2.098/15.093J: Example Solutions 2

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Example 1
$\mathbf{x}$ is a basic feasible solution to the standard form problem with corresponding basis $\mathbf{B}$. Consider a direction $\mathbf{d}$ such that $\mathbf{y} = \mathbf{x} + \mathbf{d}$ is also a feasible solution. Find the change in the objective cost.
Solution
$\triangle z = \mathbf{c}'\mathbf{d} = \mathbf{c}'_B\mathbf{d}_B + \mathbf{c}'_N\mathbf{d}_N$.
$\mathbf{Ay} = \mathbf{b}$, thus $\mathbf{Ad} = \mathbf{0}$.

$$\begin{bmatrix} \mathbf{B} & \mathbf{N} \end{bmatrix} \times \begin{bmatrix} \mathbf{d}_B \\ \mathbf{d}_N \end{bmatrix} = \mathbf{Bd}_B + \mathbf{Nd}_N = \mathbf{0}$$

Thus $\mathbf{d}_B = -\mathbf{B}^{-1}\mathbf{Nd}_N$. Therefore

$$\triangle z = \mathbf{c}'_B(-\mathbf{B}^{-1}\mathbf{Nd}_N) + \mathbf{c}'_N\mathbf{d}_N \qquad \text{or} \qquad \triangle\, z = (\mathbf{c}'_N - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{N})\mathbf{d}_N$$

Example 2
If $\mathbf{d}_N$ is a unit vector in $\mathbb{R}^{n-m}$. Find the direction $\mathbf{d}$.
Solution
$\mathbf{Ad} = \mathbf{0}$ or $\mathbf{Bd}_B + \mathbf{Nd}_N = \mathbf{0}$.
$\mathbf{Nd}_N = \mathbf{A}_j$, thus $\mathbf{Bd}_B + \mathbf{A}_j = \mathbf{0}$. Therefore, $\mathbf{d}_B = -\mathbf{B}^{-1}\mathbf{A}_j$.
Example 3 Solve the following problem using the full tableau simplex method:

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & \\ & x_1 + x_2 \leq 2 \\ & -x_1 + x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{array} \tag{1}$$

Solution
The standard form with slack variables:

$$\begin{array}{ll} \min & -x_1 - x_2 \\ \text{s.t.} & \\ & x_1 + x_2 + s_1 \quad\;\; = 2 \\ & -x_1 + x_2 \quad + s_2 = 0 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array} \tag{2}$$

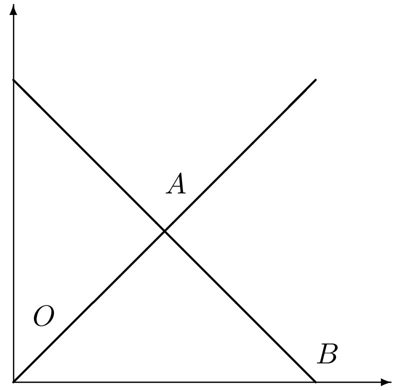

Figure 1 Feasible region of the original problem

The first tableau with slack variables as basis variables:

0	-1	-1	0	0
$s_1 = 2$	1^*	1	1	0
$s_2 = 0$	-1	1	0	1

The solution is not optimal. Select x_1 as the entering variable. $\theta^* = 2$ and s_1 will be the leaving variable. The next tableau is:

2	0	0	1	0
$x_1 = 2$	1	1	1	0
$s_2 = 2$	0	2	1	1

The solution is optimal. However, this might not be the only optimal solution. In this case, we can see that there are multiple optimal solutions. The additional basic optimal solution can be found in the next tableau:

2	0	0	1	0
$x_1 = 1$	1	0	$1/2$	$-1/2$
$x_2 = 1$	0	1	$1/2$	$1/2$

Now in the first step, choose x_2 as entering variable instead of x_1. $\theta^* = 0$ and s_2 is leaving. This is clear that degeneracy makes the simplex method chooses the same basic feasible solution (with different basis of course) in this case:

0	-2	0	0	1
$s_1 = 2$	2^*	0	1	-1
$x_2 = 0$	-1	1	0	1

The solution is not optimal. x_1 is entering the basis. We have, $\theta^* = 1$ and s_1 is leaving. The next tableau is shown below:

2	0	0	1	0
$x_1 = 1$	1	0	$1/2$	$-1/2$
$x_2 = 1$	0	1	$1/2$	$1/2$

This is the additional basic optimal solution calculated above and we can see that $\overline{c}_4 = 0$ again indicates that there might be multiple optimal solutions.

Now what if the objective function is x_1. The initial tableau will be:

0	1	0	0	0
$s_1 = 2$	1	1	1	0
$s_2 = 0$	-1	1	0	1

This solution is optimal and we have $\bar{c}_2 = 0$. However, the θ^* here is zero, which means there are not multiple optimal solutions in this case.

Example 4 Construct the dual of the following problem and then the dual of its dual:

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 + 8x_3 \\ \text{s.t.} \quad & \\ & x_1 - x_2 - x_3 \geq 1 \\ & 3x_2 + x_3 = 4 \\ & x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{3}$$

Solution

The dual problem is:

$$\begin{aligned} \max \quad & p_1 + 4p_2 + p_3 \\ \text{s.t.} \quad & \\ & p_1 \geq 0, p_2 \text{free}, p_3 \leq 0 \\ & p_1 \leq 2 \\ & -p_1 + 3p_2 \leq 3 \\ & -p_1 + p_2 + p_3 \leq 8 \end{aligned} \tag{4}$$

The minimization format of the dual problem:

$$\begin{aligned} -\min \quad & -p_1 - 4p_2 - p_3 \\ \text{s.t.} \quad & \\ & p_1 \geq 0, p_2 \text{free}, p_3 \leq 0 \\ & p_1 \leq 2 \\ & -p_1 + 3p_2 \leq 3 \\ & -p_1 + p_2 + p_3 \leq 8 \end{aligned} \tag{5}$$

The dual of the dual problem:

$$\begin{aligned} -\max \quad & 2x_1 + 3x_2 + 8x_3 \\ \text{s.t.} \quad & \\ & x_1 - x_2 - x_3 \leq -1 \\ & 3x_2 + x_3 = -4 \\ & x_3 \geq -1 \\ & x_1, x_2, x_3 \leq 0 \end{aligned} \tag{6}$$

Change the variables $y_i = -x_i$ for $i = \overline{1,3}$ and change the problem to the minimization format, we obtain the primal problem.