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1 Linear Programming

1.1 Linearity

The general optimization problem is

$$\begin{array}{ll} \min & f(x_1,\ldots,x_n) \\ \text{s.t.} & \\ & g_i(x_1,\ldots,x_n) \le b_i, \quad i=\overline{1,m} \end{array} \tag{1}$$

A linear programming problem is an optimization problem with linear functions f and g_i, $i = \overline{1,m}$. A linear function has two properties, superposition and homegeneity:

- Superposition: $f(x+y) = f(x) + f(y) \quad \forall x, y$
- Homogeneity: $f(\alpha x) = \alpha f(x) \quad \forall \alpha$

What is not a linear programming problem?

$$\begin{array}{ll} \min & x'Qx \\ \text{s.t.} & \\ & Ax = b \\ & x \ge 0 \end{array} \tag{2}$$

$$\begin{array}{ll} \min & \sum_{i=1}^n c_i|x_i| \\ \text{s.t.} & \\ & Ax \ge b \end{array} \tag{3}$$

1.2 Problem Formulation

Steps to formulate a problem:

- Analyze *input data*
- Choose *decision variables* (what is under control)
- Formulate *objective function*
- Formulate all *constraints*

Example 1 Multicommodity path-flow formulation
In a given network, there are K origin-destination(OD) pair. Each OD pair k has a set of paths P^k and a demand d^k. Each edge e in the network is associated with a cost c_e and a capacity u_e. The problem is to minimize the total cost while satisfying the demand.

Example 2 Formulations with absolute values

2 Geometry of LO

2.1 Convexity and Polyhedra

Convex functions
A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is convex if:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \qquad \forall x, y \in \mathbb{R}^n, \lambda \in [0,1] \tag{4}$$

Convex sets
A set $S \subset \mathbb{R}^n$ is convex if $\lambda x + (1-\lambda)y \in S$ for all $x, y \in S, \lambda \in [0,1]$.

Convex hulls
Let $x^1, \ldots, x^k$ be k vectors in $\mathbb{R}^n$. The convex hull of these k vectors is the set of all of their convex combinations, $\sum_{i=1}^k \lambda_i x^i$, $\lambda_i \geq 0$ for all $i = \overline{1,k}$ and $\sum_{i=1}^k \lambda_i = 1$.

Polyhedra
A polyhedron is a set $P = \{x \in \mathbb{R}^n | Ax \geq b\}$, A is a $m \times n$ matrix and $b \in \mathbb{R}^m$.
Prove that each polyhedron is a convex set in $\mathbb{R}^n$.

2.2 Extreme Points, Vertices, and Basic Feasible Solutions

Let P be a polyhedron, $P = \{x \in \mathbb{R}^n | Ax \geq b\}$.

Extreme points
$x \in P$ is an extreme point of P if there are no $y, z \in P$, $y, z \neq x$ and $\lambda \in [0,1]$ such that $x \neq \lambda y + (1-\lambda)z$.

Vertices
$x \in P$ is a vertex of P if $\exists c \in \mathbb{R}^n : \quad c'x < c'y \quad \forall y \in P, y \neq x$.

Basic feasible solutions
$x \in P$ is a basic feasible solution if:

1. All equality constraints are active
2. There are n active constraints that are linearly independent
3. All constraints are satisfied

If $P \neq \emptyset$, then extreme point, vertex, and basic feasible solution definitions are equivalent.

3 Simplex Method I

3.1 Basic solutions, basic variables, bases

How to construct a basic solution of a polyhedron in standard form, $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$?

For a standard form polyhedron constructed from $P = \{x \in \mathbb{R}^n | Ax \leq b\}$ using slack variables, how to select basic and nonbasic variables?

3.2 Optimality conditions

A basis B is said to be optimal if:

1. $B^{-1}b \geq 0$ (feasibility)
2. $\bar{c}' = c' - c'_B B^{-1} A \geq 0'$ (optimality)