2.098/15.093J Final Review

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Overview

- Pre-Midterm
- Network Optimization
- Integer Optimization
- Nonlinear Optimization
- Interior Point Methods
- Semidefinite Optimization

Pre-Midterm Topics

- Linear Programming
- Simplex Method
- Duality Theory
- Sensitivity Analysis

Network Optimization I - Formulations

- Min-cost flow formulation

$$\begin{array}{ll} \text{min} & \sum_{(i,j)\in A} c_{ij} f_{ij} \\ \text{s.t.} & b_i + \sum_{j\in I(i)} f_{ij} = \sum_{i\in O(i)} f_{ij} \\ & 0 \le f_{ij} \le u_{ij} \end{array}$$

- General feasible condition $\sum_{i=1}^{|N|} b_i = 0$

- Matrix form

$$\begin{array}{ll} \text{min} & c' f \\ \text{s.t.} & Af = b \\ & 0 \le f \le u \end{array}$$

- Special problems: shortest path, maximum flow, transportation, and assignment problems.

Network Optimization II - Network Simplex

- Basis solution
  - A spanning tree solution
  - Calculated from leaves upwards
- Reduced costs
  - $\bar{c}_{ij} = c_{ij} - (p_i - p_j)$, where p_i is node potentials
  - Node potentials are calculated from the root node using the formula $c_{ij} - (p_i - p_j) = 0$
- Simplex direction
  - Involve a cycle created by an entering arc with arcs in the current tree solution
  - θ^* is calculated by pushing as many flows as possible along the cycle in the entering arc's orientation while keeping all other flows nonnegative

Integer Optimization I - Formulation

- General integer optimization problem

$$\begin{array}{cc} \text{min} & c'x \\ \text{s.t} & Ax = b \\ & x \in \mathbf{Z}_+^n \end{array}$$

- LP relaxation

$$\begin{array}{cc} \text{min} & c'x \\ \text{s.t} & Ax = b \\ & x \geq 0 \end{array}$$

- Quality of an integer optimization formulation:
  - Closeness of the linear relaxation polyhedron to the convex hull of the set of all feasible solutions

Integer Optimization II - Algorithms

- Cutting plane method
  - Add a linear inequality constraint satisfied by all integer solutions but not current LP solution x^*
  - Gomory cut: apply rounding for $x_i + (B^{-1}A_j)_i x_N = (B^{-1}b)_i$
- Branch and bound
  - Branching: splitting the current subproblem into further subproblems
  - Bounding: calculating the lower bound for the optimal cost of a subproblem $b(F_i) \leq \min_{x \in F_i} c'x$
  - Pruning: deleting subproblem that is infeasible or $b(F_i) \geq U$, where U is the best feasible solution so far
- Heursitics methods
  - Local seach, simulated annealing, genetic algorithm

Lagrangean Relaxation I

- Primal problem

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \in X \end{aligned}$$

- Relax the constraints $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ with Lagrange multiplier $\mathbf{p}$

$$\begin{aligned} Z(\mathbf{p}) = \min \quad & \mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in X \end{aligned}$$

- Lagrange dual problem

$$Z_D = \max_{\mathbf{p} \geq \mathbf{0}} Z(\mathbf{p})$$

Lagrangean Relaxation II

- Main theorem

$$\begin{array}{ll} \text{min} & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \in \text{CH}(X) \end{array}$$

- Weak duality

$$Z_{LP} \leq Z_D \leq Z_{IP}$$

- Subgradient method
  - $Z(\mathbf{p})$ is a piecewise concave function
  - Motivated from steepest ascent method

Dynamic Programming

- Dynamic programming elements
  - State x_k, the most important element
  - Control $u_k \in U(x_k)$
  - Randomness w_k
  - Dynamic $x_{k+1} = f_k(x_k, u_k, w_k)$
  - Additive cost $E_W \left(g_N(x_N) + \sum_{i=1}^{N-1} g_k(x_k, u_k, w_k) \right)$

- Bellman's principle of optimality

$$J_k(x_k) = \min_{u_k \in U(x_k)} \{E_{w_k} (g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k)))\}$$

- Intitial condition $J_N(x_N) = g_N(x_N)$, final calculation $J_0(x_0)$

Convex Optimization

- Convex functions $f : \mathbf{R}^n \mapsto \mathbf{R}$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \qquad \forall x, y \in \mathbf{R}^n, \lambda \in [0,1]$$

- Twice differentiable convex functions

$$\nabla^2 f(x) \succeq 0 \quad \forall x$$

- Characterization

$$\nabla f(\bar{x})'(x - \bar{x}) \leq f(x) - f(\bar{x}) \quad \forall x, \overline{x}$$

- Convex optimization problem
  - Minimize a convex function over a convex set
  - x^* is a local minimum then x^* is also a global minimum

Unconstrained Optimization I

- Necessary condition
  - $f(x)$ twice differentiable
  - x^* is local minimum then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succeq 0$
  - Used to find candidates for local minima

- Sufficient condition
  - $f(x)$ twice differentiable
  - $\nabla f(x^*) = 0$ and $\nabla^2 f(x) \succeq 0 \quad \forall x \in B(x^*, \epsilon)$, then x^* is a local minimum
  - $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succ 0$, then x^* is a local minimum

- Necessary and sufficient condition for convex functions
  - $f(x)$ convex, differentiable
  - x^* is a global minimum if and only if $\nabla f(x^*) = 0$

Unconstrained Optimization II

- Steepest descent method
  - Direction $d^k = -\nabla f(x^k)$
  - Step length: bisection method, Armijo's rule
  - Linear convergence
- Newton method
  - Direction $d^k = -(\nabla^2 f(x^k))^{-1} \nabla f(x^k)$
  - Locally quadratic convergence
  - Initial solution, ill-conditioned Hessian matrix
- Conjugate method
  - Expanding subspace theorem
  - Q-conjugate $d_i' Q d_j = 0 \quad \forall i \neq j$
  - Finite convergence for quadratic case

Constrained Optimization I

- Feasible set $K = \{x \in \mathbf{R}^n | g_j(x) \leq 0, h_i(x) = 0\}$
- Karush-Kuhn-Tucker necessary condition:
- If
  - x^* is a local minimum
  - Constraint qualification condition:
    - $\nabla g_j(x)$, $j \in I = \{j | g_j(x) = 0\}$ and all $\nabla h_i(x)$ are linearly independent
- Then, $\exists (u, v)$:
  - $\nabla f(x) + \sum_j u_j \nabla g_j(x) + \sum_i v_i \nabla h_i(x) = 0$
  - $u \geq 0$
  - $g_j(x) \leq 0, h_i(x) = 0$
  - $u_j g_j(x) = 0$

Constrained Optimization II

- CQC condition can be replaced by Slater condition:
  - $\exists x \in K : g_j(x) < 0, h_i(x) = 0 \quad \forall i, j$
- KKT condition is sufficient for COP
- KKT condition is necessary and sufficient for COP under CQC
- KKT condition is necessary and sufficient for COP with linear constraints:

$$\begin{array}{ll} \text{min} & f(x) \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

Interior Point Methods I

- Affine scaling method
  - Initial interior starting point x_0
  - Direction d

$$\begin{array}{ll} \text{min} & c'd \\ \text{s.t.} & Ad = 0 \\ & d'X^{-2}d \leq 1 \end{array}$$

    where $X = \text{diag}(x_0)$
  - Step length: short and long steps

- Barrier methods
  - Remove inequality constraints using a barrier function
  - $B_\mu(x) = c'x - \mu \sum_{i=1}^{n} \log x_i$

Interior Point Methods II

- Primal path following method
  - Quadratic approximation

$$B_\mu(x+d) = B_\mu(x) + (c' - \mu e' X^{-1})d + \frac{1}{2}\mu d' X^{-2} d$$

  - Direction d

$$\begin{array}{ll} \text{min} & (c' - \mu e' X^{-1})d + \frac{1}{2}\mu d' X^{-2} d \\ \text{s.t.} & Ad = 0 \end{array}$$

- Primal-dual path following methods
  - Calculate direction for both primal and dual problems
  - Solve system of equations $F(z) = 0$, where

$$F(z) = \begin{bmatrix} Ax - b \\ A'p + s - c \\ XSe - \mu e \end{bmatrix}$$

Semidefinite Optimization I

- Formulation

$$\begin{array}{cc} \text{min} & C \circ X \\ \text{s.t.} & A_i \circ X = b_i \\ & X \succeq 0 \end{array} \quad \rightleftharpoons \quad \begin{array}{cc} \text{max} & \sum_{i=1}^m b_i y_i \\ \text{s.t.} & \sum_{i=1}^m y_i A_i + S = C \\ & S \succeq 0 \end{array}$$

- Duality
  - Weak duality: $C \circ X - \sum_{i=1}^m b_i y_i = S \circ X \geq 0$, $S \circ X = 0 \Rightarrow SX = 0$
  - Strong duality: with Slater-like condition, both dual and primal problems attain the same optimal solution

- SDO Modeling
  - Eigenvalues: $\lambda_{\text{max}}(X) \leq t \Leftrightarrow tI - X \succeq 0$
  - Matrix reformulation:

$$(Ax+b)'(Ax+b) - c'x - d \leq 0 \Leftrightarrow \begin{pmatrix} I & Ax+b \\ (Ax+b)' & c'x+d \end{pmatrix} \succeq 0$$

Semidefinite Optimization II

- Sum of squares

$$\min f(x) \Leftrightarrow \begin{array}{ll} \max & \gamma \\ \text{s.t.} & f(x) - \gamma = \tilde{x}'Q\tilde{x} \\ & Q \succeq 0 \end{array}$$

, where $\tilde{x} = (1, x, \ldots, x^k)'$.

- Primal barrier algorithm
  - Barrier cost function $B_\mu(X) = C \circ X - \mu \log(\det(X))$
  - Gradient function $\nabla B_\mu(X) = C - \mu X^{-1}$
  - KKT conditions

$$\begin{array}{c} A_i \circ X = b_i \\ C - \mu X^{-1} = \sum_{i=1}^m y_i A_i \\ X \succ 0 \end{array}$$

Good Luck!