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1 Network Flow Formulation

Given a network, a directed graph $G = (N, A)$. For each arc $(i, j) \in A$, there will be a cost c_{ij} per unit flow and a capacity u_{ij}. For each node $i \in N$, there will be a supply or demand flow b_i.

1. What conditions do we need so that there will be a feasible flow solution that satisfies all demands?
2. Formulate the problem as a linear programming problem to find the minimim flow cost.
3. Construct the dual problem for the network flow problem above.
4. Apply complementary slackness theorem for the network flow problem.

Solution

1. $\sum_{i=1}^{|N|} b_i = 0$
2. $$\begin{array}{ll} \min & \sum_{(i,j)\in A} c_{ij} f_{ij} \\ \text{s.t.} & b_i + \sum_{j\in I(i)} f_{ij} = \sum_{i\in O(i)} f_{ij} \\ & 0 \leq f_{ij} \leq u_{ij} \end{array}$$

   The matrix form

   $$\begin{array}{ll} \min & c'f \\ \text{s.t.} & Af = b \\ & 0 \leq f \leq u \end{array}$$

3. $$\begin{array}{ll} \max & \sum_{i\in N} b_i p_i + \sum_{(i,j)\in A} u_{ij} q_{ij} \\ \text{s.t.} & p_i - p_j + q_{ij} \leq c_{ij} \\ & q_{ij} \leq 0 \end{array}$$

4. The optimal dual solution will have $q_{ij} = \min\{c_{ij} - (p_i - p_j), 0\}$.
   $c_{ij} - (p_i - p_j) < 0$: $f_{ij} = u_{ij}$
   $c_{ij} - (p_i - p_j) = 0$: $0 \leq f_{ij} \leq u_{ij}$
   $c_{ij} - (p_i - p_j) > 0$: $f_{ij} = 0$

2 Network Flow Examples

Example 1
Formulate the shortest path problem as a network flow problem. How about all shortest path tree from a source node?
Solution
For each arc, the length is considered as the cost. The source node will have the supply of 1 flow unit and the destination node is the sink node with the demand of 1 flow unit. The problem is then a minimum flow problem.
The all shortest path tree is constructed similarly. The source node will have the supply of $|N| - 1$ flow units while the other nodes in the network will have the demand of 1 flow unit.
Example 2 The marriage problem
A small village has a n unmarried men and n unmarried women, m marriage brokers. Each broker knows a number of men and women and can arrange at most b_i marriages. Find the maximum number of marriages that are possible by formulating the problem as a maximum flow problem.
Solution
A source node s, a node for each man, an arc connected the source node with each man node, capacity of 1 flow unit.
A sink node t, a node for each woman, an arc connected each man woman node with the sink node, capacity of 1 flow unit.
Two nodes for each broker, which are connected with each other by an arc with the capacity of b_i. One in these two nodes is connected with man nodes that the broker knows while the other node connects to all woman nodes that the broker knows.