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We study the decoherence of a renormalized quantum field theoretical system. We consider our novel correlator approach to decoherence where entropy is generated by neglecting observationally inaccessible correlators. Using out-of-equilibrium field theory techniques at finite temperatures, we show that the Gaussian von Neumann entropy for a pure quantum state asymptotes to the interacting thermal entropy. The decoherence rate can be well described by the single particle decay rate in our model. Connecting to electroweak baryogenesis scenarios, we moreover study the effects on the entropy of a changing mass of the system field. Finally, we compare our correlator approach to existing approaches to decoherence in the simple quantum mechanical analogue of our field theoretical model. The entropy following from the perturbative master equation suffers from physically unacceptable secular growth.

I. INTRODUCTION

Recently, we have advocated a new decoherence program [1–4] particularly designed for applications in quantum field theory. Similar ideas have been proposed by Giraud and Serreau [5] independently. Older work can already be interpreted in a similar spirit [6–10]. As in the conventional approach to decoherence we assume the existence of a distinct system, environment, and observer (see e.g. [11–15]). Rather than tracing over the unaccessible environmental degrees of freedom of the density matrix to obtain the reduced density matrix $\hat{\rho}_{\rm red} = {\rm Tr}_E[\hat{\rho}]$, we use the well-known idea that loss of information about a system leads to an entropy increase as perceived by the observer. If an observer performs a measurement on a quantum system, the observer measures n-point correlators or correlation functions. Note that these n-point correlators can also be mixed and contain information about the correlation between the system and environment. A "perfect observer" would in principle be able to detect the infinite hierarchy of correlation functions up to arbitrary order. In reality, our observer is of course limited by the sensitivity of its measurement device. Also, higher order correlation functions become more and more difficult to measure due to their nonlocal character. Therefore,

neglecting the information stored in these unaccessible correlators will give rise to an increase in entropy.

In other words, our system and environment evolve unitarily, however to our observer it seems that the system evolves into a mixed state with positive entropy as information about the system is dispersed in inaccessible correlation functions. The total von Neumann entropy $S_{\rm vN}$ can be subdivided as

$$S_{\rm vN} = S^g(t) + S^{ng}(t) = S_S^g + S_E^g + S^{ng}. \qquad (1)$$

In unitary theories $S_{\rm vN}$ is conserved. In the equation above S^g is the total Gaussian von Neumann entropy, that contains information about both the system S_S^g, environment S_E^g, and their correlations at the Gaussian level S_{SE}^g (which vanish in this paper), and S^{ng} is the total non-Gaussian von Neumann entropy which consists again of contributions from the system, environment, and their correlations. Although $S_{\rm vN}$ is conserved in unitary theories, $S_S^g(t)$ can increase at the expense of other decreasing contributions to the total von Neumann entropy, such as $S^{ng}(t)$.

In the conventional approach one attempts to solve for the reduced density matrix by making use of a nonunitary perturbative "master equation" [16]. It suffers from several drawbacks. In the conventional approach to decoherence it is extremely challenging to solve for the dynamics of the reduced density matrix in a realistic interacting, out-of-equilibrium, finite-temperature quantum field theoretical setting that moreover captures perturbative corrections

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arising from renormalization. In fact, we are not aware of any solution to the perturbative master equation that meets these basic requirements.[1] Second, it is important to note that our approach does not rely on nonunitary physics. Although the von Neumann equation for the full density matrix is of course unitary, the perturbative master equation is not. From a theoretical point of view it is disturbing that the reduced density matrix should follow from a nonunitary equation despite the fact that the underlying theory is unitary and hence the implications should be carefully checked.

A. Outline

In this work, we study entropy generation in an interacting, out-of-equilibrium, finite-temperature field theory. We consider the following action [1]:

$$
\begin{aligned}
S[\phi, \chi] &= \int d^D x \mathcal{L}[\phi, \chi] \\
&= \int d^D x \mathcal{L}_0[\phi] + \mathcal{L}_0[\chi] + \mathcal{L}_{\text{int}}[\phi, \chi],
\end{aligned} \tag{2}
$$

where

$$
\mathcal{L}_0[\phi] = -\frac{1}{2}\partial_\mu \phi(x)\partial_\nu \phi(x)\eta^{\mu\nu} - \frac{1}{2}m_\phi^2(t)\phi^2(x) \tag{3a}
$$

$$
\mathcal{L}_0[\chi] = -\frac{1}{2}\partial_\mu \chi(x)\partial_\nu \chi(x)\eta^{\mu\nu} - \frac{1}{2}m_\chi^2\chi^2(x) \tag{3b}
$$

$$
\mathcal{L}_{\text{int}}[\phi, \chi] = -\frac{\lambda}{3!}\chi^3(x) - \frac{1}{2}h\chi^2(x)\phi(x), \tag{3c}
$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots)$ is the D-dimensional Minkowski metric. Here, $\phi(x)$ plays the role of the system, interacting with an environment $\chi(x)$, where we assume that $\lambda \gg h$ such that the environment is in thermal equilibrium at temperature T. In [1] we studied an environment at temperature $T = 0$, i.e., an environment in its vacuum state. In the present work, we study finite-temperature effects. We assume that $\langle \hat{\phi} \rangle = 0 = \langle \hat{\chi} \rangle$, which can be realized by suitably renormalizing the tadpoles.

Let us at this point explicitly state the two main assumptions of our work. First, we assume that the observer can only detect Gaussian correlators or two-point functions and consequently neglects the information stored in all higher order non-Gaussian correlators (of both ϕ and of the correlation between ϕ and χ). This assumption can of course be generalized to incorporate knowledge of e.g. three- or four-point functions in the definition of the

entropy [2]. Second, we neglect the backreaction from the system field on the environment field, i.e., we assume that we can neglect the self-mass corrections due to the ϕ-field on the environment χ. This assumption is perturbatively well justified [1] and thus implies that the environment remains in thermal equilibrium at temperature T. For an extensive discussion we refer to [1], but let us here just mention that this assumption is justified as we assume a hierarchy of couplings $\lambda \gg h$. This allows us to treat χ as a thermal bath, while we can take the dynamics of ϕ fully into account. The leading order self-mass correction ϕ receives is just the usual one-particle irreducible (1PI) self-mass correction, contributing at order $\mathcal{O}(h^2/\omega_\phi^2)$. The first diagram where the backreaction from ϕ on χ contributes to the ϕ propagator occurs at order $\mathcal{O}(h^4/\omega_\phi^4)$ and therefore is subleading.

Consequently, the counterterms introduced to renormalize the tadpoles do not depend on time to such an extent that we can remove these terms in a consistent manner. In fact, the presence of the $\lambda\chi^3$ interaction will introduce perturbative thermal corrections to the tree-level thermal state, which we neglect for simplicity in this work.

The calculation we are about to embark on can be outlined as follows. The first assumption above implies that we only use the three Gaussian correlators to calculate the (Gaussian) von Neumann entropy: $\langle \hat{\phi}(\vec{x}, t)\hat{\phi}(\vec{y}, t) \rangle$, $\langle \hat{\pi}(\vec{x}, t)\hat{\pi}(\vec{y}, t) \rangle$, and $\frac{1}{2}\langle\{\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)\}\rangle$. Rather than attempting to solve for the dynamics of these three correlators separately, we solve for the statistical propagator from which these three Gaussian correlators can be straightforwardly extracted. Starting from the action in Eq. (2), we thus calculate the 2PI effective action that captures the perturbative loop corrections to the various propagators of our system field. Most of our attention is thus devoted to calculating the self-masses, renormalizing the vacuum contributions to the self-masses, and dealing with the memory integrals as a result of the interaction between the two fields. Once we have the statistical propagator at our disposal, our work becomes easier. Various coincidence expressions of the statistical propagator and derivatives thereof fix the Gaussian entropy of our system field uniquely [1,2,10].

In Sec. II we recall how to evaluate the Gaussian von Neumann entropy by making use of the statistical propagator. We moreover present the main results from [1]. In Sec. III we evaluate the finite-temperature contributions to the self-masses. In Sec. IV we study the time evolution of the Gaussian von Neumann entropy in a quantum mechanical model analogous to Eq. (2). This allows us to quantitatively compare our results for the entropy evolution to existing approaches. In Sec. V we study the time evolution of the Gaussian von Neumann entropy in the field theoretic case and we discuss our main results.

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[1] For example in [17] the decoherence of inflationary primordial fluctuations is studied using the master equation however renormalization is not addressed. In [18–20] however, perturbative corrections to a density matrix are calculated in various quantum mechanical cases.

B. Applications

The work presented in this paper is important for electroweak baryogenesis scenarios. The attentive reader will have appreciated that we allow for a changing mass of the system field in the Lagrangian (3a), $m_\phi = m_\phi(t)$. Theories invoking new physics at the electroweak scale that try to explain the observed baryon-antibaryon asymmetry in the Universe are usually collectively referred to as electroweak baryogenesis. During a first-order phase transition at the electroweak scale, bubbles of the true vacuum emerge and expand in the sea of the false vacuum. Particles thus experience a rapid change in their mass as a bubble's wall passes by. Sakharov's conditions are fulfilled during this violent process such that a baryon asymmetry is supposed to be generated. The problem is to calculate axial vector currents generated by a CP-violating advancing phase interface. These currents then feed in hot sphalerons, thus biasing baryon production. The axial currents are difficult to calculate because this requires a controlled calculation of nonequilibrium dynamics in a finite-temperature plasma, taking a nonadiabatically changing mass parameter into account. In this work we do not consider fermions but scalar fields, yet the setup of our theory features many of the properties relevant for electroweak baryogenesis: our interacting scalar field model closely resembles the Yukawa part of the standard model Lagrangian, where one scalar field plays the role of the Higgs field and the other generalizes to a heavy fermion (e.g. a top quark or a chargino of a supersymmetric theory). The entropy is, just as the axial vector current, sensitive to quantum coherence. The relevance of scattering processes for electroweak baryogenesis has been treated in several papers in the 1990s [21–26], however no satisfactory solution to the problem has been found so far. Quantum coherence also plays a role in models where CP-violating particle mixing is invoked to source baryogenesis [27–31]. More recently, Herranen, Kainulainen, and Rahkila [32–34] observed that the constraint equations for fermions and scalars admit a third shell at $k_0 = 0$. The authors show that this third shell can be used to correctly reproduce the Klein paradox both for fermions and bosons in a step potential, and hope that their intrinsically off-shell formulation can be used to include interactions in a field theoretical setting for which off-shell physics is essential. The relevance of coherence in baryogenesis for a phase transition at the end of inflation has been addressed in [35–37].

A second application is of course the study of out-of-equilibrium quantum fields from a theoretical perspective. In recent years, out-of-equilibrium dynamics of quantum fields has enjoyed considerable attention as the calculations involved become more and more tractable (for an excellent review we refer to [38]). Many calculations have been performed in nonequilibrium $\lambda\phi^4(x)$, see e.g. [39–41], however also see [42–44]. The renormalization of

the Kadanoff-Baym equations has also received considerable attention [45–47]. Calzetta and Hu [48] prove an H-theorem for a quantum mechanical $O(N)$-model (also see [49]) and refer to "correlation entropy," what we would call "Gaussian von Neumann entropy." A very interesting study has been performed by Garny and Müller [50], where renormalized Kadanoff-Baym equations in $\lambda\phi^4(x)$ are numerically integrated by imposing non-Gaussian initial conditions at some initial time t_0. We differ in our approach as we include memory effects before t_0 such that our evolution, like Garny and Müller's, is divergence free at t_0. In order to more efficiently resum higher order diagrams in perturbation theory, one could consider nPI effective actions and the corresponding time evolution of the quantum corrected correlators (we refer e.g. to [51–56] regarding nPI effective actions and their applications). Although we emphasize that the 2PI formalism is sufficient to study non-Gaussian corrections to the von Neumann entropy, nPI techniques would allow us in principle to more efficiently study the time evolution of, say, the irreducible four-point function to incorporate knowledge of this function into the von Neumann entropy. These techniques are furthermore particularly useful for gauge theories, since a 2PI scheme is not gauge invariant.

Finally, we can expect that a suitable generalization of our setup in an expanding universe can also be applied to the decoherence of cosmological perturbations [9,17,57–69]. Undoubtedly the most interesting aspect of inflation is that it provides us with a causal mechanism to create the initial inhomogeneities of the Universe by means of a quantum process that later grow out to become the structure we observe today in the form of galaxies and clusters of galaxies. Decoherence should bridge the gap between the intrinsically quantum nature of the initial inhomogeneities during inflation and the classical stochastic behavior as assumed in cosmological perturbation theory.

II. KADANOFF-BAYM EQUATION FOR THE STATISTICAL PROPAGATOR

This section not only aims at summarizing the main results of [1] which we rely upon in the present paper, but we also extend the analysis of [1] to incorporate finite-temperature effects.

There is a connection between the statistical propagator and the Gaussian von Neumann entropy of a system. The Gaussian von Neumann entropy per mode S_k of a certain translationary invariant quantum system is uniquely fixed by the phase space area Δ_k the state occupies,

$$S_k(t) = \frac{\Delta_k(t) + 1}{2} \log\left(\frac{\Delta_k(t) + 1}{2}\right)$$
$$- \frac{\Delta_k(t) - 1}{2} \log\left(\frac{\Delta_k(t) - 1}{2}\right). \quad (4)$$

The phase space area, in turn, is determined from the statistical propagator $F_\phi(k, t, t')$,

$$\Delta_k^2(t) = 4[F(k, t, t')\partial_t\partial_{t'}F(k, t, t') - \{\partial_t F(k, t, t')\}^2]|_{t=t'}. \tag{5}$$

Throughout the paper we set $\hbar = 1$ and $c = 1$. The phase space area indeed corresponds to the phase space area of (an appropriate slicing of) the Wigner function, defined as a Wigner transform of the density matrix [2]. For a pure state we have $\Delta_k = 1$, $S_k = 0$, whereas for a mixed state $\Delta_k > 1$, $S_k > 0$. The expression for the Gaussian von Neumann entropy is only valid for pure or mixed Gaussian states, and not for a class of pure excited states such as eigenstates of the number operator as these states are non-Gaussian. The statistical propagator describes how states are populated and is in the Heisenberg picture defined by

$$F_\phi(x; x') = \frac{1}{2}\text{Tr}[\hat{\rho}(t_0)\{\hat{\phi}(x'), \hat{\phi}(x)\}]$$
$$= \frac{1}{2}\text{Tr}[\hat{\rho}(t_0)(\hat{\phi}(x')\hat{\phi}(x) + \hat{\phi}(x)\hat{\phi}(x'))], \tag{6}$$

given some initial density matrix operator $\hat{\rho}(t_0)$. In spatially homogeneous backgrounds, we can Fourier transform e.g. the statistical propagator as follows:

$$F_\phi(k, t, t') = \int d(\vec{x} - \vec{x}')F_\phi(x; x')e^{-\imath\vec{k}\cdot(\vec{x}-\vec{x}')}, \tag{7}$$

which in the spatially translationary invariant case we consider in this paper only depends on $k = \|\vec{k}\|$. Finally, it is interesting to note that the phase space area can be related to an effective phase space particle number density per mode or the statistical particle number density per mode as

$$n_k(t) = \frac{\Delta_k(t) - 1}{2}, \tag{8}$$

in which case the entropy per mode just reduces to the familiar entropy equation for a collection of n free Bose-particles (this is of course an effective description). The three Gaussian correlators are straightforwardly related to the statistical propagator,

$$\langle\hat{\phi}(\vec{x}, t)\hat{\phi}(\vec{y}, t)\rangle = F_\phi(\vec{x}, t; \vec{y}, t')|_{t=t'} \tag{9a}$$
$$\langle\hat{\pi}(\vec{x}, t)\hat{\pi}(\vec{y}, t)\rangle = \partial_t\partial_{t'}F_\phi(\vec{x}, t; \vec{y}, t')|_{t=t'} \tag{9b}$$
$$\langle\{\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)\}/2\rangle = \partial_{t'}F_\phi(\vec{x}, t; \vec{y}, t')|_{t=t'}. \tag{9c}$$

In order to deal with the difficulties arising in interacting nonequilibrium quantum field theory, we work in the Schwinger-Keldysh formalism [70–72], in which we can define the following propagators:

$$\imath\Delta_\phi^{++}(x; x') = \text{Tr}[\hat{\rho}(t_0)T[\hat{\phi}(x')\hat{\phi}(x)]] \tag{10a}$$
$$\imath\Delta_\phi^{--}(x; x') = \text{Tr}[\hat{\rho}(t_0)\bar{T}[\hat{\phi}(x')\hat{\phi}(x)]] \tag{10b}$$
$$\imath\Delta_\phi^{-+}(x; x') = \text{Tr}[\hat{\rho}(t_0)\hat{\phi}(x)\hat{\phi}(x')] \tag{10c}$$
$$\imath\Delta_\phi^{+-}(x; x') = \text{Tr}[\hat{\rho}(t_0)\hat{\phi}(x')\hat{\phi}(x)], \tag{10d}$$

where t_0 denotes an initial time, $\bar{T}$ and T denote the antitime ordering and time ordering operations, respectively. We define the various propagators for the χ-field analogously. In Eq. (10), $\imath\Delta_\phi^{++} \equiv \imath\Delta_\phi^F$ denotes the Feynman or time ordered propagator and $\imath\Delta_\phi^{--}$ represents the antitime ordered propagator. The two Wightman functions are given by $\imath\Delta_\phi^{-+}$ and $\imath\Delta_\phi^{+-}$. In this work, we are primarily interested in the causal propagator Δ_ϕ^c and statistical propagator F_ϕ, which are defined as follows:

$$\imath\Delta_\phi^c(x; x') = \text{Tr}(\hat{\rho}(t_0)[\hat{\phi}(x), \hat{\phi}(x')])$$
$$= \imath\Delta_\phi^{-+}(x; x') - \imath\Delta_\phi^{+-}(x; x') \tag{11a}$$
$$F_\phi(x; x') = \frac{1}{2}\text{Tr}[\hat{\rho}(t_0)\{\hat{\phi}(x'), \hat{\phi}(x)\}]$$
$$= \frac{1}{2}(\imath\Delta_\phi^{-+}(x; x') + \imath\Delta_\phi^{+-}(x; x')). \tag{11b}$$

We can express all propagators $\imath\Delta_\phi^{ab}$ solely in terms of the causal and statistical propagators,

$$\imath\Delta_\phi^{+-}(x; x') = F_\phi(x; x') - \frac{1}{2}\imath\Delta_\phi^c(x; x') \tag{12a}$$
$$\imath\Delta_\phi^{-+}(x; x') = F_\phi(x; x') + \frac{1}{2}\imath\Delta_\phi^c(x; x') \tag{12b}$$
$$\imath\Delta_\phi^{++}(x; x') = F_\phi(x; x') + \frac{1}{2}\text{sgn}(t - t')\imath\Delta_\phi^c(x; x') \tag{12c}$$
$$\imath\Delta_\phi^{--}(x; x') = F_\phi(x; x') - \frac{1}{2}\text{sgn}(t - t')\imath\Delta_\phi^c(x; x'). \tag{12d}$$

In order to study the effect of perturbative loop corrections on classical expectation values, we consider the 2PI effective action, using the Schwinger-Keldysh formalism outlined above. Variation of the 2PI effective action with respect to the propagators yields the so-called Kadanoff-Baym equations that govern the dynamics of the propagators and contain the nonlocal scalar self-energy corrections or self-mass corrections to the propagators. The Kadanoff-Baym equations for the system field read

$$(\partial_x^2 - m_\phi^2)\iota\Delta_\phi^{++}(x;x') - \int d^D y[\iota M_\phi^{++}(x;y)\iota\Delta_\phi^{++}(y;x') - \iota M_\phi^{+-}(x;y)\iota\Delta_\phi^{-+}(y;x')] = \iota\delta^D(x-x') \qquad (13a)$$

$$(\partial_x^2 - m_\phi^2)\iota\Delta_\phi^{+-}(x;x') - \int d^D y[\iota M_\phi^{++}(x;y)\iota\Delta_\phi^{+-}(y;x') - \iota M_\phi^{+-}(x;y)\iota\Delta_\phi^{--}(y;x')] = 0 \qquad (13b)$$

$$(\partial_x^2 - m_\phi^2)\iota\Delta_\phi^{-+}(x;x') - \int d^D y[\iota M_\phi^{-+}(x;y)\iota\Delta_\phi^{++}(y;x') - \iota M_\phi^{--}(x;y)\iota\Delta_\phi^{-+}(y;x')] = 0 \qquad (13c)$$

$$(\partial_x^2 - m_\phi^2)\iota\Delta_\phi^{--}(x;x') - \int d^D y[\iota M_\phi^{-+}(x;y)\iota\Delta_\phi^{+-}(y;x') - \iota M_\phi^{--}(x;y)\iota\Delta_\phi^{--}(y;x')] = -\iota\delta^D(x-x'), \qquad (13d)$$

where the self-masses at one loop have the form

$$\iota M_\phi^{ac}(x;x_1) = -\frac{\iota h^2}{2}(\iota\Delta_\chi^{ac}(x;x_1))^2 \qquad (14a)$$

$$\iota M_\chi^{ac}(x;x_1) = -\frac{\iota\lambda^2}{2}(\iota\Delta_\chi^{ac}(x;x_1))^2 - \iota h^2 \iota\Delta_\chi^{ac}(x;x_1)\iota\Delta_\phi^{ac}(x;x_1). \qquad (14b)$$

Note that we have another set of four equations of motion for the χ-field. We define a Fourier transform as

$$\iota\Delta_\phi^{ab}(x;x') = \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\iota\Delta_\phi^{ab}(\vec{k},t,t')e^{\iota\vec{k}\cdot(\vec{x}-\vec{x}')} \qquad (15a)$$

$$\iota\Delta_\phi^{ab}(\vec{k},t,t') = \int d^{D-1}(\vec{x}-\vec{x}')\iota\Delta_\phi^{ab}(x;x')e^{-\iota\vec{k}\cdot(\vec{x}-\vec{x}')}, \qquad (15b)$$

such that Eqs. (13) transform into

$$(\partial_t^2 + k^2 + m_\phi^2)\iota\Delta_\phi^{++}(k,t,t') + \int_{-\infty}^{\infty}dt_1[\iota M_\phi^{++}(k,t,t_1)\iota\Delta_\phi^{++}(k,t_1,t') - \iota M_\phi^{+-}(k,t,t_1)\iota\Delta_\phi^{-+}(k,t_1,t')] = -\iota\delta(t-t') \qquad (16a)$$

$$(\partial_t^2 + k^2 + m_\phi^2)\iota\Delta_\phi^{+-}(k,t,t') + \int_{-\infty}^{\infty}dt_1[\iota M_\phi^{++}(k,t,t_1)\iota\Delta_\phi^{+-}(k,t_1,t') - \iota M_\phi^{+-}(k,t,t_1)\iota\Delta_\phi^{--}(k,t_1,t')] = 0 \qquad (16b)$$

$$(\partial_t^2 + k^2 + m_\phi^2)\iota\Delta_\phi^{-+}(k,t,t') + \int_{-\infty}^{\infty}dt_1[\iota M_\phi^{-+}(k,t,t_1)\iota\Delta_\phi^{++}(k,t_1,t') - \iota M_\phi^{--}(k,t,t')\iota\Delta_\phi^{-+}(k,t_1,t')] = 0 \qquad (16c)$$

$$(\partial_t^2 + k^2 + m_\phi^2)\iota\Delta_\phi^{--}(k,t,t') + \int_{-\infty}^{\infty}dt_1[\iota M_\phi^{-+}(k,t,t_1)\iota\Delta_\phi^{+-}(k,t_1,t') - \iota M_\phi^{--}(k,t,t_1)\iota\Delta_\phi^{--}(k,t_1,t')] = \iota\delta(t-t'). \qquad (16d)$$

Note that we have extended the initial time $t_0 \to -\infty$ in the equation above. Again, we have an analogous set of equations of motion for the χ-field. In order to outline the next simplifying assumption, we need to Fourier transform with respect to the difference of the time variables,

$$\iota\Delta_\chi^{ab}(x;x') = \int \frac{d^D k}{(2\pi)^D}\iota\Delta_\chi^{ab}(k^\mu)e^{\iota k\cdot(x-x')} \qquad (17a)$$

$$\iota\Delta_\chi^{ab}(k^\mu) = \int d^D(x-x')\iota\Delta_\chi^{ab}(x;x')e^{-\iota k\cdot(x-x')}. \qquad (17b)$$

As already mentioned, we will not solve the dynamical equations for both the system and environment propagators, but instead we assume the following hierarchy of couplings:

$$h \ll \lambda. \qquad (18)$$

We thus assume that λ is large enough such that the χ-field is thermalized by its strong self-interaction which allows us to approximate the solutions of the dynamical equations for χ as thermal propagators [73],

$$\iota\Delta_\chi^{++}(k^\mu) = \frac{-\iota}{k_\mu k^\mu + m_\chi^2 - \iota\epsilon} + 2\pi\delta(k_\mu k^\mu + m_\chi^2)n_\chi^{eq}(|k_0|) \qquad (19a)$$

$$\iota\Delta_\chi^{--}(k^\mu) = \frac{\iota}{k_\mu k^\mu + m_\chi^2 + \iota\epsilon} + 2\pi\delta(k_\mu k^\mu + m_\chi^2)n_\chi^{eq}(|k^0|) \qquad (19b)$$

$$\iota\Delta_\chi^{+-}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2) \times [\theta(-k^0) + n_\chi^{eq}(|k^0|)] \qquad (19c)$$

$$\iota\Delta_\chi^{-+}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2) \times [\theta(k^0) + n_\chi^{eq}(|k^0|)], \qquad (19d)$$

where the Bose-Einstein distribution is given by

$$n_\chi^{eq}(k^0) = \frac{1}{e^{\beta k^0}-1}, \qquad \beta = \frac{1}{k_B T}, \qquad (20)$$

with k_B denoting the Stefan-Boltzmann constant and T the temperature. Here we use the notation $k_\mu k^\mu = -k_0^2 + k^2$ to distinguish the four-vector length from the spatial three-vector length $k = \|\vec{k}\|$. We thus neglect the backreaction of the system field on the environment field,

such that the latter remains in thermal equilibrium at temperature T. This assumption is perturbatively well justified [1]. Furthermore, we neglected for simplicity the $\mathcal{O}(\lambda^2)$ correction to the propagators above that slightly changes the equilibrium state of the environment field. Note finally that, in our approximation scheme, the dynamics of the system propagators is effectively influenced only by the 1PI self-mass corrections.

In [1], we have considered an environment field χ in its vacuum state at $T = 0$ and in the present work we investigate finite-temperature effects. Divergences originate from the vacuum contributions to the self-masses only. Since we already discussed renormalization extensively in [1], let us just state that the renormalized self-masses are given by

$$\iota M_{\phi,\text{ren}}^{ab}(k, t, t') = (\partial_t^2 + k^2)\iota Z_\phi^{ab}(k, t, t') + \iota M_{\phi,\text{th}}^{ab}(k, t, t'),$$
(21)

where the vacuum contributions $Z_\phi^{ab}(k, t, t')$ to the self-masses are given by

$$Z_\phi^{\pm\pm}(k, t, t') = \frac{h^2}{64k\pi^2}\left[e^{\mp\iota k|\Delta t|}\left(\gamma_E + \log\left[\frac{k}{2\mu^2|\Delta t|}\right]\mp\iota\frac{\pi}{2}\right) + e^{\pm\iota k|\Delta t|}(\text{ci}(2k|\Delta t|)\mp\iota\text{si}(2k|\Delta t|))\right]$$
(22a)

$$Z_\phi^{\mp\pm}(k, t, t') = \frac{h^2}{64k\pi^2}\left[e^{\mp\iota k\Delta t}\left(\gamma_E + \log\left[\frac{k}{2\mu^2|\Delta t|}\right]\mp\iota\frac{\pi}{2}\text{sgn}(\Delta t)\right) + e^{\pm\iota k\Delta t}(\text{ci}(2k|\Delta t|)\mp\iota\text{sgn}(\Delta t)\text{si}(2k|\Delta t|))\right],$$
(22b)

and where $\iota M_{\phi,\text{th}}^{ab}(k, t, t')$ are the thermal contributions to the self-masses that yet need to be evaluated. In deriving (22), we made the simplifying assumption $m_\chi \to 0$. The influence of the environment field on the system field is still perturbatively under control [1]. Furthermore, $\text{ci}(z)$ and $\text{si}(z)$ are the cosine and sine integral functions, defined by

$$\text{ci}(z) \equiv -\int_z^\infty dt\,\frac{\cos(t)}{t}$$
(23a)

$$\text{si}(z) \equiv -\int_z^\infty dt\,\frac{\sin(t)}{t}.$$
(23b)

Note that the structure of the self-mass (21) is such that we can construct relations analogous to Eq. (d12),

$$M_\phi^{+-}(k, t, t') = M_\phi^F(k, t, t') - \frac{1}{2}\iota M_\phi^c(k, t, t')$$
(24a)

$$M_\phi^{-+}(k, t, t') = M_\phi^F(k, t, t') + \frac{1}{2}\iota M_\phi^c(k, t, t')$$
(24b)

$$M_\phi^{++}(k, t, t') = M_\phi^F(k, t, t') + \frac{1}{2}\text{sgn}(t - t')\iota M_\phi^c(k, t, t')$$
(24c)

$$M_\phi^{--}(k, t, t') = M_\phi^F(k, t, t') - \frac{1}{2}\text{sgn}(t - t')\iota M_\phi^c(k, t, t').$$
(24d)

This structure applies to both the vacuum and thermal contributions separately. Thus, $Z_\phi^F(k, t, t')$ is the vacuum contribution to the statistical self-mass and $\iota Z_\phi^c(k, t, t')$ the vacuum contribution to the causal self-mass. Similarly we can define the thermal contributions to the statistical and causal self-masses $M_{\phi,\text{th}}^F(k, t, t')$ and $\iota M_{\phi,\text{th}}^c(k, t, t')$, respectively. Of course, we still need to evaluate these expressions. The vacuum contributions follow straightforwardly from Eq. (22):

$$Z_\phi^F(k, t, t') = \frac{1}{2}[Z_\phi^{-+}(k, t, t') + Z_\phi^{+-}(k, t, t')]$$
$$= \frac{h^2}{64k\pi^2}\left[\cos(k\Delta t)\left(\gamma_E + \log\left[\frac{k}{2\mu^2|\Delta t|}\right]\right.\right.$$
$$\left.+ \text{ci}(2k|\Delta t|)\right) + \sin(k|\Delta t|)\left(\text{si}(2k|\Delta t|) - \frac{\pi}{2}\right)\Big]$$
(25a)

$$Z_\phi^c(k, t, t') = \iota[Z_\phi^{+-}(k, t, t') - Z_\phi^{-+}(k, t, t')]$$
$$= \frac{h^2}{64k\pi^2}\left[-2\cos(k\Delta t)\text{sgn}(\Delta t)\left(\text{si}(2k|\Delta t|) + \frac{\pi}{2}\right)\right.$$
$$+ 2\sin(k\Delta t)\left(\text{ci}(2k|\Delta t|) - \gamma_E\right.$$
$$\left.\left.- \log\left[\frac{k}{2\mu^2|\Delta t|}\right]\right)\right].$$
(25b)

As before, we are primarily interested in the equations of motion for the causal and statistical propagators, as it turns out they yield a closed system of differential equations that can be integrated by providing appropriate initial conditions. In order to obtain the equation of motion for the causal propagator, we subtract (16b) from (16c) and use Eqs. (21) and (24) to find

$$(\partial_t^2 + k^2 + m_\phi^2)\Delta_\phi^c(k, t, t') - (\partial_t^2 + k^2)$$
$$\times \int_{t'}^t dt_1 Z_\phi^c(k, t, t_1)\Delta_\phi^c(k, t_1, t')$$
$$- \int_{t'}^t dt_1 M_{\phi,\text{th}}^c(k, t, t_1)\Delta_\phi^c(k, t_1, t') = 0.$$
(26)

In order to get an equation for the statistical propagator, we add Eq. (16b) to (16c), which we simplify to get

$$(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k, t, t') - (\partial_t^2 + k^2)\left[\int_{-\infty}^t dt_1 Z_\phi^c(k, t, t_1)F_\phi(k, t_1, t') - \int_{-\infty}^{t'} dt_1 Z_\phi^F(k, t, t_1)\Delta_\phi^c(k, t_1, t')\right]$$

$$- \int_{-\infty}^t dt_1 M_{\phi,\text{th}}^c(k, t, t_1)F_\phi(k, t_1, t') + \int_{-\infty}^{t'} dt_1 M_{\phi,\text{th}}^F(k, t, t_1)\Delta_\phi^c(k, t_1, t') = 0. \tag{27}$$

Because of the nonlocality inherent in any interacting quantum field theory, the "memory kernels," the memory integrals in Eq. (27) above, range from negative past infinity to either t or t'. To make the numerical implementation feasible, we insert a finite initial time t_0 by hand and approximate the propagators in the memory kernels from the negative past to t_0 with the free propagators inducing an error of the order $\mathcal{O}(h^4/\omega_\phi^4)$,

$$(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k, t, t') - (\partial_t^2 + k^2)\left[\int_{-\infty}^{t_0} dt_1 Z_\phi^c(k, t, t_1)F_\phi^{\text{free}}(k, t_1, t') + \int_{t_0}^t dt_1 Z_\phi^c(k, t, t_1)F_\phi(k, t_1, t')\right.$$

$$\left. - \int_{-\infty}^{t_0} dt_1 Z_\phi^F(k, t, t_1)\Delta_\phi^{c,\text{free}}(k, t_1, t') - \int_{t_0}^{t'} dt_1 Z_\phi^F(k, t, t_1)\Delta_\phi^c(k, t_1, t')\right] - \int_{-\infty}^{t_0} dt_1 M_{\phi,\text{th}}^c(k, t, t_1)F_\phi^{\text{free}}(k, t_1, t')$$

$$- \int_{t_0}^t dt_1 M_{\phi,\text{th}}^c(k, t, t_1)F_\phi(k, t_1, t') + \int_{-\infty}^{t_0} dt_1 M_{\phi,\text{th}}^F(k, t, t_1)\Delta_\phi^{c,\text{free}}(k, t_1, t') + \int_{t_0}^{t'} dt_1 M_{\phi,\text{th}}^F(k, t, t_1)\Delta_\phi^c(k, t_1, t') = 0. \tag{28}$$

Here, $F_\phi^{\text{free}}(k, t, t')$ and $\Delta_\phi^{c,\text{free}}(k, t, t')$ are the free statistical and causal propagators which, depending on the initial conditions one imposes at t_0, should either be evaluated at $T = 0$ or at some finite temperature. The memory kernels need to be included to remove the initial time singularity as discussed in [1,50]. We postpone imposing initial conditions to Sec. V, but let us at the moment just evaluate the memory kernels in these two cases. The thermal propagators read

$$F_{\phi,\text{th}}^{\text{free}}(k, t, t') = \frac{\cos(\omega_\phi(t - t'))}{2\omega_\phi}\left(1 + \frac{2}{e^{\beta\omega_\phi} - 1}\right) = \frac{\cos(\omega_\phi(t - t'))}{2\omega_\phi}\coth\left(\frac{1}{2}\beta\omega_\phi\right) \tag{29a}$$

$$\imath\Delta_\phi^{c,\text{free}}(k, t, t') = -\frac{\imath}{\omega_\phi}\sin(\omega_\phi(t - t')). \tag{29b}$$

Here, $\omega_\phi^2 = k^2 + m_{\phi,\text{in}}^2$, where in the case of a changing mass one should use the initial mass. Let us now evaluate the "infinite past memory kernels" for the vacuum contributions, i.e., the memory kernels from negative past infinity to t_0 using the two propagators above. The other memory kernels in Eq. (28) can only be evaluated numerically, as soon as we have the actual expressions of the thermal contributions to the self-masses. Let us thus evaluate

$$(\partial_t^2 + k^2)\left[\int_{-\infty}^{t_0} dt_1 Z_\phi^c(k, t, t_1)F_{\phi,\text{th}}^{\text{free}}(k, t_1, t') - \int_{-\infty}^{t_0} dt_1 Z_\phi^F(k, t, t_1)\Delta_\phi^{c,\text{free}}(k, t_1, t')\right]$$

$$= \frac{h^2}{32\pi^2}\int_{-\infty}^{t_0} dt_1\left[\frac{\cos[k(t - t_1)]}{t - t_1}\frac{\cos[\omega_\phi(t_1 - t')]}{\omega_\phi}\coth\left(\frac{1}{2}\beta\omega_\phi\right) + \frac{\sin[k(t - t_1)]}{t - t_1}\frac{\sin[\omega_\phi(t_1 - t')]}{\omega_\phi}\right]$$

$$= -\frac{h^2}{64\omega_\phi\pi^2}\left[\frac{\cos[\omega_\phi(t - t')]}{\sinh(\frac{1}{2}\beta\omega_\phi)}\{e^{(1/2)\beta\omega_\phi}\text{ci}[(\omega_\phi + k)(t - t_0)] + e^{-(1/2)\beta\omega_\phi}\text{ci}[(\omega_\phi - k)(t - t_0)]\}\right.$$

$$\left. + \frac{\sin[\omega_\phi(t - t')]}{\sinh(\frac{1}{2}\beta\omega_\phi)}\{e^{(1/2)\beta\omega_\phi}\text{si}[(\omega_\phi + k)(t - t_0)] + e^{-(1/2)\beta\omega_\phi}\text{si}[(\omega_\phi - k)(t - t_0)]\}\right]. \tag{30}$$

Because of the fact that the free thermal statistical propagator contains a temperature dependence, the corresponding "infinite past memory kernel" is of course also affected. In case we would need the $T = 0$ vacuum propagators in the memory kernels only, one can easily send $T \to 0$ in the expression above, obtaining the same memory kernels as in [1],

$$(\partial_t^2 + k^2)\left[\int_{-\infty}^{t_0} dt_1 Z_\phi^c(k, t, t_1)F_{\phi,\text{vac}}^{\text{free}}(k, t_1, t') - \int_{-\infty}^{t_0} dt_1 Z_\phi^F(k, t, t_1)\Delta_\phi^{c,\text{free}}(k, t_1, t')\right]$$

$$= -\frac{h^2}{32\omega_\phi\pi^2}\{\cos[\omega_\phi(t - t')]\text{ci}[(\omega_\phi + k)(t - t_0)] + \sin[\omega_\phi(t - t')]\text{si}[(\omega_\phi + k)(t - t_0)]\}. \tag{31}$$

III. FINITE-TEMPERATURE CONTRIBUTIONS TO THE SELF-MASSES

In this section, we evaluate all contributions to the self-masses for a finite temperature.

A. The causal self-mass

Let us first evaluate the thermal contribution to the causal self-mass. Formally, from Eq. (14), it reads

$$
M^c_{\phi,\text{th}}(k, \Delta t = t - t') = -\iota[M^{-+}_{\phi,\text{th}}(k, t, t') - M^{+-}_{\phi,\text{th}}(k, t, t')]
$$

$$
= -h^2 \int \frac{d^{D-1}\vec{k}_1}{(2\pi)^{D-1}} F^{\text{th}}_\chi(k_1, \Delta t) \Delta^c_\chi(\|\vec{k} - \vec{k}_1\|, \Delta t),
$$

$$(32)$$

where the superscript F^{th}_χ denotes that we should only keep the thermal contribution to the statistical propagator as we have already evaluated the vacuum contribution. The following change of variables is useful:

$$
\int \frac{d^{D-1}\vec{k}_1}{(2\pi)^{D-1}} = \frac{1}{(2\pi)^{D-1}} \int_0^\infty dk_1 k_1^{D-2} \int d\Omega_{D-2}
$$

$$
= \frac{\Omega_{D-3}}{(2\pi)^{D-1}} \int_0^\infty dk_1 k_1^{D-2} \int_{-1}^1 d\cos(\theta)[\sin(\theta)]^{D-4}
$$

$$
= \frac{\Omega_{D-3}}{(2\pi)^{D-1}} \int dk_1 k_1^{D-2} \int_{\omega_-}^{\omega_+} d\omega \frac{2\omega}{(2kk_1)^{D-3}}
$$

$$
\times [(\omega_+^2 - \omega^2)(\omega^2 - \omega_-^2)]^{(D-4)/2}, \quad (33)
$$

where we have chosen $\theta \equiv \angle(\vec{k}, \vec{k}_1)$. In the final line we have changed variables to $\omega \equiv \omega_\chi(\|\vec{k} - \vec{k}_1\|) = (\|\vec{k} - \vec{k}_1\|^2 + m_\chi^2)^{1/2}$, which clearly depends on θ. Furthermore $\omega_\pm^2 = (k \pm k_1)^2 + m_\chi^2$ and Ω_{D-3} denotes the area of the $D - 3$ dimensional sphere S^{D-3},

$$
\Omega_{D-3} = \frac{2\pi^{(D-2)/2}}{\Gamma(\frac{D}{2} - 1)}. \quad (34)
$$

Using Eq. (29), we have

$$
M^c_{\phi,\text{th}}(k, \Delta t) = h^2 \frac{\Omega_{D-3}}{(2\pi)^{D-1}} \int_0^\infty dk_1 k_1^{D-2} \int_{\omega_-}^{\omega_+} d\omega \frac{2\omega}{(2kk_1)^{D-3}}
$$

$$
\times [(\omega_+^2 - \omega^2)(\omega^2 - \omega_-^2)]^{(D-4)/2} \frac{1}{(k_1^2 + m_\chi^2)^{1/2}}
$$

$$
\times \frac{1}{\omega} n_\chi^{\text{eq}}(\{k_1^2 + m_\chi^2\}^{1/2}) \cos(\{k_1^2 + m_\chi^2\}^{1/2}\Delta t)
$$

$$
\times \sin(\omega\Delta t). \quad (35)
$$

This contribution cannot contain any new divergences as the latter all stem from the vacuum contribution, which allows us to let $D \to 4$. Moreover, we are interested, as in Sec. II, in the limit $m_\chi \to 0$. Equation (35) thus simplifies to

$$
M^c_{\phi,\text{th}}(k, \Delta t) = \frac{h^2}{4\pi^2} \frac{\sin(k\Delta t)}{k\Delta t} \int_0^\infty dk_1 \frac{\sin(2k_1\Delta t)}{e^{\beta k_1} - 1}
$$

$$
= \frac{h^2}{16\pi^2} \frac{\sin(k\Delta t)}{k(\Delta t)^2} \left[\frac{2\pi\Delta t}{\beta} \coth\left(\frac{2\pi\Delta t}{\beta}\right) - 1 \right].
$$

$$(36)$$

At coincidence $\Delta t \to 0$, the thermal contribution to the causal self-mass vanishes, as it should.

B. The statistical self-mass

The thermal contribution to the statistical self-mass is somewhat harder to obtain. It is given by

$$
M^F_{\phi,\text{th}}(k, \Delta t = t - t') = \frac{1}{2}[M^{-+}_{\phi,\text{th}}(k, \Delta t) + M^{+-}_{\phi,\text{th}}(k, \Delta t)]
$$

$$
= -\frac{h^2}{2} \int \frac{d^{D-1}\vec{k}_1}{(2\pi)^{D-1}} \bigg[F_\chi(k_1, \Delta t)
$$

$$
\times F_\chi(\|\vec{k} - \vec{k}_1\|, \Delta t) - \frac{1}{4}\Delta^c_\chi(k_1, \Delta t)
$$

$$
\times \Delta^c_\chi(\|\vec{k} - \vec{k}_1\|, \Delta t) \bigg] \bigg|_{\text{th}}, \quad (37)
$$

where of course we are only interested in keeping the thermal contributions. The second term in the integral consists of two causal propagators that does not contribute at all at finite temperature. It is convenient to split the thermal contributions to the statistical self-mass as

$$
M^F_{\phi,\text{th}}(k, \Delta t) = M^F_{\phi,\text{vac-th}}(k, \Delta t) + M^F_{\phi,\text{th-th}}(k, \Delta t), \quad (38)
$$

where, formally, we have

$$
M^F_{\phi,\text{vac-th}}(k, \Delta t) = -\frac{h^2}{2} \int \frac{d^{D-1}\vec{k}_1}{(2\pi)^{D-1}} \frac{\cos(\omega_\chi(k_1)\Delta t)}{\omega_\chi(k_1)}
$$

$$
\times \frac{\cos(\omega_\chi(\|\vec{k} - \vec{k}_1\|)\Delta t)}{\omega_\chi(\|\vec{k} - \vec{k}_1\|)} \frac{1}{e^{\beta\omega_\chi(k_1)} - 1}
$$

$$(39a)$$

$$
M^F_{\phi,\text{th-th}}(k, \Delta t) = -\frac{h^2}{2} \int \frac{d^{D-1}\vec{k}_1}{(2\pi)^{D-1}} \frac{\cos(\omega_\chi(k_1)\Delta t)}{\omega_\chi(k_1)}
$$

$$
\times \frac{1}{e^{\beta\omega_\chi(k_1)} - 1} \frac{\cos(\omega_\chi(\|\vec{k} - \vec{k}_1\|)\Delta t)}{\omega_\chi(\|\vec{k} - \vec{k}_1\|)}
$$

$$
\times \frac{1}{e^{\beta\omega_\chi(\|\vec{k} - \vec{k}_1\|)} - 1}. \quad (39b)
$$

Here, $M^F_{\phi,\text{vac-th}}$ is the vacuum-thermal contribution to the statistical self-mass and $M^F_{\phi,\text{th-th}}$ is the thermal-thermal contribution. As before, we let $D \to 4$ and $m_\chi \to 0$. Let us first evaluate the vacuum-thermal contribution. Equation (39a) thus simplifies to

$$M^F_{\phi,\text{vac-th}}(k, \Delta t) = -\frac{h^2}{8\pi^2 k \Delta t} \int_0^\infty dk_1 \frac{\cos(k_1 \Delta t)}{e^{\beta k_1} - 1}$$
$$\times \left[\sin((k + k_1)\Delta t) - \sin(|k - k_1|\Delta t) \right]. \tag{40}$$

We have to take the absolute values in the equation above correctly into account by making use of Heaviside step functions and we can moreover expand the exponential to find

$$M^F_{\phi,\text{vac-th}}(k, \Delta t) = -\frac{h^2}{16\pi^2 k \Delta t} \int_0^\infty dk_1 \sum_{n=1}^\infty e^{-\beta n k_1} \{ \sin((2k_1 + k)\Delta t) + 2\theta(k_1 - k)\sin(k\Delta t)$$
$$+ \{\theta(k - k_1) - \theta(k_1 - k)\} \sin((2k_1 - k)\Delta t) \}. \tag{41}$$

Integrating over k_1 and collecting the terms we get

$$M^F_{\phi,\text{vac-th}}(k, \Delta t) = -\frac{h^2}{8\pi^2 k \Delta t} \sum_{n=1}^\infty \left\{ \cos(k\Delta t)(1 - e^{-\beta nk})\frac{2\Delta t}{(\beta n)^2 + (2\Delta t)^2} + \sin(k\Delta t)e^{-\beta nk}\left[\frac{1}{\beta n} - \frac{\beta n}{(\beta n)^2 + (2\Delta t)^2} \right] \right\}. \tag{42}$$

The sum can be performed, resulting in

$$M^F_{\phi,\text{vac-th}}(k, \Delta t) = \frac{h^2}{16\pi^2 k (\Delta t)^2} \left[\sin(k\Delta t)\left\{ \frac{2\Delta t}{\beta}\log(1 - e^{-\beta k}) + e^{-\beta k}\sum_\pm \frac{\pm\frac{\iota}{2}}{1 \pm \frac{2\iota\Delta t}{\beta}}{}_2F_1\left(2, 1 \pm \frac{2\iota\Delta t}{\beta}; 2 \pm \frac{2\iota\Delta t}{\beta}; e^{-\beta k}\right) \right\} \right.$$
$$\left. - \cos(k\Delta t)\left\{ \frac{1}{2}\left(\frac{2\pi\Delta t}{\beta}\coth\left(\frac{2\pi\Delta t}{\beta}\right) - 1\right) - e^{-\beta k}\sum_\pm \frac{\pm\frac{\iota\Delta t}{\beta}}{1 \pm \frac{2\iota\Delta t}{\beta}}{}_2F_1\left(1, 1 \pm \frac{2\iota\Delta t}{\beta}; 2 \pm \frac{2\iota\Delta t}{\beta}; e^{-\beta k}\right) \right\} \right], \tag{43}$$

where ${}_2F_1$ is the Gauss hypergeometric function. For convenience we quote the low temperature ($\beta k \gg 1$) and the high temperature ($\beta k \ll 1$) limits of this expression. In the low temperature limit, (43) reduces to

$$M^F_{\phi,\text{vac-th}}(k, \Delta t) \overset{\beta k \gg 1}{\to} -\frac{h^2}{16\pi^2 k (\Delta t)^2} \left\{ \frac{\cos(k\Delta t)}{2}\left[\frac{2\pi\Delta t}{\beta}\coth\left(\frac{2\pi\Delta t}{\beta}\right) - 1 \right] \right.$$
$$\left. + e^{-\beta k}\left[\cos(k\Delta t)\frac{-(2\Delta t/\beta)^2}{1 + (2\Delta t/\beta)^2} + \sin(k\Delta t)\left(\frac{2\Delta t}{\beta} - \frac{2\Delta t/\beta}{1 + (2\Delta t/\beta)^2}\right) \right] \right\}, \tag{44}$$

and its coincidence limit is finite,

$$\lim_{\Delta t \to 0} M^F_{\phi,\text{vac-th}}(k, \Delta t) \overset{\beta k \gg 1}{\to} -\frac{h^2(\pi^2 - 6e^{-k\beta})}{24\pi^2 k \beta^2}. \tag{45}$$

In the high temperature limit, (43) reduces to

$$M^F_{\phi,\text{vac-th}}(k, \Delta t) \overset{\beta k \ll 1}{\to} \frac{h^2}{4\pi^2 \beta} \left[\cos(k\Delta t)\left\{ \log(\beta k) + \gamma_E - 1 + \frac{1}{2}\sum_\pm \psi\left(1 \pm \frac{2\iota\Delta t}{\beta}\right) \right\} - \frac{\sin(k\Delta t)}{4k\Delta t}\left\{ \sum_\pm \psi\left(1 \pm \frac{2\iota\Delta t}{\beta}\right) + 2\gamma_E \right\} \right]. \tag{46}$$

There is a mild logarithmic divergence, $M^F_{\phi,\text{vac-th}} \propto \log(\beta k)$, in the limit when $\beta k \to 0$. Also note that when we derived Eq. (46) above, we tacitly assumed that also $\Delta t/\beta \ll 1$. We however only use Eq. (46) to calculate the coincidence limit $\Delta t \to 0$ of the statistical self-mass in which case this approximation is well justified:

$$\lim_{\Delta t \to 0} M^F_{\phi,\text{vac-th}}(k, \Delta t) \overset{\beta k \ll 1}{\to} \frac{h^2}{4\pi^2 \beta}(\log(\beta k) - 1). \tag{47}$$

The final remaining contribution to the statistical self-mass is $M^F_{\phi,\text{th-th}}(k, \Delta t)$ in Eq. (39b) and is much harder to obtain. In fact, it turns out we can only evaluate its high ($\beta k \ll 1$) and low ($\beta k \gg 1$) temperature contributions in closed form. For that reason we present the calculation in Appendix A, and in the current section only state the main results. The low temperature ($\beta k \gg 1$) limit of $M^F_{\phi,\text{th-th}}(k, \Delta t)$ is given by

$$M^F_{\phi,\text{th-th}}(k, \Delta t) \stackrel{k\beta \gg 1}{\rightarrow} -\frac{h^2}{16\pi^2 k} e^{-\beta k} \left[\cos(k\Delta t) \left\{ \frac{2\pi\Delta t \coth(\frac{2\pi\Delta t}{\beta})}{\beta(\beta^2 + (\Delta t)^2)} + \frac{\beta k}{\beta^2 + (\Delta t)^2} + \frac{\beta^2(5\beta^2 + 11(\Delta t)^2)}{(\beta^2 + (\Delta t)^2)^2(\beta^2 + (2\Delta t)^2)} \right\} \right.$$

$$\left. + \sin(k\Delta t) \left\{ \frac{2\pi \coth(\frac{2\pi\Delta t}{\beta})}{\beta^2 + (\Delta t)^2} - \frac{k\Delta t}{\beta^2 + (\Delta t)^2} - \frac{2\beta\Delta t}{(\beta^2 + (\Delta t)^2)^2} - \frac{8\Delta t}{\beta(\beta^2 + (2\Delta t)^2)} \right\} \right]. \tag{48}$$

Note that this expression is finite in the limit when $\Delta t \rightarrow 0$:

$$\lim_{\Delta t \rightarrow 0} M^F_{\phi,\text{th-th}}(k, \Delta t) \stackrel{k\beta \gg 1}{\rightarrow} -h^2 \frac{3 + k\beta}{8\pi^2 k\beta^2} e^{-k\beta}. \tag{49}$$

The high temperature ($\beta k \ll 1$) limit yields

$$M^F_{\phi,\text{th-th}}(k, \Delta t) \stackrel{k\beta \ll 1}{\rightarrow} -\frac{h^2}{16\pi^2 k\beta^2} \left[\frac{\pi^2}{2} - 4(\gamma_E - \text{ci}(|k\Delta t|) + \log(|k\Delta t|)) + \frac{(k\Delta t)^2}{2} \frac{d}{d\gamma} {}_2F_3\left(1, 1; 2, 2, 1 + \gamma; -\frac{(k\Delta t)^2}{4}\right) \right.$$

$$\times \bigg|_{\gamma=1/2} + \frac{\beta \sin(k\Delta t)}{\Delta t}\left(\text{ci}(2|k\Delta t|) - \gamma_E - \log\left(\frac{2|\Delta t|}{k\beta^2}\right) - 1\right) - \frac{\beta \cos(k\Delta t)}{|\Delta t|}\left(\text{si}(2|k\Delta t|) + \frac{\pi}{2}\right)$$

$$+ k\beta \sum_{\pm} \frac{e^{-k\beta \pm \imath k\Delta t}}{2(1 \mp \imath\Delta t/\beta)} \left[{}_2F_1\left(2, 2 \mp \frac{2\imath\Delta t}{\beta}; 3 \mp \frac{2\imath\Delta t}{\beta}; e^{-k\beta/2}\right) \right.$$

$$\left. + \frac{(k\beta)^2}{12} {}_2F_1\left(4, 2 \mp \frac{2\imath\Delta t}{\beta}; 3 \mp \frac{2\imath\Delta t}{\beta}; e^{-k\beta/2}\right) \right]. \tag{50}$$

Clearly, the limit $\Delta t \rightarrow 0$ of the self-mass above is finite too:

$$\lim_{\Delta t \rightarrow 0} M^F_{\phi,\text{th-th}}(k, \Delta t)$$

$$\stackrel{k\beta \ll 1}{\rightarrow} -\frac{h^2}{32\pi^2 k\beta^2}\left(8 + \pi^2 + 4k\beta \log\left[\frac{1}{2}(k\beta)^2\right]\right), \tag{51}$$

where we ignored the subleading term in Eq. (50) to derive the coincidence limit above.

IV. ENTROPY GENERATION IN QUANTUM MECHANICS

Now that the stage is set to study entropy generation in our quantum field theoretical model, let us digress somewhat and study entropy generation in the analogous quantum mechanical model first. This allows us to quantitatively compare the evolution of the entropy resulting from the perturbative master equation and in our correlator approach. A comparison in field theory is not possible so far, due to the shortcomings of the conventional approach to decoherence using the master equation as discussed in the introduction. Let us consider the quantum mechanical system of $N + 1$ simple harmonic oscillators x and q_n, $1 \leq n \leq N$, coupled by an interaction term of the form $h_n x q_n^2$,

$$L = L_S + L_E + L_{SE} = \frac{1}{2}(\dot{x}^2 - \omega_0^2 x^2)$$

$$+ \sum_{n=1}^N \frac{1}{2}(\dot{q}_n^2 - \omega_n^2 q_n^2) - \frac{1}{2}h_n x q_n^2, \tag{52}$$

which indeed is the quantum mechanical $D = 1$ dimensional analogue of the Lagrangian density in Eq. (3) considered before. Here, ω_0 and $\{\omega_n\}$ are the frequencies of the oscillators as usual. The x oscillator is the system in a thermal environment of $\{q_n\}$ oscillators. We absorb the mass in the time in our action, and the remaining dimensionless mass dependence in the $\{q_n\}$.

A. The Kadanoff-Baym equations in quantum mechanics

The free thermal statistical and causal propagator in quantum mechanics read

$$F_{q_n}(t, t') = \frac{\cos(\omega_n(t - t'))}{2\omega_n} \coth(\beta\omega_n/2) \tag{53a}$$

$$\Delta^c_{q_n}(t, t') = \frac{-1}{\omega_n} \sin(\omega_n(t - t')). \tag{53b}$$

The statistical and causal self-energies of the x-system at lowest order in perturbation theory are defined by

$$M^F_x(t, t') = -\sum_{n=1}^N \frac{h_n^2}{4}[(\imath\Delta^{+-}_{q_n}(t, t'))^2 + (\imath\Delta^{-+}_{q_n}(t, t'))^2] \tag{54a}$$

$$M^c_x(t, t') = -\sum_{n=1}^N \frac{\imath h_n^2}{2}[(\imath\Delta^{+-}_{q_n}(t, t'))^2 - (\imath\Delta^{-+}_{q_n}(t, t'))^2], \tag{54b}$$

and are calculated as

$$M_x^F(t, t') = -\sum_{n=1}^{N} \frac{h_n^2}{2}\left[(F_{q_n}(t, t'))^2 - \frac{1}{4}(\Delta_{q_n}^c(t, t'))^2\right]$$

$$= -\sum_{n=1}^{N} \frac{h_n^2}{16\omega_n^2}\left[\left(\coth^2\left(\frac{\beta\omega_n}{2}\right) + 1\right)\cos(2\omega_n(t - t')) + \coth^2\left(\frac{\beta\omega_n}{2}\right) - 1\right] \tag{55a}$$

$$M_x^c(t, t') = -\sum_{n=1}^{N} h_n^2 F_{q_n}(t, t')\Delta_{q_n}^c(t, t') = \sum_{n=1}^{N} \frac{h_n^2}{4\omega_n^2}\sin(2\omega_n(t - t'))\coth\left(\frac{\beta\omega_n}{2}\right). \tag{55b}$$

As in our field theoretical model we neglect the backreaction from the system on the environment. The Kadanoff-Baym equations for the x-system for the statistical and causal propagators are now given by

$$(\partial_t^2 + \omega_0^2)F_x(t, t') + \int_0^{t'} dt_1 M_x^F(t, t_1)\Delta_x^c(t_1, t')$$
$$- \int_0^t dt_1 M_x^c(t, t_1)F_x(t_1, t') = 0 \tag{56a}$$

$$(\partial_t^2 + \omega_0^2)\Delta_x^c(t, t') - \int_{t'}^t dt_1 M_x^c(t, t_1)\Delta_x^c(t_1, t') = 0, \tag{56b}$$

where $\{t, t'\} \geq t_0 = 0$. An important difference compared to the field theoretical Kadanoff-Baym equations in (26) and (28) is that we do not have to renormalize them. Also, we do not have to consider any memory effects before $t_0 = 0$. We can now straightforwardly solve the Kadanoff-Baym equations above by numerical methods to find the statistical propagator and hence the quantum mechanical analogue of the phase space area (5) and entropy (4) as functions of time.

B. The master equation in quantum mechanics

In order to derive the perturbative master equation for our model, we follow Paz and Zurek [16]. The perturbative master equation is obtained straightforwardly from the Dyson series, truncated at second order, as a solution to the von Neumann equation and reads

$$\dot{\rho}_{\text{red}}(t) = \frac{1}{i}[\hat{H}_S(t), \hat{\rho}_{\text{red}}(t)] + \frac{1}{i}\sum_{n=1}^{N}\frac{h_n}{2}[\langle\hat{q}_n^2(t)\rangle\hat{x}, \hat{\rho}_{\text{red}}(t)] \tag{57}$$

$$- \sum_{\substack{n=1 \\ m=1}}^{N}\int_0^t dt_1 K_{nm}^{(3)}(t, t_1)[\hat{x}, [\hat{x}(t_1 - t), \hat{\rho}_{\text{red}}(t)]]$$

$$+ K_{nm}^{(4)}(t, t_1)[\hat{x}, \{\hat{x}(t_1 - t), \hat{\rho}_{\text{red}}(t)\}], \tag{58}$$

where we follow the notation of Paz and Zurek and define the coefficients

$$K_{nm}^{(3)}(t, t_1) = \frac{h_n h_m}{8}\langle\{\hat{q}_n^2(t), \hat{q}_m^2(t_1)\}\rangle - \frac{h_n h_m}{4}\langle\hat{q}_n^2(t)\rangle\langle\hat{q}_m^2(t_1)\rangle \tag{59a}$$

$$K_{nm}^{(4)}(t, t_1) = \frac{h_n h_m}{8}\langle[\hat{q}_n^2(t), \hat{q}_m^2(t_1)]\rangle. \tag{59b}$$

Also, note that $\hat{x}(t) = \hat{x}\cos(\omega_0 t) + \hat{p}_x\sin(\omega_0 t)/\omega_0$ due to changing back to the Schrödinger picture from the interaction picture [16]. The master equation above reduces further to

$$\dot{\rho}_{\text{red}}(t) = \frac{1}{i}[\hat{H}_S(t), \hat{\rho}_{\text{red}}(t)] - \int_0^t dt_1 \nu(t_1)[\hat{x}, [\hat{x}(-t_1), \hat{\rho}_{\text{red}}(t)]]$$
$$- i\eta(t_1)[\hat{x}, \{\hat{x}(-t_1), \hat{\rho}_{\text{red}}(t)\}]. \tag{60}$$

In the equation above, we dropped the linear term in Eq. (57) as a time dependent linear term will not affect the entropy [2]. The noise and dissipation kernels $\nu(t)$ and $\eta(t)$ are straightforwardly related to $K_{nm}^{(3)}(t, t_1)$ and $K_{nm}^{(4)}(t, t_1)$, respectively, and read at the lowest order in perturbation theory [18,74]

$$\nu(t) = \sum_{n=1}^{N}\frac{h_n^2}{16\omega_n^2}\left[\left(\coth^2\left(\frac{\beta\omega_n}{2}\right) + 1\right)\cos(2\omega_n t)\right.$$
$$\left. + \coth^2\left(\frac{\beta\omega_n}{2}\right) - 1\right] = -M_x^F(t, 0) \tag{61a}$$

$$\eta(t) = \sum_{n=1}^{N}\frac{h_n^2}{8\omega_n^2}\sin(2\omega_n t)\coth\left(\frac{\beta\omega_n}{2}\right)$$
$$= \frac{1}{2}M_x^c(t, 0). \tag{61b}$$

Note that we can easily relate the noise and dissipation kernels that appear in the master equation to the self-mass corrections in the Kadanoff-Baym equations. This is an important identity and we will return to it shortly. One thus finds

$$\dot{\rho}_{\text{red}}(t) = -i\left[\hat{H}_S(t) + \frac{1}{2}\Omega^2(t)\hat{x}^2, \hat{\rho}_{\text{red}}(t)\right]$$
$$- i\gamma(t)[\hat{x}, \{\hat{p}_x, \hat{\rho}_{\text{red}}(t)\}] - D(t)[\hat{x}, [\hat{x}, \hat{\rho}_{\text{red}}(t)]]$$
$$- f(t)[\hat{x}, [\hat{p}_x, \hat{\rho}_{\text{red}}(t)]], \tag{62}$$

where the frequency "renormalization" $\Omega(t)$, the damping coefficient $\gamma(t)$, and the two diffusion coefficients $D(t)$ and $f(t)$ are given by

$$\Omega^2(t) = -2\int_0^t dt_1 \eta(t_1)\cos(\omega_0 t_1) = \sum_{n=1}^N \frac{h_n^2}{4\omega_n^2(4\omega_n^2-\omega_0^2)}\coth\left(\frac{\beta\omega_n}{2}\right)\{-2\omega_n(1-\cos(\omega_0 t)\cos(2\omega_n t)) + \omega_0\sin(\omega_0 t)\sin(2\omega_n t)\}$$
(63a)

$$\gamma(t) = \int_0^t dt_1 \eta(t_1)\frac{\sin(\omega_0 t_1)}{\omega_0} = \sum_{n=1}^N \frac{h_n^2}{16\omega_n^2\omega_0}\coth\left(\frac{\beta\omega_n}{2}\right)\left\{\frac{\sin([\omega_0-2\omega_n]t)}{\omega_0-2\omega_n} - \frac{\sin([\omega_0+2\omega_n]t)}{\omega_0+2\omega_n}\right\}$$
(63b)

$$D(t) = \int_0^t dt_1 \nu(t_1)\cos(\omega_0 t_1) = \sum_{n=1}^N \frac{h_n^2}{16\omega_n^2}\left[\frac{1}{2}\left(\coth^2\left(\frac{\beta\omega_n}{2}\right)+1\right)\left\{\frac{\sin([\omega_0-2\omega_n]t)}{\omega_0-2\omega_n} + \frac{\sin([\omega_0+2\omega_n]t)}{\omega_0+2\omega_n}\right\} + \left(\coth^2\left(\frac{\beta\omega_n}{2}\right)-1\right)\frac{\sin(\omega_0 t)}{\omega_0}\right]$$
(63c)

$$f(t) = -\int_0^t dt_1 \nu(t_1)\frac{\sin(\omega_0 t_1)}{\omega_0} = \sum_{n=1}^N \frac{-h_n^2}{16\omega_n^2\omega_0}\left[\left(\coth^2\left(\frac{\beta\omega_n}{2}\right)+1\right)\frac{\{\omega_0(1-\cos(\omega_0 t)\cos(2\omega_n t))-2\omega_n\sin(\omega_0 t)\sin(2\omega_n t)\}}{\omega_0^2-4\omega_n^2}\right.$$
$$\left. + \left(\coth^2\left(\frac{\beta\omega_n}{2}\right)-1\right)\frac{1-\cos(\omega_0 t)}{\omega_0}\right].$$
(63d)

We are now ready to solve the master equation (62). As we are interested in the evolution of two-point functions, let us make a Gaussian ansatz and project this operator equation on the position bras and kets as follows:

$$\rho_{\rm red}(x,y;t) = \langle x|\hat\rho_{\rm red}(t)|y\rangle$$
$$= \tilde{\mathcal N}(t)\exp[-\tilde a(t)x^2 - \tilde a^*(t)y^2 + 2\tilde c(t)xy].$$
(64)

It turns out to be advantageous to directly compute the time evolution of our three nontrivial Gaussian correlators. Analogous methods have been used in [75] to analyze decoherence in an upside-down simple harmonic oscillator. We can thus derive the following set of differential equations [3]:

$$\frac{d\langle\hat x^2\rangle}{dt} = -\frac{\dot{\tilde a}_R-\dot{\tilde c}}{4(\tilde a_R-\tilde c)^2} = 2\left\langle\frac{1}{2}\{\hat x,\hat p\}\right\rangle$$
(65a)

$$\frac{d\langle\hat p^2\rangle}{dt} = -2(\omega_0^2+\Omega^2)\left\langle\frac{1}{2}\{\hat x,\hat p\}\right\rangle - 4\gamma(t)\langle\hat p^2\rangle + 2D(t)$$
(65b)

$$\frac{d\langle\frac{1}{2}\{\hat x,\hat p\}\rangle}{dt} = -(\omega_0^2+\Omega^2)\langle\hat x^2\rangle + \langle\hat p^2\rangle - f(t) - 2\gamma(t)\left\langle\frac{1}{2}\{\hat x,\hat p\}\right\rangle.$$
(65c)

These equations are completely equivalent to the master equation when one is interested in the Gaussian correlators only. Initially, we impose that the system is in a pure state,

$$\langle\hat x^2(t_0)\rangle = \frac{1}{2\omega_0}$$
(66a)

$$\langle\hat p^2(t_0)\rangle = \frac{\omega_0}{2}$$
(66b)

$$\left\langle\frac{1}{2}\{\hat x,\hat p\}(t_0)\right\rangle = 0.$$
(66c)

We can then straightforwardly find the quantum mechanical analogue of the phase space area in Eq. (5) and the von Neumann entropy for the system in Eq. (4).

Let us finally remark that our Kadanoff-Baym equations (56) can also be obtained starting from the Feynman-Vernon Gaussian path integral exponential obtained in [18] that is normally used to derive the perturbative master equation. For example, after integrating out the environment at one loop order (which is usually a first step in deriving a master equation), the 1PI equations of motion can be obtained from the effective action:

$$S_S[x^+] - S_S[x^-] - \int_0^\infty dt_1 \mathcal T[x^+(t_1)) - x^-(t_1)] + \imath\int_0^\infty dt_1 \int_0^{t_1} dt_2[x^+(t_1)-x^-(t_1)]\nu(t_1-t_2)[x^+(t_2)-x^-(t_2)]$$
$$+ \int_0^\infty dt_1\int_0^{t_1} dt_2[x^+(t_1)-x^-(t_1)]\eta(t_1-t_2)[x^+(t_2)+x^-(t_2)] = S_S[x^+] - S_S[x^-]$$
$$- \int_0^\infty dt_1 \mathcal T[x^+(t_1)-x^-(t_1)] - \frac{1}{2}\sum_{a,b=\pm}ab\int_0^\infty dt_1\int_0^\infty dt_2 x^a(t_1)\imath M^{ab}(t_1;t_2)x^b(t_2).$$
(67)

Here, M^{ab} are the self-masses that can be read off from Eqs. (55) and (24), $S_S[x^\pm]$ is the free action defined by Eq. (52) and η and ν, or, equivalently, the causal and statistical self-masses, are given in Eq. (61). In the equation above, $\mathcal T$ denotes the tadpole contribution, which does not affect the entropy [2], and reads

$$\mathcal{T} = \sum_{n=1}^{N} \frac{h_n}{4\omega_n} \coth\left(\frac{\beta\omega_n}{2}\right), \qquad (68)$$

which is easily inferred from the interaction term in (52) and (53a). The quantum corrected equation of motion for $x(t)$ follows straightforwardly by variation of Eq. (67) with respect to $x^{\pm}(t)$, and setting $x^{\pm}(t)$ equal to $x(t)$. More generally, if one would introduce nonlocal sources for two-point functions in the Feynman-Vernon path integral, one would obtain the Kadanoff-Baym equations in (56).

C. Time evolution of the entropy in quantum mechanics

Let us discuss our results. It is important to distinguish between the so-called resonant regime and nonresonant regime [3]. In the former, we have that one or more $\omega_n \simeq \omega_0/2$, with $1 \leq n \leq N$. In the nonresonant regime all environmental frequencies differ significantly from $\omega_0/2$ and are as a consequence effectively decoupled from the system oscillator. If one wants to study the efficient decoherence of such a system, the nonresonant regime is not the relevant regime to consider.

In Fig. 1 we show the Gaussian von Neumann entropy resulting from the Kadanoff-Baym equations and from the perturbative master equation as a function of time in black and gray, respectively. At the moment, we consider just one environmental oscillator $N = 1$ in the nonresonant regime. Here, the two entropies agree nicely up to the expected perturbative corrections due to the inappropriate resummation scheme of the perturbative master equation to which we will return shortly. However, let us now consider Fig. 2 where we study the resonant regime for $N = 1$. Clearly, the entropy resulting from the master equation breaks down and suffers from physically unacceptable secular growth. The behavior of the Gaussian von Neumann entropy from the Kadanoff-Baym equations is perfectly stable. Moreover, given the weak coupling $h/\omega_0^3 = 0.1$, we do not observe perfect thermalization (indicated by the dashed black line).
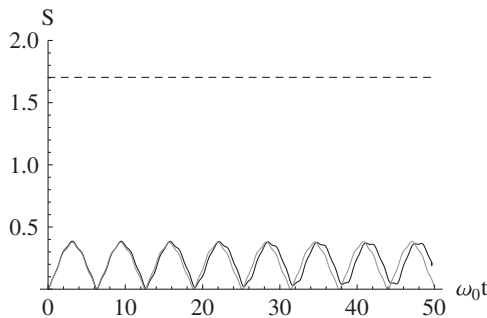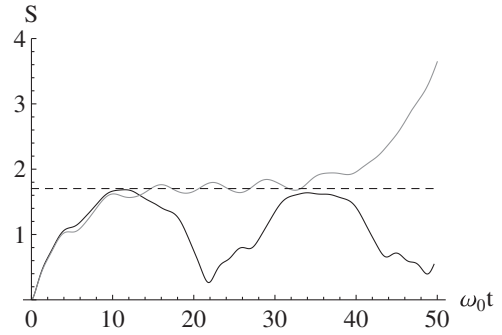

FIG. 2. Entropy as a function of time for $N = 1$ in the resonant regime. The Gaussian von Neumann entropy (black) shows a stable behavior, unlike the entropy from the perturbative master equation (gray) that reveals unphysical secular growth. The dashed line indicates full thermalization of x. We use $\omega_1/\omega_0 = 0.53$, $h/\omega_0^3 = 0.1$, and $\beta\omega_0 = 1/2$.

If we consider $N = 50$ environmental oscillators, the qualitative picture does not change. In Fig. 3 we show the evolution of the two entropies in the nonresonant regime, and in Fig. 4 in the resonant regime. The entropy from the perturbative master equation blows up as before, whereas the Gaussian von Neumann entropy is stable. In Fig. 3 we randomly select 50 frequencies in the interval [2, 4] which is what we denote by $\omega_n/\omega_0 \in [2, 4]$. In the resonant regime we use $\omega_n/\omega_0 \in [0.5, 0.6]$. The breakdown of the perturbative master equation in this regime is generic.

Just as discussed in [3], energy is conserved in our model such that the Poincaré recurrence theorem applies. This theorem states that our system will after a sufficiently long time return to a state arbitrarily close to its initial state. The Poincaré recurrence time is the amount of time this takes. Compared to the $N = 1$ case we previously considered, we observe for $N = 50$ in Fig. 4 that the Poincaré recurrence time has increased. Thus, by including more and more oscillators, decoherence becomes rapidly more irreversible, as one would expect. If we extend this discussion to
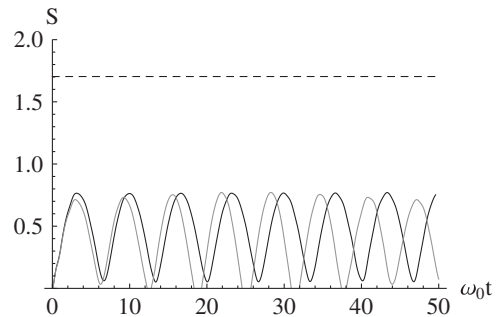

FIG. 1. Entropy as a function of time for $N = 1$ in the non-resonant regime. The Gaussian von Neumann entropy (black) agrees with the entropy from the master equation (gray) up to the expected perturbative corrections. The dashed line indicates full thermalization of x. We use $\omega_1/\omega_0 = 2$, $h/\omega_0^3 = 1$, and $\beta\omega_0 = 1/2$.

FIG. 3. Entropy as a function of time for $N = 50$ in the non-resonant regime. The Gaussian von Neumann entropy (black) agrees with the entropy from the master equation (gray) up to the expected perturbative corrections. We use $\omega_n/\omega_0 \in [2, 4]$, $1 \leq n \leq N$, $h/\omega_0^3 = 1/2$, and $\beta\omega_0 = 1/2$.
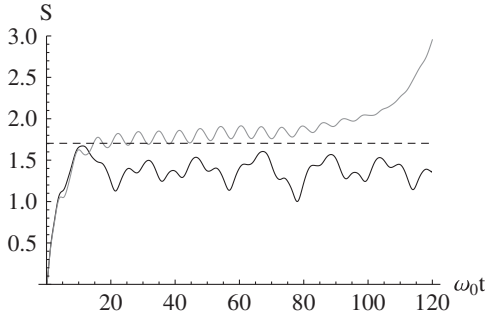
FIG. 4. Entropy as a function of time for $N = 50$ in the resonant regime. The Gaussian von Neumann entropy (black) yields a stable behavior in time, unlike the entropy from the perturbative master equation (gray) that reveals secular growth. We use $\omega_n/\omega_0 \in [0.5, 0.6]$, $1 \leq n \leq N$, $h/\omega_0^3 = 0.015$, and $\beta\omega_0 = 1/2$.

field theory, where several modes couple due to the loop integrals (hence $N \to \infty$), we conclude that clearly our Poincaré recurrence time becomes infinite. Hence, the entropy increase has become irreversible for all practical purposes and our system has (irreversibly) decohered.

In decoherence studies, one is usually interested in extracting two quantitative results: the decoherence rate and the total amount of decoherence. As emphasized before, we take the point of view that the Gaussian von Neumann entropy should be used as *the* quantitative measure of decoherence, as it is an invariant measure of the phase space occupied by a state. Hence, the rate of change of the phase space area (or entropy) is the decoherence rate and the total amount of decoherence is the total (average) amount of entropy that is generated at late times. This is to be contrasted with most of the literature [15] where noninvariant measures of decoherence are used. The statement regarding the decoherence rate we would like to make here, however, is that our Gaussian von Neumann entropy and the entropy resulting from the master equation would give the same result as their early times evolution coincides. The master equation does however not predict the total amount of decoherence accurately. In the resonant regime the entropy following from the perturbative master equation blows up at late times and, consequently, fails to accurately predict the total amount of decoherence that has taken place. Our correlator approach to decoherence does not suffer from this fatal shortcoming.

D. Deriving the master equation from the Kadanoff-Baym equations

The secular growth is caused by the perturbative approximations used in deriving the master equation (65). The coefficients appearing in the master equation diverge when $\omega_n = \omega_0/2$ which can be appreciated from Eq. (63). However, there is nothing nonperturbative about the resonant regime. Our interaction coefficient h is still very small such that the self-mass corrections to ω_0^2 are tiny.

Here we outline the perturbative approximations that cause the master equation to fail. In order to do this, we simply derive the master equation from the Kadanoff-Baym equations by making the appropriate approximations. Of course, Eq. (65a) is trivial to prove. The Kadanoff-Baym equations are given in Eq. (56) and contain memory integrals over the causal and statistical propagators. We make the approximation to use the free equation of motion for the causal propagator appearing in the memory integrals according to which

$$(\partial_t^2 + \omega_0^2)\Delta_x^{c,\text{free}}(t, t') = 0. \tag{69}$$

This equation is trivially solved in terms of sines and cosines. Let us thus impose initial conditions at $t = t'$ as follows:

$$\Delta_x^c(t, t') \simeq \Delta_x^{c,\text{free}}(t, t')$$
$$= \cos(\omega_0\Delta t)\Delta_x^c(t', t') + \frac{1}{\omega_0}\sin(\omega_0\Delta t)\partial_t\Delta_x^c(t, t')|_{t=t'}$$
$$= -\frac{1}{\omega_0}\sin(\omega_0\Delta t). \tag{70}$$

Here, $\Delta t = t - t'$. We relied upon some basic properties of the causal propagator [see e.g. Eq. (78) in the next section]. Likewise, we approximate the statistical propagator appearing in the memory integrals as

$$F_x(t, t') \simeq F_x^{\text{free}}(t, t')$$
$$= \cos(\omega_0\Delta t)F_x(t', t') + \frac{1}{\omega_0}\sin(\omega_0\Delta t)\partial_t F_x(t, t')|_{t=t'}$$
$$= \cos(\omega_0\Delta t)\langle\hat{x}^2(t')\rangle + \frac{1}{\omega_0}\sin(\omega_0\Delta t)\left\langle\frac{1}{2}\{\hat{x}, \hat{p}\}(t')\right\rangle, \tag{71}$$

where we inserted how our statistical propagator can be related to our three Gaussian correlators, from the quantum mechanical version of Eq. (9). Note that expression (71) is not symmetric under exchange of t and t', whereas the statistical propagator as obtained from e.g. the Kadanoff-Baym equations of course respects this symmetry.

Now, we send $t' \to t$ in the Kadanoff-Baym equations and carefully relate the statistical propagator and derivatives thereof to quantum mechanical expectation values. From Eq. (56a), where we change variables to $\tau = t - t_1$, it thus follows that

$$\partial_t^2 F_x(t, t')|_{t=t'} = -\omega_0^2\langle\hat{x}^2(t)\rangle - \int_0^t d\tau M_x^F(\tau, 0)\frac{\sin(\omega_0\tau)}{\omega_0}$$
$$+ \langle\hat{x}^2(t)\rangle \int_0^t d\tau M_x^c(\tau, 0)\cos(\omega_0\tau)$$
$$- \left\langle\frac{1}{2}\{\hat{x}, \hat{p}\}(t)\right\rangle \int_0^t d\tau M_x^c(\tau, 0)\frac{\sin(\omega_0\tau)}{\omega_0}. \tag{72}$$

Using Eqs. (61), (63), and (72) above reduces to (65c):

$$\frac{d\langle\frac{1}{2}\{\hat{x}, \hat{p}\}\rangle}{dt} = -(\omega_0^2 + \Omega^2)\langle\hat{x}^2\rangle + \langle\hat{p}^2\rangle - f(t)$$
$$- 2\gamma(t)\left\langle\frac{1}{2}\{\hat{x}, \hat{p}\}\right\rangle.$$

Here, we used the identities derived in Eq. (61) that relate the noise and dissipation kernels of the master equation to our causal and statistical self-mass. In order to derive the final master equation for the correlator $\langle\hat{p}^2\rangle$, we have to use the following subtle argument:

$$\partial_t^2\partial_{t'}F(t, t')|_{t=t'} = \frac{1}{2}\frac{d}{dt}\langle\hat{p}^2(t)\rangle. \tag{73}$$

In order to derive its corresponding differential equation, we thus have to act with $\partial_{t'}$ on Eq. (56a) and then send $t' \to t$. As an intermediate step, we can present

$$\frac{1}{2}\frac{d}{dt}\langle\hat{p}^2(t)\rangle$$
$$= -\omega_0^2\left\langle\frac{1}{2}\{\hat{x}, \hat{p}\}(t)\right\rangle - \int_0^t d\tau M_x^F(\tau, 0)\cos(\omega_0\tau)$$
$$+ \int_0^t d\tau M_x^c(\tau, 0)\left[-\omega_0\sin(\omega_0\tau)\langle\hat{x}^2(t')\rangle + \cos(\omega_0\tau)\right.$$
$$\left.\times\left\langle\frac{1}{2}\{\hat{x}, \hat{p}\}(t')\right\rangle - \frac{\sin(\omega_0\tau)}{\omega_0}\partial_{t'}\{\partial_tF_x(t, t')|_{t=t'}\}\right], \tag{74}$$

where we still have to send $t' \to t$ on the right-hand side. Now, one can use

$$\partial_{t'}\{\partial_tF_x(t, t')|_{t=t'}\} = \langle\hat{p}^2(t')\rangle - \omega_0^2\langle\hat{x}^2(t')\rangle. \tag{75}$$

In the light of Eqs. (61) and (63), Eq. (74) simplifies to Eq. (65b),

$$\frac{d\langle\hat{p}^2\rangle}{dt} = -2(\omega_0^2 + \Omega^2)\left\langle\frac{1}{2}\{\hat{x}, \hat{p}\}\right\rangle - 4\gamma(t)\langle\hat{p}^2\rangle + 2D(t).$$

We thus conclude that we can derive the master equation for the correlators from the Kadanoff-Baym equations using the perturbative approximation in Eqs. (70) and (71). Clearly, this approximation invalidates the intricate resummation techniques of the quantum field theoretical 2PI scheme. In the 2PI framework, one resums an infinite number of Feynman diagrams in order to obtain a stable and thermalized late time evolution. By approximating the memory integrals in the Kadanoff-Baym equations, the master equation spoils this beautiful property.

The derivation presented here can be generalized to quantum field theory. By using similar approximations, one can thus derive the renormalized correlator equations that would follow from the perturbative master equation.

V. RESULTS: ENTROPY GENERATION IN QUANTUM FIELD THEORY

Let us now return to field theory and solve for the statistical propagator and hence fix the Gaussian von Neumann entropy of our system. For completeness, let us here just once more recall Eq. (26) and (28) for the causal and statistical propagator:

$$(\partial_t^2 + k^2 + m_\phi^2)\Delta_\phi^c(k, t, t') - (\partial_t^2 + k^2)\int_{t'}^t dt_1 Z_\phi^c(k, t, t_1)\Delta_\phi^c(k, t_1, t') - \int_{t'}^t dt_1 M_{\phi,\text{th}}^c(k, t, t_1)\Delta_\phi^c(k, t_1, t') = 0 \tag{76a}$$

$$(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k, t, t') - (\partial_t^2 + k^2)\left[\int_{-\infty}^{t_0} dt_1 Z_\phi^c(k, t, t_1)F_\phi^{\text{free}}(k, t_1, t') + \int_{t_0}^t dt_1 Z_\phi^c(k, t, t_1)F_\phi(k, t_1, t')\right.$$
$$- \int_{-\infty}^{t_0} dt_1 Z_\phi^F(k, t, t_1)\Delta_\phi^{c,\text{free}}(k, t_1, t') - \left.\int_{t_0}^{t'} dt_1 Z_\phi^F(k, t, t_1)\Delta_\phi^c(k, t_1, t')\right] - \int_{-\infty}^{t_0} dt_1 M_{\phi,\text{th}}^c(k, t, t_1)F_\phi^{\text{free}}(k, t_1, t')$$
$$- \int_{t_0}^t dt_1 M_{\phi,\text{th}}^c(k, t, t_1)F_\phi(k, t_1, t') + \int_{-\infty}^{t_0} dt_1 M_{\phi,\text{th}}^F(k, t, t_1)\Delta_\phi^{c,\text{free}}(k, t_1, t') + \int_{t_0}^{t'} dt_1 M_{\phi,\text{th}}^F(k, t, t_1)\Delta_\phi^c(k, t_1, t') = 0. \tag{76b}$$

We use all self-masses calculated previously: we need the vacuum self-masses in Eq. (25), one of the two following infinite past memory kernels in Eq. (30) or (31) depending on the initial conditions chosen, the thermal causal self-mass in (36), the vacuum-thermal contribution to the statistical self-mass in Eq. (43), and finally the high temperature or low temperature contribution to the thermal-thermal statistical self-mass in Eq. (50) or (48). We are primarily interested in two cases, a constant mass for our system field and a changing one:

$$m_\phi(t) = m_0 = \text{const} \tag{77a}$$
$$m_\phi^2(t) = A + B\tanh(\rho\{t - t_m\}), \tag{77b}$$

where we let A and B take different values. Also, t_m is the time at which the mass changes, which we take to be $\rho t_m = 30$. Let us outline our numerical approach. In the code, we take $t_0 = 0$ and we let ρt and $\rho t'$ run between 0 and 100, for example. As in the vacuum case, we first need to determine the causal propagator, as it enters the equation of motion of the statistical propagator. The boundary conditions for determining the causal propagator are as follows:

$$\Delta_\phi^c(t, t) = 0 \tag{78a}$$
$$\partial_t\Delta_\phi^c(t, t')|_{t=t'} = -1. \tag{78b}$$

Condition (78a) has to be satisfied by definition and condition (78b) follows from the commutation relations.

Once we have solved for the causal propagator, we can consider evaluating the statistical propagator. As in the $T = 0$ case, the generated entropy is a constant which can be appreciated from a rather simple argument [1]. When $m_\phi = $ const, we have $F_\phi(k, t, t') = F_\phi(k, t - t')$ such that quantities like

$$F_\phi(k, 0) = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} F_\phi(k^\mu) \qquad (79a)$$

$$\partial_t F_\phi(k, \Delta t)|_{\Delta t = 0} = -\imath \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} k^0 F_\phi(k^\mu) \qquad (79b)$$

$$\partial_{t'} \partial_t F_\phi(k, \Delta t)|_{\Delta t = 0} = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} k_0^2 F_\phi(k^\mu) \qquad (79c)$$

are time independent. Consequently, the phase space area Δ_k is constant, and so is the generated entropy. If our initial conditions differ from these values, we expect to observe some transient dependence. This entropy is thus the interacting thermal entropy. The total amount of generated entropy measures the total amount of decoherence that has occurred. Given a temperature T, the thermal entropy provides a good estimate of the maximal amount of entropy that can be generated (perfect decoherence), however depending on the particular parameters in the theory this maximal amount of entropy need not always be reached (imperfect decoherence). Effectively, the interaction opens up phase space for the system field implying that less information about the system field is accessible to us and hence we observe an increase in entropy. In order to evaluate the integrals above, we need the statistical propagator in Fourier space,

$$F_\phi(k^\mu) = \frac{1}{2} \frac{\imath M^{+-}(k^\mu) + \imath M^{-+}(k^\mu)}{\imath M^{+-}(k^\mu) - \imath M^{-+}(k^\mu)}$$
$$\times \left[\frac{\imath}{k_\mu k^\mu + m_\phi^2 + \imath M^r(k^\mu)} - \frac{\imath}{k_\mu k^\mu + m_\phi^2 + \imath M^a(k^\mu)} \right]. \qquad (80)$$

Here, $\imath M^r$ and $\imath M^a$ are the retarded and advanced self-masses, respectively. All the self-masses in Fourier space in this expression are derived in Appendix B. The discussion above is important for understanding how to impose boundary conditions for the statistical propagator at t_0. We impose either so-called "pure state initial conditions" or "mixed state initial conditions." If we constrain the statistical propagator to occupy the minimal allowed phase space area initially, we impose pure state initial conditions and set

$$F_\phi(t_0, t_0) = \frac{1}{2\omega_{\text{in}}} \qquad (81a)$$

$$\partial_t F_\phi(t, t_0)|_{t = t_0} = 0 \qquad (81b)$$

$$\partial_{t'} \partial_t F_\phi(t, t')|_{t = t' = t_0} = \frac{\omega_{\text{in}}}{2}, \qquad (81c)$$

where ω_{in} refers to the initial mass $m_\phi(t_0)$ of the field if the mass changes throughout the evolution. This yields $\Delta_k(t_0) = 1$ such that

$$S_k(t_0) = 0. \qquad (82)$$

Initially, we thus force the field to occupy the minimal area in phase space. Clearly, if we constrain our field to be in such an out-of-equilibrium state initially, we should definitely not include all memory kernels pretending that our field has already been interacting from negative infinity to t_0. Otherwise, our field would have thermalized long before t_0 and could have never begun the evolution in its vacuum state. If we thus impose pure state initial conditions, we must drop the "thermal memory kernels",

$$\int_{-\infty}^{t_0} dt_1 M_{\phi, \text{th}}^c(k, t, t_1) F_\phi^{\text{free}}(k, t_1, t') \quad \text{and}$$
$$\int_{-\infty}^{t_0} dt_1 M_{\phi, \text{th}}^F(k, t, t_1) \Delta_\phi^{c, \text{free}}(k, t_1, t'), \qquad (83)$$

but rather keep the "vacuum memory kernels" in Eq. (76b), which are the other two memory kernels involving free propagators. We evaluated the relevant integrals in closed form in Eq. (31). This setup roughly corresponds to switching on the coupling h adiabatically slowly at times before t_0. At t_0, the temperature of the environment is suddenly switched on such that the system responds to this change from t_0 onward. Note that if we would not include any memory effects and switch on the coupling h nonadiabatically at t_0, the pure state initial conditions would correspond to the physically natural choice. This would however also instantaneously change the vacuum of our theory, and we would thus need to renormalize our theory both before and after t_0 separately. Including the vacuum memory kernels is thus essential, as it ensures that our evolution is completely finite at all times without the need for time dependent counterterms.[2]

Second, we can impose mixed state boundary conditions, where we use the numerical values for the statistical propagator and its derivatives calculated from Eqs. (79) and (80), such that we have $\Delta_k(t_0) = \Delta_{\text{ms}} = $ const and

$$S_k(t_0) = S_{\text{ms}} > 0, \qquad (84)$$

where we use the subscript "ms" to denote "mixed state." In other words, we constrain our system initially to be in

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[2] In [76,77] the renormalization of fermions in an expanding universe is investigated where a similar singularity at the initial time t_0 is encountered. It could in their case however be removed by a suitably chosen Bogoliubov transformation.

the interacting thermal state and S_{ms} is the value of the interacting thermal entropy. The integrals in Eq. (79) can now be evaluated numerically to yield the appropriate initial conditions. For example when $\beta\rho = 1/2$, $k/\rho = 1$, $m_\phi/\rho = 1$, and $h/\rho = 3$, we find

$$F_\phi(k/\rho = 1, \Delta t)|_{\Delta t=0} = 1.898\,85 \qquad (85a)$$

$$\partial_t F_\phi(k/\rho = 1, \Delta t)|_{\Delta t=0} = 0 \qquad (85b)$$

$$\partial_{t'}\partial_t F_\phi(k/\rho = 1, \Delta t)|_{\Delta t=0} = 2.089\,41. \qquad (85c)$$

Clearly, Eq. (85b) always vanishes as the integrand is an odd function of k^0. The numerical value of the phase space area in this case follows from Eqs. (85) and (5) as

$$\Delta_{ms} = 3.983\,71. \qquad (86)$$

The interacting thermal entropy hence reads

$$S_{ms} = 1.678\,36. \qquad (87)$$

The mixed state initial condition basically assumes that our system field has already equilibrated before t_0 such that the entropy has settled to the constant mixed state value. In this case, we include of course both the vacuum memory kernels and the thermal memory kernels.[3]

A few more words on the memory kernels for the mixed state boundary conditions are in order. For the vacuum memory kernels, we of course use Eq. (30). It is unfortunately not possible to evaluate the thermal memory kernels in closed form too. The two integrals in Eq. (83) have to be evaluated numerically as a consequence. One can numerically verify that the integrands are highly oscillatory and do not settle quickly to some constant value for each t and t' due to the competing frequencies ω and k. We chose to integrate from -300 to $t_0\rho = 0$ and smooth out the remaining oscillations of the integral by defining a suitable average over half of the period of the oscillations.

Finally, let us outline the numerical implementation of the Kadanoff-Baym equations (76). Solving for the causal propagator is straightforward as Eq. (78) provides us (for each t') with two initial conditions at $t = t'$ and at $t = t' + \Delta t$, where Δt is the numerical step size. We can thus solve the causal propagator as a function of t for each fixed t'. Solving for the statistical propagator is somewhat more subtle. The initial conditions, e.g. in Eq. (81),

[3]Let us make an interesting theoretical observation that to our knowledge would apply for any interacting system in quantum field theory. Suppose our coupling h would be time independent. Suppose also that the system field ϕ and the environment field χ form a closed system together. Now, imagine that we are interested in the time evolution of the entropy at some finite time t_0. Our system field has then already been interacting with the environment at times before t_0 such that one can expect that our system has equilibrated at t_0. Hence, to allow out-of-equilibrium initial conditions, one must always change the theory slightly. The possibility that we advocate is to drop those memory kernels that do not match the chosen initial condition. In this way, the evolution history of our field is consistent.

for a given choice of parameters only fix $F_\phi(t_0, t_0)$, $F_\phi(t_0 + \Delta t, t_0) = F_\phi(t_0, t_0 + \Delta t)$, and $F_\phi(t_0 + \Delta t, t_0 + \Delta t)$. This is sufficient to solve for $F_\phi(t, t_0)$ and $F_\phi(t, t_0 + \Delta t)$ as functions of time for fixed $t' = t_0$ and $t' = t_0 + \Delta t$. Now, we can use the symmetry relation $F_\phi(t, t') = F_\phi(t', t)$ such that we can also find $F_\phi(t_0, t')$ and $F_\phi(t_0 + \Delta t, t')$ as functions of t' for fixed $t = t_0$ and $t = t_0 + \Delta t$. The latter step provides us with the initial data that are sufficient to find $F_\phi(t, t')$ as a function of t for each fixed t'.

Once we have solved for the statistical propagator, our work becomes much easier as we can immediately find the phase space area via relation (5). The phase space area fixes the entropy.

A. Evolution of the entropy: Constant mass

Let us first turn our attention to Fig. 5. This plot shows the phase space area as a function of time at a fairly low temperature $\beta\rho = 2$. Starting at $\Delta_k(t_0) = 1$, its evolution settles precisely to Δ_{ms}, indicated by the dashed black line, as one would expect. From the evolution of the phase space area, one readily finds the evolution of the entropy as a function of time in Fig. 6.

At a higher temperature, $\beta\rho = 1/2$, we observe in Figs. 7 and 8 that the generated phase space area and
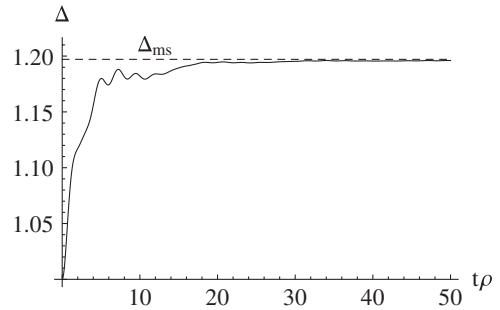

FIG. 5. Phase space area as a function of time. It settles nicely to Δ_{ms}, indicated by the dashed black line. We use $\beta\rho = 2$, $k/\rho = 1$, $m_\phi/\rho = 1$, $h/\rho = 4$, and a total number of steps $N = 2000$ up to $t\rho = 100$.
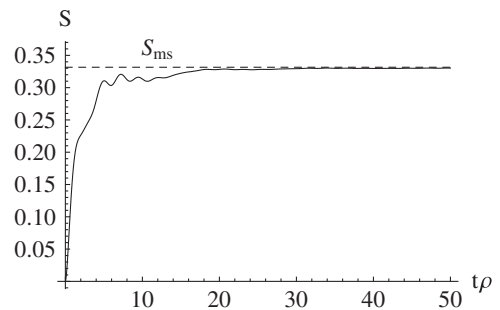

FIG. 6. Entropy as a function of time. The evolution of the entropy is obtained from the phase space area in Fig. 5. For $50 < t\rho < 100$, the entropy continues to coincide with S_{ms}.
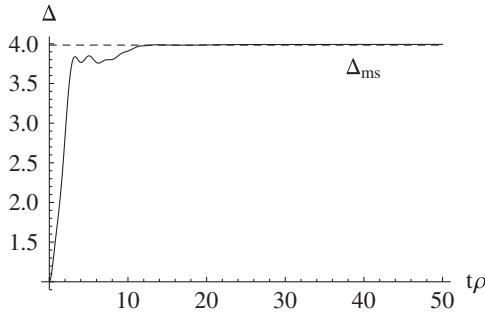
FIG. 7. Phase space area as a function of time. At high temperatures, we see that the phase space area settles quickly again to Δ_{ms}. We use $\beta\rho = 1/2$, $k/\rho = 1$, $m_\phi/\rho = 1$, $h/\rho = 3$, and $N = 2000$ up to $t\rho = 100$.
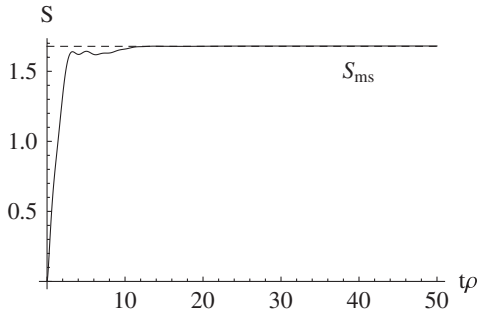

FIG. 8. Entropy as a function of time, which follows again from the evolution of the phase space area as a function of time as depicted in Fig. 7. Our pure state quickly appears to our observer as a mixed state with a large positive entropy S_{ms}.

entropy as a function of time is larger. This can easily be understood by realizing that the thermal value of the entropy, set by the environment, provides us with a good estimate of the maximal amount of decoherence that our system can experience. Again we observe an excellent agreement between Δ_{ms} or S_{ms} and the corresponding numerical evolution.

Let us now discuss Fig. 9. Here, we show two separate cases for the evolution of the entropy: one at a very low temperature $\beta\rho = 10$ (in black) and one vacuum evolution $\beta\rho = \infty$ (in gray) which we already calculated in [1]. As we would intuitively expect, we see that the former case settles to an entropy $S_{ms} = 0.045\,51$ that is slightly above the vacuum asymptote $S_{ms} = 0.043\,26$.

Finally, in Fig. 10 we show the interacting phase space area Δ_{ms} as a function of the coupling h. For $h/\rho \ll 1$, we see that Δ_{ms} approaches the free thermal phase space area $\Delta_{free} = \coth(\beta\omega/2)$. For larger values of the coupling, we see that $\Delta_{ms} > \Delta_{free}$. If these two differ significantly, we enter the nonperturbative regime. In the perturbative regime, this plot substantiates our earlier statement that the free thermal entropy Δ_{free} provides us with a good estimate of the total amount of decoherence that our system can experience. Our system however thermalizes to Δ_{ms}, and
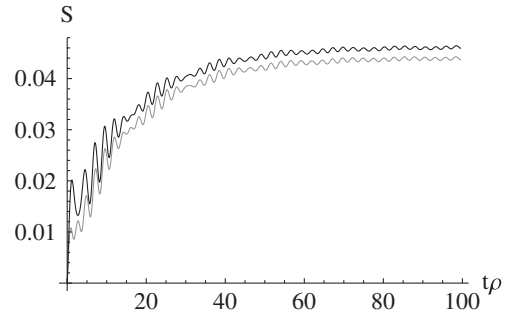

FIG. 9. Entropy as a function of time. The evolution at very low temperatures $\beta\rho = 10$ (black) resembles the vacuum evolution $\beta\rho = \infty$ (gray) from [1] as one would intuitively expect. We furthermore use $k/\rho = 1$, $m_\phi/\rho = 1$, $h/\rho = 4$, and $N = 2000$ up to $t\rho = 100$.
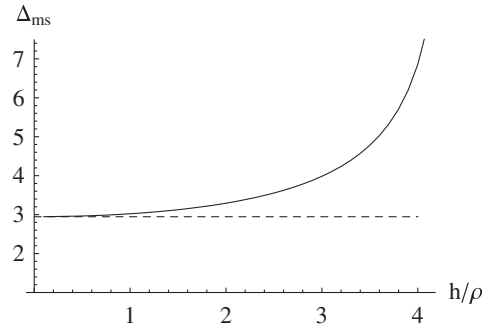

FIG. 10. We show Δ_{ms}, the interacting thermal phase space area, as a function of h/ρ. For $h/\rho \ll 1$, we see that Δ_{ms} is almost equal to the free phase space area $\Delta_{free} = \coth(\beta\omega/2)$ indicated by the dashed black line. For larger values of h/ρ we approach the nonperturbative regime. We use $\beta\rho = 1/2$, $k/\rho = 1$ and $m_\phi/\rho = 1$.

not to Δ_{free} as the interaction changes the nature of the free thermal state.

The most important point of the results shown here is that, although a pure state with vanishing entropy $S_k = 0$ remains pure under unitary evolution, we perceive this state over time as a mixed state with positive entropy $S_{ms} > 0$ as non-Gaussianities are generated by the evolution (both in the correlation between the system and environment as well as higher order correlations in the system itself) and subsequently are neglected in our definition of the Gaussian von Neumann entropy. The total amount of decoherence corresponds to the interacting thermal entropy S_{ms}.

B. Decoherence rates

As the Gaussian von Neumann entropy in Eq. (4) is the only invariant measure of the entropy of a Gaussian state, we take the point of view that this quantity, or equivalently the phase space area in Eq. (5), should be taken as the quantitative measure for decoherence. This agrees with the general view on decoherence according to which

the decoherence rate is the rate at which a system in a pure state evolves into a mixed state due to its interaction with an environment. This is to be contrasted with some of the literature where different, noninvariant measures are proposed [5,15]. For example in [15,78], the superposition of two minimum uncertainty Gaussian states located at positions x and x' is considered. The decoherence rate is defined differently, i.e., it is the characteristic time scale at which the off-diagonal contributions in the total density matrix decay and coincides with the time scale at which the interference pattern in the Wigner function decays. It is given by

$$\tau_D^{-1} = \gamma\left(\frac{x - x'}{\lambda_T}\right)^2, \qquad (88)$$

where the thermal de Broglie wavelength is given by $\lambda_T = (2mk_BT)^{-1/2}$. In other words, according to [15], the decoherence rate depends on the spatial separation $x - x'$ of the two Gaussians. Note that in quantum field theory the expression would generalize to $\tau_D^{-1} \propto (\phi - \phi')^2$. This is just one example; one can find other definitions of decoherence in the literature.

The main difference is that our decoherence rate does not depend on the configuration space variables x or ϕ but is an intrinsic property of the state. In other words, we do not look at different spatial regions of the state, but rather to the state as a whole from which we extract one decoherence rate. As we outlined in [2], a nice intuitive way to visualize the process of decoherence is in Wigner space. The Wigner transform of a density matrix coincides with the Fourier transform with respect to its off-diagonal entries. As discussed previously, the phase space area measures the area the state occupies in Wigner space in units of the minimum phase space area $\hbar/2$, which we refer to as the statistical particle number n. The pure state considered in the previous subsection decoheres and its phase space area increases to approximately its thermal value. When $\Delta \gg 1$ ($n \gg 1$), different regions in phase space of area $\hbar/2$ are, to a good approximation, not correlated and thus evolve independently. As we have considered Gaussian states only and not the superposition of two spatially separated Gaussians, which when considered together is in fact a highly non-Gaussian state, a direct comparison is not straightforward.

Let us extract the decoherence rate from the evolution of the entropy. We define the decoherence time scale to be the characteristic time it takes for the phase space area $\Delta_k(t)$ to settle to its constant mixed state value Δ_{ms}. The phase space area approaches the constant asymptotic value in an exponential manner,

$$\frac{d}{dt}\delta\Delta_k(t) + \Gamma_{\mathrm{dec}}\delta\Delta_k(t) = 0, \qquad (89)$$

where $\delta\Delta_k(t) = \Delta_{\mathrm{ms}} - \Delta_k(t)$ and where Γ_{dec} is the decoherence rate. This equation is equivalent to $\dot{n}_k = -\Gamma_{\mathrm{dec}}(n_k - n_{\mathrm{ms}})$, where n_k is defined in Eq. (8) and n_{ms} is the stationary n corresponding to Δ_{ms}. As in the vacuum
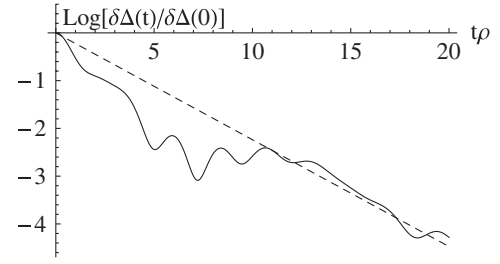

FIG. 11. Decoherence rate at low temperatures. We show the exponential approach to Δ_{ms} in solid black and the corresponding decoherence rate given in Eq. (91) (dashed line). We use the phase space area from Fig. 5.

case [1], we anticipate that the decoherence rate is given by the single particle decay rate of the interaction $\phi \to \chi^2$. The single particle decay rate reads[4]

$$\Gamma_{\phi \to \chi\chi} = -\left.\frac{\mathrm{Im}(\iota M_\phi^r)}{\omega_\phi}\right|_{k^0 = \omega_\phi}$$

$$= \frac{h^2}{32\pi\omega_\phi} + \frac{h^2}{16\pi k\beta\omega_\phi}\log\left(\frac{1 - e^{-(\beta/2)(\omega_\phi + k)}}{1 - e^{-(\beta)/(2)(\omega_\phi - k)}}\right),$$

$$\qquad (90)$$

where we used the retarded self-mass in Fourier space in Eq. (B3a) and several relevant self-masses in Appendix B. Let us briefly outline the steps needed to derive the result above. In order to calculate $\iota M_\phi^r(k^\mu)$, we use $\iota M_{\phi,\mathrm{vac}}^{++}(k^\mu)$ in Eq. (B6a) and $\iota M_{\phi,\mathrm{vac}}^{+-}(k^\mu)$ in Eq. (B12). There are no thermal-thermal contributions to $\iota M_\phi^r(k^\mu)$ which can be appreciated from Eq. (B10). Finally, in order to derive the vacuum-thermal contribution, let us recall Eq. (24c) given by $M_\phi^{++}(k, t, t') = M_\phi^F(k, t, t') + \mathrm{sgn}(t - t')\iota M_\phi^c(k, t, t')/2$. We clearly need the vacuum-thermal contribution to $M_\phi^F(k^\mu)$ which is given in Eq. (B14). The imaginary part of the second term vanishes, which can be seen by making use of an inverse Fourier transform, as in Eq. (B17) and in the first line of Eq. (B18). This fixes $\iota M_\phi^r(k^\mu)$ completely.

One should calculate the imaginary part of the retarded self-mass as it characterizes our decay process, which follows from Eq. (80). In order to calculate the decay rate, we have to project the retarded self-mass on the quasiparticle shell $k^0 = \omega_\phi$. Of course, one should really take the perturbative correction to the dispersion relation of order $\mathcal{O}(h^2/\omega_\phi^2)$ into account but this effect is rather small. Alternatively, we can project the advanced self-mass in Fourier space on $k^0 = -\omega_\phi$. We thus expect

$$\Gamma_{\mathrm{dec}} \simeq \Gamma_{\phi \to \chi\chi}. \qquad (91)$$

Let us examine Figs. 11 and 12. From our numerical calculation, we can thus easily find $\delta\Delta_k(t) = \Delta_{\mathrm{ms}} - \Delta_k(t)$ which we show in solid black. We can now compare

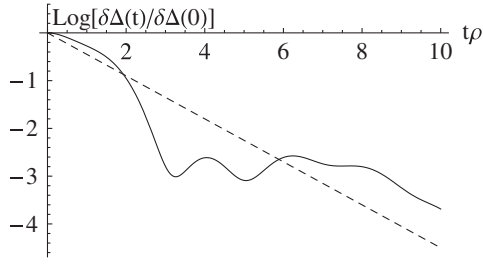---

[4]For cases where $m_\chi \neq 0$, see [79].

FIG. 12. Decoherence rate at high temperatures. We show the exponential approach to Δ_{ms} in solid black and the corresponding decoherence rate given in Eq. (91) (dashed line). We use the phase space area from Fig. 7.

with the single particle decay rate in Eq. (90) and plot $-\Gamma_{\phi \to \chi\chi} t$. We conclude that the decoherence rate can be well described by the single particle decay rate in our model, thus confirming Eq. (91) above.

C. The emerging $k^0 = 0$ shell

To develop some intuition, we depict $F_\phi(k^\mu)$ as a function of k^0 keeping various other parameters fixed. In the vacuum $\beta\rho = \infty$, it is clear from the analytic form of the statistical propagator that a $k^0 = 0$ shell does not exist. In the vacuum, we have that $F_\phi(k^\mu) = 0$ for $|k^0| \le k$. At low temperatures, $\beta\rho = 2$, we observe in Fig. 13 that two more quasiparticle peaks emerge, where $|k^0| \le k$. The original quasi particle peaks at $|k^0| \simeq \omega_\phi$ however still dominate. At high temperatures, $\beta\rho = 0.1$, we observe in Fig. 14 that the two additional quasi particle peaks already present at lower temperatures increase in size and move closer to $k^0 = 0$, where they overlap. The original quasiparticle peaks located at $|k^0| \simeq \omega_\phi$ broaden as the interaction strength h increases. Moreover, for increasing h, the original quasiparticle peaks get dwarfed by the new quasiparticle peaks at $|k^0| \le k$ that by now almost completely overlap at $k^0 = 0$.
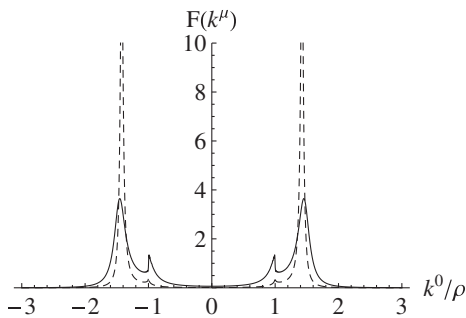

FIG. 13. The statistical propagator at low temperatures. Apart from the original quasiparticle peaks, two more peaks emerge at $|k^0| \le k$. We use $\beta\rho = 2$, $k/\rho = 1$, $m_\phi/\rho = 1$, and $h/\rho = 4$ (solid black), $h/\rho = 2$ (dashed). In the latter case, we do not show the entire original quasi particle peak for illustrative reasons.
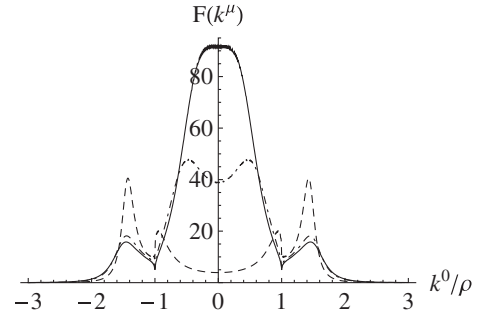

FIG. 14. The statistical propagator at high temperatures. For larger and larger coupling, we observe that the newly emerging quasiparticle peaks move closer to $k^0 = 0$, where they eventually almost completely overlap. We use $\beta\rho = 0.1$, $k/\rho = 1$, $m_\phi/\rho = 1$, and $h/\rho = 1.61$ (solid black), $h/\rho = 1.5$ (dot-dashed), and $h/\rho = 1$ (dashed).

What we observe here is related to the coherence shell at $k^0 = 0$ first introduced by Herranen, Kainulainen, and Rahkila [34,80] to study quantum mechanical reflection and quantum particle creation in a thermal field theoretical setting (and for a discussion of fermions, see [32,33]). They interpret this new spectral solution of the statistical two-point function as a manifestation of nonlocal quantum coherence. As we have just seen, the statistical propagator at late times will basically evolve to Eq. (80). We conclude that the emerging $k^0 = 0$ shell translates to large entropy generation at high temperatures. It is also clear that the naive quasiparticle picture of free thermal states breaks down in the high temperature regime.

D. Evolution of the entropy: Changing mass

Let us now study the evolution of the entropy where the mass of the system field changes according to Eq. (77b). For a constant mass m_ϕ, the statistical propagator depends only on the time difference of its arguments $F_\phi(k, t, t') = F_\phi(k, t - t')$ due to time translation invariance. This observation allowed us to find the asymptotic value of the phase space area by means of another Fourier transformation with respect to $t - t'$. When the mass of the system field changes, however, we introduce a genuine time dependence in the problem and we can only asymptotically compare the entropy to the stationary values well before and after the mass change. It is important to appreciate that the counterterms introduced to renormalize the theory do not depend on m_ϕ so we do not have to consider renormalization again [1].

Depending on the size of the mass change, we can identify the following two regimes:

$$|\beta_k|^2 \ll 1 \qquad \text{adiabatic regime} \qquad (92a)$$
$$|\beta_k|^2 \gg 1 \qquad \text{non-adiabatic regime,} \qquad (92b)$$

where β_k is one of the coefficients of the Bogoliubov transformation that relates the initial (in) vacuum to the

final (out) vacuum state. As a consequence of the mass change, the state gets squeezed [2]. If $\beta_k = 0$, the in and out vacuum states are equal such that $|\beta_k|^2$ quantifies the amount of particle creation and reads [81]

$$|\beta_k|^2 = \frac{\sinh^2(\frac{\pi\omega_-}{\rho})}{\sinh(\frac{\pi\omega_{in}}{\rho})\sinh(\frac{\pi\omega_{out}}{\rho})} \xrightarrow[\omega_{out}\gg\rho]{\omega_{in}\ll\rho} \frac{\rho}{2\pi\omega_{in}}\left(1 - \frac{\pi\omega_{in}}{2\rho}\right)^2.$$

(93)

Here, $\omega_{in}^2 = m_{\phi,in}^2 + k^2$ and $\omega_{out}^2 = m_{\phi,out}^2 + k^2$ are the initial and final frequencies. Also, $m_{\phi,in}^2 = A - B$ and $m_{\phi,out}^2 = A + B$, where we made use of Eq. (77b). Finally, we defined $\omega_\pm = \frac{1}{2}(\omega_{out} \pm \omega_{in})$. The word "particle" in particle creation is not to be confused with the statistical particle number defined by means of the phase space area in Eq. (8). Whereas the latter counts the phase space occupied by a state in units of the minimal uncertainty wave packet, the former corresponds to the conventional notion of particles in curved spacetimes, where one plane wave field excitation $\hat{a}_{\vec{k}}^\dagger|0\rangle = |\vec{k}\rangle$ is referred to as one particle (for a discussion on wave packets in quantum field theory, see [82,83]). When we consider a changing mass in the absence of any interaction terms, $|\beta_k|^2$ increases whereas the phase space area remains constant. For the parameters we consider in this paper $|\beta_k|^2 \simeq \mathcal{O}(10^{-4})$ such that we are in the adiabatic regime.

Let us consider the coherence effects due to a mass increase and decrease in Figs. 15–18. Here, we take $m_\phi/\rho = 1$ and $m_\phi/\rho = 2$ giving rise to the constant interacting thermal entropies $S_{ms}^{(1)}$ and $S_{ms}^{(2)}$, respectively. The numerical value of these asymptotic entropies is calculated just as in the constant mass case such that we find $S_{ms}^{(1)} > S_{ms}^{(2)}$. We use mixed state initial conditions as outlined in Eq. (84) and moreover we insert the initial mass in the memory kernels. In Fig. 15 we show the effects on the entropy for a mass increase at fairly low temperatures $\beta\rho = 2$. In gray we depict the two corresponding constant
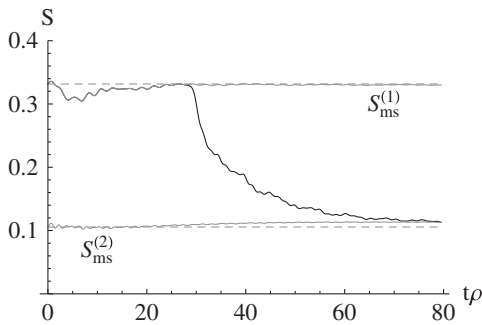

FIG. 15. Entropy as a function of time for a mass increase from $m_\phi/\rho = 1$ to $m_\phi/\rho = 2$, giving rise to the constant interacting thermal entropies $S_{ms}^{(1)}$ and $S_{ms}^{(2)}$, respectively. The mass changes rapidly at $t\rho = 30$. We use $\beta\rho = 2$, $k/\rho = 1$, $h/\rho = 4$, and $N = 1600$. The gray lines are the corresponding constant mass entropy functions where we use mixed state initial conditions.
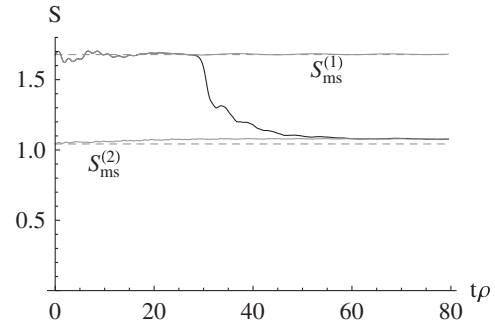

FIG. 16. Entropy as a function of time for a mass increase from $m_\phi/\rho = 1$ to $m_\phi/\rho = 2$, giving rise to the constant interacting thermal entropies $S_{ms}^{(1)}$ and $S_{ms}^{(2)}$, respectively. The mass changes rapidly at $t\rho = 30$. We use $\beta\rho = 1/2$, $k/\rho = 1$, $h/\rho = 3$, and $N = 1600$. The gray lines are the corresponding constant mass entropy functions where we used mixed state initial conditions.
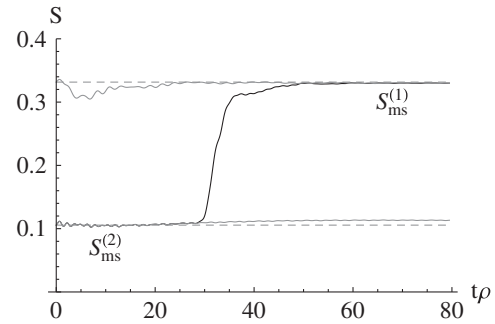

FIG. 17. Entropy as a function of time for a mass decrease where we used the same parameters as in Fig. 15.
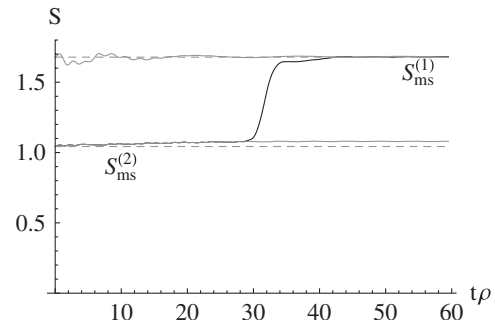

FIG. 18. Entropy as a function of time for a mass decrease where we used the same parameters as in Fig. 16.

mass entropy functions to compare the asymptotic behavior. In order to calculate the latter, we also use mixed state boundary conditions. Clearly, well before and after the mass increase, the entropy is equal to the constant interacting thermal entropy, $S_{ms}^{(1)}$ and $S_{ms}^{(2)}$, respectively. The small difference between the numerical value of the interacting thermal entropy $S_{ms}^{(2)}$ (in dashed gray) and the corresponding $m_\phi/\rho = 2$ constant mass evolution is just due to
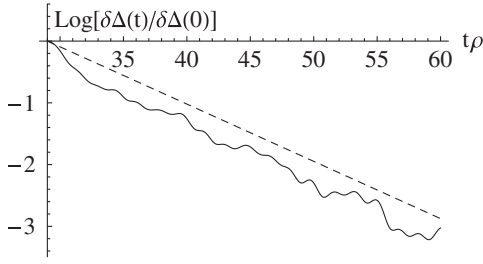
FIG. 19. Decoherence rate at low temperatures. We show the exponential approach to $\Delta_{\text{ms}}^{(2)}$ in solid black and the corresponding decoherence rate given in Eq. (91) (dashed line). We use the phase space area from Fig. 15.

numerical accuracy. It is interesting to observe that the new interacting thermal entropy is reached on a different time scale than ρ^{-1}, the one at which the system's mass has changed. Again, we verify that the rate at which the phase space area changes, defined analogously to Eq. (89), can be well described by the single particle decay rate (90). Given the fact that the mass changes so rapidly in our case, one should use the final mass $m_{\phi,\text{out}}$ in Eq. (90). In Fig. 19 we show both the exponential approach toward the constant interacting phase space area $\Delta_{\text{ms}}^{(2)}$ and the decay rate (90). In order to produce Fig. 19, we subtract the constant mass evolution of the phase space area using mixed state initial conditions rather than $\Delta_{\text{ms}}^{(2)}$ to find $\delta\Delta_k(t)$ in Eq. (89).

This qualitative picture does not change when we consider the same mass increase only now at higher temperatures $\beta\rho = 1/2$ in Fig. 16. The interacting thermal entropies in this case are larger due to the fact that the temperature is higher. Again we observe a small difference between $S_{\text{ms}}^{(2)}$ and the $m_\phi/\rho = 2$ constant mass evolution due to numerical accuracy. Also, the decoherence rate can be well described by the single particle decay rate which we depict in Figs. 19 and 20.

When we consider the "time reversed process," i.e., a mass decrease from $m_\phi/\rho = 2$ to $m_\phi/\rho = 1$, we observe an entropy increase. We show the resulting evolution of the entropy in Figs. 17 and 18 for $\beta\rho = 2$ and $\beta\rho = 1/2$, respectively. The evolution of the entropy reveals no
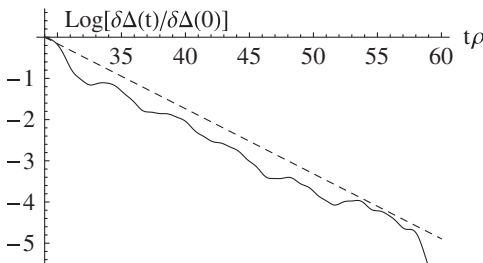

FIG. 20. Decoherence rate at high temperatures. We show the exponential approach to $\Delta_{\text{ms}}^{(2)}$ in solid black and the corresponding decoherence rate given in Eq. (91) (dashed line). We use the phase space area from Fig. 16.

further surprises and corresponds to the time reversed picture of Figs. 15 and 16. The decoherence rate for a mass decrease can again be well described by the single particle decay rate in Eq. (90).

We observe that the rate at which the mass changes is much larger than the decoherence rate. As long as this condition is satisfied, coherence effects continue to be important. Eventually though, the Gaussian von Neumann entropy settles to its new constant value and no particle creation remains as our state thermalizes again. In the context of baryogenesis, we thus expect that quantum coherence effects remain important as long as this condition persists too. Of course, one would have to generalize our model to a CP violating model in which the effects that are of relevance for coherent baryogenesis scenarios are captured.

E. Squeezed states

The effect of a large nonadiabatic mass change on the quantum state is a rapid squeezing of the state which can neatly be visualized in Wigner space. Although it is numerically challenging to implement a case where the mass changes nonadiabatically fast, we can probe its most important effect on the state by considering a state that is significantly squeezed initially. A pure and squeezed state is characterized by the following initial conditions:

$$F_\phi(k, t_0, t_0) = \frac{1}{2\omega_\phi}[\cosh(2r) - \sinh(2r)$$
$$\times \cos(2\varphi)] \tag{94a}$$

$$\partial_t\partial_{t'}F_\phi(k, t, t')|_{t=t'=t_0} = \frac{\omega_\phi}{2}[\cosh(2r) + \sinh(2r)$$
$$\times \cos(2\varphi)] \tag{94b}$$

$$\partial_t F_\phi(k, t, t_0)|_{t=t_0} = \frac{1}{2}\sinh(2r)\sin(2\varphi). \tag{94c}$$

Here, φ characterizes the angle along which the state is squeezed and r indicates the amount of squeezing. As a squeezed state is pure, we have $\Delta_k(t_0) = 1$ initially. A mixed initial squeezed state condition can be achieved by multiplying Eq. (94) by a factor.

We show the corresponding evolution for the phase space area in two cases in Figs. 21 and 22. As the squeezed state thermalizes, we observe two effects. First, there is the usual exponential approach toward the thermal interacting value Δ_{ms} we observed before. As we showed previously, this process is characterized by the single particle decay rate in Eq. (90). Second, superimposed to that behavior, we observe damped oscillatory behavior of the phase space area as a function of time that is induced by the initial squeezing.

The latter process in principle introduces a second decay rate in the evolution: one can associate a characteristic time scale at which the amplitude of the oscillations decay (superimposed on the exponential approach toward Δ_{ms}).
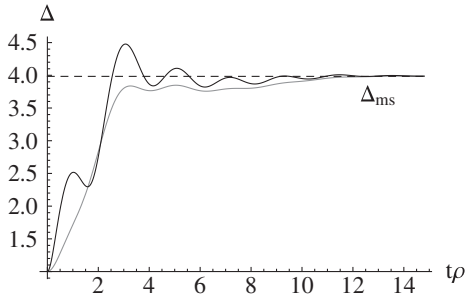
FIG. 21.   Phase space area as a function of time for a squeezed initial state. We use $\varphi = 0$, $e^{2r} = 1/5$, $\beta\rho = 0.5$, $k/\rho = 1$, $m_\phi/\rho = 1$, $h/\rho = 3$, and $N = 300$ up to $t\rho = 15$. The gray line indicates the pure state evolution previously considered in Fig. 7.
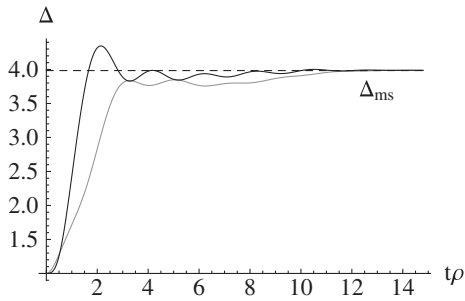

FIG. 22.   Phase space area as a function of time for a squeezed initial state. On top of the usual exponential approach toward Δ_{ms}, we observe oscillatory behavior. The amplitude of the oscillations decays with the single particle decay rate as well. We use $\varphi = 0$, $e^{2r} = 5$, and the other parameters are given in Fig. 21.

One can read off from Figs. 21 and 22 that the exponential decay of the envelope of the oscillations can also be well described by the single particle decay rate in Eq. (90). We thus observe only one relevant time scale of the process of decoherence in our scalar field model: the single particle decay rate. We thus conclude that in the case of a non-adiabatic mass change, the decay of the amplitude of the resulting oscillations will be in agreement with the single particle decay rate too.

VI. CONCLUSION

We study the decoherence of a quantum field theoretical system in a renormalized and perturbative 2PI scheme. As most of the non-Gaussian information about a system is experimentally hard to access, we argue in our "correlator approach" to decoherence that neglecting this information and, consequently, keeping only the information stored in Gaussian correlators, leads to an increase of the Gaussian von Neumann entropy of the system. We argue that the

Gaussian von Neumann entropy should be used as *the* quantitative measure for decoherence.

The most important result in this paper is shown in Fig. 8, where we depict the time evolution of the Gaussian von Neumann entropy for a pure state at a high temperature. Although a pure state with vanishing entropy $S_k = 0$ remains pure under unitary evolution, the observer perceives this state over time as a mixed state with positive entropy $S_{ms} > 0$. The reason is that non-Gaussianities are generated by the unitary evolution (both in the correlation between the system and environment as well as in higher order correlations in the system itself) and subsequently neglected in our Gaussian von Neumann entropy.

We have extracted two relevant quantitative measures of decoherence: the maximal amount of decoherence S_{ms} and the decoherence rate Γ_{dec}. The total amount of decoherence corresponds to the interacting thermal entropy S_{ms} and is slightly larger than the free thermal entropy, depending on the strength of the interaction h. The decoherence rate can be well described by the single particle decay rate of our interaction $\Gamma_{\phi\to\chi\chi}$.

This study builds the quantum field theoretical framework for other decoherence studies in various relevant situations where different types of fields and interactions can be involved. In cosmology, for example, the decoherence of scalar gravitational perturbations can be induced by e.g. fluctuating tensor modes (gravitons) [6], isocurvature modes [66], or even gauge fields. In quantum information physics it is very likely that future quantum computers will involve coherent light beams that interact with other parts of the quantum computer as well as with an environment [84,85]. For a complete understanding of decoherence in such complex systems it is clear that a quantum field theoretical framework such as developed here is necessary.

We also studied the effects on the Gaussian von Neumann entropy of a changing mass. The Gaussian von Neumann entropy changes to the new interacting thermal entropy after the mass change on a time scale that is again well described by the single particle decay rate in our model. It is the same decay rate that describes the decay of the amplitude of the oscillations for a squeezed initial state. One can view our model as a toy model relevant for electroweak baryogenesis scenarios. It is thus interesting to observe that the coherence time scale (the time scale at which the entropy changes) is much larger than the time scale ρ^{-1} at which the mass of the system field changes. We conclude that the coherent effect of a nonadiabatic mass change (squeezing) does not get immediately destroyed by the process of decoherence and thermalization.

Finally, we compared our correlator approach to decoherence to the conventional approach relying on the perturbative master equation. It is unsatisfactory that the reduced density matrix evolves nonunitarily while the

underlying quantum theory is unitary. We are not against nonunitary equations or approximations in principle, however, one should make sure that the essential physical features of the system one is describing are kept. The perturbative master equation does not break unitarity correctly, as we have shown in this paper. On the practical side, the master equation is so complex that field theoretical questions have barely been addressed: there does not exist a treatment to take perturbative interactions properly into account, nor has any reduced density matrix ever been renormalized. This is the reason for our quantum mechanical comparison, rather than a proper field theoretical study of the reduced density matrix. In Sec. IV D however, we outline the perturbative approximations used to derive the master equation from the Kadanoff-Baym equations, i.e., in the memory kernels of the Kadanoff-Baym equations we insert free propagators with appropriate initial conditions. A proper generalization to derive the renormalized perturbative master equation in quantum field theory from the Kadanoff-Baym equations should be straightforward. In the simple quantum mechanical situation, we show that the entropy following from the perturbative master equation generically suffers from physically unacceptable secular growth at late times in the resonant regime. This leads to an incorrect prediction of the total amount of decoherence that has occurred. We show that the time evolution of the Gaussian von Neumann entropy behaves well in both the resonant and in the nonresonant regime.

APPENDIX A: DERIVATION OF $M^F_{\phi,\text{th-th}}(k, \Delta t)$

Only the high and low temperature limits of $M^F_{\phi,\text{th-th}}(k, \Delta t)$ can be evaluated in closed form. We derive these expressions in this appendix.

1. Low temperature contribution

Let us recall Eq. (39b), where we can perform the ω-integral by making use of Eq. (33) and $1/(e^{\beta\omega} - 1) = \sum_{n=1}^{\infty} e^{-\beta n\omega}$,

$$
M^F_{\phi,\text{th-th}}(k, \Delta t)
$$

$$
= -\frac{h^2}{8\pi^2 k} \int_0^{\infty} dk_1 \frac{\cos(k_1\Delta t)}{e^{\beta k_1} - 1} \sum_{n=1}^{\infty} \frac{e^{-n\beta\omega}}{(\Delta t)^2 + (n\beta)^2}
$$

$$
\times [-n\beta\cos(\omega\Delta t) + \Delta t\sin(\omega\Delta t)]|_{\omega=\omega_-}^{\omega=\omega_+}, \tag{A1}
$$

where $\omega_{\pm}^2 = (k \pm k_1)^2 + m_{\chi}^2$. We now prepare this expression for k_1 integration by making use of $1/(e^{\beta k_1} - 1) = \sum_{m=1}^{\infty} e^{-\beta m k_1}$ and some familiar trigonometric identities,

$$
M^F_{\phi,\text{th-th}}(k, \Delta t) = -\frac{h^2}{16\pi^2 k} \sum_{m,n=1}^{\infty} \frac{1}{(\Delta t)^2 + (n\beta)^2} \int_0^{\infty} dk_1 e^{-\beta m k_1} \{-\beta n e^{-\beta n(k+k_1)}[\cos[(2k_1 + k)\Delta t] + \cos(k\Delta t)]
$$

$$
+ \Delta t e^{-\beta n(k+k_1)}[\sin[(2k_1 + k)\Delta t] + \sin(k\Delta t)] + \theta(k - k_1)\beta n e^{-\beta n(k-k_1)}[\cos[(2k_1 - k)\Delta t] + \cos(k\Delta t)]
$$

$$
+ \theta(k - k_1)\Delta t e^{-\beta n(k-k_1)}[\sin[(2k_1 - k)\Delta t] - \sin(k\Delta t)] + \theta(k_1 - k)\beta n e^{-\beta n(k_1-k)}[\cos[(2k_1 - k)\Delta t]
$$

$$
+ \cos(k\Delta t)] - \theta(k_1 - k)\Delta t e^{-\beta n(k_1-k)}[\sin[(2k_1 - k)\Delta t] - \sin(k\Delta t)]\}. \tag{A2}
$$

Upon integrating over k_1 and rearranging the terms we obtain

$$
M^F_{\phi,\text{th-th}}(k, \Delta t) = -\frac{h^2}{16\pi^2 k} \sum_{m,n=1}^{\infty} \frac{1}{(\Delta t)^2 + (n\beta)^2} \left\{ \sin(k\Delta t)(e^{-\beta nk} - e^{-\beta mk}) \left[\frac{\beta\Delta t(m + 3n)}{[\beta(m + n)]^2 + (2\Delta t)^2} - \frac{\Delta t}{\beta(m - n)} \right] \right.
$$

$$
+ \sin(k\Delta t)(e^{-\beta nk} + e^{-\beta mk}) \left[\frac{-\beta\Delta t(m - 3n)}{[\beta(m - n)]^2 + (2\Delta t)^2} + \frac{\Delta t}{\beta(m + n)} \right] + \cos(k\Delta t)(e^{-\beta nk} - e^{-\beta mk})
$$

$$
\times \left[\frac{-\beta^2 n(n + m) + 2(\Delta t)^2}{[\beta(m + n)]^2 + (2\Delta t)^2} + \frac{\beta^2 n(m - n) + 2(\Delta t)^2}{[\beta(m - n)]^2 + (2\Delta t)^2} - \frac{n}{m + n} + \frac{n}{m - n} \right] \right\}. \tag{A3}
$$

This expression contains two singular terms when $m = n$. By performing the integral (A2) in that case, they are to be interpreted as

$$
\frac{e^{-\beta nk} - e^{-\beta mk}}{m - n} \xrightarrow{m=n} \beta k e^{-\beta nk}. \tag{A4}
$$

This expression allows us to obtain the low temperature $\beta k \gg 1$ limit of $M^F_{\phi,\text{th-th}}(k, \Delta t)$. It then suffices to consider three contributions in Eq. (A3) only. First, there is the contribution for $m = 1 = n$, for $n = 1$ and $m \geq 2$, and finally for $m = 1$ and $n \geq 2$. The sum in the last two cases can be evaluated in closed form, such that one obtains Eq. (48).

2. High temperature contribution

Let us now consider the high temperature limit. It is clear from Eq. (A3) that when $\beta k \ll 1$ there is unfortunately no small quantity to expand about as both m and n can become arbitrarily large. Therefore, we go back to the original expression (39b), proceed as usual by making use of (33) and rewrite it in terms of new (u, v)-coordinates ("lightcone coordinates"), defined by

$$u = k_1 - \omega \tag{A5a}$$
$$v = k_1 + \omega, \tag{A5b}$$

such that of course $k_1 = (v + u)/2$ and $\omega = (v - u)/2$, in terms of which the region of integration becomes

$$-k \leq u \leq k \tag{A6a}$$
$$k \leq v < \infty. \tag{A6b}$$

Equation (39b) thus transforms into

$$
M^F_{\phi,\text{th-th}}(k, \Delta t)
$$
$$
= \frac{h^2}{32\pi^2 k} \int_{-k}^{k} du \frac{1}{2\sinh(\beta u/2)} \int_{k}^{\infty} dv [\cos(u\Delta t)
$$
$$
+ \cos(v\Delta t)] \left\{ \frac{e^{(\beta u/2)}}{e^{\beta(v+u)/2} - 1} - \frac{e^{-(\beta u/2)}}{e^{\beta(v-u)/2} - 1} \right\}, \tag{A7}
$$

where we took account of the Jacobian $J = |\partial(k_1, \omega)/\partial(u, v)| = 1/2$. One can now perform the v-integral involving the $\cos(u\Delta t)$-term. Second, since we are interested in the limit $\beta k \ll 1$ and we moreover have $|u| \leq k$, note that we also have $|\beta u| \ll 1$. The $\cos(v\Delta t)$-term can thus be expanded around $|\beta u| \ll 1$. An intermediate result reads

$$
M^F_{\phi,\text{th-th}}(k, \Delta t)
$$
$$
= \frac{h^2}{32\pi^2 k\beta} \left\{ -2 \int_{-k}^{k} du \cos(u\Delta t) \right.
$$
$$
\times \left[\frac{\log(1 - \exp[-\beta(k+u)/2])}{1 - \exp(-\beta u)} \right.
$$
$$
\left. - \frac{\log(1 - \exp[-\beta(k-u)/2])}{\exp(\beta u) - 1} \right]
$$
$$
+ \int_{-k}^{k} du \int_{k}^{\infty} dv \frac{\cos(v\Delta t)}{u} \left\{ \frac{1 + \beta u/2}{e^{\beta v/2}(1 + \beta u/2) - 1} \right.
$$
$$
\left. \left. - \frac{1 - \beta u/2}{e^{\beta v/2}(1 - \beta u/2) - 1} \right\} \right\}. \tag{A8}
$$

The reader can easily see that we deliberately do not Taylor expand the first integral fully around $|\beta u| \ll 1$. The reason is that the subsequent integration renders such a naive Taylor expansion invalid. Let us first integrate the first integral of Eq. (A8). We now expand the $\cos(u\Delta t)$-integral around $|\beta(k \pm u)| \ll 1$,

$$
\int_{-k}^{k} du \cos(u\Delta t) \left[\frac{\log(1 - \exp[-\beta(k+u)/2])}{1 - \exp(-\beta u)} \right.
$$
$$
\left. - \frac{\log(1 - \exp[-\beta(k-u)/2])}{\exp(\beta u) - 1} \right]
$$
$$
= \int_{-k}^{k} du \cos(u\Delta t) \left[\frac{1}{\beta u} \log\left(\frac{k+u}{k-u}\right) \right.
$$
$$
\left. + \frac{1}{2} \log\left(\frac{\beta^2}{4}(k^2 - u^2)\right) - \frac{1}{2} + \mathcal{O}(\beta k, \beta u) \right]. \tag{A9}
$$

Let us first evaluate the simple u-integrals in Eq. (A9),

$$
\int_{-k}^{k} du \frac{\cos(u\Delta t)}{2} \left[\log\left(\frac{\beta^2}{4}(k^2 - u^2)\right) - 1 \right]
$$
$$
= \frac{\sin(k\Delta t)}{\Delta t} \left(\text{ci}(2|k\Delta t|) - \log\left(\frac{2|\Delta t|}{k\beta^2}\right) - \gamma_E - 1 \right)
$$
$$
- \frac{\cos(k\Delta t)}{|\Delta t|} \left(\text{si}(2|k\Delta t|) + \frac{\pi}{2} \right), \tag{A10}
$$

where $\text{ci}(z)$ and $\text{si}(z)$ are the cosine and sine integral functions, respectively, defined in Eq. (23). The more complicated integral in (A9) is

$$
\int_{-k}^{k} du \frac{\cos(u\Delta t)}{\beta u} \log\left(\frac{k+u}{k-u}\right)
$$
$$
= \frac{1}{\beta} \sum_{n=0}^{\infty} \frac{2}{1 + 2n} \int_{-1}^{1} dz z^{2n} \cos(k\Delta t z)
$$
$$
= \sum_{n=0}^{\infty} \frac{\beta^{-1}}{(\frac{1}{2} + n)^2} \,_1F_2\left(\frac{1}{2} + n; \frac{1}{2}, \frac{3}{2} + n; -\frac{(k\Delta t)^2}{4}\right). \tag{A11}
$$

By making use of its definition, we expand the hypergeometric function $_1F_2$ as follows:

$$
_1F_2\left(\frac{1}{2} + n; \frac{1}{2}, \frac{3}{2} + n; -\frac{(k\Delta t)^2}{4}\right)
$$
$$
= \sqrt{\pi}\left(\frac{1}{2} + n\right) \sum_{m=0}^{\infty} \frac{1}{m!(\frac{1}{2} + n + m)\Gamma(\frac{1}{2} + m)}
$$
$$
\times \left(-\frac{(k\Delta t)^2}{4}\right)^m. \tag{A12}
$$

Inserting this into (A11) and performing the n-sum we obtain

$$
\int_{-k}^{k} du \frac{\cos(u\Delta t)}{\beta u} \log\left(\frac{k+u}{k-u}\right)
$$
$$
= \frac{1}{\beta} \left[\frac{\pi^2}{2} + \sqrt{\pi} \sum_{m=1}^{\infty} \frac{\psi(\frac{1}{2} + m) + 2\ln(2) + \gamma_E}{mm!\Gamma(\frac{1}{2} + m)} \right.
$$
$$
\left. \times \left(-\frac{(k\Delta t)^2}{4}\right)^m \right], \tag{A13}
$$

where we performed the n-sum for $m = 0$ separately. Note that

$$\frac{\psi(m + 1/2)}{\Gamma(m + 1/2)} = -\frac{d}{d\gamma} \frac{1}{\Gamma(\gamma + m)} \bigg|_{\gamma=1/2}. \tag{A14}$$

Finally, we can perform the m-sum appearing in Eq. (A13) to yield

$$\int_{-k}^{k} du \frac{\cos(u\Delta t)}{\beta u} \log\left(\frac{k + u}{k - u}\right)$$
$$= \frac{\pi^2}{2\beta} - \frac{4}{\beta}(\gamma_E - \mathrm{ci}(|k\Delta t|) + \log(|k\Delta t|))$$
$$+ \frac{(k\Delta t)^2}{2\beta} \frac{d}{d\gamma} {}_2F_3\left(1, 1; 2, 2, 1 + \gamma; -\frac{(k\Delta t)^2}{4}\right)\bigg|_{\gamma=(1/2)}. \tag{A15}$$

It is useful to know the expansions of the hypergeometric function in (A15). The large time ($k\Delta t \gg 1$) expansion of this function is

$${}_2F_3\left(1, 1; 2, 2, 1 + \gamma; -\frac{(k\Delta t)^2}{4}\right)$$
$$= \frac{\Gamma(1 + \gamma)}{\sqrt{\pi}} \frac{\cos[k\Delta t - \frac{\pi}{2}(\gamma + \frac{5}{2})]}{(k\Delta t/2)^{\gamma+(5/2)}}(1 + \mathcal{O}((k\Delta t)^{-1}))$$
$$+ 4\gamma \frac{\log((k\Delta t)^2/4) - \psi(\gamma) + \gamma_E}{(k\Delta t)^2}(1 + \mathcal{O}((k\Delta t)^{-2})), \tag{A16a}$$

whereas the small times ($k\Delta t \ll 1$) limit yields

$${}_2F_3\left(1, 1; 2, 2, 1 + \gamma; -\frac{(k\Delta t)^2}{4}\right)$$
$$= 1 - \frac{(k\Delta t)^2}{16(1 + \gamma)} + \mathcal{O}((k\Delta t)^4). \tag{A16b}$$

We still need to perform some more integrals in Eq. (A8). The second integral in Eq. (A8) can be further simplified to:

$$\int_{-k}^{k} du \int_{k}^{\infty} dv \frac{\cos(v\Delta t)e^{-(\beta v/2)}}{1 - e^{-(\beta v/2)}}$$
$$\times \left[\beta - \frac{1 + \frac{\beta u}{2}}{u + \frac{2}{\beta}[1 - e^{-(\beta v/2)}]} + \frac{1 - \frac{\beta u}{2}}{u - \frac{2}{\beta}[1 - e^{-(\beta v/2)}]}\right]. \tag{A17}$$

We can now perform the u-integral,

$$= -2\int_{k}^{\infty} dv \frac{\cos(v\Delta t)e^{-\beta v}}{1 - e^{-(\beta v/2)}}\left\{\log\left[k + \frac{2}{\beta}(1 - e^{-(\beta v/2)})\right]\right.$$
$$\left. - \log\left[-k + \frac{2}{\beta}(1 - e^{-(\beta v/2)})\right]\right\}$$
$$= -4\sum_{m=1}^{\infty} \frac{(\frac{k\beta}{2})^{2m-1}}{2m - 1} \int_{k}^{\infty} dv \frac{\cos(v\Delta t)e^{-\beta v}}{(1 - e^{-(\beta v/2)})^{2m}}$$
$$= -4\sum_{\substack{m=1 \\ n=0}}^{\infty} \frac{(\frac{k\beta}{2})^{2m-1}}{2m - 1} \frac{\Gamma(2m + n)}{\Gamma(2m)\Gamma(n + 1)}$$
$$\times \mathrm{Re} \int_{k}^{\infty} dv e^{-(\beta v/2)(n+2)+\imath v\Delta t}, \tag{A18}$$

where the reader can easily verify that the argument of both logarithms in the first line is positive. In the second and third line we have expanded the logarithm and made use of the binomial series. Because of the cosine appearing in Eq. (A18), we are only interested in the real part of the integral on the last line. The v-integral can now trivially be performed. In order to extract the high temperature limit correctly, it turns out to be advantageous to perform the m-sum in Eq. (A17) first,

$$= -4k\sum_{n=0}^{\infty} \frac{\Gamma(n + 2)}{\Gamma(n + 1)(n + 2 - 2\imath\Delta t/\beta)}$$
$$\times {}_3F_2\left(1, 1 + \frac{n}{2}, \frac{n + 3}{2}; \frac{3}{2}, \frac{3}{2}; \left(\frac{k\beta}{2}\right)^2\right)$$
$$\times \mathrm{Re}\, e^{-(k\beta/2)(n+2-2\imath(\Delta t/\beta))}. \tag{A19}$$

The hypergeometric function can be expanded in the high temperature limit as

$${}_3F_2\left(1, 1 + \frac{n}{2}, \frac{n + 3}{2}; \frac{3}{2}, \frac{3}{2}; \left(\frac{k\beta}{2}\right)^2\right)$$
$$= 1 + \frac{1}{72}(n + 2)(n + 3)(k\beta)^2 + \mathcal{O}((k\beta)^4). \tag{A20}$$

We have checked using direct numerical integration that the analytic answer improves much if we keep also the second order term in this expansion. Finally, we can perform the remaining sum over n, yielding

$$-4k\sum_{n=0}^{\infty} \frac{\Gamma(n + 2)}{\Gamma(n + 1)(n + 2 - 2\imath\Delta t/\beta)}$$
$$\times \left(1 + \frac{1}{72}(n + 2)(n + 3)(k\beta)^2\right)e^{-(k\beta/2)(n+2-2\imath(\Delta t/\beta))}$$
$$= -2k\frac{e^{-k\beta+\imath k\Delta t}}{1 - \imath\Delta t/\beta}\left[{}_2F_1\left(2, 2 - \frac{2\imath\Delta t}{\beta}; 3 - \frac{2\imath\Delta t}{\beta}; e^{-(k\beta/2)}\right)\right.$$
$$\left. + \frac{(k\beta)^2}{12} {}_2F_1\left(4, 2 - \frac{2\imath\Delta t}{\beta}; 3 - \frac{2\imath\Delta t}{\beta}; e^{-(k\beta/2)}\right)\right], \tag{A21}$$

where of course we are interested in the real part of the expression above. Having performed all the integrals needed to calculate the high temperature limit of $M^F_{\phi,\text{th-th}}(k, \Delta t)$, we can collect the results in Eqs. (A8), (A10), (A15), (A17), and (A21) above, finding precisely Eq. (50).

APPENDIX B: THE STATISTICAL PROPAGATOR IN FOURIER SPACE

This appendix is devoted to calculating the statistical propagator in Fourier space at finite temperature. The two Wightman functions, needed to calculate the statistical propagator through Eq. (11b), are given by

$$\iota\Delta^{-+}_\phi(k^\mu) = \frac{-\iota M^{-+}_\phi(k^\mu)\iota\Delta^a_\phi(k^\mu)}{k_\mu k^\mu + m^2_\phi + \iota M^r_{\phi,\text{ren}}(k^\mu)} \quad \text{(B1a)}$$

$$\iota\Delta^{+-}_\phi(k^\mu) = \frac{-\iota M^{+-}_\phi(k^\mu)\iota\Delta^a_\phi(k^\mu)}{k_\mu k^\mu + m^2_\phi + \iota M^r_{\phi,\text{ren}}(k^\mu)}, \quad \text{(B1b)}$$

where we have made use of the definition of the advanced propagator,

$$\iota\Delta^a_\phi(k^\mu) = \frac{-\iota}{k_\mu k^\mu + m^2_\phi + \iota M^a_{\phi,\text{ren}}(k^\mu)}, \quad \text{(B2)}$$

and the definitions of the advanced and retarded self-masses

$$\begin{aligned}\iota M^r_{\phi,\text{ren}}(k^\mu) &= \iota M^{++}_{\phi,\text{ren}}(k^\mu) - \iota M^{+-}_\phi(k^\mu)\\&= \iota M^{-+}_\phi(k^\mu) - \iota M^{--}_{\phi,\text{ren}}(k^\mu)\end{aligned} \quad \text{(B3a)}$$

$$\begin{aligned}\iota M^a_{\phi,\text{ren}}(k^\mu) &= \iota M^{++}_{\phi,\text{ren}}(k^\mu) - \iota M^{-+}_\phi(k^\mu)\\&= \iota M^{+-}_\phi(k^\mu) - \iota M^{--}_{\phi,\text{ren}}(k^\mu).\end{aligned} \quad \text{(B3b)}$$

Our starting point is

$$\begin{aligned}\iota M^{++}_\phi(k^\mu) &= -\frac{\iota h^2}{2}\int d^D(x-x')(\iota\Delta^{++}_\chi(x;x'))^2 e^{-\iota k(x-x')}\\&= -\frac{\iota h^2}{2}\int\frac{d^D k'}{(2\pi)^D}\iota\Delta^{++}_\chi(k'^\mu)\iota\Delta^{++}_\chi(k^\mu - k'^\mu).\end{aligned} \quad \text{(B4)}$$

The thermal propagators appearing in this equation are of course given by (19), where $m_\chi \to 0$. This calculation naturally splits again into three parts,

$$\begin{aligned}\iota M^{++}_\phi(k^\mu) &= \iota M^{++}_{\phi,\text{vac}}(k^\mu) + \iota M^{++}_{\phi,\text{vac-th}}(k^\mu)\\&\quad + \iota M^{++}_{\phi,\text{th-th}}(k^\mu),\end{aligned} \quad \text{(B5)}$$

where

$$\iota M^{++}_{\phi,\text{vac}}(k^\mu) = \frac{h^2}{32\pi^2}\left[\log\left(\frac{-k^2_0 + k^2 - \iota\epsilon}{4\mu^2}\right) + 2\gamma_E\right] \quad \text{(B6a)}$$

$$\begin{aligned}\iota M^{++}_{\phi,\text{vac-th}}(k^\mu) &= -h^2\int\frac{d^D k'}{(2\pi)^D}\frac{1}{k'^\mu k_\mu - \iota\epsilon}\\&\quad\times 2\pi\delta((k^\mu - k'^\mu)(k_\mu - k'_\mu))\\&\quad\times n^{\text{eq}}_\chi(|k^0 - k'^0|)\end{aligned} \quad \text{(B6b)}$$

$$\begin{aligned}\iota M^{++}_{\phi,\text{th-th}}(k^\mu) &= -\frac{\iota h^2}{2}\int\frac{d^D k'}{(2\pi)^D}4\pi^2\delta((k^\mu - k'^\mu)\\&\quad\times(k_\mu - k'_\mu))\delta(k'^\mu k'_\mu)n^{\text{eq}}_\chi(|k^0 - k'^0|)\\&\quad\times n^{\text{eq}}_\chi(|k'^0|),\end{aligned} \quad \text{(B6c)}$$

where the vacuum contribution (B6a) has already been evaluated and renormalized in [1]. As all thermal contributions are finite, we can safely let $D \to 4$ and make use of Eq. (33):

$$\begin{aligned}&\iota M^{++}_{\phi,\text{vac-th}}(k^\mu)\\&= -\frac{h^2}{8\pi^2 k}\int^\infty_0 dk'\int^{k+k'}_{|k-k'|}d\omega k' n^{\text{eq}}_\chi(\omega)\\&\quad\times\left(\frac{1}{-(k^0 + \omega)^2 + k'^2 - \iota\epsilon} + \frac{1}{-(k^0 - \omega)^2 + k'^2 - \iota\epsilon}\right)\end{aligned} \quad \text{(B7a)}$$

$$\begin{aligned}&\iota M^{++}_{\phi,\text{th-th}}(k^\mu)\\&= -\frac{\iota h^2}{16\pi k}\int^\infty_0 dk'\int^{k+k'}_{|k-k'|}d\omega n^{\text{eq}}_\chi(k')\sum_\pm n^{\text{eq}}_\chi(|k^0 \pm k'|)\\&\quad\times[\delta(k^0 \pm k' + \omega) + \delta(k^0 \pm k' - \omega)].\end{aligned} \quad \text{(B7b)}$$

Here, $k = \|\vec{k}\|$ as before. Transforming to (u, v)-coordinates already used in Eq. (A5) now yields

$$\begin{aligned}&\iota M^{++}_{\phi,\text{vac-th}}(k^\mu)\\&= -\frac{h^2}{32\pi^2 k}\int^k_{-k}du\int^\infty_k dv\frac{u + v}{e^{(\beta/2)(u+v)} - 1}\\&\quad\times\left(\frac{1}{(v + k^0)(u - k^0) - \iota\epsilon} + \frac{1}{(v - k^0)(u + k^0) - \iota\epsilon}\right)\end{aligned} \quad \text{(B8a)}$$

$$\begin{aligned}&\iota M^{++}_{\phi,\text{th-th}}(k^\mu)\\&= -\frac{\iota h^2}{32\pi k}\int^k_{-k}du\int^\infty_k dv n^{\text{eq}}_\chi\left(\frac{1}{2}(u + v)\right)\\&\quad\times\sum_\pm n^{\text{eq}}_\chi\left(\left|k^0 \pm \frac{1}{2}(u + v)\right|\right)[\delta(k^0 \pm v)\\&\quad + \delta(k^0 \pm u)].\end{aligned} \quad \text{(B8b)}$$

Let us first calculate $\iota M^{++}_{\phi,\text{th-th}}(k^\mu)$. The Dirac delta functions trivially reduce Eq. (B8b) further and moreover, we can make use of:

$$\int_{-k}^{k} du \frac{1}{e^{(\beta/2)(u-k^0)} - 1} \frac{1}{e^{-(\beta/2)(u+k^0)} - 1} = \frac{2}{\beta} \frac{1}{e^{-\beta k^0} - 1} \left[2\log\left(\frac{1 - e^{-(\beta/2)(k-k^0)}}{1 - e^{(\beta/2)(k+k^0)}}\right) + k\beta \right] \tag{B9a}$$

$$\int_{k}^{\infty} dv \frac{1}{e^{(\beta/2)(v-k^0)} - 1} \frac{1}{e^{(\beta/2)(v+k^0)} - 1} = \frac{2}{\beta} \left[\frac{1}{1 - e^{\beta k^0}} \log\left(1 - e^{-(\beta/2)(k-k^0)}\right) + \frac{1}{1 - e^{-\beta k^0}} \log(1 - e^{-(\beta/2)(k+k^0)}) \right]. \tag{B9b}$$

The final result for $\imath M^{++}_{\phi,\text{th-th}}(k^\mu)$ thus reads

$$\imath M^{++}_{\phi,\text{th-th}}(k^\mu)$$
$$= -\frac{\imath h^2}{16\pi k\beta} \left[\sum_{\pm} \frac{\theta(\mp k^0 - k)}{e^{\mp\beta k^0} - 1} \left\{ 2\log\left(\frac{1 - e^{-(\beta/2)(k \mp k^0)}}{1 - e^{(\beta/2)(k \pm k^0)}}\right) \right. \right.$$
$$+ k\beta \right\} + \sum_{\pm} [\theta(\mp k^0 + k) - \theta(\mp k^0 - k)]$$
$$\times \left\{ \frac{1}{1 - e^{\pm\beta k^0}} \log(1 - e^{-(\beta/2)(k \mp k^0)}) + \frac{1}{1 - e^{\mp\beta k^0}} \right.$$
$$\left. \left. \times \log(1 - e^{-(\beta/2)(k \pm k^0)}) \right\} \right]. \tag{B10}$$

Since this contribution to $\imath M^{++}_{\phi}(k^\mu)$ does not depend on the pole prescription, it completely fixes similar contributions to the other self-masses, e.g. $\imath M^{+-}_{\phi,\text{th-th}}(k^\mu) = \imath M^{++}_{\phi,\text{th-th}}(k^\mu)$. It turns out that Eq. (B8a) is not most advantageous to derive $\imath M^{++}_{\phi,\text{vac-th}}(k^\mu)$.

Let us therefore first evaluate $\imath M^{\pm\mp}(k^\mu)$. Let us thus start just as in Eq. (B4) and set

$$\imath M^{\pm\mp}_{\phi}(k^\mu) = \imath M^{\pm\mp}_{\phi,\text{vac}}(k^\mu) + \imath M^{\pm\mp}_{\phi,\text{vac-th}}(k^\mu)$$
$$+ \imath M^{\pm\mp}_{\phi,\text{th-th}}(k^\mu). \tag{B11}$$

The vacuum-vacuum contribution has been evaluated in [1] and is given by

$$\imath M^{\pm\mp}_{\phi,\text{vac}}(k^\mu) = -\frac{\imath h^2}{16\pi} \theta(\mp k^0 - k). \tag{B12}$$

The thermal-thermal contributions are given above in Eq. (B10), so we only need to determine the vacuum-thermal contributions. Hence, we perform an analogous calculation as for $\imath M^{++}$ and transform to the familiar light-cone coordinates u and v to find the following intermediate result:

$$\imath M^{\pm\mp}_{\phi,\text{vac-th}}(k^\mu)$$
$$= -\frac{\imath h^2}{16\pi k} \int_{-k}^{k} du \int_{k}^{\infty} dv n^{\text{eq}}_{\chi}(|k^0 \pm (u + v)/2|)$$
$$\times [\delta(k^0 \pm u) + \delta(k^0 \pm v)]. \tag{B13}$$

The delta functions allow us to perform one of the two integrals trivially. The remaining integral can also be obtained straightforwardly,

$$\imath M^{\pm\mp}_{\phi,\text{vac-th}}(k^\mu)$$
$$= \frac{\imath h^2}{8\pi k\beta} \left[\{\theta(k \pm k^0) - \theta(-k \pm k^0)\} \right.$$
$$\times \log(1 - e^{-(\beta/2)(k \pm k^0)}) - \theta(\mp k^0 - k)$$
$$\left. \times \log\left(\frac{1 - e^{-(\beta/2)(\mp k^0 + k)}}{1 - e^{-(\beta/2)(\mp k^0 - k)}}\right) \right]. \tag{B14}$$

By subtracting and adding the above self-masses, we can obtain the vacuum-thermal contributions to the causal and statistical self-masses in Fourier space from Eqs. (32) and (37), respectively. The vacuum-thermal contribution to the causal self-mass reads

$$M^{c}_{\phi,\text{vac-th}}(k^\mu) = \imath M^{+-}_{\phi,\text{vac-th}}(k^\mu) - \imath M^{-+}_{\phi,\text{vac-th}}(k^\mu)$$
$$= \frac{\imath h^2}{8\pi k\beta} \text{sgn}(k^0)[\log(1 - e^{-(\beta/2)(k+|k^0|)})$$
$$- \log(1 - e^{-(\beta/2)|k-|k^0||})], \tag{B15}$$

where we have made use of the theta functions to bring this result in a particularly compact form. Likewise, the vacuum-thermal contribution to the statistical self-mass now reads

$$M^{F}_{\phi,\text{vac-th}}(k^\mu) = \frac{1}{2}[M^{+-}_{\phi,\text{vac-th}}(k^\mu) + M^{-+}_{\phi,\text{vac-th}}(k^\mu)]$$
$$= \frac{h^2}{16\pi k\beta}[\text{sgn}(k - |k^0|)$$
$$\times \log(1 - e^{-(\beta/2)(k+|k^0|)})$$
$$+ \log(1 - e^{-(\beta/2)|k-|k^0||})]. \tag{B16}$$

As a check of the results above, we performed the inverse Fourier transforms of the causal and statistical self-masses in Eqs. (36) and (40), respectively, and found agreement with the results presented above.

The most convenient way of solving the vacuum-thermal contribution to $\imath M^{++}_{\phi}(k^\mu)$ is by making use of Eqs. (24) and (36). Let us set the imaginary part of $\imath M^{++}_{\phi}(k^\mu)$ equal to

$$M_{\text{sgn}}^{++}(k, \Delta t) \equiv \frac{1}{2}\text{sgn}(\Delta t)M_{\phi}^c(k, \Delta t) = \frac{h^2}{32\pi^2}\frac{\sin(k\Delta t)}{k(\Delta t)^2}\text{sgn}(\Delta t)\left[\frac{2\pi\Delta t}{\beta}\coth\left(\frac{2\pi\Delta t}{\beta}\right) - 1\right] \overset{\beta k \ll 1}{\rightarrow} \frac{h^2}{16\pi\beta}\frac{\sin(k\Delta t)}{k\Delta t}, \tag{B17}$$

where we have taken the high temperature limit. We thus have

$$M_{\text{sgn}}^{++}(k^\mu) = \int_{-\infty}^{\infty} d(\Delta t)M_{\text{sgn}}^{++}(k, \Delta t)e^{\imath k^0 \Delta t}\overset{\beta k \ll 1}{\rightarrow} \frac{h^2}{8\pi k\beta}\int_0^{\infty} d(\Delta t)\frac{\sin(k\Delta t)\cos(k^0\Delta t)}{\Delta t}$$

$$= \frac{h^2}{32\beta k}[\text{sgn}(k^0 + k) + \text{sgn}(-k^0 + k)], \tag{B18}$$

where have performed the remaining integral straightforwardly. Using Eq. (24) and (B16) we find

$$M_{\phi,\text{vac-th}}^{++}(k^\mu)\overset{\beta k \ll 1}{\rightarrow} M_{\phi,\text{vac-th}}^F(k^\mu) + \imath M_{\text{sgn}}^{++}(k^\mu) \tag{B19a}$$

$$M_{\phi,\text{vac-th}}^{--}(k^\mu)\overset{\beta k \ll 1}{\rightarrow} M_{\phi,\text{vac-th}}^F(k^\mu) - \imath M_{\text{sgn}}^{++}(k^\mu), \tag{B19b}$$

such that

$$\imath M_{\phi,\text{vac-th}}^{++}(k^\mu)\overset{\beta k \ll 1}{\rightarrow} \frac{\imath h^2}{16\pi k\beta}\left[\text{sgn}(k - |k^0|)\log(1 - e^{-(\beta/2)(k+|k^0|)}) + \log(1 - e^{-(\beta/2)|k-|k^0||})\right.$$

$$\left. + \frac{\imath\pi}{2}[\text{sgn}(k^0 + k) + \text{sgn}(-k^0 + k)]\right] \tag{B20a}$$

$$\imath M_{\phi,\text{vac-th}}^{--}(k^\mu)\overset{\beta k \ll 1}{\rightarrow} \frac{\imath h^2}{16\pi k\beta}\left[\text{sgn}(k - |k^0|)\log(1 - e^{-(\beta/2)(k+|k^0|)}) + \log(1 - e^{-(\beta/2)|k-|k^0||})\right.$$

$$\left. - \frac{\imath\pi}{2}[\text{sgn}(k^0 + k) + \text{sgn}(-k^0 + k)]\right]. \tag{B20b}$$

One can check Eq. (B18) by means of an alternative approach. The starting point is the first line of Eq. (36) and one can furthermore realize that differentiating $M_{\text{sgn}}^{++}(k^\mu)$ with respect to k^0 brings down a factor of $\imath\Delta t$ which conveniently cancels the factor of Δt that is present in the denominator. One can then integrate the resulting expressions (introducing ϵ regulators and UV cutoffs where necessary) confirming expression (B18).

In the low temperature limit, Eq. (B17) reduces to

$$M_{\text{sgn}}^{++}(k, \Delta t)\overset{\beta k \gg 1}{\rightarrow} \frac{h^2}{24k\beta^2}\sin(k\Delta t)\,\text{sgn}(\Delta t). \tag{B21}$$

We can introduce an ϵ regulator,

$$M_{\text{sgn}}^{++}(k^\mu)\overset{\beta k \gg 1}{\rightarrow} \frac{h^2}{12k\beta^2}\int_0^{\infty} d(\Delta t)\sin(k\Delta t)\cos(k^0\Delta t)e^{-\epsilon\Delta t} = \frac{h^2}{12\beta^2}\frac{1}{k^2 - k_0^2}. \tag{B22}$$

Analogously, we can derive the following expressions for the vacuum-thermal contributions to $\imath M_{\phi}^{\pm\pm}(k^\mu)$ in the low temperature limit:

$$\imath M_{\phi,\text{vac-th}}^{++}(k^\mu)\overset{\beta k \gg 1}{\rightarrow} \frac{\imath h^2}{16\pi k\beta}\left[\text{sgn}(k - |k^0|)\log(1 - e^{-(\beta/2)(k+|k^0|)}) + \log(1 - e^{-(\beta/2)|k-|k^0||}) + \frac{\imath 4\pi k}{3\beta}\frac{1}{k^2 - k_0^2}\right] \tag{B23a}$$

$$\imath M_{\phi,\text{vac-th}}^{--}(k^\mu)\overset{\beta k \gg 1}{\rightarrow} \frac{\imath h^2}{16\pi k\beta}\left[\text{sgn}(k - |k^0|)\log(1 - e^{-(\beta/2)(k+|k^0|)}) + \log(1 - e^{-(\beta/2)|k-|k^0||}) - \frac{\imath 4\pi k}{3\beta}\frac{1}{k^2 - k_0^2}\right]. \tag{B23b}$$

We have now all self-masses at our disposal necessary to calculate $F(k^\mu)$. To numerically evaluate the integrals in Eq. (79) we do not rely on the high and low temperature expressions in Eqs. (B23) or (B20) but we rather use exact numerical methods, i.e., the first line of Eq. (B17).

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