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1 Methods for Unconstrained Problems

Steepest descent method

Direction $d^k = -\nabla f(x^k)$, which is guaranteed to be a descent direction if $\nabla f(x^k) \neq 0$.
Step length α is selected so that $f(x^{k+1}) < f(x^k)$.
Linear convergence, any initial solutions

Newton method

Minimize the 2nd order Taylor expansion approximation of the function $f(x)$ around a point $\overline{x}$.
Direction $d^k = -(\nabla^2 f(\overline{x}))^{-1} \nabla f(\overline{x})$
Equally attracted to local maxima, initial solution matters, locally quadratic convergence with good intial solutions.

Conjugate gradient method

Based on expanding subspace theorem: given $d_1, \ldots, d_m \in \mathbf{R}^n$ are Q-conjugate ($d_i' Q d_j = 0, i \neq j$, $m \leq n$, fnction $f(x) = \frac{1}{2}x'Qx + c'x$, initial starting point x_0, then $x = x_0 + \sum_{i=1}^m \left(\frac{-\nabla f(x_0)' d_i}{d_i' Q d_i} \right) d_i$ is the related solution of the problem $\min_\alpha f(x_0 + \sum_{i=1}^m \alpha_i d_i)$. If $m = n$ then the solution above is the optimal solution to the problem $\min_x f(x)$
Direction $d^{k+1} = -\nabla f(x^{k+1}) + \lambda^k d^k$, where $\lambda^k = \frac{\nabla f(x^{k+1})' d^k}{(d^k)' Q d^k}$ in quadratic case and $\lambda^k = \frac{\|\nabla f(x^{k+1})\|^2}{\|\nabla f(x^k)\|^2}$ in general case, which is used to maintain the Q-conjugate property.
Finite convergence for quadratic case.

2 Constrained Optimization

KKT necessary condition

If the feasible region is $K = \{x \in \mathbf{R}^n | g_j(x) \leq 0, h_i(x) = 0\}$ then the KKT necessary condition is stated as follows:
x is a local minimum, $I = \{j | g_j(x) = 0\}$, one of the following condition holds:

1. Constraint qualification condition: all $\nabla g_j(x)$, $j \in I$ and all $\nabla h_i(x)$ are linearly independent
2. Slater condition: there exist an x such that $g_j(x) < 0$ and $h_i(x) = 0$ for all i, j.
3. All constraints are linear

then there exists u and v:

1. $\nabla f(x) + \sum_j u_j \nabla g_j(x) + \sum_i v_i \nabla h_i(x) = 0$
2. $u \geq 0$
3. $g_j(x) \leq 0, h_i(x) = 0$
4. $u_j g_j(x) = 0$

KKT sufficient condition

If K is a convex set, f is a convex function, x is a feasible solution, then if there exists $u \geq 0$ and v such that $\nabla f(x) + \sum_j u_j \nabla g_j(x) + \sum_i v_i \nabla h_i(x) = 0$ and $u_j g_j(x) = 0$ for all j then x is a global optimum of the problem.
KKT condition is both necessary and sufficient for convex optimization problems under the constraint qualification condition.

3 Interior Point Methods

Affine scaling method

An initial interior starting point x_0
A descent direction is found by solving the optimization problem

$$\begin{array}{cc} \min & c'd \\ \text{s.t.} & Ad = 0 \\ & d'X^{-2}d \leq 1 \end{array}$$

where $X = \text{diag}(x_0)$. A closed form of d could be obtained using KKT condition.
Step length can be selected to be short step or long step.

Barrier methods

A barrier function $G(x)$ is a continuous function that tends to ∞ when $g_j(x)$ tends to 0^-. The barrier method removes all inequality constraints and put them into the objective function using an appropriate barrier function with a parameter μ that tends to 0.
Primal path following algorithm uses quadratic approximation to calculate the Newton direction in each iteration:

$$\begin{array}{cc} \min & (c' - \mu e' X^{-1})d + \frac{1}{2}\mu d' X^{-2} d \\ \text{s.t.} & Ad = 0 \end{array}$$

Primal-dual path following algorithm computes the Newton direction for both primal and dual problem in each iteration by solving the system of equations $F(z) = 0$ where

$$F(z) = \begin{bmatrix} Ax - b \\ A'p + s - c \\ XSe - \mu e \end{bmatrix}$$