March 18, 2008

18.702 Problem Set 6

due monday, March 31, 2008

1. Use the method of Chapter 11, 7.9 to describe the ideals in the ring R of integers in the imaginary quadratic number field $\mathbb{Q}[\delta]$, where $\delta = \sqrt{-6}$. Draw the shapes of the ideals, and don't forget to show that the shapes you draw are represented by ideals of the ring.

2. Let $R = \mathbb{Z}[\delta]$, where $\delta = \sqrt{-3}$. (R is not the ring of integers in $\mathbb{Q}[\delta]$.) Let A be the ideal generated by the elements $2, 1+\delta$.
(a) Prove that A is a maximal ideal, and identify R/A.
(b) Prove that $\overline{A}A$ is not a principal ideal, hence the Main Lemma is not true for R.
(c) Prove that A contains the principal ideal $(2) = 2R$, but that A does not divide (2).

3. Let d be a square-free negative integer not congruent 1 (modulo 4), and let p be a prime integer. If the polynomial $x^2 - d$ factors (modulo p), then the prime p splits or ramifies in R. Suppose that this is the case, and let a be an integer whose residue $\overline{a}$ is a root of $x^2 - d$ in $\mathbb{F}_p$. Prove that $(p, a+\delta)$ is a lattice basis for a prime ideal P that divides (p).

4. Using the result of the previous problem and the method of Chapter 11, 10.25, determine the ideal class group of the ring of integers $\mathbb{Q}[\delta]$, when $\delta = \sqrt{-17}$ and when $\delta = \sqrt{-30}$.

5. Let $\delta = \sqrt{-163}$. Prove that the ring of integers in $\mathbb{Q}[\delta]$ is a unique factorization domain.