

WAVE PROPAGATION IN A TURBULENT MEDIUM:

SECOND ORDER EFFECTS

by

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ABSTRACT

M.S. Howe has developed a theory of wave propagation in a random inhomogeneous medium. His theory is applied here on the problem of wave propagation in an isotropic ideal gas with inhomogeneities due to temperature fluctuations alone. Equations for the coherent and random waves are developed, an iterative routine for their solution is described, with solutions evaluated which represent inclusion of the tertiary collision term. Phase speed and attenuation of the mean field are calculated and compared to the results of others. Tertiary results yield an expression for the random field, and it is shown that energy is not conserved.

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Chapter 1

Introduction

The problem of wave propagation in a random inhomogeneous medium is one that has been receiving ever-increasing attention in the past two decades. Classical perturbative methods performed on the total field, as used by Chernov¹ and others, prove inadequate in predicting observed behavior over long distances and where the spatial distribution of the inhomogeneities is expansive. These conventional perturbation solutions contain secular terms and predict an indefinite growth of fluctuations of the wave field with distance. Two formulations, the "smoothing" method, and the method of Howe, appear to avoid these problems by considering variations in the mean field separately from the fluctuations about this mean.

We will employ the method of Howe to examine the case of wave propagation in a turbulent isotropic gas, where turbulence is due to small random fluctuations in temperature alone. If we assure that the medium is in a state of quasi-equilibrium, and that the length scales of disturbance are small compared to those of the basic flow, then these fluctuations may be considered to be independent of time. These are the conditions assumed for the classic scattering problem treated by Chernov¹, Frisch², Howe^{3,4} and others. Fortunately, this model of "frozen" turbulence is applicable in many cases.

The Born, or binary collision, approximation is very fre-

quently used in the formulation of the scattering problem. In this paper we wish to keep an eye on the additional effects of tertiary scattering and how they modify the Born approximation results.

The approach to be used is parallel to that of (ref.3) where Howe surveys the one-dimensional analog, that of wave propagation along a stretched string of variable density. As with any method, we seek satisfactory expressions for phase speed and attenuation, in addition to expressions for the mean and random fields.

First, we outline here the essence of Howe's formulation and the "smoothing" method, and say a few words about statistical models for the medium.

1.1 The "smoothing" method

The "smoothing" method, described by Frisch², and used extensively by Karal and Keller⁵ is a perturbative analysis based on the assumption that the medium differs only slightly from a homogeneous medium. The general formulation applies to any type of linear differential or integral equation with random coefficients. The results of Karal and Keller revealed an attenuation of a plane wave due to inhomogeneities of the medium, as well as an alteration in the phase velocity.

In general, wave propagation in a homogeneous medium is governed by an equation

$$Lu_0 = 0 \tag{1.1}$$

where L is a linear non-random operator. Letting α designate a different medium characterized by the operator $L - \epsilon L_1(\alpha) - \epsilon^2 L_2(\alpha) + O(\epsilon^3)$, where ϵ is a measure of the departure of the medium from homogeneity, it can be seen that a wave $u(\alpha)$ in this medium satisfies

$$u = u_0 + \epsilon L^{-1} L_1 u_0 + \epsilon^2 (L^{-1} L_1 L^{-1} L_1 + L^{-1} L_2) u_0 + O(\epsilon^3) \quad (1.2)$$

Now we deviate from the traditional perturbation routine by taking the expectation of (1.2) and rearranging, to derive an equation for the mean wave. If we let $\langle \rangle$ denote an expected value, we find an equation for $\langle u \rangle$, correct to order ϵ^3 :

$$L \langle u \rangle - \epsilon \langle L_1 \rangle \langle u \rangle - \epsilon^2 \{ \langle L_1 L^{-1} L_1 \rangle - \langle L_1 \rangle L^{-1} \langle L_1 \rangle + \langle L_2 \rangle \} \langle u \rangle = 0 \quad (1.3)$$

In many cases the Green's function operator L^{-1} may be found explicitly. Karal and Keller used an explicitly defined Green's function in their analysis of scalar waves.

The smoothing method was used by Wenzel and Keller⁶ for several special cases, in conjunction with the Born approximation. The advantage of the "smoothing" method (and also, as we shall see, of Howe's method) over the small perturbation analysis used by Chernov is that the dispersion relation for the mean field can be derived relatively easily, yielding very important information about phase speed and damping of the coherent field. Also, the generality of the formulation allows a great deal of flexibility in the equations involved.

1.2 Howe's method

The general formulation of wave propagation in an inhomogeneous medium proposed by Howe will be summarized briefly here. Howe's method is applicable to any medium which differs slightly from a homogeneous one described by

$$L\phi = 0 \quad (1.3)$$

where L is a linear operator. In the presence of inhomogeneities the governing equation becomes

$$L\phi = G\phi \quad (1.4)$$

where G is a random linear operator, which for the sake of simplicity, we will take to have zero mean.

The scheme here is to decompose the wave field into two components, $\bar{\phi}$ and ϕ' , such that

$$\phi = \bar{\phi} + \phi' \quad (1.5)$$

and to derive coupled equations for $\bar{\phi}$ and ϕ' . Howe shows that the mean and random fields, $\bar{\phi}$ and ϕ' , can be solved for multiple scattering of any order, at least theoretically, by evaluating

$$\bar{\phi} = L^{-1}\bar{G} \sum_{n=0}^{\infty} \{L^{-1}G - L^{-1}\bar{G}\}^n L^{-1}G\bar{\phi} \quad (1.6)$$

$$\phi' = \sum_{n=0}^{\infty} \{L^{-1}G - L^{-1}\bar{G}\}^n L^{-1}G\bar{\phi} \quad (1.7)$$

The $n=0$ case represents the Born approximation. The higher order terms represent additional scattering of the scattered

wave. Though in many cases the Born approximation is a valid assumption, it neglects the effect of buffeting of coherent field energy experienced because of feedback of the scattered energy back into the mean field. Inclusion of these multiple scattering effects appears to describe a situation more consistent with reality.

In (ref.4) Howe shows that critical information may still be discerned in some cases, which include certain types of nonlinearity, where the Green's function operator L^{-1} is difficult to find explicitly.

Howe's method, like the "smoothing" method gives dispersion information fairly readily, and is adaptable to many different types of equations.

1.3 Some comments on statistical models for the medium

In the medium we are considering, fluctuations in the wave speed will be described by $\xi(\bar{x})$, a random process which is a function of the position coordinates (x,y,z) . We may characterize this process by the correlation function

$$N_{12} = \overline{\xi(\bar{x}_1)\xi(\bar{x}_2)} \quad (1.8)$$

For a spatially homogeneous process the correlation function depends only on the coordinate differences $x=x_2-x_1$, $y=y_2-y_1$, $z=z_2-z_1$. For $x=y=z=0$ the function N_{12} achieves its maximum $N_{11}=\overline{\xi^2}$, and we may write

$$N(\bar{x}) = \overline{\xi^2}R(\bar{x}) \quad (1.9)$$

The choice of the form of the correlation coefficient $R(\bar{x})$ is a difficult one to agree upon. As the distance between the points is increased, it is necessary that $R(\bar{x})$ decrease and become small compared to unity at a distance λ , called the correlation distance, i.e. the statistical dependence between fluctuations must disappear as the points move far apart, relative to the correlation distance. Chernov mentions the result, theoretically obtained by Obukhov, that in the case of homogeneous isotropic turbulence mean-square temperature fluctuations follow a law similar to the "two-thirds law." However, more often than not, a form is chosen for $R(\bar{x})$ from empirical data. Chernov uses a form for the correlation function

$$R(\bar{x}) = e^{-|\bar{x}|/\lambda} \quad (1.10)$$

while Howe, in his calculations for the random string, uses a Gaussian distribution. For the sake of exposition, in this paper we will allow the correlation coefficient to take the form in (1.9) whenever it becomes necessary to assume a particular functional dependence.

Chapter 2

2.1 Governing equation

Consider the problem of wave propagation in a turbulent non-dissipative gas, where the turbulence is due to spatial fluctuations in temperature alone. For a fluid which differs only slightly from its equilibrium state, considering propagation of sound to be an adiabatic process, the governing equation is seen to be the one derived by Chernov¹:

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p - \nabla(\log \rho) \cdot \nabla p = 0 \quad (2.1)$$

where p = acoustic pressure

ρ = density

c^2 = square of the wave speed .

We also have the isotropic relation for an ideal gas

$$c^2 = \gamma RT = \frac{\gamma p}{\rho} \quad (2.2)$$

Deviation of the medium from homogeneity is assumed to be small. In this case, small fluctuations in temperature lead to corresponding small fluctuations in pressure and density. As per Howe, we define the mean square wave speed as an ensemble average

$$a^2 = \overline{c^2} = \overline{\gamma RT} = \overline{\gamma(p/\rho)} \quad (2.3)$$

We may define $\xi(\bar{x})$, a random function of position, which

represents the variation in the wave speed as

$$a^2\{1 + \xi(\bar{x})\} = \gamma p / \rho \quad (2.4)$$

By virtue of its definition, $\overline{\xi(\bar{x})} = 0$. Also note that $\rho = \gamma p / \{a^2(1 + \xi)\}$.

If we expand the $\nabla(\log \rho) \cdot \nabla p$ term in powers of $\xi(\bar{x})$ and its gradient, and neglect terms of order $\xi^2 \nabla \xi$ and higher it is found that the equation simplifies to an inhomogeneous wave equation*

$$\frac{\partial^2 p}{\partial t^2} - a^2 \nabla^2 p = a^2 \xi \nabla^2 p \quad (2.5)$$

2.2 Equations for the mean and random fields

Now we follow precisely the method of Howe. Separate the total wave field $p(\bar{x}, t)$ into two parts: $\bar{p}(\bar{x}, t)$, the mean wave profile, and $p'(\bar{x}, t)$, the fluctuations of the field about \bar{p} .

Using

$$p = \bar{p} + p' \quad (2.6)$$

in (2.5) and taking the ensemble average of this we obtain

$$\frac{\partial^2 \bar{p}}{\partial t^2} - a^2 \nabla^2 \bar{p} = a^2 \overline{\xi \nabla^2 p'} \quad (2.7)$$

the mean field equation. To obtain an equation for p' , subtract

* Note that this result could just as well have resulted from beginning with the classical wave equation for the acoustic pressure, $p_{tt} - a^2 \nabla^2 p = 0$ and directly substituting $c^2 = a^2(1 + \xi)$.

(2.7) from the full equation (2.5):

$$\frac{\partial^2 p'}{\partial t^2} - a^2 \nabla^2 p' = a^2 \xi \nabla^2 \bar{p} + a^2 \{ \xi \nabla^2 p' - \overline{\xi \nabla^2 p'} \} \quad (2.8)$$

Equations (2.7) and 2.8) are exact, though coupled, equations for the mean and random fields. Presumably ξ and p' are small quantities, and we see from (2.7) that changes in the mean field are expected to be second order in these quantities.

The random field is seen to be due to (a) interactions between medium fluctuations $\xi(\bar{x})$ and the mean field $p(\bar{x}, t)$, and (b) interactions between fluctuations of the medium and those components of $p'(\bar{x}, t)$ which are not correlated with $\xi(\bar{x})$. The latter of these two effects is neglected in the Born approximation.

Pickett⁷ formulated a similar problem where the Born approximation was assumed, equivalent to neglecting these multiple collision terms. Chernov, in performing his small perturbation analysis, assumes the Born approximation. Wenzel and Keller also use it implicitly. Here we will attempt to formulate the scattering problem preserving these terms, to estimate the order of their effect.

Chapter 3

3.1 Analysis of the field equations

In the notation of Howe, we began with an equation for the total field in the form

$$Lp = Gp \quad (3.1)$$

where L is a linear differential operator and G is a random operator with zero mean defined by

$$L = \frac{\partial^2}{\partial t^2} - a^2 \nabla^2 \quad \text{and} \quad G = a^2 \xi \nabla^2 \quad (3.2a,b)$$

In assuming a representation of the wave field as $p = \bar{p} + p'$, we derived the equations for the mean and random waves

$$L\bar{p} = \overline{Gp'} \quad (3.3)$$

$$Lp' = G\bar{p} + \{Gp' - \overline{Gp'}\} \quad (3.4)$$

For our problem the Green's function is well known as that of the so-called "retarded potential" problem, and we may solve (3.4) for $p'(\bar{x}, t)$ in terms of $\bar{p}(\bar{x}, t)$ by iteration using as a first approximation the Born result

$$p'_1 = L^{-1}G\bar{p} \quad (3.5)$$

to derive a second approximation

$$p'_2 = L^{-1}G\bar{p} + L^{-1}GL^{-1}G\bar{p} - L^{-1}\overline{GL^{-1}G\bar{p}} \quad (3.6)$$

Formally, the total multiple scattering result is shown by

Howe to be

$$p' = \sum_{n=0}^{\infty} \{L^{-1}G - L^{-1}G\}^n L^{-1}\bar{p} \quad (3.7)$$

to yield, by formal substitution into the mean equation, the exact multiple scattering equation for the mean field

$$\bar{p} = L^{-1}\bar{G} \sum_{n=0}^{\infty} \{L^{-1}G - L^{-1}G\}^n L^{-1}Gp \quad (3.8)$$

We wish to carry through the solution to include the $n=1$ terms in the iterative scheme. Howe⁴ argues that if \bar{p} satisfies certain smoothness conditions associated with the operator G , and if G has a symmetric distribution (as is assumed here) that the tertiary collision term in (3.8), corresponding to $n=1$, automatically vanishes. Therefore, the equation for $\bar{p}(\bar{x}, t)$, valid up to third order in the random fluctuation is

$$\bar{p} = L^{-1}\overline{GL^{-1}G}\bar{p} \quad (3.9)$$

while the corresponding equation for the random field is given by (3.6)

3.1 Dependence of the random field on the mean field

If the mean wave $\bar{p}(x, t)$ is assumed to be known, the iterative outline for solution for $p'(x, t)$ proposed is as follows:

$$\frac{\partial^2 p'_1}{\partial t^2} - a^2 \nabla^2 p'_1 = a^2 \xi \nabla^2 \bar{p} \quad (3.10)$$

$$\frac{\partial^2 p'_n}{\partial t^2} - a^2 \nabla^2 p'_n = a^2 \xi \nabla^2 \bar{p} + a^2 \{ \xi \nabla^2 p'_{n-1} - \xi \nabla^2 p'_{n-1} \} \quad (3.11)$$

We will use this routine to calculate $p'(\bar{x}, t)$ up through the $n=2$ case.

To solve for p'_1 , use (3.10) and the Green's function which satisfies

$$\frac{\partial^2 g}{\partial t^2} - a^2 \nabla^2 g = \delta(\bar{x}) \delta(t) \quad :$$

$$g(\bar{x}, t) = \begin{cases} \frac{1}{4\pi a} \frac{\delta(t - |\bar{x}|/a)}{|\bar{x}|} & t > 0 \\ 0 & t < 0 \end{cases} \quad (3.12)$$

$p'_1(\bar{x}, t)$ is found by the convolution product

$$p'_1(\bar{x}, t) = a^2 \iiint_{-\infty}^{\infty} g(\bar{x} - \bar{X}, t - T) \xi(\bar{X}) \nabla^2 \bar{p}(\bar{X}, T) d^3 \bar{X} dT \quad (3.13)$$

where $d^3 \bar{X}$ is the notation we will use from now on to denote the volume integration, i.e. $d^3 \bar{X} \equiv dX dY dZ$, and a single integral is used to represent the integration over all space.

Now if we make the substitution in (3.12) $\bar{\eta} = \bar{x} - \bar{X}$, $\tau = t - T$ and perform the time integration, the result is

$$p'_1 = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\xi(\bar{x} - \bar{\eta}) \nabla^2 \bar{p}(\bar{x} - \bar{\eta}, t - |\bar{\eta}|/a)}{|\bar{\eta}|} d^3 \bar{\eta} \quad (3.14)$$

It appears that fluctuations in the wave field depend strongly on gradients of the mean field. In fact, as will be shown later, in the long wavelength limit, Howe's method predicts that p'_1 is proportional to $\nabla^2 \bar{p}$.

The next correction to the random field $\phi'_2 = p'_2 - p'_1$ satisfies

$$\frac{\partial^2 \phi'_2}{\partial t^2} - a^2 \nabla^2 \phi'_2 = a^2 \{ \xi \nabla^2 p'_1 - \overline{\xi \nabla^2 p'_1} \} \quad (3.15)$$

We calculate the righthand side of this equation by returning to the Green's function expression and recalling that

$$a^2 \nabla^2 g(\bar{x}, t) = \frac{1}{4\pi a^2 |\bar{x}|} \frac{\partial^2 \delta(t - |\bar{x}|/a) - \delta(\bar{x}) \delta(t)}{\partial t^2} \quad (3.16)$$

yielding

$$\begin{aligned} a^2 \{ \xi \nabla^2 p'_1 - \overline{\xi \nabla^2 p'_1} \} &= a^2 \{ \overline{\xi(\bar{x}) \xi(\bar{x})} - \xi(\bar{x}) \xi(\bar{x}) \} \nabla^2 \bar{p}(\bar{x}, t) \\ &+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\xi(\bar{x}) \xi(\bar{x} - \bar{\eta}) - \overline{\xi(\bar{x}) \xi(\bar{x} - \bar{\eta})}}{|\bar{\eta}|} \frac{\partial^2 \nabla^2 \bar{p}(\bar{x} - \bar{\eta}, t - |\bar{\eta}|/a) d^3 \bar{\eta}}{\partial t^2} \end{aligned} \quad (3.17)$$

Now if $\xi(\bar{x})$ is a stationary random function, i.e. if $R(\bar{x} - \bar{x}_0) = \overline{\xi(\bar{x}) \xi(\bar{x}_0)} / \bar{\xi}^2$ depends only on the distance $|\bar{x} - \bar{x}_0|$, then we can write the differential equation for ϕ'_2 in terms of the correlation function $N(\bar{x}) = \overline{\xi^2} R(\bar{x})$, noting that $R(\bar{0}) = 1$ and $\overline{\xi^2} = \text{constant}$. Then use the same Green's function as before to compute ϕ'_2 and therefore p'_2 . Following the same procedure as was done with p'_1 , the result is

$$\begin{aligned}
p_2'(\bar{x}, t) &= p_1'(\bar{x}, t) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\bar{\xi}^2 - \xi^2(\bar{x} - \bar{\eta})}{|\bar{\eta}|} \nabla^2 \bar{p}(\bar{x} - \bar{\eta}, t - |\bar{\eta}|/a) d^3 \bar{\eta} \\
&+ \frac{1}{16\pi^2} \int_{-\infty}^{\infty} \frac{\xi(\bar{x} - \bar{\beta}) \xi(\bar{\eta}) - \bar{\xi}^2 R(\bar{x} - \bar{\beta} - \bar{\eta})}{|\bar{\beta}| |\bar{\eta}|} \frac{\partial^2}{\partial t^2} \nabla^2 \bar{p}(\bar{x} - \bar{\beta} - \bar{\eta}, t - \frac{|\bar{\beta}| + |\bar{\eta}|}{a}) d^3 \bar{\eta} d^3 \bar{\beta}
\end{aligned} \tag{3.19}$$

Second order corrections to p' are now seen to depend on the variance of the fluctuations $\bar{\xi}^2 - \xi^2$, in conjunction with the Laplacian of the mean pressure field.

3.2 Integro-differential equation for the mean field

We have that, to second order in the fluctuations of the medium, the exact equation for $\bar{p}(x, t)$ is of the form

$$L\bar{p} = \overline{GL^{-1}G\bar{p}} \tag{3.20}$$

The solution $p_1' = L^{-1}G\bar{p}$ may be used to construct an equation for $\bar{p}(x, t)$:

$$\frac{\partial^2 \bar{p}}{\partial t^2} - a^2(1 - \bar{\xi}^2) \nabla^2 \bar{p} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\bar{\xi}^2 R(\bar{\eta})}{|\bar{\eta}|} \frac{\partial^2}{\partial t^2} \nabla^2 \bar{p}(\bar{x} - \bar{\eta}, t - |\bar{\eta}|/a) d^3 \bar{\eta} \tag{3.21}$$

We take the Fourier transform of (3.21), rearrange and divide by $\bar{P}(\bar{k}, \omega)$ to get

$$k^2 a^2 (1 - \bar{\xi}^2) - \omega^2 = \frac{\bar{\xi}^2 \omega^2 k^2}{4\pi} \int_{-\infty}^{\infty} \frac{R(\bar{\eta})}{|\bar{\eta}|} \exp\{-i(\bar{k} \cdot \bar{\eta}) + i\omega|\bar{\eta}|/a\} d^3 \bar{\eta} \tag{3.22}$$

where $k = |\bar{k}|$. This is the dispersion relation, which may be

evaluated as soon as a form is assumed for $R(\bar{x})$. If we assume the correlation coefficient $R(\bar{x}) = \exp(-|\bar{x}|/\lambda)$, the integration may be performed exactly by switching to a spherical coordinate system which has its polar axis aligned with the k -vector. We then integrate over all space in the spherical system. This gives, finally

$$k^2 a^2 (1 - \bar{\xi}^2) - \omega^2 = \frac{\lambda k \omega^2 \bar{\xi}^2}{2} \left[\frac{\lambda(k - k_0) - i}{1 + \lambda^2(k + k_0)^2} + \frac{\lambda(k - k_0) + i}{1 + \lambda^2(k - k_0)^2} \right] \quad (3.23)$$

where $k_0 = \omega/a$.

In the next chapter we will consider limiting forms of this dispersion relation.

Chapter 4

Limiting forms of the dispersion relation

It will prove to be enlightening to consider the forms that the dispersion relation takes in the case where the wavelength is very long compared to the correlation length (the low frequency case) or where the wavelength is very short compared to the correlation length (the high frequency case). Since any solution to the basic mean field equation (3.21) may be written as a superposition of plane wave solutions satisfying the dispersion equation (3.23), it is instructive to look closely at what happens to a plane wave traveling through the medium in these limiting cases. Comparison of the long- and short-wavelength limit results may also yield some clues as to what happens for intermediate frequency ranges.

4.1 The long-wavelength limit

We retain the assumption of small fluctuations. Mathematically the condition is that $\lambda k \ll 2\pi$, $\lambda k_0 \ll 2\pi$, $k^2 - k_0^2 = O(\overline{\xi^2})$. Expand the right side of (3.23) up to powers of $(\lambda k)^4$ and $(\lambda k_0)^4$ to get

$$k^2 a^2 (1 - \overline{\xi^2}) - \omega^2 = \lambda k \omega^2 \overline{\xi^2} \left[\lambda k + 4\lambda^3 k k_0^2 + i(2\lambda^2 k k_0 - 2\lambda^3 k^2 k_0) \right] + O(\overline{\xi^2}^2) \quad (4.1)$$

Since $k^2 - k_0^2 = O(\overline{\xi^2})$ and all terms on the right side are of

that order or smaller, we may obtain an approximation for $\omega(k)$ or $k(\omega)$ valid up to order $(\bar{\xi}^2)^2$ by a one-step iteration, setting $k_0 = k$ on the righthand side:

$$\omega = ak \left\{ 1 - \frac{\bar{\xi}^2}{2} (1 + \lambda^2 k^2 + 4\lambda^4 k^4) \right\} - ia(\lambda^3 k^4 - \lambda^4 k^5) \bar{\xi}^2 \quad (4.2)$$

$$k = \frac{\omega}{a} \left[1 + \frac{\bar{\xi}^2}{2} \left(1 + \frac{\lambda^2 \omega^2}{a^2} + 4 \frac{\lambda^4 \omega^4}{a^4} \right) + i \bar{\xi}^2 \left(\frac{\lambda^3 \omega^4}{a^4} - \frac{\lambda^4 \omega^5}{a^5} \right) \right] \quad (4.3)$$

A mean plane wave solution of the form

$$\bar{p}_0 = A_0 e^{i(k_0 x - \omega t)}$$

will exist in the medium only if $\omega = \omega(k_0)$ as defined above.

If we write this as

$$\bar{p}_0 = A_0 e^{i\{k_0 x - t \operatorname{Re}(\omega)\}} e^{-t \operatorname{Im}(\omega)} \quad (4.4)$$

we see that the negative imaginary part of ω represents the degree of attenuation of the plane wave. The characteristic time in which the wave decays to e^{-1} times its initial value is

$$T = \frac{1/\bar{\xi}^2}{a^2 \lambda^3 k_0^4 (1 - \lambda k_0)} \quad (4.5)$$

The parameter most frequently used in this connection is the attenuation coefficient $\alpha = \operatorname{Im}(k)$. In this limit

$$\alpha = \bar{\xi}^2 \lambda^3 k_0^4 (1 - \lambda k_0) \quad (4.6)$$

The characteristic length, i.e. the length over which a wave

of initial frequency ω_0 decays to e^{-1} times its initial amplitude is

$$\ell = \alpha^{-1} = \left\{ \overline{\xi}^2 \frac{\lambda^3 \omega_0^4}{a^4} \left(1 - \frac{\lambda \omega_0}{a} \right) \right\}^{-1} \quad (4.7)$$

The phase speed of the plane mean wave, c^* , is

$$c^* = \frac{\text{Re}(\omega)}{k} = a^2 \left\{ 1 - \overline{\xi}^2 (1 + \lambda^2 k_0^2 + 4\lambda^4 k_0^4) \right\} \quad (4.9)$$

Note that the phase speed of the coherent wave is less than the speed in the averaged medium, and that it decreases with increasing frequency.

Now we will calculate the development of the random field, as modeled by Howe, in the long wavelength limit. The variables of integration in (3.14) are rescaled by setting $\bar{\sigma} = \bar{\eta}/\lambda$. Relative to the new variables the correlation length is unity.

$$p'(\bar{x}, t) = \frac{\lambda^2}{4\pi} \int_{-\infty}^{\infty} \frac{\xi(\bar{x} - \lambda \bar{\sigma})}{|\bar{\sigma}|} \nabla^2 \bar{p}_0(\bar{x} - \lambda \bar{\sigma}, t - \lambda |\bar{\sigma}|/a) d^3 \bar{\sigma} \quad (4.10)$$

The operators $\lambda \frac{\partial}{\partial \bar{x}}$ and $\frac{\lambda}{a} \frac{\partial}{\partial t}$, when acting on $\bar{p}_0(x, t)$, have small results in this limit, so it is justified to expand $\nabla^2 \bar{p}_0$, as it appears in the integrand, in a Taylor series expansion about (x, t) and keep only the first order terms in these operators. The result is:

$$p'(x, t) = \frac{\lambda^2}{4\pi} \left[\begin{aligned} & \nabla^2 \bar{p}_0 \int_{-\infty}^{\infty} \frac{\xi(\bar{x} - \lambda \bar{\sigma})}{|\bar{\sigma}|} d^3 \bar{\sigma} - \lambda \frac{\partial}{\partial x} \nabla^2 \bar{p}_0 \int_{-\infty}^{\infty} \sigma_x \frac{\xi(\bar{x} - \lambda \bar{\sigma})}{|\bar{\sigma}|} d^3 \bar{\sigma} \\ & - \frac{\lambda}{a} \frac{\partial}{\partial t} \nabla^2 \bar{p}_0 \int_{-\infty}^{\infty} \xi(\bar{x} - \lambda \bar{\sigma}) d^3 \bar{\sigma} \end{aligned} \right] \quad (4.11)$$

The mean square amplitude of $p'(x, t)$ is a measure of the amount of energy stored in the random field and this is, to order $\lambda^4 k_0^4$,

$$\overline{p' p'^*} = \frac{\lambda^4 \bar{\xi}^2}{16\pi} \left[\begin{aligned} & b_0 \nabla^2 \bar{p}_0^* \nabla^2 \bar{p}_0 - 2\lambda b_1 \frac{\partial}{\partial x} \nabla^2 \bar{p}_0^* \nabla^2 \bar{p}_0 \\ & - 2\frac{\lambda}{a} b_2 \frac{\partial}{\partial t} \nabla^2 \bar{p}_0^* \nabla^2 \bar{p}_0 \end{aligned} \right] + O(\bar{\xi}^2) \quad (4.12)$$

where

$$b_0 = \iint_{-\infty}^{\infty} \frac{R\{\lambda(\bar{\beta} - \bar{\sigma})\}}{|\bar{\beta}| |\bar{\sigma}|} d^3 \bar{\beta} d^3 \bar{\sigma} \quad (4.13a)$$

$$b_1 = \iint_{-\infty}^{\infty} \beta_x \sigma_x \frac{R\{\lambda(\bar{\beta} - \bar{\sigma})\}}{|\bar{\beta}| |\bar{\sigma}|} d^3 \bar{\beta} d^3 \bar{\sigma} \quad (4.13b)$$

$$b_2 = \iint_{-\infty}^{\infty} \frac{R\{\lambda(\bar{\beta} - \bar{\sigma})\}}{|\bar{\beta}|} d^3 \bar{\beta} d^3 \bar{\sigma} \quad (4.14c)$$

The nature of our statistics demand that b_1 vanish; b_0 and

b_2 do not depend on the wavelength. If the form (4.4) is employed explicitly, the resulting mean-square amplitudes are:

$$p_0 p_0^* = A_0^2 e^{-2at\bar{\xi}^2(\lambda^3 k^3 - \lambda^4 k^4)} \quad (4.15)$$

$$\overline{p'p'^*} = \frac{A_0^2 \bar{\xi}^2 \lambda^4 k^4 b_0}{16\pi} e^{-2at\bar{\xi}^2(\lambda^3 k^3 - \lambda^4 k^4)} \quad (4.16)$$

One would hope that these expressions would imply a conservation of energy, i.e. that $p_0 p_0^* + \overline{p'p'^*} = \text{constant}$. This is most certainly not the case. Only for small times can the energy discrepancy be ignored. If $t \ll 2\pi/\omega$, then at least, $p_0 p_0^* + \overline{p'p'^*} - A_0^2 = O(\lambda^4 k^4 \bar{\xi}^2)$.

It appears that real problems occur in Howe's method in the limits of large distances and times with regard to conservation of energy of the total wave field. Either this is a result of a defect in the theory, or the multiple scattering effects that were neglected contribute significantly to the physics, and must be included.

4.2 The short-wavelength limit

In a similar manner we can use (3.22) to derive the dispersion relation in the limit $k \ll 2\pi$, $k_0 \ll 2\pi$, retaining the assumption of small fluctuations.

$$k^2 a^2 (1 - \bar{\xi}^2) - \omega^2 = \omega^2 \bar{\xi}^2 \left(\frac{1}{4} + \frac{1}{2} i \lambda k \right) \quad (4.17)$$

The equation is solved up to order $(\bar{\xi}^2)^2$ to get

$$\omega = ak(1 - \frac{5\overline{\xi^2}}{8}) - ia\frac{\overline{\xi^2}\lambda k}{4} \quad (4.13)$$

or

$$k = \frac{\omega}{a} (1 + \frac{5\overline{\xi^2}}{8}) + i\frac{\overline{\xi^2}\lambda\omega^2}{4a^2} \quad (4.14)$$

Briefly, we summarize here the values for the attenuation coefficient, characteristic time and length scales, and phase speed, correct to $O(\overline{\xi^2})$:

$$\alpha = \frac{\overline{\xi^2}\lambda k_0}{4} \quad (4.15)$$

$$T = \frac{1/\overline{\xi^2}}{a\lambda k_0^2} \quad (4.16)$$

$$\ell = \alpha^{-1} = \frac{4/\overline{\xi^2}}{\lambda k_0^2} \quad (4.17)$$

$$c^* = a(1 - \frac{5\overline{\xi^2}}{8}) \quad (4.18)$$

As might be expected, in the short-wavelength limit there is much more attenuation than in the previous case. Note that the phase speed, while still less than the phase speed in the homogeneous medium, is independent of frequency to order $\overline{\xi^2}$.

4.3 Intermediate expectations

In both the long- and short-wavelength limits, $\text{Im}(\omega)$, which represents the damping rate of the mean field, is

negative, representing a positive attenuation of the mean wave, to generate random waves at the expense of the mean wave energy. It is expected that in the intermediate range between long and short wavelength limits that we will have a continuous transition, with damping of the mean wave for all values of the wavelength. Howe presents an argument that, at least in the one-dimensional analog of this problem (the stretched string), this is true. We would expect that a parallel proof can be constructed in this three-dimensional case.

It is not a difficult task to show, algebraically from (3.26), that the phase speed of the coherent wave will always be less than the wave speed in the homogeneous medium.

If the form of the statistics involved makes the dispersion integral in (3.21) difficult to evaluate, it is possible to derive a dispersion relation valid for the range $\lambda k < 1$, a range broader than that required to assume the long wavelength limit. The integrand in (4.10) can be expanded as in section 4.1, keeping sufficiently higher order terms in $\lambda \frac{\partial}{\partial x}$ and $\frac{\lambda}{a} \frac{\partial}{\partial t}$.

Chapter 5

Conclusions

The dispersion results for mean wave propagation speed and attenuation in the limiting cases can be compared directly with results obtained by other authors. Wenzel and Keller⁶ treated a parallel problem using the "smoothing" method, and calculated both of these coefficients. Chernov¹ was able to calculate the attenuation coefficient, using the method of small perturbations. Pickett⁷ found these parameters via Howe's method in connection with wave propagation in the ocean. His assumptions differ from those here in that he assumes a constant isothermal bulk viscosity for his medium, equivalent to assuming variations in density, but not in pressure. In all of these cases, however, the same exponential correlation function is assumed.

The attenuation coefficients derived in Chapter 4 are identical in both limits to results of Wenzel-Keller and Pickett, and differ only, again in both limits, by a factor of two from the Chernov results. It appears that the theories are consistent vis-a-vis predictions of damping of the coherent wave field.

We have only Wenzel-Keller's and Pickett's results with which to compare our propagation speed. In the short wavelength limit, Pickett and Wenzel/Keller agree on a value $c^* = a(1 - \frac{1}{2}\overline{\xi^2})$ and from (4.18) we have $c^* = a(1 - (5/8)\overline{\xi^2})$.

This is a more than satisfactory agreement. In the long wavelength limit, our results compare with those of Wenzel and Keller just as satisfactorily, though Pickett predicts a wave speed much less than is calculated here or by Wenzel and Keller. Regardless, all agree that the wave speed is at all times less than that in the homogeneous medium.

With regard to the form of the random field, there are no references I can turn to which offer any help in the way of verifying or contradicting the predictions here. Small perturbation methods yield fluctuations in the total field which grow with time. Neither Pickett or Wenzel/Keller attempt to calculate, to any order, the characteristics of the random field. In section 4.1 we saw that, to the order employed here, Howe's method is less than adequate in predicting behavior comparable to observed phenomena when it comes to the random wave field. Though the law of conservation of energy is built into the basic equations from which equations for \bar{p} and p' are derived, it is apparent that it has been lost in the shuffle. If the theory is not at fault (and here I can make no such judgement) then the conclusion to draw is that the "higher order" terms that are neglected in the Born approximation have a cumulative effect which is very significant.

More work needs to be done (empirically and/or theoretically) in the way of describing more adequately relevant statistical properties of the medium itself. It is apparent

that the total field cannot be determined merely by specifying a correlation function, but it seems necessary to more clearly define the minimum amount of information required to determine uniquely the coherent field and the mean-square random field.

In more recent papers (see ref.10) Howe has been trying to develop a more wide-reaching kinetic theory of wave propagation in inhomogeneous media. Perhaps the results of this work will provide a means to better determine the exact limitations of the theory presented here.

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