THE DESIGN, CONSTRUCTION, AND APPLICATION OF A DIFFERENTIAL ANALYZER

BY

CHARLES NORTON HENSHAW

B.S. in M.E., University of Vermont
1920

S.M., Massachusetts Institute of Technology
1929

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF SCIENCE

FROM THE

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1936

Signature of Author...

Department of Mechanical Engineering, October 1, 1936

Signature of Professor

in Charge of Research.......................................

Signature of Chairman of Department

Committee on Graduate Students.

ABSTRACT

THE DESIGN, CONSTRUCTION, AND APPLICATION OF A DIFFERENTIAL ANALYZER

This discussion is concerned with the development of an apparatus for the solution of second order differential equations by means of a dynamic analogy, as proposed by Dr. N. Minorski. A torsion pendulum is so arranged that its restoring torque can be regulated according to the desired function of time and displacement. Provision is made for tracing the graph of the oscillation. In this manner, any differential equation of the general form: $\ddot{\theta} + f(t,\theta)\theta = 0$, may be solved.

The main part of the apparatus consists of two coils mounted on a vertical shaft which is suspended from a fine steel wire. Each coil is surrounded by a horizontal solenoid, rigidly mounted upon the base of the apparatus. When electric currents flow through the coils and solenoids, the moving coils tend to orient themselves with their axes parallel to the axes of the stationary solenoids. If displaced from this position and released, the moving coil assembly will oscillate according to the equation of the torsion pendulum. A recording arm attached to the vertical shaft, traces the graph of the motion upon a sheet of paper mounted on a drum rotating at a uniform rate. The recording is accomplished by means of an electric spark which produces a distinct graph upon the Stylograph paper used for this purpose.

In the preliminary investigation covered by this discussion, the apparatus was adapted to the solution of Mathieu's equation, that is, $\ddot{\theta} + (a + b \cos wt)\theta = 0$. Therefore, the auxiliary equipment was arranged to furnish a sinusoidal current to one of the moving coils, thus providing for the "b cos wt" term of the equation, while the other moving coil was supplied with direct current to provide for the "a" term. The stationary solenoids were connected to a source of direct current. A rotary type of potentiometer, with the resistance coils properly proportioned, was employed to obtain the sinusoidal current. This was driven by a variable speed motor, thus allowing any value of "w" to be obtained.

By solving several equations of the Mathieu type on the analyzer, and comparing the results with those obtained by a method of successive approximation, the apparatus was shown to be capable of a sufficient degree of accuracy for practical purposes. In the operation of the analyzer, it is necessary so to limit the amplitude of oscillation that the sine of the angular displacement may be treated as equal to the angle, within the required degree of accuracy. Errors are also introduced through damping friction, but this effect was found to be negligible in most cases. If necessary, a correction can be made. During the tests, the total error was generally less than 5 % of the initial amplitude, although at one point it amounted to 12 %. It is hoped to improve the accuracy considerably by providing more sensitive control of the currents, and otherwise refining the apparatus.

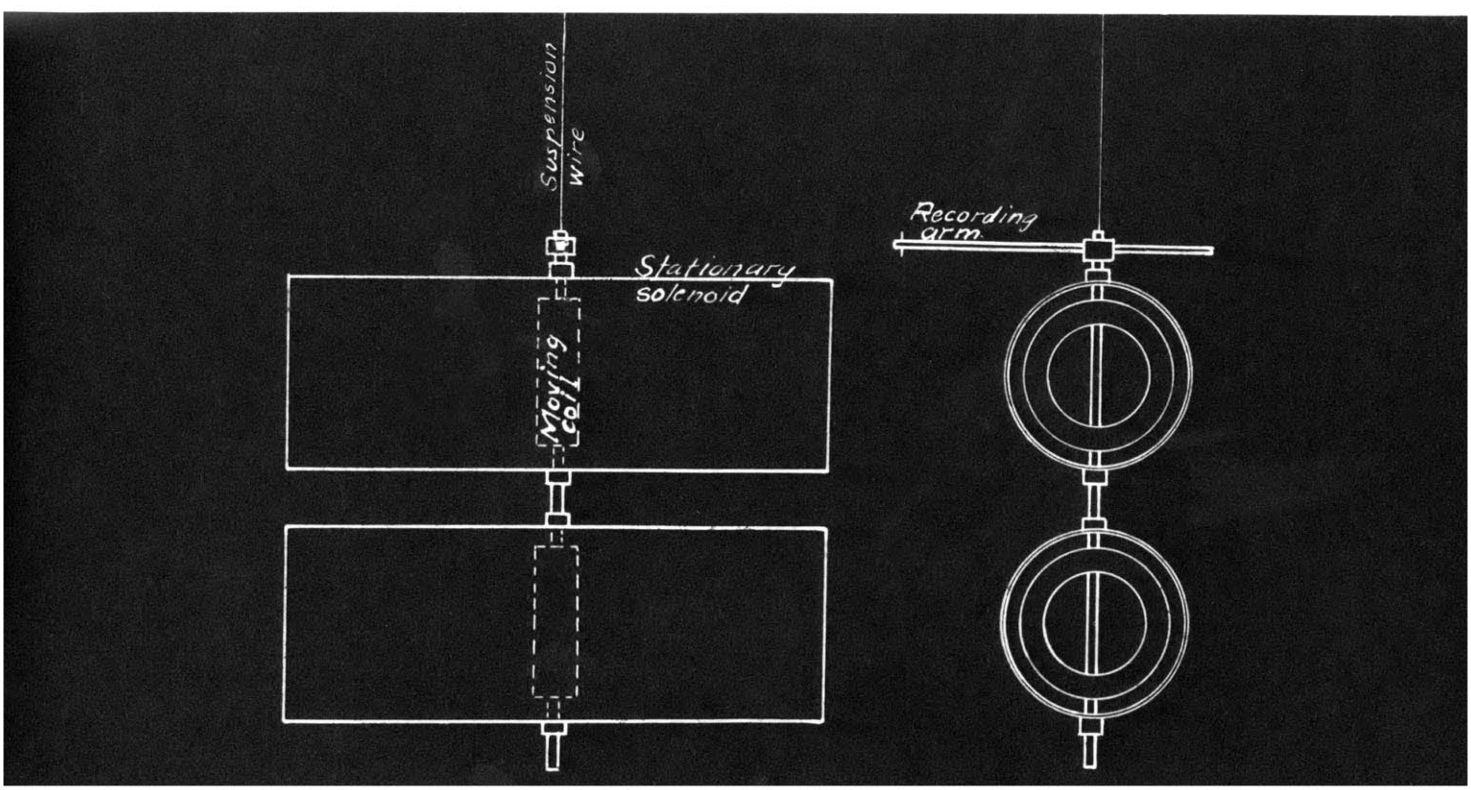
The results of the trials on the Mathieu equation have justified the further development of the apparatus to handle other types of second order equations. A potentiometer designed to furnish a current which is directly proportional to the displacement of the control from its mid position, can be arranged to produce a current varying according to any desired function of time, by utilizing a cam of the proper shape driven by a variable speed motor. In this manner, the range of the differential analyzer may be extended to include all equations of the general form: $\ddot{\theta} + f(t)\theta = 0$.

A direct current source of supply was used in these experiments, for the sake of convenience. For further development of the apparatus according to the ideas proposed by Dr. Minorski, it is necessary to adapt it to alternating current. The control of the moving coil current according to the function of time is accomplished by means of a "contour coil" device which is essentially an air core transformer with variable coupling. Alternating current of constant frequency is supplied to a long solenoid. A coil consisting of a few turns of wire wound to the required shape, slides back and forth through a slot in the middle of the solenoid. The voltage induced in this coil is proportional to that portion of its area which extends into the solenoid. By connecting to the grid of an amplifier which is included in the circuit of the moving coil of the analyzer, the current flowing will be proportional to the area of the contour coil inside the solenoid. By moving the contour coil into the solenoid at a uniform rate, the current is regulated according to the function represented by the shape of the contour. This arrangement takes the place of the cam operated potentiometer, and permits of a variety of functions of time to be generated by providing contour coils of the necessary outlines.

It is also possible to obtain functions of displacement by attaching a contour coil to the moving part of the analyzer. This will further increase the utility, by enabling the apparatus to solve equations of the form: $\ddot{\theta} + f(t,\theta)\theta = 0$.

Additional coils may be added to the rotor of the analyzer to provide a damping torque and a forced oscillation.

The utility of the analyzer was demonstrated by the solution of practical problems including an investigation of the range of stability of the Mathieu equation.



ACKNOWLEDGEMENT

The author wishes to express his thanks to all who gave assistance, especially to Professor C. E. Fuller and Dr. C. W. MacGregor, for their interest in the supervision of the research.

Valuable suggestions were given by Dr. S. H. Caldwell, Professor E. S. Taylor, and Dr. W. L. Barrow. Mr. C. E. Hentz advised as to the selection of electrical equipment, and Dr. F. L. Hitchcock gave information concerning the mathematical problems involved in the research.

The author also wishes to thank Mr. Perkins and his assistants in the machine shop, for their efforts and cooperation in the construction of the apparatus, and Mr. Broderick of the electrical engineering laboratory for his assistance in procuring electrical equipment.

CONTENTS

Note: Numbers in parentheses refer to items in the bibliography.

CHAPTER I

INTRODUCTION

The solution of differential equations by means of an apparatus may be accomplished by various methods. Among others, the following may be mentioned:

(a) In the type of differential analyzer developed by Dr. V. Bush and his associates in the Massachusetts Institute of Technology, several integrating units are employed, which may be interconnected in a variety of arrangements. (1) An integrating unit consists of two discs mounted at right angles to each other, the edge of one resting upon the polished surface of the other. The sharp edged disc can be moved axially across the surface of the polished disc. As the latter is rotated, it turns the other at a rate which is proportional to the distance from the point of contact to the center. The angular displacement of the driven disc is thus equal to a constant times the integral of f(x) dx, where dx is the increment of angular displacement of the driving disc. The axial movement is arranged to follow the curve of f(x) with respect to x. An operator follows the curve of the function by means of the pointer on the input table, thus moving the sliding disc of one of the integrators. In some cases, this operation is unnecessary, because one or more of the integrating units may be employed for the generation of the function.

An elaborate system of torque amplifiers and back-lash eliminators is provided.

(b) An apparatus has been developed by Dr. K. E. Gould, which performs the operation of integration by means of a beam of infra-red radiation passing through movable shutters and striking a thermocouple. (2). Another system, developed by Dr. T. S. Gray, utilizes visible light and a photoelectric cell. (3)

(c) The solution of certain types of second order differential equations may be accomplished more directly by means of a mechanical or electrical analogy. For instance, the equation: $\ddot{i} + (R/L)\dot{i} + (1/LC)i = 0$ applies to an oscillatory circuit. L is the inductance, R the resistance, and C the capacitance, while i represents the current in amperes. By arranging a circuit with variable L, R, and C, connected to an oscillograph, any equation of the general form: $\ddot{y} + f_1(t,y)\dot{y} + f_2(t,y)y = 0$ may be solved.

(d) The type of differential analyzer to which this discussion applies is based upon the analogy of the torsion pendulum. The equation: $(I/g)\ddot{\theta} + c\dot{\theta} + k\theta = 0$ is similar to the general form expressed in Paragraph (c). By varying the coefficients as functions of time and displacement, any equation of this form may be solved by providing a mechanism to trace the graph of angular displacement vs time.

In the development of the torsion pendulum type of differential analyzer, the basic problem is the generation and control of the restoring and damping torques. An electromagnetic method of accomplishing this result has been proposed by Dr. N. Minorski. (4). The restoring torque is produced by means of a short solenoid, free to rotate about its transverse axis, mounted inside of a long solenoid. When electric currents are passed through the coils, the movable solenoid tends to orient itself with its axis parallel to that of the surrounding coil, and if displaced from this position it will oscillate. A recording arm traces the graph of the oscillation upon a ribbon of paper which moves at a uniform speed. The currents are regulated according to the desired functions of time and displacement by auxiliary apparatus.

The development of the differential analyzer covered in this discussion is limited in application to equations of the Mathieu type, that is, of the form: $\ddot{\theta} + (a + b \cos wt)\theta = 0$. This may also include a damping term which can be removed by a change in the dependent variable, as explained in Chapter II.

Two sets of moving and stationary coils, mounted one above the other, are included in the apparatus. In solving the Mathieu equation, a continuous current is passed through both the stationary solenoids. One of the moving coils is supplied with a continuous current to provide for the "a" term of the equation, while the other is connected to a source of sinusoidal alternating current, the frequency of which may be varied at will, to take care of the "b cos wt" term.

The sinusoidal current is produced by suitable auxiliary apparatus. In the case of the analyzer under discussion, a rotary potentiometer driven by a variable speed motor is used.

If other functions of time are required instead of the cosine, a potentiometer actuated by a cam of the desired shape may be arranged.

The capacity of the analyzer is not limited to the solution of the Mathieu equation, but this preliminary investigation was made to determine whether the apparatus gave sufficient promise of success to justify further effort and expense in its development. The range of usefulness of the apparatus depends upon the variety of auxiliary apparatus which is provided, but the fundamental equipment of rotating and stationary solenoids, with recording device, serves for all applications.

CHAPTER II

APPLICATIONS OF THE MATHIEU EQUATION

The equation: $\ddot{\theta} + c\dot{\theta} + (a + b\ f(t))\theta = 0$ has many applications in physics and engineering, among which are included the following: (5, 6, 7)

1. Motion of a pendulum, the support of which is moved up and down periodically. A practical example is the vibration of the card of a magnetic compass.
2. Torsional oscillation in the drive mechanisms of electric locomotives.
3. Vibration of a stretched string with periodically varying tension.
4. Vibration of elliptic membranes.
5. Oscillations in electric circuits with periodically varying parameters. (8)
6. Torsional vibration of the crankshaft of a reciprocating engine.

The sinusoidal function of time is involved in most problems, either exactly or as a sufficiently close approximation.

The equation as stated at the top of the page may be simplified by eliminating the damping term. Let $uv = \theta$, where $v = e^{-\frac{ct}{2}}$, and substitute in the equation, obtaining:
$\ddot{u} + (a + b\ f(t) - c^2/4)u = 0.$

With the equation given in the form: $\ddot{\theta} + (a + b \cos wt)\theta = 0$, it is possible to eliminate the "w" by letting $x = wt$, $w^2 a_0 = a$, and $w^2 b_0 = b$. The equation becomes: $\ddot{\theta} + (a_0 + b_0 \cos x)\theta = 0.$

If, instead of the cosine function, the equation contains a periodic function of the independent variable with a period of 2π, it is known as "Hill's equation." An example of this type, where the function is of rectangular instead of sinusoidal form, has been mentioned in Chapter I. The general characteristics are similar to those of the Mathieu equation, but the analytical solution is much easier. (7,9)

In engineering problems involving the Mathieu equation, the object is usually to determine the conditions under which the vibration builds up to dangerous amplitudes. It may also be desired to investigate the actual form of the vibration, as in problems involving electric circuits or with reference to acoustics. In any event the analytical solution of the Mathieu equation is very difficult, while the differential analyzer can easily accomplish the result, with sufficient accuracy for practical purposes.

CHAPTER III

DESIGN AND CONSTRUCTION

The purpose is to build an apparatus for solving the Mathieu equation, but capable of being adapted to the solution of other second order equations of the general type discussed in ChapterII, by the addition of the necessary accessories.

The objects to be attained in the design are; sufficient accuracy for engineering purposes, flexibility and convenience in operation, and a minimum of expense in construction.

The general principle of the analyzer has already been discussed in Chapter I. The torque developed by the magnetic field of the moving and stationary coils is proportional to the product of the currents flowing in the coils, and the sine of the angle of displacement. Since the theory of operation is based upon a linear relationship between the torque and the angle of displacement, it is necessary so to restrict the amplitude that the angle and its sine may be considered equal, within the desired degree of accuracy. At 10 degrees the difference is about 0.5 %. It is also necessary to obtain a magnetic field of nearly uniform intensity, and of a strength that is proportional to the product of the currents. This requires a long stationary solenoid, a short moving coil, and the absence of metal within the magnetic field. On account of the small angle of displacement allowed, it is desirable to develop the maximum torque that is available.

Since the rotor is suspended from a fine steel wire, the only bearings required are guides to prevent the shaft from swinging. With the apparatus carefully leveled, the friction of these bearings is negligible. By designing for a long period of oscillation (from about 2 seconds upwards, according to the current strength in the coils),windage friction is reduced to a minimum.

The stationary solenoids are 37" long, wound on fibre tubes 16" outside diameter and about 0.1" thick. These are fastened into rectangular plywood flanges which form the vertical part of the frame. See Fig. 1, page 11. The dimensions were chosen according to the size of the fibre tube that was available, and to a reasonable compromise between length, and economy in construction. The longer the coil, the more uniform is the magnetic field produced, but more wire is required to generate a given field strength. The size of the flanges was limited by the capacity of the lathe in the machine shop which was used for winding the coils. Fibre pins are employed to fasten the flanges to the cores of the solenoids. Each solenoid is wound with approximately 1600 turns of #12 cotton covered and enameled wire. See page 11.

One moving coil consists of 4100 turns, the other of 2700 turns of #29 enameled wire. These are wound on fibre spools, and mounted on a micarta tube provided at each end with a small steel shaft which turns in agate guide bearings. See Fig. 2, page 12. Bushings are provided on the transverse center line of the stationary solenoids through which the micarta tubular shaft of the armature assembly passes.

The suspension wire is rectangular in section, .013"x .026", of Isoelastic material.* The rectangular section has a smaller torsional rigidity for a given tensile strength, than a circular section. The wire is secured at each end by a chuck made from a small steel rod with a #56 drill hole in the center. The wire was inserted in the hole, and the rod compressed in a vise, thus crushing it and holding the wire securely in place. Before assembling the suspension system, the wire was straightened by hanging a weight from it, and was heated to about 800°F. by passing an electric current through it for a few seconds. This was done to establish the "isoelastic" properties.

The suspension wire hangs inside a 1½" iron pipe, about 5 ft. long. An adjustment is provided at the upper end, which has a range of several inches in the vertical direction, and about ¾" in the horizontal direction.

Since the restoring torque of the suspension wire was found to be negligible in most cases, it is not necessary to specify Isoelastic material.

The recording arm is attached to the upper end of the armature shaft. It carries a pointed electrode which traces the graph by means of an electric spark, on a sheet of Stylograph paper secured to the drum.

The high tension current is produced by a Model T Ford ignition coil, and is conducted to the recording arm through a short spark gap between a flat electrode on the arm and

* The Isoelastic wire was supplied by Professor A. V. de Forest of the Mechanical Engineering Department of the M.I.T. See p. 74(a).

a heavy bare wire bent in the shape of a circular arc. This arrangement eliminates all sliding contacts or extra lead-in spirals. The point moves in an arc of 20" radius, and consequently the ordinates of the graph are circular arcs instead of straight lines.

The recording drum was made from a piece of 12" iron pipe. It is mounted on ball bearings, and is rotated at a speed of 1 r.p.m. by a Telechron synchronous motor.

The potentiometer, as may be seen from the photograph, Fig. 3, consists of a flat faced commutator and a rotating arm carrying two brushes. The commutator segments are connected to taps on a resistance coil so proportioned as to produce the required sinusoidal variation of current. The outfit is driven through a reduction gear by a variable speed D.C. motor, and the speed is indicated by a magneto tachometer. Fortunately, this commutator, already mounted with bearings, etc., was found in the store room of the Electrical Engineering Department. At the right hand end of the shaft, as shown in Fig. 3, are two slip rings to lead the current from the brushes to the coil of the analyzer. Also, there is an adjustable contact which actuates a magnetic release for the rotor of the analyzer. When the potentiometer arm reaches the required point on the commutator, the release mechanism operates and the arm begins to trace the graph. The release arm is located at the lower end of the armature shaft. A latch, actuated by a magnet, holds this arm stationary until the contact closes the circuit.

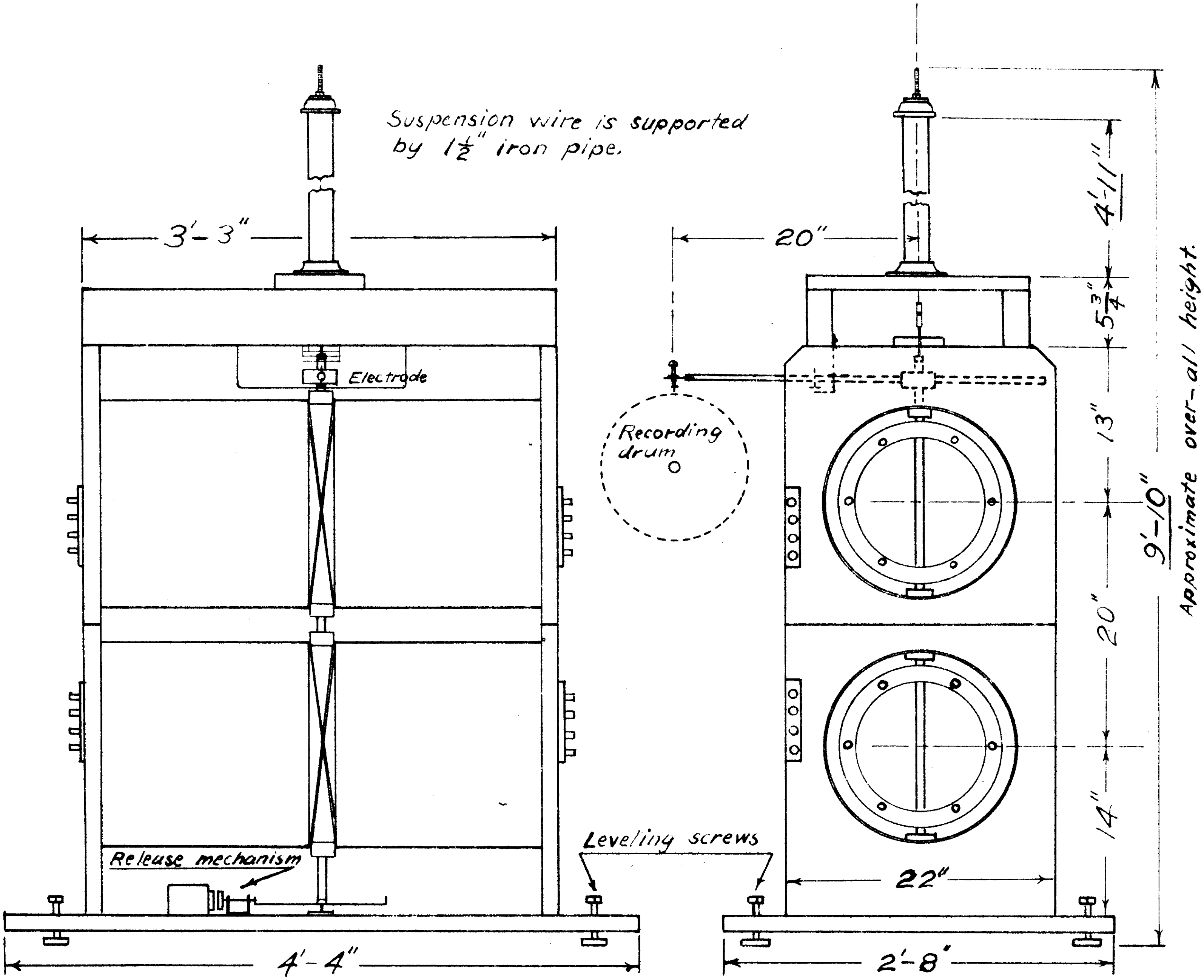


Fig. 1.

Scale 1" = 1'.

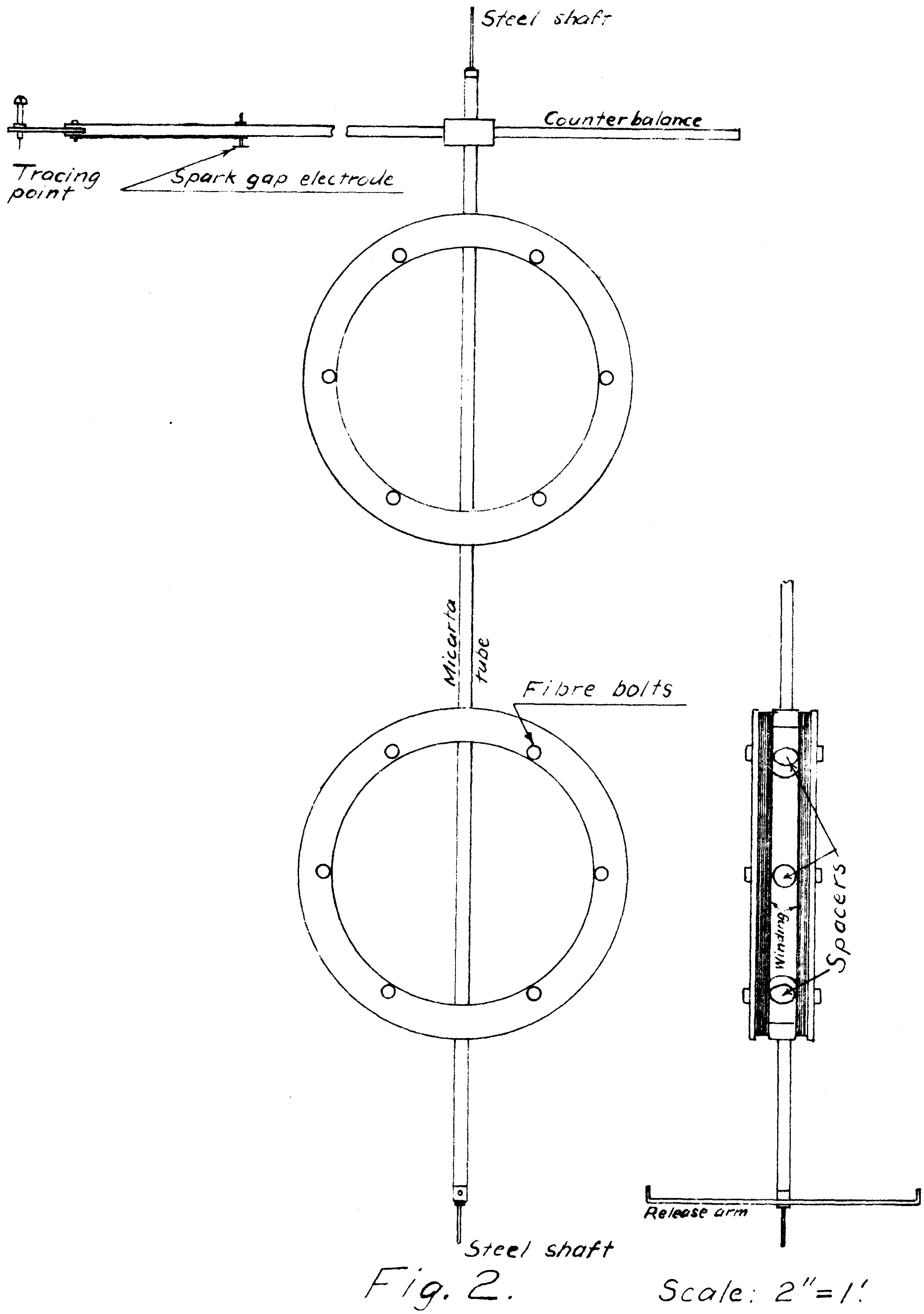


Fig. 2. Scale: 2"=1'

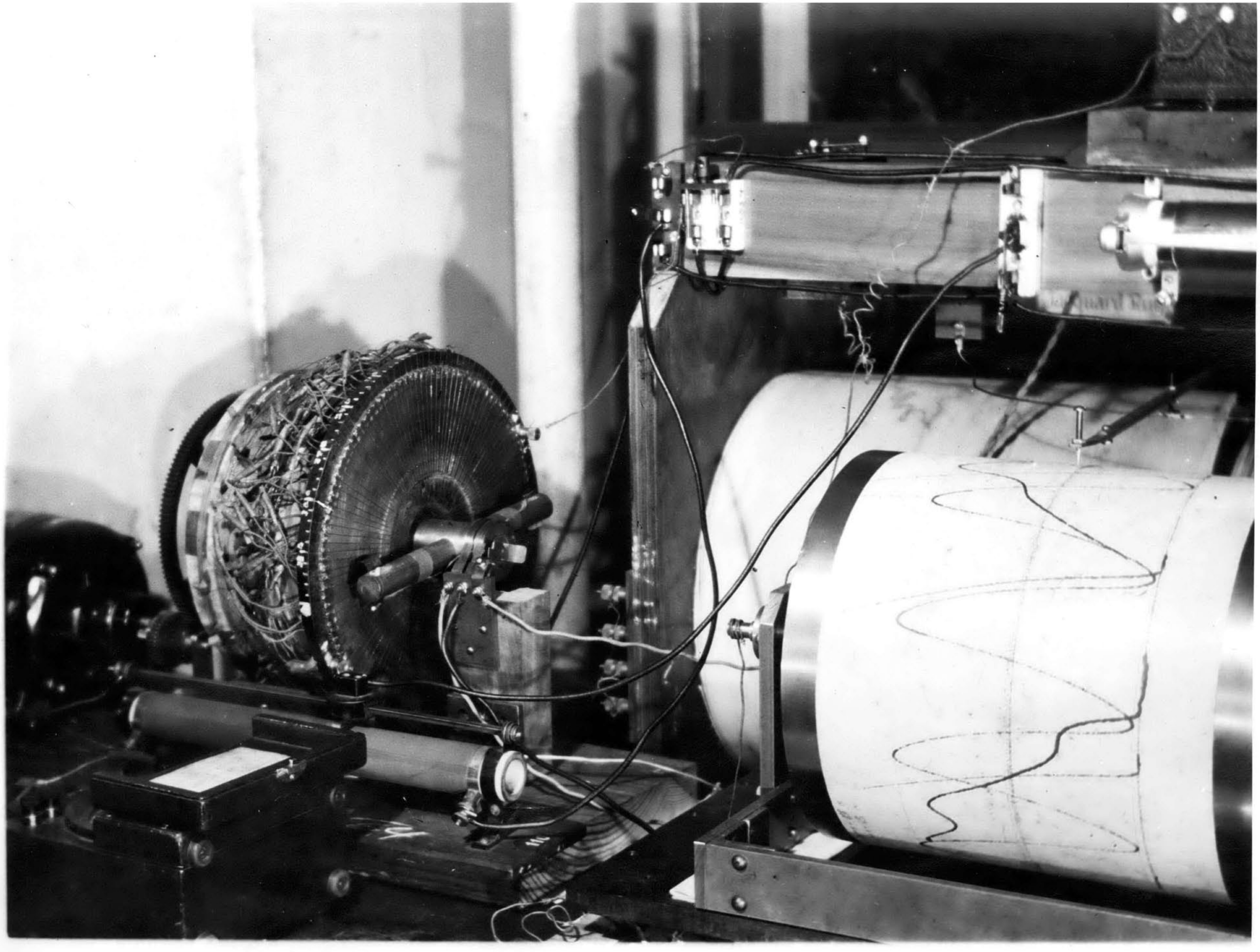


Fig. 3.

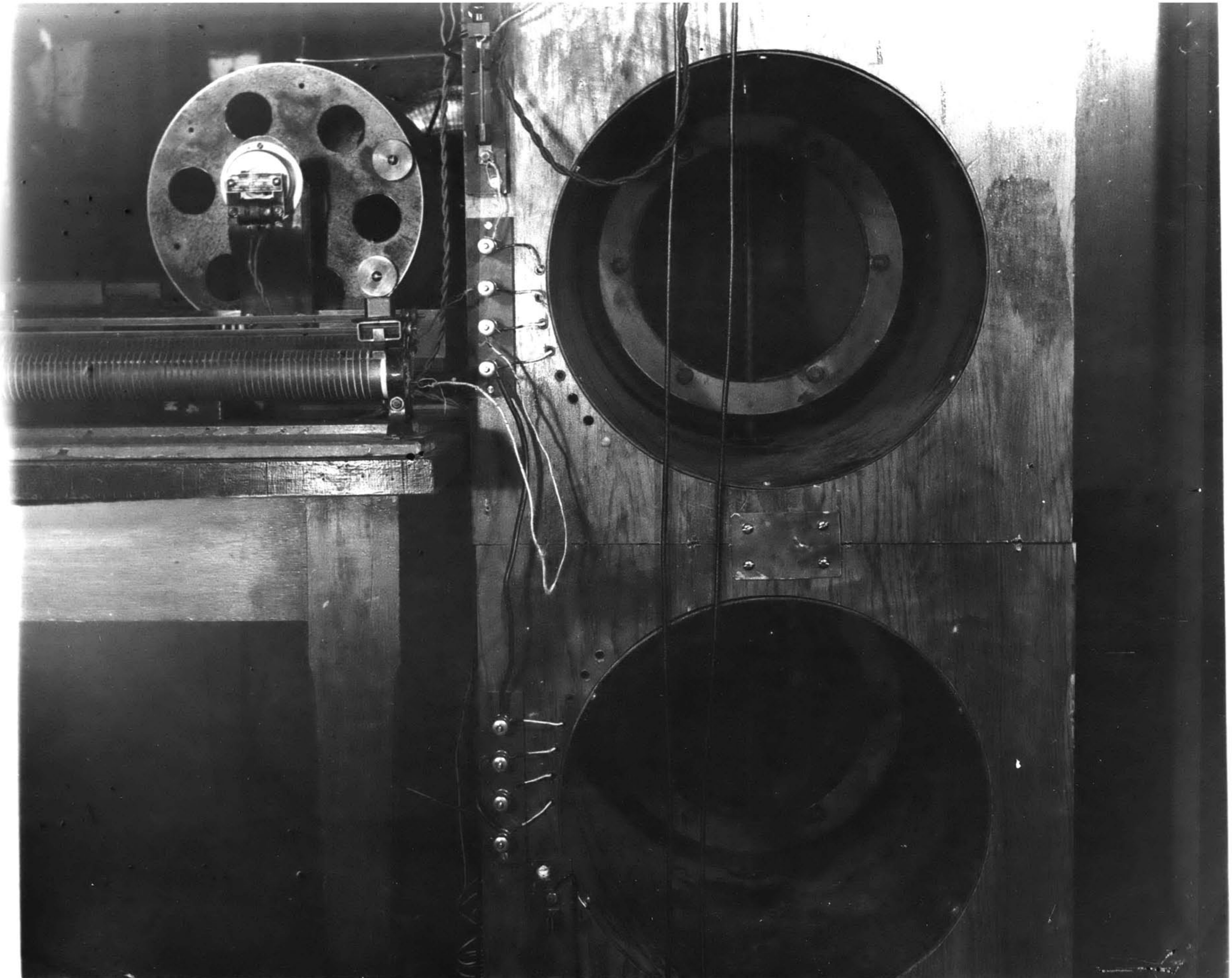


Fig. 4

14

(a) Stationary solenoids

#12 cotton and enameled wire. 0.081" bare diam.
0.087" insulated diam.
19.8 lbs per 1000 ft.
1.75 ohms " " "

Length of coil: 37"

Winding space: 37" - 1.25" = 35.75"   35.75/0.087 = 410

Use 400 turns per layer

4 layers ---- 1600 turns.

Depth of coil = 4 x 0.087 = 0.35"

Inside diam. 16"

Outside " 16.7"

Mean " 16.35"

Length of wire = 4 x 16.35π x 400/12 = 6850 ft.

Weight " " = 6850 x 0.0198 = 135 lbs.

Resistance = 6850 x 0.00175 = 12 ohms, for 4 layers in series.

Current capacity: Allow 0.5 watts per sq. in. radiating surface.

See Bibliography (10)

2 x 37 x 16.35π = 3800 sq.in. radiating surface.

37 x 0.5 = 1900 watts = $I^2R = 12\ I^2$.   I = 12.5 amps.

Note: The actual number of turns are: upper solenoid, 1580
lower " 1578

(b) Moving coils

#29 enameled wire.
- 0.0113" bare diam.
- 0.0123" insulated diam.
- 0.384 lbs per 1000 ft.
- 90. ohms per 1000 ft.

Length of coil: $1\frac{1}{4}$" 1.25/0.0123 = 101.5

Use 100 turns per layer.

4100 turns ---- 41 layers.		2700 turns --- 27 layers.
Depth of coil = 41 x 0.0123 = 0.51"		27 x 0.0123 = 0.33"
	Inside diam. = 12"	
Outside diam. = 13.02"		12.7"
Mean " = 12.51"		12.4"
Length of wire = 13450 ft.		8900 ft.
Resistance = 1220 ohms		805 ohms

Current capacity:

$2 \times 12.5\pi \times 1.25 = 100$ sq.in. radiating surface, which gives a capacity of 50 watts.

$50 = 1220\ I^2$. $I = 0.2$ amps. $50 = 805\ I^2$. $I = 0.8$ amps.

Actually, the maximum current is limited to about 0.2 amps by the capacity of the spiral lead-in wires.

By measurement, the resistances of the coils were found to be:

Cold:	1040 ohms	683 ohms
Warm:	1099 ohms	720 ohms

The resistance when warm was measured after currents of 0.2 amps., and 10 amps., respectively, had passed through the moving and stationary coils for about 30 minutes.

(c) Inductance of moving coils.

$$L_{henrys} = \frac{0.366\ (Ft/1000)^2\ F'F''}{b + c + R}$$

b = length of coil, in.
c = depth " " "
R = outside radius.
r = inside "
a = mean "
Ft = length of conductor, feet.

See Bibliography (12)

$$F' = \frac{10\ b + 12\ c + 2\ R}{10\ b + 10\ c + 1.4\ R}$$

$$F'' = 0.5 \log_{10} \left(100 + \frac{14\ R}{2\ b + 3\ c} \right)$$

For the coil with 4100 turns,

$$F' = \frac{10 \times 1.25 + 12 \times 0.51 + 2 \times 6.5}{10 \times 1.25 + 10 \times 0.51 + 1.4 \times 6.5} = 1.18$$

$$F'' = 0.5 \log_{10} \left(100 + \frac{14 \times 6.5}{2.5 + 1.5} \right) = 1.045$$

$$L = \frac{0.366(\ 13.45\ )^2}{1.25 + 0.51 + 6.5}\ 1.18 \times 1.045 = 9.9 \text{ henrys.}$$

For the coil with 2700 turns,

$$F' = \frac{10 \times 1.25 + 12 \times 0.33 + 2 \times 6.4}{10 \times 1.25 + 10 \times 0.33 + 1.4 \times 6.4} = 1.18$$

$$F'' = 0.5 \log_{10} \left(100 + \frac{14 \times 6.4}{2.5 + 0.99} \right) = 1.050$$

$$L = \frac{0.366(\ 8.9\ )^2}{1.25 + 0.33 + 6.4}\ 1.18 \times 1.05 = 4.5 \text{ henrys.}$$

(d) Inductance of stationary solenoids.

$$F' = \frac{10 \times 37 + 12 \times 0.35 + 2 \times 8.4}{10 \times 37 + 10 \times 0.35 + 1.4 \times 8.4} = 1.03$$

$$F'' = 0.5 \log_{10} \left(100 + \frac{14 \times 8.4}{74 + 1.05} \right) = 1.004$$

$$L = \frac{0.366 (6850/1000)^2}{37 + 0.35 + 8.4} \, 1.03 \times 1.004$$ = 0.39 henrys for 4 layers in series.

(e) Torque of armature.

$$\text{Torque, gram-centimeters} = 10200\ I_1 I_2 \frac{4\pi A\ c_1 c_2 \cos B}{10^9 \sqrt{L^2 + 4a^2}}$$

See Bibliography (11)

a = radius of helix, cm. = 8 x 2.54 = 20.3 cm.

A = area of moving coil = $(6 \times 2.54)^2\pi$ = 730 sq.cm.

L = length of helix = 37 x 2.54 = 94 cm.

c_1, c_2 = number of turns on helix and coil.

I_1, I_2 = current , amperes.

B = angle of inclination of coil.

Torque = $0.00092\ I_1 I_2 c_1 c_2 \cos B$, gram centimeters

Torque, ft. lbs. = the above divided by 2.54 x 12 x 454

$= 6.6 \times 10^{-8}\ I_1 I_2 c_1 c_2 \cos B$

Let $B = 90^\circ - \theta$, where θ is the angle between the axes of the helix and the coil.

Then, for small angles, Torque, ft.lbs. = $6.6 \times 10^{-8}\ I_1 I_2 c_1 c_2 \theta$

For c_2 = 4100, c_1 = 1600, Torque = $0.43\ I_1 I_2 \theta$ ft.lbs.

2700 1600, " = $0.28\ I_1 I_2 \theta$ ft.lbs.

(f) Spring constant of suspension wire.

A steel disc, 2.22" radius, 1.25" thick, weight 4.96 lbs., was suspended from the wire. The period of torsional oscillation was found to be 25.3 seconds.

$$\text{Moment of inertia} = \frac{W\ r^2}{2} = \frac{4.96}{2}\left(\frac{2.22}{12}\right)^2 = 0.0849\ \text{lb.ft}^2$$

$$\text{Period} = 2\pi\sqrt{I/gk} \qquad 25.3 = 2\pi\sqrt{0.0849/32.2k}$$

k = 0.000162 ft. lbs. per radian deflection.

(g) Moment of inertia of armature.

The period of the armature, with spiral lead-in wires disconnected and bearings removed, was found to be 125.6 seconds.

$$\left(\frac{125.6}{25.3}\right)^2 = \frac{\text{moment of inertia of armature}}{0.0849}$$

Moment of inertia of armature without release arm = 2.09 lb.ft^2
The release arm was added later. This increased the moment of inertia of the armature assembly to 2.097 lb.ft^2

(h) Restoring torque of suspension wire and spiral lead-in wires.

The electricity is conducted to the armature through three spirals of #37 enameled wire, in addition to the fourth circuit through the suspension wire.

With the lower bearing only, in place, the period of the armature with spirals connected was found to be 121 seconds.

$$\left(\frac{121}{125.6}\right)^2 = \frac{0.000162}{x} \qquad x = 0.000174$$

The restoring torque of the system = 0.000174 ft.lbs. per radian deflection.

With a current of 1 amp. in the stationary solenoid, and 0.01 amp. in the armature coil of 2700 turns, the torque per radian is approximately 0.0028 ft.lbs. Under these conditions the torque of the suspension is 6.2 % of the torque generated by the magnetic field. In most cases the torque of the suspension may be neglected.

(i) The rotary potentiometer.

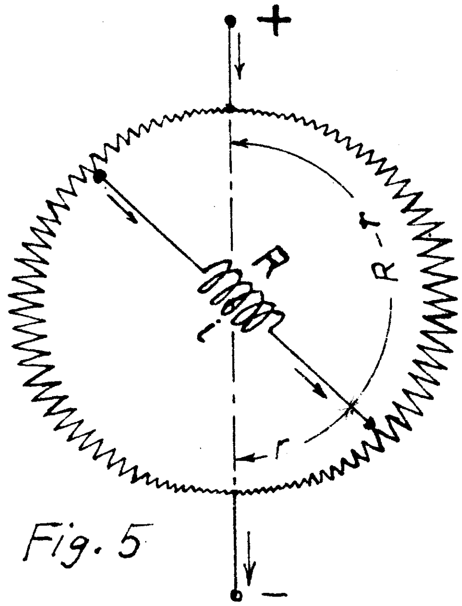


Fig. 5

E = voltage of supply.

e = " across coil.

R = resistance of coil, and also the resistance of one side of the potentiometer from the + terminal to the -.

i = current in coil.

i_1 = current in resistance r.

i_2 = " " " R - r.

The coil marked R in the center of the diagram represents the moving coil of the analyzer, and the arrows represent the brushes of the potentiometer. When the brushes are in the vertical position, maximum voltage is applied to the coil. When they are horizontal, the applied voltage is zero.

$$i = \frac{e}{R} = \frac{E - 2\,i_1 r}{R}$$

$$i_1 = \frac{E - iR}{2\,r}$$

$$i_1 r - i_2(R - r) + iR = 0$$

$$i_2 = i_1 - i$$

$$i_1 r - (i_1 - i)(R - r) + iR = 0$$

$$i_1 r - i_1 R + i_1 r + iR - ir + iR = 0$$

$$i_1(2r - R) + i(2R - r) = 0$$

$$\frac{E - iR}{2r}(2r - R) + i(2R - r) = 0$$

$$2rE - 2Rri - RE + R^2 i + 4Rri - 2r^2 i = 0$$

$$2rE - RE + 2Rri + R^2 i - 2r^2 i = 0$$

$$i = \frac{E(R - 2r)}{R^2 + 2Rr - 2r^2}$$

$$r = \tfrac{1}{2}\left(R + E/i - \frac{1}{i}\sqrt{3R^2 i^2 + E^2}\right)$$

It was originally planned to construct the potentiometer in a similar form to that of a slide wire rheostat, but a flat faced commutator was found in the storeroom of the Electrical Engineering Department, which was used for a contact surface instead of the resistance wire itself. This makes a more durable piece of equipment.

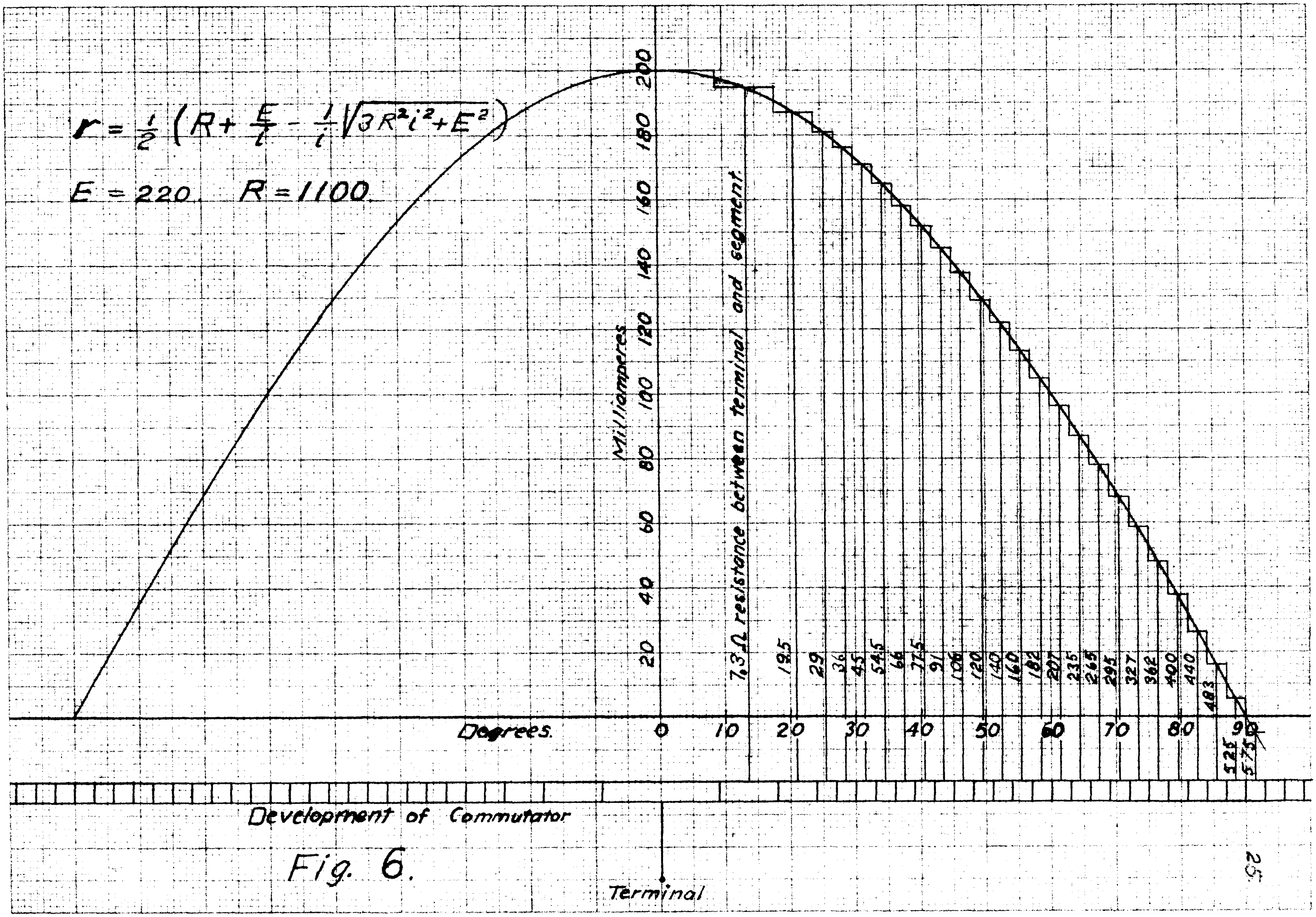
The resistance coil is wound with #29 Chromel wire, 2.37 ohms per foot. Taps leading to the commutator segments are taken off according to the diagram shown in Fig. 6, page 25, which indicates the number of ohms resistance between each commutator segment and one of the terminals connected to the supply. The commutator has 120 segments, but some of these are bridged together, as shown by the diagram.

This diagram indicates the current flowing through the moving coil of the analyzer, for a supply voltage of 220. The resistance of the coil circuit must be 1100 ohms to give the current variation shown. If the moving coil of 4100 turns is connected to the brushes, this condition is satisfied, but if the other coil (2700 turns), is used in the circuit, a resistance of 1100 - 720 = 380 ohms must be connected in series.

Actually, the corners of the current wave are rounded off, because of the inductance of the circuit.

The potentiometer is driven through a gear reduction by a $\frac{1}{4}$ h.p. direct current motor. The most satisfactory connection is to supply the field with 220 volts, and the armature with either 110 or 220 volts connected across a slide wire rheostat to form a potentiometer circuit. This arrangement provides a steadier and more flexible speed control than the use of a rheostat in series with the armature.

Gear reductions of 6.67 to 1, 22.22 to 1, and 66.67 to 1, are available.



$r = \frac{1}{2}\left(R + \frac{E}{i} - \frac{1}{i}\sqrt{3R^2i^2+E^2}\right)$
E = 220 R = 1100
Milliamperes
20 40 60 80 100 120 140 160 180 200
7.3 Ω resistance between terminal and segment.
19.5 29 36 45 54.5 66 77.5 91 106 120 140 160 182 207 235 265 295 327 362 400 440 483 525 575
Degrees.
0 10 20 30 40 50 60 70 80 90
Development of Commutator
Terminal

Fig. 6.

CHAPTER IV

TESTING FOR ACCURACY

(a) Calibration

Period $= 2\pi\sqrt{2.097/gk}$ 2.097 = moment of inertia of armature.

$k = 2.575/T^2$ k = restoring torque per radian deflection.

T = period.

With a steady current flowing through the moving coil, the torque $= C\, I_1 I_2\, c_1 c_2 \theta$, where C = a constant.

I_1, I_2 = the currents in amperes, flowing through the stationary and moving coils.

See page 19.

c_1, c_2 = the number of turns on the stationary and moving coils.

With the same current flowing through both stationary solenoids, and one moving coil at a time, the following values were found:

c_1	c_2	I_1	I_2	T	k	$C \times 10^8$
1580	2700	8.03	0.05	5.22	0.0945	5.50
			0.10	3.67	0.191	5.57
			0.15	2.99	0.287	5.58
1578	4100	8.13	0.048	4.30	0.139	5.50
			0.148	2.46	0.425	5.45
					Average	5.52

By test, Torque $= 5.52 \times 10^{-8}\, I_1 I_2 c_1 c_2 \theta$

By formula, " $= 6.6$ " " " " " " "

This calibration is valid only when the same number of amperes flows through both stationary solenoids.

(b) Consistency of results.

The following differential equations were solved by the analyzer, several graphs being obtained from each equation.

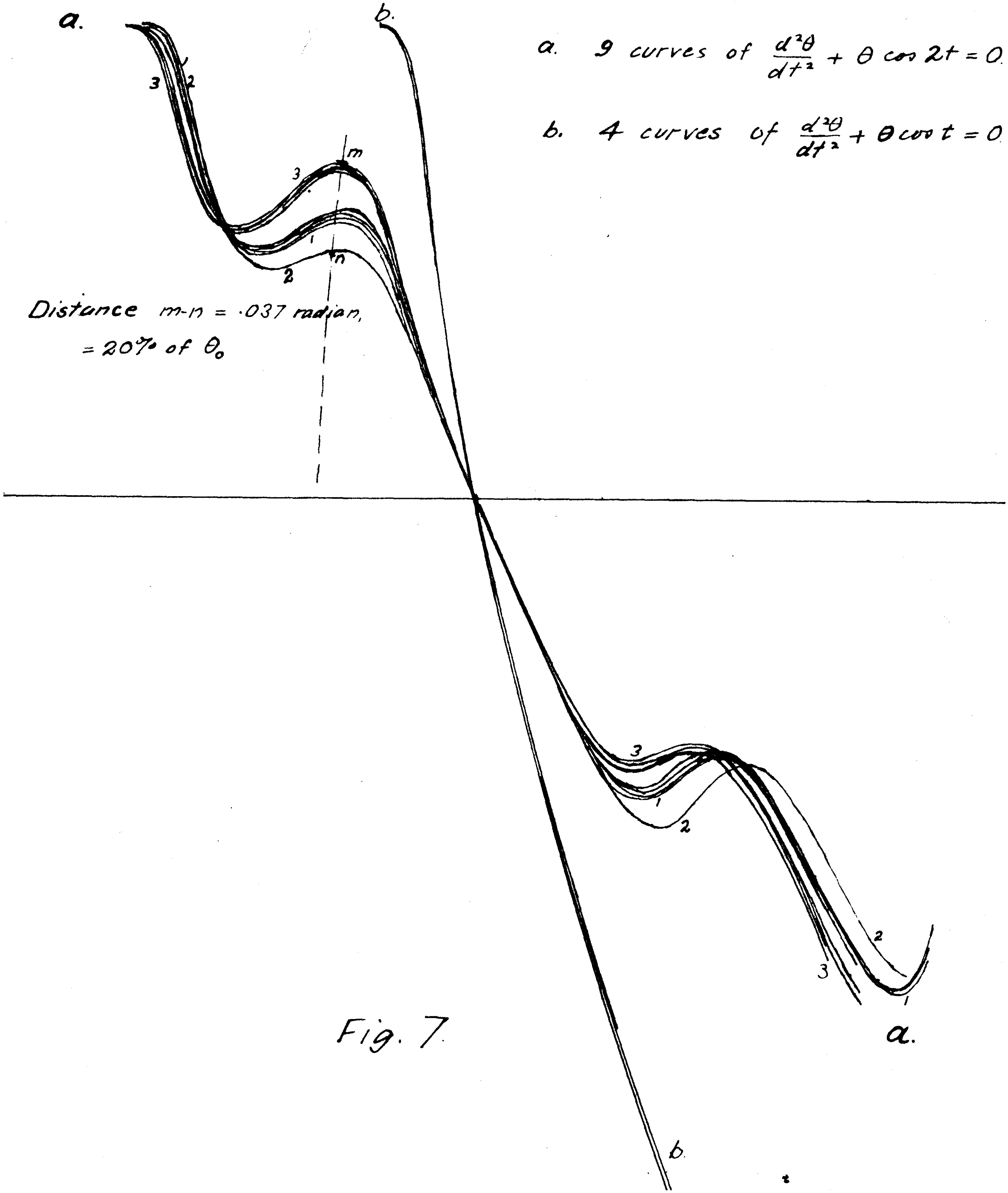
$\ddot{\theta} + \theta \cos t = 0$	$\ddot{\theta} + \theta \cos 2t = 0$
$\ddot{\theta} + \theta \cos 4t = 0$	$\ddot{\theta} + 2\theta \cos 4t = 0$
$\ddot{\theta} + (1 + \cos t)\theta = 0$	$\ddot{\theta} + (1 + \cos 2t)\theta = 0$
$\ddot{\theta} + (1 + \cos 4t)\theta = 0$	$\ddot{\theta} + (2 + \cos t)\theta = 0$
$\ddot{\theta} + (1 + 2\cos 6t)\theta = 0$	$\ddot{\theta} + (\frac{1}{2} + 2\cos 4t)\theta = 0$

The graphs of each equation were traced. In doing this, the curves obtained for the same equation were lined up under the tracing cloth with the first point where the curves cross the "t" axis, coinciding.

From the prints following this page, it may be seen that the operation of the machine is reasonably consistent, except for cases where several points of inflection occur, such as are found in the solutions of equations:

$\ddot{\theta} + \theta \cos 2t = 0$	Fig. 7a.
$\ddot{\theta} + \theta \cos 4t = 0$	Fig. 8.
$\ddot{\theta} + 2\theta \cos 4t = 0$	Fig. 9.

The graphs of these equations fail to coincide at the points of inflection by as much as 22 % of the initial amplitude. The other prints show a spreading of the graphs of from 9 % to 16 % of the initial amplitude.



a.
b.
a. 9 curves of $\frac{d^2\theta}{dt^2} + \theta \cos 2t = 0$
b. 4 curves of $\frac{d^2\theta}{dt^2} + \theta \cos t = 0$
m
n
Distance m-n = ·037 radian, = 20% of θ_0
3
2
1
a.
b.

Fig. 7.

a. Curves of $\frac{d^2\theta}{dt^2} + \theta \cos 4t = 0$

b. Continuation of same curves.

5 curves.

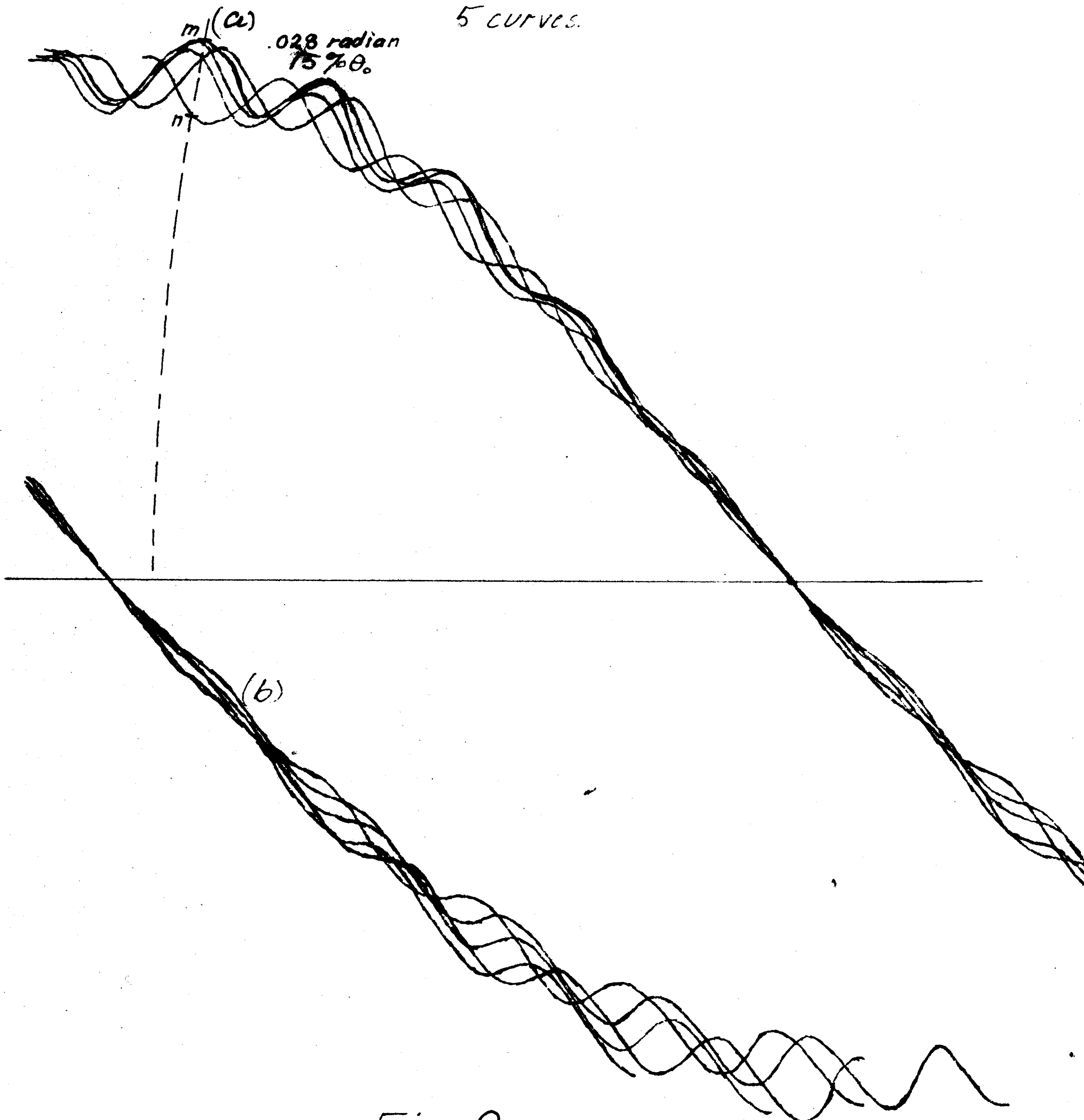


Fig. 8.

5 curves of $\frac{d^2\theta}{dt^2} + 2\theta \cos 4t = 0$

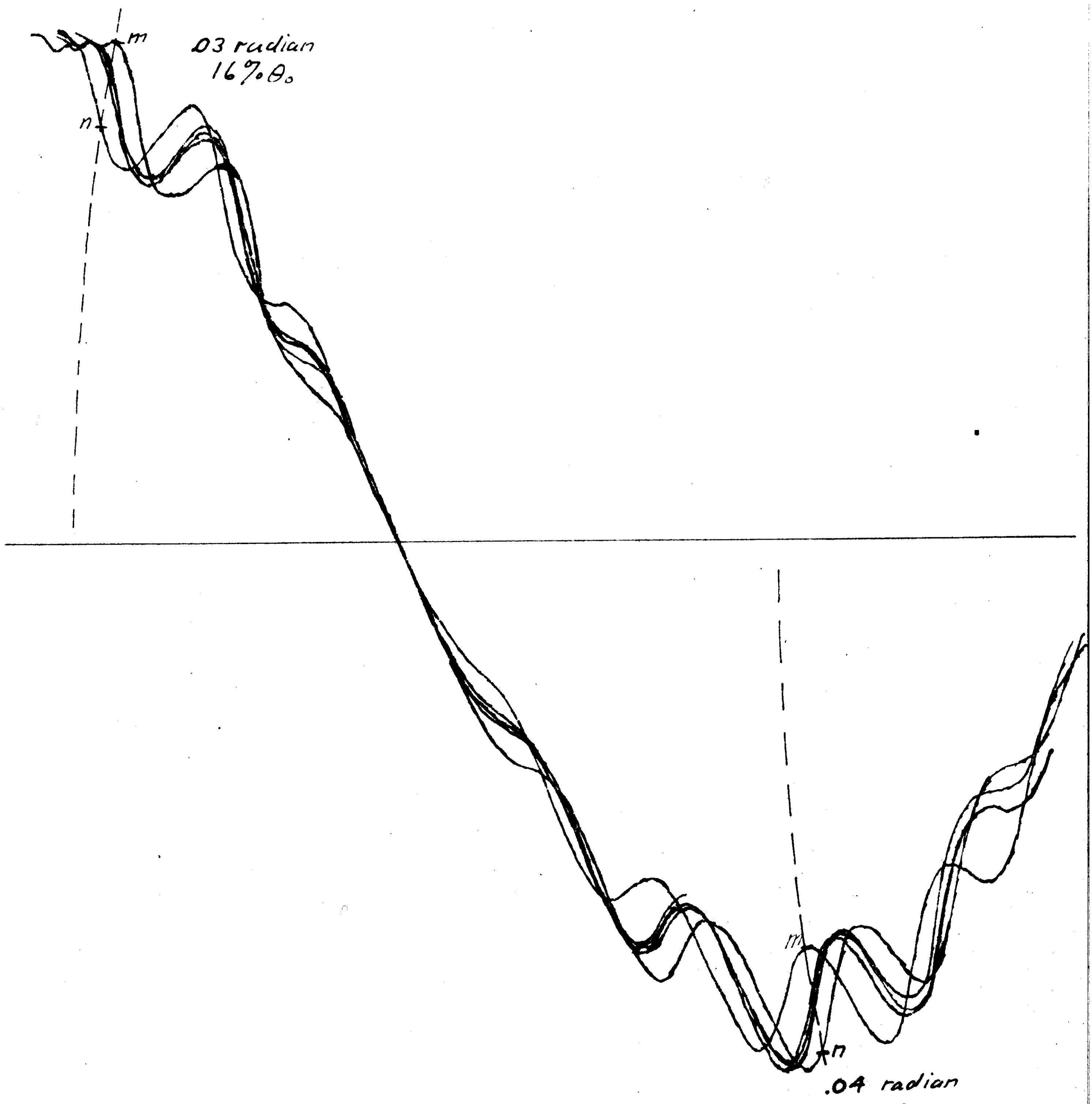


Fig. 9.

a. 5 curves of $\frac{d^2\theta}{dt^2} + (1+\cos t)\theta = 0$.

b. 4 " " $\frac{d^2\theta}{dt^2} + (1+\cos 2t)\theta = 0$.

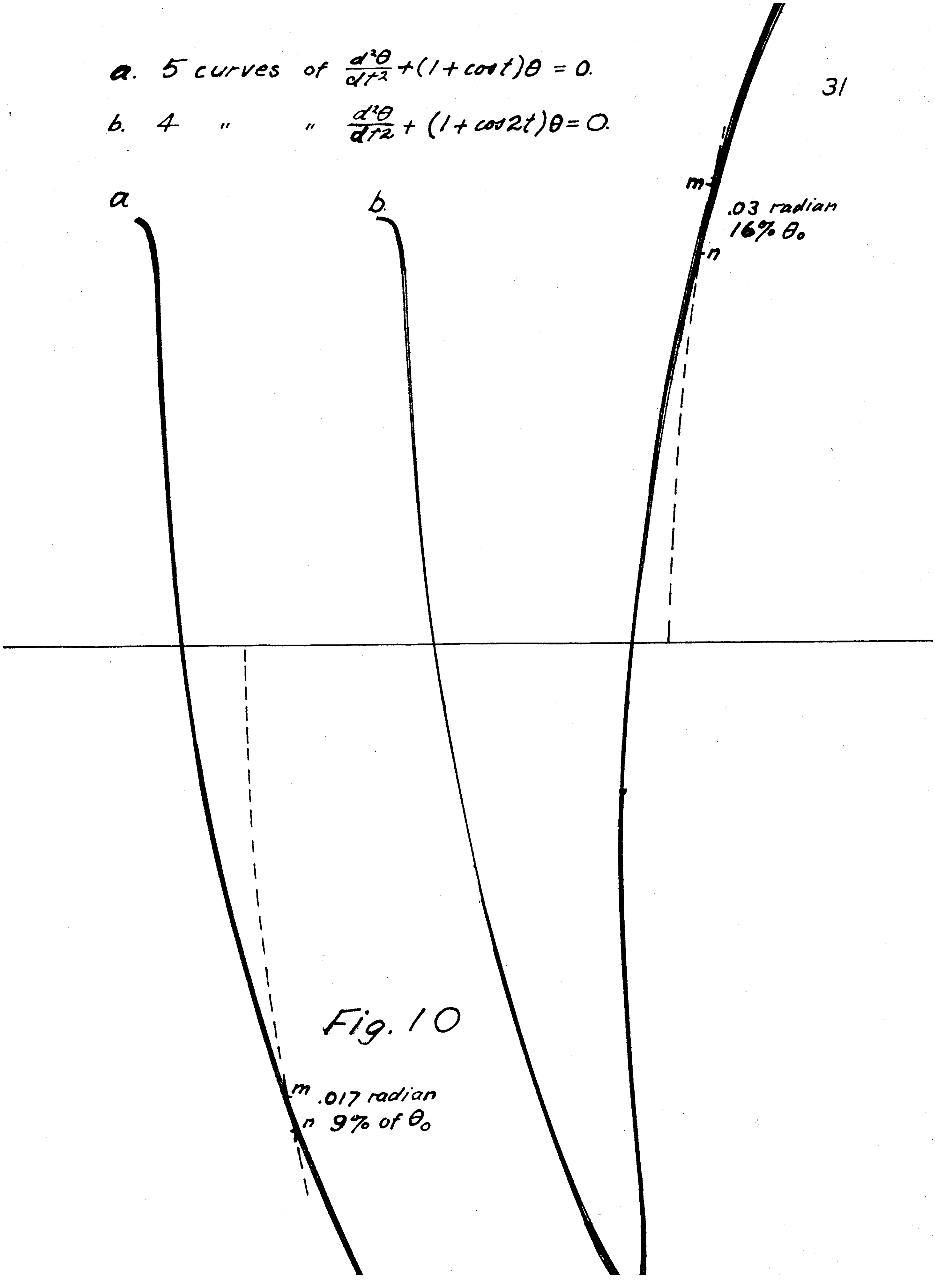


Fig. 10

5 curves of $\frac{d^2\theta}{dt^2} + (1 + \cos 4t)\theta = 0$

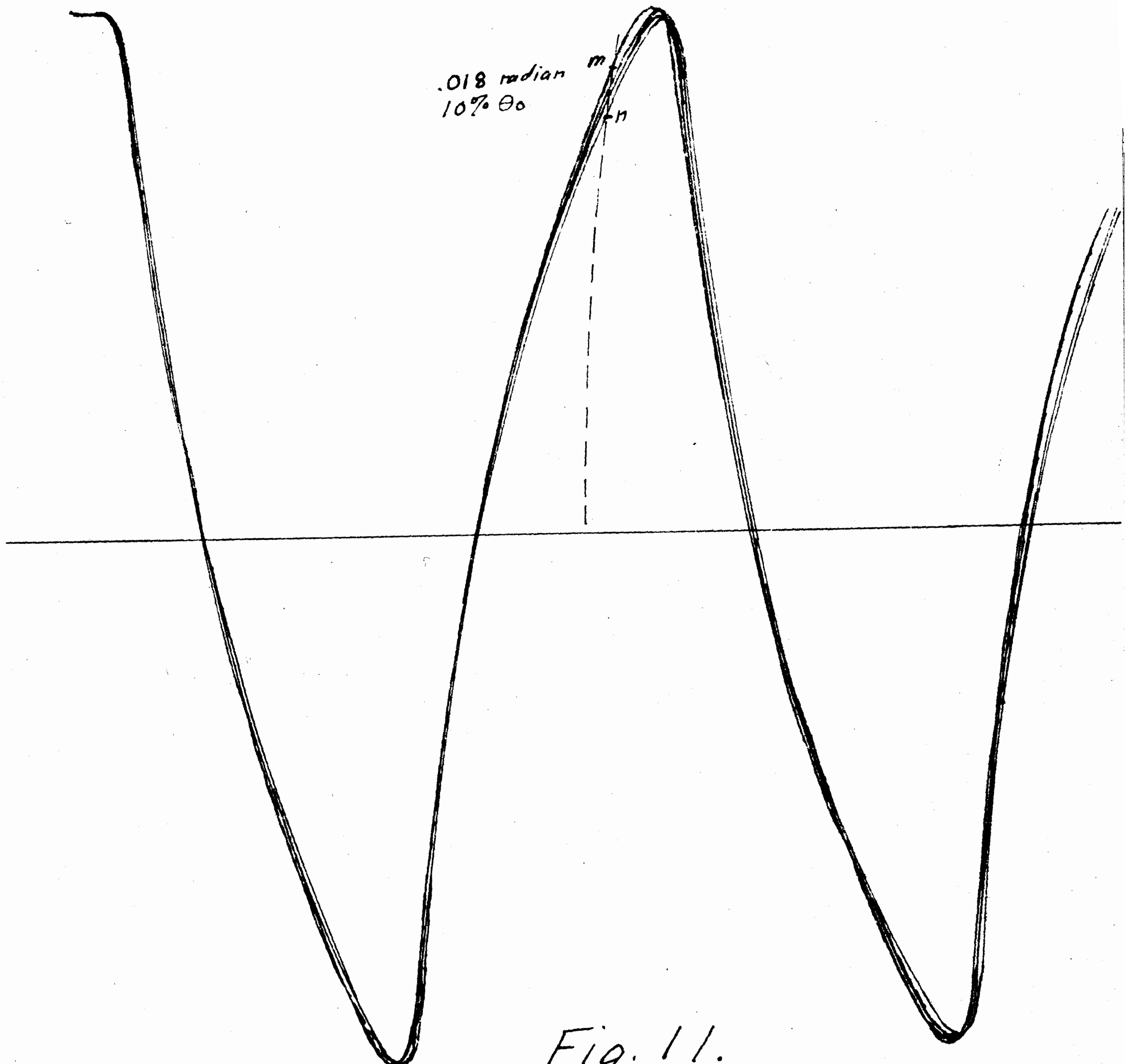


Fig. 11.

6 curves of $\frac{d^2\theta}{dt^2} + (2 + \cos t)\theta = 0.$

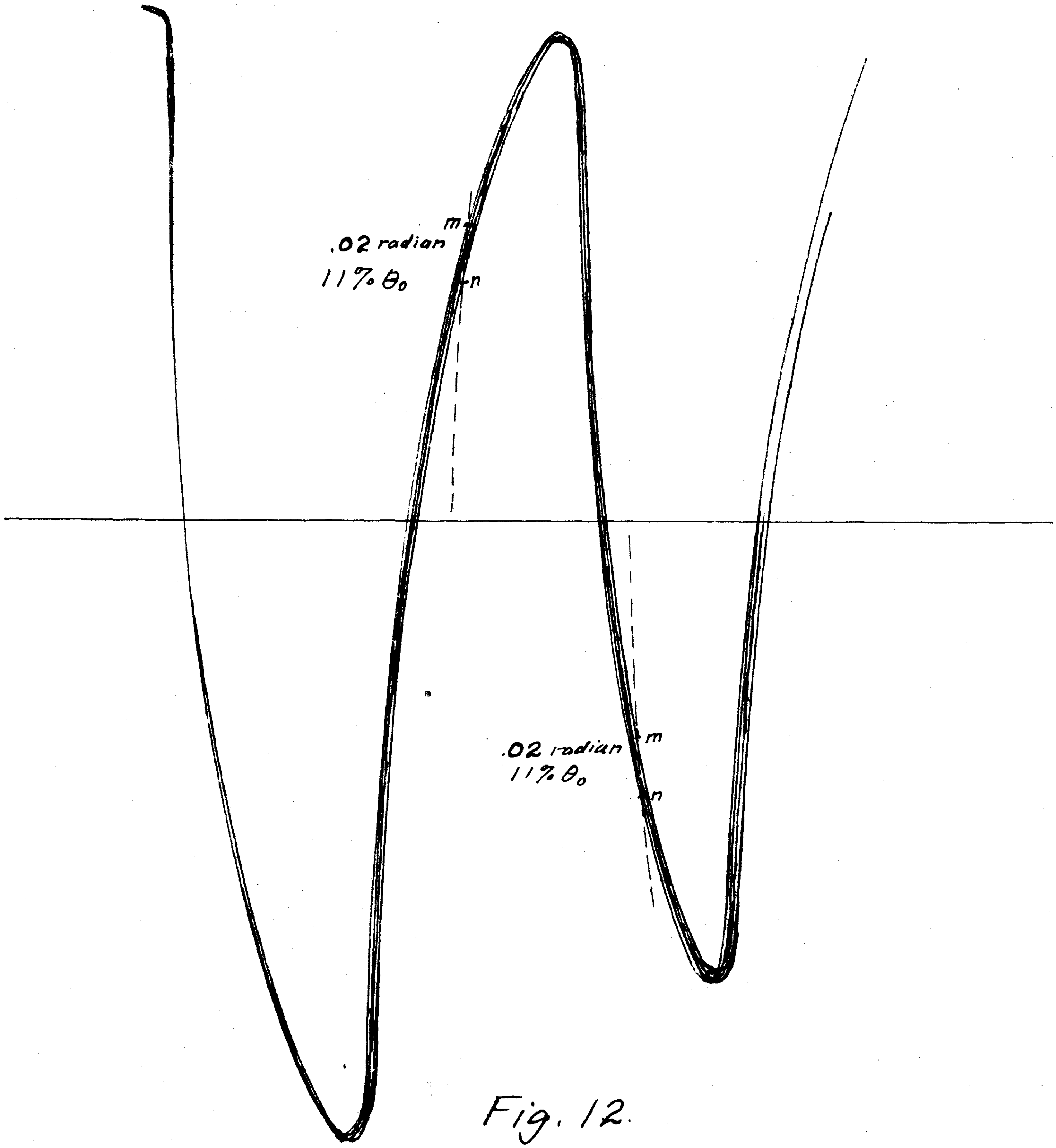


Fig. 12.

5 curves of $\frac{d^2\theta}{dt^2}+\left(\frac{1}{2}+2\cos 4t\right)\theta=0.$

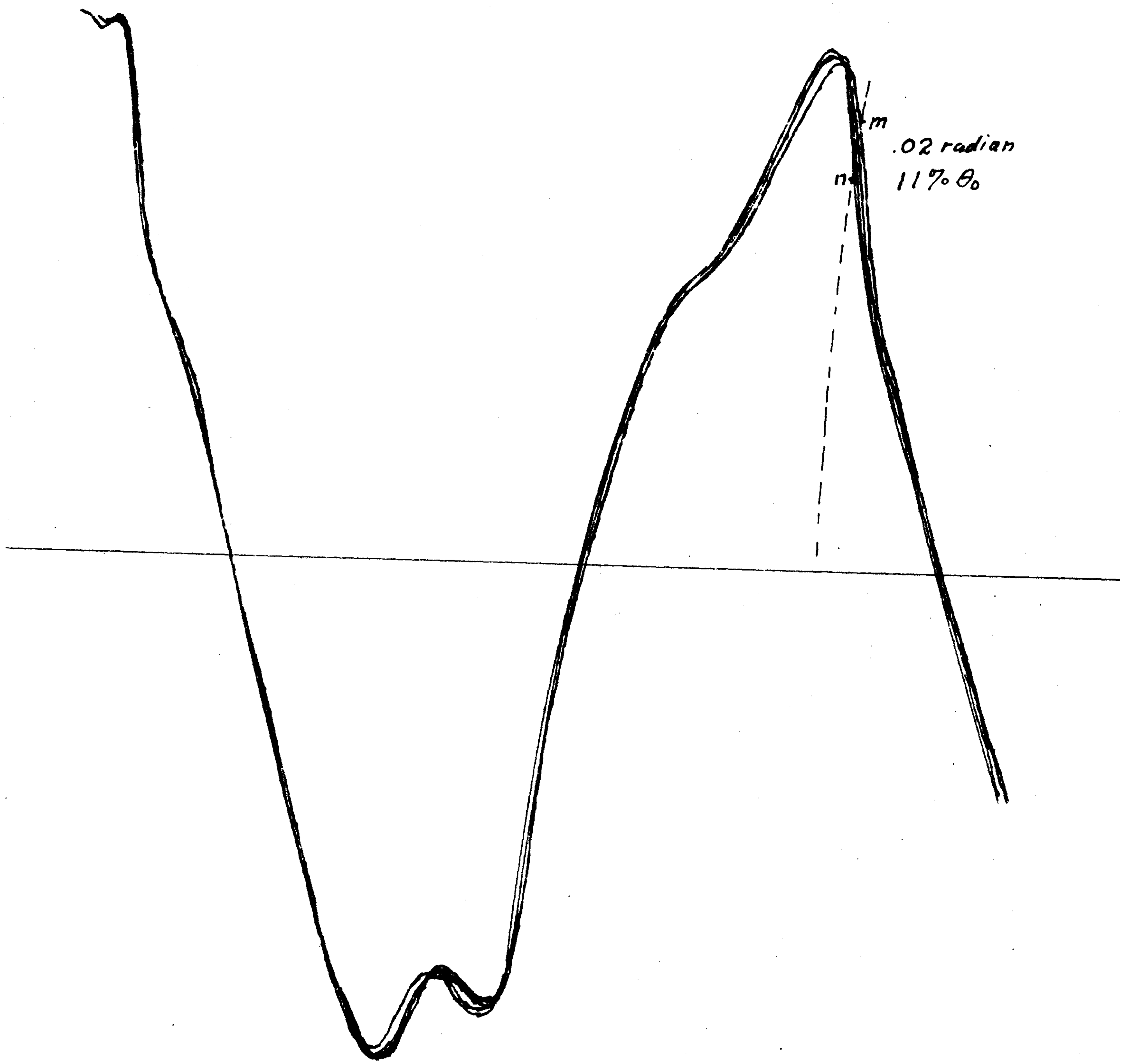


Fig. 13.

7 curves of $\frac{d^2\theta}{d\tau^2} + (1 + 2\cos 6t)\theta = 0$.

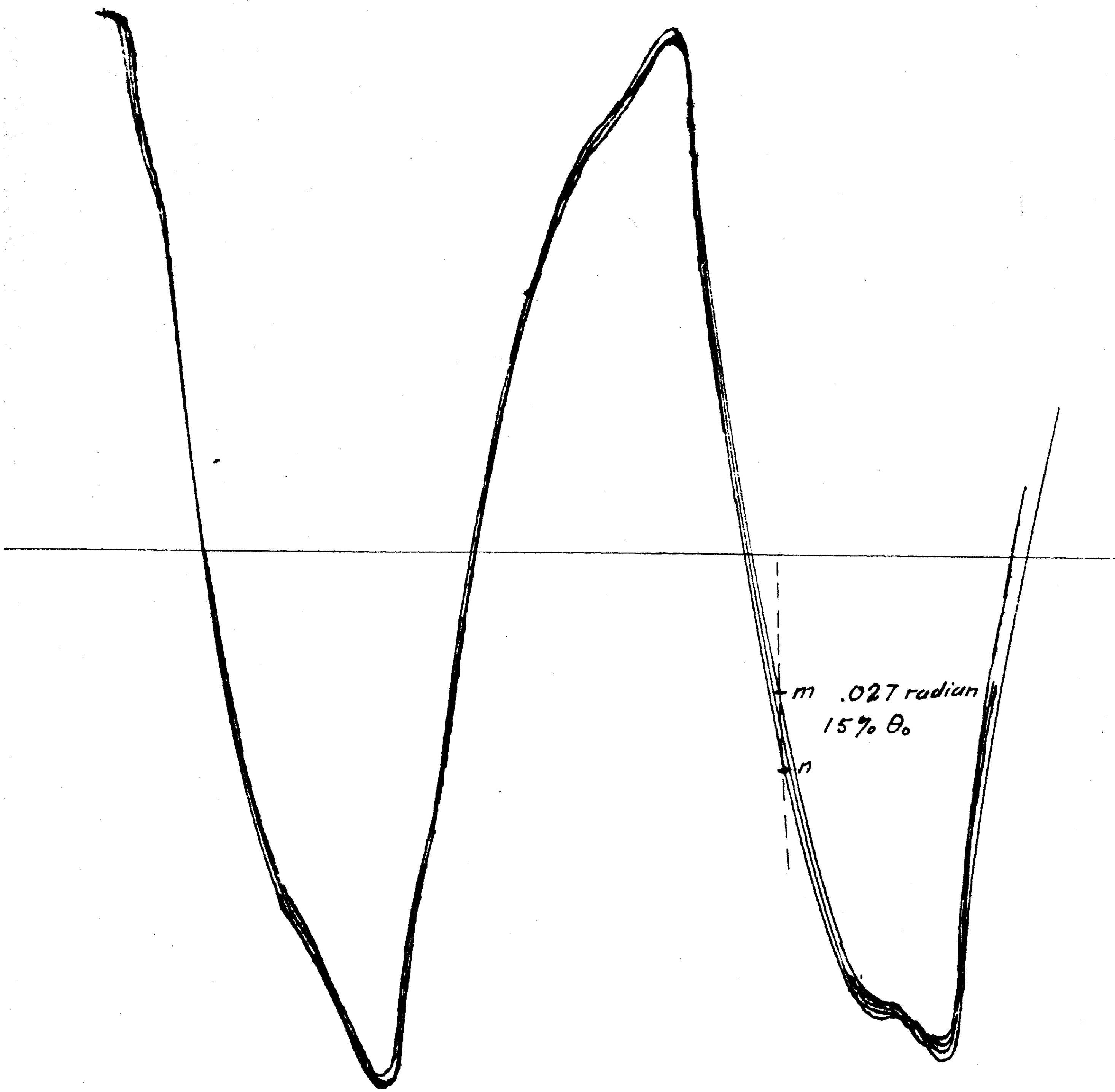


Fig. 14.

(c) Solution of differential equations

by method of successive approximation.

It was desired to compare the graphical solution obtained from the analyzer with the analytical solution for several equations of the Mathieu type. A method was used, which is a combination of Picard's and Simpson's rules. See Bibliography (13). In applying this method, it is necessary to express the equation $dy/dx = f(x,y)$, in such a form that $y = 0$ when $x = 0$. Then,

$$y_n(a + 2h) = y(a) + (h/3)(f(a) + 4f_{n-1}(a + h) + f_{n-1}(a + 2h))$$

$$y_n(a + h) = y(a) + (h/12)(5f(a) + 8f_{n-1}(a + h) - f_{n-1}(a + 2h))$$

h = the interval of x. The smaller the value of h chosen, the greater will be the accuracy.

$f(a) = f(x,y)$ at the point $x = a$.

$f(a + h) = f(x,y)$ at the point $x = a + h$.

$f(a + 2h) = f(x,y)$ at the point $x = a + 2h$

etc.

By estimating reasonable values for $f(a + h)$, $f(a + 2h)$, and substituting in the formulae, approximate values of y may be found. These are substituted in $f(x,y)$, and the new values of $f(a+ h)$, $f(a + 2h)$, are substituted in the formulae for $y(a + 2h)$, $y(a + h)$. This process is repeated until successive values of y check to the desired degree of accuracy.

Consider the differential equation: $\ddot{\theta} + (2 + \cos t)\theta = 0$. As solved on the analyzer, the initial conditions are, $\theta = \theta_0$ when $t = 0$, and $\dot{\theta} = 0$ when $t = 0$. In order to solve this equation by the method described on the last page, it is necessary to shift the origin so that $\theta = 0$ when $t = 0$. Let $\theta_0 = 1$. The new form of the equation is:

$\ddot{z} + (2 + \cos t)(z + 1) = 0$, where $\theta = z + 1$.

$f(t,z) = -(2 + \cos t)(z + 1)$

Let $y = \dot{z}$. Then $\dot{y} = -(2 + \cos t)(z + 1)$

Let $h = 0.05\pi = 0.1571$

By estimate, or from the curve obtained from the machine,

$z(0 + 0.05\pi) = -0.03$ $\qquad$ $z(0 + 0.10\pi) = -0.14$

Then, $f(0 + h) = -2.9$, $f(0 + 2h) = -2.6$, $f(0) = -3$, $y(0) = 0$

$y(0 + 2h) = 0 + .05\frac{\pi}{3}(-3 + 4(-2.9) - 2.6) = -0.9$

$y(0 + h) = 0 + .05\frac{\pi}{12}(-15 + 8(-2.9) + 2.6) = -0.47$

Using the same process to find z as a function of y and t,

$z(0 + 2h) = 0 + .05\frac{\pi}{3}(0 + 4(-0.47) - 0.9) = -0.14$

$z(0 + h) = 0 + .05\frac{\pi}{12}(0 + 8(-0.47) + 0.9) = -0.037$

Substituting these values of z in $f(t,z)$, we obtain more accurate values of y by means of the same formulae. With these new values of y, z may be determined again. This process is repeated until successive trials agree to the desired accuracy. For the above points, the final values are: $y(2h) = -0.893$, $y(h) = -0.465$, $z(2h) = -0.14$, $z(h) = -0.04$.

For the next point, repeat the process using the formula for $y(a + 2h)$, and so on.

TESTING FOR ACCURACY

It is well to check every five points by the formula:

$$y(c + 5h) = y(c) + (5h/288)(19f(c) + 75f(c + h) + 50 f(c + 2h) + 50f(c + 3h) + 75f(c + 4h) + 19f(c + 5h))$$

The next few pages contain tables and graphs, showing the comparison of the solutions by this method with those obtained from the analyzer.

In the following table are listed the average readings of θ, obtained from four graphs of $\ddot{\theta} + \theta \cos t = 0$. These are compared with the calculated values. θ is expressed as a percent of θ_0.

t	Average θ	Calculated θ	Error
0.π	100	100	0
0.1π	96	95	1
0.2π	82	82	0
0.3π	63	61	2
0.4π	38	38	0
0.5π	12.2	12	0.2
0.6π	-12.5	-12	-0.5
0.7π	-37	-39	2
0.8π	-65	-65	0
0.9π	-98	-100	2
1.0π	-134	-141	7

This table applies to the equation: $\ddot{\theta} + (2 + \cos t)\theta = 0$.

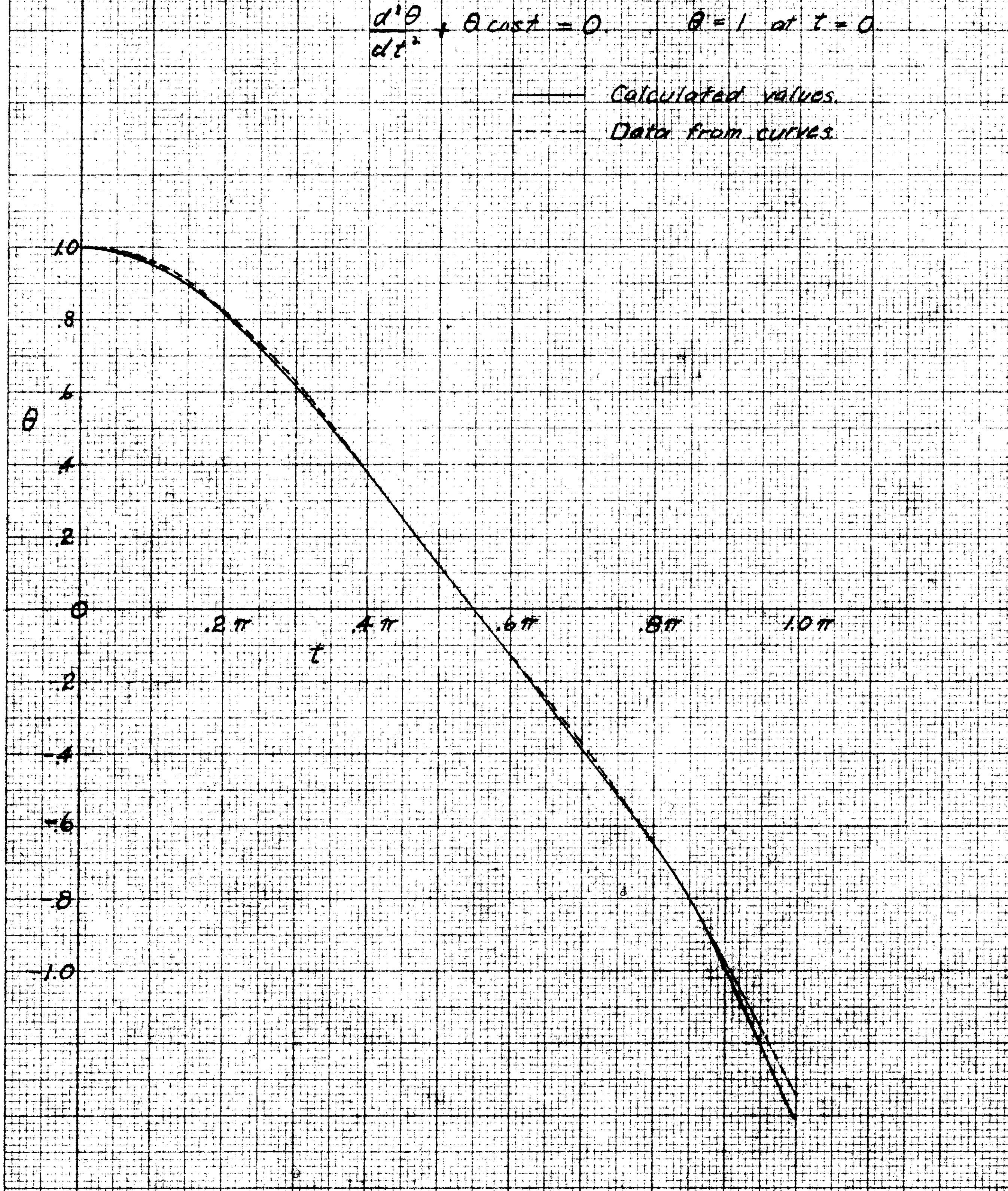
t	Average θ	Calculated θ	Error
0 π	100	100	0
0.1π	89	86	3
0.2π	48	47	1
0.3π	-5.9	-5	-0.9
0.4π	-54	-55	1
0.5π	-94	-94	0
0.6π	-116	-115	-1
0.7π	-121	-117	-4
0.8π	-109	-103	-6
0.9π	-86	-76	-10
1.0π	-49	-41	-8
1.1π	-12	-2	-10
1.2π	25	37	-12

These averages were obtained from six graphs.

This table applies to the equation: $\ddot{\theta} + (1 + \cos 4t)\theta = 0.$

t	θ from graphs	Calculated θ	Error
0 π	100	100	0
0.1π	91	91	0
0.2π	70	71	-1
0.3π	48	48	0
0.4π	22	26	-4
0.5π	-6	-2	-4
0.6π	-29	-28	-1
0.7π	-50	-52	2
0.8π	-73	-76	3
0.9π	-92	-94	2
1.0π	-100	-105	5
1.1π	-92	-91	-1
1.2π	-67	-71	4
1.3π	-42	-43	1
1.4π	-20	-17	-3
1.5π	8	8	0

The readings were obtained from the print of five graphs, on Page 32.



$\frac{d^2\theta}{dt^2} + \theta \cos t = 0$
$\theta' = 1$ at $t = 0$
Calculated values.
Data from curves.
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
θ
.2π
.4π
.6π
.8π
1.0π
t

Fig. 15(a).

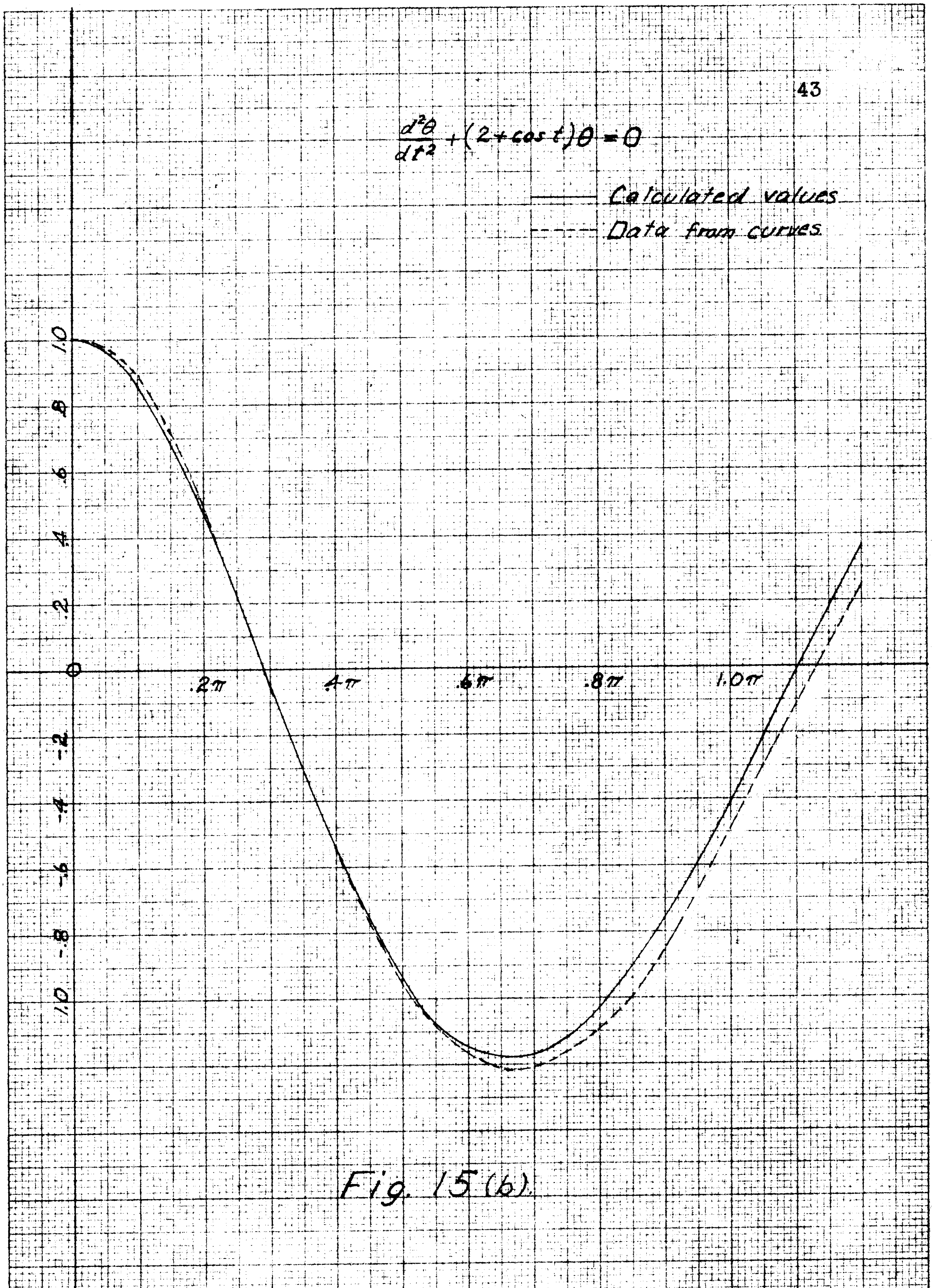


Fig. 15 (b).

FORM 1 T

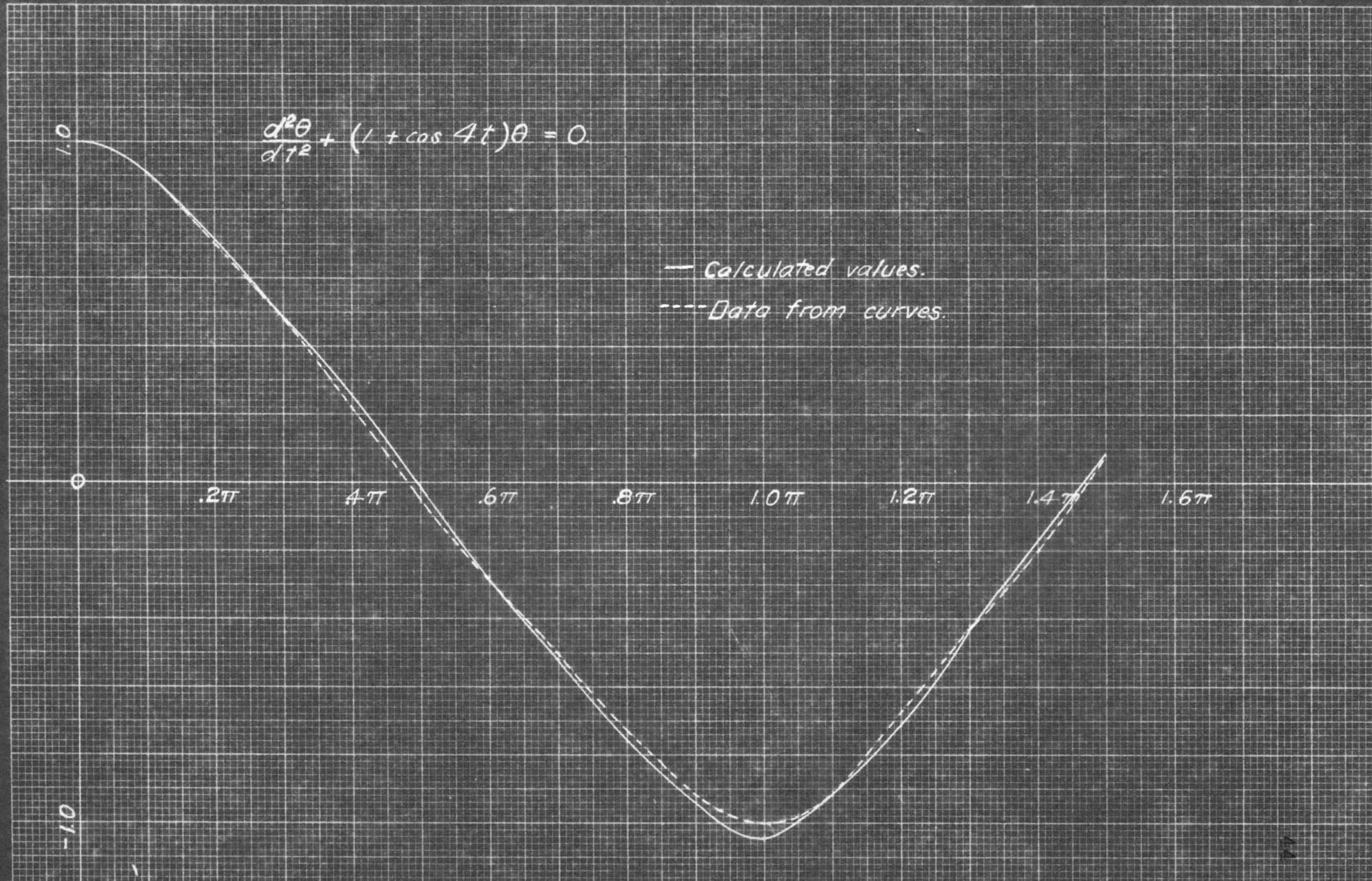

$\frac{d^2\theta}{dt^2} + (1 + \cos 4t)\theta = 0.$
— Calculated values.
---- Data from curves.
1.0
0
-1.0
.2π
.4π
.6π
.8π
1.0π
1.2π
1.4π
1.6π

Fig. 15(c)

TESTING FOR ACCURACY

(d) Sources of error.

The current flowing in the moving coil of the analyzer is appreciably decreased by the effect of inductance, when the potentiometer is rotating at a high rate of speed. The higher the frequency of the sinusoidal alternating current delivered by the potentiometer, the greater is the impedance of the coil circuit.

$I = E/Z$, where Z = the impedance which is equal to $\sqrt{R^2 + (2\pi fL)^2}$

A change of 1 % in Z causes a change of 1 % in the current I.

$$f = \frac{\sqrt{Z^2 - R^2}}{2\pi L}$$

If the moving coil with 4100 turns is connected to the potentiometer, R = 1100 ohms, L = 9.9 henrys.

If Z is increased by 1 % above R, Z = 1111.

$Z^2 - R^2 = 1236000 - 1210000 = 26000.$ $\quad f = \frac{\sqrt{26000}}{2\pi 9.9}$

f = 2.59 cycles per second, which is produced by a potentiometer speed of 155 r.p.m.

For the coil with 2700 turns, R = 1100 ohms because of the extra resistance connected in series, L = 4.5 henrys, neglecting the inductance of the extra resistance coil.

f = 5.7 cycles per second, which corresponds to a potentiometer speed of 342 r.p.m.

Therefore, the error caused by inductance will be 1% or less, if the above potentiometer speeds are not exceeded. Otherwise, the correction to be made to the value of the current can easily be calculated.

The ammeters used in the operation of the analyzer are of the portable type, with an accuracy of $\pm\frac{1}{2}$ % of the full scale reading.

On account of difficulty in controlling the speed of the potentiometer motor, it is probable that a variation in speed of about ± 5 % existed, during some of the runs. This feature can be considerably improved.

In measuring the values of θ vs t from the graph, the accuracy attainable depends upon the slope of the curve. At points where the curve is nearly horizontal, the error in measuring θ need not exceed $\pm$ 0.001 radian, or about 0.55 % of the initial amplitude. At points where the curve is nearly vertical, the error may be much greater.

The consistency of the results obtained by averaging several graphs depends upon the type of equation. Points of inflection appear to be unfavorable, as has been discussed on page 27. It is probable that outside disturbances such as air currents, and vibration of the building, are the principal causes of variation between solutions of the same equation. As shown on the prints made from tracings of graphs, pages 28 to 35, this variation sometimes amounts to about 0.04 radian, or 22 % of the initial amplitude.

In the comparison between results obtained from the analyzer, and the analytical solutions, there is a tendency for the error to increase as t is increased, as may be seen from Figs. 15(a) and 15(b). This would naturally result from inaccuracy in adjusting the currents and the motor speed. In each case, the curve obtained from the analytical solution was lined up with the curve of average values obtained from the analyzer, to coincide at the first point where the x axis is crossed.

Fig. 15(a) shows that the error in the analyzer solution of the equation: $\ddot{\theta} + \theta \cos t = 0$, is 7 % of the initial amplitude, at $t = 1.0\pi$, or at a distance of 0.45π from the point where the analytical and analyzer graphs coincide.

Fig. 15(b), referring to equation $\ddot{\theta} + (2 + \cos t)\theta = 0$, shows an error of -12 % at $t = 1.2\pi$, which is about 0.91π from the point of coincidence.

Fig. 15(c), referring to equation $\ddot{\theta} + (1 + \cos 4t)\theta = 0$, does not exhibit a cumulative error, the maximum being 5 % of the initial amplitude, at $t = \pi$.

The successive approximation method of solving the equations is subject to cumulative errors, and consequently this comparison must not be taken too seriously. It appears reasonable to conclude that the solutions obtained from the analyzer are generally as reliable as those obtained by successive approximation.

The effect of damping will be discussed in the following section.

(e) Damping.

From the graph of the equation: $\ddot{\theta} + \theta = 0$, it was found that the amplitude decreased from 3.6" to 3.33" in a distance of 20" on the t axis, or a decrease of 0.27". In a distance equivalent to $t = 2\pi$, i.e., one period, the decrease is 0.054". 3.6 - 0.054 = 3.546" = amplitude of second swing.

$$\frac{\theta_1}{\theta_2} = \frac{3.600}{3.546} = 1.015$$ =ratio of amplitudes one period apart, or the decrement.

Assuming that the damping torque is proportional to the velocity, the motion of a torsional oscillation is described by the equation: $\ddot{\theta} + c\dot{\theta} + k\theta = 0$.

The solution is: $\theta = e^{-ct/2}\,(B_1 \cos At + B_2 \sin At)$,

where $A = \frac{1}{2}\sqrt{4k - c^2}$

$$\text{Period} = T = \frac{4\pi}{\sqrt{4k - c^2}} \qquad \text{Decrement} = \frac{\theta_1}{\theta_2} = e^{\frac{cT}{2}} = e^{\frac{2\pi c}{\sqrt{4k - c^2}}}$$

In the solution of the equation: $\ddot{\theta} + \theta = 0$, on the analyzer, the decrement was found to be 1.015.

$$e^{0.0149} = 1.015. \qquad \frac{2\pi c}{\sqrt{4 - c^2}} = 0.0149 \qquad c = 0.0047$$

Therefore, the damping term $0.0047\,\dot{\theta}$ is included in the analyzer solution.

Suppose the equation: $\ddot{y} + My = 0$ is to be solved on the analyzer, where M is any function of t. It is necessary to alter this to the form including the damping term, that is, $\ddot{\theta} + c\dot{\theta} + F\theta = 0$. Let $y = \theta e^{\frac{c}{2}t}$. The equation becomes:

$\ddot{\theta} + c\dot{\theta} + (c^2/4 + M)\theta = 0$.

$c = 0.0047.$ $\ddot{\theta} + 0.0047\,\dot{\theta} + (0.0000055 + M)\theta = 0.$

This is the equation to be set up on the analyzer. The addition to M, i.e., 0.0000055, is negligible. Therefore no correction in the magnitudes of the currents is needed. The readings from the graph may be corrected by means of the equation: $y = \theta\, e^{-\frac{c}{2}t}$, if desired. At $t = 2\pi$, the decrease in the amplitude caused by damping, is only 0.054/3.6 = 1.5 %.

Therefore, the damping correction may ordinarily be neglected.

OPERATION

The process of solving a differential equation on the analyzer may be divided into three parts, namely; adjustment of the equipment, the actual operation, and the interpretation of the graph.

For a typical illustration, consider the equation:
$\ddot{\theta} + (2 + \cos t)\theta = 0$. See Fig. 12, page 33.

After carefully lining up the recording drum with reference to the horizontal center line of the analyzer, the first step is to adjust the currents flowing through the coils according to the values obtained from the formula on page 26, that is, Torque = $5.52 \times 10^{-8} I_1 I_2 c_1 c_2 \theta = k\theta$.

The differential equation applying to the oscillation of the armature of the analyzer, with steady current flowing, is:
$(I/g)\ddot{\theta} + k\theta = 0$. $I/g = 2.097/32.2 = 0.0651$
$\ddot{\theta} + 15.35k\theta = 0$.

With the arm of the rotary potentiometer turned to the position of maximum current flow, that is, $\cos wt = 1$, the rheostat in series with the corresponding stationary solenoid is so adjusted as to make $k = 1/15.35$. See the wiring diagram, page67, Appendix. Note that the only adjustment possible on the moving coil in the potentiometer circuit is the choice between 110 volts or 220 volts. See the last paragraph on page 23.

The rheostats in the circuits of the other coils are adjusted to make $k = 2/15.35$.

The equation of the analyzer is now: $\ddot{\theta} + 2\theta + \theta\cos t = 0$, which is the same as the equation to be solved.

With the switch of the potentiometer circuit turned to the 110 volt position, the maximum current flowing is 110/1100 = 0.1 amp. $c_1c_2 = 1580 \times 2700$. $5.52 \times 10^{-8} I_1 I_2 c_1 c_2 = k = 1/15.35$

$5.52 \times 10^{-8} \times 0.1 I_1 \times 1580 \times 2700 = 1/15.35$

$I_1 = 2.72$ amps. = current in upper stationary solenoid.

If the same current is used in the lower solenoid,

$5.52 \times 10^{-8} \times 2.72 I_2' \times 1580 \times 4100 = 2/15.35$

$I_2' = 0.134$ amps. = current in lower moving coil.

Before making the final adjustments, the coils should be warmed up for about 20 minutes by passing a current of approximately 10 amps. through the stationary solenoids, and 0.2 amps. through the moving coils. This is necessary in order to obtain the correct resistance in the coil connected to the potentiometer, and it also assists in maintaining steady conditions.

The current values obtained from the above calculations are only approximate, and should be checked by timing the oscillation by a stopwatch. With the upper coils alone, in the circuit, the period should be 2π for the given equation. The lower coils alone should give a period of $2\pi/\sqrt{2}$. These values are found by solving for the periods of the equations: $\ddot{\theta} + \theta = 0$, and $\ddot{\theta} + 2\theta = 0$, which describe the oscillation of the rotor under the above conditions. In the actual solution of the given equation, the corrected values of current were: 2.8 amps. for both stationary solenoids, and 0.126 amps. for the lower moving coil. The potentiometer circuit, which supplied the upper moving coil, was connected to the 110 volt source. Instead of being arranged as shown in the diagram, the control was simplified in this case by connecting both stationary solenoids in series across the 110 volt supply, with a slide wire rheostat of 3.1 amps and 44 ohms capacity included in the circuit.

The setting of the contact arm at the end of the potentiometer shaft, which actuates the release mechanism, should be checked. The release should trip when the potentiometer arm is at the position of maximum current, that is, when $\cos wt = 1$. The initial conditions for the equation as solved on the analyzer are: $\theta = \theta_0$, and $\dot{\theta} = 0$, when $t = 0$. To obtain other startingconditions, the graph may be measured from a different origin. The release contact arm may also be set at other points, according to the desired conditions.

For the given equation, $\ddot{\theta} + (2 + \cos t)\theta = 0$, the speed of the potentiometer motor should be regulated so that $w = 1$, where $w = 2\pi f$. f = frequency of the sinusoidal current delivered by the potentiometer. $f = 1/2\pi$ = revolutions per second, of the potentiometer. Therefore the potentiometer shaft must rotate at 9.55 r.p.m., which with a gear ratio of 22.22 to 1, requires a motor speed of 212 r.p.m.

After the various adjustments have been made, and a sheet of Stylograph paper fastened to the drum by means of Scotch tape, current is applied to the spark coil and to the release circuit. The spark may be continued for as many oscillations as desired. Several graphs should be obtained from the same equation.

Before removing the Stylograph paper from the drum, a sinusoidal curve should be traced on the sheet by operating the analyzer with a steady current flowing through the coils. This serves as a convenient means of locating the abscissa, by drawing a line midway between the extreme amplitudes of the sine curve. It is also advisable to trace several circular arcs by letting the recording arm oscillate when the recording drum is stationary. These furnish a check on the alignment of the recording drum.

Numerical values may be obtained from the graphs by measuring with a piece of transparent celluloid, such as a draftsman's triangle, on which is drawn a circular arc of 20" radius, divided into fractions of a radian. In case the alignment of the drum was imperfect, the measuring device may be oriented correctly with the aid of the circular arcs on the graph.

The starting point of the curve can usually be located by observing a break in the curve caused by an interruption in the spark at the instant when the release contact is made. The magnet draws a heavy current, and consequently the spark coil which is connected to the same source ceases to operate for a fraction of a second. This method of locating the point where $t = 0$, is rather crude, and a much better indicating mechanism can be designed.

On account of the limited range of the instrument, it is usually necessary to change the independent variable of the given equation so as to bring it within the capacity of the analyzer. For instance, consider the equation:
$\ddot{\theta} + (150 + 30 \cos 20 t)\theta = 0$. Let $x = 10 t$, and the equation becomes: $d^2\theta/dx^2 + (1.5 + 0.3 \cos 2x)\theta = 0$, which is within the convenient operating range of the analyzer. It is desirable so to choose the new independent variable as to give a relatively low frequency of oscillation, because under these conditions, the graph is easier to measure.

The coil circuits may be connected in a variety of arrangements, in order to operate with the measuring instruments and slide wire rheostats that are available. For instance, it is possible to use one ammeter for two different circuits, as shown on page 69 , Appendix.

SOLUTION OF PROBLEMS INVOLVING UNSTABLE RANGE OF THE MATHIEU EQUATION

(a) Electric locomotive mechanism.

As stated in Chapter II, the object of most mechanical engineering problems involving the Mathieu equation, is to determine the range of speed at which the amplitude of vibration builds up to dangerous magnitudes. For an illustration of this type of problem, consider the mechanism outlined in Fig. 16. This is similar to the arrangement used on certain electric locomotives. (5)

The cranks, at opposite sides of the locomotive, are at 90 degrees to eachother. When one of the side rods is at dead center, its resistance to torsional oscillation of the motor shaft is zero. The resistance of the combination varies approximately sinusoidally from maximum to minimum, the frequency of variation in elasticity being 4 times the r.p.m. of the drive wheels.

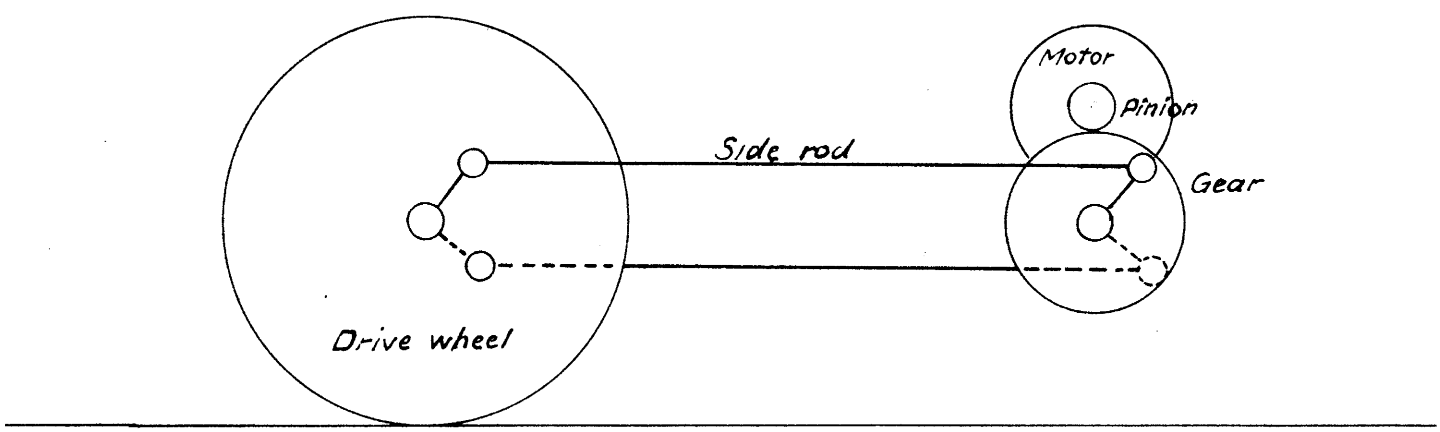


Fig. 16.

The torque applied at the armature of the motor necessary to produce a displacement of 1 radian may be determined by actual test upon the mechanism under consideration. This may be accomplished by measuring the deflection of the armature under a given static load, for the positions of maximum and minimum rigidity. Approximate figures for the torque may also be obtained by calculation.

Assume the following data for an illustrative problem:

Maximum torque per radian deflection of armature:	360000 lb.ft.
Minimum	280000
Mean	320000

$I/g = 16.$

The equation for the torsional vibration of the armature is:

$(I/g)\ddot{\theta} + (320000 + 40000 \cos wt)\theta = 0.$

The problem is to determine the speeds at which θ builds up to large magnitudes.

Dividing by I/g, $\ddot{\theta} + (20000 + 2500 \cos wt)\theta = 0.$

In order to adapt this equation to the operating range of the analyzer, it is necessary to change the independent variable. Let $x = 50\ t$. Then, $d^2\theta/dx^2 + (8 + \cos wx)\theta = 0.$

Referring to Chapter V, page 51, a current of approximately 2.7 amps. in the upper solenoid, with the upper moving coil connected through the potentiometer to the 110 volt supply is suitable. The product of the currents in the lower solenoid and corresponding moving coil should be approximately 1.5. As explained in Chapter V, the current values should be checked by timing the period of oscillation with a stopwatch. In the actual solution of this equation, the currents used were: Upper solenoid, 2.5 amps. Lower solenoid, 8.2 amps. Lower moving coil, 0.172 amps. There is a considerable variation from the calculated values, because the stronger magnetic field of the lower solenoid had an appreciable effect on the upper coil.

To find the unstable speed range, the potentiometer motor is run at a constant rate, and the recording arm of the analyzer is allowed to start from the mid-position. If the motor speed is within the critical range, the recording arm will soon build up large oscillations and hit the stops. This process is repeated for a number of different motor speeds, in order to locate the limits of unstable operation.

The solution of the given equation shows a building up of the amplitude when the speed of the potentiometer shaft is between 51 and 55 r.p.m. Since $x = 50\ t$, the actual frequencies of variation in torque are 2550 and 2750 cycles per minute, which correspond to drive wheel speeds of 640 and 690 r.p.m. Dangerous vibrations may be expected within this range.

Other critical ranges also exist, the nearest being at about one-half the above speeds. These regions are very narrow and are difficult to locate. On account of limited facilities with respect to rheostats, etc., they were not investigated.

(b) Region of instability of Mathieu equation.

The unstable range of potentiometer speed was found for the equation: $\ddot{\theta} + (a + b \cos wt)\theta = 0$, for the values of a and b given in the following table:

a	b	Motor speed	w	a_o	b_o
1.4	1	400-590	1.88-2.78	.399-.182	.285-.130
2.0	1	520-680	2.45-3.20	.333-.198	.167-.0988
3.0	1	660-800	3.11-3.77	.311-.212	.104-.0710
4.0	1	790-890	3.62-4.20	.305-.227	.0763-.0570
5.0	1	880-980	4.10-4.61	.298-.235	.0596-.0470
6.0	1	960-1060	4.52-5.00	.292-.240	.0487-.0400
7.0	1	1060-1160	5.00-5.46	.280-.233	.0400-.0333
8.0	1	1140-1220	5.37-5.74	.276-.243	.0345-.0304
1.0	0		2	.25	0

$a_o = a/w^2$ $b_o = b/w^2$ See Chapter II, page 5.

The equation becomes: $d^2\theta/dx^2 + (a_o + b_o \cos x)\theta = 0$.

Fig.17 , page60,shows b_o plotted against a_o for the limits of the region of instability. The area between the curves is the unstable region. For similar graphs, see Bibliography (5,7,8). As already mentioned on Page57 of this chapter, other regions of instability exist, but these were not located because of inadequate equipment.

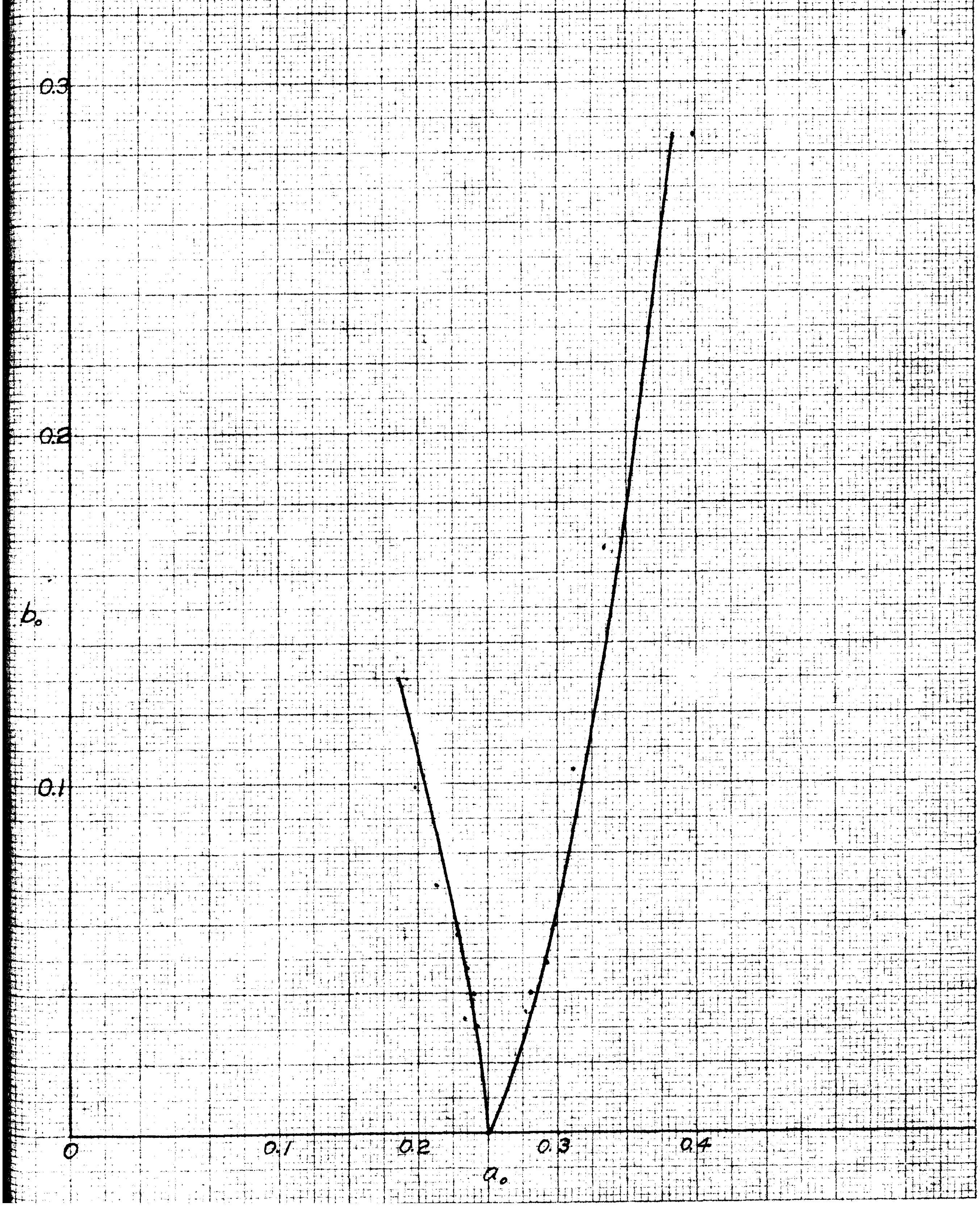
The accuracy of the above readings is questionable, because the motor speed control rheostat was incapable ofclose adjustment, and also, it was impossible for only one operator to maintain a constant speed. This experiment was included merely to show what may be done with the apparatus, and not for the purpose of obtaining precision results. With more satisfactory control equipment, and an additional operator, much greater accuracy would be possible.

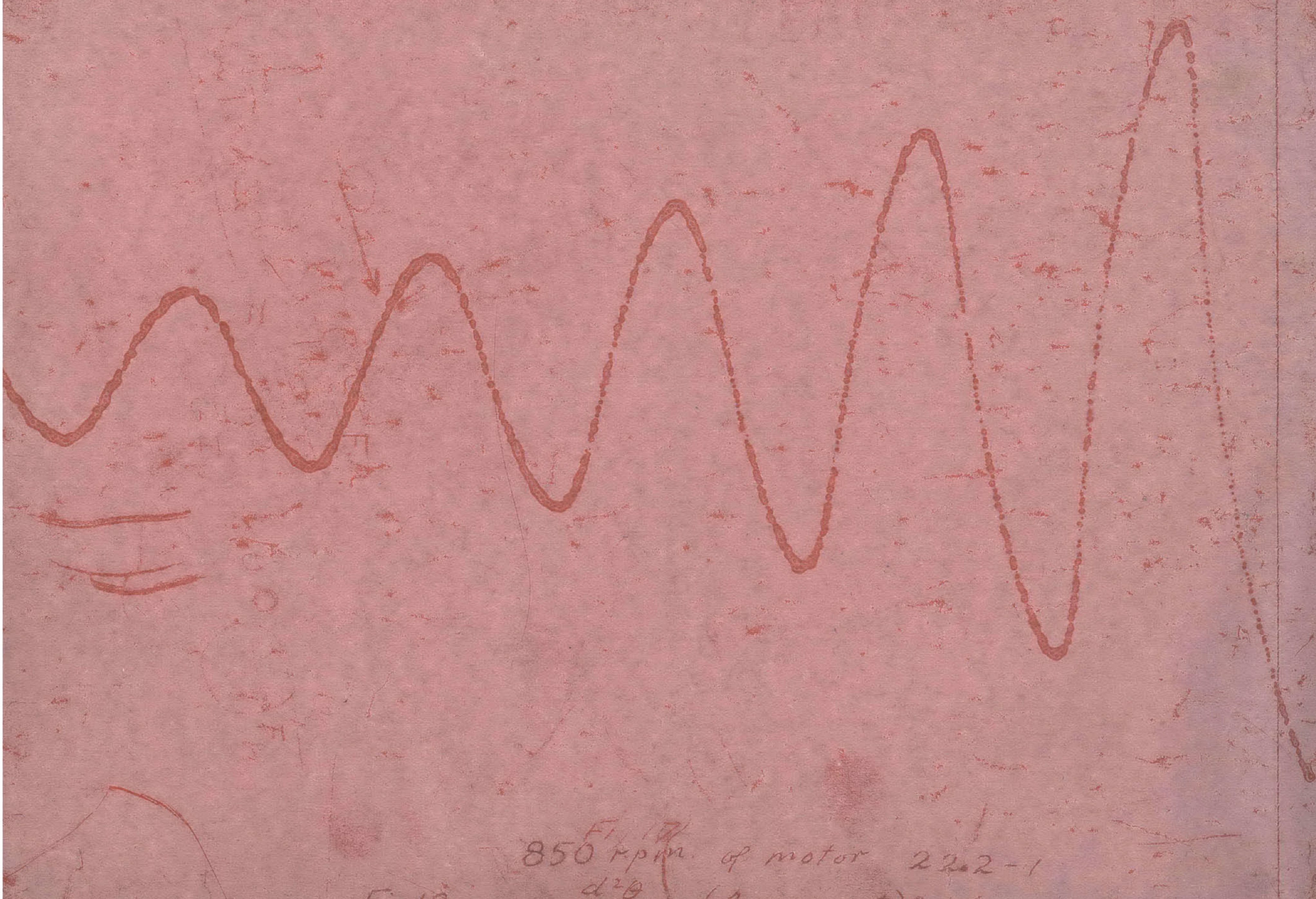
Fig. 18, page 61, shows a sample graph obtained from the analyzer, for the equation: $\ddot{\theta} + (4 + \cos wt)\theta = 0$, within the unstable range. From a small initial displacement, the amplitude of the vibration tends to build up to infinity.

Fig. 17.

Graph shows unstable range of

$\ddot{\theta} + (a_0 + b_0 \cos x)\theta = 0$





850 rpm of motor 222-1

Fig. 18 $\frac{d^2\theta}{dt^2} + (4 + \cos \omega t)\theta = 0$

CHAPTER VII

FUTURE DEVELOPMENT

Since the preliminary investigation covered by this discussion showed that the apparatus was worthy of further development, the possible improvements and additions will now be considered.

Experience with the analyzer indicates that greater accuracy in measuring the graphs could be realized by running the recording drum at a higher rate of speed, thus enlarging the scale of the abscissa. It is possible to obtain a Telechron motor giving 2 r.p.m. instead of the type giving 1 r.p.m. as is now attached to the drum. Another method is to use a regular type synchronous motor with a speed reducing gear.

A device should be provided to record the point at which the magnetic release operates.

The gear reduction arrangement between the potentiometer and the motor can be improved, in order to provide greater convenience in changing the gear ratio.

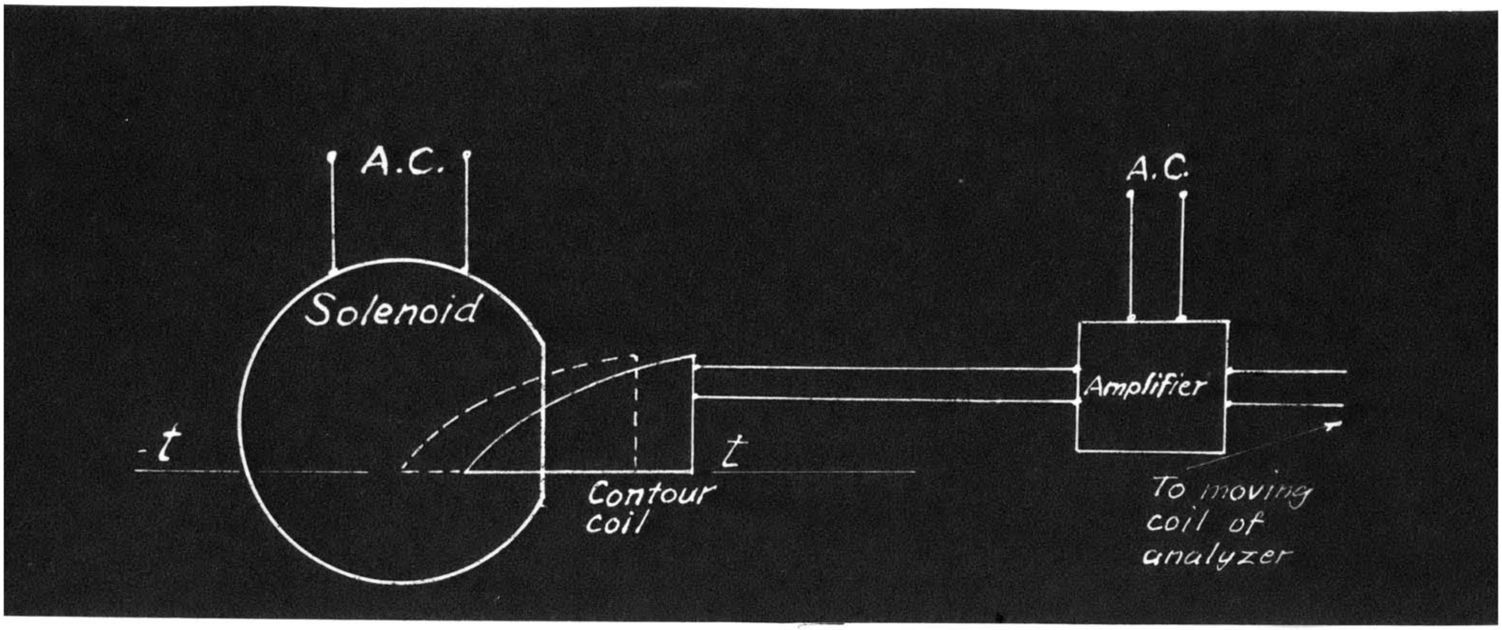
Closer speed control of the potentiometer motoris desirable. Instead of the slide wire rheostat of 84 ohms and 2.8 amps. capacity which was used for this purpose, it would be more satisfactory to provide at least 100 ohms and 5 amps. capacity.

The agate bearings at the end of the rotor shaft are not as smooth as they should be, and the clearance of the upper bearing appears to be insufficient for best results. A brass bearing surface would be easier to construct than the agate arrangement, and the friction would probably be negligible.

As already mentioned in Chapter I, Page 4 , equations involving other functions of the independent variable than the cosine, may be solved on the analyzer by providing a potentiometer actuated by a cam, instead of the type already described. In this case the sliding contact moves in a straight line instead of a circle, and the resistances are so proportioned that the current flowing through the moving coil of the analyzer is equal to a constant times the displacement of the sliding contact from the zero position. By arranging a cam of the desired profile to move the sliding contact, any equation of the general form: $\ddot{\theta} + (a + b\,f(t))\theta = 0$ may be handled by the apparatus.

The preceding discussion has been limited to the operation of the analyzer when supplied with direct current, but the original system, proposed by Dr. Minorski, is designed to use alternating current. (4)

As far as applications to the Mathieu equation are concerned, it is more convenient to adapt the apparatus to direct current. In order to extend the range of the analyzer beyond this point, the system described in Reference (4) should be considered. Alternating currents of the same frequency and phase are supplied to the stationary and moving coils of the analyzer. The current in the stationary solenoids is held at a constant effective value, while that in one of the moving coils is varied according to the desired function of time by means of a "contour coil" and an amplifier. The general principle of operation remains the same as before.



The above diagram, Fig.19, shows the arrangement of the contour coil method of control. The long solenoid shown at the left, is supplied with alternating current. The contour coil projects through a narrow slot in the center of the solenoid, and is arranged to be moved at a uniform rate of speed along the horizontal axis. The voltage induced in the contour coil is proportional to the area of that portion which projects into the solenoid, or is equal to a constant times the integral of f(t) dt, where f(t) is represented by the curve of the contour coil. By connecting to the grid of an amplifier, as shown in Fig. 19, the current flowing in the moving coil of the analyzer may be varied according to f(t). The contour coil may be wound in any desired shape, according to the function which is involved in the differential equation to be solved.

The application of the analyzer may be extended to include non-linear equations of the general form: $\ddot{\theta} + f(t,\theta)\theta = 0$ by attaching another contour coil to an arm projecting from the rotor assembly of the analyzer.

Certain difficulties are encountered in the application of alternating current to the apparatus. It is necessary to include the proper values of capacitance in the coil circuits to maintain the same phase angle of currents in both stationary and moving coils, and to reduce the impedance so as to allow a sufficient current to flow with a moderate supply voltage.

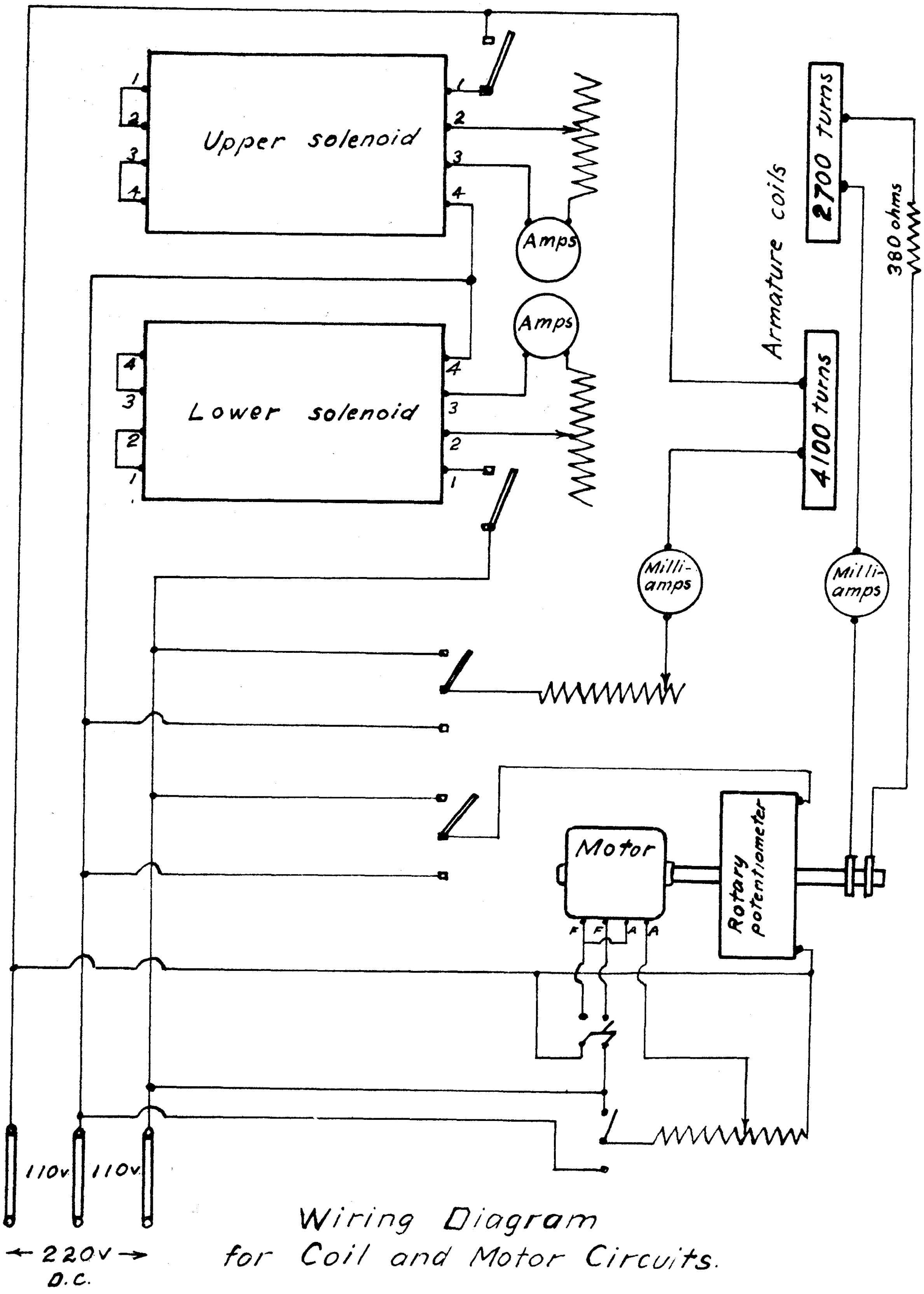
CHAPTER VIII

CONCLUSION

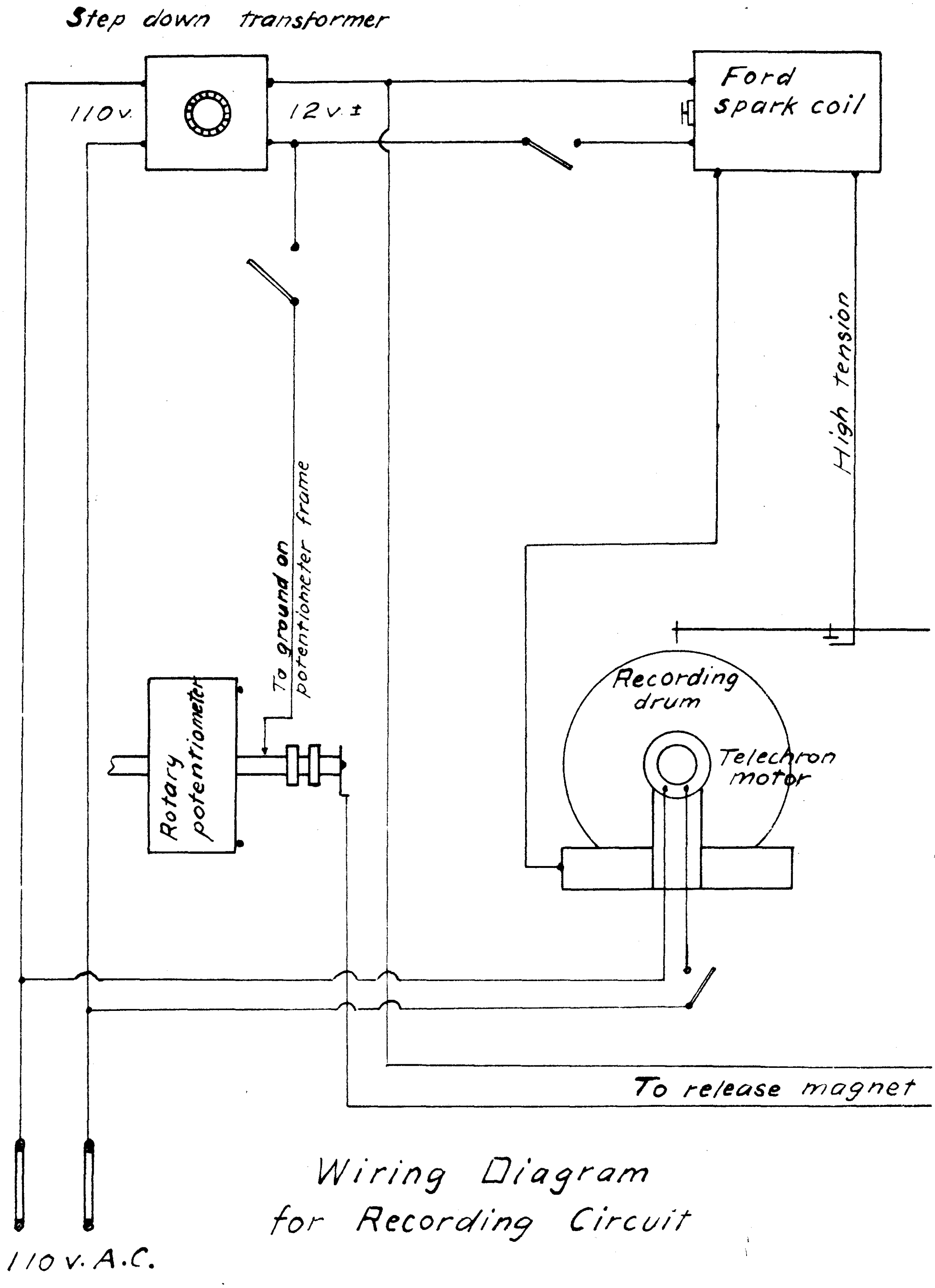
The results of this investigation show that the torsion pendulum type of differential analyzer furnishes a rapid and convenient means of solving second order equations to a sufficient degree of accuracy for most engineering purposes. The successful application of the apparatus to equations of the Mathieu type indicates that further development is worth while, both with regard to improving the accuracy and to increasing the range of utility.

The comparison between graphs obtained from the analyzer and results of computation (Chapter IV) shows an error at some points amounting to about 12 % of the initial amplitude. In most cases, the error was less than 5%. Since conditions of operation were far from ideal, it is reasonable to assume that the accuracy can be considerably improved.

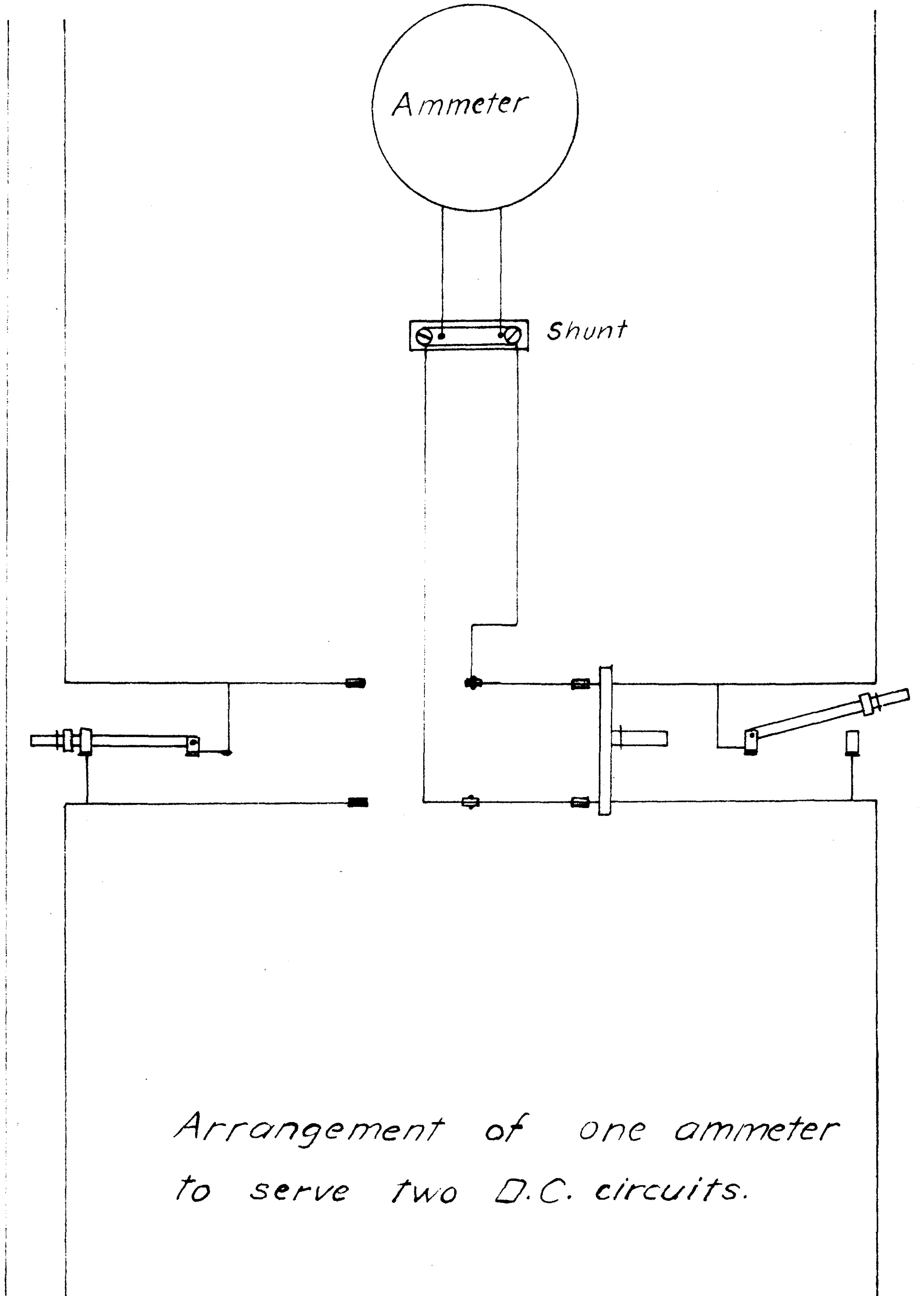
In addition to obtaining numerical results, the apparatus is well adapted to making a qualitative study of the behavior of certain types of equations under varying conditions. For instance, the investigation of the region of instability of the Mathieu equation, as illustrated in Chapter VI, is a type of problem which may be conveniently solved on the analyzer.



Wiring Diagram
for Coil and Motor Circuits.



Wiring Diagram for Recording Circuit



Arrangement of one ammeter to serve two D.C. circuits.

Readings from analyzer solutions of $\ddot{\theta} + \theta \cos t = 0$, were obtained from four graphs. They are given in the table below:

t	radians θ % θ_0	2	3	4	Average
0π	0.178 100	0.178 100	0.178 100	0.178 100	100
0.1π	0.171 96	0.172 97	0.172 97	0.170 95	96
0.2π	0.148 83	0.145 81	0.148 83	0.145 81	82
0.3π	0.110 62	0.114 64	0.110 62	0.110 62	63
0.4π	0.067 38	0.070 39	0.065 36	0.067 38	38
0.5π	0.020 11.2	0.022 12.4	0.020 11.2	0.023 13.0	12.2
0.6π	-0.027 -15.	-0.022 -12.4	-0.020 -11.2	-0.020 -11.2	-12.5
0.7π	-0.072 -40	-0.067 -38	-0.062 -35	-0.062 -35	-37
0.8π	-0.128 -72	-0.115 -65	-0.110 -62	-0.110 -62	-65
0.9π	-0.195 -110	-0.160 -90	-0.170 -95	-0.170 -95	-98
1.0π	-0.250 -140	-0.230 -130	-0.240 -135	-0.230 -130	-134

APPENDIX

Solution of differential equation: $\ddot{z} + (z + 1)\cos t = 0$

t	cos t	f(t,z)	y	z	θ = z + 1
0	1.00	-1.00	0.00	0.00	1.00
0.1π	0.951	-0.905	-0.303	-0.049	0.95
0.2π	0.809	-0.659	-0.553	-0.185	0.82
0.3π	0.588	-0.361	-0.712	-0.387	0.61
0.4π	0.309	-0.117	-0.789	-0.620	0.38
0.5π	0.000	0.000	-0.799	-0.88	0.12
0.6π	-0.309	-0.037	-0.805	-1.12	-0.12
0.7π	-0.588	-0.230	-0.839	-1.39	-0.39
0.8π	-0.809	-0.526	-0.960	-1.65	-0.65
0.9π	-0.951	-0.951	-1.17	-2.00	-1.00
1.0π	-1.00	-1.41	-1.56	-2.41	-1.41

APPENDIX

Readings from analyzer solutions of $\ddot{\theta} + (2 + \cos t)\theta = 0$, were obtained from six graphs. They are given in the table:

t	θ radians / $\%\theta_0$	1	2	3	4	5	6	Average
0	θ radians	0.183	0.183	0.183	0.183	0.183	0.183	
	$\%\theta_0$	100	100	100	100	100	100	100
0.1π	θ radians	0.163	0.161	0.161	0.160	0.168	0.163	89
	$\%\theta_0$	89	88	88	87	92	89	
0.2π	θ radians	0.080	0.090	0.085	0.090	0.092	0.092	
	$\%\theta_0$	44	49	47	49	51	51	48
0.3π	θ radians	-0.015	-0.010	-0.010	-0.010	-0.010	-0.010	
	$\%\theta_0$	-8.2	-5.5	-5.5	-5.5	-5.5	-5.5	-5.9
0.4π	θ radians	-0.110	-0.090	-0.107	-0.090	-0.100	-0.100	
	$\%\theta_0$	-60	-49	-59	-49	-55	-55	-54
0.5π	θ radians	-0.180	-0.160	-0.175	-0.170	-0.170	-0.175	
	$\%\theta_0$	-98	-87	-96	-93	-93	-96	-94
0.6π	θ radians	-0.215	-0.215	-0.210	-0.215	-0.210	-0.212	
	$\%\theta_0$	-117	-117	-115	-117	-115	-116	-116
0.7π	θ radians	-0.219	-0.222	-0.220	-0.220	-0.221	-0.221	
	$\%\theta_0$	-120	-121	-120	-120	-121	-121	-121
0.8π	θ radians	-0.190	-0.200	-0.200	-0.198	-0.205	-0.203	
	$\%\theta_0$	-104	-109	-109	-108	-112	-111	-109
0.9π	θ radians	-0.140	-0.159	-0.157	-0.160	-0.161	-0.165	-
	$\%\theta_0$	-77	-87	-86	-87	-88	-90	-86
1.0π	θ radians	-0.075	-0.100	-0.090	-0.090	-0.090	-0.090	
	$\%\theta_0$	-41	-55	-49	-49	-49	-49	-49
1.1π	θ radians	-0.01	-0.03	-0.02	-0.025	-0.-25	-0.025	-
	$\%\theta_0$	-5.5	-16	-11	-14	-14	-14	-12
1.2π	θ radians	0.06	0.04	0.05	0.04	0.045	0.04	
	$\%\theta_0$	33	22	27	22	25	22	25

Solution of differential equation: $\ddot{z} + (2 + \cos t)(z + 1) = 0$

t	cos t	f(t,z)	y	z	$\theta = z + 1$
$0\ \pi$	1.000	-3.00	0.00	0.00	1.00
$.05\pi$	0.988	-2.88	-0.465	-0.04	0.96
$.10\pi$	0.951	-2.52	-0.893	-0.14	0.86
$.15\pi$	0.891	-1.98	-1.25	-0.32	0.68
$.20\pi$	0.809	-1.32	-1.51	-0.53	0.47
$.25\pi$	0.707	-0.596	-1.66	-0.78	0.22
$.30\pi$	0.588	0.119	-1.70	-1.05	-0.05
$.35\pi$	0.454	0.760	-1.63	-1.31	-0.31
$.40\pi$	0.309	1.27	-1.47	-1.55	-0.55
$.45\pi$	0.156	1.66	-1.24	-1.77	-0.77
$.50\pi$	0.000	1.88	-0.96	-1.94	-0.94
$.55\pi$	-0.156	1.97	-0.655	-2.07	-1.07
$.60\pi$	-0.309	1.94	-0.354	-2.15	-1.15
$.65\pi$	-0.454	1.83	-0.050	-2.18	-1.18
$.70\pi$	-0.588	1.65	0.218	-2.17	-1.17
$.75\pi$	-0.707	1.44	0.466	-2.11	-1.11
$.80\pi$	-0.809	1.23	0.668	-2.03	-1.03
$.85\pi$	-0.891	1.11	0.854	-1.90	-0.90
$.90\pi$	-0.951	0.796	1.01	-1.76	-0.76
$.95\pi$	-0.987	0.595	1.11	-1.59	-0.59
1.00π	-1.000	0.410	1.19	-1.41	-0.41
1.05π	-0.988	0.212	1.24	-1.21	-0.21
1.10π	-0.951	0.021	1.26	-1.02	-0.02
1.15π	-0.891	-0.200	1.24	-0.82	0.18
1.20π	-0.809	-0.440	1.20	-0.63	0.37

APPENDIX

Solution of differential equation: $\ddot{z} + (1 + \cos 4t)(z + 1) = 0$

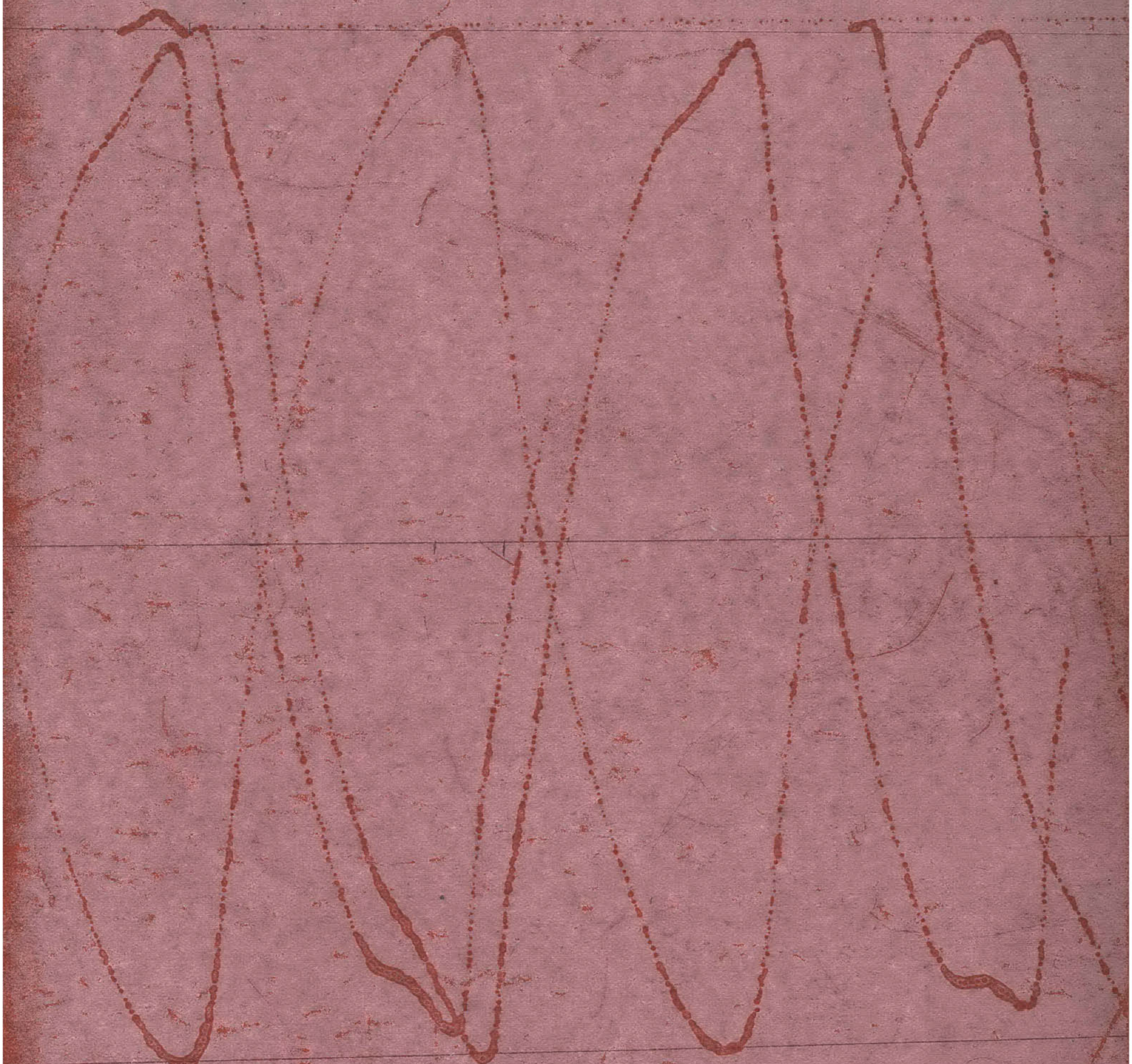
t	cos 4t	f(t,z)	y	z	$\theta = z + 1$
0	1	-2.0	0	0	1.00
0.1π	0.309	-1.19	-0.507	-0.0875	0.91
0.2π	-0.809	-0.136	-0.721	-0.288	0.71
0.3π	-0.809	-0.0925	-0.698	-0.516	0.48
0.4π	0.309	-0.340	-0.810	-0.74	0.26
0.5π	1.00	0.040	-0.895	-1.02	-0.02
0.6π	0.309	0.367	-0.793	-1.28	-0.28
0.7π	-0.809	0.0995	-0.726	-1.52	-0.52
0.8π	-0.809	0.145	-0.693	-1.76	-0.76
0.9π	0.309	1.23	-0.527	-1.94	-0.94
1.0π	1.00	2.10	0.050	-2.05	-1.05
1.1π	0.309	1.19	0.606	-1.91	-0.91
1.2π	-0.809	0.136	0.790	-1.71	-0.71
1.3π	-0.809	0.082	0.796	-1.43	-0.43
1.4π	0.309	0.262	0.866	-1.20	-0.20
1.5π	1.00	-0.22	0.900	-0.92	0.08

APPENDIX

Isoelastic material.

The suspension wire furnished by Professor V. F. de Forest is composed of an alloy containing 36 % nickel, 8 % chromium, and ½ % molybdenum. During manufacture, it was heavily cold-drawn. In order to establish the elastic properties it was necessary, after straightening the required length of wire, to heat it for a few seconds to about 800°F by passing a current of electricity through it.

After this treatment, the modulus of elasticity is constant to within 0.1 % for a temperature range from 0°F to 200°F. The creep of this material is negligible if the tensile stress is less than 55000 lbs per sq.in.



$$\frac{d^2\theta}{dt^2} + (1 + 2\cos 6t)\theta = 0$$

APPENDIX

BIBLIOGRAPHY

1. Bush, V., "The differential analyzer, a new machine for solving differential equations", Publications from the Massachusetts Institute of Technology, E. E. Dep't. Series, No. 75, Oct. 1931. Journal of the Franklin Institute, Vol. 212, No. 4, Oct., 1931.

2. Gould, K. E., "Integration of a functional product by measurement of infra-red radiation". Sc.D. thesis in E. E., 1927, M.I.T.

3. Gray, T. S., "A photoelectric integraph". Publications from the M.I.T., E. E. Dep't Series, No. 73, Aug. 1931. Journal of the Franklin Institute, Vol. 212, No. 1, July 1931.

4. Minorski, N., "Application des circuits électriques à l'integration graphique de quelques équations differentielles". R.G.E. Editeur 12, place de Laborde, 12 Paris (VIII^{e}), 1936. Extract from General Electric Review, May 30, 1936, Vol. 39, p. 787-794.

5. Den Hartog, J., "Mechanical Vibrations".

6. Timoshenko, S., "Vibration problems in engineering".

7. Van der Pol and Strutt, "On the stability of the solutions of Mathieu's equation", Philosophical Magazine, Vol. 5, p. 18, 1928.

8. Barrow, W. L., "Frequency modulation and the effects of a periodic capacity variation in a nondissipative oscillatory circuit". Proceedings of the Institute of Radio Engineers, Vol. 21, No. 8, Aug. 1933.

"On the oscillations of a circuit having a periodically varying capacitance". Proceedings of the Institute of Radio Engineers, Vol. 22, No. 2, Feb. 1934.

"A further study of oscillatory circuits having periodically varying parameters". Publications from the M.I.T., E. E. Dep't Series, No. 119, June 1936. Journal of the Franklin Institute,Vol. 221, No. 3, Mar. 1936. (W. L. Barrow, D. B. Smith, and F. W. Baumann)

9. Whittaker and Watson, "Modern analysis".

10. Marks, L. S., "Mechanical engineer's handbook".

11. Drysdale, "Electrical measuring instruments", Vol.I.

12. Fowle, F. F., "Standard handbook for electrical engineers".

13. Ford, L. R., "Differential equations".

14. Hazen, H. L., "The extension of electrical engineering analysis through the reduction of computational limitations by mechanical methods", Sc.D. thesis in E. E., , M. I. T.

APPENDIX

BIOGRAPHY

The author, Charles Norton Henshaw, was born on March 31, 1896, in New York City. He received his secondary school education at the Plattsburgh High School, Plattsburgh, N. Y. In 1920 he was graduated from the University of Vermont, with the degree of Bachelor of Science in Mechanical Engineering, and was elected to Phi Beta Kappa. In 1929 he was awarded the degree of Master of Science, by the Massachusetts Institute of Technology. During the academic year of 1933-1934, he was occupied in graduate study in physics, at Columbia University.

His experience includes one year as Instructor in Electrical Engineering, and two years as Instructor in Physics, at the University of Vermont, four years as Instructor in Mechanical Engineering at the Drexel Institute, and four years as Assistant Professor of Mechanical Engineering at the Rensselaer Polytechnic Institute.

The author is a member of the American Society of Mechanical Engineers, the Society for the Promotion of Engineering Education, and the Army Ordnance Association.