Multidisciplinary Design Optimization of the Space Shuttle External Tank
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Abstract
This study explored the multidisciplinary system design optimization of the External Fuel Tank of the Space-Transportation-System. Initially, we used the return on investment as the single objective of the system. With the simplified external tank model, a series of simulation experiments were conducted to obtain the feasible starting design. We then performed the single objective optimization using Sequential Quadratic Programming. The sensitivity of the optimal solution with respect to design variables and fixed parameters was investigated, and the improvement of our optimal design by scaling was also demonstrated. The Genetic Algorithm was attempted in this study, and the result was a significant improvement of the objective. Finally, adding the tank cost as the second objective, the Adaptive Weighted Sum Method was applied to generate the Pareto front of the two objective functions.

Introduction
The endeavor of space flight in the United Stated is becoming increasingly scrutinized in terms of both safety and profitability. While once an object of national fervor, the conclusion of the cold war and recent shuttle disasters have forced governmental decision makers to take a closer eye on the financial effectiveness of the manned space flight. The difficulty is the excess cost in sending a manned expedition into orbit using the shuttle. In part, it involves the design of the External Fuel Tank of the Space-Transportation-System (STS).

The external tank is the primary booster used to help an orbiter (space shuttle) escape the Earth’s gravitational pull and reach the terrestrial orbit. Hence, optimizing the external tank design could help convince the government leaders with the efficacy of manned spaceflight.

While there are myriad objectives that can be considered in the design of the external tank, the Return on Investment (ROI) is considered to be the overarching object encapsulating other objectives. ROI speaks to the bottom line: the profitability of putting US Astronauts in the space. However, to satisfy the bottom line is not an easy task. The design of the external tank requires a truly interdisciplinary approach. The disciplines incorporated include structures engineering, aerodynamics, operations research, finance, and industrial engineering, etc.

This study will address the following issues: (1) the simplified model of the external tank, (2) single objective optimization using gradient based method, (3) sensitivity analysis of the optimal design, (4) scaling of the problem, (5) single objective optimization using genetic algorithm, (6) multi-objective optimization and generating the Pareto front by the adaptive weighted sum method.

The Model of the External Tank
The simplified model of the external tank is in three sections: the nose cone, the cylindrical center body, and the hemisphere. There are six design variables: the height of the nose cone (H), the length of the cylindrical center body (L), the radius of the hemisphere (R), the thickness of the hemisphere (t), the thickness of the cylinder (t_c), and the thickness of the nose cone (t_n).

Nine parameters are involved in the model: (1) unit cost of material, (2) unit weight of material, (3) unit cost of seam, (4) pressure of the liquid fuel, (5) payload of oxygen, (6) payload of hydrogen, (7) profit ratio, (8) nominal payload, and (9) fixed launching cost per unit weight.

With these design variables and parameters, five constraints are constructed. The volume constraint states that the total tank volume should be greater or equal to the nominal tank volume. The three stress constraints states that the equivalent stress of the nose cone, the cylinder, or the hemisphere should be less or equal to the allowed stress. Finally, the vibration constraint states that the defined vibration factor should be greater or equal than the allowed minimum vibration factor.

Furthermore, with the above design variables, parameters, and constraints, seven possible objectives are considered: (1) total surface of the tank, (2) tank weight, (3) total seam cost, (4) tank cost, (5) delta payload, (6) payload launched, and (7) return on investment per launch.
In dealing with the design optimization, the external tank model can be decomposed into eight modules. These modules are: (1) module of surface and volume, (2) module of material weight and cost, (3) module of seam cost, (4) module of stress, (5) module of tank cost, (6) module of vibration, (7) module of aerodynamic drag penalty, and (8) module of return on investment. Figure 1 is the N² diagram of the external tank model. It represents the information flows between the modules of the model. Since in the figure there is no information feeds backward from any module, no coupling exists between modules. Therefore, the construction of the simulation architecture can be simple. For the detailed model of the external tank, please see appendix A.

**Figure 1: The N² diagram of the external tank model.**

**Initial Design**

By using the initial input for the design variables, which was available in the Shuttle Tank Spreadsheet, it was possible to further explore the design space of the external tank. Actually, the suggested design vector was only an approximated solution to the model. This design vector forced five constraints in the model to approach their bounds, but did not rigorously account for the violation of these constraints. Therefore, with this design vector, it was necessary to tune the numbers with intuition. The design vector, $x$, thus obtained is given in the following table.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Level 0</th>
<th>Level 1</th>
<th>Level 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>640</td>
<td>780</td>
<td>930</td>
</tr>
<tr>
<td>$L$</td>
<td>4770</td>
<td>4910</td>
<td>5050</td>
</tr>
<tr>
<td>$R$</td>
<td>405</td>
<td>408</td>
<td>411</td>
</tr>
<tr>
<td>$t_s$</td>
<td>0.36</td>
<td>0.78</td>
<td>1.20</td>
</tr>
<tr>
<td>$t_{cy}$</td>
<td>0.62</td>
<td>0.66</td>
<td>0.70</td>
</tr>
<tr>
<td>$t_{co}$</td>
<td>0.62</td>
<td>0.73</td>
<td>0.84</td>
</tr>
</tbody>
</table>

**Table 1: The initial design vector.**

For the above design vector, the simulation results indicated that none of the constraints were violated. Moreover, the program successfully returned the return on investment (ROI) as 0.041.

**Design of Experiments**

With the previously obtained feasible solution, we further used the design of experiments (DOE) techniques to explore the design space. To deal with the possible non-linearity of the factor effects, we decided to run a series of three-level simulation experiments. By assigning the previously obtained feasible solution as factors at level 1, the magnitudes for other factor levels are determined arbitrarily. Table 2 demonstrates the design factors and theirs levels.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Level 0</th>
<th>Level 1</th>
<th>Level 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>640</td>
<td>780</td>
<td>930</td>
</tr>
<tr>
<td>$L$</td>
<td>4770</td>
<td>4910</td>
<td>5050</td>
</tr>
<tr>
<td>$R$</td>
<td>405</td>
<td>408</td>
<td>411</td>
</tr>
<tr>
<td>$t_s$</td>
<td>0.36</td>
<td>0.78</td>
<td>1.20</td>
</tr>
<tr>
<td>$t_{cy}$</td>
<td>0.62</td>
<td>0.66</td>
<td>0.70</td>
</tr>
<tr>
<td>$t_{co}$</td>
<td>0.62</td>
<td>0.73</td>
<td>0.84</td>
</tr>
</tbody>
</table>

**Table 2: Design factors and their levels.**

By using the orthogonal array, OA (18, 7, 3, 2), a six factor, 18-run, and three-factor-level experiments was designed. Table 3 demonstrates this experimental plan.

<table>
<thead>
<tr>
<th>Treat.</th>
<th>$H$</th>
<th>$L$</th>
<th>$R$</th>
<th>$t_s$</th>
<th>$t_{cy}$</th>
<th>$t_{co}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>640</td>
<td>4770</td>
<td>405</td>
<td>0.36</td>
<td>0.62</td>
<td>0.62</td>
</tr>
<tr>
<td>2</td>
<td>780</td>
<td>4910</td>
<td>408</td>
<td>0.78</td>
<td>0.66</td>
<td>0.73</td>
</tr>
<tr>
<td>3</td>
<td>930</td>
<td>5050</td>
<td>411</td>
<td>1.20</td>
<td>0.70</td>
<td>0.84</td>
</tr>
<tr>
<td>4</td>
<td>640</td>
<td>4770</td>
<td>408</td>
<td>1.20</td>
<td>0.66</td>
<td>0.84</td>
</tr>
<tr>
<td>5</td>
<td>780</td>
<td>4910</td>
<td>411</td>
<td>0.36</td>
<td>0.70</td>
<td>0.62</td>
</tr>
<tr>
<td>6</td>
<td>930</td>
<td>5050</td>
<td>405</td>
<td>0.78</td>
<td>0.62</td>
<td>0.73</td>
</tr>
<tr>
<td>7</td>
<td>640</td>
<td>4910</td>
<td>405</td>
<td>1.20</td>
<td>0.70</td>
<td>0.73</td>
</tr>
<tr>
<td>8</td>
<td>780</td>
<td>5050</td>
<td>408</td>
<td>0.36</td>
<td>0.62</td>
<td>0.84</td>
</tr>
<tr>
<td>9</td>
<td>930</td>
<td>4770</td>
<td>411</td>
<td>0.78</td>
<td>0.66</td>
<td>0.62</td>
</tr>
<tr>
<td>10</td>
<td>640</td>
<td>5050</td>
<td>411</td>
<td>0.36</td>
<td>0.66</td>
<td>0.73</td>
</tr>
<tr>
<td>11</td>
<td>780</td>
<td>4770</td>
<td>405</td>
<td>0.78</td>
<td>0.70</td>
<td>0.84</td>
</tr>
<tr>
<td>12</td>
<td>930</td>
<td>4910</td>
<td>408</td>
<td>1.20</td>
<td>0.62</td>
<td>0.62</td>
</tr>
<tr>
<td>13</td>
<td>640</td>
<td>4910</td>
<td>411</td>
<td>0.78</td>
<td>0.62</td>
<td>0.84</td>
</tr>
<tr>
<td>14</td>
<td>780</td>
<td>5050</td>
<td>405</td>
<td>1.20</td>
<td>0.66</td>
<td>0.62</td>
</tr>
<tr>
<td>15</td>
<td>930</td>
<td>4770</td>
<td>408</td>
<td>0.36</td>
<td>0.70</td>
<td>0.73</td>
</tr>
</tbody>
</table>

**Table 3: Experimental plan for the design space search.**

In the experiment, we used the ROI as the major outputs for the simulation experiments. The experimental results indicate that the mean response for ROI was 0.038. The factor effects of ROI are presented in Table 4.

<table>
<thead>
<tr>
<th>Level</th>
<th>$H$</th>
<th>$L$</th>
<th>$R$</th>
<th>$t_s$</th>
<th>$t_{cy}$</th>
<th>$t_{co}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.007</td>
<td>0.029</td>
<td>0.017</td>
<td>0.042</td>
<td>0.050</td>
<td>0.024</td>
</tr>
<tr>
<td>1</td>
<td>0.003</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>2</td>
<td>-0.010</td>
<td>-0.029</td>
<td>-0.018</td>
<td>-0.042</td>
<td>-0.050</td>
<td>-0.025</td>
</tr>
</tbody>
</table>

**Table 4: Factor effects of the return on investment.**
From the previous table, since we desire a larger ROI, the factor effects of ROI imply that the design with every factor at its 0 level would be the best guess of an initial starting point, \( x_0 \). This starting point is given in Table 5.

<table>
<thead>
<tr>
<th>( H )</th>
<th>( L )</th>
<th>( R )</th>
<th>( t_s )</th>
<th>( t_cq )</th>
<th>( t_co )</th>
</tr>
</thead>
<tbody>
<tr>
<td>640</td>
<td>4770</td>
<td>405</td>
<td>0.36</td>
<td>0.62</td>
<td>0.62</td>
</tr>
</tbody>
</table>

Table 5: The starting point of the design optimization.

Fortunately, this treatment had been carried out in treatment one. However, its simulation result indicated violation of the volume constraint and the stress optimal design through SQP as constraint of the nose cone. Nevertheless, this initial DOE analysis was useful in elucidating the approach needed to undertake the design optimization.

**Selection of the Gradient Based Algorithm**

Based on the characteristics of the external tank model and the properties of the available algorithms, we decided to use the Sequential Quadratic Programming (SQP) algorithm as the first approach to optimize our design. The SQP algorithm is a generalization of Newton’s method in which it finds a step away from the current point by minimizing a quadratic model of the problem. A number of packages, including MATLAB and OPTIMA, use this approach in solving optimization problems.

The SQP algorithm was selected to solve our optimization problem based on four major reasons: (1) the SQP is widely used in engineering applications, (2) it is considered to be the best gradient-based algorithm, (3) it has strong theoretical basis, and (4) it can easily be incorporated in MATLAB to implement the SQP to solve the external tank optimization problem.

**Single Objective Optimization for the External Tank**

Return on investment per launch (ROI) was selected as the single objective function for which to optimize the external tank system. The reason for this choice is that, to some extent, ROI is a higher-level integration of many other objectives of our system. By using ROI as the single objective function in this part of the investigation, we hoped that it can provide us with a better design vector that will facilitate our later efforts of multi-objective optimization.

By using the SQP algorithm, successful optimization of the external tank system with respect to ROI was achieved. In this part, the `fmincon` command in MATLAB was used to conduct the calculation. The calculation converged in 15 iterations.

Starting with the design vector, \( x_0 \), obtained from the DOE investigation:

\[
\begin{bmatrix}
H \\ L \\ R \\
\end{bmatrix} = \begin{bmatrix}
640.0 \\ 4770.0 \\ 405.0 \\
\end{bmatrix},
\]

which has a ROI of 0.204, we obtained an improved optimal design through SQP as

\[
\begin{bmatrix}
H \\ L \\ R \\
\end{bmatrix} = \begin{bmatrix}
3077.3 \\ 5279.4 \\ 388.7 \\
\end{bmatrix},
\]

which has a slightly improved ROI of 0.206. Though the improvement of ROI from 0.204 to 0.206 is minute, not like \( x_0 \) that violates two constraints of the system, we actually have obtained an optimal design that satisfies all the imposed constraints.

**Sensitivity Analysis**

The MATLAB function, `fmincon`, has options that allow us to specify several outputs of sensitivity analysis. One of the useful options, in our case, is to output the gradient value of the objective function at the optimal solution. Proceed with caution, this function allows us to conduct the sensitivity analysis.

**Sensitivity Analysis with Respect to Design Variables**

It was found that the gradient value at the previous optimal design was

\[

\nabla J_* = \begin{bmatrix}
0.0001 \\
0.0002 \\
0.0045 \\
0.0931 \\
1.2639 \\
0.2742 \\
\end{bmatrix}
\]

From equation (3), the normalized sensitivity is calculated as
Sensitivity Analysis with Respect to Fixed Parameters

Since there are nine parameters in the model of the external tank system, it is useful to investigate the sensitivity of the optimal point with respect to these fixed parameters. Five of the most important parameters were identified and used to conduct sensitivity analysis. Table 6 is a list of the fixed parameters that were identified.

<table>
<thead>
<tr>
<th>Fixed Parameter</th>
<th>Initial Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit Cost of Material (dollar/kg): $c_{\text{unit}}$</td>
<td>6</td>
</tr>
<tr>
<td>Fixed launching cost per unit weight (dollar/kg): $k$</td>
<td>20000</td>
</tr>
<tr>
<td>Payload: $P_1$</td>
<td>15000</td>
</tr>
<tr>
<td>Payload: $P_2$</td>
<td>5000</td>
</tr>
</tbody>
</table>

Table 6: Fixed parameters and their values

With the parameter vector, $p$, and the optimal design, $x^*$, the sensitivities of $x^*$ with respect to $p$ were calculated as

$$\tilde{\delta}x_{\text{unit}} = \tilde{\delta}x_{\text{com}} = \tilde{\delta}_3 = \tilde{\delta}_P = \tilde{\delta}_p = 0.00$$

The above result implies that the sensitivity of $x^*$ with respect to $p$ was extremely low.

Furthermore, the sensitivities of the optimal $J^*$ with respect to $p$ were calculated as

$$p \frac{dJ}{dp} = \nabla J_p = \begin{bmatrix} \frac{dJ}{dc_{\text{unit}}} & \frac{dJ}{c_{\text{com}}} \\ \frac{dJ}{dk} & \frac{dJ}{J} \\ \frac{dJ}{dp_1} & \frac{dJ}{J} \\ \frac{dJ}{dp_2} & \frac{dJ}{J} \end{bmatrix} = \begin{bmatrix} 0.0486 \\ -0.0000 \\ -0.4371 \\ -0.0971 \end{bmatrix}$$

The above result implies that the payload ($P_1$) has the most influence on the optimal value of $ROI$. A one percent increase of $P_1$, will cause $ROI$ to increase by 0.43 percent (note that $J$ and $ROI$ have opposite signs.) In this case, with the increase of $P_1$, no constraint is violated, and the optimal design $x^*$ stays the same.

Active Constraints at Optimal Design

After obtaining the optimal design $x^*$, it is necessary to check which constraints are active by substituting $x^*$ into these constraints:

$$g_1 = 1 - \frac{V_{\text{total}}}{V_{\text{nominal}}} \leq 0$$

$$g_1 = -1.1493e - 012 \geq 0$$

$$g_2 = F_{c-o} / F_{\text{allowed}} - 1 \leq 0$$

$$g_2 = -7.2387e - 014 \geq 0$$

$$g_3 = F_{c-o} / F_{\text{allowed}} - 1 \leq 0$$


\[ g_3 = 7.9936e-015 \leq 0 \tag{9} \]

\[ g_4 = \frac{F_{co}}{F_{allowed}} - 1 \leq 0 \]

\[ g_5 = -1.3110e-012 \leq 0 \tag{10} \]

\[ g_6 = 1 - VF/VF_{allowed} \leq 0 \]

\[ g_7 = -0.1899 \tag{11} \]

The above results indicate that there are four active constraints: \( g_1, g_2, g_3, \) and \( g_4. \)

Table 7 demonstrates the effect of changing each active constraint by +1 percent and -1 percent and re-optimizing the system. For example, in the result of constraint \( g_1, \) by increasing \( V_{\text{norm, min}} \) by 1 percent, the optimal value, \( ROI, \) decreases 6.46 percent (note that \( J \) and \( ROI \) have opposite signs). At the same time, \( R \) increases 0.49 percent, \( t_s \) increases 0.47 percent, \( t_{cy} \) increases 0.48 percent, and \( t_{co} \) increases 0.45 percent.

Note in these results, \( g_1 \) is the most sensitive constraint. Perturbing constraint \( g_1 \) will cause relatively larger changes in the optimal value and the optimal design vector.

Table 7: Normalized sensitivity with respect to the constraints.

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Change (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_1 )</td>
<td>1.00</td>
</tr>
<tr>
<td>( g_2 )</td>
<td>-0.15</td>
</tr>
<tr>
<td>( g_3 )</td>
<td>0.00</td>
</tr>
<tr>
<td>( g_4 )</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Post Optimality and Scaling

The second-order Hessian approximation (using finite differencing) is given as

\[
\frac{\partial^2 J}{\partial x_i \partial x_j} \approx \frac{J(x_i + \Delta x_i, x_j + \Delta x_j, \cdots) - 2J(x_i, x_j, \cdots) + J(x_i, x_j, \cdots) - J(x_i - \Delta x_i, x_j - \Delta x_j, \cdots)}{\Delta x_i \Delta x_j}
\]

\[ (12) \]

In this part of the study, our latest optimal design vector is \( x^*, \) which is represented in equation (2). With \( x^*, \) the Hessian at can be generated by equation (12). Please note that finite differencing always has errors, and it is very dependent on the perturbation size. As a result, with each design variable, different sizes of perturbation were tested and tried to find the Nadir point at which the perturbation size gives the best estimation of the Hessian. Figure 3 is an example of the search process of the design variable: \( H. \)

![Figure 3: An example of the search for Hessian.](image)

In figure 3, the Hessian equals to 8.6844E-008. In the same fashion, we obtained the 6 diagonal entries of the Hessian at our current optimal solution:

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Hessian</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>8.68E-08</td>
</tr>
<tr>
<td>( L )</td>
<td>-4.32E-11</td>
</tr>
<tr>
<td>( R )</td>
<td>1.40E-05</td>
</tr>
<tr>
<td>( t_s )</td>
<td>2.32E-05</td>
</tr>
<tr>
<td>( t_{cy} )</td>
<td>-4.28E-04</td>
</tr>
<tr>
<td>( t_{co} )</td>
<td>4.21E-07</td>
</tr>
</tbody>
</table>

Table 8: Diagonal entries of the Hessian before scaling.

Scaling of the Problem

By using the diagonal entries of the Hessian in table 8, we determined the scaling factors, and scaled our design variables accordingly. Table 9 demonstrates the results of scaling.

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Hessian</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>8.68E-08</td>
</tr>
<tr>
<td>( L )</td>
<td>-4.32E-11</td>
</tr>
<tr>
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</tr>
<tr>
<td>( t_s )</td>
<td>2.32E-05</td>
</tr>
<tr>
<td>( t_{cy} )</td>
<td>-4.28E-04</td>
</tr>
<tr>
<td>( t_{co} )</td>
<td>4.21E-07</td>
</tr>
</tbody>
</table>

Table 9: Scaling of the Problem.
Optimization by Using Genetic Algorithms

\[ J = g_1 + g_2 + g_3 + g_4 + g_5 \]

Table 10: Diagonal entries of the Hessian after scaling

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Scaling</th>
<th>Scaled design variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>10^{-4}</td>
<td>10^{4}H'</td>
</tr>
<tr>
<td>( L )</td>
<td>10^{-5}</td>
<td>10^{5}L'</td>
</tr>
<tr>
<td>( R )</td>
<td>10^{-2}</td>
<td>10^{2}R'</td>
</tr>
<tr>
<td>( t_s )</td>
<td>10^{-2}</td>
<td>10^{2}t_s'</td>
</tr>
<tr>
<td>( t_{cy} )</td>
<td>10^{-2}</td>
<td>10^{2}t_{cy}'</td>
</tr>
<tr>
<td>( t_{co} )</td>
<td>10^{-3}</td>
<td>10^{3}t_{co}'</td>
</tr>
</tbody>
</table>

which has a ROI of 0.250.

Table 9: The scaled design variables

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Scaling</th>
<th>Scaled design variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>10^{-4}</td>
<td>10^{4}H'</td>
</tr>
<tr>
<td>( L )</td>
<td>10^{-5}</td>
<td>10^{5}L'</td>
</tr>
<tr>
<td>( R )</td>
<td>10^{-2}</td>
<td>10^{2}R'</td>
</tr>
<tr>
<td>( t_s )</td>
<td>10^{-2}</td>
<td>10^{2}t_s'</td>
</tr>
<tr>
<td>( t_{cy} )</td>
<td>10^{-2}</td>
<td>10^{2}t_{cy}'</td>
</tr>
<tr>
<td>( t_{co} )</td>
<td>10^{-3}</td>
<td>10^{3}t_{co}'</td>
</tr>
</tbody>
</table>

Table 9: The scaled design variables

Problem Redefinition and Execution

With the scaled design variables, we re-ran the optimizer and obtained the optimal solution for the redefined design variables. This optimal solution is then scaled back using the same set of factor. After scaling back, we have the new optimal design as

\[ x^* = \begin{bmatrix} H \\ L \\ R \\ t_s \\ t_{cy} \\ t_{co} \end{bmatrix} = \begin{bmatrix} 1069.6 \\ 6417.5 \\ 356.5 \\ 0.3120 \\ 0.5403 \\ 0.5502 \end{bmatrix} \]

which has a ROI of 0.250.

Table 10: Diagonal entries of the Hessian after scaling

As shown in the above table, all of the diagonal entries of the Hessian at the current improved solution are smaller than 10^2 or greater than 10^{-2}. This implies that no additional scaling is necessary.

Qualitative Assessment of Other Scaling Aspects

In the optimization problem, the Lagrange multipliers, \( \lambda_i \), represent the sensitivity of the objective function to the constraints. The \( \lambda_i \) gives information about whether or not the constraints are balanced with respect to each other, i.e. whether or not all constraints have an equal weighting in the optimization. In table 7, we have demonstrated the sensitivity of the optima \( J^* \) with respect to the four active constraints. Since these sensitivities appear to be in a reasonable range of 0.1 to 10, we are confident that these constraints are well scaled with respect to each other.

Non-Gradient Based Algorithm Selection

Based on the disaggregated characteristics of the external tank system, we decided to use the Genetic Algorithms (GAs) to search for the possible improvement of the optimal design. GAs provides a clever search strategy and is simple to implement. Through the use of a suitable variety of insertion, mating, mutation and crossover strategies, it is possible to ensure that the localized optima do not dominate the population; thus mitigating the risk of finding a sub-optimal solution.

Optimization by Using Genetic Algorithms

With the work that has been completed in the simulation design phase, it was possible to capture both the objective function and the constraints in symbolic expressions. This greatly helped in this part of the analysis, since we could export the expressions to a stand alone GA optimizer.

We encoded the constraints as penalties into the objective function. According to the analysis resulting from the use of the gradient based methods, the first constraint is the most sensitive to the optimal solution. Therefore, through trial and error, it was found that assigning a penalty multiplier of 1000 to this constraint tended to produce more reliable solutions. Later, this solution was verified using a demo version of Evolver 4.0 from Palisades Software. A simplified version of the fitness function is generated as

\[ J_{\text{fitness}} = J - (1000g_1 + g_2 + g_3 + g_4 + g_5) \]

It was also found through experimentation that an encoding size of 64 bits per design variable tended to provide more reliable results than lower bits within a relatively reasonable run time.

Initially a population size of 100 was used along with a mutation rate of 0.025 and a crossover probability of 0.5. The crossover probability is related to the mix of information that an offspring will contain after mating. In other words, a crossover probability of 0.5 will contain approximately a 50/50 mix of information from both parents. Another interpretation is that the location of the crossover is a discrete random variable normally distributed around the mid point of a given parent. Naturally, the mixes among the parents are reciprocal.
In using the GAs, the selection parameter was determined by making the selection proportional to the chromosome ranking. This criterion meant slower convergence than other selection schemes, but it was believed that the chance of mating out needed chromosomes would be less. Thus, our sacrifice in terms of convergence rate would be compensated by mitigating the risk of being trapped in local optima.

The crossover point was chosen randomly for each generation. We decide that the utility was higher to find a global solution than to speed up the execution. Over a long enough time horizon, this strategy coupled with the aforementioned strategies should insure that the maximum fitness will be achieved.

We also automated the initial starting population (evenly valued between 26 and 150 for initial population in the increments of two) and mutation rate (0.00 to 0.25 in the increments of 0.01). This amounted to a total number of 15,500,000 runs. Unfortunately, MATLAB can only store arrays that are smaller than 66464 rows. It became necessary to execute the evaluation in 234 batches. Afterwards, the top 1500 results were exported to excel for further analysis.

Our insertion and replacement techniques involved a combination of two strategies. During the actual run, each pair of parents produced two offspring. After all mating had finished, the worst member of the population was then replaced. This process was continued until either convergence was reached, or the process became an alternating series within a standard deviation of 0.01. The runs were performed using this method, utilizing a “Hall of Fame” approach (i.e. keeping track of the best individuals for each run.) The individual with the best fitness of all the runs represented our optimal solution.

After 125 hours of CPU time (about 5 days), the optimal design generated by the GAs was

$$x^* = \begin{bmatrix} H \\ L \\ R \\ t_s \\ t_{cy} \\ t_{co} \end{bmatrix} = \begin{bmatrix} 423 \\ 6927 \\ 349 \\ 0.31 \\ 0.53 \\ 0.57 \end{bmatrix}$$ \hspace{1cm} (15)

which has a ROI of 0.346.

Compared with the optimal solution obtained by SQP, which has a ROI of 0.206, we see a total improvement of about 68% percent.

### Determining the Pareto Front by Using the Adaptive Weighted Sum Method (AWSM)

In dealing with the multi-objective optimization for the external tank design, we intended to simultaneously maximize the Return on Investment (ROI) and minimize the Tank Cost (TC). To search for the Pareto front of the external tank system, we decided to use the Adaptive Weighted Sum Method (AWSM).

The AWSM has the following three advantages over other methods used to generate the Pareto front: (1) it produces evenly distributed solutions along the Pareto front, (2) it finds Pareto optimal solution in non-convex regions, and (3) it neglects non-Pareto optimal solutions in non-convex regions (de Weck and Kim, 2004).

We stated our multi-objective design optimization problem as a minimization problem. Since we wanted a larger ROI and a smaller TC, the weighted objective is to simultaneously minimize TC and –ROI.

By using the AWSM, we successfully obtained the Pareto front of the bi-objective optimization problem. Table 11 is the optimal solutions along the Pareto front, and Figure 4 illustrates the obtained Pareto front.

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**Figure 4: Pareto front of the bi-objective optimization.**
Sensitivity of Pareto Front with Respect to the Weighting Factors

To calculate the sensitivity of Pareto front with respect to the weighting factors, we begin by treating the weight, \( \alpha \), as a parameter in the objective function. We then used the general equation to calculate the sensitivity of the weighted objective function, \( J_w \), with respect to \( \alpha \).

\[
\frac{\partial J_w}{\partial \alpha} = \nabla J \delta x \tag{16}
\]

Since every point on the Pareto front represents an optimal solution, we shall evaluate \( \frac{\partial J}{\partial \alpha} \) at each point listed in the table above.

In general, it is very difficult to calculate the sensitivity, \( \frac{\partial J}{\partial \alpha} \). However, in our case, it is possible to use the primal-dual steepest descent theory to obtain the result (Brown and Smith, 2003). According to the method, suppose we have a set, \( r_s = \min\{m,n\} \), which is the linearly independent designs in the local neighborhood that have been identified. Typically it would be the closest set of design to the current design of interest. By using numerical interpolation, the differences in the design parameters and the objectives along the search directions, \( s_j \), can be measured. The following two matrices then represent the differences:

\[
X_s = \frac{\Delta x}{\Delta s_j} \quad \text{and} \quad F_s = \frac{\Delta F}{\Delta s_j}. \tag{17}
\]

The local Jacobian is estimated by

\[
J = (X_s^T X_s)^{-1} X_s^T F_s. \tag{18}
\]

The differences between the point of interest and each member of the local neighborhood is then used to estimate the gradient information, which is then utilized to determine \( \nabla J \delta x \). In our calculation, \( \frac{\partial J}{\partial \alpha} \) was determined numerically by expanding the Taylor series and evaluating first and second order terms. Please note that since the objective functions have been normalized in the objective space, the sensitivities are likewise normalized. These normalized sensitivities are presented in the table below.

<table>
<thead>
<tr>
<th>( J_1 )</th>
<th>( J_2 )</th>
<th>( J_w )</th>
<th>( \frac{\partial J_1}{\partial \alpha} )</th>
<th>( \frac{\partial J_2}{\partial \alpha} )</th>
<th>( \frac{\partial J_w}{\partial \alpha} )</th>
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Post-Optimization Benchmarking

With the real parameters of the external tank from Jenkins (2001), we conducted a primitive benchmarking for the post-optimization of the external tank.
By applying the available parameters of the external tank to our simplified model, we obtained a ROI of 0.439. Compared with the ROI of 0.206 from the SQP, the ROI of 0.250 after scaling, and the ROI of 0.346 from GA, the ROI of the external tank in use is much higher.

However, it should be emphasize that the current design of the external tank in use did not align well with our simplified, low fidelity tank model; moreover, when applying real tank design into our model, at least the volume constraint was violated.

Future work will require a model of external tank with higher fidelity, so that it can more closely resemble the performance of the real system.

Summary and Conclusions

The project started with a simplified model of the Space Shuttle External Tank. Though it is a low fidelity model, it captured the basic performance tradeoffs between the features of the external tank.

With the six design variables, an initial exploration of the design space was taken. It was discovered that the design space is highly segmented. By applying the orthogonal array, a six-factor, 18-run, and three-factor-level simulation experiment was performed. This simulation experiments provided us the starting point which had the best factor effects.

With the starting point obtained from the design of experiment, we used the Sequential Quadratic Programming (SQP) to conduct the single objective optimization of the system. The resulted optimal design had a ROI of 0.206. With this optimal solution, we discovered that the most sensitive design variable is the radius of the hemisphere (R). As to the sensitivity with respect to the fixed parameters, the oxygen payload had the most influence on the optimized ROI.

After using finite differencing to calculate the Hessian at the optimal solution, we realized the need to conduct the scaling for the problem. With the scaled design variables, we re-ran the optimizer, scaled back the optimal design, and obtained the improve ROI from 0.206 to 0.250. At this improved design solution, the Hessian indicated that no additional scaling was necessary.

Based on the segmental characteristics of the external tank system, we decided to use the Genetic Algorithms (GAs) to search for the possible improvement of the optimal design. After 15 million runs and 125 hours of CPU time, we obtained a greatly improved ROI of 0.346.

Given the confidence of our GA solution, we added the Tank Cost (TC) as the second objective and conducted the bi-objective optimization analysis. In this part, we successfully used the Adaptive Weighted Sum Method (AWSM) to identify the Pareto front of the two objectives.

The simplified external tank model illustrates that the real system is dependant and very sensitive to the initial parameters chosen. However, the Multidisciplinary Systems Design Optimization (MSDO) techniques utilized in this analysis is sound. The results of our analysis could be greatly enhanced by increasing the fidelity of the external tank model. Furthermore, integration of other shuttles systems into this analysis could yield greater insight.

Finally, any MSDO analysis should include the feedbacks from key decision makers. Unfortunately, the design of this assignment did not permit this. We believe that the future viability of space system design optimization could be greatly enhanced through the feedbacks with the NASA and governmental leaders.

References


Appendix A: Space-Transportation-System External Tank Design Modules

Mo-Han Hsieh
Engineering Systems Division, MIT.

**Design Variables**

Height of Nose Cone: $H$
Length of Cylindrical Body: $L$
Radius of Hemisphere: $R$
Thickness of Cylinder: $t_{cy}$
Thickness of Cone: $t_{co}$
Thickness of Hemisphere: $t_s$

**1. Surface and Volume:**

**1.1. Tank**

*Hemisphere*

Area: $A_s = 2\pi \times R^2$.
Volume: $V_s = \frac{2}{3}\pi \times R^3$.

*Cylinder*

Area: $A_{cy} = 2\pi \times R \times L$.
Volume: $V_{cy} = \pi \times R^2 \times L$.

*Cone*

Area: $A_{co} = 2\pi \times R \times D, \ D = \sqrt{R^2 + H^2}$.
Volume: $V_{co} = \frac{1}{3}\pi \times R^2 \times H$.

**Objective:**

*Total Surface*: $A_{total} = A_s + A_{cy} + A_{co}$
Nominal Volume: $V_{nominal} = 3,000,000,000$ (cm$^3$)
Total Volume: $V_{total} = V_s + V_{cy} + V_{co}$
**Constrain 1:**

\[ g_1 = \frac{V_{\text{total}}}{V_{\text{nominal}}} - 1 \geq 0 \]

1.2. Seam Length

- Hemisphere: \( S_s = 2\pi \times R \)
- Cylinder: \( S_{cy} = 4 \times L \)
- Cone: \( S_{co} = 4 \times D \)

- Cylinder-Hemisphere: \( S_{cns} = 2\pi \times R \)
- Cylinder-Cone: \( S_{cnc} = 2\pi \times R \)

- Total: \( S_{\text{total}} = S_s + S_{cy} + S_{co} + S_{cns} + S_{cnc} \)

2. Weight and Cost of Material:

- Unit Cost of Material (dollar/kg): \( c_{\text{unit}} = 6 \)
- Material Unit Weight (kg/cm\(^3\)): 0.0028

\[ f_{\text{weight}} = 1.15 - 0.33x + 0.165x^2 \]

- Weight Cost Function of Hemisphere: \( c_s = c_{\text{unit}} \times f_{\text{weight}}, \ x = t_s - 0.1 \)
- Weight Cost Function of Cylinder: \( c_{cy} = c_{\text{unit}} \times f_{\text{weight}}, \ x = t_{cy} - 0.1 \)
- Weight Cost Function of Cone: \( c_{co} = c_{\text{unit}} \times f_{\text{weight}}, \ x = t_{co} - 0.1 \)

**Objective:**

**Tank Weight:** \( W_{\text{tank}} = (A_s \times t_s + A_{cy} \times t_{cy} + A_{co} \times t_{co}) \times w \)

**Tank Material Cost:** \( C_m = (A_s \times t_s \times w) \times c_s + (A_{cy} \times t_{cy} \times w) \times c_{cy} + (A_{co} \times t_{co} \times w) \times c_{co} \)

3. Cost of Seams

- Unit Cost of Seam (dollar/m): \( c_{\text{seam}} = 12 \)

\[ f_{\text{seam}} = 1.2 - 0.42x + 0.25x^2 \]

- Seam Cost Function of Hemisphere: \( sc_s = c_{\text{seam}} \times f_{\text{seam}}, \ x = t_s - 0.1 \)
- Seam Cost Function of Cylinder: \( sc_{cy} = c_{\text{seam}} \times f_{\text{seam}}, \ x = t_{cy} - 0.1 \)
- Seam Cost Function of Cone: \( sc_{co} = c_{\text{seam}} \times f_{\text{seam}}, \ x = t_{co} - 0.1 \)
Seam Cost Function of Cylinder-Hemisphere: \( sc_{cns} = c_{\text{seam}} \times f_{\text{seam}}, \quad x = (t_{cy} + t_s) / 2 - 0.1 \)

Seam Cost Function of Cylinder-Cone: \( sc_{cnc} = c_{\text{seam}} \times f_{\text{seam}}, \quad x = (t_{cy} + t_{co}) / 2 - 0.1 \)

Cost of seams = (seam length)*(seam cost function):
- Hemisphere Seam Cost: \( C_{s-s} = S_s \times sc_s \)
- Cylinder Seam Cost: \( C_{s-cy} = S_{cy} \times sc_{cy} \)
- Cone Seam Cost: \( C_{s-co} = S_{co} \times sc_{co} \)
- Cylinder-Hemisphere Seam Cost: \( C_{s-cns} = S_{cns} \times sc_{cns} \)
- Cylinder-Cone Seam Cost: \( C_{s-cnc} = S_{cnc} \times sc_{cnc} \)

**Objective:**

Total Seam Cost: \( C_s = C_{s-s} + C_{s-cy} + C_{s-co} + C_{s-cns} + C_{s-cnc} \)

4. **Stress**

Stress Allowed (N/cm\(^2\)): \( F_{\text{allow}} = 40000 \)

Pressure (N/cm\(^2\)): \( p = 70 \)

**Cylinder**
- Hoop: \( F_{cy-h} = p \times R / t_{cy} \)
- Long: \( F_{cy-l} = p \times R / 2t_{cy} \)
- Equivalent Stress: \( F_{e-cy} = \sqrt{F_{cy-h}^2 + F_{cy-l}^2 - F_{cy-h} \times F_{cy-l}} \)

**Constrain 2:**
\( g_2 = F_{e-cy} / F_{\text{allow}} < 1 \)

**Hemisphere**
- Hoop: \( F_{s-h} = p \times R / 2t_s \)
- Long: \( F_{s-l} = p \times R / 2t_s \)
- Equivalent Stress: \( F_{e-s} = \sqrt{F_{s-h}^2 + F_{s-l}^2 - F_{s-h} \times F_{s-l}} \)

**Constrain 3:**
\( g_3 = F_{e-s} / F_{\text{allow}} < 1 \)
Cone

Hoop: $F_{co-h} = p \times R / t_{co}$

Long: $F_{co-l} = p \times R / 2t_{co} \times (D/H)$

Equivalent Stress: $F_{e-co} = \sqrt{F_{co-h}^2 + F_{co-l}^2 - F_{co-h} \times F_{co-l}}$

Constrain 4:

$g_4 = F_{e-co} / F_{allowed} < 1$

5. Total Tank Cost

Objective:

Total Tank Cost: $C_{tank} = C_m + C_s$

6. Vibration Constrain

$v_1 = 10000$

Allowed Minimum Vibration: $VF_{allowed} = 0.8$

Vibration Factor: $VF = v_1 \times \left( R^3 \times t_{cy} / W_{tank} \times (L + R + H) \right)^{1/2}$

Constrain 5:

$g_5 = VF / VF_{allowed} - 1 \geq 0$

7. Aerodynamic Drag Penalty

Payload (N): $p_1 = 15000$, $p_2 = 5000$

Tank Cross-section: $A_{cross} = \pi \times R^2$

$A_0 = 640000$, $(H / D)_0 = 1$

Nominal Cone Drag: $DR_{nominal} = 1.00$

Cone Drag: $DR_{cone} = 0.25 + 1.4e^{(1.6x)}$, $x = 1 + [H / D - (H / D)_0] / (H / D)_0$

Nominal Tank Surface: $A_{nominal} = 1400000$. 

Objective (Delta Payload):
\[
DP = - p_1 \times (A_{\text{cross}} - A_0) / A_0 \times DR_{\text{cone}} - p_2 \times (DR_{\text{cone}} - DR_{\text{nominal}}) / DR_{\text{nominal}} - p_2 \times (A_{\text{total}} - A_{\text{nominal}}) / A_{\text{nominal}}
\]

8. Return on Investment

Profit: \( \text{profit} = 0.05 \)

Nominal payload (N): \( PL_{\text{nominal}} = 30000 \)

Nominal Tank Weight: \( W_{\text{nominal}} = 28000 \)

Nominal Tank Cost: \( C_{\text{tank nominal}} = 510000 \).

Fixed cost to launch 1kg to orbit (dollar): \( k = 20000 \)

Cost to launch nominal payload: \( CP_{\text{nominal}} = PL_{\text{nominal}} \times k \)

Fixed cost other than tank = Cost to launch nom. payload - Cost of nom. tank:

\[
FC_{\text{other}} = CP_{\text{nominal}} - C_{\text{tank nominal}}
\]

True launch cost:

\[
C_{\text{launch}} = FC_{\text{other}} + C_{\text{tank}}
\]

Tank design influences profit in two ways:

Payload actually launched = Nominal payload - (Tank weight - Nominal Tank weight) + Delta payload due to drag:

Objective:

\[
PL_{\text{actual}} = PL_{\text{nominal}} - (W_{\text{tank}} - W_{\text{tank nominal}}) + DP
\]

Charge per kg launched = (fixed cost to launch)*(1+profit)

\[
Ch = k \times (1 + \text{profit})
\]

Customer pays = (payload actually launched)*(Charge per kg launched)

\[
PA = PL_{\text{actual}} \times Ch
\]

Return on investment per launch = (Customer pays - True launch cost) / (true launch cost):

Objective:

\[
RI = (PA - C_{\text{launch}}) / C_{\text{launch}}
\]