2.098/15.093J: Recitation 8

Xuan Vinh Doan,

November 10, 2004

1 Exact Algorithms for Integer Programming Problems

1.1 Cutting plane methods

1. Solve the linear relaxation, $\mathbf{x}^*$ is the optimal solution
2. If $\mathbf{x}^*$ is integer, then $\mathbf{x}^*$ is the optimal solution of the integer programming problem
3. If not, add a linear inequality constraint that is satisfied by all integer solutions but not $\mathbf{x}^*$

Gomory cut

Given an optimal simplex tableau of the linear relaxation, if all basic variables are integer, the optimal solution for the integer programming problem is obtained.
If not, consider the equality constraint $x_i + (\mathbf{B}^{-1}\mathbf{A}_j)_i x_N = (\mathbf{B}^{-1}\mathbf{b})_i$. The Gomory cut is then the rounding cut applied on this constraint.

Modular Arithmetic

$\sum_{i=1}^n a_i x_i = a_0$ can lead to the inequality $\sum_{i=1}^n b_i x_i \geq b_0$, where $b_i = a_i \mod d$, d is an abitrary integer and $x \in \mathbf{R}_+^n$.

Example

Consider the matching problem, $\sum_{e\in\delta(\{i\})} x_e \leq 1$. Prove that for a set S with odd cardinality, the inequality $\sum_{e\in E(S)} x_e \leq \frac{1}{2}(|S|-1)$ is valid.
Solution
Using the formula $2\sum_{e\in E(S)} x_e + \sum_{e\in\delta(S)} x_e = \sum_{i\in S}\sum_{e\in\delta(\{i\})} x_e$, we obtain the inequality $\sum_{e\in E(S)} x_e + \frac{1}{2}\sum_{e\in\delta(S)} x_e \leq \frac{1}{2}|S|$.
$|S|$ is odd and $x_e \geq 0$, thus we have: $\sum_{e\in E(S)} x_e \leq \frac{1}{2}(|S|-1)$.

1.2 Branch and bound

1. Partition: the feasible set F is partitioned into a finite collection of subsets F_i. These subsets are the feasible sets of corresponding subproblems.
2. Branching: splitting the current subproblem into further subproblems, a tree of subproblems is created.
3. Bounding: calculate the lower bound for the optimal cost of a subproblem $b(F_i) \leq \min_{\mathbf{x}\in F_i} \mathbf{c}'\mathbf{x}$. Linear relaxation or Lagrangian relaxation can be used.

4. Pruning: subproblem is infeasible or $b(F_i) \geq U$, where U is the best feasible solution so far (starting with ∞).

5. Additional cuts: valid cut could be added to the subproblem to improve the lower bound.

2 Integrer programming duality

Primal problem

$$\begin{array}{rl} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \in X \end{array}$$

Relax the constraints $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ with Lagrange multiplier $\mathbf{p}$, we obtain the problem

$$\begin{array}{rl} Z(\mathbf{p}) = \min & \mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$$

Property: $Z(\mathbf{p})$ is a piecewise concave function.
The Lagrange dual problem

$$Z_D = \max_{\mathbf{p} \geq \mathbf{0}} Z(\mathbf{p})$$

We then have:

$$\begin{array}{rl} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \in \mathrm{CH}(X) \end{array}$$

Weak duality

$$Z_D \leq Z_{IP}$$

How about strong duality for linear programming problem?
Subgradient method: motivated from the steepest ascent method, $\mathbf{p}^{t+1} = \mathbf{p}^t + \theta_t \mathbf{s}^t$, where $\mathbf{s}^t \in \partial Z(\mathbf{p}^t)$.