# ON THE BEHAVIOR OF THE HOMOGENEOUS SELF-DUAL MODEL FOR CONIC CONVEX OPTIMIZATION [1]

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October 15, 2004

**Abstract**

There is a natural norm associated with a starting point of the homogeneous self-dual (HSD) embedding model for conic convex optimization. In this norm two measures of the HSD model's behavior are precisely controlled independent of the problem instance: (i) the sizes of $\varepsilon$-optimal solutions, and (ii) the maximum distance of $\varepsilon$-optimal solutions to the boundary of the cone of the HSD variables. This norm is also useful in developing a stopping-rule theory for HSD-based interior-point methods such as SeDuMi. Under mild assumptions, we show that a standard stopping rule implicitly involves the sum of the sizes of the $\varepsilon$-optimal primal and dual solutions, as well as the size of the initial primal and dual infeasibility residuals. This theory suggests possible criteria for developing starting points for the homogeneous self-dual model that might improve the resulting solution time in practice.

**AMS Subject Classification:** 90C05, 90C22, 90C25, 90C31, 90C46, 90C51
**Keywords:** Convex Optimization, Convex Cone, Conic Optimization, Duality, Level Sets, Self-Dual Embedding, Self-scaled Cone

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[1]This research has been partially supported through the MIT-Singapore Alliance.

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# 1 Preliminaries

We consider convex optimization in conic linear form:

$$P: \quad \mathrm{VAL}_* := \min_x \quad c^T x \quad \text{s.t.} \quad Ax = b, \quad x \in C \tag{1}$$

and its dual:

$$D: \quad \mathrm{VAL}^* := \max_{y,z} \quad b^T y \quad \text{s.t.} \quad A^T y + z = c, \quad z \in C^* \ , \tag{2}$$

where $C \subset X$ is assumed to be a closed convex cone in the (finite) $n$-dimensional linear vector space $X$, and $b$ lies in the (finite) $m$-dimensional vector space $Y$. Here $C^*$ is the dual cone:

$$C^* := \{z \in X^* \mid z^T x \geq 0 \text{ for any } x \in C\} \ ,$$

where $X^*$ is the dual space of $X$ (the space of linear functionals on $X$).

We make the following assumption:

**Assumption A:** $C$ is a regular cone ($C$ is closed, convex, pointed, and has nonempty interior), whereby $C^*$ is also a regular cone.

We say that $P$ ($D$) is strictly feasible if there exists $\bar{x} \in \mathrm{int}C$ ($\bar{y}$ and $\bar{z} \in \mathrm{int}C^*$) that is feasible for $P$ ($D$).

Following [11] (also see [10]) we consider the following homogeneous self-dual (HSD) embedding of $P$ and $D$. Given initial values $(x^0, y^0, z^0)$ satisfying $x^0 \in \mathrm{int}C, z^0 \in \mathrm{int}C^*$, as well as initial constants $\tau^0 > 0, \kappa^0 > 0, \theta^0 > 0$, define the problem $H$:

$$
\begin{array}{lrrrrrrcr}
H: \ \mathrm{VAL}_H := \min_{x,y,z,\tau,\kappa,\theta} & & & & \bar{\alpha}\theta & & & & \\
\text{s.t.} & & Ax & -b\tau & +\bar{b}\theta & & & = & 0 \\
 & -A^T y & & +c\tau & +\bar{c}\theta & -z & & = & 0 \\
 & b^T y & -c^T x & & +\bar{g}\theta & & -\kappa & = & 0 \\
 & -\bar{b}^T y & -\bar{c}^T x & -\bar{g}\tau & & & & = & -\bar{\alpha} \\
 & & x \in C & \tau \geq 0 & & z \in C^* & \kappa \geq 0, & &
\end{array}
$$

where the constants $\bar{b}$, $\bar{c}$, $\bar{g}$, and $\bar{\alpha}$ are defined as follows:

$$
\begin{aligned}
\bar{b} &= \tfrac{b\tau^0 - Ax^0}{\theta^0} \\
\bar{c} &= \tfrac{A^T y^0 + z^0 - c\tau^0}{\theta^0} \\
\bar{g} &= \tfrac{c^T x^0 - b^T y^0 + \kappa^0}{\theta^0} \\
\bar{\alpha} &= \tfrac{(z^0)^T x^0 + \tau^0 \kappa^0}{\theta^0} \ .
\end{aligned} \tag{3}
$$

Note that the regular cone associated with $H$ is:

$$K_H := C \times C^* \times \mathrm{I\!R}_+ \times \mathrm{I\!R}_+^* \ , \tag{4}$$

where we distinguish between $\mathrm{I\!R}_+$ and $\mathrm{I\!R}_+^*$ only for notational consistency. Because $P$ $(D)$ can be recast equivalently as the problem of minimizing a linear function of a (regular) cone variable over the intersection of the regular cone and an affine set (see [9], [5]), we will focus on the behavior of the regular cone variables $x$ and $z$ and will effectively ignore the unrestricted variables $y$.

One natural measure of of the behavior of $P/D$ is the size of the largest $\varepsilon$-optimal solution. Define for $\varepsilon > 0$:

$$\begin{array}{llll} R_\varepsilon^P := & \max_x & \|x\| \\ & \text{s.t.} & x \text{ is feasible for } P \\ & & c^T x \le VAL_* + \varepsilon \end{array} \qquad \begin{array}{llll} R_\varepsilon^D := & \max_z & \|z\|_* \\ & \text{s.t.} & (y,z) \text{ is feasible for } D \\ & & b^T y \ge VAL^* - \varepsilon \ , \end{array} \tag{5}$$

where $\|\cdot\|$ is any given norm, and the dual norm $\|\cdot\|_*$ of $\|\cdot\|$ is:

$$\|w\|_* := \max_v \{ w^T v : \|v\| \le 1 \} \ .$$

Then $R_\varepsilon^P$ is a measure of the behavior of $P/D$: $R_\varepsilon^P$ is large to the extent that $P$ is nearly unbounded in objective value (and to the extent that $D$ is nearly infeasible), with similar remarks about $R_\varepsilon^D$. Indeed, Renegar's data-perturbation condition measure $C(d)$ must satisfy

$$C^2(d) + C(d)\frac{\varepsilon}{\|c\|_*} \ge R_\varepsilon^P$$

for $\varepsilon \le \|c\|_*$; this follows directly from Theorem 1.1 and Lemma 3.2 of [7].

A closely related measure of the behavior of $P/D$ is the maximum distance of $\varepsilon$-optimal solutions from the boundary of the cone variables:

$$\begin{array}{llll} r_\varepsilon^P := & \max_x & \mathrm{dist}(x, \partial C) \\ & \text{s.t.} & x \text{ is feasible for } P \\ & & c^T x \le VAL_* + \varepsilon \end{array} \qquad \begin{array}{llll} r_\varepsilon^D := & \max_z & \mathrm{dist}_*(z, \partial C^*) \\ & \text{s.t.} & (y,z) \text{ is feasible for } D \\ & & b^T y \ge VAL^* - \varepsilon \ , \end{array} \tag{6}$$

where $\mathrm{dist}(x, \partial C)$ denotes the minimal distance from $x$ to $\partial C$ in the norm $\|\cdot\|$ and $\mathrm{dist}_*(z, \partial C^*)$ denotes the minimal distance from $x$ to $\partial C^*$ in the dual norm $\|\cdot\|_*$.

Note that $r_\varepsilon^P$ measures the largest distance to the boundary of $C$ among all $\varepsilon$-optimal solutions $x$ of $P$. In the context of interior-point methods, $r_\varepsilon^P$ measures the extent to which near optimal-solutions are nicely bounded away from $\partial C$. Here Renegar's condition measure $C(d)$ must satisfy

$$\frac{\varepsilon \tau_C}{3\|c\|_*(C^2(d) + C(d))} \le r_\varepsilon^P$$

for $\varepsilon \le \|c\|_*$, where $\tau_C$ denotes the "min-width" constant of $C$:

$$\tau_C := \max_x \{ \mathrm{dist}(x, \partial C) : x \in C, \|x\| \le 1 \} \ ;$$

this follows directly from Theorem 1.1 of [7] and Theorem 17 of [4]. It is shown in [3] that $r_\varepsilon^P$ and $R_\varepsilon^D$ obey the following inequalities and so are nearly inversely proportional for fixed $\varepsilon > 0$:

$$\tau_C \cdot \varepsilon \le r_\varepsilon^P \cdot R_\varepsilon^D \le 2 \cdot \varepsilon \ ,$$

provided that $r_\varepsilon^P$ and $R_\varepsilon^D$ are both finite and positive, see Theorem 3.2 of [3]. Thus for a given $\varepsilon > 0$, it follows that $r_\varepsilon^P$ will be small if and only if $R_\varepsilon^D$ is large. These results can also be stated in dual forms, exchange the roles of the primal and dual problems and using the appropriate norms for the appropriate (regular) cone variables/spaces.

Herein we study the size of the largest $\varepsilon$-optimal solution $R_\varepsilon^{(\cdot)}$ and the maximum distance of $\varepsilon$-optimal solutions from the boundary of the cone $r_\varepsilon^{(\cdot)}$, as applied to the HSD model $H$ (which is also a conic optimization problem of similar conic format as $P$ and/or $D$, but with other very special structure). We denote these measures for $H$ by $R_\varepsilon^H$ and $r_\varepsilon^H$, respectively. Let $w^0 := (x^0, z^0, \tau^0, \kappa^0)$ denote the starting values of the (regular) cone variables of $H$. Our main behavioral result is that there is a natural norm $\|\cdot\|^{w^0}$ defined by $w^0$ and its regular cone $K_H$ (4), and in this norm the measures $R_\varepsilon^H$ and $r_\varepsilon^H$ are precisely controlled independent of any particular characteristics of the problem instance, as follows:

$$R_\varepsilon^H = (x^0)^T z^0 + \kappa^0 \tau^0 + \varepsilon$$

for all $\varepsilon > 0$, and

$$r_\varepsilon^H = \frac{\varepsilon}{(x^0)^T z^0 + \kappa^0 \tau^0} \ ,$$

see Theorem 3.1. Notice that $R_\varepsilon^H$ and $r_\varepsilon^H$ do not depend on the behavior of $P$, the data for $P$, the null space of $A$, etc., and only depend on the chosen starting values $x^0, z^0, \tau^0, \kappa^0$. Therefore to the extent that $R_\varepsilon^{(\cdot)}$, $r_\varepsilon^{(\cdot)}$ are relevant behavioral measures of a conic optimization problem, this indicates that $H$ is inherently well-behaved in these measures in this norm. Note also that $R_\varepsilon^H$ and $r_\varepsilon^H$ are linear in $\varepsilon$.

We also develop a stopping-rule theory for HSD-based interior-point methods such as SeDuMi [8]. Under mild assumptions, we show that a standard stopping rule implicitly involves the sum of the norms of the $\varepsilon$-optimal primal and dual solutions (where these norms are also defined by the starting points $x^0$ and $z^0$), as well as the size of the initial primal and dual infeasibility residuals. This theory suggests possible criteria for developing starting points for the homogeneous self-dual model that might improve the resulting solution time in practice.

The paper is organized as follows. In Section 2 we review the construction of a family of norms that are linear on a given regular cone. This construction is then applied in Section 3 where we present and prove the main behavioral result about $R_\varepsilon^H$ and $r_\varepsilon^H$ discussed above. Section 4 contains the analysis of a standard stopping rule for an HSD interior-point method and its connection to $R_\varepsilon^P$ and $R_\varepsilon^D$. Section 5 contains remarks and open questions.

## 2 A Family of Norms that are Linear on $K$

The norm construction presented herein is implicitly used in many results involving conic optimization and interior-point methods; we present it from first principles here for completeness. By way of motivation, consider the simple problem of computing $v \in \mathbb{R}^n$ that satisfies:

$$\begin{aligned} Mv &= 0 \\ v &\in \mathbb{R}^n_+ \\ \|v\| &= 1 \end{aligned}$$

for some norm $\|\cdot\|$. The feasible region of this problem will generally be non-convex unless $\|\cdot\|$ happens to be linear on $\mathbb{R}^n_+$, as it is in the special case when $\|v\| = \|(W^0)v\|_1$ for some $w^0 > 0$ (here $W^0$ is the diagonal matrix whose diagonal components are the corresponding components of $w^0$), in which case $\|v\| = (w^0)^T v$ for $v \in \mathbb{R}^n_+$. Conversely, suppose we have a linear function $f(v) = (w^0)^T v$ that satisfies $(w^0)^T v > 0$ for $v \in \mathbb{R}^n_+ \setminus \{0\}$. Then $w^0 \in \mathbb{R}^n_{++}$ in particular, and the following norm agrees with the linear function $f(v)$ for all $v \in \mathbb{R}^n_+$:

$$\begin{array}{rclll} \|v\| & := & \|v\|^{w^0} & := & \min_{v^1,v^2} \quad (w^0)^T(v^1+v^2) \\ & & & & \text{s.t.} \quad v^1 - v^2 = v \\ & & & & \qquad v^1 \in \mathbb{R}^n_+ \\ & & & & \qquad v^2 \in \mathbb{R}^n_+ \ . \end{array}$$

This norm is a linear function on $\mathbb{R}^n_+$, and in fact is the norm with the smallest unit ball that satisfies $\|v\| = (w^0)^T v$ for $v \in \mathbb{R}^n_+$. One can easily verify that $\|v\|^{w^0}$ corresponds to $\|(W^0)v\|_1$.

The above construction easily generalizes to an arbitrary regular cone $K$. For a regular cone $K$ in the finite-dimensional linear space $V$, let $w^0 \in \text{int}K^*$ be given, and define the following norm on $V$:

$$\begin{array}{rclll} \|v\| & := & \|v\|^{w^0} & := & \min_{v^1,v^2} \quad (w^0)^T(v^1+v^2) \\ & & & & \text{s.t.} \quad v^1 - v^2 = v \\ & & & & \qquad v^1 \in K \\ & & & & \qquad v^2 \in K \ . \end{array} \tag{7}$$

It is straightforward to verify that $\|\cdot\|$ is indeed a norm. The following result states that the restriction of $\|\cdot\|$ to $K$ is a linear function.

**Proposition 2.1** *If $v \in K$, then $\|v\|^{w^0} = (w^0)^T v$.*

**Proof:** For $v \in K$, the assignment $(v^1, v^2) \leftarrow (v, 0)$ is feasible for (7) and so $\|v\|^{w^0} \le (w^0)^T v$. However, notice that for any $(v^1, v^2)$ feasible for (7) we have $(w^0)^T(v^1 + v^2) = (w^0)^T(v + 2v^2) \ge (w^0)^T v$, showing that $\|v\|^{w^0} \ge (w^0)^T v$, and hence $\|v\|^{w^0} = (w^0)^T v$. ∎

The dual norm of $\|\cdot\|$ is readily derived as:

$$\begin{array}{rclll} \|w\|_* & := & \|w\|_*^{w^0} & := & \min_{\alpha} \quad \alpha \\ & & & & \text{s.t.} \quad w + \alpha w^0 \in K^* \\ & & & & \qquad -w + \alpha w^0 \in K^* \ . \end{array} \tag{8}$$

We now show that the norms (7) and (8) specify to well-known norms in the case of the three standard self-scaled cones $\mathbb{R}^n_+$, $S^n_+$, and $Q^n$.

**Nonnegative Orthant $\mathbb{R}^n_+$.** Let $K = K^* = \mathbb{R}^n_+$, and let $w^0 \in \text{int}K^*(= \mathbb{R}^n_{++})$ be given. We have already seen that $\|v\|^{w^0}$ works out to be $\|v\|^{w^0} = \|W^0 v\|_1$, and the dual norm works out to be:

$$\|w\|_*^{w^0} = \| \left(W^0\right)^{-1} w\|_\infty \ .$$

**Semi-Definite Cone $S^n_+$.** Let $K = K^* = S^n_+ := \{v \in S^n : v \succeq 0\}$ where $S^n$ is the space of real symmetric $n \times n$ matrices and "$\succeq$" denotes the Löwner partial ordering, namely $v \succeq u$ if and only if $v - u$ is positive semidefinite. Let $w^0 \in \text{int}K^*(= S^n_{++})$ be given. Then $\|v\|^{w^0}$ and $\|w\|_*^{w^0}$ work out to be

$$\|v\|^{w^0} = \left\|\lambda\left((w^0)^{\frac{1}{2}} v (w^0)^{\frac{1}{2}}\right)\right\|_1 \quad \text{and} \quad \|w\|_*^{w^0} = \left\|\lambda\left((w^0)^{-\frac{1}{2}} w (w^0)^{-\frac{1}{2}}\right)\right\|_\infty \ ,$$

where $\lambda(x)$ is the vector of eigenvalues of the matrix $x$. A proof of this is shown in detail in Appendix B. Note that $\|v\|^{w^0} = \text{trace}(w^0 v)$ for $v \in S^n_+$.

**Second-Order Cone $Q^n$.** Let $K = K^* = Q^n := \{v \in \mathbb{R}^n : \|(v_2, \ldots, v_n)\|_2 \le v_1\}$. Let $w^0 = e^1 := (1, 0, \ldots, 0)$, and note that $w^0 \in \text{int}Q^n$. Then $\|v\|^{w^0}$ and $\|w\|_*^{w^0}$ work out to be

$$\|v\|^{w^0} = \max\{|v_1|, \|(v_2, \ldots, v_n)\|_2\} \quad \text{and} \quad \|w\|_*^{w^0} = |w_1| + \|(w_2, \ldots, w_n)\|_2 \ .$$

Note that $\|v\|^{w^0} = (e^1)^T v$ for $v \in Q^n$. For general $w^0 \in \text{int}Q^n$, we present the following closed form expression for $\|v\|^{w^0}$ and $\|w\|_*^{w^0}$ whose proof is rather laborious, see Appendix B for details: rewrite $w^0 = (w_1^0, \bar{w})$ where $\bar{w} = (w_2^0, \ldots, w_n^0)$ and form the matrix $M$:

$$M = \begin{pmatrix} w_1^0 & (\bar{w})^T \\ \bar{w} & \left(\sqrt{(w_1^0)^2 - \bar{w}^T\bar{w}}\right) I + \frac{\bar{w}\bar{w}^T}{w_1^0 + \sqrt{(w_1^0)^2 - \bar{w}^T\bar{w}}} \end{pmatrix} \ . \tag{9}$$

Then it is shown in the Appendix that

$$\|v\|^{w^0} = \max\{|(Mv)_1|, \|((Mv)_2, \ldots, (Mv)_n)\|_2\} \ .$$

It follows directly from norm duality that

$$\|w\|_*^{w^0} = |(M^{-1}w)_1| + \|((M^{-1}w)_2, \ldots, (M^{-1}w)_n)\|_2\} \ ,$$

where $M^{-1}$ has the following direct formula:

$$M^{-1} = ((w_1^0)^2 - \bar{w}^T\bar{w})^{-1} \begin{pmatrix} w_1^0 & -(\bar{w})^T \\ -\bar{w} & \left(\sqrt{(w_1^0)^2 - \bar{w}^T\bar{w}}\right) I + \frac{\bar{w}\bar{w}^T}{w_1^0 + \sqrt{(w_1^0)^2 - \bar{w}^T\bar{w}}} \end{pmatrix} \ .$$

Returning to the case of a general regular cone $K$, we close this section with the following result which will be useful in our analysis:

**Proposition 2.2** *Suppose that $K$ is a regular cone, $w^0 \in \text{int}K^*$ is given, and $\|\cdot\|$ and $\|\cdot\|_*$ are given by (7) and (8), respectively. If $w \in K^*$, then*

$$\text{dist}_*(w, \partial K^*) \geq r \quad \Leftrightarrow \quad w - rw^0 \in K^* \ .$$

**Proof:** Suppose first that $w - rw^0 \in K^*$, and let $y \in \partial K^*$ be given. Then there exists $x \in K \setminus \{0\}$ satisfying $y^T x \leq 0$. Computing $\|w - y\|_*$ via (8) we see that any $\alpha$ that is feasible for (8) must satisfy $y - w + \alpha w^0 \in K^*$, and taking the inner product with $x$ yields $x^T(y - w + \alpha w^0) \geq 0$. Notice that $x^T(w - rw^0) \geq 0$ and recalling that $y^T x \leq 0$ yields $(\alpha - r)(w^0)^T x \geq 0$, which implies that $\alpha \geq r$ since $(w^0)^T x > 0$. Therefore $\|w - y\|_* \geq r$, and so $\text{dist}_*(w, \partial K^*) \geq r$.

Conversely, suppose that $\text{dist}_*(w, \partial K^*) \geq r$, but assume that $w - rw^0 \notin K^*$. Because $w \in K^*$ there exists $\alpha \in [0, r)$ for which $\bar{w} := w - \alpha w^0 \in \partial K^*$. Now notice that $w - \bar{w} + \alpha w^0 = 2\alpha w^0 \in K^*$ and $\bar{w} - w + \alpha w^0 = 0 \in K^*$, whereby from (8) it follows that $\|\bar{w} - w\|_* \leq \alpha < r$. And since $\bar{w} \in \partial K^*$ it follows that $\text{dist}_*(w, \partial K^*) < r$, which is contradiction. Therefore $w - rw^0 \in K^*$, completing the proof. ▮

## 3 Behavior of the HSD Model

Recall the following properties of $H$:

**Lemma 3.1** *[11], [10]*

- *$H$ is self-dual.*
- *$(x, y, z, \tau, \kappa, \theta) = (x^0, y^0, z^0, \tau^0, \kappa^0, \theta^0)$ is a strictly feasible primal (and hence dual) solution of $H$.*
- *$\text{VAL}_H = 0$ and $H$ attains its optimum.*
- *Let $(x^*, y^*, z^*, \tau^*, \kappa^*, \theta^*)$ be any optimal solution of $H$. Then $(x^*)^T z^* = 0$ and $\tau^* \cdot \kappa^* = 0$. If $\tau^* > 0$, then $x^*/\tau^*$ is an optimal solution of $P$ and $(y^*/\tau^*, z^*/\tau^*)$ is an optimal solution of $D$. If $\kappa^* > 0$, then either $c^T x^* < 0$ or $-b^T y^* < 0$ or both. The former case implies that $P$ is infeasible, and the latter case implies that $D$ is infeasible.* ▮

Pre-multiplying the four equation systems of $H$ by $y^T, x^T, \tau$, and $\theta$, respectively, and summing yields:

$$x^T z + \tau\kappa = \bar{\alpha}\theta \tag{10}$$

for any feasible solution $(x, y, z, \tau, \kappa, \theta)$ of $H$, see [11]. Pre-multiplying the four equation systems of $H$ by $(y^0)^T, (x^0)^T, \tau^0$, and $\theta^0$, respectively, summing, and using (3) yields:

$$(z^0)^T x + (x^0)^T z + \kappa^0 \tau + \tau^0 \kappa = \bar{\alpha}\theta^0 + \bar{\alpha}\theta \tag{11}$$

for any feasible solution $(x, y, z, \tau, \kappa, \theta)$ of $H$, also see [11]. We also have the following property of $H$ whose proof is deferred to the end of this section:

**Proposition 3.1** *For any $\varepsilon \geq 0$, there exists a feasible solution of $H$ with objective value (and hence optimality gap) equal to $\varepsilon$.* ▮

Let $v := (x, z, \tau, \kappa) \in K_H$ be the variables of $H$ corresponding to the cone $K_H$ (4). The dual cone of $K_H$ is:

$$K_H^* := C^* \times C \times \mathbb{R}_+^* \times \mathbb{R}_+ \ ,$$

and we write $w := (z, x, \kappa, \tau) \in K_H^*$, where the order of the variables has been amended so that variables that are dual to each other in the dual formulation of $H$ are aligned with their associated primal variables:

$$\begin{array}{rlcccccccl} \text{Primal variables}: & v = ( & x & , & z & , & \tau & , & \kappa & ) \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & \\ \text{Dual variables}: & w = ( & z & , & x & , & \kappa & , & \tau & ) \end{array}$$

Given the initial values $(x^0, y^0, z^0, \tau^0, \kappa^0, \theta^0)$ satisfying $x^0 \in \text{int}C$, $z^0 \in \text{int}C^*$, $\tau^0 > 0$, $\kappa^0 > 0$, $\theta^0 > 0$ that are used to define $H$, notice that $w^0 := (z^0, x^0, \kappa^0, \tau^0) \in K_H^*$, hence $w^0$ can be used to define the norms $\|\cdot\|^{w^0}$ and $\|\cdot\|_*^{w^0}$ on the spaces of variables $v$ and $w$ using (7) and (8), respectively:

$$\begin{array}{rclll} \|v\|^{w^0} & := & \min_{v^1, v^2} & (w^0)^T(v^1 + v^2) \\ & & \text{s.t.} & v^1 - v^2 = v \\ & & & v^1 \in K_H \\ & & & v^2 \in K_H \ . \end{array} \qquad \begin{array}{rclll} \|w\|_*^{w^0} & := & \min_\alpha & \alpha \\ & & \text{s.t.} & w + \alpha w^0 \in K_H^* \\ & & & -w + \alpha w^0 \in K_H^* \end{array} \tag{12}$$

Let us fix these norms on the spaces of variables $v = (x, z, \tau, \kappa)$ and $w = (z, x, \kappa, \tau)$, respectively.

**Remark 3.1** *Under the norms (12), suppose that $(x, y, z, \tau, \kappa, \theta)$ is a feasible solution of $H$. Then from Propositions 2.1 and 2.2 we obtain:*

$$\|(x, z, \tau, \kappa)\|^{w^0} = (z^0)^T x + (x^0)^T z + \kappa^0 \tau + \tau^0 \kappa$$

*and*

$$\text{dist}_*((z, x, \kappa, \tau), \partial K_H^*) \geq r \ \ \text{if and only if} \ \ (z, x, \kappa, \tau) - r(z^0, x^0, \kappa^0, \tau^0) \in K_H^* \ . ▮$$

For $\varepsilon > 0$ let $R_\varepsilon^H$ denote the size of the largest $\varepsilon$-optimal solution of $H$:

$$\begin{array}{rcll} R_\varepsilon^H := & \max_{x,y,z,\tau,\kappa,\theta} & \|(x, z, \tau, \kappa)\|^{w^0} \\ & \text{s.t.} & (x, y, z, \tau, \kappa, \theta) \text{ is feasible for } H \\ & & \bar{\alpha}\theta \leq \varepsilon \ , \end{array} \tag{13}$$

and let $r_\varepsilon^H$ denote the maximal distance to $\partial K_H^*$ over all $\varepsilon$-optimal solution of $H$:

$$\begin{array}{rlll} r^H_\varepsilon := & \max_{x,y,z,\tau,\kappa,\theta} & \mathrm{dist}_*((z,x,\kappa,\tau),\partial K^*_H) & \\ & \text{s.t.} & (x,y,z,\tau,\kappa,\theta) \text{ is feasible for } H & \\ & & \bar\alpha\theta \le \varepsilon\ . & \end{array} \tag{14}$$

Our main behavioral result is:

**Theorem 3.1** *Under the norms (12),*

$$R^H_\varepsilon = (x^0)^T z^0 + \kappa^0\tau^0 + \varepsilon$$

*for all* $\varepsilon \ge 0$, *and*

$$r^H_\varepsilon = \frac{\varepsilon}{(x^0)^T z^0 + \kappa^0\tau^0}$$

*for all* $\varepsilon$ *satisfying* $0 \le \varepsilon \le (x^0)^T z^0 + \kappa^0\tau^0$.

**Proof:** To prove the first assertion, let $\varepsilon \ge 0$ be given, let $\tilde\varepsilon \in [0,\varepsilon]$, and let $(x,y,z,\tau,\kappa,\theta)$ be a feasible solution of $H$ satisfying $\bar\alpha\theta = \tilde\varepsilon$ (which is guaranteed to exist by Proposition 3.1). Then $(x,z,\tau,\kappa) \in K_H$, whereby

$$\begin{array}{rcll} \|(x,z,\tau,\kappa)\|^{w^0} & = & (z^0)^T x + (x^0)^T z + \kappa^0\tau + \tau^0\kappa & \text{(from Remark 3.1)} \\ & = & \bar\alpha\theta^0 + \bar\alpha\theta & \text{(from (11))} \\ & = & (x^0)^T z^0 + \kappa^0\tau^0 + \tilde\varepsilon & \text{(from (3))} \\ & \le & (x^0)^T z^0 + \kappa^0\tau^0 + \varepsilon\ , & \end{array} \tag{15}$$

whereby it follows that $R^H_\varepsilon \le (x^0)^T z^0 + \kappa^0\tau^0 + \varepsilon$. However, simply setting $\tilde\varepsilon = \varepsilon$ shows via (15) that $R^H_\varepsilon \ge (x^0)^T z^0 + \kappa^0\tau^0 + \varepsilon$, which then proves the equality of the first assertion.

To prove the second assertion, let $(x^*,y^*,z^*,\tau^*,\kappa^*,\theta^*)$ be an optimal solution of $H$ and recall from Lemma 3.1 that $(x^0,y^0,z^0,\tau^0,\kappa^0,\theta^0)$ is feasible for $H$. Let $\lambda = \varepsilon/\bar\alpha\theta^0$, and notice that $\lambda \in [0,1]$ for $0 \le \varepsilon \le (x^0)^T z^0 + \kappa^0\tau^0 = \bar\alpha\theta^0$, whereby

$$(x,y,z,\tau,\kappa,\theta) := (1-\lambda)(x^*,y^*,z^*,\tau^*,\kappa^*,\theta^*) + \lambda(x^0,y^0,z^0,\tau^0,\kappa^0,\theta^0)$$

is a feasible solution of $H$ with objective value $\bar\alpha\theta = \varepsilon$. Then $(z,x,\kappa,\tau) - \lambda(z^0,x^0,\kappa^0,\tau^0) = (1-\lambda)(z^*,x^*,\kappa^*,\tau^*) \in K^*_H$, whereby from Remark 3.1 it follows that

$$\mathrm{dist}_*((z,x,\kappa,\tau),\partial K^*_H) \ge \lambda = \frac{\varepsilon}{\bar\alpha\theta^0} = \frac{\varepsilon}{(x^0)^T z^0 + \kappa^0\tau^0}\ ,$$

and so $r^H_\varepsilon \ge \frac{\varepsilon}{(x^0)^T z^0 + \kappa^0\tau^0}$. On the other hand, let $(x,y,z,\tau,\kappa,\theta)$ be any feasible solution of $H$ with objective value $\bar\alpha\theta \le \varepsilon$, and suppose that $\mathrm{dist}_*((z,x,\kappa,\tau),\partial K^*_H) = r$. It then follows from Remark 3.1 that

$$(z,x,\kappa,\tau) - r(z^0,x^0,\kappa^0,\tau^0) \in K^*_H\ .$$

Therefore

$$\begin{aligned}
\varepsilon &\geq \bar{\alpha}\theta = \bar{\alpha}\theta - \mathrm{VAL}_H \\
&= (x^*, z^*, \tau^*, \kappa^*)^T (z, x, \kappa, \tau) \\
&\geq (x^*, z^*, \tau^*, \kappa^*)^T (r(z^0, x^0, \kappa^0, \tau^0)) \\
&= r\bar{\alpha}\theta^0 = r((x^0)^T z^0 + \kappa^0 \tau^0) \ ,
\end{aligned}$$

which implies that $\mathrm{dist}_*((z, x, \kappa, \tau), \partial K_H^*) \leq \frac{\varepsilon}{(x^0)^T z^0 + \kappa^0 \tau^0}$ and hence the reverse inequality $r_\varepsilon^H \leq \frac{\varepsilon}{(x^0)^T z^0 + \kappa^0 \tau^0}$, completing the proof. ▮

To the extent that $R_\varepsilon^{(\cdot)}$, $r_\varepsilon^{(\cdot)}$ are relevant behavioral measures of a conic optimization problem, then Theorem 3.1 indicates that $H$ is inherently well-behaved in the norms (12). Indeed, $R_\varepsilon^H$ and $r_\varepsilon^H$ do not depend on the problem instance itself, but only on the chosen starting values $x^0, z^0, \tau^0, \kappa^0$. Note that $R_\varepsilon^H$ and $r_\varepsilon^H$ are linear in $\varepsilon$.

We close this section with the proof of Proposition 3.1, which follows as a special case of the following more general proposition.

**Proposition 3.2** *Suppose that there exist strictly feasible solutions of the given primal and dual conic optimization problems $P$ and $D$. Then for any $\varepsilon \geq 0$, there exists a feasible solution $x$ of $P$ and $(y, z)$ of $D$ with objective value gap $c^T x - b^T y = \varepsilon$.*

**Proof:** It is well-known that the supposition of strictly feasible primal and dual solutions guarantee that both $P$ and $D$ attain their optimal values with no duality gap. It therefore remains to show that there exist feasible solutions to the primal-dual pair with arbitrarily large objective value gap. By supposition, there exists $\bar{x} \in \mathrm{int}C$ satisfying $A\bar{x} = b$ and there exists $\bar{y}$ and $\bar{z} \in \mathrm{int}C^*$ satisfying $A^T\bar{y} + \bar{z} = c$. Let us first suppose that $P$ has an unbounded feasible region. Then there exists $d \in C$ satisfying $d \neq 0$ and $Ad = 0$, and it follows that $c^T d = \bar{y}^T Ad + \bar{z}^T d = \bar{z}^T d > 0$. Therefore $\bar{x} + \theta d$ is feasible for arbitrarily large $\theta \geq 0$ with arbitrarily large objective function value, proving the result in this case. If the feasible region of $P$ is bounded, it is straightforward to show that $D$ has an unbounded feasible region, and similar arguments apply. ▮

**Proof of Proposition 3.1:** Consider $H$ as a conic convex optimization problem of the form $P$. From Lemma 3.1 it follows that $H$ and its dual have strictly feasible solutions, so from Proposition 3.2 it follows that $H$ and its dual (also $H$) have feasible solutions whose objective function gap is $2\varepsilon$. But since $H$ is self-dual and $\mathrm{VAL}_H = 0$, this means that $H$ has a feasible solution with objective function value $\varepsilon$.▮

# 4 Stopping-Rule Theory for Interior-Point Methods such as SeDuMi

In this section we develop results related to a standard stopping rule used by an interior-point method for solving $P$ and $D$ via the homogeneous self-dual embedding model $H$ (such as SeDuMi developed by Jos Sturm [8]). Here the cone $C$ is the Cartesian product of self-scaled cones:

$$C = S_+^{s_1} \times \ldots \times S_+^{s_{n_s}} \times Q^{q_1} \times \ldots \times Q^{q_{n_q}} \times \Re_+^{n_l} \ .$$

(This cone notation was presented in Section 2.) We focus on norms induced by the starting points and their connection to the algorithm's stopping rule.

Consider the problems $P$ and $D$. We presume that $P$ and $D$ are both feasible and have a common optimal objective function value OPTVAL. In order to be consistent with the norm $\|v\|^{w^0} := \|(x,z,\tau,\kappa)\|^{w^0}$ on the cone variables of $H$ defined in (12), whose restriction to the cone $K_H$ takes the convenient functional form

$$\|(x,z,\tau,\kappa)\|^{w^0} = (z^0)^T x + (x^0)^T z + \kappa^0\tau + \tau^0\kappa \qquad \text{for } (x,z,\tau,\kappa) \in K_H \ , \tag{16}$$

we define the norms on $x$ and $z$ as follows:

$$\|x\| := \|x\|^{z^0} \ (= (z^0)^T x \text{ for } x \in C) \quad \text{and} \quad \|z\| := \|z\|^{x^0} \ (= (x^0)^T z \text{ for } z \in C^*) \tag{17}$$

for the variables $x$ and $z$ in $P$ and $D$, respectively. (Note that these norms are not dual to one another. We have defined the norms so that they will be consistent with (16) and in so doing we treat $P$ and $D$ and their cone variables $x$ and $z$ somewhat independently.) Using these norms and their specification (17) on the cones $C, C^*$, the sizes of the largest $\varepsilon$-optimal solutions for $P$ and $D$ are:

$$\begin{array}{rll} R_\varepsilon^P := & \max_x & (z^0)^T x \\ & \text{s.t.} & Ax = b \\ & & c^T x \leq \text{OPTVAL} + \varepsilon \\ & & x \in C \end{array} \qquad \begin{array}{rll} R_\varepsilon^D := & \max_{y,z} & (x^0)^T z \\ & \text{s.t.} & A^T y + z = c \\ & & b^T y \geq \text{OPTVAL} - \varepsilon \\ & & z \in C^* \ . \end{array} \tag{18}$$

Let $(x,y,z,\tau,\kappa,\theta)$ be an iterate generated by SeDuMi, hence $(x,y,z,\tau,\kappa,\theta)$ is feasible for $H$. In order to check whether to stop at this iterate, SeDuMi computes trial primal and dual values $(\bar{x},\bar{y},\bar{z}) := (x/\tau, y/\tau, z/\tau)$, and their residuals:

$$\begin{array}{rcl} r_p & := & b - A\bar{x} \\ r_d & := & A^T\bar{y} + \bar{z} - c \\ r_g & := & c^T\bar{x} - b^T\bar{y} \ . \end{array} \tag{19}$$

According to SeDuMi's code, the algorithm will stop at the current iterate if the following inequality is satisfied:

$$2\frac{\|r_p\|_\infty}{1 + \|b\|_\infty} + 2\frac{\|r_d\|_\infty}{1 + \|c\|_\infty} + \frac{(r_g)^+}{\max\{|c^T\bar{x}|, |b^T\bar{y}|, 0.001 \times \tau\}} \leq r_{\max} \ , \tag{20}$$

where the default is $r_{\max} = 10^{-9}$. We will analyze the slightly modified and more convenient (and perhaps more intuitive) stopping rule inequality instead:

$$2\frac{\|r_p\|_\infty}{1+\|b\|_\infty} + 2\frac{\|r_d\|_\infty}{1+\|c\|_\infty} + \frac{(r_g)^+}{\max\{|c^T\bar{x}|, |b^T\bar{y}|, 1\}} \le r_{\max} \ . \tag{21}$$

Define INITRESID ("initial residual") to be the following combined primal, dual, and gap residual of the starting point $(x^0, y^0, z^0, \tau^0, \kappa^0, \theta^0)$:

$$\text{INITRESID} := \left(2\frac{\|b - Ax^0/\tau^0\|_\infty}{1+\|b\|_\infty} + 2\frac{\|A^Ty^0/\tau^0 + z^0/\tau^0 - c\|_\infty}{1+\|c\|_\infty} + \frac{(c^Tx^0/\tau^0 - b^Ty^0/\tau^0)^+}{\max\{|\text{OPTVAL}|, 1\}}\right) , \tag{22}$$

and define the presumably similar quantity:

$$\text{QUANT} := \left( \begin{array}{c} 2\frac{\|b-Ax^0/\tau^0\|_\infty}{1+\|b\|_\infty} + 2\frac{\|A^Ty^0/\tau^0+z^0/\tau^0-c\|_\infty}{1+\|c\|_\infty} \\ \\ +\frac{\left(c^Tx^0/\tau^0 - b^Ty^0/\tau^0 + \kappa^0/\tau^0 - \frac{\theta^0}{\tau^0}(\frac{\kappa}{\theta})\right)^+}{\max\{|c^T\bar{x}|, |b^T\bar{y}|, 1\}} \end{array} \right) . \tag{23}$$

**Lemma 4.1** *Assume that $P$ and $D$ are both feasible, $(x^0, y^0, z^0, \tau^0, \kappa^0, \theta^0)$ is the starting point, and $(x, y, z, \tau, \kappa, \theta)$ is a feasible iterate of an interior-point method for solving $H$. Let $(\bar{x}, \bar{y}, \bar{z}) := (x/\tau, y/\tau, z/\tau)$ be the trial solution of $P$ and $D$. Then the stopping rule inequality (21) is equivalent to:*

$$\frac{\theta}{\theta^0} \le r_{\max}\left(\frac{\theta^0+\theta}{\theta^0}\right)\left(\frac{(x^0)^Tz^0 + \tau^0\kappa^0}{\tau^0}\right)\left(\|\bar{x}\|^{z^0} + \|\bar{z}\|^{x^0} + \kappa^0 + \frac{\tau^0\kappa}{\tau}\right)^{-1} (\text{QUANT})^{-1} \ . \tag{24}$$

**Proof:** The equations of $H$ together with (3) yield:

$$\begin{array}{rcl} r_p &=& b - A\bar{x} = \bar{b}(\theta/\tau) = (b - Ax^0/\tau^0)\left(\frac{\theta\tau^0}{\tau\theta^0}\right) \\ r_d &=& A^T\bar{y} + \bar{z} - c = \bar{c}(\theta/\tau) = \left(A^Ty^0/\tau^0 + z^0/\tau^0 - c\right)\left(\frac{\theta\tau^0}{\tau\theta^0}\right) \\ r_g &=& c^T\bar{x} - b^T\bar{y} = (\bar{g} - \kappa/\theta)(\theta/\tau) = \left(c^Tx^0/\tau^0 - b^Ty^0/\tau^0 + \frac{\kappa^0}{\tau^0} - \frac{\kappa\theta^0}{\theta\tau^0}\right)\left(\frac{\theta\tau^0}{\tau\theta^0}\right) \ , \end{array}$$

whereby (21) becomes:

$$\left(\frac{\theta\tau^0}{\tau\theta^0}\right)\text{QUANT} \le r_{\max} \ . \tag{25}$$

Next observe that

$$\begin{array}{rcll} (z^0)^T\bar{x} + (x^0)^T\bar{z} + \kappa^0 + \frac{\tau^0\kappa}{\tau} &=& \frac{(z^0)^Tx + (x^0)^Tz + \kappa^0\tau + \tau^0\kappa}{\tau} & \\ &=& \frac{\bar{\alpha}\theta^0 + \bar{\alpha}\theta}{\tau} \ , & \text{(from (11))} \end{array}$$

which yields

$$\frac{1}{\tau} = \frac{(z^0)^T\bar{x} + (x^0)^T\bar{z} + \kappa^0 + \frac{\tau^0\kappa}{\tau}}{\bar{\alpha}(\theta^0+\theta)} = \frac{\theta^0\left(\|\bar{x}\|^{z^0} + \|\bar{z}\|^{x^0} + \kappa^0 + \frac{\tau^0\kappa}{\tau}\right)}{((z^0)^Tx^0 + \tau^0\kappa^0)(\theta^0+\theta)} \ ,$$

using (17) and the definition of $\bar{\alpha}$ in (3). Substituting the above in (25) and rearranging terms yields the result. ▮

Let $T$ denote the total number of iterations performed by an interior-point method for solving $H$, and let $\beta$ denote the (geometric) average decrease in the duality gap of $H$ over all iterations:

$$\beta := \sqrt[T]{\frac{2\bar{\alpha}\theta}{2\bar{\alpha}\theta^0}} = \sqrt[T]{\frac{\theta}{\theta^0}} \ . \tag{26}$$

The following corollary follows by taking the logarithm of both sides of (24) and using (26).

**Corollary 4.1**

$$T = \left\lceil \frac{\ln\left(\frac{\theta^0}{\theta^0+\theta}\right) + \ln\left(\|\bar{x}\|^{z^0} + \|\bar{z}\|^{x^0} + \kappa^0 + \frac{\tau^0\kappa}{\tau}\right) + \ln\left(\text{QUANT}\right) + \ln\left(\frac{\tau^0}{(x^0)^T z^0 + \tau^0\kappa^0}\right) + |\ln(r_{\max})|}{|\ln(\beta)|} \right\rceil \ ▮ \tag{27}$$

We now try to simplify this expression by making a few reasonable presumptions. As the algorithm gets closer to stopping we have $\theta \to 0$ and $\kappa \to 0$. Furthermore, so long as $P$ and $D$ are not nearly-infeasible, $\tau$ will stay bounded away from 0, i.e., there exists $\tilde{\tau} > 0$ such that $\tau \geq \tilde{\tau}$ for all late iterates. Let us also presume that as the algorithm gets closer to stopping that $\bar{x} = (x/\tau)$ is sufficiently close to the set of primal $\varepsilon$-optimal solutions and $(\bar{y}, \bar{z}) = (y/\tau, z/\tau)$ is sufficiently close to the set of dual $\varepsilon$-optimal solutions, and that these level sets are not large (which will be the case if the primal and dual optima are unique or are nearly so), whereby

$$\|\bar{x}\|^{z^0} \approx R^P_\varepsilon \quad \text{and} \quad \|\bar{z}\|^{x^0} \approx R^D_\varepsilon \ . \tag{28}$$

These presumptions allow us to simplify (27) to:

$$T \approx \frac{\ln\left(R^P_\varepsilon + R^D_\varepsilon + \kappa^0\right) + \ln\left(\text{QUANT}\right) + \ln\left(\frac{\tau^0}{(x^0)^T z^0 + \tau^0\kappa^0}\right) + |\ln(r_{\max})|}{|\ln(\beta)|} \ . \tag{29}$$

Finally, let us presume that INITRESID $\approx$ QUANT. Notice from (22) and (23) that these two quantities differ only in their third term, and that the denominators of the third terms of each are nearly identical so long as $c^T\bar{x} \approx b^T\bar{y} \approx$ OPTVAL. Therefore INITRESID $\approx$ QUANT is valid to the extent that the difference between the numerators of the third terms of INITRESID and QUANT is dominated by the other numbers in their expressions. Notice that although the numerator of the third term of QUANT contains the fraction $-\kappa/\theta$ and both $\kappa$ and $\theta$ are typically close to zero for near-optimal solutions of $H$, the effect on the overall expression is muted somewhat since the numerator of the third term uses only the positive part of expression therein. In the Appendix we present some computational evidence that indicates that the presumption that INITRESID $\approx$ QUANT is probably reasonable. This final presumption allows (29) to be rewritten as:

$$T \approx \frac{\ln\left(R_\varepsilon^P + R_\varepsilon^D + \kappa^0\right) + \ln(\text{INITRESID}) + \ln\left(\frac{\tau^0}{(x^0)^T z^0 + \tau^0\kappa^0}\right) + |\ln(r_{\max})|}{|\ln(\beta)|} . \quad (30)$$

**Remark 4.1 Theoretical Algorithm with Constant Rate of Convergence.** *If the interior-point method for solving $H$ is implemented with a constant rate of convergence as would be the case for a theoretical polynomial-time algorithm, then $\beta$ is pre-specified independent of the problem instance; for example one can use $\beta = 1 - \frac{1}{8\sqrt{\vartheta}}$ where $\vartheta$ is the complexity parameter of the self-concordant barrier of the cone $K_H$, see [5]. In this case (30) simplifies to*

$$T \approx 8\sqrt{\vartheta}\left(\ln\left(R_\varepsilon^P + R_\varepsilon^D + \kappa^0\right) + \ln(\text{INITRESID}) + \ln\left(\frac{\tau^0}{(x^0)^T z^0 + \tau^0\kappa^0}\right) + |\ln(r_{\max})|\right) . \quad (31)$$

*Notice that the number of iterations is fairly precisely predicted by five quantities: (i) the complexity value $\vartheta$ of the self-concordant barrier for the cone $K_H$, (ii) the initial feasibility and optimality gap measure INITRESID, (iii) the size of the largest solutions measured in the norms induced by the starting point, (iv) the initial optimality gap measure of $H$ scaled by $\tau^0$, and (v) the pre-specified tolerance $r_{\max}$.*

**Remark 4.2 Factors Affecting the Average Convergence Rate.** *Not much is known or understood about the actual factors that influence the average convergence rate $\beta$. We computed $\beta$ for* 77 *problem instances in the SDPLIB suite solved via SeDuMi using the stopping rule inequality (21). We observed $\beta$ in the range* 0.12-0.66*, see Table 1. We also computed $\beta$ for a set of* 144 *second-order cone problem instances that were generated specifically to have have a wide range of condition measure values $C(d)$, in the range $10^2$-$10^9$, see [2] for details how these problems were generated. Here we observed $\beta$ in the range* 0.06-0.55*, see Table 2, with larger values roughly corresponding to problems with larger values of $R_\varepsilon^P + R_\varepsilon^D$ and with larger condition measure $C(d)$ (see [2] for details). This indicates that the average convergence rate may itself be partially influenced by at least one other quantity in (30).*

**Remark 4.3 Scale-Invariance.** *Note that the numerator of (30) is invariant under positive scaling of the starting values $(x^0, y^0, z^0, \tau^0, \kappa^0)$. To see this, suppose that these values are rescaled by some scalar $\alpha > 0$. Then $(R_\varepsilon^P + R_\varepsilon^D + \kappa^0) \leftarrow \alpha\left(R_\varepsilon^P + R_\varepsilon^D + \kappa^0\right)$ and $\frac{\tau^0}{(x^0)^T z^0 + \tau^0\kappa^0} \leftarrow \alpha^{-1}\frac{\tau^0}{(x^0)^T z^0 + \tau^0\kappa^0}$, and the other quantities in the numerator of (30) are unchanged.*

**Remark 4.4 Strategies for Reducing IPM Iterations.** *While the numerator of (30) is invariant under positive scalings, in general different choices of*

$(x^0, y^0, z^0, \tau^0, \kappa^0)$ *can lead to different values of* $\left(R_\varepsilon^P + R_\varepsilon^D + \kappa^0\right)$, *INITRESID, and* $\beta$, *suggesting the possibility of developing heuristics to choose* $(x^0, y^0, z^0, \tau^0, \kappa^0)$ *based, perhaps, on solutions to related versions of the problem that might yield smaller values of some these quantities. One can easily compute INITRESID and hence try to heuristically reduce its value. However,* $R_\varepsilon^P$, $R_\varepsilon^D$ *and* $\beta$ *are in general not known* a priori*, so it is not such a simple matter to develop heuristics to reduce their values. It is nevertheless an interesting line of research inquiry to try to develop ways to reduce these values either in theory or in practice.*

## 5 Conclusions and Open Questions

Theorem 3.1 shows that if one measures distance using using the primal/dual norms (12) induced by the starting point of the HSD embedding, then the behavioral measures $R_\varepsilon^H$ and $r_\varepsilon^H$ are precisely controlled independent of any particular characteristics of the problem instance $P/D$, indicating that $H$ is inherently well-behaved in these measures in this norm.

Furthermore, the primal norm of (12) is implicitly involved in the standard stopping criterion for an IPM for solving $P/D$ via the HSD embedding model: under mild assumptions, the stopping rule implicitly involves the sum of the norms of the $\varepsilon$-optimal primal and dual solutions (where these norms are also defined by the starting points $x^0$ and $z^0$), as well as the size of the initial primal and dual infeasibility residuals. This theory suggests possible criteria for developing starting points for the homogeneous self-dual model that might improve the resulting solution time in practice.

The analysis of the stopping criterion herein is valid for the case when $P$ and $D$ both have solutions. It would be interesting to extend this line of analysis to the case where $P$ and/or $D$ are infeasible, to answer the question: what are the relevant behavioral measures and possibly associated norms that capture the stopping criterion for an instance of $P/D$ in which one or both problems are infeasible?

**Acknowledgement.** I am grateful to Kim Chuan Toh and Cai Zhi for stimulating discussions and computational testing of the homogeneous self-dual model, and to Kim Chuan Toh for reading an early draft of this paper.

## Appendix A: On the Presumption that INITRESID $\approx$ QUANT

We tested the presumption that INITRESID $\approx$ QUANT on two data sets of conic problems: (i) the SDPLIB test set of semidefinite programming problems archived at `http://www.nmt.edu/~sdplib/`, and (ii) a set of 144 second-order-cone problems generated to have a wide range of condition measures $C(d)$, see [2]. We used SeDuMi to solve these problems, modified to use the amended stopping rule (21). Table 1 shows the values of the ratio INITRESID/QUANT as well as the average decrease in the duality gap $\beta$ (26) for 77 problems in the SDPLIB test set (we did not compute these values for the following 15 problems due to their size and/or infeasibility: equalG51, infd1 (dual infeasible), infd2 (dual infeasible), infp1 (primal infeasible), infp2, (primal infeasible) maxG32, maxG51, maxG55, maxG60, qpG11, qpG51, theta5, theta6, thetaG11, thetaG51). The computational results show that INITRESID/QUANT is consistently close to 1.0 (to one decimal place) for these problems, except for the problem qap5, for which the ratio is 0.9. We also solved a set of 144 second-order cone problems that were generated specifically to have have a wide range of condition measure values $C(d)$, in the range $10^2 - 10^9$, see [2] for details how these problems were generated. Table 2 shows the values of the ratio INITRESID/QUANT as well as the average decrease in the duality gap $\beta$ (26) for these 144 second-order cone problems. Here we observed INITRESID/QUANT in the range $0.9 - 3.9$. These results indicate that the presumption that INITRESID $\approx$ QUANT is quite reasonable.

Table 1: The ratio INITRESID/QUANT and $\beta$ for 77 Problems in the SDPLIB Test Set.

| Problem | INITRESID/QUANT | $\beta$ | Problem | INITRESID/QUANT | $\beta$ |
|---|---|---|---|---|---|
| arch0 | 1.0 | 0.30 | hinf11 | 1.0 | 0.35 |
| arch2 | 1.0 | 0.28 | hinf12 | 1.0 | 0.29 |
| arch4 | 1.0 | 0.30 | hinf13 | 1.0 | 0.29 |
| arch8 | 1.0 | 0.28 | hinf14 | 1.0 | 0.39 |
| control1 | 1.0 | 0.21 | hinf15 | 1.0 | 0.33 |
| control2 | 1.0 | 0.25 | maxG11 | 1.0 | 0.22 |
| control3 | 1.0 | 0.28 | mcp100 | 1.0 | 0.20 |
| control4 | 1.0 | 0.30 | mcp124-1 | 1.0 | 0.22 |
| control5 | 1.0 | 0.31 | mcp124-2 | 1.0 | 0.20 |
| control6 | 1.0 | 0.34 | mcp124-3 | 1.0 | 0.23 |
| control7 | 1.0 | 0.34 | mcp124-4 | 1.0 | 0.23 |
| control8 | 1.0 | 0.33 | mcp250-1 | 1.0 | 0.25 |
| control9 | 1.0 | 0.31 | mcp250-2 | 1.0 | 0.24 |
| control10 | 1.0 | 0.36 | mcp250-3 | 1.0 | 0.24 |
| control11 | 1.0 | 0.39 | mcp250-4 | 1.0 | 0.22 |
| equalG11 | 1.0 | 0.25 | mcp500-1 | 1.0 | 0.27 |
| gpp100 | 1.0 | 0.66 | mcp500-2 | 1.0 | 0.25 |
| gpp124-1 | 1.0 | 0.63 | mcp500-3 | 1.0 | 0.25 |
| gpp124-2 | 1.0 | 0.64 | mcp500-4 | 1.0 | 0.23 |
| gpp124-3 | 1.0 | 0.64 | qap5 | 0.9 | 0.12 |
| gpp124-4 | 1.0 | 0.65 | qap6 | 1.0 | 0.33 |
| gpp250-1 | 1.0 | 0.66 | qap7 | 1.0 | 0.34 |
| gpp250-2 | 1.0 | 0.56 | qap8 | 1.0 | 0.36 |
| gpp250-3 | 1.0 | 0.57 | qap9 | 1.0 | 0.35 |
| gpp250-4 | 1.0 | 0.56 | qap10 | 1.0 | 0.34 |
| gpp500-1 | 1.0 | 0.60 | ss30 | 1.0 | 0.35 |
| gpp500-2 | 1.0 | 0.57 | theta1 | 1.0 | 0.15 |
| gpp500-3 | 1.0 | 0.59 | theta2 | 1.0 | 0.15 |
| gpp500-4 | 1.0 | 0.54 | theta3 | 1.0 | 0.16 |
| hinf1 | 1.0 | 0.30 | theta4 | 1.0 | 0.17 |
| hinf2 | 1.0 | 0.35 | truss1 | 1.0 | 0.14 |
| hinf3 | 1.0 | 0.38 | truss2 | 1.0 | 0.23 |
| hinf4 | 1.0 | 0.35 | truss3 | 1.0 | 0.21 |

| Problem | INITRESID/QUANT | $\beta$ | Problem | INITRESID/QUANT | $\beta$ |
|---|---|---|---|---|---|
| hinf5 | 1.0 | 0.35 | truss4 | 1.0 | 0.17 |
| hinf6 | 1.0 | 0.33 | truss5 | 1.0 | 0.27 |
| hinf7 | 1.0 | 0.25 | truss6 | 1.0 | 0.35 |
| hinf8 | 1.0 | 0.32 | truss7 | 1.0 | 0.31 |
| hinf9 | 1.0 | 0.23 | truss8 | 1.0 | 0.32 |
| hinf10 | 1.0 | 0.33 | | | |

Table 2: The ratio INITRESID/QUANT and $\beta$ for 144 Second-Order Cone Problems.

| Problem | INITRESID/QUANT | $\beta$ | Problem | INITRESID/QUANT | $\beta$ |
|---|---|---|---|---|---|
| sm_18 | 1.0 | 0.22 | md_2 | 1.0 | 0.23 |
| sm_18_1 | 1.0 | 0.21 | md_2_1 | 1.0 | 0.22 |
| sm_18_5 | 1.0 | 0.22 | md_2_5 | 1.0 | 0.23 |
| sm_18_75 | 1.0 | 0.23 | md_2_75 | 1.0 | 0.27 |
| sm_18_9 | 1.0 | 0.24 | md_2_9 | 1.0 | 0.28 |
| sm_18_95 | 1.0 | 0.27 | md_2_95 | 1.0 | 0.26 |
| sm_18_97 | 1.0 | 0.28 | md_2_97 | 1.0 | 0.26 |
| sm_18_99 | 1.0 | 0.29 | md_2_99 | 1.0 | 0.28 |
| sm_18_995 | 1.0 | 0.34 | md_2_995 | 1.0 | 0.30 |
| sm_18_999 | 1.0 | 0.37 | md_2_999 | 1.0 | 0.39 |
| sm_18_9995 | 1.0 | 0.37 | md_2_9995 | 1.0 | 0.41 |
| sm_18_9999 | 1.0 | 0.39 | md_2_9999 | 1.0 | 0.55 |
| sm_19 | 1.0 | 0.18 | sm2_1 | 1.0 | 0.28 |
| sm_19_1 | 1.0 | 0.18 | sm2_1_1 | 1.0 | 0.25 |
| sm_19_5 | 1.0 | 0.21 | sm2_1_5 | 1.0 | 0.36 |
| sm_19_75 | 1.0 | 0.21 | sm2_1_75 | 1.0 | 0.27 |
| sm_19_9 | 1.0 | 0.25 | sm2_1_9 | 1.0 | 0.43 |
| sm_19_95 | 1.0 | 0.24 | sm2_1_95 | 1.0 | 0.44 |
| sm_19_97 | 1.0 | 0.27 | sm2_1_97 | 1.0 | 0.43 |
| sm_19_99 | 1.0 | 0.27 | sm2_1_99 | 1.0 | 0.45 |
| sm_19_995 | 1.0 | 0.28 | sm2_1_995 | 1.0 | 0.46 |
| sm_19_999 | 1.0 | 0.29 | sm2_1_999 | 1.0 | 0.49 |
| sm_19_9995 | 1.0 | 0.30 | sm2_1_9995 | 1.0 | 0.50 |
| sm_19_9999 | 1.0 | 0.31 | sm2_1_9999 | 1.0 | 0.49 |
| sm2_3 | 0.9 | 0.06 | md_3 | 0.9 | 0.23 |
| sm2_3_1 | 0.9 | 0.06 | md_3_1 | 0.9 | 0.22 |
| sm2_3_5 | 0.9 | 0.10 | md_3_5 | 0.9 | 0.22 |
| sm2_3_75 | 0.9 | 0.14 | md_3_75 | 0.9 | 0.24 |
| sm2_3_9 | 0.9 | 0.19 | md_3_9 | 0.9 | 0.25 |
| sm2_3_95 | 0.9 | 0.19 | md_3_95 | 0.9 | 0.25 |
| sm2_3_97 | 0.9 | 0.24 | md_3_97 | 0.9 | 0.24 |
| sm2_3_99 | 1.0 | 0.22 | md_3_99 | 0.9 | 0.27 |
| sm2_3_995 | 1.0 | 0.21 | md_3_995 | 0.9 | 0.28 |
| sm2_3_999 | 1.3 | 0.26 | md_3_999 | 1.0 | 0.31 |
| sm2_3_9995 | 1.5 | 0.31 | md_3_9995 | 1.0 | 0.34 |
| sm2_3_9999 | 3.9 | 0.35 | md_3_9999 | 1.0 | 0.36 |
| sm_5 | 1.0 | 0.22 | md_5 | 0.9 | 0.23 |
| sm_5_1 | 1.0 | 0.21 | md_5_1 | 0.9 | 0.22 |
| sm_5_5 | 1.0 | 0.19 | md_5_5 | 0.9 | 0.22 |
| sm_5_75 | 1.0 | 0.23 | md_5_75 | 0.9 | 0.24 |
| sm_5_9 | 1.0 | 0.26 | md_5_9 | 0.9 | 0.25 |

| Problem | INITRESID/QUANT | $\beta$ | Problem | INITRESID/QUANT | $\beta$ |
|---|---|---|---|---|---|
| sm_5_95 | 1.0 | 0.26 | md_5_95 | 0.9 | 0.25 |
| sm_5_97 | 1.0 | 0.27 | md_5_97 | 0.9 | 0.25 |
| sm_5_99 | 1.0 | 0.29 | md_5_99 | 0.9 | 0.28 |
| sm_5_995 | 1.0 | 0.30 | md_5_995 | 0.9 | 0.28 |
| sm_5_999 | 1.0 | 0.34 | md_5_999 | 0.9 | 0.31 |
| sm_5_9995 | 1.0 | 0.35 | md_5_9995 | 0.9 | 0.33 |
| sm_5_9999 | 1.0 | 0.38 | md_5_9999 | 1.0 | 0.37 |
| md_1 | 1.0 | 0.22 | md_4 | 1.0 | 0.27 |
| md_1_1 | 1.0 | 0.24 | md_4_1 | 1.0 | 0.26 |
| md_1_5 | 1.0 | 0.26 | md_4_5 | 1.0 | 0.27 |
| md_1_75 | 1.0 | 0.24 | md_4_75 | 1.0 | 0.28 |
| md_1_9 | 1.0 | 0.29 | md_4_9 | 1.0 | 0.25 |
| md_1_95 | 1.0 | 0.28 | md_4_95 | 1.0 | 0.29 |
| md_1_97 | 1.0 | 0.33 | md_4_97 | 1.0 | 0.30 |
| md_1_99 | 1.0 | 0.32 | md_4_99 | 1.0 | 0.35 |
| md_1_995 | 1.0 | 0.32 | md_4_995 | 1.0 | 0.37 |
| md_1_999 | 1.0 | 0.35 | md_4_999 | 1.0 | 0.42 |
| md_1_9995 | 1.0 | 0.36 | md_4_9995 | 1.0 | 0.46 |
| md_1_9999 | 1.0 | 0.40 | md_4_9999 | 1.0 | 0.46 |
| lg_1 | 0.9 | 0.14 | md_6 | 0.9 | 0.20 |
| lg_1_1 | 0.9 | 0.14 | md_6_1 | 0.9 | 0.20 |
| lg_1_5 | 0.9 | 0.16 | md_6_5 | 0.9 | 0.19 |
| lg_1_75 | 0.9 | 0.16 | md_6_75 | 0.9 | 0.19 |
| lg_1_9 | 1.0 | 0.17 | md_6_9 | 0.9 | 0.21 |
| lg_1_95 | 1.0 | 0.17 | md_6_95 | 0.9 | 0.21 |
| lg_1_97 | 0.9 | 0.19 | md_6_97 | 0.9 | 0.22 |
| lg_1_99 | 1.0 | 0.17 | md_6_99 | 0.9 | 0.24 |
| lg_1_995 | 1.0 | 0.19 | md_6_995 | 0.9 | 0.24 |
| lg_1_999 | 1.5 | 0.23 | md_6_999 | 0.9 | 0.29 |
| lg_1_9995 | 1.6 | 0.25 | md_6_9995 | 0.9 | 0.29 |
| lg_1_9999 | 1.7 | 0.26 | md_6_9999 | 0.9 | 0.32 |

## Appendix B: On Norms that are Linear on $S^n_+$ and $Q^n$

**The Positive Semi-definite Cone.** We first prove that for $w^0 \in \mathrm{int}S^n_+$ the norm (7) has the form $\|v\|^{w^0} = \left\|\lambda\left((w^0)^{\frac{1}{2}}v(w^0)^{\frac{1}{2}}\right)\right\|_1$. To show this, we convert to the more standard matrix and trace notation used for semidefinite optimization, see [1] for example. To avoid confusion with roots of semidefinite matrices, let us instead use $\bar{W}$ for the given positive definite matrix in $\mathrm{int}S^n_+$ and write (7) and its conic dual as:

$$\begin{array}{rlllll} \|V\|^{\bar{W}} = & \min_{V^1,V^2} & \bar{W}\bullet(V^1+V^2) & \quad (\mathrm{Dual}): & \max_X & V\bullet X \\ & \text{s.t.} & V^1-V^2=V & & \text{s.t.} & \bar{W}+X\in S^n_+ \\ & & V^1,V^2\in S^n_+ & & & \bar{W}-X\in S^n_+ \ . \end{array}$$

For $V \in S^n$, consider the eigendecomposition of $\bar{W}^{\frac{1}{2}}V\bar{W}^{\frac{1}{2}} = P(D-E)P^T$ where $P$ is orthonormal, and $D, E$ are nonnegative diagonal matrices corresponding to the nonnegative and nonpositive eigenvalues of $\bar{W}^{\frac{1}{2}}V\bar{W}^{\frac{1}{2}}$, respectively. Let $S$ denote the diagonal matrix whose diagonal is composed of the sign of the diagonal of $D-E$, and let $V^1 = \bar{W}^{-\frac{1}{2}}PDP^T\bar{W}^{-\frac{1}{2}}$, $V^2 = \bar{W}^{-\frac{1}{2}}PEP^T\bar{W}^{-\frac{1}{2}}$, and $X = \bar{W}^{\frac{1}{2}}PSP^T\bar{W}^{\frac{1}{2}}$. Then it is relatively easy to check that $V^1, V^2, X$ are primal and dual feasible in the above conic programs with common objective function value $I \bullet (D+E) = \left\|\lambda\left(\bar{W}\right)^{\frac{1}{2}}V\bar{W}^{\frac{1}{2}}\right)\right\|_1$, proving the result.

**The Second-Order Cone.** We now prove that for $w^0 \in \text{int}Q^n$ the norm (7) has the closed-form $\|v\|^{w^0} = \max\{|(Mv)_1|, \|(Mv)_2, \ldots, (Mv)_n\|\}$ where $v = (v_1, \bar{v})$ and $M$ is given by (9). Suppose first that $w^0 = e^1 := (1, 0, \ldots, 0)$; in this case $M = I$ and we need to show that $\|v\|^{e^1} = \max\{|v_1|, \|\bar{v}\|\}$. To show this, write (7) and its conic dual as:

$$\begin{array}{rlllll} \|v\|^{w^0} = & \min_{v^1,v^2} & (e^1)^T(v^1+v^2) & \quad (\text{Dual}): & \max_x & v^Tx \\ & \text{s.t.} & v^1 - v^2 = v & & \text{s.t.} & e^1 + x \in Q^n \\ & & v^1, v^2 \in Q^n & & & e^1 - x \in Q^n \ , \end{array}$$

and consider three cases:

**Case 1:** $v_1 \geq \|\bar{v}\|$**.** Here $v^1 = v, v^2 = 0, x = e^1$ are primal and dual feasible in the above conic programs with common objective value $v_1^1 = \max\{|v_1|, \|\bar{v}\|\}$, proving the result in this case.

**Case 2:** $-v_1 \geq \|\bar{v}\|$**.** Here $v^1 = 0, v^2 = -v, x = -e^1$ are primal and dual feasible in the above conic programs with common objective value $-v_1^1 = \max\{|v_1|, \|\bar{v}\|\}$, proving the result in this case.

**Case 3:** $-\|\bar{v}\| < v_1 < \|\bar{v}\|$**.** Let $\beta := (v_1 + \|\bar{v}\|)/(2\|\bar{v}\|)$. Then $\beta \in (0,1)$ and define $v^1 = (\beta\|\bar{v}\|, \beta\bar{v}), v^2 = ((1-\beta)\|\bar{v}\|, (\beta-1)\bar{v}, ), x = (0, \bar{v}/\|\bar{v}\|)$. Then $v^1, v^2, x$ are primal and dual feasible in the above conic programs with common objective function value $\|\bar{v}\| = \max\{|v_1|, \|\bar{v}\|\}$, proving the result in this case.

Now consider an arbitrary given $w^0 \in \text{int}Q^n$. Let $v = (v_1, \bar{v}) \in Q^n$, and consider the self-scaled (see [6]) barrier function

$$f(v) := -\ln(v_1^2 - \bar{v}^T\bar{v})$$

for $v \in \text{int}Q^n$. The Hessian of $f(\cdot)$ is given by:

$$H(v) = \frac{1}{(v_1^2 - \bar{v}^T\bar{v})^2}\begin{pmatrix} 2v_1^2 + 2\bar{v}^T\bar{v} & -4v_1\bar{v}^T \\ -4v_1\bar{v} & 2(v_1^2 - \bar{v}^T\bar{v})I + 4\bar{v}\bar{v}^T \end{pmatrix} \ ,$$

and it follows from the definition of a self-scaled barrier [6] that $H(v)$ maps $Q^n$ onto $Q^n$ for $v \in \text{int}Q^n$.

Now let $w^0 = (w_1^0, \bar{w}) \in \text{int}Q^n$ be given, and define $M$ as in (9). Then it is laborious but straightforward to check that $M = H(\tilde{v})$ for

$$\tilde{v} = \left(\frac{\sqrt{w_1^0 + \tilde{\gamma}}}{\tilde{\gamma}}, \frac{-\bar{w}}{\tilde{\gamma}\sqrt{w_1^0 + \tilde{\gamma}}}\right) \ ,$$

where $\tilde{\gamma} = \sqrt{(w_1^0)^2 - \bar{w}^T\bar{w}}$, and so $M$ maps $Q^n$ onto $Q^n$, i.e., $Mv \in Q^n$ if and only if $v \in Q^n$. Then notice that:

$$\begin{array}{rccccc} \|v\|^{w^0} = & \min_{v^1,v^2} & (w^0)^T(v^1+v^2) & = & \min_{v^1,v^2} & (e^1)^T M(v^1+v^2) \\ & \text{s.t.} & v^1 - v^2 = v & & \text{s.t.} & Mv^1 - Mv^2 = Mv \\ & & v^1, v^2 \in Q^n & & & Mv^1, Mv^2 \in Q^n \ , \end{array}$$

since $Me^1 = w^0$, $M$ is invertible, and $Mv \in Q^n$ if and only if $v \in Q^n$. But substituting $y^1 = Mv^1, y^2 = Mv^2$ the rightmost program above can be rewritten as:

$$\begin{array}{rcc} \|v\|^{w^0} = & \min_{y^1,y^2} & (e^1)^T(y^1+y^2) \\ & \text{s.t.} & y^1 - y^2 = Mv \\ & & y^1, y^2 \in Q^n \ , \end{array}$$

which we have already seen is just $\max\{|(Mv)_1|, \|(Mv)_2, \ldots, (Mv)_n\|\}$.

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