Applications of Probability to Partial Differential Equations and Infinite Dimensional Analysis

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by

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B.S. Mathematics, Tsinghua University (2006)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

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Abstract

This thesis consists of two parts. The first part applies a probabilistic approach to the study of the Wright-Fisher equation, an equation which is used to model demographic evolution in the presence of diffusion. The fundamental solution to the Wright-Fisher equation is carefully analyzed by relating it to the fundamental solution to a model equation which has the same degeneracy at one boundary. Estimates are given for short time behavior of the fundamental solution as well as its derivatives near the boundary. The second part studies the probabilistic extensions of the classical Cauchy functional equation for additive functions both in finite and infinite dimensions. The connection between additivity and linearity is explored under different circumstances, and the techniques developed in the process lead to results about the structure of abstract Wiener spaces. Both parts are joint work with Daniel W. Stroock.

Thesis Supervisor: Daniel W. Stroock
Title: Emeritus Professor of Mathematics

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Introduction

The first part of the thesis is a study of the fundamental solution for the following Wright-Fisher equation:

$$\begin{gathered}\partial_t u\,(x,\,t) = x\,(1-x)\,\partial_x^2 u\,(x,\,t) \text{ in } (0,\,1)\times(0,\,\infty) \quad \text{with boundary values} \\ \lim_{t\searrow 0} u\,(x,\,t) = \varphi\,(x) \text{ for } x\in(0,\,1) \text{ and } u\,(0,\,t) = 0 = u\,(1,\,t) \text{ for } t\in(0,\,\infty).\end{gathered} \tag{0.0.1}$$

This is joint work with Stroock ([1]). Our interest in this problem was the outgrowth of questions asked by Nick Patterson, who uses it at the Broad Institute to model the distribution and migration of genes in his work. Using much more purely analytic technology, the same quesitons have been addressed by C. Epstein and R. Mazzeo ([4]). Earlier work on this topic can be found in [2], [18] and [10]. In particular, Crow and Kimura ([2]) constructed a fundamental solution for (0.0.1) as an expansion of eigenfunctions given by polynomials, which is a very useful expression as $t\to\infty$ but provides little information when t is small. In Chapter 1, we will give a careful analysis of (0.0.1), with particular emphasis on the behavior of its fundamental solution for short time near the boundary.

We start our analysis by noticing that close to the origin, $x\,(1-x)\,\partial_x^2$ has the same degeneracy as $x\partial_x^2$. Just as every uniformly elliptic differential equation can be considered as a pertubation of the heat equation, we shall treat (0.0.1) as a pertubation of

$$\begin{gathered}\partial_t u(\xi,\,t) = \xi\partial_\xi^2 u(\xi,\,t) \text{ in } (0,\infty)\times(0,\infty) \quad \text{with boundary values} \\ u(0,\,t) = 0 \text{ and } \lim_{t\searrow 0} u(\xi,\,t) = \varphi(\xi) \text{ for } \xi\in(0,\,\infty).\end{gathered} \tag{0.0.2}$$

The fundamental solution $q\,(\xi,\,\eta,\,t)$ for (0.0.2) has an explicit expression, and, apart from a factor $\frac{1}{\eta}$, it is analytic all the way to the spacial boundary. Our goal is to set up the connection between $q(\xi, \eta, t)$ and the fundamental solution $p(x, y, t)$ for (0.0.1) so that we can transfer the regularity properties from $q(\xi, \eta, t)$ to $p(x, y, t)$.

For this purpose, we first treat $p(x, y, t)$ as the density of the distribution of the diffusion process associated with (0.0.1). Then the Markov property of the diffusion process allows us to carry out a "localization" procedure on $p(x, y, t)$, which enables us to focus on its behavior only near the boundary at the origin. To be precise, given $\beta \in (0, 1)$, let $p_\beta(x, y, t)$ be the fundamental solution to

$$\partial_t u(x, t) = x(1-x)\partial_x^2 u(x, t) \text{ in } (0, \beta) \times (0, \infty) \quad \text{with boundary values}$$
$$u(0, t) = 0 = u(\beta, t) \text{ and } \lim_{t \searrow 0} u(x, t) = \varphi(x) \text{ for } x \in (0, \beta),$$

then one can express $p(x, y, t)$ in terms of $p_\beta(x, y, t)$ when both x and y are less than β. At the same time, close to the origin, (0.0.1) can be converted into a perturbation of (0.0.2) through the change of variable $x \mapsto \psi(x) = (\arcsin\sqrt{x})^2$. In fact, there is a choice of function V such that, if $q^V_{\psi(\beta)}(\xi, \eta, t)$ is the fundamental solution to

$$\partial_t u(\xi, t) = \xi\partial_\xi^2 u(\xi, t) + V(\xi)\, u(\xi, t) \text{ in } (0, \psi(\beta)) \times (0, \infty) \quad \text{with boundary values}$$
$$u(0, t) = 0 = u(\psi(\beta), t) \text{ and } \lim_{t \searrow 0} u(\xi, t) = \varphi(\xi) \text{ for } x \in (0, \psi(\beta)),$$

then

$$y(1-y)\, p_\beta(x, y, t) = \frac{\psi(y)\, q^V_{\psi(\beta)}(\psi(x), \psi(y), t)}{\sqrt{\psi'(x)\,\psi'(y)}} \text{ for } (x, y) \in (0, \beta)^2.$$

Finally, by combining Duhamel's perturbation formula with an idea similar to "localizing" $p(x, y, t)$ but in reverse, we can relate $q^V_{\psi(\beta)}(\xi, \eta, t)$ to $q(\xi, \eta, t)$.

Via the procedure outlined above, not only are we able to construct $p(x, y, t)$ out of $q(\xi, \eta, t)$, we also show that the regularity properties of $q(\xi, \eta, t)$ get inherited by $p(x, y, t)$. Theorem 1.8 gives an estimate (1.6.16) of $p(x, y, t)$ in terms of $q(\xi, \eta, t)$ for short time near the origin. In fact, the rather explicit estimate in (1.6.17) shows that the difference between 1 and the ratio

$$\frac{y(1-y)\sqrt{\psi'(x)\psi'(y)}\,p(x, y, t)}{\psi(y)\, q(\psi(x), \psi(y), t)}$$

is of order t when t is small and x, y are close to the origin. Theorem 1.11 contains the results on the derivatives of $p(x, y, t)$. It says that $y(1-y)\,p(x, y, t)$ is smooth with bounded derivatives of all order for $(x, y) \in (0, 1)^2$ and $t > 0$, and the mth spacial derivatives of $y(1-y)\,p(x, y, t)$ can be bounded in terms of the lth spacial derivatives of $q(\xi, \eta, t)$ with $l = 1, \cdots, m$. Finally, it is worth pointing out that one can use the results on the derivatives to further refine the estimates on $p(x, y, t)$ (§6 in [1]). However, the calculation involved are tedious.

The second part of the thesis deals with problems arising from Gaussian measures on infinite dimensional vector spaces. It is well known that although a Gaussian measure fits "naturally" in $\mathbb{R}^N$, or equivalently, any real Hilbert space H with $\dim H = N$, it doesn't when $\dim H = \infty$. The canonical example of an infinite dimensional Gaussian measure is the distribution of a Brownian motion, which lives on the classical Wiener space, the space of continuous paths on $\mathbb{R}$ that vanish both at the origin and at infinity. A generalization of this classical case, known as an abstract Wiener space, was first introduced by L. Gross ([8]). To explain Gross's idea, suppose E is a real separable Banach space, $\mathcal{W}$ is a probability measure on E and H is a real Hilbert space. Then, the triple $(H, E, \mathcal{W})$ forms an ***abstract Wiener space***, if H is continuously embedded as a dense subset in E, and for every $x^* \in E^* \setminus \{0\}$, $x \in E \mapsto \langle x, x^* \rangle$ under $\mathcal{W}$ is a non-degenerate centered Gaussian random variable with variance $\|h_{x^*}\|_H^2$, where $h_{x^*} \in H$ is determined by $(h, h_{x^*})_H = \langle h, x^* \rangle$ for all $h \in H$. The theory of abstract Wiener space provides a powerful way of thinking about Gaussian measures on a Banach space, one which underlies the contents of Chapter 2.

Chapter 2 begins with a probabilistically natural variation on the classical Cauchy functional equation

$$f(x+y) = f(x) + f(y) \quad \text{for all} \quad x, y \in \mathbb{R}, \tag{0.0.3}$$

with particular emphasis on its infinite dimensional analog. The study of (0.0.3) has a rich history, and its solutions, so called additive functions are well understood. People know that if an additive function is also Lebesgue measurable, then it must be continuous, and hence linear. This statement remains true when $\mathbb{R}$ is replaced by any finite dimensional space.

We consider variations of (0.0.3) of different sorts. One such variation is a measure theoretical generalization

$$f(x+y) = f(x) + f(y) \quad \text{for Lebesgue-almost every} \quad (x, y) \in \mathbb{R}^2, \tag{0.0.4}$$

which has obvious analog in any finite dimensional space. Moving to the infinte dimensional case, if E is an infinite dimensional Banach space, then the analog of (0.0.3) is just

$$f(x+y) = f(x) + f(y) \quad \text{for all} \quad x,\, y \in E. \tag{0.0.5}$$

However, since Lebesgue measure does not exist in E, a naïve generalization of (0.0.4) makes no sense. Instead, we take a Gaussian measure as the reference measure. Namely, suppose E is a real, separable Banach space, and $\mathcal{W}$ is with a non-degenerate centered Gaussian measure on E, then we formulate the analog of (0.0.4) as

$$f(x+y) = f(x) + f(y) \quad \text{for } \mathcal{W}^2\text{-almost every} \quad (x,\, y) \in E^2. \tag{0.0.6}$$

In all the preceding generalizations, the question we want to answer is what further condition needs to be added in order for the solution to be linear, or almost surely linear with respect to the reference measure.

The same question for (0.0.4) was asked by Erdös ([5]), and an answer was given independently by N.G. de Brujin ([3]) and W.B. Jurkat ([9]). They showed that, even if it is not Lebesgue measurable, every solution to (0.0.4) is almost everywhere equal to an additive function, and therefore every Lebesgue measurable solution to (0.0.4) is almost everywhere equal to a linear function. We produce a proof in the measurable case from a different perspective (Lemma 2.1 and Corollary 2.6). The question for (0.0.5) can be answered using the results from the real valued case. That is, if $f:\ E \to \mathbb{R}$ is Borel measurable and additive, then for any $x \in E$, $g:\ \mathbb{R} \to \mathbb{R}$ defined by $g(t) \equiv f(tx)$ is Lebesgue measurable and additive. Therefore, g is linear, and hence $f(tx) = g(t) = tg(1) = tf(x)$, which implies f is linear. If one applies a theorem of L. Schwartz in [11], then one will see that f is even continuous. In other words, if $f:\ E \to \mathbb{R}$ is Borel measurable and additive, then $f \in E^*$. In fact, Schwartz's theorem in [11], the one that implies Borel measurable linear maps are continuous, applies to a class of vector spaces that are more general than separable Banach spaces. Stroock studied the same problem in [16] using the sort of arguments that we will develop in §2.3.2. One can find a proof to the Schwartz's theorem in Theorem 2.7 when the Borel measurable linear maps are from a separable Banach space to the real line. The answer to the question for (0.0.6) may be the most interesting. In fact, we have shown that there is a solution to (0.0.6) which is $\mathcal{W}-$almost surely NOT linear (Example 2.9), and conversely, a $\mathcal{W}-$almost surely linear function may violate (0.0.6) almost everywhere (Example 2.10).

Clearly, (0.0.6) is not a good generalization of (0.0.3) if linearity is one's goal. It turns out that a better candidate is provided by the following equation, whose solutions we will call *Wiener maps*:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \text{for } \mathcal{W}^2\text{-almost every} \quad (x, y) \in E^2, \tag{0.0.7}$$

where α and β are some positive real numbers such that $\alpha^2 + \beta^2 = 1$. The notion of Wiener map was introduced by Stroock in [12] under the condition that $\alpha = \beta = \frac{1}{\sqrt{2}}$, and a lot of the arguments and results about Wiener maps in §2.3.2 are based on the ideas from [12] and [13]. In particular, Theorem 2.4 and Theorem 2.5 show that if $(H, E, \mathcal{W})$ is an abstract Wiener space, then the set of Wiener maps is the same as the set of all Paley-Wiener integrals on $(H, E, \mathcal{W})$, and every Wiener map is $\mathcal{W}-$almost surely equal to a linear functional that is defined up to a null set on E. The reason why Wiener maps are "closer" than solutions to (0.0.6) to being linear lies in the "singular/equivalence" dichotomy of Gaussian measures on E. Namely, for a pair (α, β) as described above, the distribution of $\alpha x + \beta y$ under $\mathcal{W}^2$ is exactly $\mathcal{W}$, whereas, by contrast, the distribution of $x + y$ under $\mathcal{W}^2$ is singular to $\mathcal{W}$. Therefore, it should not be surprising that (0.0.7) captures linearity of f better than (0.0.6) can.

The study of the connection between additivity and linearity leads to a discussion about the structure of abstract Wiener spaces. As an extension of the theory of Wiener maps, Lemma 2.11 revisits a result from [13], which is based on the idea originating in [8], and Theorem 2.12 and Theorem 2.13 examine the relationship between the "underlying" Hilbert spaces and the "housing" Banach spaces.

Chapter 1

Wright-Fisher Equation

1.1 Introduction

The equation:

$$\begin{gathered}\partial_t u\,(x,\,t) = x\,(1-x)\,\partial_x^2 u\,(x,\,t) \text{ in } (0,\,1)\times(0,\,\infty) \quad \text{with boundary values} \\ \lim_{t\searrow 0} u\,(x,\,t) = \varphi\,(x) \text{ for } x\in(0,\,1) \text{ and } u\,(0,\,t) = 0 = u\,(1,\,t) \text{ for } t\in(0,\,\infty),\end{gathered} \tag{1.1.1}$$

referred as the Wright-Fisher equation, was introduced by Wright and Fisher as a model to study demography in the presence of diffusion. It has had a renaissance as a model for the migration of alleles in the genome. The goal in Chapter 1 is to study the fundamental solution $p\,(x,\,y,\,t)$ to (1.1.1). The difficulty comes from the degeneracy of the elliptic operator $x\,(1-x)\,\partial_x^2$ at the boundary $\{0,\,1\}$. Since close to the origin, $x\,(1-x)\,\partial_x^2$ has the same degeneracy with $x\partial_x^2$, we will first solve the model equation

$$\begin{gathered}\partial_t u(x,\,t) = x\partial_x^2 u(x,\,t) \text{ in } (0,\,\infty)\times(0,\,\infty) \quad \text{with boundary values} \\ u(0,\,t) = 0 \text{ and } \lim_{t\searrow 0} u(x,\,t) = \varphi(x) \text{ for } x\in(0,\,\infty).\end{gathered} \tag{1.1.2}$$

To develop some intuition, we investigate two related Cauchy initial value problems. First, we consider

$$\begin{gathered}\partial_t u(x,\,t) = x^2\partial_x^2 u(x,\,t) \text{ in } (0,\,\infty)\times(0,\,\infty) \quad \text{with boundary values} \\ u(0,\,t) = 0 \text{ and } \lim_{t\searrow 0} u(x,\,t) = \varphi(x) \text{ for } x\in(0,\,\infty).\end{gathered} \tag{1.1.3}$$

The fundamental solution to (1.1.3) is the density for the distribution of the solution to the corresponding Itô stochastic differential equation

$$dX(t,\, x) = \sqrt{2}X(t,\, x)dB(t) \text{ with } X(0,\, x) = x.$$

Thus,

$$X\,(t,\, x) = x \exp\left(\sqrt{2}B\,(t) - t\right) \tag{1.1.4}$$

where $\{B\,(t) \,:\, t \geq 0\}$ is a standard Brownian motion on the real line. Therefore, the fundamental solution to (1.1.3) is

$$p\,(x,\, y,\, t) = \frac{d}{dy}\mathbb{P}\big(X(t,x) \leq y\big) = y^{-2}\bar{p}\,(x,\, y,\, t)\,,$$

where

$$\bar{p}\,(x,\, y,\, t) = \sqrt{\frac{xy}{4\pi t}} \exp\left(-\frac{1}{4t}\left[\left(\log\frac{y}{x}\right)^2 + t^2\right]\right).$$

It's natural to isolate the factor y^{-2} because the operator $x^2\partial_x^2$ is formally self-adjoint with respect to $y^{-2}dy$, and therefore it makes sense that $\bar{p}\,(x,\, y,\, t) = y^2 p\,(x,\, y,\, t)$ is symmetric. Furthermore, one sees from (1.1.4) that the boundary condition at the origin is invisible in this case because, for all $t \geq 0$,

$$X\,(t,\, 0) \equiv 0 \text{ and } X\,(t,\, x) > 0 \text{ if } x > 0.$$

Finally, $\bar{p}\,(x,\, y,\, t)$ is smooth on $(0,\, \infty)^3$, but the spacial derivatives become unbounded along the diagonal as $x = y \searrow 0$.

Second, we look at

$$\begin{gathered}\partial_t u(x,\, t) = x\partial_x^2 u(x,\, t) + \tfrac{1}{2}\partial_x u(x,\, t) \text{ in } (0,\, \infty) \times (0,\, \infty) \quad \text{with boundary values} \\ u(0,\, t) = 0 \text{ and } \lim_{t \searrow 0} u(x,\, t) = \varphi(x) \text{ for } x \in (0,\, \infty).\end{gathered} \tag{1.1.5}$$

Now, the associated Itô stochastic differential equation is

$$dX(t,\, x) = \sqrt{2X(t,\, x)}dB(t) + \tfrac{1}{2}dt \quad \text{with } X(0,\, x) = x,$$

whose solution (the existence and the uniqueness of the solution are guaranteed by a theorem of Watanabe and Yamada ([17])) is given by

$$X(t,\,x)=\left(\sqrt{x}+\frac{1}{\sqrt{2}}B(t)\right)^2.$$

Taking the boundary condition of (1.1.5) into account, one sees that the solution for (1.1.5) is given by the density of the distribution

$$y\mapsto\mathbb{P}\left(B(t)\le 2^{\frac12}(y^{\frac12}-x^{\frac12})\ \&\ B(\tau)>-\sqrt{2x}\text{ for }\tau\in[0,t]\right),$$

and an application of the reflection principle shows that the density is

$$p(x,\,y,\,t)=y^{-\frac12}\bar{p}(x,y,t)\quad\text{where }\bar{p}(x,y,t)=(\pi t)^{-\frac12}e^{-\frac{x+y}{t}}\sinh\left(2\sqrt{\frac{xy}{t^2}}\right).$$

Here we isolate the factor $y^{-\frac12}$ for the same reason as alluded above. Then $\bar{p}(x,\,y,\,t)$ is smooth on $(0,\,\infty)^3$ but has spacial derivatives that are unbounded near the origin.

However, if one ignores the boundary condition in (1.1.5), then the fundamental solution is just the density of $X(t,\,x)$, which is

$$y^{-\frac12}(\pi t)^{-\frac12}e^{-\frac{x+y}{t}}\cosh\left(2\sqrt{\frac{xy}{t^2}}\right).$$

Apart from the factor $y^{-\frac12}$, it is analytic all the way to the boundary. We will see in the next section that the fundamental solution for the model equation (1.1.2) has the same regular properties.

1.2 The Model Equation

In this section, we solve the one-point analog of the Wright-Fisher equation,

$$\begin{gathered}\partial_t u(x,\,t)=x\partial_x^2 u(x,\,t)\text{ in }(0,\,\infty)\times(0,\,\infty)\quad\text{with boundary values}\\ u(0,\,t)=0\text{ and }\lim_{t\searrow0}u(x,\,t)=\varphi(x)\text{ for }x\in(0,\,\infty)\end{gathered}\tag{1.2.1}$$

for $\varphi \in C_b((0, \infty); \mathbb{R})$. Taking a hint from (1.1.3) and (1.1.5), we seek the fundamental solution to (1.2.1) in the form of

$$q(x, y, t) = y^{-1}\bar{q}(x, y, t) = y^{-1}e^{-\frac{x+y}{t}}q\left(\frac{xy}{t^2}\right) \tag{1.2.2}$$

for some appropriate function $q: (0, \infty) \to \mathbb{R}$. Plugging (1.2.2) into (1.2.1), one sees that

$$\partial_t q(x, y, t) = y^{-1}e^{-\frac{x+y}{t}}\left[\frac{x+y}{t^2}q\left(\frac{xy}{t^2}\right) - 2\frac{xy}{t^3}q'\left(\frac{xy}{t^2}\right)\right], \text{ and}$$

$$x\partial_x^2 q(x, y, t) = y^{-1}e^{-\frac{x+y}{t}}\left[\frac{x}{t^2}q\left(\frac{xy}{t^2}\right) - 2\frac{xy}{t^3}q'\left(\frac{xy}{t^2}\right) + \frac{xy^2}{t^4}q''\left(\frac{xy}{t^2}\right)\right].$$

Therefore, given $y \in (0, \infty)$, for $q(\cdot, y, \star)$ to be a solution for (1.2.1) on $(0, \infty)^2$, it is necessary that q solves

$$\xi q''(\xi) - q(\xi) = 0, \text{ for } \xi \in (0, \infty).$$

By standard results about Bessel functions (e.g., Chapter 4 in [19]), $q(\xi)$ must be of the form

$$q(\xi) = c_1\sqrt{\xi}I_1\left(2\sqrt{\xi}\right) + c_2\sqrt{\xi}K_1\left(2\sqrt{\xi}\right),$$

where $c_1, c_2 \in \mathbb{R}$, I_1 and K_1 are the modified Bessel functions with pure imaginary arguments. To determine c_1 and c_2, we first notice that by standard estimates on solutions to parabolic equations (e.g., §5.2.2 in [15]), there exists an $\alpha_0 \in (0, 1)$ such that

$$\frac{\alpha_0}{t^{\frac{1}{2}}} \le q(1, 1, t) = e^{-\frac{2}{t}}q\left(\frac{1}{t^2}\right) \le \frac{1}{\alpha_0 t^{\frac{1}{2}}} \quad \text{for } t \in (0, 1]. \tag{1.2.3}$$

However, the asymptotic behavior of $\frac{1}{t}K_1\left(\frac{2}{t}\right)$ as $t \searrow 0$ clearly violates (1.2.3), which implies c_2 has to be zero. Furthermore, when written as a power series ([19]), $\sqrt{\xi}I_1\left(2\sqrt{\xi}\right) =$

$\sum_{n=1}^{\infty} \frac{\xi^n}{n!(n-1)!}$. By the Monotone Convergence Theorem, one has

$$\begin{aligned}
\int_{(0,\infty)} q(x,y,t)\,dy &= c_1 e^{-\frac{x}{t}} \int_{(0,\infty)} \left(e^{-\frac{y}{t}} \sum_{n=1}^{\infty} \frac{x^n}{t^{2n}} \frac{y^{n-1}}{n!\,(n-1)!} \right) dy \\
&= c_1 e^{-\frac{x}{t}} \sum_{n=1}^{\infty} \frac{x^n}{t^{2n}} \frac{1}{n!\,(n-1)!} \left(\int_{(0,\infty)} e^{-\frac{y}{t}} y^{n-1} dy \right) \\
&= c_1 e^{-\frac{x}{t}} \left(e^{\frac{x}{t}} - 1 \right) \\
&= c_1 - c_1 e^{-\frac{x}{t}}.
\end{aligned}$$

If $q(x,y,t)$ is a fundamental solution, then for any $x > 0$, $\int_{(0,\infty)} q(x,y,t)\,dy$ should tend to 1 as $t \searrow 0$, which implies $c_1 = 1$. Therefore, we should take

$$q(\xi) = \sqrt{\xi} I_1 \left(2\sqrt{\xi} \right) = \sum_{n=1}^{\infty} \frac{\xi^n}{n!\,(n-1)!}. \tag{1.2.4}$$

Defining $q(x,y,t)$ as in (1.2.2), we want to show that $q(x,y,t)$ is indeed the fundamental solution to (1.2.1). To this end, we first notice from the right hand side of (1.2.4) that for all $\xi \in (0,\infty)$,

$$\begin{gathered}
\frac{1}{2} \sum_{n=1}^{\infty} \xi^n \frac{2^{2n}}{(2n)!} \le q(\xi) \le \xi \sum_{n=0}^{\infty} \xi^n \frac{2^{2n}}{(2n)!}, \text{ and hence} \\
\frac{\cosh\left(2\sqrt{\xi}\right) - 1}{2} \le q(\xi) \le \xi e^{2\sqrt{\xi}}.
\end{gathered}$$

Thus, for $(x,y,t) \in (0,\infty)^3$,

$$e^{-\frac{x+y}{t}} \frac{\cosh\left(2\frac{\sqrt{xy}}{t}\right) - 1}{2y} \le q(x,y,t) \le \frac{x}{t^2} e^{-\frac{(\sqrt{y}-\sqrt{x})^2}{t}}, \tag{1.2.5}$$

from which it is clear that, as $t \searrow 0$,

$$\int_{(0,\infty)\setminus(x-\delta,\,x+\delta)} q(x,y,t)dy \to 0 \quad \text{for each } \delta > 0$$

uniformly for x in compact subsets of $(0,\infty)$. Hence, for each $\varphi \in C_b\left((0,\infty);\mathbb{R}\right)$,

$$u_\varphi(x,t) \equiv \int_{(0,\infty)} \varphi(y) q(x,y,t) dy \to \varphi(x) \tag{1.2.6}$$

uniformly for x in compact subsets of $(0, \infty)$. Meanwhile, from (1.2.5), it is also clear that as $x \searrow 0$, $u_\varphi(x, t) \to 0$ uniformly for $t \in [\delta, \infty)$ for every $\delta > 0$. Summarizing, we have shown that $q(x, y, t)$ is a fundamental solution to (1.2.1).

It is useful for us to notice that the estimates in (1.2.5) can be improved when $xy \geq t^2$. In fact, if $\xi \geq 1$, then standard results on the asymptotic behaviors of the Bessel function I_1 (e.g., Chapter 7 in [19]) shows that

$$\delta_0 \xi^{\frac{1}{4}} e^{2\xi^{\frac{1}{2}}} \leq q(\xi) \leq \delta_0^{-1} \xi^{\frac{1}{4}} e^{2\xi^{\frac{1}{2}}}$$

for some $\delta_0 \in (0, 1)$. Thus, when $xy \geq t^2$,

$$\delta_0 y^{-1} e^{-\frac{x+y}{t}} \frac{(xy)^{\frac{1}{4}}}{t^{\frac{1}{2}}} \exp\left(2\frac{x^{\frac{1}{2}} y^{\frac{1}{2}}}{t}\right) \leq q(x, y, t) \leq \delta_0^{-1} y^{-1} e^{-\frac{x+y}{t}} \frac{(xy)^{\frac{1}{4}}}{t^{\frac{1}{2}}} \exp\left(2\frac{x^{\frac{1}{2}} y^{\frac{1}{2}}}{t}\right),$$

which leads to

$$\frac{\delta_0}{y} \frac{(xy)^{\frac{1}{4}}}{t^{\frac{1}{2}}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{t}} \leq q(x, y, t) \leq \frac{1}{\delta_0 y} \frac{(xy)^{\frac{1}{4}}}{t^{\frac{1}{2}}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{t}}. \tag{1.2.7}$$

Similarly with the probabilistic considerations we adopted for (1.1.3) and (1.1.5), $q(x, y, t)$ can be interpreted as the density of the distribution of the diffusion process associated with (1.2.1). To be precise, we consider the following Itô stochastic differential equation

$$dY(t, x) = \sqrt{2Y(t, x)}\, dB(t), \quad \text{with } Y(0, x) = x. \tag{1.2.8}$$

Again, by the Watanabe-Yamada theorem alluded in §1.1, for each $x \in [0, \infty)$, there exists an almost surely unique $Y(t, x)$ satisfying (1.2.8). Set $\zeta_0^Y(x) \equiv \inf\{t \geq 0 : Y(t, x) = 0\}$, then $Y(t, x) = 0$ for all $t \geq \zeta_0^Y(x)$. Then $q(x, y, t)$ is the density of the distribution $y \mapsto \mathbb{P}\left(Y(t, x) \leq y,\ t < \zeta_0^Y(x)\right)$. To better describe this, we introduce the σ-algebra $\mathcal{F}_t$ which is generated by $\{B(\tau) : 0 \leq \tau \leq t\}$. Then $Y(t, x)$ is $\mathcal{F}_t$-measurable. In addition, by Itô's formula and Doob's Stopping Time Theorem, given $t > 0$, for any $w \in C^{2,1}((0, \infty) \times (0, t)) \cap C_b([0, \infty) \times [0, t])$, if $f = \partial_t w + x\partial_x^2 w$ is bounded, then

$$\left(w(Y(\tau \wedge t \wedge \zeta_0^Y(x), x), \tau \wedge t \wedge \zeta_0^Y(x)) - \int_0^{\tau \wedge t \wedge \zeta_0^Y(x)} f(Y(s, x), s)\, ds,\ \mathcal{F}_\tau,\ \mathbb{P}\right)$$

is a martingale. In particular, if $\varphi \in C_b((0, \infty); \mathbb{R})$, and $u_\varphi(t, x)$ is defined as in (1.2.6), then by taking $w(\tau, x) = u_\varphi(t-\tau, x)$, we have

$$u_\varphi(x, t) = \int_{(0,\infty)} \varphi(y) q(x, y, t) dy = \mathbb{E}\big[\varphi(Y(t, x)), \zeta_0^Y(x) > t\big]. \tag{1.2.9}$$

This proves that $u_\varphi(x, t)$ is the one and only solution to (1.2.1). Furthermore, as a consequence of the uniqueness of $Y(t, x)$, one knows that $\left\{Y(t, x) : (t, x) \in (0, \infty)^2\right\}$ satisfies the Markov property. In particular, this implies $q(x, y, t)$ satisfies the Chapman-Kolmogorov equation

$$q(x, y, s+t) = \int_{(0,\infty)} q(x, \xi, s) q(\xi, y, t) d\xi \text{ for } (x, y) \in (0, \infty)^2 \text{ and } s, t > 0. \tag{1.2.10}$$

1.3 Localization of the Wright-Fisher Equation

As mentioned earlier, the elliptic operator $x(1-x)\partial_x^2$ has the same degeneracy with $x\partial_x^2$ near the origin. In this section, we will introduce a "localization" procedure which allows us to relate the model equation (1.2.1) to the original Wright-Fisher equation:

$$\begin{aligned} &\partial_t u(x, t) = x(1-x)\partial_x^2 u(x, t) \text{ in } (0, 1) \times (0, \infty) \quad \text{with boundary values} \\ &\lim_{t \searrow 0} u(x, t) = \varphi(x) \text{ for } x \in (0, 1) \text{ and } u(0, t) = 0 = u(1, t) \text{ for } t \in (0, \infty) \end{aligned} \tag{1.3.1}$$

for $\varphi \in C_b((0, 1); \mathbb{R})$. Our goal is to transfer the properties of $q(x, y, t)$ (as defined in (1.2.2)) to the fundamental solution $p(x, y, t)$ for (1.3.1). Because we cannot simply write it down, proving that $p(x, y, t)$ even exists requires some thought. Crow and Kimura constructed in [2] a fundamental solution as an expansion of eigenfunctions which are given by polynomials. Their expression for $p(x, y, t)$ is very useful as $t \to \infty$, but gives little information for small t. To better understand the short time behavior of $p(x, y, t)$, we are going to adopt a different approach.

To get started, we will begin with the diffusion process corresponding to (1.3.1). Namely, for $x \in [0, 1]$, let $\{X(t, x) : t \geq 0\}$ be the solution to the Itô stochastic differential equation

$$dX(t, x) = \sqrt{2X(t, x)(1 - X(t, x))} dB(t), \quad \text{with } X(0, x) = x. \tag{1.3.2}$$

Again, the existence and uniqueness of $X(x, t)$ are guranteed by the Watanabe-Yamada theorem. Set $\zeta_\xi^X(x) \equiv \inf\{t \geq 0 : X(t, x) = \xi\}$ for $\xi \in [0, 1]$ and $\zeta^X(x) \equiv \zeta_0^X(x) \wedge \zeta_1^X(x)$, then $X(t, x) = X(\zeta^X(x), x)$ for $t \geq \zeta^X(x)$. Similarly, given $t > 0$, denote $\mathcal{F}_t$ the σ-algebra generated by $\{B(\tau) : 0 \leq \tau \leq t\}$. Then $X(t, x)$ is $\mathcal{F}_t$-measurable. Furthermore, by the Markov property of $X(t, x)$, for any $\Gamma \in \mathcal{B}_{(0,1)}$ where $\mathcal{B}_{(0,1)}$ is the Borel σ-algebra on $(0, 1)$, if

$$P(t, x, \Gamma) = \mathbb{P}\left(X(t, x) \in \Gamma \text{ and } \zeta^X(x) > t\right),$$

then $P(t, x, \cdot)$ satisfies the Chapman-Kolmogorov equation

$$P(s+t, x, \Gamma) = \int_{(0,1)} P(t, y, \Gamma)\, P(s, x, dy), \quad \text{for } x \in (0,1) \text{ and } \Gamma \in \mathcal{B}_{(0,1)}. \tag{1.3.3}$$

In addition, for any $\varphi \in C_c^2((0, 1); \mathbb{R})$, set $f \equiv x(1-x)\varphi''$, then, by Itô's formula and Doob's Stopping Time Theorem,

$$\left(\varphi(X(\tau \wedge t \wedge \zeta^X(x), x)) - \int_0^{\tau\wedge t\wedge\zeta^X(x)} f\big(X(s, x)\big)ds,\ \mathcal{F}_\tau,\ \mathbb{P}\right)$$

is a martingale. Also,

$$\int_{(0,1)} \varphi(y)\, P(t, x, dy) = \varphi(x) + \int_0^t \left(\int_{(0,1)} f(y)\, P(\tau, x, dy)\right) d\tau,$$

which proves that as $t \searrow 0$, $P(t, x, \cdot)$ tends to the unit point mass at $x \in (0, 1)$ weakly. In other words, as a distribution, $P(\star, x, \cdot)$ is a solution to the Kolmogorov equation $\partial_t u = \partial_y^2(y(1-y)u)$. By standard hypoellipticity results for parabolic operators (e.g., §3.4.2 in [15]), for each $x \in (0, 1)$, there exists a smooth function $(y, t) \in (0, 1) \times (0, \infty) \mapsto p(x, y, t) \in \mathbb{R}^+$ such that $P(t, x, dy) = p(x, y, t)\, dy$ and

$$\partial_t p(x, y, t) = \partial_y^2\big(y(1-y)p(x, y, t)\big) \quad \text{in } (0, 1) \times (0, \infty) \text{ with } \lim_{t\searrow 0} p(x, \cdot, t) = \delta_x.$$

Furthermore, (1.3.3) implies that $p(x, y, t)$ also satisfies the Chapman-Kolmogorov equation

$$p(x, y, s+t) = \int_{(0,1)} p(\xi, y, t)p(x, \xi, s)d\xi, \quad \text{for } x, y \in (0, 1)^2 \text{ and } s, t > 0.$$

Finally, one notices that by uniqueness, $1-X(t,\, x)$ has the same distribution with $X(t,\, 1-x)$, and hence, it is clear that

$$p(x,\, y,\, t) = p(1-x,\, 1-y,\, t)\,,$$

Therefore, it is sufficient for us to study $p(x,\, y,\, t)$ near only one boundary, say, the origin.

Now we are ready to carry out the "localization" procedure. In order to focus attention to what is happening at the origin, let $\beta \in (0,\, 1)$ be fixed, and consider the fundamental solution $p_\beta(x,\, y,\, t)$ to

$$\partial_t u(x,\, t) = x(1-x)\partial_x^2 u(x,\, t) \text{ in } (0,\, \beta)\times(0,\, \infty) \quad \text{with boundary values}$$
$$u(0,\, t) = 0 = u(\beta,\, t) \text{ and } \lim_{t\searrow 0} u(\cdot,\, t) = \varphi$$

for $\varphi \in C_b((0,\, \beta)\,;\, \mathbb{R})$. Next, we focus on the time that $X(t,\, x)$ spends inside the region $(0,\, \beta)$ by recording the moments when $X(t,\, x)$ "travels across" a small interval to the left of β. To be precise, given $0<\alpha<\beta<1$, for $x \in (0,\, \alpha)$, define $\eta^X_{0,[\alpha,\beta]}(x) \equiv 0$, and for $n \geq 1$, set

$$\begin{aligned} \eta^X_{2n-1,[\alpha,\beta]}(x) &\equiv \inf\{t \geq \eta^X_{2(n-1),[\alpha,\beta]}(x) :\ X(t,\, x) = \beta\} \wedge \zeta^X(x) \quad \text{and} \\ \eta^X_{2n,[\alpha,\beta]}(x) &\equiv \inf\{t \geq \eta^X_{2n-1,[\alpha,\beta]}(x) :\ X(t,\, x) = \alpha\} \wedge \zeta^X(x) \end{aligned} \tag{1.3.4}$$

for $n \geq 1$. Then, if $\varphi \in C_c((0,\, \alpha)\,;\, \mathbb{R})$,

$$\begin{aligned} \int_{(0,\,1)} \varphi(y)\, p(x,\, y,\, t)\, dy &= \mathbb{E}\left[\varphi(X(t,\, x))\,,\, t \leq \zeta^X(x)\right] \\ &= \sum_{n=0}^{\infty} \mathbb{E}\left[\varphi(X(t,\, x))\,,\ \eta^X_{2n,[\alpha,\beta]}(x) < t < \eta^X_{2n+1,[\alpha,\beta]}(x)\right]. \end{aligned}$$

By the definition of $p_\beta(x,\, y,\, t)$ and the fact that φ is supported on $(0,\, \alpha)$,

$$\mathbb{E}\left[\varphi(X(t,\, x))\,,\ 0 < t < \eta^X_{1,[\alpha,\beta]}(x)\right] = \int_{(0,\,\alpha)} \varphi(y)\, p_\beta(x,\, y,\, t)\, dy.$$

Furthermore, by the Markov property of $X(t, x)$, for $n \geq 1$,

$$\begin{aligned}&\mathbb{E}\left[\varphi\left(X\left(t,\, x\right)\right),\ \eta^X_{2n,[\alpha,\beta]}(x) < t < \eta^X_{2n+1,[\alpha,\beta]}(x)\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[\varphi\left(X\left(t-\eta^X_{2n,[\alpha,\beta]}(x),\, \alpha\right)\right),\ t-\eta^X_{2n,[\alpha,\beta]}(x) < \eta^X_{1,[\alpha,\beta]}\left(\alpha\right)\right],\ \eta^X_{2n,[\alpha,\beta]}(x) < t\right]\\ =&\mathbb{E}\left[\int_{(0,\,\alpha)}\varphi\left(y\right)p_\beta\left(\alpha,\, y,\, t-\eta^X_{2n,[\alpha,\beta]}(x)\right)dy,\ \eta^X_{2n,[\alpha,\beta]}(x) < t\right].\end{aligned}$$

Collecting all the terms from the right hand side, one sees that

$$\begin{aligned}\int_{(0,\,1)}\varphi\left(y\right)p\left(x,\, y,\, t\right)dy =& \int_{(0,\,\alpha)}\varphi\left(y\right)p_\beta\left(x,\, y,\, t\right)dy\\ &+\sum_{n=1}^{\infty}\mathbb{E}\left[\int_{(0,\,\alpha)}\varphi\left(y\right)p_\beta\left(\alpha,\, y,\, t-\eta^X_{2n,[\alpha,\beta]}(x)\right)dy,\ \eta^X_{2n,[\alpha,\beta]}(x) < t\right].\end{aligned}$$

Since this is true for all $\varphi \in C_c(0,\, \alpha)$, then we have an expression for $p(x,\, y,\, t)$ in terms of p_β:

$$p(x,\, y,\, t) = p_\beta(x,\, y,\, t) + \sum_{n=1}^{\infty}\mathbb{E}\left[p_\beta\left(\alpha,\, y,\, t-\eta^X_{2n,[\alpha,\beta]}(x)\right),\ \eta^X_{2n,[\alpha,\beta]}(x) < t\right], \qquad (1.3.5)$$

for every $(x,\, y,\, t) \in (0,\, \alpha)^2 \times (0,\, \infty)$. Thus, to understand $p(x,\, y,\, t)$, it suffices to study $p_\beta(x,\, y,\, t)$.

1.4 Change of Variables

To make use of the results from the previous section, we are going to treat $p_\beta(x,\, y,\, t)$ as a perturbation of $q(x,\, y,\, t)$ and make a change of variables, one which was used also by Wm. Feller ([6] and [7]). For this purpose, we choose a ψ such that $\psi(0) = 0$, and $X^\psi(t,\, x) = \psi \circ X(t,\, x)$ is the diffusion corresponding to an equation of the form $\partial_t u = x\partial_x^2 u + b(x)\,\partial_x u$ for some coefficient function b. In other words, we want $X^\psi(t,\, x)$ to satisfy the following Itô stochastic equation:

$$dX^\psi(t,\, x) = \sqrt{2X^\psi(t,\, x)}dB(t) + b\big(X^\psi(t,\, x)\big)dt \text{ with } X^\psi(0,\, x) = \psi(x)$$

for $t < \zeta^{\psi}(x) \equiv \inf\{\tau \geq 0 : X^{\psi}(\tau, x) = \psi(1)\}$. Because, by Itô's formula and (1.3.2), $X^{\psi}(t, x)$ also satisfies

$$dX^{\psi}(t, x) = \psi'(X(t, x))\sqrt{2X(t, x)(1 - X(t, x))}dB(t) + \psi''(X(t, x))X(t, x)(1 - X(t, x))dt,$$

ψ and b should be chosen so that

$$\begin{aligned} \psi'(x)\sqrt{2x(1-x)} &= \sqrt{2\psi(x)}, \text{ and} \\ b \circ \psi(x) &= \psi''(x)x(1-x) \text{ for } x \in (0, 1). \end{aligned} \tag{1.4.1}$$

Solving (1.4.1), we know that

$$\begin{aligned} \psi(x) = (\arcsin\sqrt{x})^2, \quad \psi(1) = \frac{\pi^2}{4}, \quad \psi^{-1}(x) = (\sin\sqrt{x})^2, \\ \text{and } b(x) = \frac{\psi\psi''}{(\psi')^2} \circ \psi^{-1} = \frac{\sin(2\sqrt{x}) - 2\sqrt{x}\cos(2\sqrt{x})}{\sin(2\sqrt{x})}. \end{aligned} \tag{1.4.2}$$

Clearly $b(x)$ blows up at the right end point $\psi(1) = \frac{\pi^2}{4}$, but this causes no problem because $p_\beta(x, y, t)$ is "localized" near 0. In fact, we only have to analyze the fundamental solution $r_\theta(x, y, t)$ to the following equation:

$$\begin{aligned} \partial_t u(x, t) = x\partial_x^2 u(x, t) + b(x)\partial_x u(x, t) \text{ in } (0, \theta) \times (0, \infty) \\ \text{with } u(0, t) = 0 = u(\theta, t) \text{ and } \lim_{t \searrow 0} u(\cdot, t) = \varphi \end{aligned} \tag{1.4.3}$$

for $\theta = \psi(\beta) \in \left(0, \frac{\pi^2}{4}\right)$ and $\varphi \in C_b((0, \theta); \mathbb{R})$. Indeed, since $r_\theta(x, y, t)$ does not "feel" b off of $(0, \theta)$, we can replace b by any $b_\theta \in C_c^\infty(\mathbb{R})$ such that $b_\theta = b$ on $(0, \theta)$. Furthermore, $b(x)$ can be extended as a real analytic function on $\left(-\frac{\pi^2}{4}, \frac{\pi^2}{4}\right)$ which vanishes at $\psi(0) = 0$. Therefore, we can write $b(x) = xc(x)$ for some $c(x) \in C_c^\infty(\mathbb{R})$.

Finally we convert (1.4.3) to a perturbation of (1.2.1) by a potential. The conversion is described in the following lemma:

Lemma 1.1. *Set*

$$C(x) \equiv \frac{1}{2}\int_0^x c(\xi)\,d\xi \quad \textit{and} \quad V^c(x) \equiv -x\left(\frac{c'(x)}{2} + \frac{c(x)^2}{4}\right). \tag{1.4.4}$$

Then $u(x,\, t)$ *is a solution to* $\partial_t u(x,\, t) = x\partial_x^2 u(x,\, t) + xc(x)\,\partial_x u(x,\, t)$ *if and only if* $u(x,\, t) = e^{-C(x)} w(x,\, t)$*, where* $w(x,\, t)$ *is a solution to*

$$\partial_t w(x,\, t) = x\partial_x^2 w(x,\, t) + V^c(x)\, w(x,\, t)\,.$$

Proof. The easiest way to prove this is by direct calculation. Suppose $u(x,\, t),\, w(x,\, t) \in C^{2,1}\left((0,\, \infty)^2;\, \mathbb{R}\right)$ satisfy $u(x,\, t) = e^{-C(x)} w(x,\, t)$, and $C(x)$ and $V^c(x)$ are defined according to (1.4.4). Then,

$$\partial_t u = e^{-C(x)} \partial_t w,\ \partial_x u = e^{-C(x)} \left(\partial_x w - \frac{c(x)}{2} w\right),\ \text{and}$$
$$\partial_x^2 u = e^{-C(x)} \left[\partial_x^2 w - c(x)\,\partial_x w + \left(\frac{c^2(x)}{4} - \frac{c'(x)}{2}\right) w\right].$$

Therefore, if $\partial_t u(x,\, t) = x\partial_x^2 u(x,\, t) + xc(x)\,\partial_x u(x,\, t)$, then

$$\begin{aligned}
\partial_t w &= x\partial_x^2 w - xc(x)\,\partial_x w + x\left(\frac{c^2(x)}{4} - \frac{c'(x)}{2}\right) w + xc(x)\left(\partial_x w - \frac{c(x)}{2} w\right) \\
&= x\partial_x^2 w - x\left(\frac{c^2(x)}{4} + \frac{c'(x)}{2}\right) w \\
&= x\partial_x^2 w + V^c(x)\, w.
\end{aligned}$$

On the other hand, if $\partial_t w(x,\, t) = x\partial_x^2 w(x,\, t) + V^c(x)\, w(x,\, t)$, by reversing the steps above, one sees that $u(x,\, t)$ satisfies the desired equation. □

As Lemma 1 shows, it is sufficient for us to study the following equation:

$$\partial_t w(x,\, t) = x\partial_x^2 w(x,\, t) + V^c(x)\, w(x,\, t) \text{ in } (0,\, \theta) \times (0,\, \infty)$$
$$\text{with } w(0,\, t) = 0 = w(\theta,\, t) \text{ and } \lim_{t \searrow 0} w(\cdot,\, t) = \varphi$$

for some $\varphi \in C_b((0,\, \theta);\, \mathbb{R})$.

For the moment, we only take into account the boundary condition at the origin, and will worry about the boundary at θ in the next section. Namely, we consider equations of the following form:

$$\begin{gathered}\partial_t w(x,\,t) = x\partial_x^2 w(x,\,t) + V(x)w(x,\,t) \text{ in } (0,\infty)^2 \\ \text{with } w(0,\,t) = 0 \text{ and } \lim_{t\searrow 0} w\,(t,\,\cdot) = \varphi\end{gathered} \tag{1.4.5}$$

for some $\varphi \in C_b\left((0,\,\theta)\,;\,\mathbb{R}\right)$ and denote its fundamental solution by $q^V\,(x,\,y,\,t)$. Because (1.4.5) is a pertubation of (1.2.1) with a potential term, $q^V\,(x,\,y,\,t)$ can be directly related to $q\,(x,\,y,\,t)$. To be precise, by Duhamel's perturbation formula, $q^V\,(x,\,y,\,t)$ is the solution to the integral equation

$$q^V(x,\,y,\,t) = q(x,\,y,\,t) + \int_0^t \int_{(0,\,\infty)} q(x,\,\xi,\,t-\tau)V(\xi)q^V(\xi,\,y,\,\tau)d\xi d\tau. \tag{1.4.6}$$

(1.4.6) can be obtained by considering

$$\Phi_{x,\,y,\,t}\,(\tau) \equiv \int_{(0,\,\infty)} q(x,\,\xi,\,t-\tau)q^V(\xi,\,y,\,\tau)d\xi.$$

Clearly, $\Phi_{x,\,y,\,t}\,(0) = q\,(x,\,y,\,t)$, $\Phi_{x,\,y,\,t}\,(t) = q^V\,(x,\,y,\,t)$, and because q and q^V are fundamental solutions to (1.2.1) and (1.4.5) respectively, $\frac{d}{d\tau}\Phi_{x,\,y,\,t}\,(\tau)$ equals

$$\begin{aligned}&-\frac{d}{dt}\left(\int_{(0,\,\infty)} q(x,\,\xi,\,t-\tau)q^V(\xi,\,y,\,\tau)d\xi\right) + \int_{(0,\,\infty)} q(x,\,\xi,\,t-\tau)\frac{d}{d\tau}q^V(\xi,\,y,\,\tau)d\xi \\ &= -\int_{(0,\,\infty)} q(x,\,\xi,\,t-\tau)\xi\partial_\xi^2 q^V(\xi,\,y,\,\tau)d\xi + \int_{(0,\,\infty)} q(x,\,\xi,\,t-\tau)\left(V\,(\xi) + \xi\partial_\xi^2\right)q^V(\xi,\,y,\,\tau)d\xi \\ &= \int_{(0,\,\infty)} q(x,\,\xi,\,t-\tau)V\,(\xi)\,q^V(\xi,\,y,\,\tau)d\xi,\end{aligned}$$

which leads immediately to (1.4.6). To solve (1.4.6), set $q_0^V \equiv q$, and

$$q_n^V(x,\,y,\,t) \equiv \int_0^t \int_{(0,\,\infty)} q(x,\,\xi,\,t-\tau)V(\xi)q_{n-1}^V(\xi,\,y,\,\tau)d\xi d\tau \quad \text{for } n \geq 1. \tag{1.4.7}$$

We summarize the results on q_n^V and the construction of q^V in the following lemma.

Lemma 1.2. *For each $n \geq 0$, q_n^V's definition is equivalent to*

$$q_{n+1}^V(x,\,y,\,t) = \int_0^t \int_{(0,\,\infty)} q_n^V(x,\,\xi,\,t-\tau)V(\xi)q(\xi,\,y,\,\tau)d\xi d\tau. \tag{1.4.8}$$

In addition, $q_n^V(x, y, t)$ *is smooth on* $(0, \infty)^3$,

$$|q_n^V(x, y, t)| \leq \frac{(\|V\|_u t)^n}{n!} q(x, y, t) \tag{1.4.9}$$

where $\|V\|_u$ *is the uniform norm of* V. *Set* $\bar{q}_n^V(x, y, t) \equiv y\, q_n^V(x, y, t)$, *then*

$$\bar{q}_n^V(x, y, t) = \bar{q}_n^V(y, x, t). \tag{1.4.10}$$

Furthermore, if

$$q^V(x, y, t) \equiv \sum_{n=0}^{\infty} q_n^V(x, y, t), \tag{1.4.11}$$

then $q^V(x, y, t)$ *solves (1.4.6). Finally,* $q^V(x, y, t)$ *is the fundamental solution to (1.4.5).*

Proof. We prove the statements about q_n^V by induction. When $n = 0$, they follow directly from the Chapman-Kolmogorov equation (1.2.10) satisfied by $q(x, y, t)$. Given the inductive hypothesis on $n-1$, the results for n can be verified using (1.4.7), (1.2.10) and Fubini's theorem. First,

$$\begin{aligned} q_n^V(x, y, t) &= \int_0^t \int_{(0,\infty)} q(x, \xi, t-\tau) V(\xi) q_{n-1}^V(\xi, y, \tau) d\xi d\tau \\ &= \int_0^t \int_{(0,\infty)} q(x, \xi, t-\tau) V(\xi) \int_0^\tau \int_{(0,\infty)} q_{n-2}^V(\xi, \eta, \tau - s) V(\eta) q(\eta, y, s) d\eta ds d\xi d\tau \\ &= \int_0^t \int_{(0,\infty)} q_{n-1}^V(x, \eta, t-s)\, V(\eta)\, q(\eta, y, s)\, d\eta ds. \end{aligned}$$

Next,

$$\begin{aligned} |q_n^V(x, y, t)| &\leq \int_0^t \int_{(0,\infty)} q(x, \xi, t-\tau)\, |V(\xi)|\, \left|q_{n-1}^V(\xi, y, \tau)\right| d\xi d\tau \\ &\leq \|V\|_u \int_0^t \int_{(0,\infty)} q(x, \xi, t-\tau) \frac{(\|V\|_u \tau)^{n-1}}{(n-1)!} q(\xi, y, \tau) d\xi d\tau \\ &= \frac{(\|V\|_u t)^n}{n!} q(x, y, t). \end{aligned}$$

In addition, by (1.2.2) and (1.4.8),

$$\begin{aligned}\bar{q}_n^V(x,\,y,\,t) &= \int_0^t \int_{(0,\infty)} \bar{q}(x,\,\xi,\,t-\tau)\xi^{-1}V(\xi)\bar{q}_{n-1}^V(\xi,\,y,\,\tau)d\xi d\tau \\ &= x\int_0^t \int_{(0,\infty)} q(\xi,\,x,\,t-\tau)\,V(\xi)q_{n-1}^V(y,\,\xi,\,\tau)d\xi d\tau \\ &= xq_n^V(y,x,t) = \bar{q}_n^V(y,\,x,\,t)\,.\end{aligned}$$

Finally, the smoothness of q_n^V follows directly from (1.4.7), (1.4.9) and the estimate (1.2.5) for q.

Assuming $q^V(x,\,y,\,t)$ is defined as in (1.4.11), one can easily check that $q^V(x,\,y,\,t)$ is continuous on $(0,\,\infty)^3$, q^V solves (1.4.6) and satisfies

$$\left|q^V(x,\,y,\,t)\right| \le e^{t\|V\|_u}q(x,\,y,\,t). \tag{1.4.12}$$

In addition, by (1.4.10), if $\bar{q}^V(x,\,y,\,t) \equiv yq^V(x,\,y,\,t)$, then

$$\bar{q}^V(x,\,y,\,t) = \bar{q}^V(y,\,x,\,t)\,, \tag{1.4.13}$$

which along with (1.4.6) and the properties of $q(x,\,y,\,t)$ implies the smoothness of $q^V(x,\,y,\,t)$ on $(0,\,\infty)^3$.

To verify that q^V is the fundamental solution to (1.4.5), we take $\varphi \in C_b((0,\,\infty)\,;\,\mathbb{R})$, and define

$$w_\varphi(x,\,t) \equiv \int_{(0,\infty)} \varphi(y)\,q^V(x,\,y,\,t)\,dy.$$

It is clear that $w_\varphi(0,\,t) = 0$ and because of (1.4.9),

$$\left|w_\varphi(x,\,t) - \int_{(0,\infty)} \varphi(y)\,q(x,\,y,\,t)\,dy\right| \to 0 \text{ as } t \searrow 0$$

uniformly for x in compact subsets of $(0,\,\infty)$. Therefore, by (1.2.6), $\lim_{t\searrow 0} w_\varphi(\cdot,\,t) = \varphi$ uniformly for x in compact subsets of $(0,\,\infty)$. What remains is to show that w_φ is a smooth solution to (1.4.5). Given the estimate (1.7.12) from §1.7, this is obvious, but here we will present another proof. Set

$$w_{\varphi}^{\epsilon}(x,t) \equiv \int_{(0,\,\infty)} q(x,\,y,\,\epsilon)w_{\varphi}(y,\,t)dy,$$

then as $\epsilon \searrow 0$, $w_{\varphi}^{\epsilon} \to w_{\varphi}$ on $(0,\,\infty)^2$. Furthermore, by (1.4.6) and (1.2.10), we have

$$w_{\varphi}^{\epsilon}(x,t) = \int_{(0,\,\infty)} \varphi\,(y)\,q\,(x,\,y,\,t+\epsilon)\,dy + \int_0^t \int_{(0,\,\infty)} q(x,\,\xi,\,t+\epsilon-\tau)V(\xi)w_{\varphi}(\xi,\,\tau)d\xi d\tau.$$

Therefore,

$$\begin{aligned}(x\partial_x^2 + V - \partial_t)w_{\varphi}^{\epsilon}(x,\,t) &= V\,(x)\,w_{\varphi}^{\epsilon}\,(x,\,t) - \int_{(0,\,\infty)} q(x,\,\xi,\,\epsilon)V(\xi)w_{\varphi}(\xi,\,t)d\xi \\ &= \int_{(0,\,\infty)} q(x,\,\xi,\,\epsilon)\,(V\,(x) - V\,(\xi))\,w_{\varphi}(\xi,\,t)d\xi \\ &\longrightarrow 0 \text{ uniformly for } x \text{in compact subsets as } \epsilon \searrow 0,\end{aligned}$$

which implies w_{φ} solves (1.4.5) in the sense of distributions and hence is a smooth, classical solution ([15]). □

1.5 Localization of q^{V}

In this section we will carry out a "localization" procedure for $q^V\,(x,\,y,\,t)$ similar to the one we did for $p\,(x,\,y,\,t)$ in §1.3. To start, we first examine the relation between $q^V\,(x,\,y,\,t)$ and the diffusion process $Y\,(t,\,x)$ as defined in section 2, which corresponds to the model equation (1.2.1). Denote by $\mathcal{B}_{(0,\,\infty)}$ the Borel σ-algebra on $(0,\,\infty)$, and for any $\Gamma \in \mathcal{B}_{(0,\,\infty)}$, set

$$Q^V\,(t,\,x,\,\Gamma) \equiv \mathbb{E}\left[e^{\int_0^t V(Y(\tau,\,x))d\tau},\,Y\,(t,\,x) \in \Gamma\right].$$

Given $\varphi \in C_b\,((0,\,\infty)\,;\,\mathbb{R})$, recall that $w_{\varphi}\,(x,\,t)$ is a smooth solution to (1.4.5), and furthermore, it is clear from (1.2.5) and (1.4.12) that w_{φ} is bounded on $(0,\,\infty) \times [0,\,t]$ for each $t > 0$. Applying Itô's formula, one sees that

$$\left(e^{\int_0^s V(Y(\tau,\,x))d\tau} w_{\varphi}\,(Y\,(s,\,x)\,,\,t-s)\,,\,\mathcal{F}_s,\,\mathbb{P}\right)$$

is a martingale for $s \in [0, t]$. Hence, at times $s = 0$ and $s = t$,

$$\begin{aligned} w_\varphi(x, t) &= \mathbb{E}\left[e^{\int_0^t V(Y(\tau, x))d\tau}\varphi(Y(t, x))\right] \\ &= \int_{(0, \infty)} \varphi(y) Q^V(t, x, dy). \end{aligned}$$

In other words, $q^V(x, y, t)$ is the density of the distribution $Q^V(t, x, dy)$, and hence $q^V(x, y, t) \geq 0$. Furthermore, by an elementary application of the Markov property of $Y(t, x)$,

$$Q^V(s+t, x, \Gamma) = \int_{(0, \infty)} Q^V(t, \xi, \Gamma) Q^V(s, x, d\xi) \quad \text{for } \Gamma \in \mathcal{B}_{(0,\infty)}.$$

As a consequence, q^V satisfies the Chapman-Kolmogorov equation

$$q^V(x, y, s+t) = \int_{(0,\infty)} q^V(x, \xi, s) q^V(\xi, y, t) d\xi \ \text{ for } (x, y) \in (0, \infty)^2 \text{ and } s, t > 0. \tag{1.5.1}$$

Given $\theta \in (0, \infty)$, the next step is to produce from $q^V(x, y, t)$ the fundamental solution $q^V_\theta(x, y, t)$ to

$$\begin{gathered} \partial_t w(x, t) = x\partial_x^2 w(x, t) + V(x)w(x, t) \text{ in } (0, \theta) \times (0,\infty) \\ \text{with } w(0, t) = 0 = w(0, \theta) \text{ and } \lim_{t \searrow 0} w(\cdot, t) = \varphi. \end{gathered} \tag{1.5.2}$$

To this end, set $\zeta^Y_\theta(x) \equiv \inf\{t \geq 0 : Y(t, x) = \theta\}$, and define

$$\begin{aligned} q^V_\theta(x, y, t) &\equiv q^V(x, y, t) \\ &\quad - \mathbb{E}\left[e^{\int_0^{\zeta^Y_\theta(x)} V(Y(\tau, x))d\tau} q^V(\theta, y, t - \zeta^Y_\theta(x)), \zeta^Y_\theta(x) \leq t\right] \end{aligned} \tag{1.5.3}$$

Lemma 1.3. *$q^V_\theta(x, y, t)$ is a non-negative, smooth function on $(0, \theta)^2 \times (0, \infty)$, and*

$$q^V_\theta(x, y, t) \leq e^{t\|V\|_u} q(x, y, t). \tag{1.5.4}$$

Moreoever, for any $(x, y) \in (0, \theta)^2$ and $s, t > 0$, q^V_θ satisfies

$$y q^V_\theta(x, y, t) = x q^V_\theta(y, x, t), \ \textit{and} \tag{1.5.5}$$

$$q_\theta^V(x,\,y,\,s+t)=\int_{(0,\,\theta)}q_\theta^V(x,\,\xi,\,s)q_\theta^V(\xi,\,y,\,t)\,d\xi. \tag{1.5.6}$$

Finally, for each $y\in(0,\,\theta)$, $(x,\,t)\mapsto q_\theta^V(x,\,y,\,t)$ solves (1.5.2) with $\varphi=\delta_y$.

Proof. First we notice from the Markov property of $Y(t,\,x)$ that for any $\varphi\in C_b((0,\,\theta)\,;\mathbb{R})$,

$$\begin{aligned}
&\mathbb{E}\left[e^{\int_0^t V(Y(\tau,x))d\tau}\varphi\left(Y\left(t,\,x\right)\right),\,\zeta_\theta^Y(x)\le t\right]\\
=&\mathbb{E}\left[e^{\int_0^{\zeta_\theta^Y(x)} V(Y(\tau,x))d\tau}e^{\int_0^{t-\zeta_\theta^Y(x)} V(Y(\tau,\theta))d\tau}\varphi\left(Y\left(t-\zeta_\theta^Y\left(x\right),\,\theta\right)\right),\,\zeta_\theta^Y(x)\le t\right]\\
=&\mathbb{E}\left[e^{\int_0^{\zeta_\theta^Y(x)} V(Y(\tau,x))d\tau}\int_{(0,\,\theta)}\varphi\left(y\right)q^V\left(\theta,\,y,\,t-\zeta_\theta^Y\left(x\right)\right)dy,\,\zeta_\theta^Y(x)\le t\right].
\end{aligned}$$

At the same time, recall that

$$\begin{aligned}
&\int_{(0,\,\theta)}\varphi\left(y\right)q^V\left(x,\,y,\,t\right)dy\\
=&\mathbb{E}\left[e^{\int_0^t V(Y(\tau,x))d\tau}\varphi\left(Y\left(t,\,x\right)\right)\right]\\
=&\mathbb{E}\left[e^{\int_0^t V(Y(\tau,x))d\tau}\varphi\left(Y\left(t,\,x\right)\right),\,\zeta_\theta^Y(x)>t\right]+\mathbb{E}\left[e^{\int_0^t V(Y(\tau,x))d\tau}\varphi\left(Y\left(t,\,x\right)\right),\,\zeta_\theta^Y(x)\le t\right].
\end{aligned}$$

Therefore, if q_θ^V is defined as in (1.5.3), then

$$\int_{(0,\,\theta)}\varphi\left(y\right)q_\theta^V\left(x,\,y,\,t\right)dy=\mathbb{E}\left[e^{\int_0^t V(Y(\tau,x))d\tau}\varphi\left(Y\left(t,\,x\right)\right),\,\zeta_\theta^Y(x)>t\right] \tag{1.5.7}$$

for all $\varphi\in C_b((0,\,\theta)\,;\mathbb{R})$, which implies that $q_\theta^V(x,\,y,\,t)$ is the density of the distribution

$$y\mapsto\mathbb{E}\left[e^{\int_0^t V(Y(\tau,x))d\tau},\;Y\left(t,\,x\right)\le y\text{ and }\zeta_\theta^Y\left(x\right)>t\right].$$

Hence, q_θ^V is non-negative on $(0,\,\theta)^2\times(0,\,\infty)$, and (1.5.4) follows directly from (1.4.12). (1.5.6) is another consequence of the Markov property of $Y(t,\,x)$. In fact, for any $\varphi\in C_b((0,\,\theta)\,;\mathbb{R})$,

$$\begin{aligned}
\int_{(0,\,\theta)}\varphi(y)q_\theta^V(x,\,y,\,s+t)dy=&\mathbb{E}\left[e^{\int_0^{s+t} V(Y(\tau,x))d\tau}\varphi\big(Y(s+t,\,x)\big),\,\zeta_\theta^Y(x)>s+t\right]\\
=&\mathbb{E}\left[e^{\int_0^s V(Y(\tau,x))d\tau}\int_{(0,\,\theta)}\varphi(y)q_\theta^V\big(Y(s,\,x),\,y,\,t\big)dy,\,\zeta_\theta^Y(x)>s\right]\\
=&\int_{(0,\,\theta)}\varphi(y)\left(\int_{(0,\,\theta)}q_\theta^V(\xi,\,y,\,t)q_\theta^V(x,\,\xi,\,s)d\xi\right)dy.
\end{aligned}$$

The proof of (1.5.5) requires some work. Notice that it is sufficient for us to show that for every $t > 0$,

$$\iint\limits_{(0,\theta)^2} \varphi_0(x) q_\theta^V(x, y, t)\varphi_1(y)\frac{dxdy}{x} = \iint\limits_{(0,\theta)^2} \varphi_0(x) q_\theta^V(y, x, t)\varphi_1(y)\frac{dxdy}{y}$$

for all $\varphi_0, \varphi_1 \in C_b((0, \theta); \mathbb{R}^+)$, which is equivalent to

$$\begin{aligned}\int_{(0,\theta)} &\mathbb{E}\left[\varphi_0(Y(0, x))\, e^{\int_0^t V(Y(\tau, x))d\tau}\varphi_1(Y(t, x)),\, \zeta_\theta^Y(x) > t\right]\frac{dx}{x}\\ &= \int_{(0,\theta)} \mathbb{E}\left[\varphi_0(Y(t, y))\, e^{\int_0^t V(Y(\tau, y))d\tau}\varphi_1(Y(0, y)),\, \zeta_\theta^Y(y) > t\right]\frac{dy}{y}.\end{aligned} \tag{1.5.8}$$

Set $\check{Y}(\tau, y) \equiv Y(t - \tau, y)$ for $\tau \in [0, t]$, and denote F the non-negative, Borel measurable mapping on $C([0, t]; \mathbb{R}^+)$ given by

$$\omega \in C\left([0, t]; \mathbb{R}^+\right) \mapsto F(\omega) \equiv \varphi_0(\omega(0))\, e^{\int_0^t V(\omega(\tau))d\tau}\varphi_1(\omega(t)),$$

then (1.5.8) is further equivalent to

$$\begin{aligned}\int_{(0,\infty)} &\mathbb{E}\left[F \circ \left(Y(\cdot, x) \restriction [0, t]\right),\, \zeta_\theta^Y(x) > t\right]\frac{dx}{x}\\ &= \int_{(0,\infty)} \mathbb{E}\left[F \circ \left(\check{Y}(\cdot, y) \restriction [0, t]\right),\, \zeta_\theta^Y(y) > t\right]\frac{dy}{y}.\end{aligned} \tag{1.5.9}$$

To prove (1.5.9), it suffices to show that it is true if F is of the form $F(\omega) = f_0(\omega(\tau_0)) \cdots f_n(\omega(\tau_n))$ for some $n \geq 1$, $f_0, \ldots, f_n \in C_c((0, \infty); [0, \infty))$ and $0 = \tau_1 < \cdots < \tau_n = t$. But this is just an application of the symmetry of $\bar{q}(x, y, t)$ as defined in (1.2.2) and again the Markov property of $Y(t, x)$. Hence, what still remains is to show that q_θ^V is a smooth solution to (1.5.2).

It is clear that for every $(y, t) \in (0, \theta) \times (0, \infty)$, $q_\theta^V(0, y, t) = 0 = q_\theta^V(\theta, y, t)$, and moreover, by (1.5.7),

$$\lim_{t \searrow 0} \int_{(0,\theta)} \varphi(y)\, q_\theta^V(x, y, t)\, dy = \varphi(x)$$

uniformly for x in compact subsets of $(0, \theta)$. Thus, due to (1.5.5), (1.5.6) and standard hypoellipticity results, we only need to show q_θ^V satisfies $\partial_t q_\theta^V(x, y, t) = \partial_y^2\left(y q_\theta^V(x, y, t)\right) +$

$V(y)\, q_\theta^V(x,\, y,\, t)$ in the sense of distributions. To see this, take $\varphi \in C_c^2((0,\, \theta)\,;\, \mathbb{R})$, and set $f(x) \equiv x\varphi''(x) + V(x)\,\varphi(x)$, then by Itô's formula and Doob's Stopping Time Theorem, for each $t > 0$,

$$e^{\int_0^{\tau\wedge t\wedge\zeta_\theta^Y(x)} V(Y(s,x))ds}\varphi\left(Y\left(\tau\wedge t\wedge\zeta_\theta^Y(x)\,,x\right)\right) \\ - \int_0^{\tau\wedge t\wedge\zeta_\theta^Y(x)} \left(e^{\int_0^s V(Y(p,x))dp} f\left(Y\left(s,x\right)\right)\right) ds$$

is a martingale with respect to $\{\mathcal{F}_\tau :\ 0 \le \tau \le t\}$ under $\mathbb{P}$. Hence,

$$\mathbb{E}\left[e^{\int_0^{t\wedge\zeta_\theta^Y(x)} V(Y(s,x))ds}\varphi\left(Y\left(t\wedge\zeta_\theta^Y(x)\right)\right)\right] \\ = \varphi(x) + \mathbb{E}\left[\int_0^{t\wedge\zeta_\theta^Y(x)} \left(e^{\int_0^s V(Y(p,x))dp} f\left(Y\left(s,\, x\right)\right)\right) ds\right],$$

and then

$$\int_{(0,\theta)} \varphi(y)\, q_\theta^V(x,\, y,\, t)\, dy = \varphi(x) + \int_0^t \left(\int_{(0,\theta)} f(y)\, q_\theta^V(x,\, y,\, \tau)\, dy\right) d\tau,$$

which proves the desired result. □

1.6 Back to Wright-Fisher Equation

We are now ready to return to the Wright-Fisher equation. Let ψ and b be the functions defined in (1.4.2) , and set $c(x) = \frac{b(x)}{x}$. Given $\beta \in (0,\, 1)$, let c_β be a smooth, compactly supported function which coincides with c on $(0,\, \psi(\beta)]$. For $(x,\, y,\, t) \in (0,\, \beta)^2 \times (0,\, \infty)$, define

$$p_\beta(x,\, y,\, t) = \frac{1}{y(1-y)} \frac{\psi(y)\, q_{\psi(\beta)}^V(\psi(x)\,,\, \psi(y)\,,\, t)}{\sqrt{\psi'(x)\,\psi'(y)}}, \tag{1.6.1}$$

where $V_\beta(x) = -x\left(\frac{c_\beta'(x)}{2} + \frac{c_\beta^2(x)}{4}\right)$. As an immediate consequence of (1.5.5) and (1.5.6) , for all $(x,\, y) \in (0,\, \beta)^2$ and $s,\, t > 0$, one has

$$y(1-y)\, p_\beta(x,\, y,\, t) = x(1-x)\, p_\beta(y,\, x,\, t)\,, \tag{1.6.2}$$

and p_β satisfies the Chapman-Kolmogorov equation

$$p_{\beta}(x,\, y,\, s+t) = \int_{(0,\,\beta)} p_\beta(\xi,\, y,\, t) p_\beta(x,\, \xi,\, s) d\xi. \tag{1.6.3}$$

As one would hope, p_β defined in (1.6.1) is indeed the p_β we introduced in §1.2. In fact, we have the following lemma:

Lemma 1.4. *For each $\beta \in (0,\, 1)$ and $\varphi \in C_c((0,\, \beta)\,;\, \mathbb{R})$,*

$$\mathbb{E}\big[\varphi(X(t,\, x)),\, \zeta_\beta^X(x) > t\big] = \int_{(0,\,\beta)} \varphi(y) p_\beta(x,\, y,\, t) dy \tag{1.6.4}$$

for $(x,\, t) \in (0,\, \beta)\times(0,\, \infty)$. Furthermore, if $0 < \alpha < \beta < \gamma < 1$, $\zeta_\gamma^X(x) \equiv \inf\{t \ge 0 :\ X(t,\, x) = \gamma\}$ and $\left\{\eta^X_{n,[\alpha,\beta]} :\, n \ge 0\right\}$ is defined as in (1.3.4), then

$$\begin{aligned} p_\gamma(x,\, y,\, t) &= p_\beta(x,\, y,\, t) \\ &\quad + \sum_{n=1}^{\infty} \mathbb{E}\left[p_\beta\big(\alpha,\, y,\, t - \eta^X_{2n,[\alpha,\beta]}(x)\big),\, \eta^X_{2n,[\alpha,\beta]}(x) < t \wedge \zeta_\gamma^X(x)\right] \end{aligned} \tag{1.6.5}$$

for $(x,\, y,\, t) \in (0,\, \alpha)^2 \times (0,\, \infty)$.

Proof. To prove (1.6.4), we first note that by (1.4.2), for $x \in (0,\, \beta)$, $c_\beta(x) = \frac{b(x)}{x} = \left(\frac{\psi''}{(\psi'^2)} \circ \psi^{-1}\right)(x)$. So, if $C_\beta(x) = \frac{1}{2}\int_0^x c_\beta(\xi)\, d\xi$, then

$$e^{-C_\beta(x)} = \exp\left(-\frac{1}{2}\int_0^x \frac{\psi''\left(\psi^{-1}(\xi)\right)}{\left(\psi'\left(\psi^{-1}(\xi)\right)\right)^2} d\xi\right) = \left(\psi' \circ \psi^{-1}\right)^{-\frac{1}{2}}. \tag{1.6.6}$$

Now we define

$$u(x,\, t) \equiv e^{-C_\beta \circ \psi} \int_{(0,\,\psi(\beta))} \tilde{\varphi}(\xi)\, q^{V_\beta}_{\psi(\beta)}\left(\psi(x),\, \xi,\, t\right) d\xi \tag{1.6.7}$$

where $\tilde{\varphi} = e^{C_\beta}\left(\varphi \circ \psi^{-1}\right)$, and set $w(x,\, t) \equiv u\left(\psi^{-1}(x),\, t\right)$. Then, by Lemma 1 and Lemma 3, $w(x,\, t)$ is a smooth solution to $\partial_t w(x,\, t) = x\partial_x^2 w(x,\, t) + xc(x)\,\partial_x w(x,\, t)$ on $(0,\, \psi(\beta)) \times (0,\, \infty)$, with boundary conditions $w(0,\, t) = 0 = w(\psi(\beta),\, t)$ and $\lim_{t\searrow 0} w(x,\, t) = e^{-C_\beta}\tilde{\varphi} = \varphi \circ \psi^{-1}$. Thus, $u(x,\, t)$ is a smooth solution to $\partial_t u(x,\, t) = x(1-x)\,\partial_x^2 u(x,\, t)$ on $(0,\, \beta) \times (0,\, \infty)$, with boundary conditions $u(0,\, t) = 0 = u(\beta,\, t)$ and $\lim_{t\searrow 0} u(x,\, t) = \varphi$. Hence, just as in the proof of (1.2.9), one can apply Itô's formula and Doob's Stopping Time Theorem to see that $u(x, t)$ equals to the left hand side of (1.6.4). On the other hand, by (1.4.1) and (1.6.6), the right hand side of (1.6.7) is also equal to

$$\begin{aligned}
&(\psi'(x))^{-\frac{1}{2}} \int_{(0,\,\psi(\beta))} \left(\psi'\left(\psi^{-1}\xi\right)\right)^{-\frac{1}{2}} \left(\varphi\left(\psi^{-1}(\xi)\right)\right) q^{V_\beta}_{\psi(\beta)}(\psi(x),\, \xi,\, t)\, d\xi \\
&= (\psi'(x))^{-\frac{1}{2}} \int_{(0,\,\beta)} (\psi'(y))^{\frac{1}{2}} \varphi(y)\, q^{V_\beta}_{\psi(\beta)}(\psi(x),\, \psi(y),\, t)\, dy \\
&= \int_{(0,\,\beta)} \varphi(y)\, p_\beta(x,\, y,\, t)\, dy,
\end{aligned}$$

which proves (1.6.4).

Given (1.6.4), the proof of (1.6.5) is essentially the same as that of (1.3.5). The only change is that one has to take into account the condition $\zeta^X_\gamma(x) > t$, but this causes no serious difficulty. □

The following estimates will be useful to us later .

Lemma 1.5. *Let* $0 < \alpha < \beta < 1$ *and* $t \in (0, 1]$. *Then*

$$\mathbb{P}\left(\zeta^Y_{\psi(\beta)}(\psi(x)) \le t\right) \le \frac{\psi(x)}{\psi(\alpha)} e^{-\frac{(\psi(\beta)-\psi(\alpha))^2}{4\psi(\beta)t}}$$

and

$$\mathbb{P}\left(\eta^X_{2n,[\alpha,\beta]}(x) \le t\right) \le \frac{x}{\alpha} e^{-4n^2\frac{(\beta-\alpha)^2}{t}} \quad \textit{for } n \ge 1.$$

Proof. Because $Y(\cdot, \psi(x))$ and $X(\cdot, x)$ get absorbed at 0 and, before they hit $\{0, 1\}$, are martingales, one knows that for $x \in (0, \alpha]$,

$$\mathbb{P}\left(\zeta^Y_{\psi(\alpha)}(\psi(x)) < \infty\right) = \frac{\psi(x)}{\psi(\alpha)} \text{ and } \mathbb{P}\left(\zeta^X_\alpha(x) < \infty\right) = \frac{x}{\alpha}.$$

Since $\alpha < \beta$ and therefore $\psi(\alpha) < \psi(\beta)$, then $Y(\cdot, \psi(x))$ has to hit $\psi(\alpha)$ before it hits $\psi(\beta)$. Furthermore, by the Markov property, given that $\zeta^Y_{\psi(\alpha)}(\psi(x)) < \infty$, $\zeta^Y_{\psi(\beta)}(\psi(x)) - \zeta^Y_{\psi(\alpha)}(\psi(x))$ is independent of $\zeta^Y_{\psi(\alpha)}(\psi(x))$ and has the same distribution as $\zeta^Y_{\psi(\beta)}(\psi(\alpha))$. Therefore,

$$\mathbb{P}\left(\zeta^Y_{\psi(\beta)}(\psi(x)) \le t\right) \le \frac{\psi(x)}{\psi(\alpha)} \mathbb{P}\left(\zeta^Y_{\psi(\beta)}(\psi(\alpha)) \le t\right).$$

Similarly,

$$\mathbb{P}\left(\eta^X_{2n,[\alpha,\beta]}(x) \le t\right) \le \frac{x}{\alpha} \mathbb{P}\left(\eta^X_{2n,[\alpha,\beta]}(\alpha) \le t\right).$$

To complete the first estimate, we use Itô's formula and Doob's Stopping Time Theorem to see that for any $\lambda \in \mathbb{R}$,

$$\left(\exp\left[\lambda Y\left(t \wedge \zeta^{Y}_{\psi(\beta)}(\psi(\alpha)),\, \psi(\alpha)\right) - \lambda^2 \int_0^{t\wedge\zeta^{Y}_{\psi(\beta)}(\psi(\alpha))} Y(s,\, \psi(\alpha))\, ds\right],\, \mathcal{F}_t,\, \mathbb{P}\right)$$

is a martingale, which implies,

$$\mathbb{E}\left[\exp\left(\lambda Y\left(t \wedge \zeta^{Y}_{\psi(\beta)}(\psi(\alpha)),\, \psi(\alpha)\right) - \lambda^2 \int_0^{t\wedge\zeta^{Y}_{\psi(\beta)}(\psi(\alpha))} Y(s,\, \psi(\alpha))\, ds\right)\right] = e^{\lambda\psi(\alpha)}.$$

Further, by Fatou's Lemma,

$$\begin{aligned}
&e^{\lambda\psi(\beta)}\mathbb{E}\left[e^{-\lambda^2\psi(\beta)\zeta^{Y}_{\psi(\beta)}(\psi(\alpha))}\right] \\
&\le \liminf_{t\to\infty} \mathbb{E}\left[\exp\left(\lambda Y\left(t \wedge \zeta^{Y}_{\psi(\beta)}(\psi(\alpha)),\, \psi(\alpha)\right) - \lambda^2 \int_0^{t\wedge\zeta^{Y}_{\psi(\beta)}(\psi(\alpha))} Y(s,\, \psi(\alpha))\, ds\right)\right].
\end{aligned}$$

Hence, we have

$$e^{\lambda\psi(\beta)}\mathbb{E}\left[e^{-\lambda^2\psi(\beta)\zeta^{Y}_{\psi(\beta)}(\psi(\alpha))}\right] \le e^{\lambda\psi(\alpha)} \quad \text{for all } \lambda \in \mathbb{R}.$$

Applying Markov's inequality, one sees that

$$\mathbb{P}\left(\zeta^{Y}_{\psi(\beta)}(\psi(\alpha)) \le t\right) \le \exp\left(\lambda^2\psi(\beta)\, t - \lambda\left(\psi(\beta) - \psi(\alpha)\right)\right),$$

which leads to the asserted estimate when $\lambda = \frac{\psi(\beta)-\psi(\alpha)}{2\psi(\beta)}$.

The argument for the second estimate is similar. For any $n \ge 0$, given that $\eta^{X}_{n,[\alpha,\beta]}(\alpha) < \infty$, $\eta^{X}_{n+1,[\alpha,\beta]}(\alpha) - \eta^{X}_{n,[\alpha,\beta]}(\alpha)$ is independent of $\eta^{X}_{n,[\alpha,\beta]}(\alpha)$, and has the same distribution as, depending on whether n is even or odd, $\zeta^{X}_{\beta}(\alpha)$ or $\zeta^{X}_{\alpha}(\beta)$. Hence, for $n \ge 1$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda^2\eta^{X}_{2n,[\alpha,\beta]}(\alpha)}\right] &= \mathbb{E}\left[\exp\left(-\lambda^2\sum_{m=0}^{2n-1}\left(\eta^{X}_{m+1,[\alpha,\beta]}(\alpha) - \eta^{X}_{m,[\alpha,\beta]}(\alpha)\right)\right)\right] \\
&= \left(\mathbb{E}\left[\exp\left(-\lambda^2\zeta^{X}_{\beta}(\alpha)\right)\right]\right)^n \left(\mathbb{E}\left[\exp\left(-\lambda^2\zeta^{X}_{\alpha}(\beta)\right)\right]\right)^n.
\end{aligned}$$

Next, if we take into account the fact that $x(1-x) \le \frac{1}{4}$ for all $x \in [0,\, 1]$, the same reasoning which we used above shows that

$$\mathbb{E}\left[e^{-\frac{\mu^2}{4}\zeta_\beta^X(\alpha)}\right] \le e^{\mu(\alpha-\beta)} \text{ and } \mathbb{E}\left[e^{-\frac{\lambda^2}{4}\zeta_\alpha^X(\beta)}\right] \le e^{\lambda(\beta-\alpha)} \text{ for all } \mu, \lambda \in \mathbb{R},$$

which leads to

$$\mathbb{E}\left[e^{-\frac{c^2}{4}\zeta_\beta^X(\alpha)}\right] \vee \mathbb{E}\left[e^{-\frac{c^2}{4}\zeta_\alpha^X(\beta)}\right] \le e^{-c(\beta-\alpha)} \text{ for all } c>0,$$

Finally,

$$\mathbb{P}\left(\eta_{2n,[\alpha,\beta]}^X(\alpha) \le t\right) \le \exp\left(\frac{c^2}{4}t - 2nc(\beta-\alpha)\right) \text{ for all } c>0,$$

and by taking $c = \frac{4n(\beta-\alpha)}{t}$, we have

$$\mathbb{P}\left(\eta_{2n,[\alpha,\beta]}^X(\alpha) \le t\right) \le \exp\left(-\frac{4n^2(\beta-\alpha)^2}{t}\right).$$

□

We now have got everything we need to make the construction of $p(x, y, t)$ outlined in §1.3, and are ready to summarize the main result in the following theorem.

Theorem 1.6. *There is a unique continuous function* $(x, y, t) \in (0, 1)^2 \times (0, \infty) \mapsto p(x, y, t) \in (0, \infty)$ *such that*

$$\int_{(0,1)} \varphi(y) p(x, y, t)\, dy = \mathbb{E}[\varphi(X(t, x))] \textit{ for } (x, t) \in (0, 1) \times (0, \infty), \tag{1.6.8}$$

where $\varphi \in C_b([0, 1]; \mathbb{R})$ *and* φ *vanishes on* $\{0, 1\}$*. Moreover,* $p(x, y, t)$ *satisfies*

$$p(x, y, t) = p(1-x, 1-y, t), \tag{1.6.9}$$

$$y(1-y) p(x, y, t) = x(1-x) p(y, x, t), \tag{1.6.10}$$

and the Chapman-Kolmogorov equation

$$p(x, y, s+t) = \int_{(0,1)} p(\xi, y, t) p(x, \xi, s) d\xi \quad \textit{for } x, y \in (0, 1)^2 \textit{ and } s, t > 0, \tag{1.6.11}$$

In addition, for each $0 < \alpha < \beta < 1$*,*

$$p(x, y, t) = p_\beta(x, y, t) + \sum_{n=1}^{\infty} \mathbb{E}\left[p_\beta\left(\alpha, y, t - \eta^X_{2n,[\alpha,\beta]}(x)\right), \eta^X_{2n,[\alpha,\beta]}(x) < t\right] \tag{1.6.12}$$

for all $(x, y, t) \in (0, \alpha)^2 \times (0, \infty)$*, where* $p_\beta(x, y, t)$ *is defined as in (1.6.1). Finally,* $p(x, y, t)$ *is smooth and, for each* $y \in (0, 1)$*, satisfies*

$$\partial_t p(x, y, t) = x(1-x)\,\partial_x^2 p(x, y, t) \text{ on } (0, 1) \times (0, \infty)$$
$$\text{with } p(0, y, t) = 0 = p(1, y, t) \text{ and } \lim_{t \searrow 0} p(\cdot, y, t) = \delta_y$$

Proof. For each $(x, t) \in (0, \alpha) \times (0, \infty)$, (1.6.5) implies that $\gamma \in (\beta, 1) \mapsto p_\gamma(x, \cdot, t)$ is non-decreasing and, from Lemma 5, it is clear that the family $\{p_\gamma(x, \cdot, t) : \beta < \gamma < 1\}$ is locally equicontinuous. Hence, there exists a unique continuous function $y \in (0, 1) \mapsto p(x, y, t) \in (0, \infty)$ such that $p_\gamma(x, y, t) \nearrow p(x, y, t)$ as $\gamma \nearrow 1$.

Now it is clear that (1.6.8) is a direct consequence of (1.6.4) and the Dominated Convergence Theorem, and, from (1.6.5) and the Monotone Convergence Theorem,

$$\begin{aligned}
p(x, y, t) &= \lim_{\gamma \nearrow 1} p_\gamma(x, y, t) \\
&= p_\beta(x, y, t) + \lim_{\gamma \nearrow 1} \sum_{n=1}^{\infty} \mathbb{E}\left[p_\beta\left(\alpha, y, t - \eta^X_{2n,[\alpha,\beta]}(x)\right), \eta^X_{2n,[\alpha,\beta]}(x) < t \wedge \zeta^X_\gamma(x)\right] \\
&= p_\beta(x, y, t) + \sum_{n=1}^{\infty} \lim_{\gamma \nearrow 1} \mathbb{E}\left[p_\beta\left(\alpha, y, t - \eta^X_{2n,[\alpha,\beta]}(x)\right), \eta^X_{2n,[\alpha,\beta]}(x) < t \wedge \zeta^X_\gamma(x)\right] \\
&= p_\beta(x, y, t) + \sum_{n=1}^{\infty} \mathbb{E}\left[p_\beta\left(\alpha, y, t - \eta^X_{2n,[\alpha,\beta]}(x)\right), \eta^X_{2n,[\alpha,\beta]}(x) < t\right].
\end{aligned}$$

In addition, one can easily check that (1.6.11) and (1.6.10) follow from (1.6.3) and (1.6.2) respectively. Since $1 - X(t, x)$ has the same distribution as $X(t, 1-x)$, one knows that (1.6.9) holds.

Knowing (1.6.10), (1.6.11) and (1.6.8), the continuity of $y \mapsto p(x, y, t)$ leads to the continuity of $(x, y, t) \mapsto p(x, y, t)$. Finally, using the same sort of reasoning which we used in proving the smoothness of q^V_θ in Lemma 3, we can show that $p(x, y, t)$ is smooth and solves the desired equation. □

We now have completed the construction of $p(x, y, t)$. Our next objective is to "transfer" to $p(x, y, t)$ the properties of the fundamental solution to the model equation (1.2.1). We will achieve this by "comparing" $\bar{p}(x, y, t)$ with $\bar{q}(\psi(x), \psi(y), t)$. We start with the following lemma.

Lemma 1.7. *Set* $\bar{p}(x, y, t) \equiv y(1-y)p(x, y, t)$*, and for* $\beta \in (0, 1)$, $V_\beta(x) \equiv -x\left(\frac{c_\beta'(x)}{2} + \frac{c_\beta^2(x)}{4}\right)$. *Then, for each* $0 < \alpha < \beta < 1$ *and* $\rho \in (0, 1)$,

$$\begin{aligned} &\left|\sqrt{\psi'(x)\psi'(y)}\bar{p}(x, y, t) - \bar{q}^{V_\beta}(\psi(x), \psi(y), t)\right| \\ &\qquad \leq \frac{K(\alpha, \beta, \rho)e^{t\|V_\beta\|_u}}{\beta - \alpha}\psi(x)\psi(y)e^{-\frac{\mu_\beta(\beta-\alpha)^2}{t}} \end{aligned} \tag{1.6.13}$$

for all $(x, y, t) \in (0, \rho\alpha]^2 \times (0, 1]$*, where*

$$\mu_\beta = \left(\frac{1 + 4\psi(\beta)}{4\psi(\beta)}\right) \wedge 4, \text{ and } K(\alpha, \beta, \rho) = \frac{2^{\frac{1}{2}}\pi^{\frac{3}{2}}\psi(\beta)}{\psi(\alpha)\alpha^4(1-\rho)^4}.$$

If in addition, $\left|\sqrt{\psi(y)} - \sqrt{\psi(x)}\right| \leq \beta - \alpha$*, then*

$$\begin{aligned} &\left|\frac{\sqrt{\psi'(x)\psi'(y)}\bar{p}(x, y, t)}{\bar{q}^{V_\beta}(\psi(x), \psi(y), t)} - 1\right| \\ &\qquad \leq \frac{K(\alpha, \beta, \rho)e^{t\|V_\beta\|_u}}{\delta_0(\beta - \alpha)}\left(e^2t^2 \vee (\psi(x)\psi(y))\right)e^{-\frac{(\mu_\beta - 1)(\beta-\alpha)^2}{t}}, \end{aligned} \tag{1.6.14}$$

where δ_0 *is chosen as in (1.2.7).*

Proof. Because, by (1.6.12),

$$\bar{p}(x, y, t) = \bar{p}_\beta(x, y, t) + \sum_{n=1}^{\infty}\mathbb{E}\left[\bar{p}_\beta\big(\alpha, y, t - \eta^X_{2n,[\alpha,\beta]}(x)\big), \eta^X_{2n,[\alpha,\beta]}(x) < t\right],$$

we know that $\sqrt{\psi'(x)\psi'(y)}\bar{p}(x, y, t)$ equals

$$\begin{aligned} &\bar{q}^{V_\beta}_{\psi(\beta)}(\psi(x), \psi(y), t) \\ &\qquad + \sqrt{\frac{\psi'(x)}{\psi'(\alpha)}}\sum_{n=1}^{\infty}\mathbb{E}\left[\bar{q}^{V_\beta}_{\psi(\beta)}\left(\psi(\alpha), \psi(y), t - \eta^X_{2n,[\alpha,\beta]}(x)\right), \eta^X_{2n,[\alpha,\beta]}(x) < t\right]. \end{aligned}$$

Thus, $\sqrt{\psi'(x)\psi'(y)}\bar{p}(x,\,y,\,t)$ is dominated by

$$\bar{q}^{V_\beta}\left(\psi(x),\,\psi(y),\,t\right)+\sqrt{\frac{\psi'(x)}{\psi'(\alpha)}}\sup_{\tau\in(0,\,t]}\bar{q}^{V_\beta}_{\psi(\beta)}\left(\psi(\alpha),\,\psi(y),\,\tau\right)\sum_{n=1}^{\infty}\mathbb{P}\left(\eta^X_{2n,[\alpha,\beta]}(x)<t\right)$$

and $\sqrt{\psi'(x)\psi'(y)}\bar{p}(x,\,y,\,t)$ dominates $\sqrt{\psi'(x)\psi'(y)}\bar{p}_\beta(x,\,y,\,t)=\bar{q}^{V_\beta}_{\psi(\beta)}\left(\psi(x),\,\psi(y),\,t\right)$, which by Lemma 3, is equal to

$$\bar{q}^{V_\beta}\left(\psi(x),\,\psi(y),\,t\right)-\mathbb{E}\Big[e^{\int_0^{\zeta^Y_{\psi(\beta)}(\psi(x))}V(Y(\tau,\psi(x)))d\tau}$$
$$\times\,\bar{q}^{V_\beta}\left(\psi(\beta),\,\psi(y),\,t-\zeta^Y_{\psi(\beta)}\left(\psi(x)\right)\right),\,\zeta^Y_{\psi(\beta)}(\psi(x))<t\Big].$$

By the second part of Lemma 5,

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}\left(\eta^X_{2n,[\alpha,\beta]}(x)<t\right) &\leq \sum_{n=1}^{\infty}\frac{x}{\alpha}e^{-4n^2\frac{(\beta-\alpha)^2}{t}}\\ &\leq \frac{x}{\alpha}\left(1+\frac{\sqrt{\pi t}}{4(\beta-\alpha)}\right)e^{-\frac{4(\beta-\alpha)^2}{t}} \qquad (1.6.15)\\ &\leq \frac{2x}{\alpha(\beta-\alpha)}e^{-\frac{4(\beta-\alpha)^2}{t}}.\end{aligned}$$

In addition, when $\xi\in(0,\,1)$,

$$(1-\xi)^{\frac{1}{2}}\,\psi'\left(\xi\right)=\xi^{-\frac{1}{2}}\left(\arcsin\sqrt{\xi}\right)\in\left[1,\,\frac{\pi}{2}\right],$$

so

$$\frac{\psi'\left(x\right)}{\psi'\left(\alpha\right)}\leq\frac{\pi}{2}\frac{\sqrt{1-\alpha}}{\sqrt{1-x}}\leq\frac{\pi}{2}\text{ for }x\in(0,\,\rho\alpha].$$

Thus, the right hand side of the upper bound can be replaced by

$$\bar{q}^{V_\beta}\left(\psi(x),\,\psi(y),\,t\right)+\sqrt{\frac{\pi}{2}}\sup_{\tau\in(0,\,t]}\bar{q}^{V_\beta}_{\psi(\beta)}\left(\psi(\alpha),\,\psi(y),\,\tau\right)\frac{2x}{\alpha(\beta-\alpha)}e^{-\frac{4(\beta-\alpha)^2}{t}}.$$

Meanwhile, because

$$\left(\sqrt{\psi\left(\xi\right)}\right)'=\left(\arcsin\sqrt{\xi}\right)'=\frac{1}{2\sqrt{\xi(1-\xi)}}\geq 1,$$

the upper bounds in (1.2.5) and (1.4.12) lead to

$$\begin{aligned}\bar{q}^{V_\beta}_{\psi(\beta)}\left(\psi(\alpha),\,\psi(y),\,\tau\right) &\le e^{t\|V_\beta\|_u}\frac{\psi(\alpha)\psi(y)}{\tau^2}e^{-\frac{(\sqrt{\psi(y)}-\sqrt{\psi(\alpha)})^2}{\tau}}\\ &\le e^{t\|V_\beta\|_u}\sup_{s\in(0,t]}\frac{\psi(\alpha)\psi(y)}{s^2}e^{-\frac{\alpha^2(1-\rho)^2}{s}}\\ &\le e^{t\|V_\beta\|_u}\frac{\psi(\alpha)\psi(y)}{\alpha^4(1-\rho)^4}\end{aligned}$$

for all $\tau\in(0,\,t]$. Summarizing the preceding, one has

$$\sqrt{\psi'(x)\psi'(y)}\bar{p}(x,\,y,\,t)\le\bar{q}^{V_\beta}\left(\psi(x),\,\psi(y),\,t\right)+\frac{\sqrt{2\pi}x\psi(\alpha)\psi(y)e^{t\|V_\beta\|_u}}{\alpha^5(1-\rho)^4(\beta-\alpha)}e^{-\frac{4(\beta-\alpha)^2}{t}},$$

and, to achieve the upper bound in (1.6.13), one just needs to notice that $\xi\le\psi\left(\xi\right)\le\pi\xi$ for $\xi\in[0,\,1]$.

To prove the lower bound in (1.6.13), it is clear that one needs to estimate

$$\mathbb{E}\left[e^{\int_0^{\zeta^Y_{\psi(\beta)}(\psi(x))}V(Y(\tau,\psi(x)))d\tau}\bar{q}^{V_\beta}\left(\psi(\beta),\,\psi(y),\,t-\zeta^Y_{\psi(\beta)}\left(\psi(x)\right)\right),\,\zeta^Y_{\psi(\beta)}(\psi(x))<t\right],$$

which, by Lemma 5, (1.4.12) and (1.2.5), is dominated by

$$\begin{aligned}&\frac{\psi(x)e^{t\|V_\beta\|_u}}{\psi(\alpha)}e^{-\frac{(\psi(\beta)-\psi(\alpha))^2}{4\psi(\beta)t}}\sup_{\tau\in(0,\,t]}\bar{q}\left(\psi(\beta),\,\psi(y),\,\tau\right)\\ &\le e^{t\|V_\beta\|_u}e^{-\frac{(\psi(\beta)-\psi(\alpha))^2}{4\psi(\beta)t}}\frac{\psi(x)\psi(y)\psi(\beta)}{\psi(\alpha)}\sup_{\tau\in(0,\,t]}\frac{\exp\left(-\frac{(\sqrt{\psi(\beta)}-\sqrt{\psi(y)})^2}{\tau}\right)}{\tau^2}\\ &\le e^{t\|V_\beta\|_u}e^{-\frac{(\beta-\alpha)^2}{4\psi(\beta)t}}\frac{\psi(x)\psi(y)\psi(\beta)}{\psi(\alpha)}e^{-\frac{(\beta-\alpha)^2}{t}}\sup_{\tau\in(0,\,t]}\frac{\exp\left(-\frac{(\alpha-\alpha\rho)^2}{\tau}\right)}{\tau^2}\\ &\le\frac{\psi(x)\psi(y)\psi(\beta)e^{t\|V_\beta\|_u}}{\psi(\alpha)\alpha^4(1-\rho)^4}\exp\left(-\frac{(1+4\psi(\beta))(\beta-\alpha)^2}{4\psi(\beta)t}\right).\end{aligned}$$

The second inequality comes from the facts that $\psi'\ge1$ and $\left(\sqrt{\psi\left(\beta\right)}-\sqrt{\psi\left(y\right)}\right)^2\ge(\alpha-\rho\alpha)^2+(\beta-\alpha)^2$. Hence, we have proved the (1.6.13).

Given (1.6.13), to prove an estimate of the sort of (1.6.14), we only need to get a lower bound for $\bar{q}^{V_\beta}\left(\psi\left(x\right),\,\psi\left(y\right),\,t\right)$. To this end, recalling from (1.2.4) , (1.2.7) and the construction of q^{V_β}, one has $\bar{q}^{V_\beta} \geq \bar{q}$, and

$$\bar{q}\left(\psi(x),\, \psi(y),\, t\right) \geq \begin{cases} \frac{\psi(x)\psi(y)}{t^2} e^{-\frac{\psi(x)+\psi(y)}{t}} & \text{if } \psi(x)\psi(y) \leq t^2, \\ \delta_0 \frac{\psi(x)^{\frac{1}{4}}\psi(y)^{\frac{1}{4}}}{t^{\frac{1}{2}}} e^{-\frac{\left(\sqrt{\psi(y)}-\sqrt{\psi(x)}\right)^2}{t}} & \text{if } \psi(x)\psi(y) \geq t^2. \end{cases}$$

Furthermore, when $\psi(x)\psi(y) \leq t^2$,

$$e^{-\frac{\psi(x)+\psi(y)}{t}} = e^{-\frac{(\sqrt{\psi(y)}-\sqrt{\psi(x)})^2}{t}} e^{-\frac{2\sqrt{\psi(x)\psi(y)}}{t}} \geq e^{-2} e^{-\frac{\left(\sqrt{\psi(y)}-\sqrt{\psi(x)}\right)^2}{t}},$$

and so

$$\begin{aligned} &\left| \frac{\sqrt{\psi'(x)\psi'(y)}\bar{p}(x,\, y,\, t)}{\bar{q}^{V_\beta}\left(\psi(x),\, \psi(y),\, t\right)} - 1 \right| \\ &\quad \leq \frac{K(\alpha,\, \beta,\, \rho) e^{t\|V_\beta\|_u}}{\beta - \alpha} e^2 t^2 \exp\left[-\frac{\mu_\beta(\beta-\alpha)^2}{t} + \frac{\left(\sqrt{\psi(y)} - \sqrt{\psi(x)}\right)^2}{t} \right]. \end{aligned}$$

When $\psi(x)\psi(y) \geq t^2$,

$$\begin{aligned} &\left| \frac{\sqrt{\psi'(x)\psi'(y)}\bar{p}(x,\, y,\, t)}{\bar{q}^{V_\beta}\left(\psi(x),\, \psi(y),\, t\right)} - 1 \right| \\ &\quad \leq \frac{K(\alpha,\, \beta,\, \rho) e^{t\|V_\beta\|_u}}{\delta_0\left(\beta - \alpha\right)} \psi\left(x\right) \psi\left(y\right) \exp\left[-\frac{\mu_\beta(\beta-\alpha)^2}{t} + \frac{\left(\sqrt{\psi(y)} - \sqrt{\psi(x)}\right)^2}{t} \right]. \end{aligned}$$

Taking into account the condition that $\left|\sqrt{\psi\left(y\right)} - \sqrt{\psi\left(x\right)}\right| \leq \beta - \alpha$, one sees that (1.6.14) follows from above. □

Finally, we state the estimates on $\bar{p}\left(x,\, y,\, t\right)$ in terms of $\bar{q}\left(\psi\left(x\right),\, \psi\left(y\right),\, t\right)$ in the following theorem.

Theorem 1.8. *For each* $0 < \alpha < \beta < 1$ *and* $\rho \in (0,\, 1)$*,*

$$\begin{aligned} &\left|\sqrt{\psi'(x)\psi'(y)}\bar{p}(x,\, y,\, t) - \bar{q}\left(\psi(x),\, \psi(y),\, t\right)\right| \\ &\leq e^{t\|V_\beta\|_u} \left\{ t\left\|V_\beta\right\|_u \bar{q}\left(\psi\left(x\right),\, \psi\left(y\right),\, t\right) + \frac{K(\alpha,\, \beta,\, \rho)}{\beta - \alpha} \psi(x)\psi(y) e^{-\frac{\mu_\beta(\beta-\alpha)^2}{t}} \right\} \end{aligned} \tag{1.6.16}$$

for all $(x, y, t) \in (0, \rho\alpha]^2 \times (0, 1]$, *where* μ_β *and* $K(\alpha, \beta, \rho)$ *are defined as in Lemma 7. If in addition,* $\left|\sqrt{\psi(y)} - \sqrt{\psi(x)}\right| \leq \beta - \alpha$, *then*

$$\begin{aligned}
&\left|\frac{\sqrt{\psi'(x)\psi'(y)}\bar{p}(x, y, t)}{\bar{q}(\psi(x), \psi(y), t)} - 1\right| \\
&\leq e^{t\|V_\beta\|_u}\left\{t\|V_\beta\|_u + \frac{K(\alpha, \beta, \rho)}{\delta_0(\beta-\alpha)}\left(e^2t^2 \vee (\psi(x)\psi(y))\right)e^{-\frac{(\mu_\beta-1)(\beta-\alpha)^2}{t}}\right\}.
\end{aligned} \tag{1.6.17}$$

Proof. As in Lemma 7, (1.6.17) is a direct consequence of (1.6.16) and the lower bound on $\bar{q}(\psi(x), \psi(y), t)$ that we used in the proof of Lemma 7. To show (1.6.16), it is sufficient to notice that

$$\begin{aligned}
\left|\bar{q}^{V_\beta}(\xi, \eta, t) - \bar{q}(\xi, \eta, t)\right| &\leq \left(\sum_{n=1} \frac{(t\|V_\beta\|_u)^n}{n!}\right)\bar{q}(\xi, \eta, t) \\
&\leq t\|V_\beta\|_u e^{t\|V_\beta\|_u}\bar{q}(\xi, \eta, t),
\end{aligned}$$

which follows trivially from (1.4.9) and (1.4.11). □

1.7 Derivatives

Our goal in this section is to get regularity estimates on $p(x, y, t)$, and we will start by examining the derivatives of $q(x, y, t)$. To this end, one should first notice that if u is a solution to (1.2.1) and for integer $m \geq 1$, $u^{(m)}$ is its mth derivative with respect to x, then $u^{(m)}$ satisfies

$$\partial_t w(x, t) = x\partial_x^2 w(x, t) + m\partial_x w(x, t). \tag{1.7.1}$$

Thus, we should expect that

$$u^{(m)}(x, t) = \int_{(0,\infty)} \varphi^{(m)}(y)\, q^{(m)}(x, y, t)\, dy,$$

where $\varphi^{(m)}$ is the mth derivative of the initial data φ and $q^{(m)}(x, y, t)$ is the fundamental solution to (1.7.1).

Next, in order to learn more about $q^{(m)}(x, y, t)$, we will adopt the same approach as we used in studying $q(x, y, t)$ in §1.2. We first notice that if $\varphi \in C_c^\infty((0, \infty); \mathbb{R})$, then

$$\left(\frac{d}{dx}\right)^m \int_{(0,\infty)} \varphi(y)\, q(x, y, t)\, dy = \int_{(0,\infty)} \varphi^{(m)}(y)\, q^{(m)}(x, y, t)\, dy, \tag{1.7.2}$$

which suggests that $q^{(m)}(x, y, t)$ satisfies

$$\begin{aligned}
\partial_y^m q^{(m)}(x, y, t) &= (-1)^m \partial_x^m q(x, y, t) \\
&= (-1)^m e^{-\frac{y}{t}} \partial_x^m \left(e^{-\frac{x}{t}} \sum_{n=1} \frac{x^n y^{n-1}}{t^{2n} n!\,(n-1)!} \right) \\
&= \sum_{k=0}^m \binom{m}{k} e^{-\frac{x+y}{t}} (-1)^k \sum_{n=k} \frac{x^{n-k} y^{n-1}}{t^{2n+m-k}(n-1)!\,(n-k)!} \\
&= \sum_{k=0}^m \binom{m}{k} e^{-\frac{x+y}{t}} (-1)^k \sum_{j=0} \frac{x^j y^{j+k-1}}{t^{2j+k+m} j!\,(j+k-1)!} \\
&= \partial_y^m \left(e^{-\frac{x+y}{t}} \sum_{j=0} \frac{x^j y^{j+m-1}}{t^{2j+m} j!\,(j+m-1)!} \right).
\end{aligned} \tag{1.7.3}$$

Taking a hint from the above, we will take $q^{(m)}(x, y, t)$ to be

$$q^{(m)}(x, y, t) \equiv \frac{y^{m-1} e^{-\frac{x+y}{t}}}{t^m} q^{(m)}\left(\frac{xy}{t^2}\right), \text{ where } q^{(m)}(\xi) = \sum_{n=0}^\infty \frac{\xi^n}{n!\,(n+m-1)!}, \tag{1.7.4}$$

and show that $q^{(m)}(x, y, t)$ is indeed the fundamental solution to (1.7.1). It is clear that for any $y \in (0, \infty)$, $q^{(m)}(\cdot, y, \star)$ is a solution to (1.7.1) on $(0, \infty)^2$. Moreover, for any $(x, t) \in (0, \infty)^2$,

$$\begin{aligned}
\int_{(0,\infty)} q^{(m)}(x, y, t)\, dy &= \frac{e^{-\frac{x}{t}}}{t^m} \sum_{n=0} \frac{x^n t^{-2n}}{n!\,(n+m-1)!} \left(\int_{(0,\infty)} e^{-\frac{y}{t}} y^{n+m-1} dy \right) \\
&= \frac{e^{-\frac{x}{t}}}{t^m} \sum_{n=0} \frac{x^n t^{-2n}}{n!\,(n+m-1)!} t^{n+m} (n+m-1)! \\
&= 1.
\end{aligned}$$

From (1.2.4) and (1.2.5), one knows that

$$0 \leq q^{(m)}(x, y, t) \leq \frac{y^{m-1}}{t^m}\left(\frac{e^{-\frac{x+y}{t}}}{(m-1)!} + \bar{q}(x, y, t)\right) \\ \leq \frac{y^{m-1}}{t^m}\left(1+\frac{xy}{t^2}\right)e^{-\frac{(\sqrt{y}-\sqrt{x})^2}{t}} \tag{1.7.5}$$

for all $(x, y, t) \in (0, \infty)^3$. Thus, for any $\epsilon > 0$,

$$\int_{(0,\infty)\setminus(x-\epsilon,\, x+\epsilon)} q^{(m)}(x, y, t)\, dy \to 0 \text{ as } t \searrow 0$$

uniformly for x in compact subsets of $(0, \infty)$.

As with $q(x, y, t)$, when $xy \geq t^2$, we can improve the estimates in (1.7.5). In fact, because

$$q^{(m)}(\xi) = \xi^{-\frac{m-1}{2}} I_{m-1}(\xi)$$

where I_{m-1} is the Bessel function of order $m-1$ with purely imaginary argument, then by the results from [19], there exists $\delta_m \in (0, 1)$ such that

$$\delta_m \xi^{\frac{1}{4}-\frac{m}{2}} e^{2\sqrt{\xi}} \leq q^{(m)}(\xi) \leq \delta_m^{-1} \xi^{\frac{1}{4}-\frac{m}{2}} e^{2\sqrt{\xi}} \quad \text{for } \xi \geq 1.$$

Hence, when $xy \geq t^2$,

$$\delta_m \frac{y^{m-1}(xy)^{\frac{1}{4}-\frac{m}{2}}}{\sqrt{t}} e^{-\frac{(\sqrt{y}-\sqrt{x})^2}{t}} \leq q^{(m)}(x, y, t) \leq \delta_m^{-1} \frac{y^{m-1}(xy)^{\frac{1}{4}-\frac{m}{2}}}{\sqrt{t}} e^{-\frac{(\sqrt{y}-\sqrt{x})^2}{t}}. \tag{1.7.6}$$

In addition, an immediate consequence of (1.7.4) is

$$\frac{y^{m-1}e^{-\frac{x+y}{t}}}{t^m(m-1)!} \leq q^{(m)}(x, y, t) \leq \frac{y^{m-1}e^{-\frac{x+y}{t}}}{t^m} \cdot e \text{ for } xy \leq t^2. \tag{1.7.7}$$

Next, we will state a generalization of (1.7.2) in the following lemma.

Lemma 1.9. *Suppose that* $\varphi \in C^m((0, \infty); \mathbb{R})$, *set* $\varphi^{(l)} \equiv \partial_x^l \varphi$, *and assume* $\varphi^{(l)}$ *is bounded on* $(0, 1)$ *and has sub-exponential growth at* ∞ *for each* $0 \leq l \leq m$. *If* $\lim_{x \searrow 0} \varphi(x) = 0$, *then*

$$\partial_x^m \int_{(0,\infty)} \varphi(y) q(x, y, t) dy = \int_{(0,\infty)} \varphi^{(m)}(y) q^{(m)}(x, y, t) dy. \tag{1.7.8}$$

Moreover, for any $l \geq 1$,

$$\partial_x^m \int_{(0,\infty)} \varphi(y)q^{(\ell)}(x, y, t)dy = \int_{(0,\infty)} \varphi^{(m)}(y)q^{(m+\ell)}(x, y, t)dy \tag{1.7.9}$$

even if φ *does not vanish at* 0.

Proof. We notice that for any φ that satisfies our condition, we can find $\Phi \in C^{m+l}((0, \infty); \mathbb{R})$ such that $\Phi^{(l)} = \varphi$ and $\lim_{x \searrow 0} \Phi(x) = 0$. Hence, if (1.7.8) is true, then the right hand side of (1.7.9) is equal to

$$\begin{aligned} \partial_x^m \int_{(0,\infty)} \Phi^{(l)}(y)\, q^{(l)}(x, y, t)\, dy &= \partial_x^{m+l} \int_{(0,\infty)} \Phi(y)\, q(x, y, t)\, dy \\ &= \int_{(0,\infty)} \varphi^{(m)}(y)q^{(m+\ell)}(x, y, t)dy. \end{aligned}$$

Thus, we only need to prove (1.7.8). But it is clear from the computation in (1.7.3) that (1.7.8) holds if $\varphi \in C_c^m((0, \infty); \mathbb{R})$. Given a $\varphi \in C^m((0, \infty); \mathbb{R})$ that vanishes at 0, by (1.7.6) and (1.7.7), without loss of generality, we can assume φ is supported on a compact subset of $[0, \infty)$. For $\epsilon > 0$, set $\varphi_\epsilon(y) \equiv \chi_{[2\epsilon, \infty)}(y)\, \varphi(y - 2\epsilon)$, choose $\rho \in C_c^\infty((-1, 1); \mathbb{R}^+)$ with total integral 1, and set $\rho_\epsilon \equiv \frac{1}{\epsilon}\rho\left(\frac{\cdot}{\epsilon}\right)$. Then $\varphi_\epsilon \star \rho_\epsilon \in C_c^\infty((0, \infty); \mathbb{R})$, and so

$$\partial_x^m \int_{(0,\infty)} \varphi_\epsilon \star \rho_\epsilon(y)q(x, y, t)dy = \int_{(0,\infty)} (\varphi_\epsilon \star \rho_\epsilon)^{(m)}(y)q^{(m)}(x, y, t)dy$$

for each $(x, t) \in (0, \infty)^2$. Clearly, the left hand side of above tends to the left hand side of (1.7.8) as $\epsilon \searrow 0$, and the right hand side can be written as

$$\int_{(0, 3\epsilon]} \rho_\epsilon^{(m)} \star \varphi_\epsilon(y)q^{(m)}(x, y, t)dy + \int_{(3\epsilon,\infty)} \rho_\epsilon \star \varphi_\epsilon^{(m)}(y)q^{(m)}(x, y, t)dy.$$

As $\epsilon \searrow 0$, the first term satisfies

$$\int_{(3\epsilon,\infty)} \rho_\epsilon \star \varphi^{(m)}(y - 2\epsilon)q^{(m)}(x, y, t)dy \longrightarrow \int_{(0,\infty)} \varphi^{(m)}(y)q^{(m)}(x, y, t)dy.$$

Meanwhile, by (1.7.5), the second term is bounded by some contant times

$$\begin{aligned}
&\int_{(-\epsilon,\,\epsilon)}\left|\rho_\epsilon^{(m)}(\xi)\right|\int_{[\xi+2\epsilon,\,3\epsilon]}|\varphi(y-\xi-2\epsilon)|\,y^{m-1}dyd\xi\\
\le&\int_{(-1,\,1)}\left|\rho^{(m)}(\eta)\right|d\eta\cdot\int_{[0,\,2\epsilon]}|\varphi(y)|\left(\frac{y}{\epsilon}+3\right)^{m-1}\epsilon^{-1}dy\\
\le&\int_{(-1,\,1)}\left|\rho^{(m)}(\eta)\right|d\eta\cdot\int_{[0,\,2]}|\varphi(\epsilon y)|\,(y+3)^{m-1}\,dy,
\end{aligned}$$

which tends to 0 because φ vanishes at 0. Hence, (1.7.8) is proved. □

It will be useful for us to notice a few identities involving $q^{(m)}(x,\,y,\,t)$. Set $q^{(0)}(x,\,y,\,t)\equiv q(x,\,y,\,t)$, then it is easy to check that,

$$\partial_x q^{(m)}(x,\,y,\,t)=\frac{1}{t}\left(q^{(m+1)}(x,\,y,\,t)-q^{(m)}(x,\,y,\,t)\right)\text{ for all } m\ge 0,$$

and therefore,

$$\partial_x^m q^{(k)}(x,\,y,\,t)=\frac{1}{t^m}\sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}q^{(m+k-\ell)}(x,\,y,\,t)\quad\text{for all } k,m\in\mathbb{N}. \tag{1.7.10}$$

Furthermore, for $m\in\mathbb{N}$, $(s,\,t)\in(0,\,\infty)^2$, if $k\in\mathbb{N}$, $0\le k\le m$, and

$$q^{(m,\,k)}(x,\,y,\,s,\,t)\equiv\int_{(0,\,\infty)}q^{(m)}(x,\,\xi,\,s)\,q^{(k)}(\xi,\,y,\,t)\,d\xi,$$

then we claim that

$$q^{(m,\,k)}(x,\,y,\,s,\,t)=\frac{1}{(s+t)^{m-k}}\sum_{j=0}^{m-k}\binom{m-k}{j}s^j t^{m-k-j}q^{(k+j)}(x,\,y,\,s+t). \tag{1.7.11}$$

We will prove (1.7.11) by induction. First notice that when $m=k$, by Lemma 9 and (1.2.10),

$$\begin{aligned}
\int_{(0,\,\infty)}q^{(m)}(x,\,\xi,\,s)\,q^{(m)}(\xi,\,y,\,t)\,d\xi &= \partial_x^m\int_{(0,\,\infty)}q(x,\,\xi,\,s)q(\xi,\,y,\,t)dy\\
&= q^{(m)}(x,\,y,\,s+t)\,.
\end{aligned}$$

In other words, $q^{(m)}(x,\,y,\,t)$ also satisfies Chapman-Kolmogorov equation. Now assume that

(1.7.11) is true for some m and all $0 \le k \le m$. Then, for $0 \le k \le m$, by (1.7.10) ,

$$\partial_x q^{(m,\,k)}(x,\,y,\,s,\,t) = \frac{1}{s} q^{(m+1,\,k)}(x,\,y,\,s,\,t) - \frac{1}{s} q^{(m,\,k)}(x,\,y,\,s,\,t).$$

On the other hand, by the inductive hypothesis,

$$\begin{aligned}\partial_x q^{(m,\,k)}(x,\,y,\,s,\,t) \quad &= \quad \frac{1}{(s+t)^{m+1-k}} \sum_{j=0}^{m-k} \binom{m-k}{j} s^j t^{m-k-j} q^{(k+j+1)}(x,\,y,\,s+t) \\ &\quad - \frac{1}{(s+t)^{m+1-k}} \sum_{j=0}^{m-k} \binom{m-k}{j} s^j t^{m-k-j} q^{(k+j)}(x,\,y,\,s+t).\end{aligned}$$

Therefore, $q^{(m+1,\,k)}(x,\,y,\,s,\,t)$ equals

$$\begin{aligned}&\frac{1}{(s+t)^{m+1-k}} \Bigg(\sum_{j=1}^{m+1-k} \binom{m-k}{j-1} s^j t^{m+1-k-j} q^{(k+j)}(x,y,s+t) \\ &\qquad + \sum_{j=0}^{m-k} \binom{m-k}{j} s^j t^{m+1-k-j} q^{(k+j)}(x,y,s+t) \Bigg) \\ &= \frac{1}{(s+t)^{m+1-k}} \sum_{j=0}^{m+1-k} \binom{m+1-k}{j} s^j t^{m+1-k-j} q^{(k+j)}(x,y,s+t).\end{aligned}$$

Hence, one sees that (1.7.11) is true for $m+1$ and all $0 \le k \le m+1$.

The reason why we are interested in the preceding is that they allow us to produce rather sharp estimates on the derivatives of $q^V(x,\,y,\,t)$.

Lemma 1.10. *Set*

$$C_m(V) \equiv \frac{5^m}{2} \max \left\{ \|\partial^\ell V\|_u : 0 \le \ell \le m \right\}$$

and

$$S_m(x,\,y,\,t) \equiv \sum_{\ell=0}^{m} \binom{m}{\ell} q^{(\ell)}(x,\,y,\,t).$$

Then

$$\left|\partial_x^m q^V(x,\,y,\,t)\right| \le A(t,\,m,\,V)\, \frac{S_m(x,\,y,\,t)}{t^m} \tag{1.7.12}$$

for $(x,\,y,\,t) \in (0,\,\infty)^2 \times (0,\,1]$, *where*

$$A(t,\,m,\,V) \equiv \frac{1 + \left(tC_m(V)\right)^m \left(e^{t2^m\|V\|_u} - 1\right) + \left(tC_m(V)\right)^{m+1} e^{2^m\|V\|_u t}}{1 - tC_m(V)}.$$

Proof. Recalling the construction of $q^V(x, y, t)$ in §1.4, one sees that it is sufficient to show

$$\left|\partial_x^m q_n^V(x, y, t)\right| \le C_m(V)^{m\wedge n}\left(2^m\|V\|_u\right)^{(n-m)^+}\frac{t^n}{(n-m)^+!}\frac{S_m(x, y, t)}{t^m}, \tag{1.7.13}$$

where $(x, y, t) \in (0, \infty)^3$ and $n \in \mathbb{N}$. Indeed, if (1.7.13) is true, then by (1.4.11) and (1.4.7),

$$\begin{aligned}\left|\partial_x^m q^V(x, y, t)\right| &\le \frac{S_m(x, y, t)}{t^m}\left[\sum_{n\le m-1}(tC_m(V))^n + (tC_m(V))^m\sum_{n=0}\frac{(2^m\|V\|_u t)^n}{n!}\right] \\ &\le \frac{S_m(x, y, t)}{t^m}\left[\frac{1-(tC_m(V))^m}{1-tC_m(V)} + (tC_m(V))^m e^{2^m\|V\|_u t}\right],\end{aligned}$$

which is the desired result.

When $n = 0$, (1.7.13) follows directly from (1.7.10) with $k = 0$. When $m = 0$, (1.7.13) simply reduces to (1.4.9). Now we will concentrate on the case when $m = 1$. Using (1.4.7) and Lemma 9, one can write $\partial_x q^{(1)}(x, y, t)$ as the sum of

$$\int_0^{\frac{t}{2}}\left(\int_{(0,\infty)}\partial_x q(x, \xi, t-\tau)V(\xi)q(\xi, y, \tau)d\xi\right)d\tau$$

and

$$\int_0^{\frac{t}{2}}\left(\int_{(0,\infty)}q^{(1)}(x, \xi, \tau)\partial_\xi\big(V(\xi)q(\xi, y, t-\tau)\big)d\xi\right)d\tau.$$

By (1.7.10) , (1.2.10) and (1.7.11), the first term is bounded by $\|V\|_u$ times

$$\begin{aligned}&\int_0^{\frac{t}{2}}(t-\tau)^{-1}\left(\int_{(0,\infty)}\left(q(x, \xi, t-\tau)+q^{(1)}(x, \xi, t-\tau)\right)q(\xi, y, \tau)d\xi\right)d\tau \\ &= q(x, y, t)\left(\int_0^{\frac{t}{2}}(t-\tau)^{-1}d\tau\right) + \frac{1}{t}\int_0^{\frac{t}{2}}\frac{\left(\tau q(x, y, t)+(t-\tau)q^{(1)}(x, y, t)\right)}{t-\tau}d\tau \\ &= \tfrac{1}{2}q^{(1)}(x, y, t) + \frac{q(x, y, t)}{t}\int_0^{\frac{t}{2}}\frac{t+\tau}{t-\tau}d\tau \\ &\le \tfrac{1}{2}q^{(1)}(x, y, t) + \tfrac{5}{4}q(x, y, t).\end{aligned}$$

As for the second, it can be dominated by

$$\|V'\|_u \int_0^{\frac{t}{2}} q^{(1,0)}(x,\, y,\, \tau,\, t-\tau)d\tau + \|V\|_u \int_0^{\frac{t}{2}} \int_{(0,\infty)} q^{(1)}(x,\, \xi,\, \tau)\frac{\left(q+q^{(1)}\right)(x,\, \xi,\, t-\tau)}{t-\tau} d\xi d\tau$$

$$= \int_0^{\frac{t}{2}} \left(\|V'\|_u + \|V\|_u (t-\tau)^{-1}\right) q^{(1,0)}(x,\, y,\, \tau,\, t-\tau)d\tau + \log 2 \|V\|_u q^{(1)}(x,\, y,\, t)$$

$$\leq \left(\frac{1}{2}\|V\|_u + \frac{3t}{8}\|V'\|_u\right) q(x,\, y,\, t) + \left(\frac{t}{8}\|V'\|_u + \frac{3}{2}\|V\|_u\right) q^{(1)}(x,\, y,\, t).$$

After combining these, one gets that $\partial_x q_1^V(x,\, y,\, t) \leq C_1(V)\, S_1(x,\, y,\, t)$ for all $t \in (0,\, 1]$. Hence (1.7.13) holds for $m = n = 1$. The proof of (1.7.12) for $m = 1$ and $n \geq 2$ is by induction. Specifically, by (1.4.8),

$$\begin{aligned}
\left|\partial_x q_{n+1}^V(x,\, y,\, t)\right| &= \left|\int_0^t \left(\int_{(0,\infty)} \partial_x q_n^V(x,\, \xi,\, t-\tau) V(\xi) q(\xi,\, y,\, \tau) d\xi\right) d\tau\right| \\
&\leq \frac{C_1(V) 2^{n-1} \|V\|_u^n}{(n-1)!} \int_0^t \int_{(0,\infty)} (t-\tau)^{n-1} \left(q+q^{(1)}\right)(x,\, \xi,\, t-\tau) q(\xi,\, y,\, \tau) d\xi d\tau \\
&= \frac{C_1(V) 2^{n-1} \|V\|_u^n}{(n-1)!} \left(\frac{t^n}{n} q(x,\, y,\, t) + \frac{1}{t}\int_0^t \frac{\left(\tau q(x,\, y,\, t) + (t-\tau) q^{(1)}(x,\, y,\, t)\right)}{(t-\tau)^{1-n}} d\tau\right) \\
&\leq \frac{C_1(V) 2^{n-1} \|V\|_u^n}{(n-1)!} \left(\frac{2t^n}{n} q(x,\, y,\, t) + \frac{t^n}{n+1} q^{(1)}(x,\, y,\, t)\right) \\
&\leq \frac{C_1(V) \left(2t\|V\|_u\right)^n}{n!} S_1(x,\, y,\, t).
\end{aligned}$$

The strategy for general $m \geq 1$ should be clear now. One should apply the argument used to estimate $\partial_x q_1^V$ not just once but m times. After the mth repetition, one arrives at

$$\left|\partial_x^m q_n^V(x,\, y,\, t)\right| \leq C_m(V)^n t^{n-m} S_m(x,\, y,\, t) \text{ for all } 0 \leq m \leq n.$$

For $n \geq m+1$, one uses (1.4.8) to proceed inductively as above. □

Finally, we have reached our goal for this section of getting estimates on the derivatives of $\bar{p}(x,\, y,\, t)$.

Theorem 1.11. *For each* $0 < \alpha < \beta < 1$, $\rho \in (0,\, 1)$, *and* $m \in \mathbb{N}$, *there exists a* $K_m(\alpha,\, \beta,\, \rho) < \infty$, *such that*

$$\left|\partial_x^m \bar{p}(x,\, y,\, t)\right| \leq K_m(\alpha,\, \beta,\, \rho)\psi(y) \left(\frac{S_m(\psi(x),\, \psi(y),\, t)}{t^m} + e^{-\frac{4(\beta-\alpha)^2}{t}}\right) \tag{1.7.14}$$

for all $(x, y, t) \in (0, \rho\alpha]^2 \times (0, 1]$. *In particular, for each* $\epsilon > 0$, $\bar{p}(x, y, t)$ *has bounded derivatives of all orders on* $(0, 1)^2 \times [\epsilon, \infty)$.

Proof. Recall the function $p_\beta(x, y, t)$ defined in §1.6 (1.6.1), and set $\bar{p}_\beta(x, y, t) \equiv y(1-y)\,p_\beta(x, y, t)$. Then by (1.6.2), $\bar{p}_\beta$ is symmetric and hence, from (1.6.12),

$$\partial_x^m \bar{p}(x, y, t) = \partial_x^m \bar{p}_\beta(x, y, t) + \sum_{n=1}^{\infty} \mathbb{E}\Big[\partial_x^m \bar{p}_\beta\big(x, \alpha, t - \eta^X_{2n,[\alpha,\beta]}(y)\big),\, \eta^X_{2n,[\alpha,\beta]}(y) < t\Big].$$

Therefore, by (1.6.15),

$$\left|\partial_x^m \bar{p}(x, y, t)\right| \le \left|\partial_x^m \bar{p}_\beta(x, y, t)\right| + \frac{2y}{\alpha(\beta-\alpha)} \sup_{\tau\in(0,t]} \left|\partial_x^m \bar{p}_\beta(x, \alpha, \tau)\right| e^{-\frac{4(\beta-\alpha)^2}{t}}.$$

If $\bar{q}^{V_\beta}(\xi, \eta, t) \equiv \eta q^{V_\beta}(\xi, \eta, t)$, then by (1.4.13), $\bar{q}^{V_\beta}(\xi, \eta, t)$ is also symmetric. By (1.5.3) and (1.6.1), one knows that $\bar{p}_\beta(x, y, t)$ equals

$$\frac{\bar{q}^{V_\beta}(\psi(x), \psi(y), t)}{\sqrt{\psi'(x)\psi'(y)}} - \mathbb{E}\left[e^{\int_0^{\zeta^Y_{\psi(\beta)}(\psi(y))} V(Y(\tau, \psi(y)))d\tau} \frac{\bar{q}^{V_\beta}\left(\psi(x), \psi(\beta), t - \zeta^Y_{\psi(\beta)}(\psi(y))\right)}{\sqrt{\psi'(x)\psi'(y)}},\, \zeta^Y_{\psi(\beta)}(\psi(y)) < t\right].$$

Thus, by Lemma 5, $\partial_x^m \bar{p}_\beta(x, y, t)$ is bounded by a constant times

$$\max_{0\le\ell\le m}\left|\left(\partial_x^\ell \bar{q}^{V_\beta}\right)(\psi(x), \psi(y), t)\right| + \frac{\psi(y)}{\psi(\alpha)} e^{-\frac{(\psi(y)-\psi(\beta))^2}{4\psi(\beta)t}} \sup_{\tau\in(0,t]} \max_{0\le\ell\le m}\left|\left(\partial_x^\ell \bar{q}^{V_\beta}\right)(\psi(x), \psi(\beta), \tau)\right|.$$

By combining these with Lemma 10, (1.7.6), (1.7.7) and the fact that $(\psi(\beta) - \psi(\alpha))^2 \ge \psi(\beta)(\beta-\alpha)^2$, one gets the desired result. □

1.8 Refinements

This section deals with possible improvement for the estimate 1.6.16 in Theorem 1.8. If one takes a look at the arguments there, one would notice that the weak link is the replacement of $\bar{q}^{V_\beta}$ by $\bar{q}$ using 1.6.13. The problem with $\bar{q}^{V_\beta}$ is that we can only estimate it but cannot give an explicit expression for it in terms of familiar quantities. Nonetheless, if one is interested in $\bar{p}(x, y, t)$ when every variable is small, then one can make a modest improvement by using a

Taylor approximation for V_β.

In order to carry this out, we will need the following computation.

Lemma 1.12. *Define* $\{C_{k,j}:\ k\in\mathbb{N},\ 1\le j\le k\}$ *so that* $C_{1,1}=1$, *and for* $k\ge 2$,

$$C_{k,1}=k!,\ C_{k,k}=1,\quad \textit{and}\ \ C_{k+1,j}=(k+j)\,C_{k,j}+C_{k,j-1}\ \textit{for}\ 2\le j\le k.$$

Then

$$q(x,\,y,\,t)y^k=\sum_{j=1}^{k}C_{k,j}t^{k-j}x^jq^{(k+j)}(x,\,y,\,t)\quad \textit{for}\ k\ge 1. \tag{1.8.1}$$

In particular, for $k\ge 1$,

$$\int_0^t\left(\int_{(0,\,\infty)}q(x,\,\xi,\,t-\tau)\xi^kq(\xi,\,y,\,\tau)d\xi\right)d\tau=tx\sum_{j=1}^{k}Q^{(k,j)}(x,\,y,\,t), \tag{1.8.2}$$

where

$$Q^{(k,j)}(x,\,y,\,t)=\frac{(k+j)!C_{k,j}}{(2k+1)!}t^{k-j}x^{j-1}\sum_{i=0}^{k+j}\frac{(k-j+i)!}{i!}q^{(i)}(x,\,y,\,t).$$

Proof. For $m\ge 0$, recalling from (1.7.4) and (1.7.1), one sees that

$$x\partial_x^2q^{(m)}(x,\,y,\,t)+m\partial_xq^{(m)}(x,\,y,\,t)=\left(\frac{x+y}{t^2}-\frac{m}{t}\right)q^{(m)}(x,\,y,\,t)-\frac{2x}{t^2}q^{(m+1)}(x,\,y,\,t).$$

At the same time, by (1.7.10), $x\partial_x^2q^{(m)}(x,\,y,\,t)+m\partial_xq^{(m)}(x,\,y,\,t)$ also equals

$$\frac{x}{t^2}q^{(m+2)}(x,\,y,\,t)+\left(\frac{m}{t}-\frac{2x}{t^2}\right)q^{(m+1)}(x,\,y,\,t)+\left(\frac{x}{t^2}-\frac{m}{t}\right)q^{(m)}(x,\,y,\,t).$$

Thus,

$$yq^{(m)}(x,\,y,\,t)=xq^{(m+2)}(x,\,y,\,t)+mtq^{(m+1)}(x,\,y,\,t)\quad \text{for all } m\ge 0. \tag{1.8.3}$$

Now we can prove (1.8.1) by induction. When $k=1$, this is just (1.8.3) with $m=0$. Assume it holds for k, then

$$\begin{aligned} q(x,\,y,\,t)\,y^{k+1} &= \sum_{j=1}^{k} C_{k,j} t^{k-j} x^j q^{(k+j)}(x,\,y,\,t) y \\ &= \sum_{j=1}^{k} C_{k,j} t^{k-j} x^j \left(x q^{(k+j+2)}(x,\,y,\,t) + (k+j)\, t q^{(k+j+1)}(x,\,y,\,t) \right) \\ &= \sum_{j=2}^{k+1} \left(C_{k,j-1} + (k+j)\, C_{k,j} \right) t^{k+1-j} x^j q^{(k+j+1)}(x,\,y,\,t) + C_{k,1}\,(k+1)\, t^k x q^{(k+2)}(x,\,y,\,t) \\ &= \sum_{j=1}^{k+1} C_{k+,j} t^{k+1-j} x^j q^{(k+1+j)}(x,\,y,\,t). \end{aligned}$$

Given (1.8.1), by (1.7.11), we know that

$$\begin{aligned} &\int_0^t \left(\int_{(0,\,\infty)} q(x,\,\xi,\,t-\tau) \xi^k q(\xi,\,y,\,\tau)\, d\xi \right) d\tau \\ =\; & \sum_{j=1}^{k} C_{k,j} x^j \int_0^t (t-\tau)^{k-j} q^{(k+j,0)}(x,\,y,\,t-\tau,\,\tau) d\tau \\ =\; & \sum_{j=1}^{k} C_{k,j} t^{-k-j} x^j \left(\sum_{i=0}^{k+j} \binom{k+j}{i} \left(\int_0^t (t-\tau)^{k-j+i} \tau^{k+j-i} d\tau \right) q^{(i)}(x,\,y,\,t) \right) \\ =\; & tx \sum_{j=1}^{k} C_{k,j} t^{k-j} x^{j-1} \sum_{i=0}^{k+j} \frac{(k+j)!(k-j+i)!}{i!(2k+1)!} q^{(i)}(x,\,y,\,t), \end{aligned}$$

which is the expression we want. □

It is worth noticing that the numbers $\{C_{k,j} :\ 1 \le j \le k\}$ are the coefficients for the polynomials which gives the kth moment of $q\,(x,\,y,\,t)$. That is,

$$\int_{(0,\infty)} y^k q(x,\,y,\,t) dy = \sum_{j=1}^{k} C_{k,j} t^{k-j} x^j,$$

which follows immediately from (1.8.1) and the fact that $q^{(m)}\,(x,\,\cdot,\,t)$ has total mass 1 for all $m \ge 1$.

With Lemma 1.12, we can implement the idea mentioned at the beginning of this section. Suppose $0 < \beta < 1$ and $(x,\,y) \in (0,\,\beta)^2$. First, by the definition of q^{V_β} and (1.4.9), one can easily check that,

$$\left|q^{V_\beta}(x,\, y,\, t) - q(x,\, y,\, t) - \int_0^t \left(\int_{(0,\,\infty)} q(x,\, \xi,\, t-\tau) V_\beta(\xi) q(\xi,\, y,\, \tau) d\xi\right) d\tau\right| \\ \leq \frac{(t\|V_\beta\|_u)^2 e^{t\|V_\beta\|_u}}{2} q(x,\, y,\, t).$$

Second, use Taylor's theorem and the fact that $V_\beta(0) = 0$, we have

$$\left|\int_0^t \left(\int_{(0,\,\infty)} q(x,\, \xi,\, t-\tau) V_\beta(\xi) q(\xi,\, y,\, \tau) d\xi\right) d\tau - V_\beta'(0) \int_0^t \left(\int_{(0,\,\infty)} q(x,\, \xi,\, t-\tau) \xi q(\xi,\, y,\, \tau) d\xi\right) d\tau\right| \\ \leq \frac{\|V_\beta''\|_u}{2} \int_0^t \left(\int_{(0,\,\infty)} q(x,\, \xi,\, t-\tau) \xi^2 q(\xi,\, y,\, \tau)\, d\xi\right) d\tau.$$

Next, combine the two inequalities with (1.12) to see that

$$\left|q^{V_\beta}(x,\, y,\, t) - q(x,\, y,\, t) - \frac{V_\beta'(0)}{3} tx \sum_{i=0}^{3} q^{(i)}(x,\, y,\, t)\right| \leq \frac{\|V_\beta\|_u^2}{2} e^{t\|V_\beta\|_u} t^2 q(x,\, y,\, t) \\ + \frac{\|V_\beta''\|_u}{10} tx \left[t \sum_{j=0}^{3} (j+1)\, q^{(j)}(x,\, y,\, t) + 2x \sum_{j=0}^{4} q^{(j)}(x,\, y,\, t)\right]. \tag{1.8.4}$$

After putting the preceding together with (1.6.13), we have a slight refinement of the first estimate in Theorem 1.8 as the following.

Theorem 1.13. *Given* $0 < \alpha < \beta < 1$ *and* $\rho \in (0,\, 1)$. *For* $(x,\, y,\, t) \in (0,\, \rho\alpha)^2 \times (0,\, \infty)$,

$$\left|\sqrt{\psi'(x)\psi'(y)}\bar{p}(x,\, y,\, t) - \bar{q}(\psi(x),\, \psi(y),\, t) - \frac{V_\beta'(0)\, t\psi(x)\psi(y)}{3} \sum_{j=0}^{3} q^{(j)}(\psi(x),\, \psi(y),\, t)\right| \\ \leq \frac{K(\alpha,\, \beta,\, \rho)}{\beta - \alpha} e^{t\|V_\beta\|_u} \psi(x)\psi(y) e^{-\frac{\mu_\beta(\beta-\alpha)^2}{t}} + \frac{\|V_\beta\|_u^2}{2} e^{t\|V_\beta\|_u} t^2 \bar{q}(\psi(x),\, \psi(y),\, t) \\ + \frac{\|V_\beta''\|_u}{10} t\psi(x)\psi(y) \left[t \sum_{j=0}^{3} (j+1)\, q^{(j)}(\psi(x),\, \psi(y),\, t) + 2\psi(x) \sum_{j=0}^{4} q^{(j)}(\psi(x),\, \psi(y),\, t)\right],$$

where $K(\alpha,\, \beta,\, \rho)$ *and* μ_β *are as defined in Lemma 1.7.*

Proof. This is a direct consequence of Lemma 1.7 and (1.8.4). □

It is clear that the preceding line of reasoning can be repeated to get better and better approximations to $p(x, y, t)$. However, the computation for even the next step is tedious, since it involves dealing with the integrals of the form

$$\iint\limits_{0\le\tau_1<\tau_2\le t} \left(\iint\limits_{(0,\infty)^2} q(x,\, \xi_2,\, t-\tau_2)\xi_2 q(\xi_2,\, \xi_1,\, \tau_2-\tau_1)\xi_1 q(\xi_1,\, y,\, \tau_1) d\xi_1 d\xi_2 \right) d\tau_1 d\tau_2.$$

Chapter 2

Gaussian Measures on Separable Banach Spaces

2.1 Introduction

In this chapter we will look at probability measures on a separable Banach space that are centered Gaussian. Namely, suppose $(E, \|\cdot\|_E)$ is a real separable Banach space with dual space E^*, then the probability measure $\mathcal{W}$ on E is a *centered Gaussian measure* if, for any $x^* \in E^*$, the random variable $x \in E \mapsto \langle x, x^* \rangle \in \mathbb{R}$ has centered Gaussian distribution under $\mathcal{W}$. Throughout this chapter, we will also assume $\mathcal{W}$ is non-degenerate in the sense that, $\mathbb{E}^{\mathcal{W}}\left[\langle \cdot, x^* \rangle^2\right] = 0$ if and only if $x^* = 0$.

Although E will have infinite dimensions in general, it will be helpful to first make the following observation for the finite dimensional case when E can be identified as $\mathbb{R}^N$. Namley, if $\gamma_{0,C}$ is the Gaussian measure on $\mathbb{R}^N$ with mean 0 and non-degenerate covariance C, then $\mathbb{R}^N$ can be turned into a Hilbert space H by taking the inner product to be $(h, g)_H \equiv (h, C^{-1}g)_{\mathbb{R}^N}$ for all $h, g \in H$. Furthermore, if λ_H is the Lebesgue measure that assigns measure 1 to the unit cube in H, then we have

$$\begin{aligned} \gamma_{0,C}(dh) &= \frac{1}{\sqrt{2\pi}^N} \exp\left(-\frac{\|h\|_H^2}{2}\right) \lambda_H(dh), \\ \widehat{\gamma_{0,C}}(g) &= \exp\left(-\frac{\|g\|_H^2}{2}\right) \text{ for all } g \in H^* = H. \end{aligned} \tag{2.1.1}$$

Set $\mathcal{W}_H(dh) \equiv \gamma_{0,C}(dh)$, then (2.1.1) shows that $\mathcal{W}_H$ is the standard Gaussian measure on H. In this sense, one sees that in the finite dimensional case, the natural home for a Gaussian measure is a Hilbert space. However, the construction above clearly fails when the dimension is infinite. In fact, given a real, separable Hilbert space H, if $\dim H = \infty$, then no such measure $\mathcal{W}_H$ can exist. The reason is well known: if it did, then for any orthonormal basis $\{h_m : m \geq 1\}$ of H, one can easily check that $h \in H \mapsto X_m(h) = (h, h_m)_H \in \mathbb{R}$ would be independent, standard Gaussian random variables, and therefore by the strong law of large numbers, $\|h\|_H^2 = \sum_{m=0}^{\infty} X_m^2(h)$ would be infinite for almost every $h \in H$ under $\mathcal{W}_H$. In other words, H is too small to accommodate $\mathcal{W}_H$.

It might be helpful to recognize that an analogous problem occurs in study of partial differential equations. Namely, although a natural place to look for solutions to PDE's is in the class of continuously differentiable functions, one often has to work with a more general family, say, Sobolev spaces. Similarly, to overcome the problem described above, we must consider objects that are more general than elements of H and allow Gaussian measures to live on a larger space. The idea introduced by Leonard Gross in [8] is to complete H with respect to a more forgiving norm than $\|\cdot\|_H$, so that the resulting Banach space E is large enough to house $\mathcal{W}_H$. The resulting triple $(H, E, \mathcal{W}_H)$ is referred as ***an abstract Wiener space.***

The existence and construction of abstract Wiener spaces were first discussed by Gross ([8]), and then formulated by Stroock ([14], [13]) in a different way, the formulation which will be used here. §2.2 briefly introduces the construction of abstract Wiener spaces and some important structural properties of infinite dimensional Gaussian measures. In §2.3, we study the probabilistic extensions of the classical Cauchy functional equation which defines the notion of additivity. We first review the results in finite dimensional spaces, and then examine the infinite dimensional analog under Gaussian measures. The various techniques developed in the process lead naturally to results about the structure of abstract Wiener spaces.

2.2 Abstract Wiener Space

As explained in the previous section, an infinite dimensional Gaussian measure does not live on the natural Hilbert space but does on a larger Banach space. To make this precise, we first consider the following properties of abstract spaces (Lemma 8.2.3 in [13]). Given a real, separable Banach space $(E, \|\cdot\|_E)$, suppose $(H, \|\cdot\|_H)$ is a real Hilbert space that is continuously embedded as a dense subspace of E. Then for any $x^* \in E^*$, one can use the Riesz representation to find a unique element $h_{x^*} \in H$ such that $\langle h, x^* \rangle = (h, h_{x^*})_H$ for all $h \in H$. In fact, if one equips E^* with the weak topology and H with the strong topology, then the mapping $x^* \in E^* \mapsto h_{x^*} \in H$ is linear, continuous, one-to-one and onto a dense subspace of H. Thus, if $x \in E$, then $x \in H$ if and only if there exists a $K < \infty$ such that $|\langle x, x^* \rangle| \leq K \|h_{x^*}\|_H$ for all $x^* \in E^*$. Without loss of generality, from now on we will assume that

$$\|h\|_E \leq \|h\|_H \quad \text{for all} \quad h \in H, \quad \text{and therefore} \quad \|h_{x^*}\|_H \leq \|x^*\|_{E^*} \quad \text{for all} \quad x^* \in E^*. \tag{2.2.1}$$

Then, for each $h \in H$, $\|h\|_H = \sup\{\langle h, x^* \rangle : x^* \in E^* \text{ and } \|x^*\|_{E^*} \leq 1\}$. Finally, there exists a sequence $\{x_n^* : n \geq 1\} \subseteq E^*$ such that $\{h_n \equiv h_{x_n^*} : n \geq 1\}$ forms an orthonormal basis in H. In particular, for $x \in E$,

$$x \in H \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \langle x, x_n^* \rangle^2 < \infty, \tag{2.2.2}$$

and

$$(h, g)_H = \sum_{n=1}^{\infty} \langle h, x_n^* \rangle \langle g, x_n^* \rangle \quad \text{for all} \quad h, g \in H.$$

Now assume $\mathcal{W}$ is a non-degenerate centered Gaussian measure on E, and therefore, for each $x^* \in E^* \backslash \{0\}$, the mapping $x \in E \mapsto \langle x, x^* \rangle$ has non-degenerate centered Gaussian distribution under $\mathcal{W}$. Furthermore, assume $\mathcal{W}$ has the "right" characteristic function, i.e.,

$$\mathbb{E}^{\mathcal{W}}[\exp(i\langle \cdot, x^* \rangle)] = \exp\left(-\frac{\|h_{x^*}\|_H^2}{2}\right) \quad \text{for all} \quad x^* \in E^*, \tag{2.2.3}$$

or equivalently, under $\mathcal{W}$, for each $x^* \in E^*$, $\langle \cdot, x^* \rangle$ is a centered Gaussian random variable with variance $\|h_{x^*}\|_H^2$. If we denote by $\mathcal{I}$ the mapping $\mathcal{I}: h_{x^*} \in H \mapsto \langle \cdot, x^* \rangle \in L^2(\mathcal{W}; \mathbb{R})$, then clearly

$$\|h_{x^*}\|_H^2 = \mathbb{E}^{\mathcal{W}}\left[\mathcal{I}(h_{x^*})^2\right],$$

and hence $\mathcal{I}$ is a linear isometry from $\{h_{x^*} : x^* \in E^*\}$ to $L^2(\mathcal{W}; \mathbb{R})$. In addition, since

$\{h_{x^*} : x^* \in E^*\}$ is dense in H, $\mathcal{I}$ admits a unique extension as a linear isometry from H to $L^2(\mathcal{W}; \mathbb{R})$, so we have

$$\|h\|_H^2 = \mathbb{E}^{\mathcal{W}}\left[\mathcal{I}(h)^2\right] \quad \text{for} \quad h \in H.$$

Finally, $\{\mathcal{I}(h) : h \in H\}$ forms a family of centered Gaussian random variables with covariance

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h)\mathcal{I}(g)] = (h, g)_H \quad \text{for} \quad h, g \in H.$$

A triple $(H, E, \mathcal{W})$ as described above is an ***abstract Wiener space***, the mapping $\mathcal{I} : H \to L^2(\mathcal{W}; \mathbb{R})$ is referred as the ***Paley-Wiener map***, and the random variables $\mathcal{I}(h)$'s are the ***Paley-Wiener integrals.*** We will introduce some structural properties of abstract Wiener spaces from different perspectives.

First suppose E and $\mathcal{W}$ are given as above, i.e., E is a real separable Banach space and $\mathcal{W}$ is a non-degenerate centered Gaussian measure on E. If, for every $x^* \in E^*$, we define

$$h_{x^*} \equiv \int_E x\,\langle x, x^*\rangle\,\mathcal{W}(dx) \in E, \quad \text{and}$$
$$\|h_{x^*}\|_H^2 \equiv \int_E \langle x, x^*\rangle^2\,\mathcal{W}(dx),$$

then one can easily check that

$$H \equiv \overline{\{h_{x^*} \in E : x^* \in E^*\}}^{\|\cdot\|_H}$$

is the one and only Hilbert space which makes $(H, E, \mathcal{W})$ into an abstract Wiener space. H is also called the ***Cameron-Martin space*** for $(H, E, \mathcal{W})$.

Next, given a real, separable Hilbert space H, the results from [8] and [13] (Corollary 8.3.2) guarantee that there exist a real, separable Banach space E and a Borel probability measure $\mathcal{W}$ on E, such that $(H, E, \mathcal{W})$ is an abstract Wiener space, i.e., H is continuously embedded as a dense subset of E, and $\mathcal{W}$ satisfies (2.2.3). However, the choice of such E is not "canonical". An important result from [8] says that, if $(H, E, \mathcal{W})$ is an abstract Wiener space, we can always find a real, separable Banach space E_0 which is continuously embedded in E as a measurable subset with $\mathcal{W}(E_0) = 1$, such that $(H, E_0, \mathcal{W} \upharpoonright E_0)$ is an abstract Wiener space (Corollary 8.3.10 in [13]). In other words, if we consider H as the "underlying"

Hilbert space of $\mathcal{W}$, we can always make the "housing" Banach space E smaller and closer to H. We will revisit this result under a specific setting developed in §2.3, and show that, the intersection of all the possible "housing" spaces is precisely H itself.

Before closing this section, we still need to discuss the following question: given a real separable Hilbert space H, and a real separable Banach space E in which H is continuously embedded as a dense subset, when and how we can get a probability measure $\mathcal{W}$ on E such that $(H,\, E,\, \mathcal{W})$ is an abstract Wiener space (Theorem 8.3.3 in [13]). Let's first assume such $\mathcal{W}$ exists. Given any orthonormal basis $\{h_m :\, m \geq 1\}$ in H, denote by $\mathcal{F}_n$ the σ-algebra generated by $\{\mathcal{I}\,(h_m) :\, 1 \leq m \leq n\}$ and set $\mathcal{F} \equiv \vee_{n=1}^{\infty} \mathcal{F}_n$. Then, since $\sum_{m=1}^{n} (h,\, h_m)_H \, \mathcal{I}\,(h_m) \to \mathcal{I}\,(h)$ in $L^2\,(\mathcal{W};\, \mathbb{R})$, for every $h \in H$, $\mathcal{I}\,(h)$ is measurable with respect to $\overline{\mathcal{F}}^{\mathcal{W}}$, the completion of $\mathcal{F}$ under $\mathcal{W}$. In particular, $\langle \cdot,\, x^* \rangle = \mathcal{I}\,(h_{x^*})$ is measurable with respect to $\overline{\mathcal{F}}^{\mathcal{W}}$ for every $x^* \in E^*$, and hence the Borel σ-algebra of E is contained in $\overline{\mathcal{F}}^{\mathcal{W}}$. Further, by applying the martingale convergence theorem, one sees that

$$S_n\,(x) \equiv \sum_{m=1}^{n} \mathcal{I}\,(h_m)\,(x)\, h_m \to x \;\text{ as }\; n \to \infty \;\text{ almost surely under }\; \mathcal{W}. \tag{2.2.4}$$

Morever, because h_m's are orthonormal, if $(H,\, E,\, \mathcal{W})$ is an abstract Wiener space, then $\{\mathcal{I}\,(h_m) :\, m \geq 1\}$ is a sequence of independent standard Gaussian random variables under $\mathcal{W}$. Therefore, (2.2.4) implies

$$\sum_{m=1}^{\infty} \xi_m h_m \;\text{ converges in } E \;\text{ for almost every } \boldsymbol{\xi} \equiv (\xi_1,\, \cdots,\, \xi_m,\, \cdots) \;\text{ under }\; \gamma^{\mathbb{N}}, \tag{2.2.5}$$

where γ is the standard Gaussian distribution on $\mathbb{R}$.

Conversely, given H and E, if there exists an orthonormal basis $\{h_m :\, m \geq 1\}$ such that (2.2.5) is true, then by considering

$$S\,(\boldsymbol{\xi}) \equiv \begin{cases} \sum_{m=1}^{\infty} \xi_m h_m, & \text{when } \sum_{m=1}^{\infty} \xi_m h_m \text{ converges in } E, \\ 0, & \text{elsewhere,} \end{cases}$$

one can easily verify that $(H,\, E,\, \mathcal{W})$, where $\mathcal{W} \equiv S_* \gamma^{\mathbb{N}}$, is an abstract Wiener space.

2.3 Additive Functions and Gaussian Measures

In this section we examine a probabilistically natural extension of the classical Cauchy functional equation both in finite and infinite dimensions. In particular we will show that the naïve generalization of the finite dimensional result fails in infinite dimensions. The way we find the alternative can be viewed as an application of the theory of Gaussian measures on Banach spaces introduced in §2.2.

We start with a brief review of the results about additive functions in the classical setting. A function $f : \mathbb{R} \to \mathbb{R}$ is called ***additive*** if and only if f solves the following Cauchy functional equation:

$$f(x+y) = f(x) + f(y) \quad \text{for all } x,\, y \in \mathbb{R}. \tag{2.3.1}$$

Given an additive function f , (2.3.1) clearly implies

$$f(px+qy) = pf(x) + qf(y) \qquad \text{for any } x,\, y \in \mathbb{R} \text{ and } p,\, q \in \mathbb{Q}, \tag{2.3.2}$$

and hence, if f is in addition continuous, then f must be linear, i.e., there exists a $c \in \mathbb{R}$, such that $f(x) = cx$ for $x \in \mathbb{R}$.

In fact, it is well-known that if f is additive and λ−measurable, where λ denotes the Lebesgue measure on $\mathbb{R}$, then f is linear. One proof of this statement is based on a result of Vitali, which says that if $\Gamma \subseteq \mathbb{R}$ is λ−measurable with $\lambda(\Gamma) > 0$, then for some $\delta > 0$,

$$(-\delta,\, \delta) \subseteq \Gamma - \Gamma \equiv \{x - y :\ x,\, y \in \Gamma\}\,. \tag{2.3.3}$$

If $M > 0$ is large enough that the measurable set $\Gamma \equiv \{x \in \mathbb{R} :\ |f(x)| \le M\}$ has positive Lebesgue measure, then by (2.3.2) one can easily check that $|f| \le 2M$ on $\Gamma - \Gamma$. Choose $\delta > 0$ so that (2.3.3) is true, and for any $x \in \mathbb{R}$, take q_x to be a positive rational number so that $\frac{\delta}{2} \le \frac{|x|}{q_x} \le \delta$. Again, using (2.3.2), one sees that

$$|f(x)| = q_x \left| f\left(\frac{x}{q_x}\right) \right| \le q2M \le \frac{4\,|x|}{\delta} \ \text{ for all } \ x \in \mathbb{R}.$$

Therefore, f is continuous and hence linear.

If one is willing to use the axiom of choice, one can construct a counterexample in the absense of Lebesgue measurability. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$, and for each $x \in \mathbb{R}$, expand x using the Hamel basis $\mathcal{B}$ in $\mathbb{R}$. That is,

$$x = \sum_{b \in \mathcal{B}} a_b(x)\, b,$$

where the choice of $a_b(x)$'s is unique and there are only finitely many of them being non-zero. It is easy to verify that, for each b, a_b is an additive function, but cannot be linear because it takes only rational values.

The result above, which says that a Lebesgue measurable additive function is linear and continuous, holds in any finite dimensional vector space. One may want to ask whether there is an analog in infinite dimensions. The answer is yes. To formulate it, let's consider the following obvious extension of additivity to infinite dimensions. If E is a real separable Banach space, then we say $f : E \to \mathbb{R}$ is additive if f satisfies

$$f(x+y) = f(x) + f(y) \quad \text{for all } x,\, y \in E.$$

Now assume $f : E \to \mathbb{R}$ is in addition Borel measurable, then for any $x \in E$, $g : t \in \mathbb{R} \mapsto g(t) \equiv f(tx)$ is Lebesgue measurable and additive. Therefore, by the result from the real valued case, g is linear and hence $f(tx) = g(t) = tg(1) = tf(x)$. That is, a Borel measurable additive function on a real separable Banach space must be linear. If one combines this with a theorem of Schwartz ([11], or for a proof that is more in keeping with this chapter, [16], or Theorem 2.7), then one arrives at the conclusion that if $f : E \to \mathbb{R}$ is Borel measurable and additve, then $f \in E^*$. In other words, the extension of the classical additivity to infinite dimensions will lead to the same class of functions, the one of linear continuous functions, as in finite dimensions, provided the presence of Borel measurability.

2.3.1 Almost Everywhere Additivity

A similar question was asked by Erdös in [5] when (2.3.1) is replaced by

$$f(x+y) = f(x) + f(y) \quad \text{for } \lambda^2 - \text{almost every } (x,\, y) \in \mathbb{R}^2, \tag{2.3.4}$$

where λ^2 is the Lebesgue measure on $\mathbb{R}^2$. Specifically, Erdös asked what could be said about the linearity of function $f: \mathbb{R} \to \mathbb{R}$ if f is a solution to (2.3.4). N.G. de Brujin ([3]) and W.B. Jurkat ([9]) independently showed that, even if f is not $\lambda-$measurable, every solution to (2.3.4) is almost everywhere equal to an additive function, and therefore every $\lambda-$measurable solution to (2.3.4) is almost everywhere equal to a linear function. In this and the next subsection, we will present a proof to the statement for the measurable case using different considerations.

To get started, we define the "almost everywhere additivity" in the following sense: suppose E is a real, separable Banach space with Borel $\sigma-$algebra $\mathcal{B}_E$, and $\mathcal{W}$ is a non-degenerate centered Gaussian measure on E. Given a $\mathcal{B}_E-$measurable $f: E \to \mathbb{R}$, we say that$f$ is *almost everywhere additive* if and only if

$$f(x+y) = f(x) + f(y) \quad \text{for } \mathcal{W}^2 - \text{almost every } (x, y) \in E^2. \tag{2.3.5}$$

Note that when $\dim E = N < \infty$, E can be identified with $\mathbb{R}^N$, and $\mathcal{W}$ can be taken as γ^N which is equivalent to the Lebesgue measure on $\mathbb{R}^N$. Therefore, (2.3.5) is equivalent to (2.3.4) for any finite dimensional space. However, when $\dim E = \infty$, because there is no nontrivial translation invariant measure on E, a definition of the sort of (2.3.4) makes no sense, but (2.3.5) avoids that problem by taking the Gaussian measure as the reference measure.

For later convenience, we need to introduce some notation. Given a measure μ and a real constant $c \neq 0$, denote by μ_c the distribution of $\xi \to c\xi$ when ξ has distribution μ. If two measures μ and ν are equivalent, then we write $\mu \sim \nu$. For $x, y \in E$, and $a, b \in \mathbb{R}$, define the mapping $S_{a,b}: E^2 \to E$ by $S_{a,b}(x, y) \equiv ax + by$. Clearly $S_{a,b}$ is $\mathcal{B}_E^2-$measurable.

Lemma 2.1. *Given a real, separable Banach space E with $\dim E < \infty$, if $f: E \to \mathbb{R}$ is almost everywhere additive, then for any $p, q \in \mathbb{Q}^+$,*

$$f(px+qy) = f(px) + f(qy) \quad \textit{for } \mathcal{W}^2 - \textit{almost every } (x, y) \in E^2. \tag{2.3.6}$$

Proof. Clearly, this result stands as the analog to (2.3.2) in the probabilistic setting, but the proof is less trivial. For convenience, we will assume that $E = \mathbb{R}$ and $\mathcal{W} = \gamma$, even though the proof can be generalized to any finite dimensional space with no difficulty. Furthermore, notice that it is sufficient to show that, for any $n \in \mathbb{N}$,

$$f(nx) = nf(x) \quad \text{for } \gamma - \text{almost every } x \in \mathbb{R}. \tag{2.3.7}$$

Indeed, if (2.3.7) is true, then, since $\gamma_m \sim \gamma \sim \gamma_{\frac{1}{m}}$ for any $m \in \mathbb{N}$, $m \geq 1$, (2.3.7) is true when n is replaced by any $q \in \mathbb{Q}$, $q > 0$, which, together with the fact that f is almost everywhere additive, implies (2.3.6).

To prove (2.3.7), we use induction. There is nothing to be done when $n = 1$. Now assume it holds for n. Because $\gamma_{n+1} \times \gamma \sim \gamma^2$, we have that

$$f((n+1)x + y) = f((n+1)x) + f(y) \quad \text{for } \gamma^2 - \text{almost every } (x, y) \in \mathbb{R}^2.$$

On the other hand, we also know that $\gamma_n \times (S_{1,1})_* \gamma \sim \gamma^2$, and therefore, by the inductive hypothesis and (2.3.5),

$$\begin{aligned} f((n+1)x + y) &= f(nx) + f(x+y) \\ &= nf(x) + f(x) + f(y) \quad \text{for } \gamma^2 - \text{almost every } (x, y) \in \mathbb{R}^2. \end{aligned}$$

Comparing the right hand sides of the two equations above, we see that (2.3.7) also holds for $n + 1$. □

In fact, as we will see later, Lemma 2.1 is the "link" between the almost everywhere additivity and the "almost everywhere linearity", which will be defined in the next subsection. However, we want to point out that the infinite dimensional analog of Lemma 2.1 fails. That is, in an infinite dimensional space equipped with a Gaussian measure, we need to find an alternative generalization of (2.3.1), so that the solutions will satisfy equations of the sort of (2.3.6).

2.3.2 Wiener Maps

Again, assume E is a real, separable Banach space with Borel σ−algebra $\mathcal{B}_E$, and $\mathcal{W}$ is a non-degenerate centered Gaussian measure on E. If $\alpha, \beta \in (0, 1)$, and $\alpha^2 + \beta^2 = 1$, then we say that (α, β) is a *Pythagorean pair*. Let $f : E \to \mathbb{R}$ be $\mathcal{B}_E$−measurable, then f is called *a*

Wiener map if and only if

$$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y) \quad \text{for } \mathcal{W}^2-\text{almost every } (x,\, y)\in E^2 \tag{2.3.8}$$

for some Pythagorean pair $(\alpha,\, \beta)$.

Note that, when $\dim E<\infty$, if $f:\ E\to\mathbb{R}$ is almost everywhere additive, then f is a Wiener map. To see this, simply take $(p,\, q)$ from (2.3.6) to be $\left(\frac{3}{5},\, \frac{4}{5}\right)$, which is a Pythagorean pair. However, when $\dim E=\infty$, as alluded to at the end of §2.3.1, Wiener maps are the appropriate generalizations of almost everywhere additive functions in infinite dimensions. The reason is the following. It is clear that $\left(S_{\alpha,\beta}\right)_* \mathcal{W}^2=\mathcal{W}$ for any Pythagorean pair $(\alpha,\, \beta)$ while $\left(S_{1,1}\right)_* \mathcal{W}^2$ is singular to $\mathcal{W}$ when $\dim E=\infty$. Therefore, under $\mathcal{W}^2$, the right hand side of (2.3.8) has exactly the same distribution as f, while the right hand side of (2.3.5) does not provide any information about the distribution of f.

Throughout the section, unless otherwise stated, we will always assume $\dim E=\infty$, although all the following results hold for finite dimensional cases, only with easier proofs. A lot of the arguments and results about Wiener maps here are the outgrowth of [12]. Although in [12], the Wiener map is defined with $\alpha=\beta=\frac{1}{\sqrt{2}}$, it turns out to be equivalent to the definition given by (2.3.8) (Corollary 2.6).

Lemma 2.2. *If $f:\ E\to\mathbb{R}$ is a Wiener map, then $f\in L^2(\mathcal{W};\,\mathbb{R})$.*

Proof. Denote $\mu\equiv f_*\mathcal{W}$. Since $\left(S_{\alpha,\beta}\right)_* \mathcal{W}^2=\mathcal{W}$, (2.3.8) implies $\mu=\mu_\alpha\star\mu_\beta$. Taking the characteristic functions of both sides and using induction on n, we have

$$\hat{\mu}(\xi)=\hat{\mu}(\alpha\xi)\,\hat{\mu}(\beta\xi)=\prod_{m=0}^{n}\left(\hat{\mu}\left(\alpha^m\beta^{n-m}\xi\right)\right)^{\binom{n}{m}} \quad \text{for all } \ \xi\in\mathbb{R},\ n\geq 1. \tag{2.3.9}$$

Assume for the moment that μ is a symmetric distribution, in which case $\hat{\mu}(\xi)=\int_{\mathbb{R}}\cos(\xi y)\,\mu(dy)$. Using the inequality $-\log t\geq 1-t$ for $t\in(0,\, 1]$, one sees that (2.3.9) implies

$$\begin{aligned} -\log\hat{\mu}(1) &= -\sum_{m=0}^{n}\binom{n}{m}\log\hat{\mu}-m\left(\alpha^m\beta^n\right) \\ &\geq \int_{\mathbb{R}}\sum_{m=0}^{n}\left(1-\cos\left(\alpha^m\beta^{n-m}y\right)\right)\mu(dy)\,. \end{aligned}$$

Furthermore, since $\alpha^2 + \beta^2 = 1$,

$$0 \leq \sum_{m=0}^{n} \left(1 - \cos\left(\alpha^m \beta^{n-m} y\right)\right) \to \frac{y^2}{2} \text{ as } n \to \infty.$$

Thus, by Fatou's Lemma, we know that $\frac{1}{2}\int_{\mathbb{R}} y^2 \mu(dy) \leq -\log \hat{\mu}(1) < \infty$.

Now for general μ, define $\nu \equiv \mu_{-1} \star \mu$. It is easy to check that ν is a symmetric distribution, and (2.3.9) also holds when μ is replaced by ν, and hence $\int_{\mathbb{R}} y^2 \nu(dy) < \infty$. Furthermore, if a is a median of μ, i.e., $\mu([a, \infty) \wedge \mu((-\infty, a]) \geq \frac{1}{2}$, then for any $t \geq 0$,

$$\mu(\{y : |y - a| \geq t\}) \leq 2\nu((t, \infty) \cup (-\infty, -t)),$$

which implies $\int_{\mathbb{R}} y^2 \mu(dy) \leq 2a^2 + \int_{\mathbb{R}} y^2 \nu(dy) < \infty$. □

Recall from §2.2 that if E is a real, separable space E with a non-degenerate centered Gaussian measure $\mathcal{W}$, then there exists a real Hilbert space H which makes $(H, E, \mathcal{W})$ into an abstract Wiener space. Also, for each $x^* \in E^*$, $h_{x^*} \in H$ is the element in H such that $(h, h_{x^*})_H = \langle h, x^* \rangle$ for all $h \in H$, and, there exists $\{x_n^* : n \geq 1\} \subseteq E^*$, such that $\{h_n \equiv h_{x_n^*} : n \geq 1\}$ is an orthonormal basis in H. In particular, since the Paley-Wiener map $\mathcal{I}$ is a linear isometry from H to $L^2(\mathcal{W}; \mathbb{R})$, $\{\mathcal{I}(h_n) : n \geq 1\}$ forms a sequence of independent standard Gaussian random variables under $\mathcal{W}$. In other words, $\boldsymbol{\mathcal{I}} \equiv (\mathcal{I}(h_1), \cdots, \mathcal{I}(h_n), \cdots)$ has the distribution $\gamma^{\mathbb{N}}$ under $\mathcal{W}$.

Before proceeding, we need to review some properties of Hermite polynomials. Denote by $H_n(s)$ the nth order Hermite polynomials for $n \geq 0$, i.e. $H_n(s) \equiv (-1)^n e^{\frac{s^2}{2}} \left(\frac{d}{ds}\right)^n \left(e^{-\frac{s^2}{2}}\right)$, whose generating function is given by

$$e^{\lambda s - \frac{\lambda^2}{2}} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(s) \text{ for } \lambda \in \mathbb{R}. \tag{2.3.10}$$

For $\boldsymbol{s} = (s_1, \cdots, s_n, \cdots) \in \mathbb{R}^{\mathbb{N}}$ and a multi-index $\boldsymbol{a} = (a_1, \cdots, a_n, \cdots)$ which only has fintely many non-zero a_n's, define $H_{\boldsymbol{a}}(\boldsymbol{s}) \equiv \prod_{n=1}^{\infty} H_{a_n}(s_n)$. Then, for any $m \geq 1$,

$$(\partial_m H_{\boldsymbol{a}})(\boldsymbol{s}) = a_m H_{a_m - 1}(s_m) \prod_{n=1, n \neq m}^{\infty} H_{a_n}(s_n).$$

Furthermore, $\left\{\frac{H_{\boldsymbol{a}}(s)}{\sqrt{\boldsymbol{a}!}} :\ |\boldsymbol{a}| = n,\ n \geq 0\right\}$ forms an orthonormal basis in $L^2\left(\gamma^{\mathbb{N}};\ \mathbb{R}\right)$, and

$$L^2\left(\gamma^{\mathbb{N}};\ \mathbb{R}\right) = \oplus_{n=0}^{\infty}\overline{\operatorname{span}\left\{H_{\boldsymbol{a}}(s)\,;\ |\boldsymbol{a}| = n\right\}}^{L^2(\gamma^{\mathbb{N}};\mathbb{R})}.$$

In terms of the abstract Wiener space $(H,\ E,\ \mathcal{W})$, the preceding says that $\left\{\frac{H_{\boldsymbol{a}}(\boldsymbol{\mathcal{I}})}{\sqrt{\boldsymbol{a}!}} :\ |\boldsymbol{a}| = n,\ n \geq 0\right\}$ forms an orthonormal basis for $L^2(\mathcal{W};\ \mathbb{R})$, and if

$$Z^{(n)}(E) \equiv \overline{\operatorname{span}\left\{H_{\boldsymbol{a}}(\boldsymbol{\mathcal{I}})\,;\ |\boldsymbol{a}| = n\right\}}^{L^2(\mathcal{W};\mathbb{R})},$$

then $L^2(\mathcal{W};\ \mathbb{R}) = \oplus_{n=0}^{\infty}Z^{(n)}(E)$. Since, clearly, $\left(H^2,\ E^2,\ \mathcal{W}^2\right)$ is also an abstract Wiener space, we can look at the same structure of $L^2\left(\mathcal{W}^2;\ \mathbb{R}\right)$ of the product space. To be specific, define $\boldsymbol{\mathcal{J}} :\ E^2 \to \mathbb{R}^2$ by

$$\boldsymbol{\mathcal{J}}(x,\ y) \equiv \left(\boldsymbol{\mathcal{J}}_1(x,\ y),\ \boldsymbol{\mathcal{J}}_2(x,\ y)\right) = \left(\boldsymbol{\mathcal{I}}(x),\ \boldsymbol{\mathcal{I}}(y)\right),$$

and $H_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{\mathcal{J}}) \equiv H_{\boldsymbol{a}}(\boldsymbol{\mathcal{J}}_1)\,H_{\boldsymbol{b}}(\boldsymbol{\mathcal{J}}_2)$, then $\left\{\frac{H_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{\mathcal{J}})}{\sqrt{\boldsymbol{a}!\boldsymbol{b}!}} : |\boldsymbol{a}| + |\boldsymbol{b}| = n,\ n \geq 0\right\}$ is an orthonormal basis for $L^2\left(\mathcal{W}^2;\ \mathbb{R}\right)$. Similarly, if

$$Z^{(n)}\left(E^2\right) \equiv \overline{\operatorname{span}\left\{H_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{\mathcal{J}}) : |\boldsymbol{a}| + |\boldsymbol{b}| = n\right\}}^{L^2(\mathcal{W}^2;\mathbb{R})},$$

then $L^2\left(\mathcal{W}^2;\ \mathbb{R}\right) = \oplus_{n=0}^{\infty}Z^{(n)}\left(E^2\right)$. Finally, if X is an element in $L^2(\mathcal{W};\ \mathbb{R})$ (or $L^2\left(\mathcal{W}^2;\ \mathbb{R}\right)$), then we use P_nX to denote the projection of X onto $Z^{(n)}(E)$ (or $Z^{(n)}\left(E^2\right)$).

Lemma 2.3. *Suppose $f :\ E \to \mathbb{R}$ is Borel measurable and $f \in L^2(\mathcal{W};\ \mathbb{R})$, then for any Pythagorean pair $(\alpha,\ \beta)$, $f \circ S_{\alpha,\beta} \in L^2\left(\mathcal{W}^2;\ \mathbb{R}\right)$. Moreover,*

$$(P_nf) \circ S_{\alpha,\beta} = P_n\left(f \circ S_{\alpha,\beta}\right). \tag{2.3.11}$$

Proof. The first statement is an immediate consequence of the fact that $\left(S_{\alpha,\beta}\right)_*\mathcal{W}^2 = \mathcal{W}$. Since $f \circ S_{\alpha,\beta} \in L^2\left(\mathcal{W}^2;\ \mathbb{R}\right)$, it is clear that

$$\sum_n (P_nf) \circ S_{\alpha,\beta} = \sum_n P_n\left(f \circ S_{\alpha,\beta}\right).$$

In view of this, it is enough for us to prove $(P_nf) \circ S_{\alpha,\beta} \in Z^{(n)}\left(E^2\right)$, which comes down to showing $H_{\boldsymbol{a}} \circ S_{\alpha,\beta} \in Z^{(n)}\left(E^2\right)$ for every multi-index $\boldsymbol{a}$ with $|\boldsymbol{a}| = n$. But, by (2.3.10) and the fact that $\alpha^2 + \beta^2 = 1$, for any λ, s, $t \in \mathbb{R}$

$$\begin{aligned}\sum_n \frac{\lambda^n}{n!} H_n(\alpha s + \beta t) &= e^{\lambda(\alpha s+\beta t) - \frac{\lambda^2}{2}(\alpha^2+\beta^2)} \\ &= \left(\sum_n \frac{\lambda^n \alpha^n}{n!} H_n(s)_n\right)\left(\sum_m \frac{\lambda^m \beta^m}{m!} H_n(t)\right).\end{aligned}$$

It is clear that

$$H_n(\alpha s + \beta t) = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} H_k(s) H_{n-k}(t),$$

which is sufficient for us to get the conclusion $H_{\boldsymbol{a}} \circ S_{\alpha,\beta} \in Z^{(n)}\left(E^2\right)$. □

Now for each $h \in H$, we consider the differential operator ∂_h on $\cup_n Z^{(n)}(E)$, which is the linear map that takes $Z^{(n)}(E)$ to $Z^{(n-1)}(E)$ and determined by

$$\begin{aligned}(\partial_h H_{\boldsymbol{a}})(\boldsymbol{\mathcal{I}}) &= \frac{d}{dt}\left(H_{\boldsymbol{a}}(\boldsymbol{\mathcal{I}})(\cdot + th)\right)\Big|_{t=0} \\ &= \sum_{m=0}^{\infty} (h, h_m)_H (\partial_m H_{\boldsymbol{a}})(\boldsymbol{\mathcal{I}})\end{aligned}$$

for any multi-index $\boldsymbol{a}$ with $|\boldsymbol{a}| = n$. Note that,

$$\|(\partial_h H_{\boldsymbol{a}})(\boldsymbol{\mathcal{I}})\|^2_{L^2(\mathcal{W};,\mathbb{R})} = \boldsymbol{a}! \sum_{m=0}^{\infty} (h, h_m)_H^2 \, a_m \leq n\boldsymbol{a}! \, \|h\|_H^2 .$$

Similarly, if $\partial_{(h,g)}$ is the differential operator on $\cup_n Z^{(n)}\left(E^2\right)$, then for any multi-indices $\boldsymbol{a}$, $\boldsymbol{b}$ with $|\boldsymbol{a}| + |\boldsymbol{b}| = n$,

$$\begin{aligned}\left(\partial_{(h,g)} H_{\boldsymbol{a},\boldsymbol{b}}\right)(\boldsymbol{\mathcal{J}}_1, \boldsymbol{\mathcal{J}}_2) &= \frac{d}{dt}\left(H_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{\mathcal{J}}_1, \boldsymbol{\mathcal{J}}_2)(\cdot + th, \star + tg)\right)\Big|_{t=0} \\ &= \sum_{m=0}^{\infty} (h, h_m)_H (\partial_m H_{\boldsymbol{a}})(\boldsymbol{\mathcal{J}}_1) H_{\boldsymbol{b}}(\boldsymbol{\mathcal{J}}_2) + \sum_{k=0}^{\infty} (g, h_k)_H (\partial_k H_{\boldsymbol{b}})(\boldsymbol{\mathcal{J}}_2) H_{\boldsymbol{a}}(\boldsymbol{\mathcal{J}}_1) \\ &= (\partial_h H_{\boldsymbol{a}})(\boldsymbol{\mathcal{J}}_1) H_{\boldsymbol{b}}(\boldsymbol{\mathcal{J}}_2) + H_{\boldsymbol{a}}(\boldsymbol{\mathcal{J}}_1)(\partial_g H_{\boldsymbol{b}})(\boldsymbol{\mathcal{J}}_2),\end{aligned}$$

and therefore,

$$\begin{aligned}\left\|\left(\partial_{(h,g)}H_{\boldsymbol{a},\boldsymbol{b}}\right)(\mathcal{J}_1,\,\mathcal{J}_2)\right\|^2_{L^2(\mathcal{W}^2;\,\mathbb{R})} &= \boldsymbol{a}!\boldsymbol{b}!\left(\sum_m (h,\,h_m)_H^2\,a_m + \sum_k (g,\,h_k)_H^2\,b_k\right)\\ &\leq n\boldsymbol{a}!\boldsymbol{b}!\left(\|h\|_H^2 + \|g\|_H^2\right).\end{aligned}$$

Finally, we are ready to state the main result on Wiener maps.

Theorem 2.4. *Suppose $(H,\,E,\,\mathcal{W})$ is an abstract Wiener space, and $f:\,E\to\mathbb{R}$ is a Wiener map, then there exists an $h\in H$, such that $f(x)=\mathcal{I}(h)(x)$ for $\mathcal{W}-$almost every $x\in E$.*

Proof. If $(\alpha,\,\beta)$ is the Pythagorean pair for which f satisfies (2.3.8), then, by Lemma 2.3,

$$P_n\left(f\circ S_{\alpha,\beta}\right)(x,\,y) = \alpha P_n f(x) + \beta P_n f(y) \quad \text{for } \mathcal{W}^2 - \text{almost every } (x,\,y)\in E^2.$$

On the other hand, by (2.3.11),

$$P_n\left(f\circ S_{\alpha,\beta}\right) = \left(P_n f\right)\circ S_{\alpha,\beta}.$$

So we know that $P_n f$ is also a solution to (2.3.8), and therefore, for any $h\in H$,

$$\begin{aligned}0=\partial_{(\beta h,\,-\alpha h)}\left(\left(P_n f\right)\circ S_{\alpha,\beta}(x,\,y)\right) &= \partial_{(\beta h,\,-\alpha h)}\left(\alpha P_n f(x) + \beta P_n f(y)\right)\\ &= \alpha\beta\left(\partial_h\left(P_n f\right)(x) - \partial_h\left(P_n f\right)(y)\right)\end{aligned}$$

for $\mathcal{W}^2-$almost every $(x,\,y)\in E^2$. Because $\partial_h\left(P_n f\right)(x)$ is independent of $\partial_h\left(P_n f\right)(y)$ under $\mathcal{W}^2$, the only way this can be true is if $\partial_h\left(P_n f\right)$ is a constant $\mathcal{W}-$almost surely, and hence in $Z^{(0)}(E)$. Since, $\partial_h\left(P_n f\right)\in Z^{(n-1)}(E)$, this means that when $n\geq 2$, $\partial_h\left(P_n f\right)=0$ $\mathcal{W}-$almost surely. In particular, $\partial_{h_m}\left(P_n f\right)=0$ $\mathcal{W}-$almost surely for $m\geq 1$ and $n\geq 2$, which implies

$$\|P_n f\|^2_{L^2(\mathcal{W};\,\mathbb{R})} = \frac{1}{n}\sum_{m=1}^{\infty}\|\partial_{h_m}\left(P_n f\right)\|^2_{L^2(\mathcal{W};\,\mathbb{R})} = 0.$$

That is, when $n\geq 2$, $P_n f = 0$ $\mathcal{W}-$almost surely. In addition, by (2.3.8),

$$\mathbb{E}^{\mathcal{W}}[f] = \mathbb{E}^{\mathcal{W}^2}\left[f\circ S_{\alpha,\beta}\right] = (\alpha+\beta)\,\mathbb{E}^{\mathcal{W}}[f]$$

for some α, $\beta \in (0, 1)$ and $\alpha^2 + \beta^2 = 1$. Hence, $P_0 f = \mathbb{E}^{\mathcal{W}}[f] = 0$. Hence, we know that $f = P_1 f = \mathcal{I}(h)$ $\mathcal{W}-$almost surely for some $h \in H$. □

Our next goal is to show that a Wiener map is $\mathcal{W}-$almost everywhere equal to a linear function which is defined up to a null set on E. In fact, this is an immediate consequence of the following theorem.

Theorem 2.5. *If $(H, E, \mathcal{W})$ is an abstract Wiener space, then, for every $h \in H$, there exists a Banach space E_h, which is embedded in E as a measurable set that satisfies $\mathcal{W}(E_h) = 1$, and a $\Phi \in E_h^*$, such that $\mathcal{I}(h)(x) = \Phi(x)$ for $\mathcal{W}-$almost everywhere $x \in E_h$. In particular, Φ satisfies (2.3.8) with any Pythagorean pair.*

Proof. The proof to this theorem is inspired by Theorem 8.3.9 in [13], which is based on the idea from [8] that is alluded in §2.2. Suppose $\{x_n^* : n \geq 1\} \subseteq E^*$ is the sequence which makes $\{h_n \equiv h_{x_n^*} : n \geq 1\}$ an orthonormal basis for H. Given $x \in E$ and $n \in \mathbb{N}$, we introduce the following notation:

$$S_n(x) \equiv \sum_{m=1}^{n} \langle x, x_m^* \rangle h_m,$$

$$s_n(x) \equiv \sum_{m=1}^{n} \langle x, x_m^* \rangle (h_m, h)_H .$$

Since H is ly embedded into E, without loss of generality, we will assume $\|\cdot\|_E \leq \|\cdot\|_H$. By Theorem 8.3.9 in [13], we can choose an increasing subsequence $\{n_l : l \geq 1\} \subseteq \mathbb{N}$, such that the following two inequalities,

$$\mathbb{E}^{\mathcal{W}}\left[\left\|S_{n_{l+1}} - S_{n_l}\right\|_E^2\right] \leq 2^{-l} \text{ and} \tag{2.3.12}$$

$$\mathbb{E}^{\mathcal{W}}\left[\left|s_{n_{l+1}} - s_{n_l}\right|^2\right] = \sum_{m=n_l+1}^{n_{l+1}} (h, h_m)^2 \leq 2^{-l} \|h\|_H^2 \tag{2.3.13}$$

are both satisfied for $l \geq 1$. Set $n_0 = 1$, denote by E_h^0 the subset

$$\left\{x \in E : S_{n_l}(x) \to x \text{ in } E \text{ as } l \to \infty, \sum_{l=0}^{\infty} \left\|S_{n_{l+1}}(x) - S_{n_l}(x)\right\|_E < \infty, \text{ and } \sum_{l=0}^{\infty} \left|s_{n_{l+1}}(x) - s_{n_l}(x)\right| < \infty\right\},$$

and, for $x \in E_h^0$, define

$$\|x\|_{E_h} \equiv \sum_{l=0}^{\infty} \left\|S_{n_{l+1}}(x) - S_{n_l}(x)\right\|_E + \sum_{l=0}^{\infty} \left|s_{n_{l+1}}(x) - s_{n_l}(x)\right|.$$

Denote by E_h the closure of E_h^0 in E under $\|\cdot\|_{E_h}$. Clearly, E_h is a measurable subspace of E and we will show that E_h is furthermore complete under $\|\cdot\|_{E_h}$. To this end, consider a Cauchy sequence $\{x_j \in E_h : j \geq 1\}$ under $\|\cdot\|_{E_h}$. Then both $\left\{\sum_{l=0}^{\infty} \left\|S_{n_{l+1}}(x_j) - S_{n_l}(x_j)\right\|_E : j \geq 1\right\}$ and $\left\{\sum_{l=0}^{\infty} \left|s_{n_{l+1}}(x_j) - s_{n_l}(x_j)\right| : j \geq 1\right\}$ are Cauchy sequences in $\mathbb{R}$. So there exist real numbers R and r, such that

$$\lim_{j\to\infty} \sum_{l=0}^{\infty} \left\|S_{n_{l+1}}(x_j) - S_{n_l}(x_j)\right\|_E = R, \text{ and } \lim_{j\to\infty} \sum_{l=0}^{\infty} \left|s_{n_{l+1}}(x_j) - s_{n_l}(x_j)\right| = r.$$

Notice that

$$\begin{aligned} \|x_j - x_i\|_E &= \lim_{l\to\infty} \|S_{n_l}(x_j) - S_{n_l}(x_i)\|_E \\ &\leq \sum_{l=0}^{\infty} \left\|S_{n_{l+1}}(x_j) - S_{n_l}(x_j) - \left(S_{n_{l+1}}(x_i) - S_{n_l}(x_i)\right)\right\|_E \\ &\leq \|x_j - x_i\|_{E_h}. \end{aligned}$$

Hence $\{x_j : j \geq 1\}$ is also a Cauchy sequence in E, and therefore there exists $x \in E$, such that $x_j \to x$ in E as $j \to \infty$. If we can prove $x \in E_h$, and $x_j \to x$ also in E_h, then we would have the completeness of E_h.

We start proving $x \in E_h$ by noticing that

$$\begin{aligned} \sum_{l=0}^{L} \left\|S_{n_{l+1}}(x) - S_{n_l}(x)\right\|_E &= \lim_{j\to\infty} \sum_{l=0}^{L} \left\|S_{n_{l+1}}(x_j) - S_{n_l}(x_j)\right\|_E \\ &\leq \lim_{j\to\infty} \sum_{l=0}^{\infty} \left\|S_{n_{l+1}}(x_j) - S_{n_l}(x_j)\right\|_E, \end{aligned}$$

which implies

$$\sum_{l=0}^{\infty} \left\|S_{n_{l+1}}(x) - S_{n_l}(x)\right\|_E \leq R.$$

In addition,

$$\begin{aligned}
\left\|S_{n_l}(x)-x\right\|_E &\leq \left\|S_{n_l}(x)-S_{n_l}(x_j)\right\|_E+\left\|S_{n_l}(x_j)-x_j\right\|_E+\left\|x_j-x\right\|_E \\
&= \lim_{i\to\infty}\left\|S_{n_l}(x_i)-S_{n_j}(x_j)\right\|_E+\left\|S_{n_l}(x_j)-x_j\right\|_E+\left\|x_j-x\right\|_E \\
&\leq \sup_{i\geq j}\left\|S_{n_l}(x_i)-S_{n_j}(x_j)\right\|_E+\left\|S_{n_l}(x_j)-x_j\right\|_E+\left\|x_j-x\right\|_E \\
&\leq \sup_{i\geq j}\left\|x_i-x_j\right\|_{E_h}+\left\|S_{n_l}(x_j)-x_j\right\|_E+\left\|x_j-x\right\|_E.
\end{aligned}$$

Given any $\epsilon > 0$, we choose a j that is large enough, such that $\|x_j - x\|_E \leq \epsilon/3$ and $\sup_{i\geq j}\|x_i - x_j\|_{E_h} \leq \epsilon/3$. Then, for the fixed j, choose l large enough, so that $\|S_{n_l}(x_j) - x_j\|_E \leq \epsilon/3$. Then, $\|S_{n_l}(x) - x\|_E \leq \epsilon$, and so $\lim_{l\to\infty} S_{n_l}(x) = x$ in E. Moreover, we have

$$\begin{aligned}
\sum_{l=0}^{L}\left|s_{n_{l+1}}(x)-s_{n_l}(x)\right| &= \lim_{j\to\infty}\sum_{l=0}^{L}\left|s_{n_{l+1}}(x_j)-s_{n_l}(x_j)\right| \\
&\leq \lim_{j\to\infty}\sum_{l=0}^{\infty}\left|s_{n_{l+1}}(x_j)-s_{n_l}(x_j)\right|,
\end{aligned}$$

which implies $\sum_{l=0}^{\infty}\left|s_{n_{l+1}}(x) - s_{n_l}(x)\right|$ is convergent. Therefore, we know that $x \in E_h$.

Verifying $\|x_j - x\|_{E_h} \to 0$ comes down to showing

$$\sum_{l=0}^{\infty}\left\|S_{n_{l+1}}(x_j)-S_{n_l}(x_j)-\left(S_{n_{l+1}}(x)-S_{n_l}(x)\right)\right\|_E \to 0,$$

and

$$\sum_{l=0}^{\infty}\left|s_{n_{l+1}}(x)-s_{n_l}(x)-\left(s_{n_{l+1}}(x_j)-s_{n_l}(x_j)\right)\right| \to 0.$$

We will only prove the first statement, since the second follows from similar considerations. By Fatou's Lemma,

$$\begin{aligned}
&\sum_{l=0}^{\infty}\left\|S_{n_{l+1}}(x)-S_{n_l}(x)-\left(S_{n_{l+1}}(x_j)-S_{n_l}(x_j)\right)\right\|_E \\
\leq\ &\liminf_{i\to\infty}\sum_{l=0}^{\infty}\left\|S_{n_{l+1}}(x_i)-S_{n_l}(x_i)-\left(S_{n_{l+1}}(x_j)-S_{n_l}(x_j)\right)\right\|_E \\
\leq\ &\sup_{i\geq j}\left\|x_i-x_j\right\|_{E_h} \to 0 \text{ as } j\to\infty.
\end{aligned}$$

Summarizing, we have proved that E_h is a Banach space with the norm $\|\cdot\|_{E_h}$.

Now, consider the following functional on E_h:

$$\Phi:\ x \in E_h \mapsto \Phi(x) \equiv \sum_{l=0}^{\infty}\left(s_{n_{l+1}}(x) - s_{n_l}(x)\right) = \lim_{l\to\infty} s_{n_l}(x) = \lim_{l\to\infty}\sum_{m=1}^{n_l}(h,\ h_m)_H\langle x,\ x_m^*\rangle.$$

Clearly, Φ is a Borel measurable linear map and because,

$$|\Phi(x)| \le \sum_{l=0}^{\infty}\left|s_{n_{l+1}}(x) - s_{n_l}(x)\right| \le \|x\|_{E_h},$$

Φ is a continuous linear functional on E_h. The reason why $\mathcal{W}(E_h) = 1$ lies in the choice of $\{n_l:\ l \ge 1\}$. Namely, by (2.3.12) and (2.3.13), for $\mathcal{W}-$almost every $x \in E$, $\|x\|_{E_h} < \infty$. Moreover, (2.2.4) shows that $S_n(x) \to x$ and hence $s_n(x) \to \mathcal{I}(h)(x)$ as $n \to \infty$ for $\mathcal{W}-$almost every $x \in E$. Therefore, we know that $\mathcal{W}(E_h) = \mathcal{W}(E_h^0) = 1$, and $\Phi(x) = \mathcal{I}(h)(x)$ for $\mathcal{W}-$almost every $x \in E$.

To complete the proof, we still need to show that Φ is a Wiener map for any Pythagorean pair $(\alpha,\ \beta)$, but this is obvious since $\mathcal{W}(E_h) = 1$ and Φ on E_h is linear. □

Corollary 2.6. *Given an abstract Wiener space $(H,\ E,\ \mathcal{W})$ and a Borel measurable functional $f:\ E \to \mathbb{R}$, then f is a Wiener map, if and only if f is $\mathcal{W}-$almost everywhere linear in the sense that, there exists a measurable subset $L \subseteq E$ with $\mathcal{W}(L) = 1$, and a linear function Φ defined on span(L), such that $f = \Phi$ on L. In particular, f is a solution to (2.3.8) with any Pythagorean pair.*

Proof. This is an immediate consequence of Theorem 2.4 and Theorem 2.5. □

As mentioned in the introduction, it is worth pointing out that the result from Lemma 2.2, which says that all Wiener maps are square integrable, leads to a result ([16]) closely related to the theorem of Schwartz in [11].

Theorem 2.7. *If E is a real, separable Banach space, and $f:\ E \to \mathbb{R}$ is Borel measurable and linear, then $f \in E^*$.*

Proof. To prove $f \in E^*$, we need to show that if $x_n \to 0$ in E as $n \to \infty$, then $f(x_n) \to 0$. Assume not, then we can find a sequence $\{x_n \in E:\ n \ge 1\}$ such that $\|x_n\|_E = \frac{1}{n^2}$, and $|f(x_n)| \ge n$. Let $E_0 \equiv \overline{\text{span}\{x_n:\ n \ge 1\}}^E$. Since $\sum_{n=1}^{\infty}\|x_n\|_E < \infty$, $\sum_{n=1}^{\infty}|\xi_n|\,\|x_n\|_E < \infty$ for $\gamma^{\mathbb{N}}-$almost every $\boldsymbol{\xi} \equiv (\xi_1, \cdots, \xi_n, \cdots) \in \mathbb{R}^{\mathbb{N}}$. Thus, there exists a random variable

$X : \mathbb{R}^{\mathbb{N}} \to E_0$ such that $X(\boldsymbol{\xi}) \equiv \sum_{n=1}^{\infty} \xi_n x_n \in E_0$ for $\gamma^{\mathbb{N}}$−almost every $\boldsymbol{\xi} \in \mathbb{R}^{\mathbb{N}}$. Denote by $\mathcal{W}$ the distribution of X under $\gamma^{\mathbb{N}}$. Furthermore, because f is linear, $f \restriction E_0$ is a Wiener map, and $\mathbb{E}^{\mathcal{W}}\left[f^2\right] < \infty$. However,

$$\mathbb{E}^{\mathcal{W}}\left[f^2\right] = \mathbb{E}^{\gamma^{\mathbb{N}}}\left[f^2(X)\right] \geq f^2(x_n) \geq n^2 \text{ for all } n \geq 1.$$

Hence, the contradiction shows that such a sequence $\{x_n : n \geq 1\}$ cannot exist, which implies f is bounded and hence $f \in E^*$. □

Remark 2.8. As one can see, Borel measurable additive functions on either finite or infinite dimensional spaces must be linear and continuous. However, the preceding results about Wiener maps reveal that it is more complicated when one deals with probabilistic generalizations of additive functions. It is clear from above that, every Wiener map is almost everywhere equal to some linear function, which is, continuous if everything is in finite dimensions, or, defined up to a zero-measure set if in infinite dimensions . However, when it comes to almost everywhere additive functions, the differences between finite and infinite dimensions are more prominent. To explaint this, let's first consider the finite dimensional case. When $\dim E = N$, whence $H = E = \mathbb{R}^N$ and $\mathcal{W} = \gamma^N$, if $f : E \to \mathbb{R}$ is almost everywhere additive, then f is a Wiener map, and therefore there exists a $\xi \in \mathbb{R}^N$ such that $f(x) = (x, \xi)_{\mathbb{R}^N}$ for γ^N−almost every $x \in \mathbb{R}^N$. At the same time, if measurable function $f : E \to \mathbb{R}$ is equal to $(\cdot, \xi)_{\mathbb{R}^N}$ for some $\xi \in \mathbb{R}^N$ on a measurable set L with $\gamma^N(L) = 1$, then, because $(S_{1,1})_* \gamma^{2N} \sim \gamma^N$, $x + y \in L$ for γ^{2N}−almost every $(x, y) \in L^2$, and hence f satisfies (2.3.5). Therefore, in finite dimensions, an almost everywhere additive function is almost everywhere linear, and vice versa. On the contrary, when $\dim E = \infty$, the following two examples illustrate that an almost additive function can be "far away" from being a Wiener map, or being linear, and vice versa.

Example 2.9. Given an abstract Wiener space $(H, E, \mathcal{W})$ with $\dim E = \infty$, again take a sequence $\{x_m^* : m \geq 1\} \subseteq E^*$ such that $\left\{h_m \equiv h_{x_m^*} : m \geq 1\right\}$ is an orthonormal basis of H. Set

$$N \equiv \left\{ x \in E : \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \langle x, x_m^* \rangle^2 \text{ exists} \right\},$$

and define $f:\ E \to \mathbb{R}$ to be

$$f(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \langle x,\, x_m^* \rangle^2, & \text{if } x \in N; \\ \pi, & \text{otherwise.} \end{cases}$$

Clearly, f is Borel measurable, and by the Law of Large Number, one sees that $f(x) = 1$ for $\mathcal{W}$–almost every $x \in E$, and,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \langle x,\, x_m^* \rangle \langle y,\, x_m^* \rangle \to 0 \ \text{ for } \ \mathcal{W}^2 - \text{almost every} \ \ (x,\, y) \in E^2.$$

Therefore, for $\mathcal{W}^2$–almost every $(x,\, y) \in E^2$,

$$\begin{aligned} f(x+y) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \langle x+y,\, x_m^* \rangle^2 &= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \left(\langle x,\, x_m^* \rangle + \langle y,\, x_m^* \rangle \right)^2 \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \left(\langle x,\, x_m^* \rangle^2 + 2 \langle x,\, x_m^* \rangle \langle y,\, x_m^* \rangle + \langle y,\, x_m^* \rangle^2 \right) \\ &= f(x) + f(y). \end{aligned}$$

That is, f is almost eveywhere additive. However, with similar arguments, one can easily check that,

$$f\left(\frac{x+y}{\sqrt{2}} \right) = 1 \neq \sqrt{2} = \frac{f(x) + f(y)}{\sqrt{2}} \ \text{ for } \ \mathcal{W}^2 - \text{almost every} \ \ (x,\, y) \in E^2,$$

$$\text{and } f(2x) = 4 \neq 2 = 2f(x) \ \text{ for } \ \mathcal{W}^2 - \text{almost every} \ \ x \in E.$$

Therefore, f is NOT a Wiener map, and f is $\mathcal{W}$–almost NOWHERE linear.

Conversely, the following example shows that Wiener maps are not necessarily almost everywhere additive.

Example 2.10. Assume $(H,\, E,\, \mathcal{W})$ and the sequence $\{x_m^* :\ m \geq 1\} \subseteq E^*$ are as in Exmaple 2.9, and define $f:\ E \to \mathbb{R}$ to be

$$f(x) = \begin{cases} \langle x,\, x_1^* \rangle, & \text{if } \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \langle x,\, x_m^* \rangle^2 = 1; \\ \pi, & \text{otherwise.} \end{cases}$$

By the same arguments as in Example 2.9, we know that for any Pythagorean pair (α, β),

$$f(\alpha x + \beta y) = \langle \alpha x + \beta y, x_1^* \rangle = \alpha f(x) + \beta f(y) \quad \text{for } \mathcal{W}^2 - \text{almost every} \quad (x, y) \in E^2,$$

so f is a Wiener map. However,

$$f(x+y) = \pi \neq \langle x, x_1^* \rangle + \langle y, x_1^* \rangle = f(x) + f(y) \quad \text{for } \mathcal{W}^2 - \text{almost every} \quad (x, y) \in E^2.$$

which means f is almost NOWHERE additive.

2.3.3 Back to Abstract Wiener Spaces

We will now justify the statement mentioned in §2.2 that, the choice of the "housing" E for an abstract Wiener space $(H, E, \mathcal{W})$ is not canonical, and we can always make E smaller. In fact, if one follows the original arguments given by Gross in [8], one could haven chosen the "smaller" E_0 such that E_0 is even compactly embedded in E. In connection with the theory of Wiener maps, we have the following observations.

Lemma 2.11. *Let* $(H, E, \mathcal{W})$, $\{x_n^* \in E^* : n \geq 1\}$, $h \in H$ *and* $\left(E_h, \|\cdot\|_{E_h}\right)$ *as in Theorem 2.5, then* $(H, E_h, \mathcal{W} \upharpoonright E_h)$ *forms an abstract Wiener space.*

Proof. We first show that H is continuously embedded in E_h as a dense subspace under $\|\cdot\|_{E_h}$. Take $\{n_l : l \geq 1\}$ and define S_{n_l} and s_{n_l} as in the proof of Theorem 2.5, then, for any $x \in E_h$,

$$\begin{aligned} \|S_{n_l}(x) - x\|_{E_h} &= \sum_{j=l}^{\infty} \left\|S_{n_{j+1}}(x) - S_{n_j}(x)\right\|_E + \sum_{j=l}^{\infty} \left|s_{n_{j+1}}(x) - s_{n_j}(x)\right| \\ &\to 0 \text{ as } l \to \infty. \end{aligned}$$

Therefore, the density of H in E_h is obvious. In addition, by Theorem 8.3.9 in [13], for any $g \in H$,

$$\begin{aligned} \|g\|_{E_h} &= \sum_{l=0}^{\infty} \left\|S_{n_{l+1}}(g) - S_{n_l}(g)\right\|_E + \sum_{l=0}^{\infty} \left|s_{n_{l+1}}(g) - s_{n_l}(g)\right| \\ &\leq \|g\|_H \left(\sum_{l=1}^{\infty} 2^{-l} + \sum_{m=1}^{n_1} \|h_m\|_H^2 + \|h\|_H \right) \\ &\leq (1 + n_1 + \|h\|_H) \|g\|_H. \end{aligned}$$

Hence, H is continuously embedded in E_h.

To complete the proof, we still need to show that for any $y^* \in E_h^*$, $x \in E_h \mapsto_{E_h} \langle x, y^* \rangle_{E_h^*}$ has centered Gaussian distribution with variance $\|h_{y^*}\|_H^2$ under $\mathcal{W} \upharpoonright E_h$, where $h_{y^*} \in H$ is determined by $(g, h_{y^*})_H = {}_{E_h}\langle g, y^* \rangle_{E_h^*}$ for all $g \in H$. In fact, since $S_{n_l}(x) \to x$ as $l \to \infty$ in E_h, then

$$\begin{aligned} {}_{E_h}\langle x, y^* \rangle_{E_h^*} &= \lim_{l \to \infty} {}_{E_h}\langle S_{n_l}(x), y^* \rangle_{E_h^*} \\ &= \sum_{m=1}^{\infty} {}_{E_h}\langle h_m, y^* \rangle_{E_h^*} \, {}_E\langle x, x_m^* \rangle_{E^*} \\ &= \sum_{m=1}^{\infty} (h_m, h_{y^*})_H \, {}_E\langle x, x_m^* \rangle_{E^*} . \end{aligned}$$

Therefore, ${}_{E_h}\langle \cdot, y^* \rangle_{E_h^*}$ has the same distribution with $\mathcal{I}(h_{y^*})$ under $\mathcal{W} \upharpoonright E_h$. □

The preceding shows that given an infinite dimensional Gaussian measure $\mathcal{W}$ with underlying Hilbert space H, we can always get smaller housing space of $\mathcal{W}$ by "shrinking" the Banach space E in the triple $(H, E, \mathcal{W})$. One should expect that as this process goes on, E is getting closer and closer to H. Indeed, we have the following result.

Theorem 2.12. *Suppose H is a real, separable Hilbert space and $\mathcal{W}$ is a non-degenerate centered Gaussian measure, such that $(H, E, \mathcal{W})$ is an abstract Wiener space for some real, separable Banach space E, then*

$$H = \cap \{ E : \ (H, E, \mathcal{W} \upharpoonright E) \ \textit{ is an abstract Wiener space } \} .$$

Proof. Starting with any E such that $(H, E, \mathcal{W})$ is an abstract Wiener space, take a sequence $\{x_m^* : m \geq 1\} \subseteq E^*$ so that $\{h_m \equiv h_{x_m^*} : m \geq 1\}$ is an orthonormal basis in H. For $h \in H$, denote by E_h the Banach space which we found in Theorem 2.5. Note that we will be done if we can show $H = \cap \{E_h : h \in H\}$. Clearly, $H \subseteq \{E_h : h \in H\}$, so we only need to prove the other direction. Given $g \in \cap \{E_h : h \in H\}$, by Theorem 2.5, for any $h \in H$, there exsits a subsequence $\{n_l^h : l \geq 1\}$ such that

$$\lim_{l \to \infty} \sum_{m=1}^{n_l^h} (h, h_m)_H \langle g, x_m^* \rangle \quad \text{exists.} \tag{2.3.14}$$

In fact, we claim that

$$\lim_{n\to\infty}\sum_{m=1}^{n}(h,\, h_m)_H\langle g,\, x_m^*\rangle \quad \text{exists for every } h\in H. \tag{2.3.15}$$

If not, there must be an $h_0 \in H$, such that $\sum_{m=1}^n |(h_0,\, h_m)_H \langle g,\, x_m^*\rangle| \to \infty$. Choose $\tilde{h}_0 \in H$ that satisfies $\left(\tilde{h}_0,\, h_m\right)_H = \pm(h_0,\, h_m)_H$ for all $m \geq 1$, and the plus or minus signe is determined so that

$$\left(\tilde{h}_0,\, h_m\right)\langle g,\, x_m^*\rangle = |(h_0,\, h_m)_H\langle g,\, x_m^*\rangle|\,.$$

Thus, $\sum_{m=1}^n \left(\tilde{h}_0,\, h_m\right)_H \langle g,\, x_m^*\rangle$ will diverge to infinity along any subsequence, which violates (2.3.14). The contradiction shows that (2.3.15) must be true.

Moreover, if $\Phi_n(h) \equiv \sum_{m=1}^n (h,\, h_m)_H\langle g,\, x_m^*\rangle$ for $h \in H$, then Φ_n's are continuous linear functionals on H, and Φ_n converges to $\Phi \equiv \sum_{m=1}^\infty (\cdot,\, h_m)_H\langle g,\, x_m^*\rangle$ weakly. By Uniform Boundness Principle, Φ is also a continuous linear functional on H. In particular, there exists a constant $C_g < \infty$, such that,

$$\left|\sum_{m=1}^{\infty}(h,\, h_m)_H\langle g,\, x_m^*\rangle\right| \leq C_g\,\|h\|_H \quad \text{for all } \ h\in H,$$

which implies $\sum_{m=1}^\infty \langle g,\, x_m^*\rangle^2 < \infty$, and hence by (2.2.2), $g \in H$. □

We close this section by a statement, which, in some sense, complements Theorem 2.12.

Theorem 2.13. *If E is a real, separable Banach space, then*

$$E = \cup\{H:\ H \textit{ is the Cameron-Martin space for some Gaussian measure } \mathcal{W} \textit{ on } E\}\,.$$

Proof. We first use induction to produce sequences $\{x_n \in E : n \geq 1\}$ and $\{x_n^* \in E^* : n \geq 1\}$ such that for every $n,\, m \geq 1$, $x_1, \cdots, x_n$ are linearly independent, $\|x_n\|_E = \frac{1}{n^2}$, and $\langle x_m,\, x_n^*\rangle = \delta_{m,n}$. We start with an arbitrary $x_1 \in E$ with $\|x_1\|_E = 1$, and $x_1^* \in E^*$ such that $\langle x_1,\, x_1^*\rangle = 1$. Suppose we have got $\{x_1, \cdots, x_n\}$ and $\{x_1^*, \cdots, x_n^*\}$. Choose $y_{n+1} \in E \cap \left(\overline{\text{span}\{x_1, \cdots, x_n\}}\right)^{\complement}$, and set

$$x_{n+1} \equiv \frac{y_{n+1} - \sum_{m=1}^n \langle y_{n+1},\, x_m^*\rangle x_m}{(n+1)^2\,\|y_{n+1} - \sum_{m=1}^n\langle y_{n+1},\, x_m^*\rangle x_m\|_E}.$$

It is clear that $\langle x_{n+1}, x_m^* \rangle = 0$ for $1 \le m \le n$. By Finally, by Hahn-Banach Theorem, there exists $x_{n+1}^* \in E^*$ such that $\langle x_{n+1}, x_{n+1}^* \rangle = 1$ and $\langle x_m, x_{n+1}^* \rangle = 0$ for $1 \le m \le n$. Furthermore, since E is separable, then $\overline{\text{span}\{x_n : n \ge 1\}} = E$.

Since $\sum_{n=1}^{\infty} \|x_n\|_E < \infty$, $\sum_{n=1}^{\infty} |\xi_n| \, \|x_n\|_E < \infty$ for $\gamma^{\mathbb{N}}$– almost every $\boldsymbol{\xi} \equiv (\xi_1, \cdots, \xi_n, \cdots) \in \mathbb{R}^{\mathbb{N}}$. Thus, there exists a random variable $X : \mathbb{R}^{\mathbb{N}} \to E$ such that $X(\boldsymbol{\xi}) = \sum_{n=1}^{\infty} \xi_n x_n \in E$ for $\gamma^{\mathbb{N}}$–almost every $\boldsymbol{\xi} \in \mathbb{R}^{\mathbb{N}}$. Denote by $\mathcal{W}$ the distribution of X under $\gamma^{\mathbb{N}}$. Since $\text{span}\{x_n : n \ge 1\}$ is dense in E, then one can easily verify that $\mathcal{W}$ is a non-degenerate centered Gaussian measure on E, and $\mathbb{E}^{\mathcal{W}}\left[\langle \cdot, x^* \rangle^2\right] = \sum_{n=1}^{\infty} \langle x_n, x^* \rangle^2$. Therefore, by the considerations in §2.2, the Cameron-Martin space for $\mathcal{W}$ contains the following elements:

$$h_{x^*} = \int x \langle x, x^* \rangle \mathcal{W}(dx) = \sum_{n=1}^{\infty} \langle x_n, x^* \rangle x_n \quad \text{for all } x^* \in E^*.$$

In particular, $h_{x_1^*} = x_1 \in H$, which is sufficient for us to draw the desired conclusion. $\square$

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