# 15.093J/2.098J Optimization Methods
## Assignment 1 Solutions

**Exercise 1.1** BT, Exercise 1.10.
We use the following decision variables:

$$\begin{aligned} x_i &= \text{units produced in month } i, \\ s_i &= \text{inventory carried from month } i \text{ to month } i+1, \\ y_i &= |x_i - x_{i+1}|. \end{aligned}$$

$$\begin{array}{lll} \text{minimize} & c_1 \sum_{i=1}^{11} s_i + c_2 \sum_{i=1}^{11} y_i & \\ \text{subject to} & s_1 = x_1 - d_1 & \\ & s_i = s_{i-1} + x_i - d_i & i = 2, \ldots, 11 \\ & y_i \geq x_i - x_{i+1} & i = 1, \ldots, 11 \\ & y_i \geq x_{i+1} - x_i & i = 1, \ldots, 11 \\ & x_i, y_i, s_i \geq 0. & \end{array}$$

**Exercise 1.2** BT, Exercise 1.14.

**(a)** Decision variables:
$x$ is the number of units of product 1,
$y$ is the number of units of product 2.
Formulation:

$$\text{maximize } (6-3)x + (5.40-2)y$$

subject to the constraints

$$\begin{aligned} 3x + 4y &\leq 20{,}000 \\ 3x + 2y &\leq 4{,}000 + 0.45 \times 6x + 0.30 \times 5.4y \\ x, y &\geq 0. \end{aligned}$$

Equivalently,

$$\begin{array}{ll} \text{maximize} & 3x + 3.4y \\ \text{subject to} & 3x + 4y \leq 20{,}000 \\ & 0.3x + 0.38y \leq 4{,}000 \\ & x, y \geq 0. \end{array}$$

**(b)** Optimal solution is $x = 6{,}666.67$ and $y = 0$. The optimal integer solution is then $x = 6{,}666$ and $y = 0$.

**(c)** If the right-hand side of the constraint $3x + 4y \leq 20{,}000$ changes to $22{,}000$, then the optimal solution becomes $x = 7{,}333.33$ and $y = 0$ and the optimal integer solution becomes $x = 7{,}333$ and $y = 0$. Since the profit will increase by \$2,001, which is more than \$400, the investment should be made.

**Exercise 1.3** BT, Exercise 1.16.
Decision variables:
$x_j$ is the number of processes (in millions) of type $j$ used, $j = 1, 2, 3$.
Formulation:

$$\begin{array}{ll} \text{maximize} & 38(4x_1 + x_2 + 3x_3) + 33(3x_1 + x_2 + 4x_3) - (51x_1 + 11x_2 + 40x_3) \\ \text{subject to} & 3x_1 + x_2 + 5x_3 \leq 8 \\ & 5x_1 + x_2 + 3x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

The optimal solution is $x_1 = 0$, $x_2 = 0.5$ million and $x_3 = 1.5$ million.

**Exercise 1.4** BT, Exercise 1.17.
Decision variables:

$x_i$ is the number of shares of stock $i$ sold.
Formulation:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} r_i(s_i - x_i) \\ \text{subject to} & \sum_{i=1}^{n} q_i x_i - 0.3 \sum_{i=1}^{n} (q_i - p_i) x_i - 0.01 \sum_{i=1}^{n} q_i x_i \geq K \\ & 0 \leq x_i \leq s_i, \qquad \forall i. \end{array}$$

The optimal solutions for the given data are:

For $K = 100,000$:

Objective value $= 365,480$
$x_1 = 0, x_2 = 1061.864, x_3 = 800, x_4 = 0, x_5 = 0$
Integrality constraint: $x_2 = 1062$

For $K = 140,000$:

Objective value $= 324,400$
$x_1 = 0, x_2 = 1693.776, x_3 = 800, x_4 = 0, x_5 = 0$
Integrality constraint: $x_2 = 1694$

**Exercise 1.5**

**(1)** Yes. $(0, 1, 3)$ is a BFS. The number of nonzero elements is less than or equal to 4, or the number of tight constraints is greater than or equal to 3.

**(2)** Yes. $(0, 1, 3)$ is degenerate. The number of nonzero elements is less than 4, or it has more than 3 tight constraints. The possible corresponding basis are: $[A_2 A_3 A_5 A_i]$ where $i \in \{1, 4, 6, 7\}$, since they are all linearly independent with with the previous columns.

**(3)** Let $\bar{c}^1$ and $\bar{c}^2$ be the reduced cost associated to $B_1$ and $B_2$, respectively. $\bar{c}^1 = c - c_{B_1} B_1^{-1} A = (0, 0, 0, -3, 0, 8/3, 4/3)$, thus $B_1$ does not satisfy the optimality condition because $\bar{c}^1_3 < 0$. However, $\bar{c}^2 = c - c_{B_2} B_2^{-1} A = (3, 0, 0, 0, 0, 2/3, 1/3)$, thus $B_2$ satisfies the optimality condition because $\bar{c}^2_i \geq 0, \quad \forall i$.

**(4)** The simplex tableau for $B_1$ looks as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 0 | 0 | 0 | -3 | 0 | 8/3 | 4/3 |
| 0 | 1 | 0 | 0 | 1* | 0 | -2/3 | -1/3 |
| 1 | 0 | 1 | 0 | 0 | 0 | -1/3 | 1/3 |
| 3 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 0 | 0 | -1 | 1 | 2/3 | 1/3 |

$x_4$ will enter the basis and $x_1$ leaves the basis, pivoting on element (1,4) of the tableau. The basis will change, but the solution remains the same at $(0, 1, 3)$. $B_2$ is the new basis.

**(5)** Since $(1, 0, 3)$ is nondegenerate, we can find a unique basis corresponding to it, namely, $B_3 = [A_1 A_3 A_5 A_7]$. The corresponding simplex tableau is the following:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | -4 | 0 | -3 | 0 | 4 | 0 |
| 1 | 1 | 1* | 0 | 1 | 0 | -1 | 0 |
| 3 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | -1 | 0 | -1 | 1 | 1 | 0 |
| 3 | 0 | 3 | 0 | 0 | 0 | -1 | 1 |

$x_2$ enters the basis and $x_1$ leaves the basis, pivoting on (1,2). The new BFS is (0,1,3). The method moved from (1,0,3) to (0,1,3).

**(6)** Since $(0, 0, 3)$ is nondegenerate, we can find a unique basis corresponding to it, namely, $B_4 = [A_3 A_4 A_5 A_7]$. The corresponding simplex tableau is the following:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | -2 | -2 | 0 | 0 | 0 | 2 | 0 |
| 3 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 1 | 1* | 1 | 0 | 1 | 0 | -1 | 0 |
| 2 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 3 | 0 | 3 | 0 | 0 | 0 | -1 | 1 |

$x_1$ enters and x4 leaves the basis, pivoting on (2,1). The basis becomes $B_3$, as in part (5). We'll move from (0,0,3) to (1,0,3) in Figure 1. The reduced cost corresponding to $B_3$ is $\bar{c}^3 = c - c_{B_3} B_3^{-1} A = (0,0,0,2,0,0,0)$, thus the optimality conditioned is satisfied with this basis. There are multiple optima to this new cost vector. $x_2$ can enter the basis where $\bar{c}_2^3 = 0$, thus it won't change the objective value and the solution is not degenerate.

**Exercise 1.6** BT, Exercise 2.10.

**(a)** True. The set $P$ lies in an affine subspace defined by $m = n - 1$ linearly independent constraints, that is, of dimension one. Hence, every solution of $\mathbf{Ax} = \mathbf{b}$ is of the form $\mathbf{x}^0 + \lambda \mathbf{x}^1$, where $\mathbf{x}^0$ is an element of $P$ and $\mathbf{x}^1$ is a nonzero vector. Thus, $P$ is contained in a line and cannot have more than two extreme points. (If it had three, the one "in the middle" would be a convex combination of the other two, hence not an extreme point.)

**(b)** False. Consider minimizing $x_1$ subject to $x_1 = 1$, $(x_1, x_2) \geq (0,0)$. The optimal solution set is unbounded.

**(c)** False. Consider a standard form problem with $\mathbf{c} = \mathbf{0}$. Then, any feasible $\mathbf{x}$ is optimal, no matter how many positive components it has.

**(d)** True. If $\mathbf{x}$ and $\mathbf{y}$ are optimal, so is any convex combination of them.

**(e)** False. Consider the problem of minimizing $x_2$ subject to $(x_1, x_2) \geq (0,0)$ and $x_2 = 0$. Then the set of all optimal solutions is the set $\{(x_1, 0) \mid x_1 \geq 0\}$. There are several optimal solutions, but only one optimal basic feasible solution.

**(f)** False. Consider the problem of minimizing $|x_1 - 0.5| = \max\{x_1 - 0.5, -x_1 + 0.5\}$ subject to $x_1 + x_2 = 1$ and $(x_1, x_2) \geq (0,0)$. Its unique optimal solution is $(x_1, x_2) = (0.5, 0.5)$, which is not an extreme point of the feasible set.