18.702 Algebra II
Spring 2008

FUNCTION FIELDS

We denote the complex t-plane by T, and the field $\mathbb{C}(t)$ of rational functions in t by F. The elements of F are fractions $p(t)/q(t)$ of complex polynomials p and q with $q \neq 0$. Function fields are finite field extensions of F.

Let K be a function field of degree n over the field $F = \mathbb{C}(t)$ and let α be a primitive element for the extension. Let f be the irreducible polynomial for α over F, so that $K = F(\alpha)$ is isomorphic to the field $F[x]/(f)$. By clearing denominators in the coefficients, we make f into an irreducible polynomial in $\mathbb{C}[t, x]$, and we write it as

$$f(t, x) = a_n(t)x^n + a_{n-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t), \tag{2}$$

where $a_i(t)$ are polynomials in t with greatest common divisor 1. It is important to remember that the polynomial f depends on the choice of the primitive element α.

The *Riemann surface* X of a polynomial $f(t, x)$ is the locus of zeros of f in complex (t, x)-space $\mathbb{C}^2$. The Riemann surface of a polynomial $f(t, x)$ of degree n in x is an n-sheeted branched covering of T. Its branch points are the points $t = t_0$ above which there are fewer than n points of X. These are the points at which f and $\frac{\partial f}{\partial x}$ have a common solution.

Essentially everything we say about Riemann surfaces is valid only outside of a finite set of points. So we use this notation: If X is an infinite set, then X' denotes the set obtained by deleting an unspecified finite subset Δ from X:

$$X' = X - (\text{variable finite set}). \tag{1}$$

We will carry along the simple example

$$f(t, x) = x^2 - t. \tag{3}$$

The Riemann surface of this polynomial is depicted on page 518.

A *isomorphism* of extension fields K and L is an isomorphism of fields $\varphi : K \to L$ that restricts to the identity on F.

An *isomorphism* $X \to Y$ of branched coverings of T is a continuous, bijective map $\theta : X' \to Y'$ that is compatible with the projections of these surfaces to T. The primes indicate that we expect to delete finite sets of points from X and Y in order that the map θ be defined.

Speaking loosely, we call a branched covering $\pi : S \to T$ is *path connected* if S' is path connected, meaning that for every sufficiently large finite subset Δ of S, the set $S - \Delta$ is path connected.

The next theorem describes the extension fields of F. Unfortunately the proof is too long to present here.

Theorem 4. *[Riemann Existence Theorem]*
(a) *Let $f(t, x)$ be an irreducible polynomial. The Riemann surface of f is a path connected branched covering of T.*
(b) *Let S be a path connected branched covering of T. There is an irreducible polynomial $f(t, x)$ whose Riemann surface is isomorphic to S.*
(c) *Let $f(t, x)$, $g(t, x)$ be irreducible polynomials, let the field extensions they define be K, L and let their Riemann surfaces be X, Y. Then K and L are isomorphic field extensions of F if and only if X and Y are isomorphic branched coverings of T.*

The theorem provides a method of deciding whether or not two polynomials define isomorphic field extensions. A simple criterion that can often be used is that their branch points must coincide. However, if we are given a branched covering S of T, it may be difficult to find a polynomial with Riemann surface S. Many polynomials define isomorphic field extensions, and contrary to what one might expect, finding something often becomes difficult when there are many choices.

There is one part of **(c)** that is rather easy to verify:

Proposition 5. *Let $f(t,x)$ and $g(t,y)$ be irreducible polynomials, and let X and Y be the Riemann surfaces that correspond to the field extensions $K = F[x]/(f)$ and $L = F[y]/(g)$. An isomorphism $\varphi : K \to L$ of field extensions induces an isomorphism $\theta : X \to Y$ of Riemann surfaces.*

Proof. We drop the symbol t temporarily, and write $f(x)$ for $f(t,x)$, etc. The isomorphism φ gives us a root β of $g(y)$ in K, which can be represented as the residue modulo (f) of a polynomial $b(x)$. The map $\theta : X \to Y$ is defined by $\theta(t,x) = (t, b(x))$.

Since $g(\beta) = 0$, the polynomial $g(b(x))$ is in the ideal (f) of $F[x]$. So there is a polynomial $h(x)$ such that $g(b(x)) = h(x)f(x)$. Putting the variable t back into the notation,

$$g(t, b(t,x)) = h(t,x)f(t,x). \tag{6}$$

If (t,x) is a point of X, then $f(t,x) = 0$, so $g(t,b(t,x)) = h(t,x)f(t,x) = 0$, and therefore $\theta(t,x)$ is a point of Y. The inverse function is obtained by interchanging the roles of K and L.

Because $b(x)$ and $h(x)$ are polynomials with coefficients in F, there may be denominators that are polynomials in t, so θ may be undefined at some points.

Cut and Paste

"Cut and paste" is a procedure to construct or deconstruct a branched covering.

Let's go back to our example of the Riemann surface X of the polynomial $x^2 - t$. If we cut X open along the double locus of of the figure on page 518, the negative real axis, it decomposes into the two parts re $x > 0$ and re $x < 0$. Each of these parts projects to the t-plane T bijectively, provided that we disregard what happens along the cut.

Turning this procedure around, we can construct a branched covering isomorphic to X in the following way: We stack two copies S_1, S_2 of the complex plane over T and cut them open along the negative real axis $(-\infty, 0]$. These copies of T will be called *sheets*. Then we glue "side A" of S_1, to "side B" of S_2 and vice versa (see the figure on page 520). We need four dimensions to do this gluing without allowing the surface to cross itself.

Suppose we are given an n-sheeted branched covering $S \to T$, and let $p_1, ..., p_k$ be the set of its branch points in T. For $\nu = 1, ..., k$, we choose non-intersecting half lines C_ν that lead from p_ν to infinity. We cut T open along the half lines C_ν and we also cut S open at all points that lie over these half lines (see page 521).

We should be specific about what we mean by cutting. Let's agree that cutting T open means removing all points of the half lines C_ν, including p_ν, and cutting S open means removing all points that lie over those half lines.

Lemma 7. *When S is cut open above the half lines C_ν, it decomposes as a union of n "sheets" $S_1, ..., S_n$, numbered arbitrarily, each of which projects bijectively to the cut plane T.*

This is true because first, the cut surface S is an unbranched covering space of the cut plane T, and second, the complement of the half lines in T is a simply connected set. It is intuitively plausible that an unbranched covering of s simply connected set decomposes. The sheet that contains a point p consists of all points that can be joined to p by a path without crossing the cuts. The proof is given as an exercise in [Munkres, *Topology* p. 342, exc. 8].

Now to reconstruct the surface S we take n copies of the cut plane T, we call them "sheets" and label them as $S_1, ..., S_n$. We stack them up over T. Except for the cuts, the union of these sheets is our branched covering, and to reconstruct S we must describe the rule for gluing the sheets back together along the cuts.

On T, we circle a point p_ν in a counterclockwise direction (see page 521), and we call the side of C_ν we pass through before crossing C_ν as "side A" and the side we pass through after crossing as "side B". We label the corresponding sides of the sheet S_i as A_i and B_i respectively. Then the rule for gluing S amounts to instructions that A_i is glued to B_j for some j. This rule is described by the permutation σ_ν of the indices $1, ..., n$ that sends $i \mapsto j$.

It is fairly clear that we can construct a covering using an arbitrary set of permutations σ_ν, except that what should happen above the branch points themselves will not be clear. To remove ambiguity, we simply omit all branch points and all points that lie over them.

Branching Data. For $\nu = 1, ..., k$, a permutation σ_ν of the indices $1, ..., n$.
Gluing Instructions. If $\sigma_\nu(i) = j$, glue side A_i to side B_j along the cut C_ν,

When the gluing is done no cuts remain, and the union of the sheets is our covering. As is true of the Riemann surface depicted on page 518, four dimensions are needed to embed the resulting surface without self crossings.

We remark that if σ_ν is the trivial permutation, then each sheet is glued to itself above C_ν. So the cut C_ν is not needed. In this case we say that p_ν is not a "true" branch point.

The next corollary sums up this discussion:

Corollary 8. *Every n-sheeted branched covering $S \to T$ is isomorphic to one constructed by the cut and paste process.* □

It is important to note that the numbering of the sheets is arbitrary, and that the concept of a "top sheet" has no intrinsic meaning for the Riemann surface. If there were a top sheet, we could define x as a single valued function by choosing the value on that sheet. One can do this only after the Riemann surface has been cut open. This is the whole point: Wandering around on the surface leads us from one sheet to another.

Except for the arbitrary numbering of the sheets, the permutations σ_ν are uniquely determined by the covering S. A change of numbering by a permutation ρ will change each σ_ν to the conjugate $\rho\sigma_\nu\rho^{-1}$.

Corollary 9. *Let X and Y be branched coverings constructed by cut and paste, using the same points p_ν and half lines C_ν. Let the permutations defining their gluing data be σ_ν and τ_ν respectively. Then X and Y are isomorphic branched coverings if and only if there is a permutation ρ such that $\tau_\nu = \rho\sigma_\nu\rho^{-1}$ for each ν.* □

Proposition 10. *The branched covering S constructed by cut and paste is path connected if and only if the permutations $\sigma_1, ..., \sigma_k$ generate a subgroup H of the symmetric group that operates transitively on the set of sheets.*

Proof. Each sheet S_i is path connected. If one of the permutations σ_ν sends the index i to j, the sheets S_i and S_j are glued together along the cut C_ν. In that case, there will be a short path across the cut that leads from a point of S_i to a point of S_j. Then because the sheets themselves are path connected, any points of S_i and S_j can be connected by a path. So S is path connected if and only if, for every pair of indices i, j, there is a sequence of the permutations σ_ν that carries $i = i_0 \mapsto i_1 \mapsto \cdots \mapsto i_r = j$. This will be true if and only if the group H operates transitively. □

Computing the Permutations

Determining the permutations σ_ν that define the gluing data of a Riemann surface presents two problems. First the "local problem". At each branch point p we must determine the permutation σ of the sheets that occurs when one circles that point. As we have seen, σ depends on the numbering

of the sheets. Second, we must make sure that we number the sheets in the same way for each branch point.

The local problem can be solved, but in complicated cases controling the numbering is hard to do by hand, though a computer has no problem. Here is what the computer does. It chooses a "base point" b in the cut plane T and computes the n distinct roots of the polynomial $f(b,x)$ numerically, with a suitable accuracy. It numbers these roots arbitrarily, say $\gamma_1, ..., \gamma_n$, and labels the sheets by calling S_i the sheet that contains the root γ_i. Then it walks to a point b_ν in the vicinity of a branch point p_ν, taking care not to cross any of the cuts. The roots γ_i vary continuously, and the computer can follow this variation by recomputing roots every time it takes a small step. This tells it how to label the sheets at the point b_ν. Then to determine the permutation σ_ν, the computer follows a counterclockwise loop around p_ν, again recomputing roots as it goes along. Because the loop crosses the cut C_ν, the roots will have been permuted by σ_ν when the path returns to b_ν. Needless to say, doing this by hand is incredibly tedious. We will find ways to get around it in the examples we present later on.

We give an incomplete analysis of the local problem. Our method is to determine the permutation by relating the Riemann surface to one which we know, namely the Riemann surface Y of the polynomial $y^k - t$. This is a k-sheeted covering of T branched only at the origin, and with suitable numbering of the sheets, the permutation will be cyclic. To see this, we parametrize a circular path around the origin in T, as $t = re^{i\theta}$. Since $y^k = t$, we can parametrize the locus of Y that lies over this path as $y = \rho e^{i\eta}$, where $\rho^k = r$ and $k\eta = \theta$. Then if we start at $t = r$, $y = \rho$, the circular path in T loops k times around the origin as η runs from 0 to 2π. This shows that the permutation of the sheets is cyclic.

It is a fact that a Riemann surface can always be parametrized locally as a power series in a fractional power of t, a *Puiseux series*. These series can get fairly complicated, so we content ourselves with a special case that can often be used.

Let X be the Riemann surface $f(t,x) = 0$, and let t_0 be a branch point of X. Substituting $t = t_0$, we obtain a polynomial in one variable $f^0(x) = f(t_0, x)$.

Proposition 11. *With the above notation, let x_0 be a root of $f^0(x)$. Suppose that*

- *x_0 is a k-fold root of $f^0(x)$, and*
- *the partial derivative $\frac{\partial f}{\partial t}$ is not zero at the point (t_0, x_0).*

Then the permutation of the sheets at the point t_0 contains a k-cycle.

Proof. We change variables to move the point (t_0, x_0) to the origin, and we write

$$f(t,x) = \sum a_{ij} t^i x^j. \tag{12}$$

The first bullet tells us that the coefficients a_{0j} vanish for $j < k$ and that $a_{0k} \neq 0$. The second bullet tells us that $a_{10} \neq 0$. So f has the form

$$f(t,x) = x^k u(x) - tv(t,x), \tag{13}$$

where $u(0) \neq 0$ and $v(0,0) \neq 0$.

For any complex number s, $(1+z)^s$ has a binomial series expansion

$$(1+z)^s = 1 + \binom{s}{1} z + \binom{s}{2} z^2 + \cdots, \tag{14}$$

where

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k(k-1)\cdots 1}, \tag{15}$$

which converges for $|x| < 1$.

Let $\varphi(t,x) = \frac{u}{v}$. Then $\varphi(0,0) \neq 0$, so we can write $\varphi = c(1+z)$ for suitable scalar c and function $z = z(t,x)$. Then with $s = \frac{1}{k}$ and $\psi(t,x) = \varphi(t,x)^s$, the equation (9) can be written as

$$x^k \psi^k = t. \tag{16}$$

There is an ambiguity of a root of unity, but choosing roots suitably and with $y^k = t$, we will have $y = x\psi(t,x)$. This equation defines a local isomorphism from the Riemann surface Y to X. □

Example 17. $f(t,x) = x^2 - t^3 + t$ and $\frac{\partial f}{\partial x} = 2x$. So X is a two-sheeted covering of T. There are three branch points $t = 0,\ t = 1,\ t = -1$. We have $\frac{\partial f}{\partial t} = -3t^2 + 1$, and Proposition 13 applies at all of the branch points. So the permutation of the sheets at each of these points is the transposition $(\mathbf{1\,2})$. We don't have to be careful about the numbering of the two sheets in this case, because interchanging them won't change the transposition.

Example 18. $f(t,x) = x^3 - 3x + t$, and $\frac{\partial f}{\partial x} = 3x^2 - 3$. Here X is a three-sheeted covering. The branch points are $p_1 : t = 2$ and $p_2 : t = -2$. Since $\frac{\partial f}{\partial t} = 1$, Proposition 13 applies again.

To determine the permutation σ_1 at the point p_1, we set $t = 2$ in the polynomial f, obtaining $x^3 - 3x + 2 = (x+1)^2(x-2)$. There is a double root at the point $(2,-1)$. Since $\frac{\partial f}{\partial t}$ is nowhere zero, the permutation σ_1 contains 2-cycle, and since it is a permutation of three indices, it is a transposition. Similarly, σ_2 is a transposition.

We can number the sheets so that $\sigma_1 = (\mathbf{1\,2})$. Because X is path connected, σ_1 and σ_2 generate a transitive group of permutations. So σ_2 must be either $(\mathbf{2\,3})$ or $(\mathbf{1\,3})$. Switching the sheets called S_1 and S_2 doesn't affect σ_1, but it interchanges the two other transpositions. So with sheets numbered suitably, we will have $\sigma_1 = (\mathbf{1\,2})$ and $\sigma_2 = (\mathbf{2\,3})$.

Example 19 $f(t,x) = x^3 - t(t-1)^2$, $\frac{\partial f}{\partial x} = 3x^2$. Again, X is a three-sheeted covering. The branch points are at $t = 0$ and 1. Both $f(0,x)$ and $f(1,x)$ have triple roots. The partial derivative $\frac{\partial f}{\partial t} = 3t^2 - 4t + 1$ is not zero at $t = 0$, so the three sheets are permuted cyclically at that point. With suitable numbering, the permutation at the point $t = 0$, we'll call it σ_0, will be $(\mathbf{1\,2\,3})$.

But the point $t = 1$ presents problems. First, $\frac{\partial f}{\partial t}$ vanishes at $t = 1$. Second how can we control the numbering of the sheets at the second point? In the previous example, knowing that the Riemann surface is path connected was enough to determine the permutations. This fact gives us no information here because σ_0 operates transitively on the sheets by itself.

We use a trick that works only in the simplest cases. The trick is to compute the permutation that we get by walking around a very large circle Γ. Looking back at the figure on page 521, we see that a large circular path will cross each of the cuts once. So provided that we start to the right of the cut C_1, the sheets will be permuted by the product permutation $\sigma_r \cdots \sigma_2\sigma_1$, which, in our case, is $\sigma_1\sigma_0$. If we can determine that permutation, then since we know σ_0, we will be able to recover σ_1.

We make the change of variable $u = t^{-1}$. The *point at infinity* of the plane T is the point $u = 0$. The path $t = re^{i\theta}$, with r large and $0 \le \theta \le 2\pi$ is the same as a small circlular path $u = r^{-1}e^{-i\theta}$ about ∞, except that the orientation is reversed. The path circles ∞ in the clockwise direction. The effect of this change of direction is to replace the product permutation $\sigma_1\sigma_0$ by its inverse.

We substitute $t = u^{-1}$ into $f(t,x)$:

$$f(u^{-1},x) = x^3 - u^{-1}(u^{-1}-1)^2. \tag{20}$$

Clearing the denominator, we obtain $(ux)^3 = (1-u)^2$. We substitute $x = u^{-1}y$:

$$f(t^{-1},t^{-1}x) = h(u,y) = y^3 - (1-u)^2. \tag{21}$$

For u different from 0 and ∞, the substitution $(t, x) = (u^{-1}, u^{-1}y)$ is invertible. So the Riemann surfaces defined by $f = 0$ and $h = 0$ are isomorphic.

Looking above the point $u = 0$, we see that $h(0, y)$ has three roots $1, \omega, \omega^2$. Therefore ∞ is not a branch point. This shows that $(\sigma_1\sigma_0)^{-1} = 1$. Since $\sigma_0 = (\mathbf{1\,2\,3})$, $\sigma_1 = (\mathbf{3\,2\,1})$.