15.093J/2.098J Optimization Methods
Assignment 4 Solutions

Exercise 4.1 BT, Exercise 7.1.
Construct the network as follows.
For each day i, create two nodes: node d_i for dirty tablecloths and c_i for clean tablecloths. There is a supply of r_i for node d_i and similarly, there is a demand of r_i for node c_i.
The node c_i can receive new tablecolths from purchasing; create a node n for this new tablecoth source.
For each node c_i, there is an incoming arc from node n with arc cost of p (and unlimited capacity). There is also an incoming arc from node d_{i-n-j} with arc cost of f if $i > n + j$ and an incoming arc from node d_{i-m-j} with arc cost of g if $i > m + j$ (launder tablecloths can be distributed for many days, not one).
Not all the laundered tablecloths will be reused , especially at the very end of the period. These tablecloth are wasted. We could create a waste node for this purpose; however, the network flow needs to be balanced and therefore, we could put all wasted tablecolths back to the new tablecloth source. All these arcs will have zero cost and again, unlimited capacity. The network is then balanced with the equal supply and demand of $\sum_{i=1}^{N} r_i$.
Exercise 4.2 BT, Exercise 7.2.
Construct the network as follows.
For each wood unit i, create a node w_i. For each year j, create two nodes: node h_j to collect all harvested wood and node y_j for wood demand.
For each node w_i, we have the supply of $\sum_{j=1}^{K} a_{ij}$ due to the assumption of possible late harvesting. There is an outgoing arc from this node to each node h_j with zero cost and capacity of a_{ij}.
For each node h_j, there is an outgoing arc to the node y_j with zero cost and the capacity of u_j.
For each node y_j, there is a demand of d_j. We also have an outgoing arc to the node y_{j+1} with arc cost of c_j and unlimited capacity if $j < K$.
Due to the harvesting limits, we might need a node that collects all remained supplies so that the network is balanced.
Exercise 4.3 BT, Exercise 7.3.
Construct the network as follows.
For each team i, create a node t_i. For each unordered pair (i, j), $i \neq j$, create a node $g_{(i,j)}$.
For each node t_i, we have a supply of x_i number of wins. There is an outgoing arc to each node $g_{(i,j)}$ with zero cost and capacity of k.
For each node $g_{(i,j)}$, we have a demand of k.
If there is a feasible flow solution for the network problem above, the given set of numbers of wins is a possible outcome.
Exercise 4.4 BT, Exercise 7.5.
Given a directed graph (V, E) with a set of supply nodes S_+ and set of demand nodes S_-, we construct a new directed graph as follows.
For a pair of nodes (i, j), $i \in S_+$, $j \in S_-$, construct all directed paths that connect i to j.
For each arc $e \in E$, considered all paths that used e, assume there are k of them. We create k arcs with the same arc cost, one for each path so that after considering all arcs in E, we will have a network with all node and arc disjoint paths that connect every pair of nodes (i, j), $i \in S_+$, $j \in S_-$. It is possible that there is no direct path between a specific pair of nodes.
Given this new network, the optimal solution of network flow problem will use only shortest paths between each pair of nodes (i, j), $i \in S_+$, $j \in S_-$. If there is a longer path in the solution, we can shift all the flow from this path to the corresponding shortest path and the total cost is reduced (contradiction). We can do this operation because all arcs have umlimited capacity and all arc costs are nonnegative (which means all path costs are also nonnegative). Thus we can see that the optimal cost of the network flow problem in this new network is equal to the optimal cost of the transportation problem constructed using the shortest direct paths.
We now prove that the new networks has the same optimal cost as the original one $Z_P = Z_A$. Consider the optimal solution of the network flow problem in the new network. We can construct a feasible solution for the network problem in the original network as follows.
For each arc in the original problem, we assign a flow that is equal to the total sum of all flows on the associated arcs in the new network. Clearly, this is a feasible solution; thus $Z_P \geq Z_A$.
Now consider an optimal basic feasible solution of the network flow problem in the original network. This is a

tree solution and we always have a unique path between every pair of nodes in the network. Construct disjoint direct paths for each pairs of node (i, j), $i \in S_+$, $j \in S_-$ if there is one using arcs in the solution. We can see that a feasible path solution for the new network can be calculated, thus $Z_A \geq Z_P$.
So we have $Z_A = Z_P$. From these two results, we can conclude that the transportation problem constructed in the problem has the same optimal cost as the original network flow problem.
Exercise 4.5 BT, Exercise 7.9.

(a) The matrix A will depend on how you index the arcs; however, no matter how the arcs are ordered, each column of A must have one entry 1, one entry -1 and other entries 0.

(b) You should be able to calculate the solution given a tree by starting from its leaves. The reduced costs of nonbasic variables are calculated using the cycle created by the corresponding arc and the current tree arcs. The new solution is created by pushing maximum possible flow around the cycle mentioned above.
It turns out that after two iterations, we get the optimal solution by letting arc a_{85} leave and arc a_{87} enter the tree; arc a_{25} leave and arc a_{23} enter. The solution is $f_{32} = 2$, $f_{21} = 1$, $f_{54} = 2$, $f_{74} = 5$, and $f_{41} = 1$; other arcs have zero flows.