2.098/15.093J Midterm Review

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Overview

- Linear Programming
- Simplex Method
- Duality Theory
- Sensitivity Analysis
- Column Generation Method

Linear Programming I

- Standard form of linear programming problems

$$\begin{array}{ll} \text{min} & c'x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

- Standard form reduction
  - Free variables $x_i = x_i^+ - x_i^-$, $x_i^+, x_i^- \geq 0$
  - Slack and surpus variables
  - max $c'x = -$min $-c'x$

Linear Programming II

- Piecewise linear convex objective

$$\begin{array}{ll} \min & \max_{i=1,\ldots,m} c_i' x + d_i \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & z \\ \text{s.t.} & z \geq c_i' x + d_i \\ & Ax = b \\ & x \geq 0 \end{array}$$

- Absolute values

$$\begin{array}{ll} \min & \sum_{i=1}^n c_i |x_i| \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

  - $z_i = |x_i| \Leftrightarrow x_i \leq z_i$ and $-x_i \leq z_i$
  - $x_i = x_i^+ - x_i^-$, $x_i^+ - x_i^- \geq 0 \Rightarrow |x_i| = x_i^+ + x_i^-$ (assume $c \geq 0$)

Geometry of Linear Programming I

- Convexity
  - Convex function f:
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall x, y, \lambda \in [0,1]$$
  - Convex set S:
$$\lambda x + (1-\lambda)y \in S \text{ for all } x, y \in S, \lambda \in [0,1]$$
  - Convex hull of $x^1, \ldots, x^k$:
$$C = \left\{ \sum_{i=1}^{k} \lambda_i x^i | \lambda_i \geq 0 \quad \forall i = \overline{1,k}, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$
- Polyhedra
  - $P = \{x \in R^n | Ax \geq b\}$, A is a $m \times n$ matrix and $b \in R^m$
  - Standard form representation $P = \{x \in R^n | Ax = b, x \geq 0\}$

Geometry of Linear Programming II

- Extreme points
  - $x \in P$ is an extreme point of P if $\nexists y, z \in P$, $y, z \neq x$, $\lambda \in [0, 1]$:
$$x = \lambda y + (1 - \lambda) z$$
- Vertices
  - $x \in P$ is a vertex of P if $\exists c$:
$$c'x < c'y \quad \forall y \in P, y \neq x$$
- Basic feasible solutions
  - $x \in P$ is a basic feasible solution if:
    1. All equality constraints are active
    2. There are n active constraints that are linearly independent
    3. All constraints are satisfied

Simplex Method I

- Optimality conditions for a basis B
  - $B^{-1}b \geq 0$ (feasibility)
  - $\overline{c}' = c' - c'_B B^{-1} A \geq 0'$ (optimality)

- Simplex development
  - d: feasible direction, then $\triangle z = c'd = (c' - c'_B B^{-1} N) d_N$
  - Simplex feasible direction: $d_B = -B^{-1}A_j$, $d_j = 1$, then $\triangle z = c' - c'_B B^{-1} A_j = \overline{c}_j$
  - $y = x + \theta d$ feasible: $\theta \leq \min_{i | d_{B(i)} < 0} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right)$

Simplex Method II

- Unboundedness (objective)
  - $\overline{c}_j < 0$
  - $B^{-1}A_j \leq 0$

- Degeneracy (BFS)
  - More than n tight constraints
  - More than one basis
  - $x_B \not> 0$

- Multiple optimal solutions
  - $\exists j$ non basic: $\overline{c}_j = 0$
  - $\theta^* = \min_{i | d_{B(i)} < 0} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right) > 0$ ($\theta^* = \infty$ if $d_B \geq 0$)

Simplex Method III

- Revised simplex method
  - Elementary row operation: $[B^{-1}|B^{-1}A_j] \to [\overline{B}^{-1}|e_l]$

- Full tableau implementation

$-c'_B B^{-1} b$	$c' - c'_B B^{-1} A$
$B^{-1} b$	$B^{-1} A$

- Initial tableau
  - Solve the problem with auxiliary variables ($b \geq 0$)

$$\begin{array}{cc} Z = \min & \sum_{i=1}^{m} y_i \\ \text{s.t} & Ax + y = b \\ & x, y \geq 0 \end{array}$$

  - Infeasiblity condition: $Z > 0$

Duality Theory I

- Primal and dual problem

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \leftrightarrow \begin{array}{ll} \max & p'b \\ \text{s.t.} & p'A \leq c' \end{array}$$

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax \geq b \end{array} \leftrightarrow \begin{array}{ll} \max & p'b \\ \text{s.t.} & p'A = c' \\ & p \geq 0 \end{array}$$

- Duality theorem
  - Weak duality: x, p are feasible solution to primal and dual problem, then $p'b \leq c'x$
  - Strong duality: primal problem has an optimal solution $\Leftrightarrow$ dual problem has an optimal solution and $c'x^* = b'p^*$
  - Complementary slackness: x, p are feasible solution to primal and dual problem, then x, p are optimal solution if and only if $p_i(a_i'x - b_i) = 0 \quad \forall i$ and $(c_j - p'A_j)x_j = 0 \quad \forall j$

Duality Theory II

- Solution possibilities

	Finite Optimum	Unboundedness	Infeasible
Finite Optimum	$*$	–	–
Unbounded	–	–	$*$
Infeasible	–	$*$	$*$

- Dual simlex method
  - Dual basis solution $p' = c'_B B^{-1}$
  - Dual feasibility condition $c' - p'A \geq 0$: primal optimality condition
  - Dual simplex: maintaining dual feasibility, working towards primal feasibility
  - Optimality conditions: both primal and dual feasibility
  - Primal infeasibility (dual unboundedness) detection
- Farkas's lemma

$$\nexists x \geq 0 : Ax = b \Leftrightarrow \exists p : p'A \geq 0', p'b < 0$$

Sensitivity Analysis I

- Basic question

  When input data (A, b, c) change, can we still use the current plan (current optimal basis)?

- Optimality conditions need to be checked:
  - $B^{-1}b \geq 0$ (feasibility)
  - $\overline{c}' = c' - c'_B B^{-1} A \geq 0'$ (optimality)

- Changes in b
  - Feasibility condition needs to be checked
  - Allowable range calculation

- Changes in c
  - Optimality condition needs to be checked
  - Allowable range calculation (basic and non-basic cases)

Sensitivity Analysis II

- A new variable added
  - Optimality condition needs to be checked
  - Primal simplex method
- A new inequality constraint added
  - The current solution needs to be checked against the new constraint
  - Construct new tableau for new problem if new constraint is violated
- Global dependence on b
  - The optimal cost $F(b)$ is a convex function of b
- Global dependence on c
  - The optimal cost $G(c)$ is a concave function of c

Column Generation Method

- Problem

  The number of variables (number of columns of A) is very large and we cannot store the matrix in memory

- Fact

  The optimal basic solution of a linear programming problem in standard form has at most m variables with positive values, where m is the number of rows of the matrix A

- Idea

  Starting with a small number of variables and introducing (or replacing) new variables when needed

- Implementation

  Solving the problem $Z = \min_j (c_j - p' A_j)$. If $Z \geq 0$, optimal solution is obtained; otherwise, introducing (or replacing one old column by) the column $j^* = \text{argmin}_j (c_j - p' A_j)$.

Good Luck!