline{\text{th}}}$ element of $\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})$ is still in the form of $\phi_i(x_j)$, and therefore all the results of the paper can be applied.

### *C. Matrix $\mathbf{M}^\dagger \mathbf{M}$ with arbitrary rank*

The joint PDF of the eigenvalues of an uncorrelated non-central Wishart matrix when $\mathbf{M}^\dagger \mathbf{M}$ is allowed arbitrary rank (say $m$) is given by [31]

$$f_{\boldsymbol{\lambda}}(\mathbf{x}) = K_{\text{un}}^{'} |\mathbf{W}(\mathbf{x})| \cdot |\mathbf{G}(\mathbf{x})| \prod_{l=1}^{q} x_l^{p-q} e^{-x_l} \tag{5}$$

where $K_{\text{un}}^{'}$ is a normalizing constant, the $(i,j)^{\underline{\text{th}}}$ element of $\mathbf{W}(\mathbf{x})$ is $x_i^{q-j}$, and the $(i,j)^{\underline{\text{th}}}$ element of $\mathbf{G}(\mathbf{x})$ is

$$\{\mathbf{G}(\mathbf{x})\}_{i,j} = \begin{cases} \frac{{}_0\mathcal{F}_1(p-q+1, \mu_j x_i)}{(p-q)!} & j = 1, \ldots, m \\ x_i^{q-j} & j = m+1, \ldots, q. \end{cases} \tag{6}$$

It is straightforward to observe that (5) is in the form of (1). Finally, since $K_{\text{un}}^{'}$ contains the factor $1/|\mathbf{V}_1(\mu)|$, Lemma 2 of [29], [30] can still be applied to obtain a friendlier expression for the joint PDF of the eigenvalues in the case of coincident eigenvalues (for instance when $\mu_1 = \mu_2 = \cdots = \mu_\ell$).

### *D. The CDF of the Extreme Eigenvalues*

Here, we provide expressions for the CDF's of the extreme eigenvalues of a Wishart matrix. The CDF of the smallest eigenvalue $\lambda_q$ in the case of correlated central Wishart can be derived as follows. We start from (7) shown at the top of the next page. Now, using Corollary 2 of [8] with $\xi(x) = x^{p-q}$, $\phi_i(x_j) = e^{-x_j/\sigma_i}$, $\psi_i(x_j) = x_j^{i-1}$, and $\infty > x_1 > x_2 > \ldots > x_q > u$, we get

$$F_{\lambda_q}(u) \;\; = \;\; 1 - K_{\text{cc}} \left| \tilde{\mathbf{S}}_{\text{cc}}(u) \right| \tag{8}$$

where $K_{\text{cc}}$ is given in Table I, and the $(i,j)^{\underline{\text{th}}}$ element of $\tilde{\mathbf{S}}_{\text{cc}}(u)$ can be derived as

$$\begin{aligned} s_{ij}(u) &= \int_u^\infty x^{p-q+j-1} e^{-x/\sigma_i} dx \\ &= \sigma_i^{p-q+j} \, \Gamma\left(p-q+j, \frac{u}{\sigma_i}\right). \end{aligned} \tag{9}$$

In (9), we have used the following identity

$$\int_u^\infty x^{a-1} e^{-x/b} dx = b^a \, \Gamma\left(a, \frac{u}{b}\right) \tag{10}$$

which is valid for $u > 0$, $\Re\{a\} > 0$, and $\Re\{b\} > 0$, with $\Gamma(k,u) \triangleq \int_u^\infty x^{k-1} e^{-x} dx$ [32, pp. 949, 8.350.2].[4]

[4] $\Re\{x\}$ denotes the real part of $x$.

In the case of uncorrelated noncentral Wishart ($\boldsymbol{\Sigma} = \mathbf{I}$ and $\mathbb{E}\{\mathbf{A}\} \neq \mathbf{0}$), we can follow similar steps as above to obtain the following

$$F_{\lambda_q}(u) = 1 - K_{\text{un}} \left| \tilde{\mathbf{S}}_{\text{un}}(u) \right| \tag{11}$$

where $K_{\text{un}}$ is given in Table I, and the $(i,j)^{\underline{\text{th}}}$ element of $\tilde{\mathbf{S}}_{\text{un}}(u)$ can be derived as

$$\begin{aligned} s_{ij}(u) &= \int_u^\infty x^{p-q+i-1} e^{-x} {}_0\mathcal{F}_1(p-q+1; x\mu_j) dx \\ &= \sum_{l=0}^{\infty} \frac{(p-q+1)_l \, \mu_j^l}{l!} \Gamma\left(p-q+l+i, u\right) \end{aligned} \tag{12}$$

with $(b)_\ell$ defining the Pochhammer symbol [32]. In (12) we have used the identity [32, eq. (9.19), pp. 1084].

In the case of uncorrelated central Wishart, the CDF of the smallest and of the largest eigenvalue of $\mathcal{W}_q(p, \mathbf{I})$ has been derived in [20, eq. (5)] and [20, eq. (6)], respectively. The distribution of the largest eigenvalue in case of correlated central Wishart ($\boldsymbol{\Sigma} \neq \mathbf{I}$ and $\mathbb{E}\{\mathbf{A}\} = \mathbf{0}$) is given in [19, eq. (34)]. In the case of uncorrelated noncentral Wishart ($\boldsymbol{\Sigma} = \mathbf{I}$ and $\mathbb{E}\{\mathbf{A}\} \neq \mathbf{0}$), the expression for the CDF of the largest eigenvalue $\lambda_1$ is given in [18, eqs. (2-4)].

## III. Some Useful Theorems

In this Section we provide two theorems which represent the main contribution of the paper. Theorem 1 is used in Sec. IV.A to obtain the distribution for $k$ extreme eigenvalues. Theorem 2 is used in Sec. IV.B to obtain the distribution for an arbitrary number of consecutive eigenvalues.

*Theorem 1:* Consider two $(N \times N)$ matrices $\boldsymbol{\Phi}(\mathbf{x})$ and $\boldsymbol{\Psi}(\mathbf{x})$ with $(i,j)^{\underline{\text{th}}}$ elements $\phi_i(x_j)$ and $\psi_i(x_j)$, respectively, an arbitrary function $\xi(x)$, and two arbitrary real numbers $a$, $b$, with $a \leq b$. Defining $\varphi(n,m,x) \triangleq \phi_n(x)\,\psi_m(x)\,\xi(x)$, and $M < N$, the three identities (13)-(15), shown at the top of the next page, hold.

Note that the size of the matrices in the right hand side of (13)-(15) is $(N-M) \times (N-M)$. The sum given in (13)-(15) is defined as

$$\overline{\sum}_{\mathbf{n},N,M} \triangleq \sum_{n_1=1}^{N} \sum_{n_2=1, n_2 \neq n_1}^{N} \cdots \sum_{n_M=1, n_M \neq \{n_1, \ldots, n_{M-1}\}}^{N} . \tag{16}$$

The function $\mathrm{s}(\boldsymbol{n}, \boldsymbol{m})$ takes values in the set $\{-1, +1\}$ and can be evaluated using the following formula

$$\mathrm{s}(\boldsymbol{n}, \boldsymbol{m}) = (-1)^{\sum_{l=1}^{M} (i_{n_l} + i_{m_l})} \tag{17}$$

where $i_{n_l}$ and $i_{m_l}$ give the position of the elements $n_l$ and $m_l$ in the ordered sets $\mathcal{A}_{\mathbf{n}}^{(l-1)} = \{1, \cdots, N\} \setminus \{n_1, \cdots, n_{l-1}\}$ and $\mathcal{A}_{\mathbf{m}}^{(l-1)} = \{1, \cdots, N\} \setminus \{m_1, \cdots, m_{l-1}\}$, respectively.

The $(i,j)^{\underline{\text{th}}}$ element of $\boldsymbol{\Xi}(\mathbf{n}, \mathbf{m}, a, x_M)$ is

$$\omega_{ij}(\mathbf{n}, \mathbf{m}, a, x_M) = \int_a^{x_M} \varphi(r_{i,\mathbf{n}}, r_{j,\mathbf{m}}, x) \, dx \tag{18}$$

and $r_{i,\mathbf{n}}$ is the $i^{\underline{\text{th}}}$ element of the ordered set $\mathcal{A}_{\mathbf{n}}^{(M)}$. Note that $r_{i,\mathbf{n}}$ is invariant with respect to a permutation of $\mathbf{n}$; that is, if $\tilde{\mathbf{n}}$ contains the same elements of $\mathbf{n}$ (although in a different order) we have $r_{i,\mathbf{n}} = r_{i,\tilde{\mathbf{n}}}$.

$$F_{\lambda_q}(u) = 1 - \mathbb{P}\{\lambda_q > u\} = 1 - \int_u^\infty \int_{x_q}^\infty \cdots \int_{x_2}^\infty f_{\boldsymbol{\lambda}}(\mathbf{x}) dx_1 \cdots dx_{q-1} dx_q = 1 - K_{\text{cc}} \int_u^\infty \int_{x_q}^\infty \cdots \int_{x_2}^\infty |\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})| \cdot |\mathbf{V}_1(\mathbf{x})| \prod_{j=1}^{q} x_j^{p-q} dx_1 \cdots dx_{q-1} dx_q \quad (7)$$

$$\int_a^{x_M} \cdots \int_a^{x_{N-2}} \int_a^{x_{N-1}} |\mathbf{\Phi}(\mathbf{x})| \cdot |\mathbf{\Psi}(\mathbf{x})| \prod_{l=1}^{N} \xi(x_l)\, dx_N dx_{N-1} \cdots dx_{M+1} = \overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} \mathrm{s}(\boldsymbol{n}, \boldsymbol{m})\, |\mathbf{\Xi}(\mathbf{n}, \mathbf{m}, a, x_M)| \prod_{l=1}^{M} \varphi(n_l, m_l, x_l) \quad \text{for } b \geq x_M \geq x_{M+1} \geq \cdots \geq x_N \geq a \quad (13)$$

$$\int_{x_{N-M+1}}^b \cdots \int_{x_3}^b \int_{x_2}^b |\mathbf{\Phi}(\mathbf{x})| \cdot |\mathbf{\Psi}(\mathbf{x})| \prod_{l=1}^{N} \xi(x_l)\, dx_1 dx_2 \cdots dx_{N-M} = \overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} \mathrm{s}(\boldsymbol{n}, \boldsymbol{m})\, |\mathbf{\Xi}(\mathbf{n}, \mathbf{m}, x_{N-M+1}, b)| \prod_{l=N-M+1}^{N} \varphi(n_l, m_l, x_l) \quad \text{for } b \geq x_1 \geq \cdots \geq x_{N-M} \geq x_{N-M+1} \quad (14)$$

$$\int_a^b \cdots \int_a^b \int_a^b |\mathbf{\Phi}(\mathbf{x})| \cdot |\mathbf{\Psi}(\mathbf{x})| \prod_{l=1}^{N} \xi(x_l) dx_{M+1} dx_{M+2} \cdots dx_N = (N-M)! \overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} \mathrm{s}(\boldsymbol{n}, \boldsymbol{m})\, |\mathbf{\Xi}(\mathbf{n}, \mathbf{m}, a, b)| \prod_{l=1}^{M} \varphi(n_l, m_l, x_l) \quad \text{for } b \geq x_{M+1} \geq a, \cdots, b \geq x_N \geq a \quad (15)$$

*Proof:* The proof of Theorem 1 is given in the Appendix. □

*Corollary 1:* Consider two $(N \times N)$ matrices $\mathbf{\Phi}(\mathbf{x})$ and $\mathbf{\Psi}(\mathbf{x})$ with $(i,j)^{\underline{\text{th}}}$ elements $\phi_i(x_j)$ and $\psi_i(x_j)$, respectively, an arbitrary function $\xi(x)$, and $a$, $b$ two arbitrary real numbers with $a \leq b$. The following three $(N-1)$-fold integrals can be simplified as

$$\int_a^{x_1} \cdots \int_a^{x_{N-2}} \int_a^{x_{N-1}} |\mathbf{\Phi}(\mathbf{x})| \cdot |\mathbf{\Psi}(\mathbf{x})| \prod_{l=1}^{N} \xi(x_l)\, dx_N dx_{N-1} \cdots dx_2 = \sum_{n=1}^{N} \sum_{m=1}^{N} (-1)^{n+m} \varphi(n, m, x_1)\, |\mathbf{\Xi}(n, m, a, x_1)| \quad (19)$$

$$\int_{x_N}^b \cdots \int_{x_3}^b \int_{x_2}^b |\mathbf{\Phi}(\mathbf{x})| \cdot |\mathbf{\Psi}(\mathbf{x})| \prod_{l=1}^{N} \xi(x_l)\, dx_1 dx_2 \cdots dx_{N-1} = \sum_{n=1}^{N} \sum_{m=1}^{N} (-1)^{n+m} \varphi(n, m, x_N)\, |\mathbf{\Xi}(n, m, x_N, b)| \quad (20)$$

and

$$\int_a^b \cdots \int_a^b \int_a^b |\mathbf{\Phi}(\mathbf{x})| \cdot |\mathbf{\Psi}(\mathbf{x})| \prod_{l=1}^{N} \xi(x_l)\, dx_2 dx_3 \cdots dx_N = (N-1)! \sum_{n=1}^{N} \sum_{m=1}^{N} (-1)^{n+m} \varphi(n, m, x_1)\, |\mathbf{\Xi}(n, m, a, b)| \quad (21)$$

where the size of the matrices in (19)-(21) are $(N-1) \times (N-1)$ and

$$r_{n,m} \triangleq \begin{cases} n & \text{if } \ n < m \\ n+1 & \text{if } \ n \geq m. \end{cases} \quad (22)$$

*Proof:* See Theorem 1 with $M = 1$. □

*Lemma 1:* Let $g(x_1, \ldots, x_N)$ be a symmetric function in the variables $x_1, x_2, \ldots, x_L$ and let $\mathcal{D}$ be a domain of integration for $x_{M+1}, x_{M+2}, \ldots, x_N$, with $L < M < N$. Then, the function $h(x_1, x_2, \ldots, x_L, \ldots, x_M)$ shown in (23) at the top of the next page is symmetric in the variables $x_1, x_2, \ldots, x_L$.

$$h(x_1, x_2, \dots, x_L, \dots, x_M) \triangleq \int\int \cdots \int_{\mathcal{D}} g(x_1, x_2, \dots, x_N) dx_{M+1}\, dx_{M+2} \cdots dx_N \tag{23}$$

*Proof:* Since $g(\cdot)$ is a symmetric function, $\forall \ell, k \leq L$, we can write (23) as

$$\begin{aligned} &h(\dots, x_\ell, \dots, x_k, \dots, x_L, \dots, x_M) \\ &= \int \cdots \int_{\mathcal{D}} g(\dots, x_\ell, \dots, x_k, \dots) dx_{M+1} \cdots dx_N \\ &= \int \cdots \int_{\mathcal{D}} g(\dots, x_k, \dots, x_\ell, \dots) dx_{M+1} \cdots dx_N \\ &= h(\dots, x_k, \dots, x_\ell, \dots, x_L, \dots, x_M). \end{aligned} \tag{24}$$

Thus, the function $h(x_1, x_2, \dots, x_L, \dots, x_M)$ in (23) is symmetric in the variables $x_1, x_2, \dots, x_L$. □

*Theorem 2:* Define the multiple integral (25) as shown at the top of the next page, where

$$g^{(\mathbf{n})(\mathbf{m})}(x_1, \dots, x_M) \triangleq \mathrm{s}(\boldsymbol{n}, \boldsymbol{m})\, |\mathbf{D}(x_M)| \prod_{l=1}^{M} \varphi(n_l, m_l, x_l) \tag{26}$$

and the $(i, j)^{\underline{\text{th}}}$ element of $\mathbf{D}(x_M)$ is given by

$$d_{i,j}^{(\mathbf{n})(\mathbf{m})}(x_M) \triangleq \int_a^{x_M} \varphi(r_{i,\mathbf{n}}, r_{j,\mathbf{m}}, x)\, dx \tag{27}$$

with $b \geq x_1 \geq x_2 \geq \cdots \geq x_L \geq \alpha$ and $L < M$. (25) can be simplified as

$$\begin{aligned} \mathcal{J} \;=\; & \frac{1}{L!} \overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} \mathrm{s}(\boldsymbol{n}, \boldsymbol{m})\, |\mathbf{D}(x_M)| \\ & \times \prod_{l=L+1}^{M} \varphi(n_l, m_l, x_l) \prod_{l=1}^{L} \int_\alpha^b \varphi(n_l, m_l, x)\, dx. \end{aligned} \tag{28}$$

*Proof:* Note that the integrand in (25) is the result of (13). Since the integrand in (13) is symmetric in the variables $x_1, x_2, \dots, x_N$, by Lemma 1 the integrand in (25) is also symmetric in $x_1, x_2, \dots, x_L$. Therefore, (25) becomes (29), shown at the top of the next page, which gives (28). □

Theorems 1 and 2 can be applied to matrices in the form of (2). In this case, the proof of Theorem 1 is essentially the same except for the use of Lemma 2 of [10] instead of Corollary 2 of [8].

## IV. Analysis of Some Distributions of Interest

### A. Marginal PDF for the Extreme Eigenvalues

The expressions for the CDF of $\lambda_1$ and $\lambda_q$ seen in Sec. II-D can be used to obtain the corresponding PDF. Recalling that these CDF's are in the form $K|\mathbf{A}(u)|$ and that the derivative of the determinant of a matrix can be written as [28], [33]

$$\frac{d}{du}|\mathbf{A}(u)| = |\mathbf{A}(u)| \cdot \mathrm{tr}\left\{\mathbf{A}^{-1}(u) \frac{d}{du}\mathbf{A}(u)\right\} \tag{30}$$

one can obtain the expressions for the PDF of $\lambda_1$ and $\lambda_q$. This approach has been used for example in [22] to derive the PDF of $\lambda_1$ for uncorrelated noncentral Wishart. Unfortunately, the expression obtained by such approach does not lend itself for further analysis. To alleviate this problem, in the following we propose an alternative approach, leading to friendlier expressions. Specifically, using the theorems in Sec. III, we derive the PDF of the extreme eigenvalues as well as that of an unordered eigenvalue of a Wishart matrix.

Let us start with $f_{\lambda_1}(x_1)$: it can be obtained by integrating the joint PDF of $\boldsymbol{\lambda}$ over $\lambda_2, \dots, \lambda_q$

$$f_{\lambda_1}(x_1) = \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_{q-1}} f_{\boldsymbol{\lambda}}(\mathbf{x})\, dx_q \dots dx_3 dx_2. \tag{31}$$

By substituting (1) in (31) and applying (19) of Corollary 1 with $a = 0$, we get

$$f_{\lambda_1}(x_1) = K \sum_{n=1}^{q} \sum_{m=1}^{q} (-1)^{n+m} \varphi(n, n, x_1)\, |\,\boldsymbol{\Xi}(n, m, 0, x_1))|\,. \tag{32}$$

To derive $f_{\lambda_q}(x_q)$, we recall that

$$f_{\lambda_q}(x_q) = \int_{x_q}^{\infty} \dots \int_{x_3}^{\infty} \int_{x_2}^{\infty} f_{\boldsymbol{\lambda}}(\mathbf{x})\, dx_1 dx_2 \dots dx_{q-1} \tag{33}$$

and applying (20) of Corollary 1 with $b \to \infty$ we get

$$f_{\lambda_q}(x_q) = K \sum_{n=1}^{q} \sum_{m=1}^{q} (-1)^{n+m} \varphi(n, m, x_q)\, |\,\boldsymbol{\Xi}(n, m, x_q, \infty)|\,. \tag{34}$$

Note that in the case of unordered eigenvalues, the PDF of a generic eigenvalue $\lambda$ can be written as

$$f_\lambda(u) = \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{f_{\boldsymbol{\lambda}}(\mathbf{x})}{q!} dx_q \dots dx_3 dx_2 \tag{35}$$

where $f_{\boldsymbol{\lambda}}(\mathbf{x})/q!$ is the joint PDF of the unordered eigenvalues. In (35), we have used the property that the normalizing constant for the case of the ordered eigenvalues is $q!$ times that of the unordered eigenvalues. Applying (21) of Corollary 1 with $a = 0$ and $b \to \infty$, we get

$$\begin{aligned} &f_\lambda(u) \\ &= (q-1)! K \sum_{n=1}^{q} \sum_{m=1}^{q} (-1)^{n+m} \varphi(n, m, u)\, |\,\boldsymbol{\Xi}(n, m, 0, \infty)|\,. \end{aligned} \tag{36}$$

In general, (32), (34), and (36) are valid when the joint PDF of the eigenvalues can be written in the form of (1). These expressions can be specialized for the following cases:

*1) Uncorrelated Central Wishart:* In the case of uncorrelated central Wishart $\phi_i(x_j) = \psi_i(x_j) = x_j^{i-1}$ and $\xi(x) = e^{-x} x^{p-q}$. It is straightforward to show that the product $\phi_{r_{i,n}}(x) \psi_{r_{j,m}}(x)$ is equal to $x^{\alpha_{i,j}^{(n)(m)}}$, where

$$\alpha_{i,j}^{(n)(m)} \triangleq \begin{cases} i + j - 2 & \text{if } i < n \text{ and } j < m \\ i + j & \text{if } i \geq n \text{ and } j \geq m \\ i + j - 1 & \text{otherwise.} \end{cases} \tag{37}$$

Using (37) and the identities [32, eqs. 3.381.1 and 3.381.3], (32), (34), and (36) can be simplified and the PDF's of $\lambda_1$,

$$\mathcal{J} = \int_\alpha^b \cdots \int_{x_3}^b \int_{x_2}^b \overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} g^{(\mathbf{n})(\mathbf{m})}(x_1, \ldots, x_M)\, dx_1 dx_2 \cdots dx_L \tag{25}$$

$$\begin{aligned} &\frac{1}{L!}\int_\alpha^b \cdots \int_\alpha^b \int_\alpha^b \overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} g^{(\mathbf{n})(\mathbf{m})}(x_1, \ldots, x_M)\, dx_1 dx_2 \cdots dx_L \\ &\quad = \frac{1}{L!}\overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} \mathrm{s}(\boldsymbol{n},\boldsymbol{m})\, |\mathbf{D}(x_M)| \prod_{l=L+1}^{M} \varphi(n_l, m_l, x_l) \int_\alpha^b \cdots \int_\alpha^b \int_\alpha^b \prod_{l=1}^{L} \varphi(n_l, m_l, x_l)\, dx_1 dx_2 \cdots dx_L \end{aligned} \tag{29}$$

$\lambda_q$, and $\lambda$ can be written in the following concise way

$$f_{\lambda_{(\cdot)}}(u) = K_{\text{uc}} \sum_{n=1}^{q} \sum_{m=1}^{q} (-1)^{n+m} u^{n+m-2+p-q} e^{-u} \left|\mathbf{\Omega}_{(\cdot)}^{(\text{uc})}\right| \tag{38}$$

where $\lambda_{(\cdot)} \in \{\lambda_1, \lambda_q, \lambda\}$, and the $(i,j)^{\underline{\text{th}}}$ element of $\mathbf{\Omega}_{(\cdot)}^{(\text{uc})}$ is given by

$$\omega_{i,j}^{(\text{uc})} = \begin{cases} \gamma\left(\alpha_{i,j}^{(n)(m)} + p - q + 1, u\right) & \text{for } \lambda_{(\cdot)} = \lambda_1 \\ \Gamma\left(\alpha_{i,j}^{(n)(m)} + p - q + 1, u\right) & \text{for } \lambda_{(\cdot)} = \lambda_q \\ \left(\alpha_{i,j}^{(n)(m)} + p - q\right)!\, \zeta_{q,1} & \text{for } \lambda_{(\cdot)} = \lambda \end{cases} \tag{39}$$

and

$$\zeta_{a,b} \triangleq \prod_{\ell=0}^{b-1} (a-\ell)^{-\frac{1}{a-b}} \tag{40}$$

with $\gamma(k,u) \triangleq \int_0^u x^{k-1} e^{-x} dx$ denoting the incomplete Gamma function [32, pp. 949, 8.350.1].

Note that a first expression for the PDF of an unordered eigenvalues was obtained in [7] in terms of Laguerre polynomials. That expression was then simplified by [34] to avoid the necessity to compute Laguerre polinomials.

*2) Correlated Central Wishart:* In the case of correlated central Wishart, $\phi_i(x_j) = x_j^{i-1}$, $\psi_i(x_j) = e^{-x_j/\sigma_i}$ and $\xi(x) = x^{p-q}$. The PDF's $\lambda_1$, $\lambda_q$, and $\lambda$ can be written as

$$f_{\lambda_{(\cdot)}}(u) = K_{\text{cc}} \sum_{n=1}^{q} \sum_{m=1}^{q} (-1)^{n+m} u^{p-q+n-1} e^{-u/\sigma_m} \left|\mathbf{\Omega}_{(\cdot)}^{(\text{cc})}\right| \tag{41}$$

where the $(i,j)^{\underline{\text{th}}}$ element of $\mathbf{\Omega}_{(\cdot)}^{(\text{cc})}$ is given by

$$\omega_{i,j}^{(\text{cc})} = \begin{cases} \left(\sigma_{r_{j,m}}\right)^{p-q+r_{i,n}} \gamma\left(p - q + r_{i,n}, \frac{u}{\sigma_{r_{j,m}}}\right) & \text{for } \lambda_{(\cdot)} = \lambda_1 \\ \left(\sigma_{r_{j,m}}\right)^{p-q+r_{i,n}} \Gamma\left(p - q + r_{i,n}, \frac{u}{\sigma_{r_{j,m}}}\right) & \text{for } \lambda_{(\cdot)} = \lambda_q \\ \left(\sigma_{r_{j,m}}\right)^{p-q+r_{i,n}} (p - q + r_{i,n} - 1)!\, \zeta_{q,1} & \text{for } \lambda_{(\cdot)} = \lambda. \end{cases} \tag{42}$$

To give an example, when $q = 2$, the PDF of $\lambda_1$ for correlated central Wishart can be written as a sum of four incomplete Gamma functions as shown in (43) at the top of the next page.[5] This can be used to derive a closed-form expression for the moment generating function of $\lambda_1$.

[5]A similar expression for the uncorrelated central Wishart with $q = 2$ can be found in [22, eq. (12)].

To the best of the authors' knowledge the PDF's for the largest and smallest eigenvalues provided here are new. For the unordered case only, an alternative expression for $f_\lambda(u)$ has been obtained, using a different approach, in [35].

*3) Uncorrelated Noncentral Wishart:* In the case of uncorrelated noncentral Wishart, $\phi_i(x_j) = x_j^{i-1}$, $\Psi_i(x_j) = {}_0\mathcal{F}_1(p-q+1; x_j\mu_i)$ and $\xi(x) = x^{p-q}e^{-x}$. The PDF's of $\lambda_1$, $\lambda_q$, and $\lambda$ can be written as (44), shown at the top of the next page, where the $(i,j)$ element of $\mathbf{\Omega}_{(\cdot)}^{(\text{un})}$ is given by

$$\omega_{i,j}^{(\text{un})} = \begin{cases} \mathcal{I}^{(\text{I})}(\mu_{r_{j,m}}, p-q+1, p-q+r_{i,n}, u) & \text{for } \lambda_{(\cdot)} = \lambda_1 \\ \mathcal{I}^{(\text{II})}(\mu_{r_{j,m}}, p-q+1, p-q+r_{i,n}, u) & \text{for } \lambda_{(\cdot)} = \lambda_q \\ \mathcal{I}^{(\text{III})}(\mu_{r_{j,m}}, p-q+1, p-q+r_{i,m})\, \zeta_{q,1} & \text{for } \lambda_{(\cdot)} = \lambda \end{cases} \tag{45}$$

with

$$\mathcal{I}^{(\text{I})}(a,b,c,u) \triangleq \sum_{l=0}^{\infty} \frac{a^l \gamma(c+l,u)}{(b)_l\ l!} \tag{46}$$

$$\mathcal{I}^{(\text{II})}(a,b,c,u) \triangleq \sum_{l=0}^{\infty} \frac{a^l \Gamma(c+l,u)}{(b)_l\ l!} \tag{47}$$

and

$$\mathcal{I}^{(\text{III})}(a,b,c) \triangleq (c-1)!\,{}_1\mathcal{F}_1(c,b,a). \tag{48}$$

In deriving $f_\lambda(u)$ we have used the identity in [32, eq. (7.522.5), pp. 855].[6]

### *B. Joint PDF of an Arbitrary Number of Consecutive Eigenvalues*

To evaluate the joint PDF of $k$ consecutive eigenvalues (i.e., from $\lambda_\ell$ to $\lambda_{\ell+k-1}$), we can use Theorems 1 and 2 with $N = q$, $M = \ell + k - 1$, $L = \ell - 1$, $\alpha = x_\ell$, $a = 0$, $b \to \infty$ to get

$$\begin{aligned} &f_{\lambda_\ell \cdots \lambda_{\ell+k-1}}(x_\ell, \ldots, x_{\ell+k-1}) \\ &\quad = \frac{K}{(\ell-1)!} \overline{\sum}_{\mathbf{n},q,\ell+k-1} \overline{\sum}_{\mathbf{m},q,\ell+k-1} \mathrm{s}(\boldsymbol{n},\boldsymbol{m})\, |\mathbf{D}(x_{\ell+k-1})| \\ &\quad \times \left[\prod_{l=\ell}^{\ell+k-1} \varphi(n_l, m_l, x_l)\right] \prod_{l=1}^{\ell-1} \int_{x_\ell}^{\infty} \varphi(n_l, m_l, x)\, dx \end{aligned} \tag{49}$$

where the elements of $\mathbf{D}(\cdot)$ are defined in (27).

[6]The PDF's of the unordered and the largest eigenvalue for the uncorrelated noncentral Wishart have also been obtained in [36] and [18], respectively.

$$f_{\lambda_1}(u) = K_{\text{cc}} \sum_{n=1}^{2} \sum_{m=1}^{2} (-1)^{n+m} \left(\sigma_{r_{1,m}}\right)^{p-2+r_{1,n}} u^{p-3+n} e^{-u/\sigma_m} \gamma \left(p-2+r_{1,n}, \frac{u}{\sigma_{r_{1,m}}}\right) \tag{43}$$

$$f_{\lambda_{(\cdot)}}(u) = K_{\text{un}} \sum_{n=1}^{q} \sum_{m=1}^{q} (-1)^{n+m} u^{p-q+n-1} e^{-u} {}_0\mathcal{F}_1(p-q+1; u\mu_m) \left|\boldsymbol{\Omega}_{(\cdot)}^{(\text{un})}\right| \tag{44}$$

As a special case, we can derive the PDF of the $\ell^{\text{th}}$ eigenvalue:

$$\begin{aligned} f_{\lambda_\ell}(x_\ell) &= \frac{K}{(\ell-1)!} \overline{\sum}_{\mathbf{n},q,\ell} \overline{\sum}_{\mathbf{m},q,\ell} \mathrm{s}(\boldsymbol{n},\boldsymbol{m}) \left|\mathbf{D}(x_\ell)\right| \\ &\times \varphi(n_\ell, m_\ell, x_\ell) \prod_{l=1}^{\ell-1} \int_{x_\ell}^{\infty} \varphi(n_l, m_l, x)\, dx. \end{aligned} \tag{50}$$

In the case of a Wishart matrix, the distribution of $\lambda_\ell$ can be written as

$$\begin{aligned} f_{\lambda_\ell}(x_\ell) = \frac{K}{(\ell-1)!} \overline{\sum}_{\mathbf{n},q,\ell} \overline{\sum}_{\mathbf{m},q,\ell} \mathrm{s}(\boldsymbol{n},\boldsymbol{m}) \\ \times \det \boldsymbol{\Delta}\, x_\ell^{p-q+n_\ell+\epsilon} e^{-\varrho} \prod_{l=1}^{\ell-1} \eta \end{aligned} \tag{51}$$

where $K$, $\epsilon$, $\varrho$ and $\eta$ are given by

$$K = \begin{cases} K_{\text{uc}} & \text{for UCW} \\ K_{\text{cc}} & \text{for CCW} \\ K_{\text{un}} & \text{for UNW} \end{cases} \tag{52}$$

$$\epsilon = \begin{cases} m_k - 2 & \text{for UCW} \\ -1 & \text{for CCW} \\ -1 & \text{for UNW} \end{cases} \tag{53}$$

$$\varrho = \begin{cases} 1 & \text{for UCW} \\ 1/\sigma_{m_k} & \text{for CCW} \\ 1 & \text{for UNW} \end{cases} \tag{54}$$

$$\eta = \begin{cases} \Gamma(p-q+n_l+m_l-1, x_k) & \text{for UCW} \\ (\sigma_{m_l})^{p-q+n_l}\, \Gamma\left(p-q+n_l, \frac{x_k}{\sigma_{m_l}}\right) & \text{for CCW} \\ \mathcal{I}^{(\text{II})}(\mu_{m_l}, p-q+1, p-q+n_l, x_k) & \text{for UNW} \end{cases} \tag{55}$$

with UCW, CCW and UNW denoting uncorrelated central Wishart, correlated central Wishart and uncorrelated noncentral Wishart, respectively. The $(i,j)^{\text{th}}$ element of the matrix $\boldsymbol{\Delta}$ is given by

$$\delta_{i,j} = \begin{cases} \gamma\left(p-q+r_{i,\mathbf{n}}+r_{j,\mathbf{m}}-1, x_k\right) & \text{for UCW} \\ \left(\sigma_{r_{j,\mathbf{m}}}\right)^{p-q+r_{i,\mathbf{n}}} \gamma\left(p-q+r_{i,\mathbf{n}}, \frac{x_k}{\sigma_{r_{j,\mathbf{m}}}}\right) & \text{for CCW} \\ \mathcal{I}^{(\text{I})}(\mu_{r_{j,\mathbf{m}}}, p-q+1, p-q+r_{i,\mathbf{n}}, x_k) & \text{for UNW.} \end{cases} \tag{56}$$

To give an example, let us consider the uncorrelated noncentral case with $q=3$ and $k=2$. In this case the matrix $\mathbf{D}(x_k)$ in (50) is a scalar and $r_{1,\mathbf{n}} = 6 - n_1 - n_2$. The PDF of $\lambda_2$ becomes (57) as shown at the top of the next page. Expressions for the CDF of the $\ell^{\text{th}}$ eigenvalue are given in [37, eq. (4.31)] for the correlated central and in [37, eq. (4.33)] for the uncorrelated central case. In both cases the expressions are written in recursive form and do not lead to an easy derivation of the corresponding PDF apart from numerical differentiation. Furthermore, the expression [37, eq. (4.31)] contains an infinite series and is written in terms of zonal polynomials. An expression for the CDF of the $\ell^{\text{th}}$ eigenvalue can be found in a recursive form [24, eq. (17)]. Due to the inherent complexity of the recursive expression, only a first order expansion of the marginal PDF was obtained in [24, eq. (22)]. An alternative expression for the joint PDF of a subset of eigenvalues of a Wishart matrix has been recently given in [38].

As a special case of (49), we can also obtain simplified expressions for the joint PDF of $k$ largest or smallest eigenvalues of a Wishart matrix. For brevity this derivation is omitted here.

All the functions included in this section can be easily computed by using standard software packages such as Matlab or Mathematica. For the reader's convenience, the elements of $\boldsymbol{\Omega}_{(\cdot)}$, $\boldsymbol{\Delta}$ and $\epsilon$, $\varrho$ and $\eta$ are reported in Tables II and III for the different cases.

TABLE II
THE ELEMENTS OF THE MATRIX $\boldsymbol{\Omega}_{(\cdot)}$ FOR EACH CASE

| | $\lambda_1$ |
|---|---|
| $\omega_{i,j}^{(\text{uc})}$ | $\gamma\left(\alpha_{i,j}^{(n)(m)}+p-q+1, u\right)$ |
| $\omega_{i,j}^{(\text{cc})}$ | $\left(\sigma_{r_{j,m}}\right)^{p-q+r_{i,n}} \gamma\left(p-q+r_{i,n}, \frac{u}{\sigma_{r_{j,m}}}\right)$ |
| $\omega_{i,j}^{(\text{nc})}$ | $\mathcal{I}^{(\text{I})}(\mu_{r_{j,m}}, p-q+1, p-q+r_{i,n}, u)$ |
| | $\lambda_q$ |
| $\omega_{i,j}^{(\text{uc})}$ | $\Gamma\left(\alpha_{i,j}^{(n)(m)}+p-q+1, u\right)$ |
| $\omega_{i,j}^{(\text{cc})}$ | $\left(\sigma_{r_{j,m}}\right)^{p-q+r_{i,n}} \Gamma\left(p-q+r_{i,n}, \frac{u}{\sigma_{r_{j,m}}}\right)$ |
| $\omega_{i,j}^{(\text{nc})}$ | $\mathcal{I}^{(\text{II})}(\mu_{r_{j,m}}, p-q+1, p-q+r_{i,n}, u)$ |
| | $\lambda$ |
| $\omega_{i,j}^{(\text{uc})}$ | $\left(\alpha_{i,j}^{(n)(m)}+p-q\right)!\, \zeta_{q,1}$ |
| $\omega_{i,j}^{(\text{cc})}$ | $\left(\sigma_{r_{j,m}}\right)^{p-q+r_{i,n}} (p-q+r_{i,n}-1)!\, \zeta_{q,1}$ |
| $\omega_{i,j}^{(\text{un})}$ | $\mathcal{I}^{(\text{III})}(\mu_{r_{j,m}}, p-q+1, p-q+r_{i,m})\, \zeta_{q,1}$ |

## V. NUMERICAL EXAMPLES

In this section we give some numerical examples related to the PDF of the largest, of the smallest, and of a randomly chosen eigenvalue of a Wishart matrix. Fig. 1 shows the PDF of the largest eigenvalue of a correlated central Wishart matrix with $p=q=5$. The $(i,j)^{\text{th}}$ element of the correlation matrix $\boldsymbol{\Sigma}$ is taken here as $\rho^{|i-j|}$ with $\rho \in [0,1)$ (exponential correlation case). The figure, where the correlation coefficient $\rho$ ranges from 0 to 1, clearly shows that the correlation increases the spread of the random variable around the mean

$$f_{\lambda_2}(x_2) = K_{\text{uc}} \overline{\sum}_{\mathbf{n},3,2} \overline{\sum}_{\mathbf{m},3,2} \mathrm{s}(\boldsymbol{n},\boldsymbol{m})\, \gamma\left(p+8-n_1-n_2-m_1-m_2, x_2\right) \times x_2^{p-5+n_2+m_2} e^{-x_2} \Gamma\left(p-4+n_1+m_1, x_2\right) \tag{57}$$

TABLE III
THE ELEMENTS OF THE MATRIX $\mathbf{\Delta}$ AND $\epsilon$, $\varrho$ AND $\eta$ FOR THE EACH CASE

| | Uncorrelated Central | Correlated Central | Uncorrelated Noncentral |
|---|---|---|---|
| $\delta_{i,j}$ | $\gamma\left(p-q+r_{i,\mathbf{n}}+r_{j,\mathbf{m}}-1, x_k\right)$ | $\left(\sigma_{r_{j,\mathbf{m}}}\right)^{p-q+r_{i,\mathbf{n}}} \gamma\left(p-q+r_{i,\mathbf{n}}, \frac{x_k}{\sigma_{r_{j,\mathbf{m}}}}\right)$ | $\mathcal{I}^{(\mathrm{I})}(\mu_{r_{j,\mathbf{m}}}, p-q+1, p-q+r_{i,\mathbf{n}}, x_k)$ |
| $\epsilon$ | $m_k-2$ | $-1$ | $-1$ |
| $o$ | $1$ | $1/\sigma_{m_k}$ | $1$ |
| $\eta$ | $\Gamma(p-q+n_l+m_l-1, x_k)$ | $(\sigma_{m_l})^{p-q+n_l}\, \Gamma\left(p-q+n_l, \frac{x_k}{\sigma_{m_l}}\right)$ | $\mathcal{I}^{(\mathrm{II})}(\mu_{m_l}, p-q+1, p-q+n_l, x_k)$ |

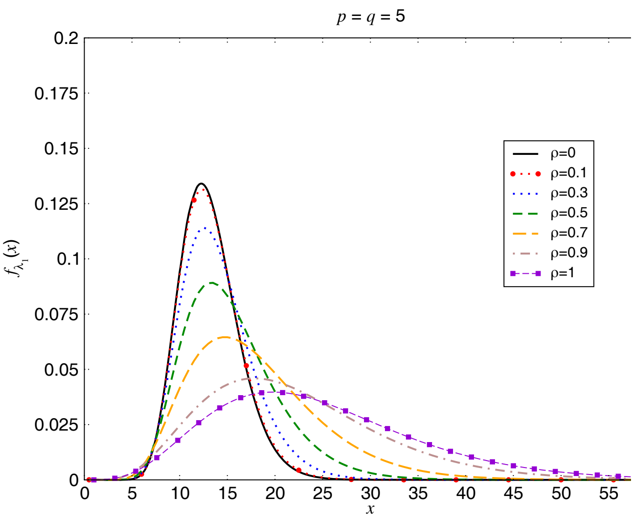


Fig. 1. Probability density function of $\lambda_1$ of the correlated central Wishart matrix, parametrized by $\rho$.

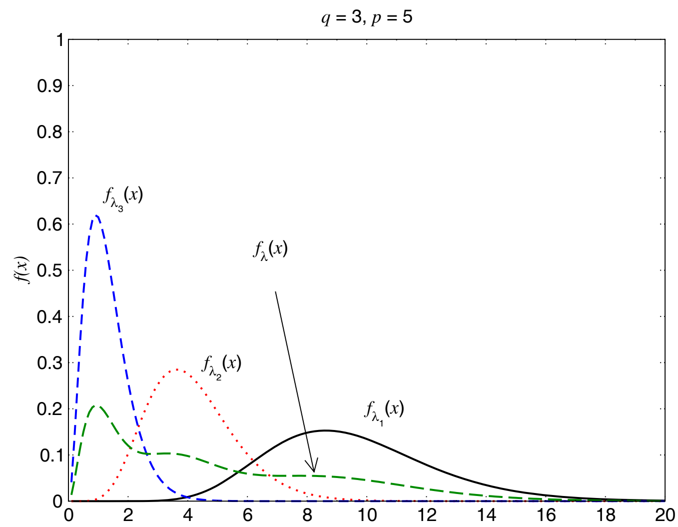


Fig. 2. Probability density function of $\lambda_1, \lambda_2, \lambda_3$ and $\lambda$ (unordered case) of the uncorrelated central Wishart matrix.

value. These results have direct application to the performance analysis of MIMO-MRC systems. In particular, the role played by correlation on the error probability has been investigated in [25].

The next three figures show the PDF's of the eigenvalues for central uncorrelated (Fig. 2), and central correlated Wishart (Figs. 3 and 4). These results have direct application to the analysis of MIMO-SVD systems [7], [24]. Furthermore, the distribution of the generic unordered eigenvalue has been extensively used in the past to analyze the MIMO capacity [7], [35]. Fig. 2 shows the PDF's of the various eigenvalues of an uncorrelated central Wishart matrix $\mathcal{W}_3(5, \mathbf{I})$. We observe that the distribution of $\lambda_3$ is concentrated around its mean ($\mathbb{E}\{\lambda_3\} = 1.32$), whereas $\lambda_1$ is quite spread around its mean ($\mathbb{E}\{\lambda_1\} = 9.52$). The comparison between their variances confirms this behavior: $\mathbb{V}\{\lambda_1\} = 7.57$, $\mathbb{V}\{\lambda_2\} = 2.18$ and $\mathbb{V}\{\lambda_3\} = 0.53$.

Similarly, in Figs. 3 and 4 we consider the correlated central Wishart with $\rho = 0.3$ and $\rho = 0.9$, respectively. Comparing Figs. 2 and 3 we see, as expected, that the PDF of the eigenvalues for correlated central Wishart behaves similarly as that of the uncorrelated case when the correlation is low. On the contrary, a large value of the correlation coefficient $\rho$ strongly reduces both the mean ($\mathbb{E}\{\lambda_q\} = 0.175$) and variance ($\mathbb{V}\{\lambda_q\} = 9.8 \cdot 10^{-3}$) of the smallest eigenvalue.

## VI. CONCLUSIONS

In this paper, we first proposed a general methodology for the evaluation of multiple nested integrals that can be applied to eigenvalues of Wishart matrices. We then derived the cumulative density function of the smallest eigenvalue, a concise representation for the PDF of the extreme eigenvalues, the joint PDF of $k$ unordered eigenvalues, the joint PDF of $k$ consecutive ordered eigenvalues, and the PDF of the $\ell^{\text{th}}$ ordered eigenvalue.

The results, obtained in closed-form for the cases of both uncorrelated and correlated central, as well as uncorrelated noncentral Wishart matrices, can be used to investigate the performance of MIMO systems in the presence of Rayleigh (both correlated and uncorrelated) as well as uncorrelated Rician fading.

For brevity, this paper focused on the analysis of full-rank Wishart matrices. Nonetheless, our results can be applied to

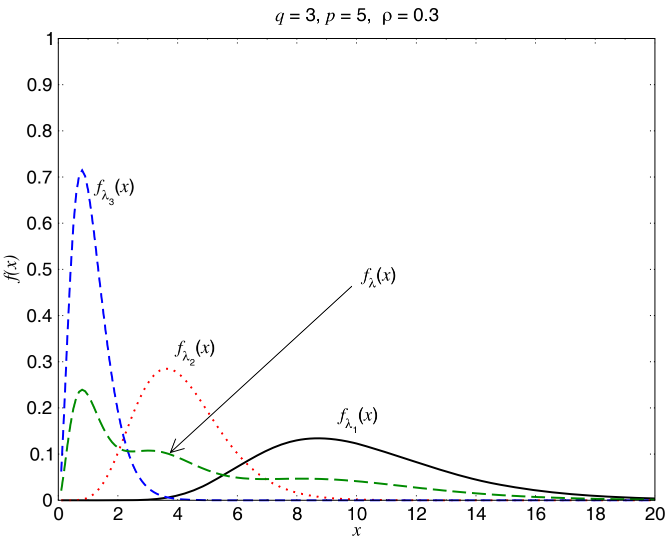


Fig. 3. Probability density function of $\lambda_1, \lambda_2, \lambda_3$ and $\lambda$ (unordered case)

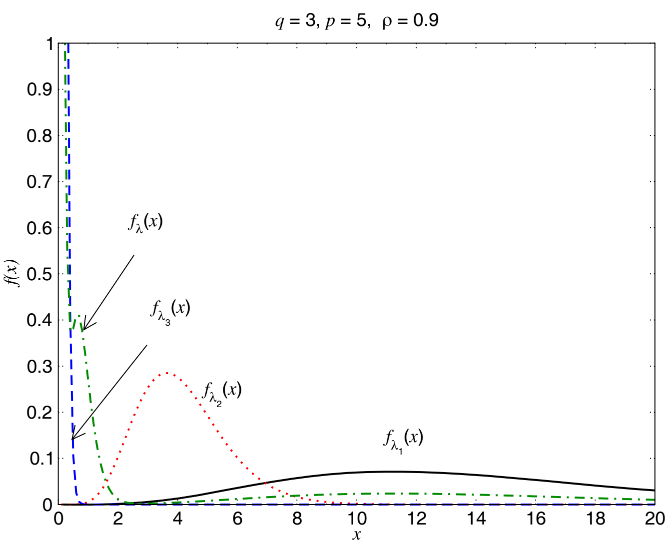


Fig. 4. Probability density function of $\lambda_1, \lambda_2, \lambda_3$ and $\lambda$ (unordered case) of the correlated central Wishart matrix with $\rho = 0.9$.

all cases in which the joint probability density function can be written in the form of (1), in particular this includes non-full rank (also denoted as singular) Wishart matrices.

## Appendix: Proof of Theorem 1

*Proof:* Let $\mathbf{A}(\mathbf{x})$ be a $(N \times N)$ complex matrix with $(i,j)^{\underline{\text{th}}}$ element denoted by $a_i(x_j)$. It is straightforward to see that [28, pp. 7]

$$|\mathbf{A}(\mathbf{x})| = \sum_{n=1}^{N} (-1)^{n+k} a_n(x_k) |\mathbf{A}^{(n)(k)}(\mathbf{x})| \;\; \forall\, k \in \{1, \ldots, N\} \tag{58}$$

where $\mathbf{A}^{(n)(k)}(\mathbf{x})$ is the $(N-1 \times N-1)$ matrix obtained by deleting of the $n^{\underline{\text{th}}}$ row and $k^{\underline{\text{th}}}$ column of the matrix $\mathbf{A}(\mathbf{x})$. The previous equation can be easily generalized as shown in (59) at the top of the next page, where now $\hat{\mathbf{A}}^{(\mathbf{n})(M)}(\mathbf{x})$ represents the matrix we obtain from $\mathbf{A}(\mathbf{x})$ by deleting the first $M$ columns and the rows $n_1, n_2, \ldots, n_M$. The function $\text{sgn}(\mathbf{n}) = \text{sgn}(n_1, \ldots, n_M)$ gives 1 or $-1$ according to the position assumed by the terms $a_{n_l}(x_l)$ in the corresponding submatrix and can be derived as $\text{sgn}(\mathbf{n}) = (-1)^{M + \sum_{l=1}^{M} i_{n_l}}$, where $i_{n_l}$ is defined as (17). The $(i,j)^{\underline{\text{th}}}$ element of the submatrix $\hat{\mathbf{A}}^{(\mathbf{n})(M)}(\mathbf{x})$ can be written as $a_{r_{i,\mathbf{n}}}(x_{j+M})$ where $r_{i,\mathbf{n}}$ has been defined previously in Sec. III. Below we will consider the three cases separately.

- Proof of eq. (13): by using (59) and the definition (16), the left hand side of (13) can be written as shown in (60). Now, using $z_j = x_{j+M}$ with $j = 1, \ldots, N-M$, $b_i^{(\mathbf{n})}(z_j) \triangleq \phi_{r_{i,\mathbf{n}}}(z_j)$, $c_i^{(\mathbf{m})}(z_j) \triangleq \psi_{r_{i,\mathbf{m}}}(z_j)$, the right hand side of (60) becomes (61) where the $(i,j)^{\underline{\text{th}}}$ elements of $\mathbf{B}^{(\text{n})}(\mathbf{z})$ and $\mathbf{C}^{(\text{m})}(\mathbf{z})$ are $b_i^{(\mathbf{n})}(z_j)$ and $c_i^{(\mathbf{m})}(z_j)$, respectively. Applying the results of Corollary 2 of [8] to the $N-M$ multiple nested integrals of (61), we obtain (13).
- Proof of eq. (14): Recalling that

$$|\mathbf{A}(\mathbf{x})| = \sum_{n_1=1}^{N} \sum_{n_2=1, n_2 \neq n_1}^{N} \cdots \sum_{n_M=1, n_M \neq n_1, \ldots n_M \neq n_{M-1}}^{N} \text{sgn}(\mathbf{n}) \times \left( \prod_{l=N-M+1}^{N} a_{n_l}(x_l) \right) |\tilde{\mathbf{A}}^{(\mathbf{n})(M)}(\mathbf{x})| \tag{62}$$

where now $\tilde{\mathbf{A}}^{(\mathbf{n})(M)}(\mathbf{x})$ is the submatrix we obtain from $\mathbf{A}(\mathbf{x})$ by deleting the last $M$ columns and the rows $n_1, n_2, \ldots, n_M$, and $\text{sgn}(\mathbf{n})$ can be evaluated by means of the following relation

$$\begin{aligned} \text{sgn}(\mathbf{n}) &= (-1)^{\sum_{l=1}^{M} i_{n_l} + \sum_{l=1}^{M} (N-l+1)} \\ &= (-1)^{\sum_{l=1}^{M} i_{n_l} + \frac{M(2N-M+1)}{2}}. \end{aligned} \tag{63}$$

The $(i,j)$ element of $\tilde{\mathbf{A}}^{(\mathbf{n})(M)}(\mathbf{x})$ can be written as $a_{r_{i,\mathbf{n}}}(x_j)$. Using (62) and (63), the left part of (14) becomes (64). Again, using Corollary 2 of [8] we thus prove the second part of Theorem 1.
- Proof of eq. (15): The proof is the same as for the proof of (13) but here we use Corollary 1 of [8] which is valid for the domain of integration $\mathcal{D} = \{a \leq z_1 \leq b, \cdots, a \leq z_{N-M} \leq b\}$.

□

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$$|\mathbf{A}(\mathbf{x})| = \sum_{n_1=1}^{N} \sum_{n_2=1, n_2 \neq n_1}^{N} \cdots \sum_{n_M=1, n_M \neq n_1, \ldots n_M \neq n_{M-1}}^{N} \operatorname{sgn}(\mathbf{n}) \left( \prod_{l=1}^{M} a_{n_l}(x_l) \right) |\hat{\mathbf{A}}^{(\mathbf{n})(M)}(\mathbf{x})| \quad (59)$$

$$\overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} (-1)^{M+\sum_{l=1}^{M} i_{n_l}} (-1)^{M+\sum_{l=1}^{M} i_{m_l}} \prod_{l=1}^{M} \varphi(n_l, m_l, x_l)$$
$$\times \int_a^{x_M} \cdots \int_a^{x_{N-2}} \int_a^{x_{N-1}} \left| \hat{\mathbf{\Phi}}^{(\mathbf{n})(M)}(\mathbf{x}) \right| \cdot \left| \hat{\mathbf{\Psi}}^{(\mathbf{m})(M)}(\mathbf{x}) \right| \prod_{l=M+1}^{N} \xi(x_l)\, dx_N dx_{N-1} \cdots dx_{M+1}$$
$$= \overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} \mathrm{s}(\boldsymbol{n}, \boldsymbol{m}) \prod_{l=1}^{M} \varphi(n_l, m_l, x_l)$$
$$\times \int_a^{x_M} \cdots \int_a^{x_{N-2}} \int_a^{x_{N-1}} \left| \hat{\mathbf{\Phi}}^{(\mathbf{n})(M)}(\mathbf{x}) \right| \cdot \left| \hat{\mathbf{\Psi}}^{(\mathbf{m})(M)}(\mathbf{x}) \right| \prod_{l=M+1}^{N} \xi(x_l) dx_N dx_{N-1} \cdots dx_{M+1} \quad (60)$$

$$\overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} \mathrm{s}(\boldsymbol{n}, \boldsymbol{m}) \prod_{l=1}^{M} \varphi(n_l, m_l, x_l)$$
$$\times \int_a^{x_M} \int_a^{z_1} \cdots \int_a^{z_{N-M-1}} |\mathbf{B}^{(\mathrm{n})}(\mathbf{z})| \cdot |\mathbf{C}^{(\mathrm{m})}(\mathbf{z})| \prod_{l=1}^{N-M} \xi(z_l)\, dz_{N-M} \cdots dz_2 dz_1 \quad (61)$$

$$\overline{\sum}_{\mathbf{n},N,M} \overline{\sum}_{\mathbf{m},N,M} \mathrm{s}(\boldsymbol{n}, \boldsymbol{m}) \prod_{l=N-M+1}^{N} \varphi(n_l, m_l, x_l)$$
$$\times \int_{x_{N-M+1}}^{b} \cdots \int_{x_2}^{b} \left| \tilde{\mathbf{\Phi}}^{(\mathbf{n})(M)}(\mathbf{x}) \right| \cdot \left| \tilde{\mathbf{\Psi}}^{(\mathbf{m})(M)}(\mathbf{x}) \right| \prod_{l=1}^{M} \xi(x_l) dx_1 \cdots dx_{N-M} \quad (64)$$


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