Optimal Control with Structure Constraints and its Application to the Design of Passive Mechanical Systems

by

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Abstract

Structured control (static output feedback, reduced-order control, and decentralized feedback) is one of the most important open problems in control theory and practice. In this thesis, various techniques for synthesis of structured controllers are surveyed and investigated, including H_2 optimization, H_∞ optimization, L_1 control, eigenvalue and eigenstructure treatment, and multiobjective control. Unstructured control—full-state feedback and full-order control—is also discussed. Riccati-based synthesis, linear matrix inequalities (LMI), homotopy methods, gradient- and subgradient-based optimization are used. Some new algorithms and extensions are proposed, such as a subgradient-based method to maximize the minimal damping with structured feedback, a multiplier method for structured optimal H_2 control with pole regional placement, and the LMI-based H_2/H_∞/pole suboptimal synthesis with static output feedback. Recent advances in related areas are comprehensively surveyed and future research directions are suggested.

In this thesis we cast the parameter optimization of passive mechanical systems as a decentralized control problem in state space, so that we can apply various decentralized control techniques to the parameter design which might be very hard traditionally. More practical constraints for mechanical system design are considered; for example, the parameters are restricted to be nonnegative, symmetric, or within some physically-achievable ranges. Marginally statable systems and hysterically damped systems are also discussed. Numerical examples and experimental results are given to illustrate the successful application of decentralized control techniques to the design of passive mechanical systems, such as multi-degree-of-freedom tuned-mass dampers, passive vehicle suspensions, and others.

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Contents

Chapter 1

Introduction

1.1 Background

In the design of passive mechanical systems, such as tuned mass dampers, vehicle suspensions, vibration isolators, structures and machine elements, we must often choose the parameters of stiffness and damping to achieve some performance objectives, such as vibration suppression and disturbance rejection. Usually the design is accomplished through trial-and-error procedures based on the designer's experience. There are also numerous optimization methods proposed for some special problems in this catalog, yet they are not widely adopted due to their complexity and numerical inefficiency. And many problems still remain challenging.

For example, it is known that multi-degree-of-freedom tuned mass dampers have the potential capacity to damp more than one mode, yet there is no available approach for their design, other than in the simple case where the motions are decoupled in space ([OsD98] [Whi00]). Another example is vehicle suspensions. Although active suspensions have been investigated for three decades ([EKE95] [Hro97]), passive suspensions are still dominant in the automotive industry since they are simple, reliable and economical. But design methods are still based on the simplified quarter-car or half-car model, and there is no efficient method to take account of the full car model or all of the performance requirements.

However, it is very interesting to view these problems from the view of feedback control. *The springs feed back the relative displacements locally, the damping elements feed back the relative velocities locally*, and the control forces with decentralized feedback take the role of stiffness and damping connections. This means that parameter selection and optimization of a huge class of passive mechanical systems becomes a structured control problem. Thus many difficult problems in passive mechanical systems become tractable in the framework of structured control.

1.2 Problem Formulation

For a mechanical system, we can write the equations of motion in matrix form as

$$M_q\ddot{q} + C_q\dot{q} + K_q q = B_u u + B_d d + B_p q_0 + B_v \dot{q}_0 \tag{1.1}$$

where M_q, C_q and K_q are the mass (positive definite), damping, and stiffness matrices, respectively. The "control force" vector input u is generated by the springs and damping elements to be designed, d is the exogenous disturbance force input, q_0 and $\dot{q}_0$ are the exogenous displacement and velocity disturbances.

For simple systems, the motion equation (1.1) can be obtained with Newton's law. For complicated systems, we can use the Lagrange-Hamilton principle. Structural matrix analysis [Gha97] is a very powerful method to model large three-dimensional structural systems. The basic idea is to assemble the individual mass, damping, and stiffness matrices by taking every node as independent, and then to use a constraint condensation technique to get the system motion matrix.

Defining the state variables as

$$x = \begin{bmatrix} q \\ \dot{q} - M_q^{-1} B_v q_0 \end{bmatrix}$$

we can write the equation of motion (1.1) as

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & I \\ -M_q^{-1}K_q & -M_q^{-1}C_q \end{bmatrix} x + \begin{bmatrix} 0 & M_q B_v \\ M_q^{-1}B_d & M_q^{-1}B_p - M_q^{-1}C_q M_q^{-1}B_v \end{bmatrix} \begin{bmatrix} d \\ q_0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ M_q^{-1}B_u \end{bmatrix} u \\
&\stackrel{\text{def}}{=} Ax + B_1 w + B_2 u
\end{aligned} \tag{1.2}
$$

Using geometric information, we can write the output vector of relative displacements and velocities between connection points as

$$y = C_2 x + D_{21} w \tag{1.3}$$

in which the coefficient of u D_{22} is zero naturally. Also, we can take the the displacements and velocities at the critical nodes together with the the control force u as the cost output z

$$z = C_1 x + D_{11} w + D_{12} u \tag{1.4}$$

Thus parameter design of passive mechanical systems is cast as a decentralized control problem in state space, as shown in Figure 1-1. Our goal is to determine the feedback law

$$u = Fy \tag{1.5}$$

where the feedback gain F is a decentralized (block-diagonal) matrix composed of

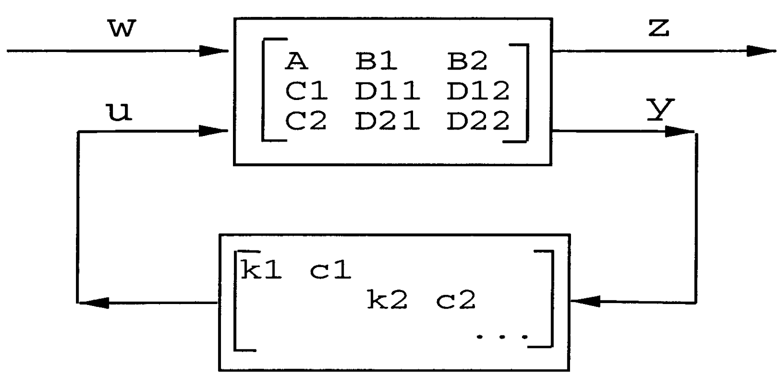

Figure 1-1: Decentralized control block diagram of passive mechanical systems

the parameters to be designed:

$$F = \begin{bmatrix} k_1 & c_1 & & & & & & \\ & & k_2 & c_2 & & & & \\ & & & & \cdots & \cdots & & \\ & & & & & & k_m & c_m \end{bmatrix} \tag{1.6}$$

where $k_i \geq 0$ and $c_i \geq 0$, $i = 1, 2, ...m$. For systems with hysteretic damping, F is a complex decentralized matrix:

$$F = \begin{bmatrix} k_1(1 + i\frac{w}{|w|}\eta_1) & 0 & & & & & & \\ & & k_2(1 + i\frac{w}{|w|}\eta_2) & 0 & & & & \\ & & & & \cdots & \cdots & & \\ & & & & & & k_n(1 + i\frac{w}{|w|}\eta_n) & 0 \end{bmatrix}$$

Therefore, state-space control techniques can be used to design passive systems. We should bear in mind that for a practical design, the parameters k_i and c_i should be be nonnegative or reside in some reasonable internals.

With system augmentation, this above setup is still valid for the case where the disturbance w includes velocities or accelerations and the cost output z includes accelerations. We can also handle weighted costs or weighted disturbances by taking shape filters into account in the generalized plant.

1.3 State-Space Control

Optimal and robust control has achieved great progress since the 1960s thanks to the contributions of Kalman, Athans [LeA70], Anderson and Moore [AnM90], Doyle and Glover [DGK89], Boyd [BEF94], Gahinet [GaP94], Patton [LiP98], and many others. The most popular techniques are H_2/LQR, H_∞ optimization, and to a lesser extent, eigenstructure treatment. Multi-objective control has also been developed recently. As we know, H_2/LQR or H_∞ controllers with full-state feedback or full-order output feedback can be obtained nicely by solving one or two Riccati equations or by solving linear matrix inequality (LMI) problems. Eigenstructure assignment software packages are also available now [LiP98].

However, modern control has not been widely adopted in industry. Besides the difficulty of modelling the plant uncertainties, an important reason is that unstructured controllers are often physically impossible or impractical, because they required high-order implementation or full-state sensing. Structured controller design—reduced order, static output-feedback control, and especially decentralized control—is one of the most important open problems in modern control theory and practice. Structured control problems usually require a non-convex optimization, and some problems turn out to be NP hard [BIT97]. Recently, structured control has been investigated extensively, and some encouraging progress has been achieved for static output-feedback control [EOA97]. More information can be found in the recent surveys [Sil96] [SAD97].

We note that state-space control techniques have already been utilized for the design of passive systems in the past decade. Lin *et al.* [LLZ90] adopted Kosut's sub-optimal LQG [Kos70] to the design of passive vibration isolation system. Projective control has been used to design an SDOF TMD [Ste94]. Static output H_2 was also formulated in [HaA98] for a passive system and was solved via a genetic algorithm. A Kalman filter was used in [SuT00] to trade off the rejection of disturbance force and ground vibration, but this approach only yields a practical design for an SDOF main system. Camino *et al.* [KZP99] used the static output LQG algorithm developed by Geromel [GSS98] in the design of a quarter-car suspension. Iterative LMI for static output H_∞ was adopted by Poncela and Schmitendorf [PonS98] to the design of the SDOF TMD. In the above literature, only (centralized) static output feedback control is related. Cluck *et al.* [GRG96] proposed some engineering approximation methods to the full-state feedback LQG, so as to obtain the passive parameters of supplemental dampers. Kosut's suboptimum also has been used by Karnopp [EKE96] for design of decentralized passive suspension. The application of gradient-based decentralized LQR optimization to the design of passive dampers can be found in [AgY99].

1.4 Thesis Outline

In this thesis, various techniques of state-space controller design are examined, including structured and unstructured H_2, H_∞, L_1, and multi-objective optimal control, and eigenvalue & eigenstructure treatment. Riccati-based, LMI-based, gradient- and subgradient-based optimization, and homotopy methods are used to solve these problems. Recent advances in related areas are comprehensively surveyed, including structured and unstructured cases. Some new algorithms and extensions are proposed. Numerical examples and experimental results are given to illustrate their successful application to the design of passive mechanical systems, such as tuned mass dampers, passive vehicle suspensions, and others.

In Chapter 2, the concept and physical meaning of the H_2 norm are introduced, and the synthesis approach of H_2 optimal control with full-state and full-order dynamic output feedback are briefly reviewed. Then we focus on structured H_2 control—static output feedback, lower-order control, decentralized control—and the associated computational methods. Examples are given to illustrate the application to passive mechanical systems, such as tuned-mass dampers. Practical constraints for passive mechanical systems are considered, such as nonnegative or symmetric parameters.

In Chapter 3, the concept of the H_∞ norm and robustness are briefly introduced, then we discuss Riccati-based and LMI-based H_∞ synthesis for full-state feedback and full-order control. H_∞ control with lower-order output feedback or static output feedback is solved with iterative LMI techniques. Decentralized H_∞ control is also examined. Numerical examples are given to show the application to the parameter design of passive mechanical systems, and the efficiency of various algorithms are compared.

In Chapter 4, the physical meaning of the eigenvalues and eigenvectors are highlighted, and important results and techniques of eigenvector and eigenstructure assignment are reviewed, including full-state feedback, static output feedback, constrained output feedback, and regional pole placement. A new approach is proposed to treat the poles of decentralized (and other architecture constrained) systems, so as to maximize the minimal damping. Hysteretically damped systems (complex feedback) are considered. Practical examples are presented to illustrate the application to the design of passive mechanical systems. The performance of the closed-loop systems produced with H_2, H_∞, and eigenvalue treatment are compared in the examples.

After we investigate the individual control techniques, we turn to multi-objective synthesis in Chapter 5. There we discuss multi-objective control in the classification of H_2/H_∞, H_2/regional pole placement, and H_∞ or H_2/H_∞/regional pole placement. Unstructured control and structured control are covered. L_1 optimization and L_1 associated multi-objective control are briefly reviewed. In this chapter, we also extend the cone-complementary linearization algorithm [EOA97] to general multiobjective suboptimal control with static output feedback; we propose a new approach for decentralized H_2 control with arbitrary pole regional constraints.The advantages of multi-objective control are highlighted in the example of a mechanical system design.

Chapter 6 presents the conclusions and the main contributions of this thesis. Some future research directions are suggested.

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Chapter 2

Optimal H_2 Control

H_2 optimal control, or equivalent to linear quadratic regulator, is one of the most widely used and elegant control techniques. For full-state feedback and full-order dynamic output feedback, we have an almost "analytic" solution by solving one or two matrix Riccati equations. However, if the controller has some structure constraints, such as static output feedback, fixed order, or decentralized architecture, Riccati equations turn out to be useless for direct H_2 synthesis. Nonlinear programming or linear matrix inequalities are used to seek for structured H_2 or LQR control.

In this chapter, the concept and physical meaning of the H_2 norm are introduced, then synthesis approaches of H_2 optimal control with full-state and full-order dynamic output feedback are briefly discussed. And we focus on structured H_2 control — static output feedback, lower-order control, decentralized control, and the associated computational methods. Examples are given to illustrate their application to the design of passive mechanical systems, such as tuned mass dampers, and passive isolation and suspensions.

2.1 Norm and System H_2 Norm

2.1.1 Norm, Signal Norm, and System Norm

A **norm** is a map from a space (vector, matrix, function space, etc.) to the real numbers. A norm must be nonnegative, homogeneous, and satisfy the triangle inequality.

For example,

- the 1 norm of an n-dimensional vector: $\|x\|_1 = \max_{1\leq i\leq n} |x_i|$;
- the 2 norm of an n-dimensional vector: $\|x\|_2 = \sqrt{(x'x)}$;
- the ∞ norm of an n-dimensional vector: $\|x\|_\infty = \sum_1^n |x_i|$;
- the 2 norm of an $m \times n$ matrix: $\|A\|_2 = \max_{\|x\|_2=1} \sqrt{(x'A'Ax)} = \sqrt{\lambda_{\text{max}}(A'A)} = \sqrt{\lambda_{\text{max}}(AA')} = \sigma_{\text{max}}(A)$, where λ_{max} means the largest eigenvalues of a matrix, σ_{max} is the largest singular value of a matrix;
- the ∞ norm of a $m \times n$ matrix: $\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^n |a_{ij}|$.

where the prime indicates the complex conjugate transpose of a vector or matrix.

In modern control techniques, a **signal norm** is often used as a measure of signals. For example, the L_2 norm of a continuous-time vector signal $f(t)$ is defined as the square root of the signal's energy:

$$\|f\|_2 = \left[\int_{-\infty}^{\infty} f'(t)f(t)dt\right]^{1/2} = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F'(\omega)F(\omega)d\omega\right]^{1/2} \tag{2.1}$$

where $F(\omega)$ is the Fourier transform of $f(t)$. The L_∞ norm of a continuous-time vector signal $f(t)$ is defined as the peak amplitude, evaluated over all signal components and all time:

$$\|f\|_\infty = \sup_t \max_i |f_i(t)| = \sup_t \|f(t)\|_\infty \tag{2.2}$$

Based on a given signal norm, we can define a **system induced norm**, which is the upper bound of the gain of a system H_{zw} (supposed unbiased). That is,

$$\|H_{zw}\|_{pi} = \sup_{w\neq 0} \frac{\|z\|_p}{\|w\|_p} \tag{2.3}$$

where $\|w\|_p$ is the signal p-norm of the input $w(t)$, $\|z\|_p$ is the signal p-norm of the corresponding output $z(t)$. The index p is often chosen to be 1, 2, or ∞. If the signal p-norm of the system output is finite for any input with finite p-norm, we say the system is p-stable. Further if $\|H\|_{pi}$ is finite, we say this system is p-stable with finite gain. For systems that can be represented by a finite-dimensional state-space model,

p stability coincides with BIBO stability, meaning that any bounded input results in a bounded output [MMV99].

The L_∞ induced system norm is the upper bound of the $L_\infty \to L_\infty$ gain, and is sometimes called the system H_1 norm, since for LTI systems the L_∞ induced norm turns out to be the 1 norm of the system transfer matrix. The L_2 induced norm is the upper bound of the $L_2 \to L_2$ gain, called the system H_∞ norm because for LTI systems this bound turns out to be the ∞ norm of the system transfer matrix. The system H_2 norm is not an induced norm, but is also important and meaningful. In the following, we focus mainly on the system H_2 norm and optimal H_2 control. We will discuss system H_∞ norms in the next chapter.

2.1.2 System H_2 Norm

The system H_2 norm is *the upper bound of the peak amplitude of the output with unit energy input.* Therefore, it is also called the L_2 to L_∞ gain.

$$\|z\|_\infty^2 = (\sup_t \max_i |z_i(t)|)^2 \leq \text{const} + \gamma^2 \|w\|_2^2 = \text{const} + \gamma^2 \int_0^\infty w(t)'w(t)dt \quad (2.4)$$

Consider a causal finite-dimensional continuous-time LTI system, represented in state-space form by the matrices (A, B, C, D) so that its transfer matrix H_{zw} is given by

$$H_{zw}(s) = C(sI - A)^{-1}B + D \quad (2.5)$$

The L_2 to L_∞ gain is the H_2 norm of H_{zw}. Physically, it is *the signal energy of the impulse response with initial states set to zero*:

$$\begin{aligned} \|H_{zw}\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}[H'_{zw}(jw)H_{zw}(jw)]dw \\ &= \int_{0^-}^{\infty} \text{trace}[h'_{zw}(t)h_{zw}(t)]\, dt \end{aligned} \quad (2.6)$$

Before we obtain the analytical expression for the system H_2 norm, we introduce one identity:

Lemma 2.1 If M is symmetric and A is Hurwitz (that is, all eigenvalues are in the open right half plane), then $K = \int_0^\infty e^{A't}Me^{At}\, dt$ if and only if $KA + A'K + M = 0$.

Proof: "$\Longleftarrow$", post-multiply $K = \int_0^\infty e^{A't} M e^{At}\, dt$ by A, integrate by parts, notice $lim_{t\to\infty} e^{At} = 0$, we can get the Lyapunov equation.

"$\Longrightarrow$", pre-multiply and post-multiply the Lyapunov equation by $e^{A't}$ and e^{At} and integrate.

For the casual LTI system $H(s) = C(sI - A)^{-1}B + D$, the impulse response is

$$h_{zw}(t) = Ce^{tA}B + D$$

so to obtain a finite H_2 norm, D in equation (2.5) has to be zero. Further, if A is Hurwitz, according to equation (2.6) and Lemma 2.1, we have:

$$\begin{aligned}
\|H_{zw}\|_2^2 &= \int_0^\infty \text{trace}[h'_{zw}(t)h_{zw}(t)]\, dt \\
&= \int_0^\infty \text{trace}[B'e^{tA'}C'Ce^{tA}B]\, dt = \text{trace}(B'QB) \\
&= \int_0^\infty \text{trace}[Ce^{tA}BB'e^{tA'}C']\, dt = \text{trace}(CPC')
\end{aligned}$$

where P,Q are the unique solutions of Lyapunov equations $AP + PA' + BB' = 0$, $A'Q + QA + C'C = 0$.

This result is restated in the following theorem [Meg01] [ZDG95]:

Theorem 2.1 The H_2 norm of system (A,B,C,D) is infinite if A is not Hurwitz or $D \neq 0$, otherwise

$$\|H\|_2^2 = \text{trace}(CPC') = \text{trace}(B'QB) \tag{2.7}$$

where P or Q can be computed by solving Lyapunov equations

$$AP + PA' + BB' = 0 \tag{2.8}$$

$$A'Q + QA + C'C = 0 \tag{2.9}$$

Remark: Take $V(x) = x'Px$ as a Lyapunov fuction, then $\dot{V}(x) = \dot{x}'Px + x'P\dot{x} = x'(PA + A'P)x = -x'BB'x$. So if B has full row rank, then $P > 0$ is sufficient and necessary to guarantee that A is Hurwitz, ie asymptotically stable. Similarly if C has full column rank, then $Q > 0$ is sufficient and necessary to guarantee that A is Hurwitz.

Theorem 2.1 provides an approach to evaluate the system H_2 norm. More important, it also provides a start point to derivate the Riccati equation for unstructured H_2 control or to obtain the conditions for structured H_2 control.

If a symmetric Q exists, then for any X such that $X > Q$, we have $A'X + XA + C'C < 0$ and $\text{trace}(B'XB) > \text{trace}(B'QB)$. Thus we can get a matrix inequality to compute the system H_2 norm if A is Hurwitz [BEF94]:

$$\|H\|_2^2 = \inf\{\text{trace}(B'XB) : \ A'X + XA + C'C < 0\} \tag{2.10}$$

Or similarly for any Y such that $Y > P$, we have

$$\|H\|_2^2 = \inf\{\text{trace}(CYC') : \ AY + YA' + BB' < 0\} \tag{2.11}$$

Further, it can be shown [GNL95] that this inferior in (2.11) can be calculated as:

$$\begin{aligned} \|H\|_2^2 = \inf_{S,Y} \quad & \text{trace}(S) \\ \text{s.t.} \quad & AY + PY' + BB' < 0 \\ & \begin{bmatrix} S & CY \\ YC' & Y \end{bmatrix} > 0 \end{aligned} \tag{2.12}$$

This matrix inequality is also very useful in LMI-based H_2 synthesis

White Noise to Variance Gain, another view of H2 norm.

White noise is the most important stochastic model in LTI system design. In mathematics, a scalar white noise signal $f(t)$ is a weird set of generalized functions, equipped with a probability measure such that

$$Eg = 0 \text{ and } E|g|^2 = \int_a^b |h(t)|^2 dt \text{ for } g = \int_a^b h(t) f(t) dt \tag{2.13}$$

for any function $h(t)$ [Meg01], where E is the average or expected value. This definition can be extended to the vector case.

So, if a vector $z(t)$ is the output of a causal LTI system with (unit) white noise input under zero initial condition, we have

$$\lim_{T\to\infty} E\frac{1}{T}\int_0^T z(t)'z(t)dt = \lim_{t\to\infty}\int_0^t \text{trace}(h(t)'h(t)dt = \|H\|_2^2 \tag{2.14}$$

This means that H_2 norm is *the asymptotic value of system output variance with (unit) white noise input.*

2.2 H_2 Control with Full Sate Feedback

In this section, Riccati equation for H_2 control with full-state feedback is developed, and more equivalent statements are introduced. More detail can be found in the lecture notes by Megretski [Meg01] and the book of Anderson and Moore [AnM90].

2.2.1 H_2/LQR and Riccati Equation

Full-state feedback problem: given a continuous-time LTI model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \end{aligned} \tag{2.15}$$

All states are available for control, and we are required to design a full-state feedback controller

$$u = Fx$$

to minimize the H_2 norm of the closed-loop system $w(t) \rightarrow z(t)$.

We make the following four assumptions

1) (A, B_2) is stabilizable;

2) (C_1, A) is detectable;

3) D_{12} has full column rank;

4) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

where "stabilizable" means all unstable modes of A are controllable under B_2. Assumption (2) is not essential. It enforces that unconditional optimization will result a stabilizing control law [ZFDG95]. Assumption (2) together with assumption (1) guarantees that the BIBO stability is equivalent to internal (Lyapunov) stability. Assumptions (3) and (4) are used to avoid the singularity of Riccati-based algorithm, but they are not required for LMI-based algorithm, as in H_∞ optimization [GaP94].

With the definition of system H_2 norm, we have

$$\begin{aligned}
J(u) &= \|H_{zw}\|_2^2 = \lim_{T\to\infty} E\frac{1}{T}\int_0^T z(t)'z(t)dt \\
&= \lim_{T\to\infty} E\{\frac{1}{T}\int_0^T (C_1x + D_{12}u)'(C_1x + D_{12}u)dt\} \\
&= \lim_{T\to\infty} E\{\frac{1}{T}\int_0^T \begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} C_1'C_1 & C_1'D_{12} \\ D_{12}'C_1 & D_{12}'D12 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt\} \\
&= E\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} Q & N \\ N' & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \text{ with } x(t)|_{t=0} = x_0 \qquad (2.16)
\end{aligned}$$

in which $Q = C_1'C_1, R = D_{12}'D_{12}, N = C_1'D_{12}$.

This is a standard LQR problem with initial state x_0 satisfying $E(x_0x_0') = \text{trace}(B_1'B_1)$. (For the LQR problem, generally x_0 can be any given state.) Further, with the idea of "completion of square", we have

$$\begin{aligned}
J(u) &= E\int_0^\infty \left\{ \begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} Q & N \\ N' & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{d}{dt}(x'Kx) - \frac{d}{dt}(x'Kx) \right\} dt \\
&= E\int_0^\infty \left\{ \begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} Q & N \\ N' & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right. \\
&\quad \left. + [(Ax + B_2u)'Kx + x'K(Ax + B_2u)] - \frac{d}{dt}(x'Kx) \right\} dt \\
&= E\int_0^\infty \{(u - Fx)'R(u - Fx) - \frac{d}{dt}(x'Kx)\}dt \qquad (2.17) \\
&\geq -\int_0^\infty \frac{d}{dt}(x'Kx)dt \\
&= E(x_0'Kx_0) = \text{trace}(B_1'KB_1)
\end{aligned}$$

where K is a symmetric matrix.

By comparing the two sides of the above equation (2.17) we can obtain

$$Q + A'K + KA = F'RF$$

$$-RF = N' + B_2'K$$

Since $R > 0$, we can further obtain the **algebraic Riccati equation** for the matrix K:

$$(A - B_2R^{-1}N')'K + K(A - B_2R^{-1}N') - KB_2R^{-1}B_2'K + Q - NR^{-1}N' = 0 \quad (2.18)$$

and the stabilizing optimal control $u = Fx$:

$$F = -R^{-1}(B_2'K + N') \tag{2.19}$$

where K is the stabilizing solution to Riccati equation (2.18). (Note: for the general Riccati equation $A'X + XA - XBB'X + Q = 0$, "stabilizing" means $A - BB'X$ is Hurwitz.) Consequently the minimum H_2 norm is given by

$$\|H\|_2^2 = \min_u J(u) = E(x_0'Kx_0) = \text{trace}(B_1KB_1) \tag{2.20}$$

Remark: From (2.18) and (2.19), we can see that full-state feedback H_2/LQR optimal gain F doesn't depend on the initial state x_0 or B_1. This is quite different from static output feedback.

If the above four assumptions are satisfied, the algebraic Riccati equation (2.18) will have a unique stabilizing solution K, and then we can solve for the optimal controller. Numerically efficient algorithms to solve the algebraic Riccati equations (and Lyapunov equations) were developed around 1970 and standard codes are available [ZDK95].

The above equations (2.18) and (2.20) can also be obtained from Theorem 2.1, by substituting the closed-loop description $\left[\begin{array}{c|c} A + B_2F & B_1 \\ \hline C_1 + D_{12}F & 0 \end{array}\right]$ into the Lyapunov equation therein. We will see this type of procedure in next section.

2.2.2 KYP Lemma

There are also other equivalent statements for H_2 optimal control with full-state feedback. They are summarized in the KYP lemma. For the proof of KYP lemma, please refer to the lecture notes [Meg01].

Theorem 2.2 (Kalman, Yakubovich, Popov, etc) The following statements are equivalent:

1) Minimum in the H_2/LQR problem (2.15) and (2.16) exists and is unique for any x_0;

2) (A, B_2) is stabilizable, $R > 0$, and the algebraic Riccati equation (2.18) has a stabilizing solution $K = K'$;

3) (A, B_2) is stabilizable, $R > 0$, and the algebraic Riccati inequality

$$(A - B_2R^{-1}N')'L + L(A - B_2R^-1N') - LB_2R^{-1}B_2'L + Q - NR^{-1}N' > 0$$

has a solution $L = L'$;

4) (A, B_2) is stabilizable, $R > 0$, and the Hamiltonian matrix

$$H = \begin{bmatrix} A - B_2R^{-1}N' & B_2R^{-1}B_2' \\ Q - NR^{-1}N' & -(A - BR^{-1}N')' \end{bmatrix}$$

doesn't have any eigenvalue on the imaginary axis;

5) (A, B_2) is stabilizable, and there exists a scalar $\epsilon > 0$, such that

$$\begin{bmatrix} (j\omega I - A)^{-1}B_2 \\ I \end{bmatrix}' \begin{bmatrix} Q & N \\ N' & R \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B_2 \\ I \end{bmatrix} \geq \epsilon \begin{bmatrix} (j\omega I - A)^{-1}B_2 \\ I \end{bmatrix}' \begin{bmatrix} (j\omega I - A)^{-1}B_2 \\ I \end{bmatrix}$$

for all $\omega \in \mathbf{R}$ except those for which $j\omega$ is an eigenvalue of A.

If these conditions hold, then

a) The minimum $\|H\|_2^2$ equal to trace$(B_1'KB_1)$ (or equavalently the minimum cost of LQR equals $x_0'Kx_0$), and the optimal state-feedback control is

$$u(t) = Fx(t) = -R^{-1}(B_2'K + N')x(t), \text{ for } \forall t$$

b) K equals $-\Psi\Phi^{-1}$, where the columns of $[\Phi', \Psi']'$ form a base in the stable invariant subspace of the Hamiltonian matrix.

Remark: The algebraic Riccati inequality (in KYP statement (3) doesn't seem to be linear about L, however using the **Schur complement**

$$\begin{bmatrix} Q(x) & S(x) \\ S'(x) & R(x) \end{bmatrix} > 0 \Longleftrightarrow \begin{cases} R(x) > 0 \\ Q(x) - S(x)R^{-1}(x)S'(x) > 0 \end{cases} \tag{2.21}$$

We can cast it in an equavalent linear matrix Inequality (LMI) form,

$$\begin{bmatrix} (A - B_2R^{-1}N')'L + L(A - B_2R^{-1}N') + Q - NR^{-1}N' & LB_2 \\ B_2'L & R \end{bmatrix} > 0 \qquad (2.22)$$

Thus, full-state feedback H_2 optimization (or LQR) problem also can be handled with the LMI techniques.

$$\min_{L} \ \text{trace}(B_1'LB_1) \text{ subject to LMI (2.22)} \qquad (2.23)$$

And to solve this LMI problem, we don't have the singularity limitation, unlike Riccati-based method. Another LMI relation to solve full-state feedback H_2 control also can be derived based on (2.12), see [GNL95].

2.2.3 Phase Margin and Gain Margin

Phase margins and gain margins play a critical role in the classical control systems design. One attracting feature of H_2 optimal controllers with full-state feedback is that they can guarantee certain stability robustness with respect to uncertainties in the gain margin or phase margin of the feedback loop.

For $\dot{x} = Ax + B_2u, u = Fx$, the *upper gain* margin is defined as the scaler factor g_u by which the gain F can increase before the closed system $A + B_2g_uF$ goes unstable. The *lower gain* margin g_d is the scaler factor by which the gain F can decrease before the closed system $A + B_2g_dF$ go unstable. The *phase margin* is the smallest ψ such that $A + B_2e^{j\psi}F$ and $A + B_2e^{-j\psi}F$ are still stable.

Corollary 2.1: For the optimal H_2 controller with full-state feedback, the upper gain margin is infinity, lower gain margin is at least 0.5, and phase margin is at least $\frac{\pi}{3}$.

Taking $V(x) = x'Kx$ as the Lyapunov function for the "perturbed" system, and use the Reccati equation (2.18), we can prove the above corollary.

2.3 Full Order Optimal H_2 Control with Output Feedback

From the previous section, we see that the optimal H_2 controller with full-state feedback can be obtained just by solving the Riccati equation. However for most cases, it is impractical or impossible to sense the full-states. So it is more important to be able to design a controller that feedback only the output signals. In this section, we will examine H_2 optimization to design the full-order compensator with output feedback. Then in the next sections we will discuss static output feedback and lower-order output-feedback controllers with H_2 optimization.

2.3.1 Kalman State Estimator and Kalman filter

First let's see how to estimate the state of the plant with process noise from the measured output signals. Given a continuous time LTI plant model

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ y_v &= Cx + Du + v \end{aligned} \tag{2.24}$$

where the process noise w and measurement noise v satisfying $E(w) = E(v) = 0$, , $E[w(t)w'(\tau)] = Q\delta(t-\tau)$, $E[v(t)v'(\tau)] = R\delta(t-\tau)$, and $E[w(t)v'(\tau)] = N\delta(t-\tau)$, this means that w and v are zero-mean white noises. We are required to estimate the state x with $\hat{x}(t)$ using $y_v(t)$ and u(t), so as to minimize the steady-state error covariance

$$J = \lim_{t\to\infty} E[e(t)e'(t))], \text{ where } e(t) = x(t) - \hat{x}(t)$$

Suppose the estimator has the form

$$\dot{\hat{x}} = A\hat{x} + Bu - H(y_v - C\hat{x} - Du) \tag{2.25}$$

From equation (2.25) and equation (2.24), we obtain the differential equation for estimation error

$$\dot{e} = (A + HC)e + [G\ H] \begin{bmatrix} w \\ v \end{bmatrix} \tag{2.26}$$

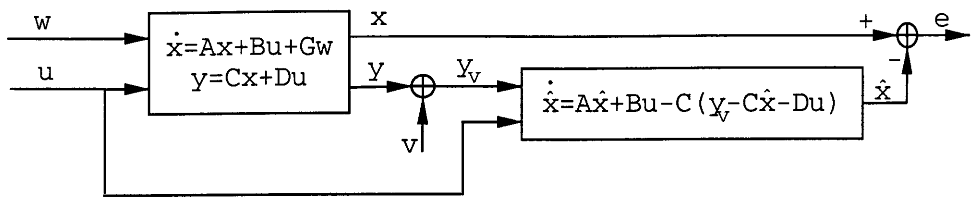

Figure 2-1: Kalman state estimator

We can see that minimization of the error covariance is a linear quadric problem. Following the idea of 'completion of square' similar as in Section 2.2, we can get another Riccati equation. Below we use another approach starting from Theorem 2.1.

From equation (2.26), we write the closed-loop system

$$\begin{bmatrix} w \\ v \end{bmatrix} \rightarrow e : \left[\begin{array}{c|c} A - HC & \text{[G,H]} \\ \hline \text{I} & 0 \end{array}\right]$$

Rescaling the intensity of the white noise, and substituting the closed-loop into the Laypunov equation in Theorem 2.1, we can express the cost $J = \text{trace}(L)$, where L is a symmetric matrix satisfying

$$(A + HC)L + L(A + HC)' + [G, H] \begin{bmatrix} Q & N \\ N' & R \end{bmatrix} [G\ H]' = 0 \tag{2.27}$$

Define

$$\hat{J}(L, H, M) = J + \text{trace}\{[(A + HC)L + L(A + HC)' + [G, H] \begin{bmatrix} Q & N \\ N' & R \end{bmatrix} [G, H]']S\}$$

where S is a Lagrange matrix. To minimize trace(L) subjecte to (2.27), we require $\partial\hat{J}/\partial H = 0$, that is $L'C' + GN + HR = 0$. If $R > 0$, we write $H = -(GN + LC')R^{-1}$. Substitute it into (2.27), we obtain the **Riccati equation** about L

$$(A - GNR^{-1}C)L + L(A - GNR^{-1}C) - LC'R^{-1}CL + G(Q - NR^{-1}N')G' = 0 \tag{2.28}$$

If the pair (C, A) is detectable, $R > 0, Q - NR^{-1}N' \geq 0$, and $[A - NR^{-1}C, Q - NR^{-1}N']$ has no uncontrollable mode on the imaginary axis, then Riccati equation

(2.32) has a unique solution L [ZDK95]. Thus we can solve for the residue matrix H for state estimator and the minimum cost J_{min}

$$H = -(GN + LC')R^{-1} \tag{2.29}$$

$$J_{min} = \text{trace}(L) \tag{2.30}$$

The above is design of Kalman state estimator. From equation (2.32) and (2.29) we can see that the residue matrix H is independent of B and D.

More generally, instead of estimating state x, if we would like to estimate some output $z = M'x$ with the minimum error in the sense of H_2 norm. The cost $J = \lim_{t\to\infty} E[(z(t) - \dot{\hat{z}}(t))'(z(t) - \dot{\hat{z}}(t))']$. The optimal H_2 estimator looks like a filter to obtain the concerned signal z from the signal with process noise w and sensor noise v, as shown in Figure 2-2. That is why it is also called Kalman filter.

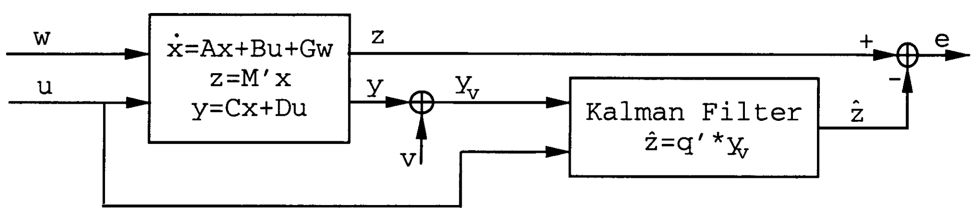

Figure 2-2: Kalman filter

The estimator is constructed as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + H(y_v - C\hat{x} - Du) \\ \dot{\hat{z}} &= M'\hat{x} \end{aligned} \tag{2.31}$$

As in the Kalman state estimator, we can solve for the optimal problem

$$\min_{H} \; J = \text{trace}(MLM'), \text{ s.t. } () \tag{2.32}$$

Further the residual gain matrix H is exactly the same as that in the state estimator and can be obtained from equation (2.32) and (2.29). The corresponding minimum cost of H_2 estimation is trace(MLM'). It is interesting to observer that the residual matrix H DOES NOT depend on which component $M'x$ of the state is being estimated. This is different from H_∞ optimal estimation.

2.3.2 Output Feedback

Now let's consider the general full-order H_2 control problem with output feedback. Given an LTI plant model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x \ + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \tag{2.33}$$

we are required to design a proper rational controller $K_{opt}(s)$ which stabilizes the plant internally and minimizes the H_2 norm of the closed-loop system from the input w to cost z.

In the plant model, D_{11} is assumed to be zero so as to guarantee that the H_2 problem is well posed. And since the case D_{22} nonzero can be recovered from zero case [ZhD95], without loss of generality, we can assume D_{22} to be zero in the following.

The H_2 optimal control problem can be taken as an LQR problem involving the cost

$$\begin{aligned} J(u) &= \lim_{T\to\infty} E\{\frac{1}{T}\int_0^T z(t)'z(t)dt\} \\ &= \lim_{T\to\infty} E\{\frac{1}{T}\int_0^T \begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} C_1'C_1 & C_1'D_{12} \\ D_{12}'C_1 & D_{12}'D12 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt\} \end{aligned}$$

with correlated process white noise $\xi(t)$ and sensor white noise $\theta(t)$ entering the system via the channels $[B_1', \ D_{21}']'$,

$$E\{\begin{bmatrix} \xi(t) \\ \theta(t) \end{bmatrix}\begin{bmatrix} \xi(\tau) \\ \theta(\tau) \end{bmatrix}'\} = \begin{bmatrix} B_1 \\ D_{12} \end{bmatrix}\begin{bmatrix} B_1 \\ D_{12} \end{bmatrix}' \delta(t-\tau)$$

And the H_2 optimal control problem can be realized as a full-state feedback controller and a Kalman state estimator, as shown in Figure (2-3). One remarkable feature of the H_2 optimal compensator is that we can design control gain and estimator separately, which is not the case for the H_∞ optimization.

Theorem 2.3 For the H_2 optimal control problem (2.33), if

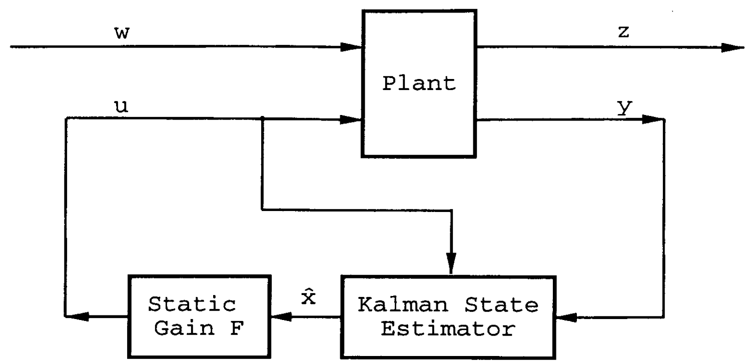

Figure 2-3: Full-order dynamic output feedback H2 controller

i) (A, B_2) is stabilizable, and (C_2, A) is detectable;

ii) D_{12} has full column rank and D_{21} has full row rank;

iii) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

iv) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full column rank for all ω,

then there exists a unique optimal H_2 controller

$$K_{opt}(s) := \left[\begin{array}{c|c} A - HC_2 + B_2F & \mathrm{H} \\ \hline \mathrm{F} & 0 \end{array}\right] \tag{2.34}$$

where F is the H_2 optimal gain matrix with full-state feedback, H is the residual gain matrix of Kalman state estimator, and the cost

$$\begin{aligned} \min \|H_{zw}\|_2^2 &= \mathrm{trace}(B_1'KB1) + \mathrm{trace}((B_2'F + D_{12}'C1)L(B_2'F + D_{12}'C1)') \\ &= \mathrm{trace}((HC_2' + B_1D_{12}')'P(HC_2' + B_1D_{12}')) + \mathrm{trace}(C_1LC_1') \end{aligned} \tag{2.35}$$

in which K and L are respectively the solutions of the Riccati equations associated with the full-state feedback and the Kalman state estimator.

For a detailed proof of the above separation theorem of H_2 optimal control, please refer to [ZhD95] or [DGK89]. Further more, all the stabilizing γ-suboptimal H_2 controllers such that $\|H_{zw}\|_2 < \gamma$ can be expressed as

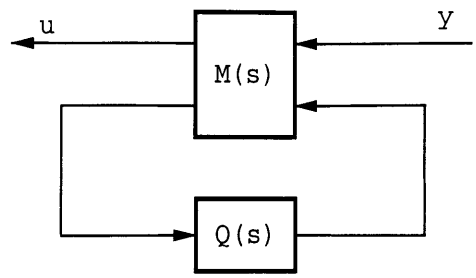

Figure 2-4: Q parameterization of suboptimal H2 controller

where $M(s)$ is

$$M(s) := \left[\begin{array}{c|cc} A - HC_2 + B_2F & H & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

and Q is any stable transfer function, and $\|Q\|_2^2 < \gamma^2 - (\min \|H_{zw}\|_2)^2$ [DGK89].

Finally, we should point out that, H_2 control with full order output feedback also can be handled with LMI techniques, even for singular plants that don't meet the assumptions ii)-iv) in Theorem 2.3 ([GaP94] [MOS98]).

2.4 H_2 Control With Static Output Feedback

In Sections 2.2 and 2.3, we showed that optimal H_2 control with full-state feedback or full-order output feedback can be obtained simply by solving one or two decoupled Reccati equations. However, in most application to sense the full-states is not practical or impossible, and it's also difficult and not robust to implement full-order controller (the order of the original plant + the order of shape filters). That is one of the most important reasons that modern control is not widely adopted by the engineers in industry. Generally the design of a lower-order optimal controller leads to a untractable problem. We will discuss some approaches available for lower-order control the next section. In this section, we will focus on H_2 optimal control with static output feedback.

The static output problem is one of the most important problems in control engineering. It has been brought to the attention of the control community in 1970 by Levine and Athans [LeA70], and has been investigated intensively in the past three decades. However, many problems still remain open. (More information can be found in the recent review paper [SAD97]).

In this section, we will introduce static output stabilization, and and the static output H_2 optimization and sub-optimization. Other related topics will be discussed in the following chapters, such as H_∞ optimization and eignstructure assignment with static output feedback.

2.4.1 Static Output Stabilization Problem

Given a plant model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x \ + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{2.36}$$

with the static output feedback control $u(t) = Fy(t)$, the closed-loop would be:

$$\begin{aligned} \dot{x} &= (A + B_2 F C_2)x + (B_1 + B_2 F D_{21})w \\ z &= (C_1 + D_{12} F C_2)x + D_{12} F D_{21} w \end{aligned} \tag{2.37}$$

Before we come to optimal control, we need to consider a more basic problem: Does there exist a stabilizing static gain F? In another words, is there an F such that $A + B_2 F C_2$ is Hurwitz?

Theorem 2.4 There exists a stabilizing static output feedback gain if and only if there exists some $P = P' > 0$ such that

$$B_{2\perp}(AP + PA')B'_{2\perp} < 0 \tag{2.38}$$

$$C'_{2\perp}(A'P^{-1} + P^{-1}A)(C'_{2\perp})' < 0 \tag{2.39}$$

where $B_{2\perp}$ and $C'_{2\perp}$ are full-rank matrices orthogonal to B_2 and C'_2, respectively.

The above theorem follows from Lyapunov stability theorem. $V(x) = x'Px$, $\dot{V}(x) = x'[(A + B_2FC_2)P + P(A + B_2FC_2)']x$ without w. So $A + B_2FC_2$ is Hurwitz if and only if there exists some symmetric matrix P such that

$$(A + B_2FC_2)P + P(A + B_2FC_2)' < 0, \text{ and } P = P' > 0 \tag{2.40}$$

With elimination lemma [BEF94] and considering $P > 0$, we can obtain Theorem 2.4 from (2.40).

Lemma 2.2 (Elimination Lemma) Given matrices $G \in R^{n\times n}$, $U \in R^{n\times m}$ and $V \in R^{n\times r}$, there exists an $S \in R^{m\times r}$ such that $G + USV' + VS'U' > 0$ if and only if $U'_\perp GU_\perp > 0$ and $V'_\perp GV_\perp > 0$

Theorem 2.4 gives the sufficient and necessary condition for the existence of a stabilizing static output feedback gain. However, to check this condition or solve for such gain is not easy, because the condition given by (2.38) and (2.39) is not convex for P. This problem remain open for long time and different algorithms have been presented. Recently, two nice algorithms ([OlG97], [GSS98], [EOA97]) have been proposed to solve it.

The first is the **Min/Max Algorithm** proposed by Geromel *et al.* [OlG97]. Their ideas are summarized as:

1. Inequality (2.40) is equivalent to

$$(A + B_2FC_2)P + P(A + B_2FC_2)' + B_1B_1' < 0, \text{ and } P = P' > 0 \tag{2.41}$$

With the Schur complement, (2.41) can be written as,

$$\begin{bmatrix} (A + B_2FC_2)P + P(A + B_2FC_2)' & B_1 \\ B_1' & -I \end{bmatrix} < 0, \text{ and } P = P' > 0 \tag{2.42}$$

Then by eliminating the matrix F, we can say that $A + B_2FC_2$ is Hurwitz if and only if there exists some matrix pair (P, Q), such that

$$B_{2\perp}(AP + PA' + B_1B_1)B_{2\perp}' < 0, P = P' > 0 \tag{2.43}$$

$$\begin{bmatrix} C_{2\perp}' & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} QA + A'Q & QB_1 \\ B_1'Q & -I \end{bmatrix} \begin{bmatrix} (C_{2\perp}')' & 0 \\ 0 & I \end{bmatrix} < 0, Q = Q' > 0 \tag{2.44}$$

$$P = Q^{-1} \tag{2.45}$$

2. Observe that (2.43) is in LMI form (convex in P), and if P satisfies (2.43) then ρP will also satisfy (2.43) for $\rho \geq 1$; (2.44) is also in LMI form (convex in Q), and if Q satisfies (2.44) then θQ will also satisfy (2.44) for $\theta \leq 1$. So a min/max algorithm is presented: Starting from Q_0 satisfying the LMI (2.44), iteratively solve the two convex problems until convergence.

- Solve for (ρ_k, P_k):

$$\min_{\rho, P} \rho, \text{ s.t. Q satisfying LMI (2.43) and } Q_k^{-1} \leq P \leq \rho Q_k^{-1}$$

- Solve for (θ_k, Q_k):

$$\max_{\theta, Q} \theta, \text{ s.t. Q satisfying LMI (2.44) and } \theta P_k^{-1} \leq Q \leq P_k^{-1}$$

Remark 1: The min/max algorithm above will generate a monotonically decreasing sequence ρ_k with the lower bound of 1, and a monotonically increasing sequence θ_k with the upper bound of 1. It also generates a increasing sequence P_k and a decreasing sequence Q_k. If the sequence P_k is bounded, then $P_\infty Q_\infty = I$. Numeric experiments showed ([OlG97] [EOA97]) that this algorithm failed to converge for less than 5% of 1000 randomly generated stabilization problems satisfying Kimura's sufficient condition [Kim75].

Remark 2: In (2.41), B_1B_1' is an immaterial item for stability study, but it plays an important role in for the convergence of the algorithm. Without it we can't guarantee that θQ also satisfies (2.44) if Q satisfies (2.44) and $\theta \leq 1$. B_1B_1' can also be replaced by any symmetric negative semi-definite matrix with appropriate dimension.

El Ghaoui *et al.* ([EQA97]) proposed another better approach to solve the static output stabilization problem. Assuming $Q = P^{-1}$, with Lemma 2.3 we can obtain Corollary 2.3.

Lemma 2.3 For any pair of symmetric $n \times n$ matrices (X, Y), if $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$, then $\mathrm{trace}(XY) \geq n$, and the equality holds iff $YX = I$.

Corollary 2.2 Stabilizing static output feedback gain exists if and only if

$$\begin{aligned} B_{2\perp}(AP + PA')B_{2\perp}' &< 0, P > 0 \\ C'_{2\perp}(A'Q + QA)(C'_{2\perp})' &< 0, Q > 0 \\ \begin{bmatrix} P & I \\ I & Q \end{bmatrix} &\geq 0 \\ \text{and } \mathrm{trace}(PQ) &= n \end{aligned}$$

where n is the order of the plant.

El Ghaoui's approach is to minimize a bilinear objective function $\mathrm{trace}(PQ)$ subject to three LMI constraints with the **cone complementarily linearization algorithm**. Thus this method is more general than the min/max algorithm. The efficiency of El Ghaoui's approach is extremely satisfactory. We will discuss the details in next chapter.

Once we get the feasible pair (P, Q), we can reconstruct the static gain F with the closed-form formula in [IwS94] or get F by solving the LMI feasible problem (2.40) or (2.42) with the solved P.

2.4.2 Static Output H_2 Optimization

Checking the closed-loop system (2.37) with Theorem 2.1, we can see that $D_{12}FD_{21}$ has to be zero, otherwise the optimal H_2 problem is not well posed. For convenience, we assume $D_{21} = 0$ or $D_{12} = 0$. Since in most applications sensor noise is ignorable compared with process noise, below we assume $D_{21} = 0$. For the case D_{12}=0 and $D_{21} \neq 0$, we can get similar result.

From Theorem 2.1, we can obtain the Theorem 2.5 for static output H_2 optimization.

Theorem 2.5 Given the LTI plant model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x \ + D_{12} u \\ y &= C_2 x \end{aligned} \tag{2.46}$$

With the stabilizing static output feedback $u(t) = Fy(t)$, the H_2 norm $\|H_{zw}\|_2^2$ of the closed loop would be trace$(B_1' K B_1)$, where K satisfies

$$K(A + B_2 F C_2) + (A + B_2 F C_2)' K + (C_1 + D_{12} F C_2)'(C_1 + D_{12} F C_2) = 0 \tag{2.47}$$

or is trace$((C1 + D_{12} F C_2) P (C1 + D_{12} F C_2)')$, where P satisfying

$$(A + B_2 F C_2) P + P(A + B_2 F C_2)' + B_1 B_1' = 0 \tag{2.48}$$

Remark: If B_1 is full row rank, $A + B_2 F C_2$ is (asymptotically) stable if and only if $P > 0$. If C_1 is full column rank and $D_{12}' C_1 = 0$, then $K > 0$ is sufficient (not necessary) to guarantee the stability of $A + B_2 F C_2$.

So static output H_2 control becomes a constrained optimization:

$$\begin{aligned} &\min_F \operatorname{trace}(B_1' K B_1) \qquad (2.49) \\ &\text{s.t. } K(A + B_2 F C_2) + (A + B_2 F C_2)' K + (C_1 + D_{12} F C_2)'(C_1 + D_{12} F C_2) \\ &\quad A + B_2 F C_2 \text{ is Hurwitz} \end{aligned}$$

Or

$$\begin{aligned} &\min_F \operatorname{trace}((C1 + D_{12} F C_2) P (C1 + D_{12} F C_2)') \qquad (2.50) \\ &\text{s.t. } (A + B_2 F C_2) P + P(A + B_2 F C_2)' + B_1 B_1' = 0 \\ &\quad A + B_2 F C_2 \text{ is Hurwitz} \end{aligned}$$

Levine and Athans [LeV70] derived the necessary condition for simplified LQR (C_2 is full row rank, and $C_1'D_{12} = 0$) for the random initial states. Extending their approach to general H_2 control with static output feedback, we can arrive The Corollary 2.3.

Corollary 2.3 If C_2 is full row rank, the necessary condition for static output feedback H_2 optimal control is:

$$F = -(D_{12}'D_{12})^{-1}(D_{12}'C_1 + B_2'K)LC_2'(C_2LC_2')^{-1} \tag{2.51}$$

$$K(A + B_2FC_2) + (A + B_2FC_2)'K + (C_1 + D_{12}FC_2)'(C_1 + D_{12}FC_2) \tag{2.52}$$

$$L(A + B_2FC_2)' + (A + B_2FC_2)L + B_1'B_1 \tag{2.53}$$

However, to solve the above highly coupled nonlinear matrix equations (2.51)-(2.53) is not a trivial procedure. Levine and Athans proposed an iterative procedure (but convergence is not guaranteed): Start with a stabilizing F_0 and update F with equation (2.51) $F_{k+1} = -(D_{12}'D_{12})^{-1}(D_{12}'C_1+B_1'K_k)L_kC_1'(CLC')^{-1}$, where K_k and L_k are obtained by solving the Lyapunov function (2.52) and (2.53) with the previous F_k. Later the other gradient based algorithms, such as Newton method, have also been successfully used, please see the survey paper by Makila and Toivonen ([MaT85]). In addition, **homotopy** has also proved to be an efficient method to search for static output feedback gain ([Mer91] [CoS98]). We will discuss the computational methods in Section 2.7.

It's worthy to note that the optimal gain F will depend on the initial state x_0 (LQR) or B_1 (H_2), which is different from the case of full-state feedback.

2.4.3 γ-Suboptimal H_2 Problem with Static Output Feedback

Suppose we would like to design a static output feedback controller $u = Fy$, subject to some performance of the H_2 norm $\gamma(> \gamma_{opt})$

$$||H_{zw}||_2 < \gamma \tag{2.54}$$

This problem is usually called **γ-suboptimal H_2 problem** with static output feedback. It is also interesting, since in most applications, the H_2 norm is not the only objective, we might also want to achieve some other performance goal, such as H_∞ norm or damping ratio. The γ-suboptimal H_2 problem can be handled with the LMI techneques

Substituting the closed-loop description $\left[\begin{array}{c|c} A+B_2FC_2 & B_1 \\ \hline C_1+D_{12}FC_2 & 0 \end{array}\right]$ into (2.10), we can get

$$\|H\|_2^2 = \inf \operatorname{trace}(B_1'XB_1)$$

$$\text{s.t. } A+B_2FC_2 \text{ is Hurwitz} \tag{2.55}$$

$$(A+B_2FC_2)'X + X(A+B_2FC_2) + (C_1+D_{12}FC_2)'(C_1+D_{12}FC_2) < 0 \tag{2.56}$$

With the Schur complement (2.56) can be written as

$$\begin{bmatrix} (A+B_2FC_2)'X + X(A+B_2FC_2) & (C_1+D_{12}FC_2)' \\ (C_1+D_{12}FC_2) & -I \end{bmatrix} < 0 \tag{2.57}$$

If $(C_1+D_{12}FC_2)'(C_1+D_{12}FC_2) > 0$, from the Lyapunov stability theorem we know that $A+B_2FC_2$ is Hurwitz iff $X > 0$. Since $(C_1+D_{12}FC_2)'(C_1+D_{12}FC_2) \geq 0$ for whatever F, $X > 0$ is the necessary condition for F to be optimal. Eliminating F with Lemma 2.2, we can obtain Theorem 2.6.

Theorem 2.6 Given the plant model (2.46), the necessary condition for the existence of a stabilizing static gain $u = Fy$ subject to $\|H_{zw}\|_2 < \gamma$ is that there exist positive matrix pair (X,Y) such that

$$\operatorname{trace}(B_1'XB_1) < \gamma^2 \tag{2.58}$$

$$N_y' \begin{bmatrix} A'X + XA & C_1' \\ C_1 & -I \end{bmatrix} N_y < 0 \tag{2.59}$$

$$N_u' \begin{bmatrix} AY + YA' & YC_1' \\ C_1Y & -I \end{bmatrix} N_u < 0 \tag{2.60}$$

$$X = Y^{-1} > 0 \tag{2.61}$$

where N_y and N_u are respectively an orthogonal basis to the null space of $[C_2, 0]$ and $[B_2', D_{12}']$. If some feasible pair (X,Y) is found, we can reconstruct F by solving the

LMI feasible problem (2.57). Further, if such F is stabilizing, then it is a γ-optimal H_2 solution.

Similar to the static stabilization problem, condition (2.61) $X = Y^{-1}$ destroys the convexity. However Geromel's min/max algorithm ([OlG97],[GSS98]) can't be extended for checking the static H_2 conditions (2.58)-(2.61), since (2.58) destroys the feasibility of generating a monotonically decreasing sequence, so convergence is no guaranteed. El Ghaoui's cone complementary algorithm ([EOA97]) can be extended to the static H_2 problem to handle $X = Y^{-1}$. We will see this algorithm in next chapter.

Once the feasible pair (X, Y) is solved, (2.57) becomes an LMI form in F. And there always exists some feasible F, since (2.59)-(2.61) hold iff (2.57) holds and $X > 0$.

Remark 1: The above (2.58)-(2.61) are necessary conditions, and they will become sufficient if $(C_1 + D_{12}FC_2)'(C_1 + D_{12}FC_2) > 0$. So this F obtained in the above procedure might not be the γ-optimal H_2 solution, since the closed loop $A + B_2FC_2$ might have pure imaginary eigenvalue. So we need to check: If F is stabilizing, then it is a γ-optimal H_2 solution. (This point has been ignored in most of the literature.) Particularly, if $C_1'D_{12} = 0$ and C_1 has column rank, then the conditions are necessary and sufficient.

Remark 2: If F has additional architecture constraints, such as decentralization, ,we can't guarantee the reconstruction of a structured F with pair (X, Y). Since the elimination lemma doesn't hold for structured F: (2.59)-(2.61) $\Rightarrow$ (2.57) and $X > 0$, but vice versa is not true.

2.4.4 Suboptimal H_2 Control

As we've already seen above, we need iterative computational methods to get the optimal or γ-suboptimal H_2 gain F. There is also some non-iterative approaches to obtain a suboptimal controller. The idea for the suboptimum is, to find a output feedback gain 'closest' (in some sense) to the full-state feedback gain $F^\star$.

Kosut ([Kos70]) proposed an approximation of output feedback to minimize the effect of error excitation for the deterministic case with $C_1'D_{12} = 0$. Similar ideas can be extended to the stochastic case and without limit on $C_1'D_{12}$.

Suppose $F^\star$ is the optimal H_2 gain with full-state feedback (which can be obtained by the Riccati equations in Section 2.2, $x^\star$ is the state corresponding its closed loop $\left[\begin{array}{c|c} A+B_2F^\star & B_1 \\ \hline C_1+D_{12}F^\star & 0 \end{array}\right]$ and x is the state of the closed-loop $\left[\begin{array}{c|c} A+B_2FC_2 & B_1 \\ \hline C_1+D_{12}FC_2 & 0 \end{array}\right]$ corresponding to the output feedback gain F. One approximation is to find some F that *minimizes the control force error variance* $q(t) = (F^\star - FC_2)x^\star$ with some weight $R^\star > 0$. This turn out to be a problem similar to the Kalman estimator. We can get

$$\min_F \lim_{T\to\infty} E\{\frac{1}{T}\int_0^T q'(t)R^\star q(t)dt\} = \min \ \text{trace}(B_1'VB_1)$$

$$\text{s.t. } (A+B_2FC_2)'V + V(A+B_2FC_2) + (FC_2-F^\star)'R^\star(FC_2-F^\star)$$

Then by introducing matrix Lagrangian L, we can get the necessary condition for the case of full-row-rank C_2:

$$F = F^\star LC_2'(C_2LC_2')^{-1} \tag{2.62}$$

where L is got by solving the Lyapunov equation

$$(A+B_2FC_2)L + L(A+B_2FC_2)' + B_1B_1' = 0 \tag{2.63}$$

So, we can solve the Riccati equation (2.18) for $F^\star$, then solve another Lyapunov equation (2.63) then get a "suboptimal" H_2 control F with static output feedback.

Although the above design procedure is quite concise, we can only get an approximation. Worse, there is no guarantee of the stability of the close-loop system, since the cost function doesn't contain the dynamic information about the closed-loop system. However, it is still worthy to mention, since it us concise and can be used as an initial start point for other iterative procedures.

Another H_2 suboptimal approach is **projective control**. we will discuss it in the section of lower-order control, since it can also be used to design the suboptimal reduced-order controller.

2.5 Lower-Order H_2 Control with Output Feedback

The order of a system is the minimal number of state in its state-space realization. Design of lower-order optimal controllers remains as a challenging job in the last two decades. Till now it is still one of the main focuses in the community of control. Generally speaking, there are indirect approaches (suboptimal) and direct approaches. It's very interesting to note that lower-order H_2 control problem can be cast as a static output feedback problem. And so the methods for optimal static output control, such as gradient-based optimization, homotopy methods and LMI, can be used to directly design lower-order controller.

2.5.1 Lower-Order H_2 Control Problem and Lower-Order Stabilization

Lower-order H2 Control Problem: Given a n-th order LTI plant model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x \ + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{2.64}$$

design a k-th order controller $(k < n)$

$$\begin{aligned} \dot{x_K} &= A_K x_K + B_K y \\ u &= C_K x_K + D_K y \end{aligned} \tag{2.65}$$

to stabilize the plant internally and minimize the the H_2 norm of the closed-loop system from w to z.

Before we solve the lower-order H2 control, we need to know that the existence of lower-order stabilizing control. This is **Lower-Order Stabilization Problem**. The result is given in Theorem 2.7. The proof is similar to that of static output feedback, or see reference [IwK94] for details.

Theorem 2.7 There exists a k-th order output feedback stabilizing control for n-th order LTI plant model (2.64) if and only if there exists symmetric matrices R and S, such that

$$\begin{aligned} B_{2\perp}(AR + RA')B'_{2\perp} < 0, R > 0 \\ C'_{2\perp}(A'S + SA)(C'_{2\perp})' < 0, S > 0 \\ \begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \\ \text{and trace}(RS) \leq n + k \end{aligned}$$

where $B_{2\perp}$ and $C'_{2\perp}$ are full-rank matrices orthogonal (null space) to B_2 and C'_2 respectively.

The cone complementary linearization algorithm ([EOA97]) or the min/max algorithm ([GSS98]) can be used to check the above condition and to solve for the stabilizing reduced-order control.

2.5.2 Indirect Design of Lower-Order Controller

Model order reduction is the most widely used indirect method to design for lower-order H_2 controllers as well as lower-order H_∞ controllers. Projective control is also an approximation to keep some dynamic properties with the reduced-order realization.

Model Order Reduction

The idea of model reduction is to use a lower-order system to approximate a high order system in some distance measure. Balance model reduction and Hankel norm approximation are two good ones among the numerous approaches proposed in the past years.

Balance mode reduction was first introduced by Moore (1981), and was contributed to by many other researchers after that. As we know, the realization of a system is not unique. Any stable LTI system has a balance realization (A,B,C,D), whose controllability and observability Gramians are equal and diagonal ([ZDG95], [Meg01]).

$$A\Sigma + \Sigma A' + BB' = 0 \tag{2.66}$$

$$A'\Sigma + \Sigma A + C'C = 0 \tag{2.67}$$

where Gramian $\Sigma = \text{diag}([\sigma_1, \sigma_2, ...\sigma_r, \sigma_{r1}, ...\sigma_n])$, $\sigma_1 \geq \sigma_2 \geq ...\sigma_r \geq \sigma_{r1} \geq ...\sigma_n$.

Partition A,B,C as $\begin{bmatrix} A_{r\times r} & * \\ * & * \end{bmatrix}$, $\begin{bmatrix} B_{r:} \\ * \end{bmatrix}$ and $[C_{:r}, *]$. Then the truncated r-th order system $(A_{r\times r}, B_{r:}, C_{:r}, D)$ is an approximation whose H_∞ error bound is $2(\sigma_{r1} + \sigma_{r2}... + \sigma_n)$

For the proof, please refer to [ZhD95].

Hankel norm approximation is another mode reduction approach. It is based on Hankel operator and yield an 'analytical' solution which minimizes the Hankel norm. Please see the reference [ZhD95] for details. Hankel norm approximation results in a lower-order system whose H_∞ error bound is $(\sigma_{r1} + \sigma_{r2}... + \sigma_n)$, which is smaller than that of balance model reduction.

To get a lower-order optimal controller, we can first reduce the order of the plant (2.64) from n to m, then design a m-order optimal controller for the reduced-order plant via the approach in section 2.3. Or we can design an optimal controller with the same order of the original plant (**??**), then approximate it as a lower-order controller. In both case, the lower controller is not really the H_2 or H_∞ optimum; worse, generally it can't guarantee the stability of the closed loop, since we didn't consider the closed loop information when we reduced the order.

Projective Control

Projective control can also be interpreted as a controller order reduction approach. It tries to keep the dominant behavior (eigenstructures) of the full sate or full order optimal (H_2 or H_∞) controller.

Static Output Feedback via Projective Control. Suppose F_f is an optimal control with full-state feedback, then the closed loop $A_c = A + B_2F_f$. For the corresponding eigen-structure matrix (X, Λ), we have

$$(A + B_2F_f)X = X\Lambda \tag{2.68}$$

With the output feedback $y \in R^r$, $u = Fy = FC_2x$, static projective control tries to preserve the r number of eigen-structures (X_r, Λ_r) of the closed-loop of optimal full-state feedback.

$$(A + B_2FC_2)X_r = X_r\Lambda_r = (A + B_2F_f)X_r \tag{2.69}$$

Then we obtain:

$$F = F_fX_r(C_2X_r)^{-1} \tag{2.70}$$

Lower-Order Feedback via Projective Control We discuss the case of strictly proper controller, which was first introduced in [NaV93]. Suppose we want to use a p-th order controller $(A_K, B_K, C_K, 0)$ to preserve the p eigen-structures (X_p, Λ_p) of the plant with a strict-proper optimal H_2 or H_∞ controller $(A_f, B_f, C_f, 0)$.

$$\begin{bmatrix} A & B_2C_f \\ B_fC_2 & A_f \end{bmatrix} \begin{bmatrix} X_p \\ W_{pf} \end{bmatrix} = \begin{bmatrix} X_p \\ W_{pf} \end{bmatrix} \Lambda_p \tag{2.71}$$

$$\begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \begin{bmatrix} X_p \\ W_p \end{bmatrix} = \begin{bmatrix} X_p \\ W_p \end{bmatrix} \Lambda_p \tag{2.72}$$

Thus we can obtain

$$A_K = \Lambda_p - L_pC_2X_p,\ B_K = L_p,\ C_K = C_fW_{pf} \tag{2.73}$$

where L_p is free parameter matrix.

As we have seen, projective control can preserve some dominant eigenvalues and eigenvectors and thus keep some dominant behavior of the optimal controller. However, the the procedure the other poles are not considered, and thus might yield a unstable system.

2.5.3 Direct Design of Lower-Order H_2 Controller

To get better performance, we can design lower-order H_2 controller directly.

Assume

$$\tilde{x} = \begin{bmatrix} x \\ x_K \end{bmatrix}, \tilde{u} = \begin{bmatrix} \dot{x}_K \\ u \end{bmatrix}, \text{ and } \tilde{y} = \begin{bmatrix} x_K \\ y \end{bmatrix}$$

Then from the plant model (2.64) and the controller equation (2.65) we can get an augmented plant description

$$\dot{\tilde{x}} = \begin{bmatrix} A & 0 \\ 0 & 0_{k\times k} \end{bmatrix} \tilde{x} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 & B_2 \\ I_{k\times k} & 0 \end{bmatrix} \tilde{u} \tag{2.74}$$

$$z = [C_1, 0]\tilde{x} + [0, D_{12}]\tilde{u} \tag{2.75}$$

$$\tilde{y} = \begin{bmatrix} 0 & I_{k\times k} \\ C_2 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w \tag{2.76}$$

and a "static" output controller

$$\tilde{u} = \tilde{F}\tilde{y} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \tilde{y} \tag{2.77}$$

So we can see that that the optimal lower-order control can be cast as a static output feedback problem with the augmented plant

$$\left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{array}\right] := \left[\begin{array}{c|cc} \begin{bmatrix} A & 0 \\ 0 & 0_{k\times k} \end{bmatrix} & \begin{bmatrix} B_1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & B_2 \\ I_{k\times k} & 0 \end{bmatrix} \\ \hline [C_1, 0] & 0 & [0, D_{12}] \\ \begin{bmatrix} 0 & I_{k\times k} \\ C_2 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} & 0 \end{array}\right] \tag{2.78}$$

and the "static" output feedback gain $\tilde{F}$ for the augmented plant is composed of the system matrix of reduced-order controller $\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$.

Therefore, the techniques used in static output feedback can be used directly for the design of optimal lower-order control, such as gradient-based methods, LMI-based methods, and homotopy.

Similarly, we also can cast the dynamic decentralized control as a static structured control with output feedback. The system augmentation is the same as (3.33) and (2.78), except that A_K, B_K, C_K and D_K in (3.33) are in block diagonal form with appropriate size.

2.6 Decentralized H_2 Control With Static Output Feedback

In the previous sections about H_2 optimal control, we assume the information from each sensor is available for each actuator. This is sometimes called centralized control. However, this is not the case for many real applications, such as power network system, flexible manufacturing systems, where the controller architecture is structured: the outputs of certain group of sensor are only available for certain actuators.

Static decentralized control is a typical example, and most of methods developed for decentralized control can be extended to arbitrary architectured control. Decentralized control has attracted attention since the 1970s. The early work can be seen in the survey paper by Sandell, Varaiya, Athans and Safonov ([SVA78]), and later work can be found in the book by Siljak (1991) or the survey [Sil96]. Before the 1980s, the main technique for decentralized control is decomposition, which is problem dependent. Then Wenk and Knapp (1980), Geromel and Bernussou (1982) extended Athans's work([LeA70]) about centralized LQR to decentralized. And more efficient convergent algorithms were adopted after that, including Newton's method([ToM87]) and Homotopy method([Mer91]).

To help the understanding, we will introduce some relevant knowledge about matrix calculus. Kronecker product is also mentioned, since it is useful in eigenstructure assignment and to solve the modified Lyapunov equation in multi-objective control.

2.6.1 Matrix Calculus

Matrix Calculus is a set of differentiation formulas which preserve the matrix notation during the operation of differentiation. So it is very useful in optimization and system theory.

The *Kronecker product* of A $(p \times q)$ and B $(m \times n)$ is denoted $A \otimes B$ and is a $pm \times qn$ matrix defined by

$$A \otimes B \triangleq \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B \\ a_{21}B & a_{22}B & \dots & a_{2q}B \\ \vdots & & & \\ a_{p1}B & a_{p2}B & \dots & a_{pq}B \end{bmatrix} \tag{2.79}$$

The *Kronecker sum* of N $(n \times n)$ and M $(m \times m)$ is defined as

$$N \oplus M \triangleq N \otimes I_m + I_n \otimes M \tag{2.80}$$

An important vector-valued function of the matrix $(p \times q)$ defined as

$$\text{vec}(A) \triangleq \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.q} \end{bmatrix} \tag{2.81}$$

There are number of associated operation rules, such as

$$\text{vec}(ADB) = B' \otimes A)\text{vec}(D)$$

$$\text{trace}(ADB) = (\text{vec}(A'))'(I \otimes D)\text{vec}(B)$$

please refer to [Bre78] for details.

The derivative of a matrix A with respect to a scalar b is done term by term:

$$\frac{\partial A}{\partial b} = (\frac{\partial a_{ij}}{\partial b}) \tag{2.82}$$

The derivative $\frac{\partial A}{\partial B}$ of a matrix A with the respect to a matrix B is defined to be a partitioned matrix whose ijth partition is $\frac{\partial A}{\partial b_{ij}}$. There are also associated operation rules. Please refer to [Bre78]. The derivative of a scalar function $f(X)$ with respect to a matrix X is also done term by term:

$$\frac{\partial f}{\partial X} = (\frac{\partial f}{\partial x_{ij}}) \tag{2.83}$$

Based on this definition, we can obtain some useful formulas about the trace function

$$\text{trace}[(\frac{\partial f}{\partial X})'\delta X] = \lim_{\epsilon \to 0} \frac{f(X + \epsilon\delta X) - f(X)}{\epsilon}, \forall \delta X$$

$$\frac{\partial}{\partial X}\text{trace}(NXL) = N'L', \quad \frac{\partial}{\partial X}\text{trace}(NX'L) = LN$$

$$\frac{\partial}{\partial X}\text{trace}(NXLX') = N'XL' + NXL$$

2.6.2 Decentralized H_2 Optimal Control with Static Output Feedback

The static decentralized H2 problem can be stated as: Given a LTI plant model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + \sum_{i=1}^{N} B_{2i} u_i \\ z &= C_1 x + \sum_{i=1}^{N} D_{12i} u_i \\ y_i &= C_{2i} x, i = 1, 2, ..., N \end{aligned} \tag{2.84}$$

where y_i is the ith measurement available for the ith control vector u_i. Design the decentralized static output feedback controller $u_i = F_{di} y_i$, $i = 1, 2, ..., N$, such that the system is stable and the H_2 norm $||H_{zw}||_2^2$ of closed loop $w \to z$ is minimized.

Let $B_2 = [B_{21}, B_{22}, ...B_{2N}]$, $C_2' = [C_{21}', C_{22}', ...C_{2N}']$, $D_{12} = [D_{121}, D_{122}, ...B_{12N}]$, $u = [u_1', u_2', ...u_N']'$, and $y = [y_1', y_2', ...y_N']'$. The controller can be written as

$$u = F_d \; y = \begin{bmatrix} F_{d1} & & & \\ & F_{d2} & & \\ & & \ddots & \\ & & & F_{dN} \end{bmatrix} y$$

With Theorem 2.1, we can formulate the decentralized problem as a constrained optimization problem:

$$\min J(F_d) = \|H_{zw}\|_2^2 = \text{trace}(B_1'KB_1) \tag{2.85}$$
$$\text{s.t. } K(A+B_2F_dC_2)+(A+B_2F_dC_2)'K+(C_1+D_{12}F_dC_2)'(C_1+D_{12}F_dC_2)=0$$
$$F_d \in S_f$$

where S_f is the set of matrices which have the prescribed decentralized structure and stabilizes the closed-loop system. Define the Lagrange function as

$$\begin{aligned} \pounds(F_d, K, L) &= \text{trace}\{(B_1'KB_1) + [K(A+B_2F_dC_2) \\ &\quad +(A+B_2F_dC_2)'K + (C_1+D_{12}F_dC_2)'(C_1+D_{12}F_dC_2)]L\} \end{aligned} \tag{2.86}$$

where L is a symmetric Lagrange multiplier matrix, then, using matrix calculus we reviewed previously, we can obtain

$$\partial\pounds/\partial F_d = 2(D_{12}'D_{12}F_dC_2 + D_{12}'C_1 + B_2'K)LC_2' \tag{2.87}$$
$$\partial\pounds/\partial L = K(A+B_2F_dC_2)+(A+B_2F_dC_2)'K+(C_1+D_{12}F_dC_2)'(C_1+D_{12}F_dC_2 \tag{2.88}$$
$$\partial\pounds/\partial K = L(A+B_2F_dC_2)' + (A+B_2F_dC_2)L + B_1B_1' \tag{2.89}$$

The meaning of $\partial\pounds/\partial F_d$ is $(\partial\pounds/\partial F_{dij})$, so $2(D_{12}'D_{12}F_dC_2 + B_2'K)LC_2'$ in the right side of (2.87) is not exactly the derivative of $\pounds$ with respect to the *design variables* in F_d. So we need to pick out the entries corresponding to the free design variables,

$$\partial\pounds/\partial F_d = 2(D_{12}'D_{12}F_dC_2 + D_{12}'C_1 + B_2'K)LC_2'.F_p \tag{2.90}$$

where F_p is a matrix with an 0 in the positions corresponding to the prescribed entries in F_d and 1 corresponding to the free design entries in F_d, and $M.F_p$ denote multiplication of M and F_p entry by entry.

The necessary conditions of optimization are each of the expressions in Equations (2.90), (2.88) and (2.89) to be zero

$$\partial\pounds/\partial F_d = 0,\ \partial\pounds/\partial L = 0,\ \partial\pounds/\partial K = 0 \tag{2.91}$$

and

$$A + B_2 F_d C_2 \text{ is Hurwitz} \tag{2.92}$$

But it is not easy to solve these nonlinear equations. We will discuss computational methods in Section 2.7.

2.6.3 Decentralized Suboptimal H_2 Control

Kosut's suboptimal ([Kos70]) can also be applied to the static decentralized case. Following a procedure similar to the centralized case, the result obtained is

$$F_{di} = F^{\star} L C_{2i}' (C_{2i} L C_{2i}')^{-1} \tag{2.93}$$

Where $F^{\star}$ is obtained by solving the Riccati equation (2.18) for full-state feedback, and L is obtained by solving the Lyapunov equation (2.63)

As in the case of static output feedback, Kosut's suboptimum (2.93) can't guarantee the stability of the closed-loop system.

Finally, we need to point out that the LMI-like conditions in Theorem 2.6 can't be extended to the decentralized output feedback case, since the given conditions on input and output null spaces lose the sufficiency for the decentralized case, and we can't reconstruct F_d with the matrix pair (X, Y) therein. Optimal decentralized control with output feedback is essentially a bilinear matrix inequality (BMI) problem. However, some particular decentralized H_2 problems can be handled within the framework of LMI, such as decentralized state feedback ([BCG98]), or some part of the decentralized (strict proper) dynamic output controller are fixed ([OGB00]). For the case of state feedback decentralized H_2, parameter space optimization [GBP94] and Hamilton–Jacobi-style equations [SaS94] are also reported.

2.7 Computational Methods for Structured Optimal H2 Control

In the previous discussion, we've seen that the H_2 control for full-state feedback and full-order output feedback can be obtained easily by solving Riccati equations. Model reduction, Kosut's suboptimum, and projective approximation can also be obtained concisely. However, with structural constraint (such as static output feedback, decentralized static-output feedback, lower-order optimal control), H_2 control will result in a constrained optimization problem (such as (2.50), (2.91), or (2.49)), or a nonconvex matrix inequality problem (such as (2.59)–(2.61)). In this section, we will focus on how to solve these problems iteratively.

2.7.1 Direct Constrained optimization

Let's propose the iterative procedure for constrained H_2/LQR from the view of nonlinear programming. We will take the static decentralized control as an example, static output feedback and lower-order optimal control follow the same procedure.

Nonlinear Programming

Constrained optimization problem is usually solved with unconstrained optimization techniques by introducing Lagrange multipliers or penalty functions. Generally for **Unconstrained Optimization**

$$\min f(x), \; x \in R^n, f(x) \in R$$

there are non-gradient-based and gradient-based methods. The **Simplex method** starts with multi-dimensional points, then follows the procedure of reflection and expansion. It only requires to evaluate the function values, not derivatives, but it converges very slowly. **Powell's method** searches successively in a prescribed set of directions, and converges faster than the simplex method. If we use the gradient at the current point of the iteration to update the searching direction, we can expect more efficiency. Steepest descent, conjugate gradient and BFGS are such methods.

Any smooth function $f(x)$ can be approximated by a Taylor series

$$f(x) = f(x_k) + \nabla f(x_k)'(x - x_k) + (x - x_k)'\nabla^2 f(x_k)(x - x_k) + ... \tag{2.94}$$

If we update x according to

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k) \tag{2.95}$$

then $f(x_{k+1}) = f(x_k) - \alpha_k \|\nabla f(x_k)\|^2 + o(\alpha_k)$. So with properly chosen $\alpha_k \geq 0$, searching in the direction of $-\nabla f(x_k)$ will result in a descent sequence. Further, if $f(x)$ is smooth, this sequence will converge to a local minimum eventually. This is the **steepest descent method**.

For the functions whose contours form a narrow valley near the minimum, the steepest descent method might converge very slowly because of the so-called zip-zag phenomenon. The conjugate method has been proposed to accelerate the convergence.

Suppose we have moved along some direction $\mathbf{u}$ and now intend to move along some new direction $\mathbf{v}$. The motion along $\mathbf{v}$ not to spoil our minimization along $\mathbf{u}$ is just that direction staying perpendicular to $\mathbf{u}$, i.e. that the change in gradient be perpendicular to $\mathbf{u}$.

$$0 = \mathbf{u}'\delta(\nabla f) = \mathbf{u}'\nabla^2 f \mathbf{v} \tag{2.96}$$

If so, we say the $\mathbf{u}$ and $\mathbf{v}$ are conjugate with respect to matrix $\nabla^2 f$. For an N dimensional quadric function, minimizing along N linearly independent mutually conjugate directions will result in exactly the minimum. In practice, the search directions of **conjugate gradient method** are generated by

$$d_0 = -\nabla f(x_0) \tag{2.97}$$

$$d_k = -\nabla f(x_k) + \frac{(\nabla f(_k) - \nabla f(x_{k-1}))'\nabla f(x_k)}{\nabla f(x_{k-1})'\nabla f(x_{k-1})} d_{k-1} \tag{2.98}$$

Newton's method tries to use the second-order derivative (Hessian matrix), and uses quadratic function to approximate the objective function. Newton's method converges very fast typically. However, usually it is hard to get the Hessian matrix analytically. Quasi-Newton methods were proposed to construct a sequences of matrices to approximate the inverse of the Hessian matrix. The **BFGS algorithm** (named after Broyden, Fletcher, Goldfarb and Shanno) is an efficient Quasi-Newtown method. It is is not so sensitive to the accuracy as the conjugate method, and usually doesn't need to be periodically restarted as the steepest gradient method. The BFGS algorithm iterates according to:

$$x_{k+1} = x_k + \alpha_k d_k \tag{2.99}$$

$$d_k = -D_k \nabla f(x_k) \tag{2.100}$$

$$D_{k+1} = D_k + \frac{p_k p_k'}{p_k Q_k} - \frac{D_k d_k d_k' D_k}{q_k' D_k q_k} + \tau_k v_k v_k' \tag{2.101}$$

where $p_k = x_{k+1} - x_k$, $q_k = \nabla f(x_{k+1}) - \nabla f(x_k)$, $\tau_k = q_k' D_k q_k$, $v_k = \frac{p_k}{p_k q_k} - \frac{D_k q_k}{\tau_k}$, and D_0 can be chosen as any positive definite matrix, such as I.

After we obtain the searching direction d_k, we need to choose the proper step size α_k to update the point x_k. There are a number of rules for step size selection, please refer to [Ber95].

Minimization Rule:

$$\alpha_k = \arg \min_{\alpha \geq 0} f(x_k + \alpha d_k) \tag{2.102}$$

The golden section search method, parabolic interpolation, or any other one dimensional minimization can be used to solve for α_k.

Armijo Rule: The Armijo rule is a successive stepsize reduction to avoid the considerable computation for one dimensional minimization. For the fixed scalar s, $\beta \in (0,1)$, $\epsilon \in (0,1)$, set $\alpha_k = s\beta^{m_k}$, where m_k is the first nonnegative integer m such that

$$f(x_k) - f(x_k + s\beta^m d_k) \geq -\epsilon s \beta^m \nabla f(x_k)' d_k \tag{2.103}$$

s can be chosen as 1, β is usually $0.1 \sim 0.5$, and ϵ is usually close to zero. The Armijo rule required that the cost improvement e sufficiently large in each step.

Adaptive Rule: Try the step size at some length (such as 1), if the cost improves, we can extend the step size by a fixed factor, until the cost can't be improved any more; if the first step length fails, we decrease the step size by a fixed factor, until a local 'best' cost is achieved.

The above is unconstrained optimization, and more information about it can be found in [Ber95] [PFT88]. As we mentioned, by introducing Lagrange multipliers (under some conditions) or a penalty function, we can change a **constrained optimization** problem into an unconstrained optimization problem. The necessary condition can be seen from Kuhn-Tucker's Theorem, please refer to [Ber95] for details.

Theorem 2.8 (Kuhn-Tucker Necessary Conditions) Let x^* be a local minimum of the constrained optimization problem:

$$\begin{aligned} &\min_{x \in R^n} f(x) \\ &\quad \text{s.t.} \quad h_i(x) = 0, i = 1, 2, ...m \\ &\quad\quad\quad\quad g_j(x) \leq 0, j = 1, 2, ...r \end{aligned} \tag{2.104}$$

and assume $\nabla h_i(x^*)$, $i = 1, 2, ...m$, and $\nabla g_j(x^*)$, $j \in \{j | g_j(x^*) = 0\}$, are linearly independent, then there exist unique **Lagrange multipliers** $\lambda^* = (\lambda_1^*, ...\lambda_m^*)$, $\mu^* = (\mu_1^*, ...\mu_r^*)$, such that

$$\nabla_x L(x, \lambda, \mu) = 0 \tag{2.105}$$

$$\mu_j^* \geq 0, j = 1, 2, ...r \tag{2.106}$$

$$\mu_j^* = 0, \forall j \text{ with } g_j(x^*) < 0 \tag{2.107}$$

where $L(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^r \mu_j g_j(x)$
If so, $f(x*) = \min_{x \in R^n} L(x, \lambda^*, \mu^*)$

The Lagrange multipliers λ^* and μ^* are usually obtained by solving a dual problem, which is non-smooth. **Penalty function** or **multipliers methods** can change a constrained optimization into an unconstrained problem directly. We will see this application in Chapter 4.

Gradient-based Optimization for Structured Control

Let's consider the problem:

$$\min f(y)$$
$$\text{s.t. } h(x,y) = 0$$

where $x \in R^r$, $y \in R^n$, $f(y) : R^n \to R$, and $h(x,y) : R^r \times R^n \to R^m$.

Define the Lagrange function $\pounds(x,y,\lambda) = f(y) + \lambda' h(x,y)$, $\lambda \in R^m$. Then

$$\frac{\partial \pounds(x,y,\lambda)}{\partial x} = \lambda' \frac{\partial h(x,y)}{\partial x}, \text{ and } \frac{\partial \pounds(x,y,\lambda)}{\partial y} = \frac{\partial f(y)}{\partial y} + \lambda' \frac{\partial h(x,y)}{\partial y}$$

If $\frac{\partial \pounds(x,y,\lambda)}{\partial y} = 0$, then $\frac{\partial f(y)}{\partial y} = -\lambda' \frac{\partial h(x,y)}{\partial y}$; and if $h(x,y)$=0, then $\frac{\partial h(x,y)}{\partial x} + \frac{\partial h(x,y)}{\partial y}\frac{\partial y}{\partial x} = 0$. So

$$\frac{\partial f(y(x))}{\partial x} = \frac{\partial f(y)}{\partial y}\frac{\partial y}{\partial x} = -\lambda' \frac{\partial h(x,y)}{\partial y}\{-\frac{\partial h(x,y)}{\partial x} / \frac{\partial h(x,y)}{\partial y}\} = \lambda' \frac{\partial h(x,y)}{\partial x} = \frac{\partial L(x,y,\lambda)}{\partial x}$$

This means, $\frac{\partial f(y(x))}{\partial x} = \frac{\partial \pounds(x,y,\lambda)}{\partial x}$ under the constraint $h(x,y) = 0$ and $\frac{\partial \pounds(x,y,\lambda)}{\partial y} = 0$.

Now come back to structure constrained H_2 problems. Take decentralized case (2.85) and (2.86) as an example. Map the free design variables in F_d as the vector y, and map K as the vector x, L as the vector λ, we can see that $\partial \pounds / \partial F_d$ *is actually the gradient matrix of* $\partial \|H_{zw}\|^2 / \partial F_d$ *if* K *and* L *are evaluated via* $\partial \pounds / \partial L = 0$ *and* $\partial \pounds / \partial K = 0$. And it's also interesting to note, for fixed F_d, $\partial \pounds / \partial L = 0$ and $\partial \pounds / \partial K = 0$ are two decoupled Lyapunov equations, which can be solved easily. So we can used gradient-based methods to solve for the optimal H_2 gain F_d. However, we should keep in mind that we have another constrain: $A + B_2 F_d C_2$ is Hurwitz.

The **gradient-based algorithm** is summarized as

Step 1: Find an initial stabilizing F_d with structure constraints.

Step 2: Solve the decoupled Lyapunov equations for K and L

$$K(A + B_2 F_d C_2) + (A + B_2 F_d C_2)' K + (C_1 + D_{12} F_d C_2)'(C_1 + D_{12} F_d C_2) = 0$$

$$L(A + B_2 F_d C_2)' + (A + B_2 F_d C_2) L + B_1 B_1' = 0$$

Evaluate the gradient

$$\partial J(F_d)/\partial F_d = 2(D_{12}'D_{12}F_dC_2 + D_{12}'C_1 + B_2'K)LC_2'.F_p$$

If $\|\partial J(F_d)/\partial F_d\|$ is small enough, stop, otherwise go to step 3.

Step 3: Based on the gradient $\partial J(F_d)/\partial F_d$, calculate the search direction D (steepest descent, conjugate gradient, or FBGS). Choose the step size α (minimization rule, Armijo rule, or adaptive rule, etc.) with the additional requirement that $A + B_2(F_d + \alpha D)C_2$ is Hurwitz. Update F_d with $F_d + \alpha D$. Go to Step 2.

Remark 1: Gradient-based algorithms require a stabilizing initial F_d. We can use Cao's iterative LMI method to get such an initial F_d, see [CSM98] or next chapter. Kosut's suboptimum or projective control might also be a good initial guess for F_d, if they yield a stabilizing F_d.

Remark 2: Note that in step 3 when we choose the stepsize α, we also require that $A+B_2(F_d+\alpha D)C_2$ is Hurwitz in addition to cost descent. Such an α exists under mild conditions ([ToM85], [MoC85], [MaT87]), and if so the gradient-based algorithm will converge to a stationary point. Generally, if the sensors or actuators are redundant, step 3 might be infeasible. That means we can't find the next stabilizing point in in the steepest gradient direction, conjugate direction, or FBGS direction. If so, we can project the search direction into the stabilizing set, and continue the iteration in the *feasible projection direction*. Or more simply and practically, we can choose another initial point to restart the iteration. Numerical experience indicates that it only happens sporadically, and the algorithm works well practically.

Remark 3: Gradient-based method can only converge to a local minimum. So as to have more chance to find the global minimum, it's necessary to choose more stabilizing initial points, restart the iteration, and compare the local minima.

Toivonen and Makila ([ToM85]) adopted the Anderson–Moore algorithm for optimal static output decentralized control with $D_{12}'D_{12} > 0$, $C_1'D_{12} = 0$, and C_{2i} full row rank, and they claimed its convergence under certain conditions. There the search direction is obtain with second order Taylor series approximation of the the cost function, $D = -(D'_{12}D_{12})^{-1}(D'_{12}C_1 + B'_2K)LC'_2(C_2LC'_2)^{-1}.F_p - F_d$, and the step size is determined using Armijo's rule. Levine–Athans-type method [LaA70] (no converge guaranteed) in fact searches in the same direction with a fixed step size 1.

2.7.2 LMI-based Approaches

As we have seen in Sections 2.4, static output stabilization and γ-suboptimal H_2 control with static output feedback (or with lower-order dynamic feedback) can be described with LMI constraints and a nonlinear equality or rank constraint. (See the expression in Theorem 2.4 and Theorem 2.6).

The non-convex constraint $X = Y^{-1}$ (Equation (2.60)) makes this problem impossible to solve directly in the framework of LMI. However, there are several methods proposed to handle it with an iterative LMI approach. The most numerically efficient two are proposed recently by Geromel *et al.* [GSS98] and El Ghaoui *et al.* [EOA97]. We already discussed Geromel's min/max algorithm for the static output stabilization problem and pointed out that it can't used in H_2 γ-suboptimal control. El Ghaoui's cone complementary linearization algorithm is essentially a method for bilinear objective optimization subject to LMI constraints. So it can be extended to H_2 and H_∞ problems with (centralized) static output feedback or lower-order control. We will discuss this efficient algorithm in the next chapter.

We also note that **homotopy/continuation method** has also been successfully applied to solve the structure constraint problem by Mercadal [1991], and it is observed that its performance is better than that of the conjugate gradient, and a little less efficient then FBGS method [CoS98].

2.8 Applications in the Design of Passive Mechanical Systems

As we have introduced in the Chapter of introduction, many passive mechanical systems can be cast as structured (decentralized) control. Tuned-mass dampers, passive vibration isolators, and vehicle suspensions are such practical examples. Since it is hard to implement a negative stiffness or dashpot, in the design for these passive mechanical systems, we have an additional constraint: we require the parameters to be nonnegative, or more practically fall into some reasonable range. Further, for some design like vehicle suspensions, we also require that the parameters in left and right sides are symmetric. We note that Blondel and Tsitsiklis [BlT97] have proved that static centralized output-feedback stabilization problem with gain intervals constraints, and static decentralized output-feedback stabilization with identical blocks are NP hard. In the following examples, we would like to emphasize how to modify the previous algorithms, so as to handle these additional constraints.

2.8.1 Application 1: Multi-Degree-of-Freedom Tuned Mass Damper Design

Tuned-mass dampers (TMD)— often called dynamic vibration absorbers (DVA) — are efficient passive vibration suppression devices comprising of a mass, springs, and viscous or hysteretic dampers. Since proposed in 1909, they have been widely used in machinery, buildings, and structures. A great deal of research has been carried out since Den Hartog presented his "equal peaks" method method for design of SDOF tuned-mass dampers in 1928. Many methods have been developed for the design of a single-degree-of-freedom (SDOF) absorber to damp SDOF vibration. Yet there are very few studies for the case where both the damper and the main system have multiple degrees of freedom.

If TMD has more then one degree of freedom, we can expect to make full use of the mass inertia to damp more than one mode of the main system. If the movements are decoupled in space, we can design the parameters individually, just by taking it as individual SDOF TMDs. However, this occurs rarely in practice. Structured H_2 is one good solution for this problem, as well as the the multiple SDOF TMDs design.

Consider an aluminium block supported with six flexures, as shown in Figure 2-5. The cube dimensions is 150 mm $\times$ 170 mm $\times$ 200 mm. To keep the cube free of deformation, kinematic supports are used, resulting in a very lightly damped system. And worrying about material creep, we can't use viscoelastic material. Moreover, the space available for damping treatment is very limited. So we intend to design a MDOF TMD.

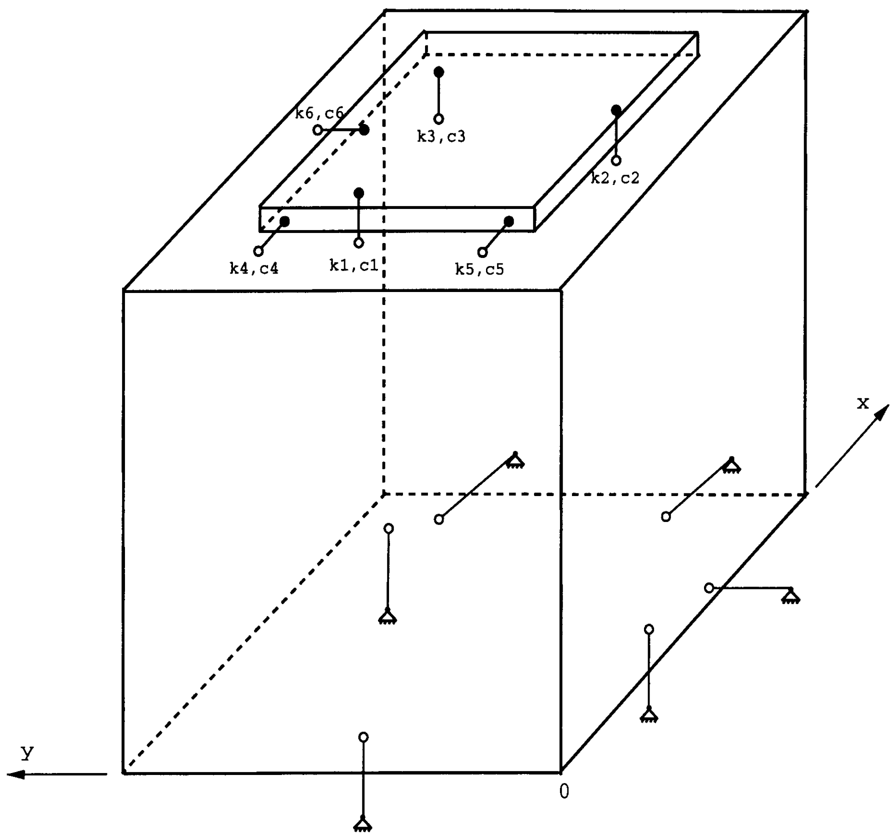

Figure 2-5: Multi-Degree-of-Freedom Tuned Mass Damper

Figure 2-6 shows the six modes consisting of motions of the rigid block (without TMD) relative to the ground. We can see that the movements are coupled in space.

Suppose we attach another 110 mm × 125 mm × 6.35 mm steel tuning mass on the top of cube (4.8% of the aluminium block mass). Take the ground excitation $(x_g, y_g, z_g)'$ as the noise input w, and the velocities $(\dot{x}_c, \dot{y}_c, \dot{z}_c, \dot{\theta}_x, \dot{\theta}_y, \dot{\theta}_z)'$ at the cube mass center as the cost output z. Now our task is to design the six spring/dashpot parameters connecting the TMD and the cube, so as the minimize the variance of the motion of cube mass center subject to the white noise input w.

Replace the flexures between the TMD and cube with control forces generated by

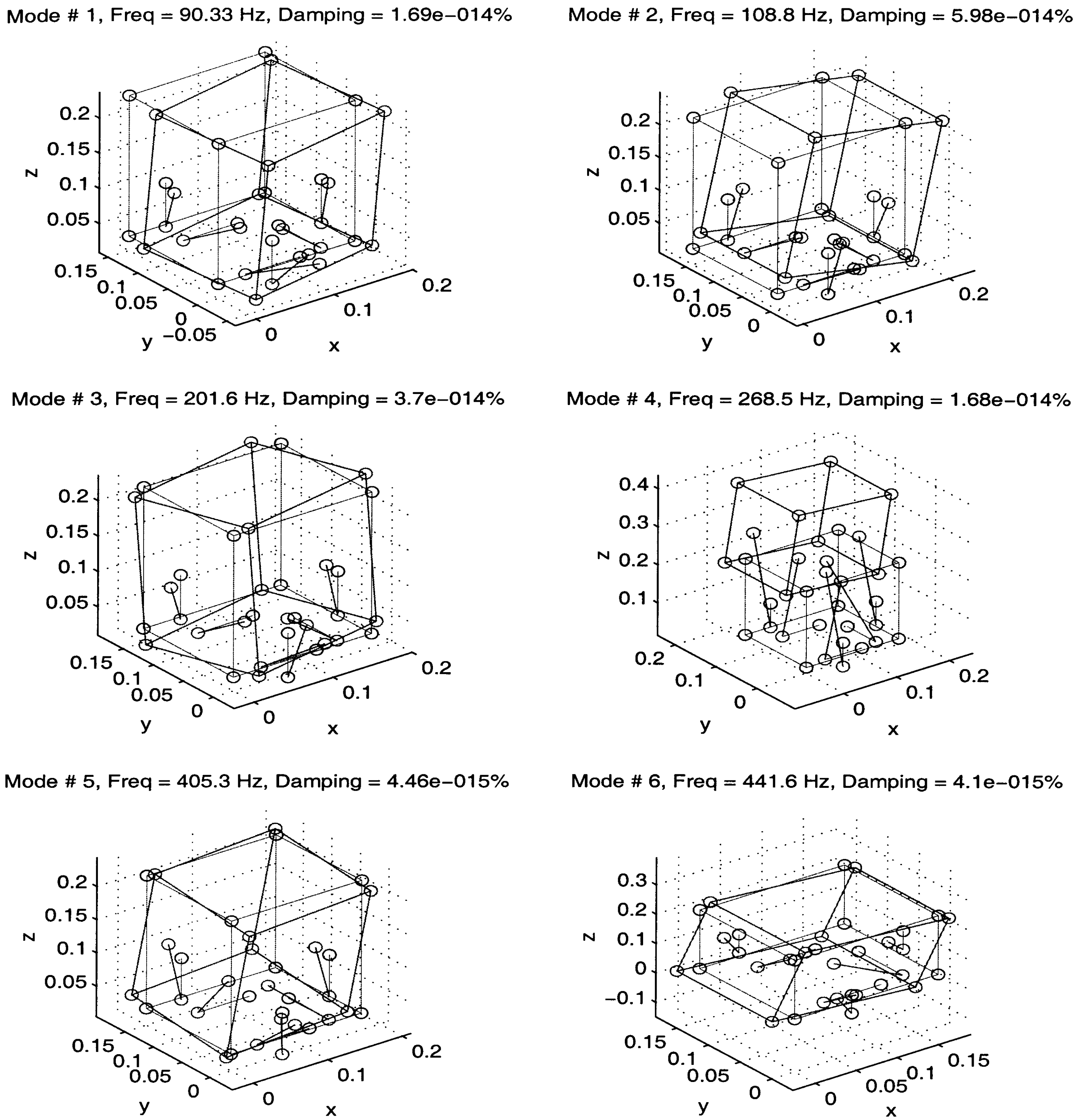

Figure 2-6: Modal shapes without damper

k and c. With the structural matrix analysis, we can get the the the plant model and the static decentralized controller to be designed as

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & 0 \end{array}\right]$$

$$F_d = \begin{bmatrix} k_1 & c_1 & & & & & & \\ & & k_2 & c_2 & & & & \\ & & & & \cdots & \cdots & & \\ & & & & & & k_6 & c_6 \end{bmatrix}$$

where $k_i \geq 0$, $c_i \geq 0$, $i = 1, 2, ...6$.

As in [BlT97], it not easy to handle the problem that the elements of gain F_d in constrained in some interval. However, if we replace F_d with $F_d.F_d$, the **constraint of nonnegative parameters** is satisfied automatically. With this replacement, we need to modify the gradient-based algorithm.

Now the Lagrange function (2.86) becomes $\pounds(F_d.F_d, K, L)$. With

$$\partial\pounds(F_d.F_d, K, L)/\partial L = 0$$
$$\partial\pounds(F_d.F_d, K, L)/\partial K = 0$$

$\partial J(F_d)/\partial F_d$ would be equal to $\partial\pounds(F_d.F_d, K, L)/\partial F_d$. Thus we have

$$K(A + B_2(F_d.F_d)C_2) + (A + B_2(F_d.F_d)C_2)'K + (C_1 + D_{12}(F_d.F_d)C_2)'(C_1 + D_{12}(F_d.F_d)C_2) = 0 \quad (2.108)$$

$$L(A + B_2(F_d.F_d)C_2)' + (A + B_2(F_d.F_d)C_2)L + B_1B_1' = 0 \quad (2.109)$$

$$\partial J(F_d)/\partial F_d = 4[(D_{12}'D_{12}(F_d.F_d)C_2 + D_{12}'C_1 + B_2'K)LC_2'].F_d \quad (2.110)$$

Note: with array multiplication of F_d, the derivatives $J(F_d)$ to free design variables are also picked out automatically. The other part is the same as what we already discussed in sections 2.6 and 2.7. More general, if we would like to **constrain some**

parameter F_{dij} in some internal $[r_1, r_2]$ for some physical reason, we can specify F_{dij} with one parameter r,

$$0.5(r_1 + r_2) + 0.5(r_2 - r_1)sinr \tag{2.111}$$

and make the corresponding modification the Equation (2.110).

With initial parameters $k_i = 5 \times 10^5$ N/m, and $c_i = 20$ N·s/m, $i = 1, 2, ...6$, the FBGS method converged very quickly. We also tried around 20 groups of initial parameters F_d randomly, and compared the the results. One local minimum was found as:

i	k_i, N/m	c_i, N·s/m
1	1.9763×10^5	104.090
2	1.3878×10^6	884.522
3	8.9773×10^4	33.6856
4	3.6651×10^5	39.5086
5	1.3392×10^6	62.8814
6	4.1487×10^6	2.86×10^{-12}

Since there are always uncertainties between the model and plant, adjustable springs and dashpots are necessary for the implementation. We created one type such flexures, where the stiffness and damping in axile direction is adjustable around the designed values, and the stiffness in other directions are ignorable, as shown in Figure 2-7. The damping is generated by squeezing film, and the stiffness is dominated by the Belleville springs. For the sake of brevity we omit the details about the determination of geometrical dimensions and the test of prototypes.

With the above parameter design, we built an experiment to implement the 6DOF TMD, using the adjustable spring/daspot elements we created. Figure 2-8 is the experiment setup.

After several turns of adjustment around the designed parameter, the whole system was well tuned. one typical transfer function was shown in figure 2-9. All six modes are damped well.

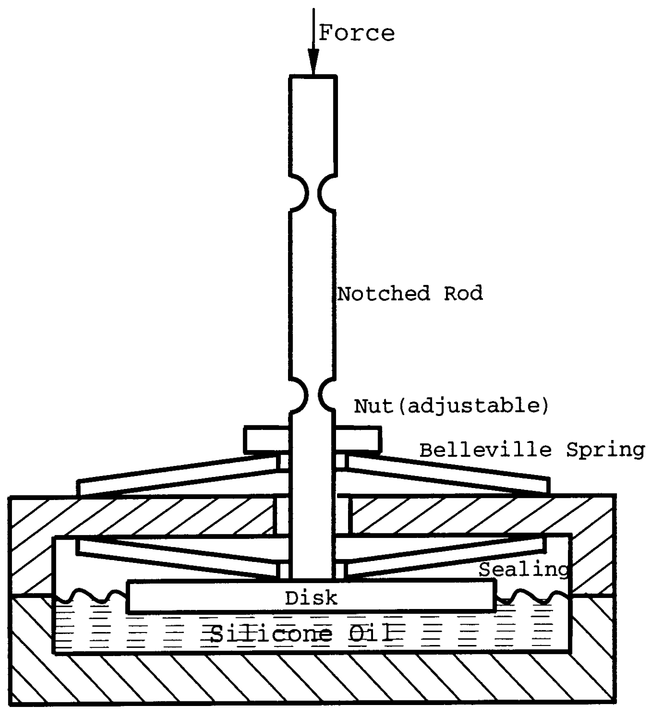

Figure 2-7: Flexure with adjustable one-directional stiffness and damping

2.8.2 Application 2: Passive Vehicle Suspension Design

As we mentioned in Chapter 1, optimal design of passive vehicle suspensions is another application of decentralized control techniques. Figure 2-10 shown an 8-DOF full car model including the passenger dynamics.

The stiffness and damping of suspensions play a critical role in four important performance measures of the vehicle: ride comfort, body motion, road handling, and suspension travel. The ride comfort is measured by the accelerations of the passenger with the human-vibration sensitivity filter according to the ISO2631 standard. The body motion includes height, pitch, and roll velocities. Road handling requires that the dynamic contact force between the ground and tires large enough. Suspension

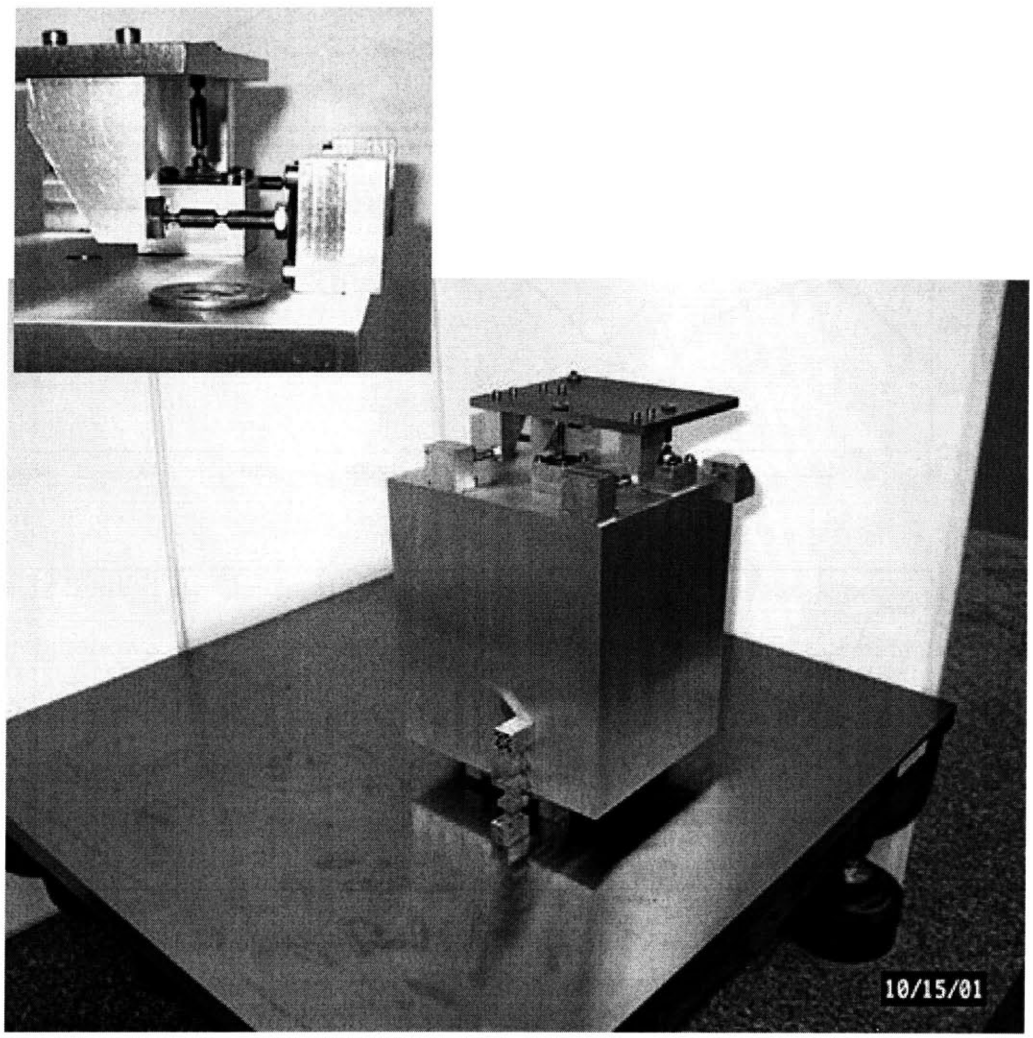

Figure 2-8: 6DOF TMD experiment setup

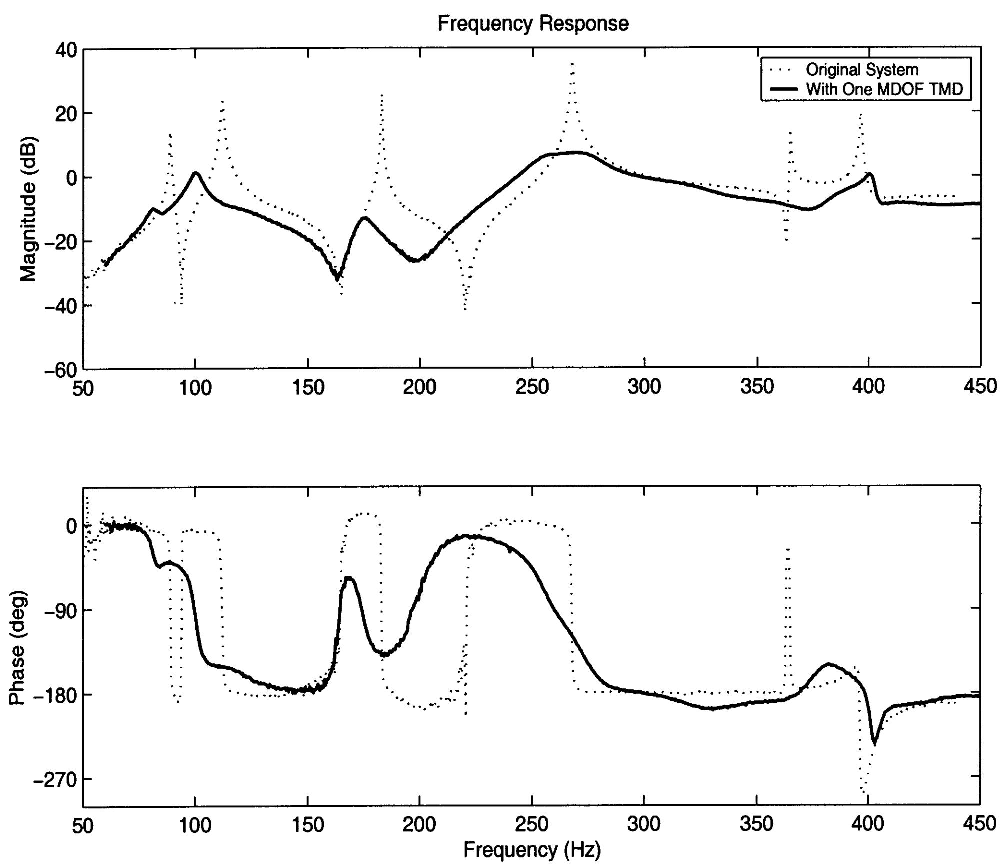

Figure 2-9: One typical transfer function with/without one 6DOF TMD

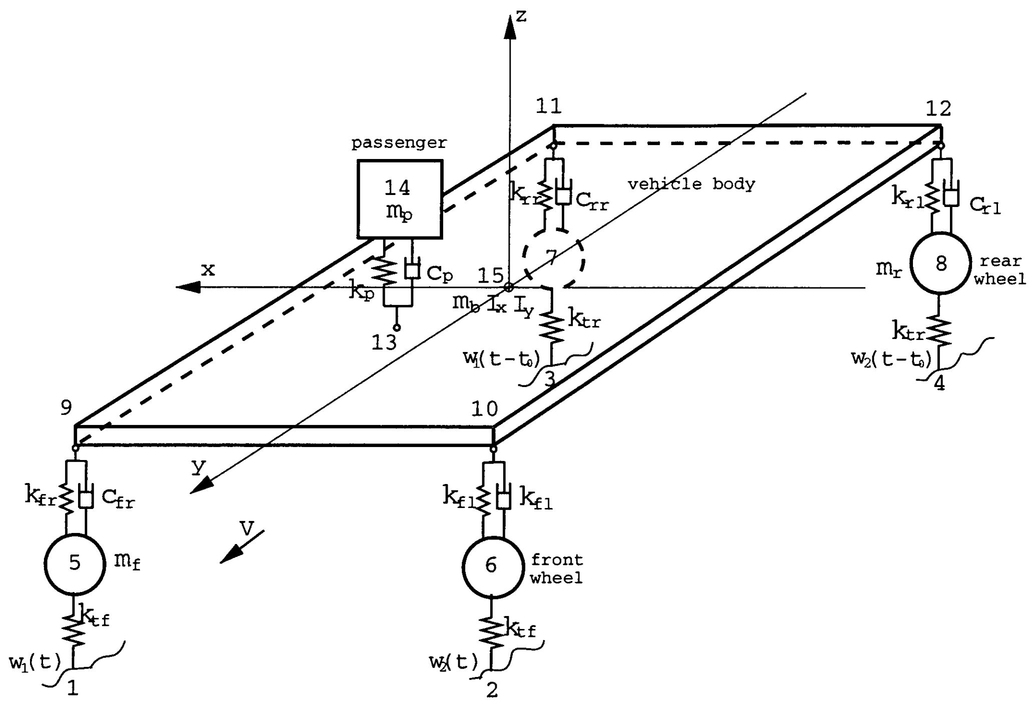

Figure 2-10: Passive vehicle suspension

travel is limited by the bump and rebound stops. These four aspects conflict each other somehow. As it is well-known to the automotive engineers, the typical velocity excitation from the ground can be modelled as white noise whose intensity depends on the roughness of road surface and the vehicle speed. Thus LQR or H_2 active control can make a meaningful trade off for the four requirements [TaE98] [Hro97]. However, since active or semi-active suspensions increase the cost and complexity, the simple and reliable passive suspensions still dominate in the automobile industry. Here we apply the decentralized H_2 optimization to the parameter design of the passive suspension for a full car.

The system modelling is shown in Figure 2-11, in which the four requirements are specified as the cost output. The road inputs at the left track and the right track are assumed to be independent, and the excitation on the real wheels is taken as a delay of the excitation on the front wheels.

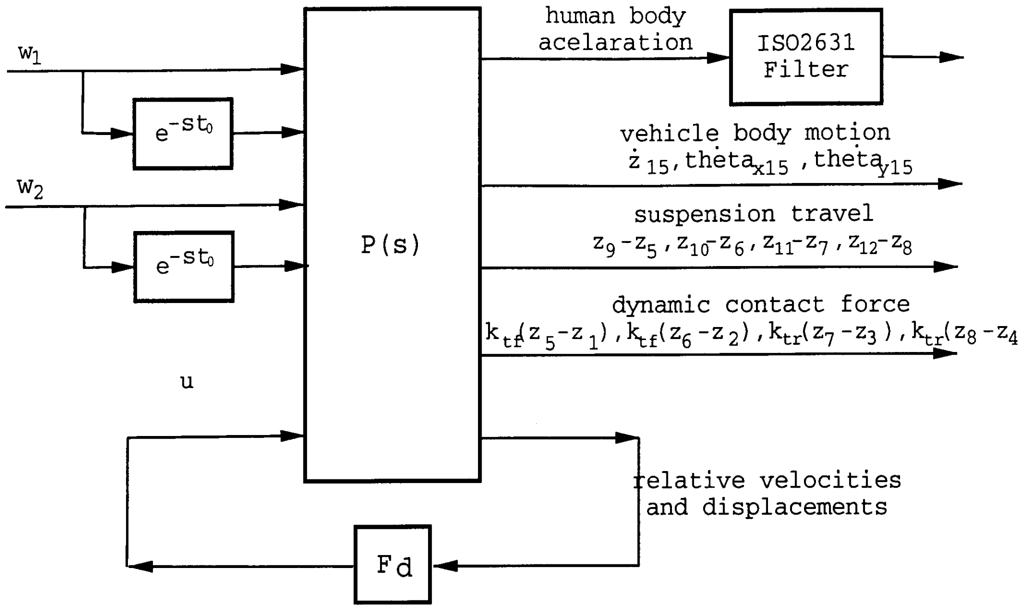

Figure 2-11: Decentralized control modelling of passive vehicle suspension

Using Pade functions to approximate the delay, we can get a rational generalized plant. And the feedback gain is decentralized. We can use the gradient-based methods to solve for the optimal H_2 parameters F_d.

$$F_d = \begin{bmatrix} k_{fl} & c_{fl} & & & & & & \\ & & k_{fr} & c_{fr} & & & & \\ & & & & k_{rl} & c_{rl} & & \\ & & & & & & k_{rr} & c_{rr} \end{bmatrix} \tag{2.112}$$

A detailed design based on a real car model is currently under investigation and will be reported later. Here we would like to highlight something. As we know, for a real car the mass is not symmetric about the x axis, but in the suspension design, we would like to make the parameters symmetric:

$$k_{fl} = k_{fr} := k_f$$

$$k_{rl} = k_{rr} := k_r$$

$$c_{fl} = c_{fr} := c_f$$

$$c_{rl} = c_{rr} := c_r$$

Usually this kind of constraint will yield a very hard problem. Blondel and Tsitsiklis [BIT97] proved that the decentralized stabilization problem with identical blocks is NP-hard. However, our problem is still tractable. In the decentralized control we proposed before, (2.90) is used to generate the gradient of cost the J with respect to the free parameters. With the basic gradient chain rule, we can get:

$$\partial J/\partial k_f = \partial J/\partial k_{fl} + \partial J/\partial k_{fr}$$

Thus we can get the the gradient of J to the true free design parameters subject to the symmetry constraints. The gradient-based procedure is still valid with this modification.

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Chapter 3

Optimal H_∞ Control

As we have seen from the previous chapter, the H_2 norm is meaningful performance measure, and H_2 optimization is an efficient method to design an unstructured or structured controller, which will minimize the output peak magnitude with unit energy input, or in other words, minimize the output variance with white noise input. But sometimes, we care more about the peak magnitudes of a system's steady-state response to the worst case sinusoid disturbance. Moreover, the H_2 norm is not a system induced norm, and it is hard to handle system robust stability and robust performance under uncertainties. That is why H_∞ has attracted more attention in the control community since the 1980s, especially after LMI (linear matrix inequality) techniques developed in the 1990s.

In this chapter, the concept of the H_∞ norm and robustness are briefly introduced, then we discuss Riccati-based and LMI-based H_∞ synthesis for full state feedback and full order control. H_∞ control with lower order output feedback or static output feedback is solved with iterative LMI techniques. Decentralized H_∞ control is also examined. Numerical examples are given to show the application in parameter design of passive mechanical systems.

3.1 H_∞ Norm, Uncertainty, and H_∞ Problems

3.1.1 System H_∞ Norm

A causal system has finite $L_2 \to L_2$ gain if there exists some finite constant γ such that

$$\begin{aligned} \|z\|_2^2 &= \int_0^\infty z(t)'z(t)dt \\ &\leq \text{const} + \gamma^2 \|w\|_2^2 = \text{const} + \gamma^2 \int_0^\infty w(t)'w(t)dt \end{aligned} \tag{3.1}$$

for any input w(t) and any output z(t), where the "const" can be dependent on the initial status [Meg01] [ZGD95].

A finite order LTI system has finite $L_2 \to L_2$ gain if all of its poles are in the open left half plane. The $L_2 \to L_2$ gain of a stable causal LTI system turns out to be the H_∞ norm of the transfer matrix [DDV99]:

$$\|H_{zw}\|_{2i} = sup_{w\neq 0}\frac{\|z\|_2}{\|w\|_2} = \sup_w \sigma_{max}[H_{zw}(\omega)] \tag{3.2}$$

Proof:

$$\begin{aligned} \|z\|_2^2 &= \int_0^\infty z(t)'z(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty Z(\omega)'Z(\omega)dw \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty w(\omega)'H_{zw}(\omega)'H_{zw}(\omega)w(\omega)dw \\ &\leq \frac{1}{2\pi}\int_{-\infty}^\infty \sigma_{max}^2[H_{zw}(\omega)]w(\omega)'w(\omega)dw \\ &\leq \sup_w \sigma_{max}^2[H_{zw}(\omega)]\|z\|_2^2 \end{aligned}$$

and the upper bound can be achieved by choosing the input as the singular vector associated with the largest singular value of $H_{zw}(\omega_0)$

For a SISO system, the H_∞ norm is the peak magnitude of Bode plot: the maximum steady-state response to the worst-case sinusoid inputs. So it's also a meaningful performance measure in many applications. Furthermore, unlike H_2 norm, the H_∞ norm is an induced norm (with the important property $||H_1(\omega)H_2(\omega)||_\infty \leq ||H_1(\omega)||_\infty ||H_2(\omega)||_\infty$), and it can be used in robust control design.

The theorem below gives one approach (γ iteration) to computate the $||H||_\infty$ norm for continuous-time systems.

Theorem 3.1 Let A be Hurwitz, then the L_2 gain of the LTI system (A,B,C,D) is less than γ if and only if $\gamma > \sigma(D)$, and the matrix

$$\begin{bmatrix} A + B(\gamma^2 I - D'D)^{-1}D'C & B(\gamma^2 I - D'D)^{-1}B' \\ -C'C - C'D(\gamma^2 I - D'D)^{-1}D'C & -A' - C'D(\gamma^2 I - D'D)^{-1}B \end{bmatrix} \quad (3.3)$$

has no eigenvalues on the imaginary axis.

Proof: First observe that $||H||_\infty < \gamma \iff I - \frac{1}{\gamma^2}H'(j\omega)H(j\omega)$ is invertible for all $\omega \in R \iff [I - \frac{1}{\gamma^2}H'(s)H(s)]^{-1}$ has no poles on imaginary axis. Construct a realization of $[I - \frac{1}{\gamma^2}H^T(-s)H(s)]^{-1}$ with feedback $\frac{1}{\gamma}H^T(-s)$ and $\frac{1}{\gamma}H(s)$, and write down the state equation of this realization ($2n$-order) [DDA99].

From Theorem 3.1 we also can get an equivalent description in matrix inequality via KYP Lemma. We will introduce it later in Theorem 3.6.

3.1.2 Uncertainty and Robustness

One of the most important things in control is robustness. In classical control we use gain margin and phase margin to ensure some performance and robustness. Modern control is some kind of optimization. How well the "optimal" controller can work in reality greatly depends on how close the model is to the physical plant. Unfortunately, we can never model exactly the physical plant. And because of time variance, nonlinearity, lumping of parameters, and ignorance of high order modes, there are always some differences or errors between our model and reality. Because of this it's more practice to represent the physical system as a set of plant with *model uncertainty.* The task of robust control is to make the controller designed for the nominal plant be able to stabilize the whole set of plants, or be able to achieve the required performance for the whole set of plants. This is known as *robust stability* and *robust* *performance.*

Usually the uncertainty is represented as

1) Additive perturbation

$$\Omega = \{P(s)|P(s) = P_0(s) + W(s)\Delta(s)), \|\Delta\|_\infty \leq 1\}$$

2) Multiplicative perturbation

$$\Omega = \{P(s)|P(s) = P_0(s)(I + W(s)\Delta(s)), \|\Delta\|_\infty \leq 1\}$$

3) Feedback perturbation

$$\Omega = \{P(s)|P(s) = P_0(s)(I + W(s)\Delta(s))^{-1}, \|\Delta\|_\infty \leq 1\}$$

Where $\Delta(s)$ is an arbitrary stable transfer matrix with L_2 gain no greater than 1, $W(s)$ is a stable proper rational transfer matrix to represent the frequency characteristics of uncertainty. The three uncertainty representations are shown in Figure 3-1.

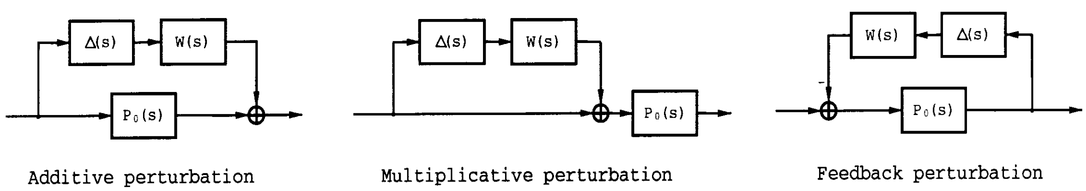

Figure 3-1: Representations of uncertainty

We can describe the connection with uncertainty as linear fraction model, as seen in Figure 3-2.

$$\begin{bmatrix} \Psi_1(s) \\ \Psi_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22} \end{bmatrix} \begin{bmatrix} \Phi_1(s) \\ \Phi_2(s) \end{bmatrix}$$

where $G_{11}(s)$, $G_{12}(s)$, $G_{21}(s)$ and $G_{22}(s)$ are stable. Then the closed loop of $G_{\psi_2\phi_2}$ is obtained as

$$G_{\psi_2\phi_2}(s) = G_{22}(s) + G_{21}(s)\Delta(s)(I - G_{11}(s)\Delta(s))^{-1}G_{12}(s)$$

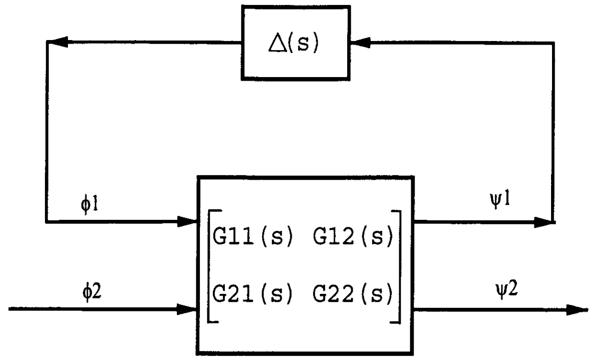

Figure 3-2: Standard model for uncertainty

So the stability of the closed loop will depend on $(I - G_{11}(s)\Delta(s))^{-1}$. If $\|G_{11}\Delta\|_\infty < 1$, then $(I - G_{11}(s)\Delta(s))^{-1}$ will have no poles, and thus we can achieve robust stability for the closed loop under uncertainty $\Delta(s)$. This is the small gain theorem.

Theorem 3.2. (Small Gain Theorem) Define the set of stable uncertainty matrices as $\{\Delta(s) | \|\Delta\|_\infty \leq 1\}$, if $M(s)$ is stable, then $(I - M(s)\Delta(s))^{-1}$ and $\Delta(s)(I - M(s)\Delta(s))^{-1}$ are stable if and only if $\|M\|_\infty < 1$.

If the uncertainty Δ has some structure, for example it is diagonal, it will result a *structured singular value* problem, which is known as *μ-synthesis*, (handled by D-K iteration) contributed by Doyle, Safonov, *et al.*.

3.1.3 H_∞ Problems

Consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{3.4}$$

Optimal H_∞ Problem: Design a stabilizing controller Ctrl(s) such that the closed loop $\|H_{zw}\|_\infty$ is minimized.

Suboptimal H_∞ Problem: Given $\gamma > 0$, find a stabilizing controller Ctrl(s) such that the closed loop $\|H_{zw}\|_\infty < \gamma$.

There might also be some structural constraints on $Ctrl(s)$, such as static output feedback, fixed order, or decentralized structure. This is called structured H_∞ control.

For a MIMO system, the optimal H_∞ controller is generally not unique. From the small gain theorem, we can see it is that often good enough to get an suboptimal controller: design a controller such that the close loop L_2 gain is less than one over the H_∞ norm of the uncertainty. In this chapter, we will discuss both optimal and suboptimal H_∞ problems

3.2 Riccati-based Full-State and Full-Order H_∞ Control

After the paper [DGK89] by Doyle *et al.*, Riccati-based approach came to be widely accepted as an efficient H_∞ synthesis method for full state and full order feedback. In some of the literature, it is called DGKF's solution.

3.2.1 Full-State Feedback

Given a plant model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \end{aligned} \tag{3.5}$$

design a full state stabilizing gain F, such that the closed loop $\|H_{zw}\|_\infty < \gamma$.

To simplify the theorem statement and proof, the following assumptions are made. For the general case, please refer to [ZDG95].

i) (C_1, A) is detectable;

ii) (A, B_2) is stabilizable;

iii) $D_{12}'[C_1, D_{12}] = [0, I]$.

With those assumptions, by introducing a symmetric matrix X we have

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 = \int_0^\infty [(C_1x + D_{12}u)'(C_1x + D_{12}u) - \gamma^2 w'w]dt$$
$$= \int_0^\infty [\frac{d}{dt}(x'Xx) + \|C_1x + D_{12}u)\|^2 - \gamma^2 w'w]dt - \int_0^\infty \frac{d}{dt}(x'Xx)dt$$
$$= \int_0^\infty [2x'X(Ax + B_1w + B_2u) + \|C_1x + D_{12}u\|^2 - \gamma^2 w'w]dt$$
$$- x'(\infty)Xx(\infty) + x'(0)Xx(0)$$
$$= \int_0^\infty x'(A'X + XA + C_1'C_1 + X(\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X)xdt$$
$$+ \int_0^\infty \|u + B_2'Xx\|^2 dt - \int_0^\infty \gamma^2\|w - \frac{1}{\gamma^2}B_1'Xx\|^2 dt - x'(\infty)Xx(\infty) + x'(0)Xx(0)$$

Thus, under zero initial state $x(0) = 0$ and with a stabilizing ($x(\infty) = 0$) controller $u = -B_2'Xx$, where X satisfies the Riccati equation $XA + A'X + X(\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X + C_1C_1' = 0$, we can arrive $\|z\|_2^2 - \gamma^2\|w\|_2^2 = -\int_0^\infty \gamma^2\|w - \frac{1}{\gamma^2}B_1'Xx\|^2 dt < 0$. Therefore, we have Theorem 3.3. In the theorem, $X \geq 0$ is necessary and sufficient to guarantee the stability [ZDG95].

Theorem 3.3 (Suboptimal H_∞ with full state feedback) If the assumptions $(i) \sim (iii)$ hold, then $\|H_{zw}\|_\infty < \gamma$ with full state feedback if and only if there exists some symmetric matrix $X \geq 0$ such that

$$XA + A'X + X(\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X + C_1C_1' = 0 \tag{3.6}$$

one such controller is $u = -B_2'Xx$, and all the stabilizing controllers satisfying $\|H_{zw}\|_\infty < \gamma$ can be parameterized as

$$u = -B_2'X - \frac{1}{\gamma^2}Q(s)B_1'XQ(s) \tag{3.7}$$

where Q is stable and $\|Q\|_\infty < \gamma$

Theorem 3.3 gives an approach to solve suboptimal H_∞ problem. To find the optimal H_∞ control, we can use γ iteration: solve a series of suboptimal problems until we know that γ can not decrease any more.

3.2.2 H_∞ Filter

Like the Kalman filter with H_2 measure, the H_∞ filter is an estimator with the H_∞ measure. Suppose a dynamic system is described as

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \ x(0) = 0 \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{3.8}$$

find a causal estimate $\hat{z}$ of z using u and the measurement y, such that the L_2 gain of w to the error $z - \hat{z}$ is less than γ. See the Figure 3-3.

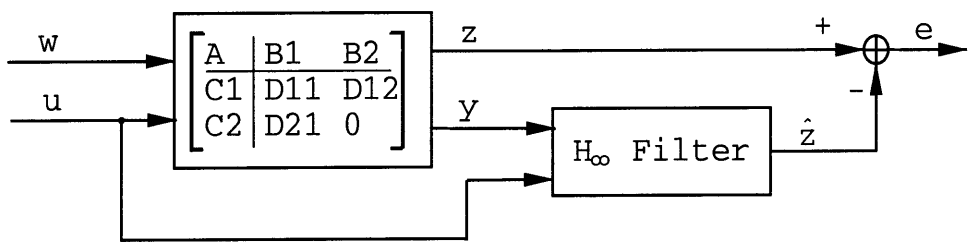

Figure 3-3: H_∞ filter problem

For the sake of brevity, we make some assumptions as below and set $D_{11} = 0$. For relaxed case please refer to [ZDG95] .

i) (C_2, A) is detectable;

ii) (A, B_1) is stabilizable;

iii) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Suppose the filter is in the form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_2 u + L(y - C_2\hat{x}) \\ \hat{z} &= C_1\hat{x} + D_{12} u \end{aligned} \tag{3.9}$$

Then the error $e = z - \hat{z}$ satisfies

$$\dot{e} = (A - LC_2)e + (B_1 - LD_{21}w \tag{3.10}$$

Following a procedure similar to that in full-state feedback, we can get the Riccati equation for the H_∞ filter, as seen in Theorem 3.4.

Theorem 3.4 (H_∞ Filter) If the assumptions $(i) \sim (iii)$ hold, then there exists a causal filter such that the L_2 gain of w to error $z - \hat{z}$ is less than γ if and only if there is some symmetric matrix $Y \geq 0$ satisfying the Riccati equation

$$AY + YA' + Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2)Y + B_1B_1' = 0 \tag{3.11}$$

and one such filter is given by equation (3.9) where $L = -YC_2'$.

Remark: Unlike H_2 Kalman filter, where the residual gain matrix doesn't depend on which component x of the state is being estimated, the residual gain matrix L in the H_∞ filter depends on C_1.

3.2.3 Full-Order H_∞ Control (Suboptimal)

As we discussed in Chapter 2, full order H_2 control is composed of a full-state feedback gain and a H_2 state estimator, and the two associated Riccati equations are *decoupled.* For full order H_∞ control, we also can construct same structure: full-state gain and a H_∞ filter. However, we arrive two *coupled* Riccati equations, as stated in Theorem 3.5. Please refer to [DGK89] for proof.

Theorem 3.5 (full-order H_∞ suboptimal control.) For LTI system,

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u \\ z &= C_1x + D_{11}w + D_{12}u \\ y &= C_2x + D_{21}w \end{aligned} \tag{3.12}$$

Assume:

i) (A, B_1) is stabilizable and (C_1, A) is detectable;

ii) (A, B_2) is stabilizable and (C_1, A) is detectable;

iii) $D'_{12}[C_1, D_{12}] = [0, I]$;

iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D'_{21} = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Then there exists a stabilizing output-feedback controller such that the L_2 gain bound of $w \to z$ $\|H_{zw}\|_\infty < \gamma$ if and only if there exist symmetric matrices $X \geq 0$ and $Y \geq 0$, such that

$$XA + A'X + X(\frac{1}{\gamma^2}B_1B'_1 - B_2B'_2)X + C_1C'_1 = 0 \tag{3.13}$$

$$AY + YA' + Y(\frac{1}{\gamma^2}C'_1C_1 - C'_2C_2)Y + B_1B'_1 = 0 \tag{3.14}$$

$$\rho(XY) < \gamma^2 \tag{3.15}$$

And one such controller is

$$\left[\begin{array}{c|c} A + \frac{1}{\gamma^2}B_1B'_1X + B_2F + (I - \frac{1}{\gamma^2}YX)^{-1}LC_2 & -(I - \frac{1}{\gamma^2}YX)^{-1}L \\ \hline \text{F} & 0 \end{array}\right] \tag{3.16}$$

where $F = -B'_2X$ and $L = -YC'_2$.

For the relaxed cases, such as $D_{11} \neq 0$, or $D'_{12}[C_1, D_{12}] \neq [0, I]$, or $[B'_1,\ D'_{21}]'D'_{21} \neq [0,\ I]'$, we also can obtain two coupled Riccati equations, and all the suboptimal controller can be described by a free parameter transmission $Q(s)$, please see the book by Zhou *et al.* for details [ZDG95].

3.3 LMI-based H_∞ Synthesis

Section 3.3 is an outline of H_∞ synthesis by solving Riccati equations for the case of full-state feedback or full-order output feedback. It has been widely accepted as an efficient H_∞ synthesis method. However, Riccati equation turns out to be helpless to design structured controller, and it is unclear how to exploit the Q parameterization for the design proposes, such as multi-objective control [GaP94]. Linear matrix inequalities (LMI) has emerged recently as a powerful approach in linear system analysis and synthesis. In this section, after a brief introduction to LMI, we will show its application in state feedback and full-order H_∞ control. In the following section, we will show how to handle structured H_∞ control in the framework of LMI. More applications of LMI in multi-objective control will be shown in Chapter 5.

3.3.1 Linear Matrix Inequalities

An LMI is any constraint of the form

$$A(x) = A_0 + x_1 A_1 + x_2 A_2 + ... + x_N A_N < 0 \tag{3.17}$$

where $x = (x_1, x_2, ...x_N)$ are scalar variables, $A_0, ...A_N$ are given symmetric matrices, and $A(x) < 0$ means negative definite.

The LMI (3.17) is a convex constraint on x, and the feasible set of x is a convex set. So if the objective function (such as $c'x$) is also convex, then the optimization is a convex problem which can be handled in polynomial time. Multiple LMIs can be regarded as a single LMI: $A^{(1)}(x) < 0, ...A^{(k)}(x) < 0 \Longleftrightarrow \text{diag}(A^{(1)}(x), ...A^{(k)}(x)) < 0$. This property is useful to describe the various specifications individually in LMI form without destroying convexity.

In control applications, mostly the decision variables come out as matrix variables $A(X) < 0$, such as Lyapunov inequality $A'X + XA < 0$, X is symmetric. These kind inequalities can still be written in the standard form (3.17), so it is LMI in matrix variables X. Moreover, X can have some structure, for example, some entries of X are prescribed.

There are three generic LMI problems,

1. Feasible problem, find a solution such that $A(x) < 0$;

2. minimizing of a convex objective under LMI constraints. In particular, the linear objective $\min_x c'x$ s.t. $A(x) < 0$;

3. Generalized eigenvalue minimization: $\min_x \lambda$ s.t. $\begin{cases} A(x) < \lambda B(x) \\ B(x) > 0 \\ C(x) < 0 \end{cases}$

All of the above three generic LMI problems are tractable in polynomial time with the efficient interior point algorithm, please see the book by Boyd, El Ghaoui *et al.* [EGF94]. [Vab00] is also a good review of LMI in control. Several software packages were developed for solving LMI problems. Contributed by Gahinet and Nemirovisji, LMI control toolbox is available in the environment of Matlab [GNL95]. In this tool box, there are functions *feasp mincx* and *gevp* to handle the three generic LMI problems.

For example, in Chapter 2 Theorem 2.1, the H_2 norm of the system (A, B, C, D) is computed by solving Lyapunov equation $AP+PA'+BB'=0$, $\|H\|_2^2 = \text{trace}(CPC')$. It also can be shown that $\|H\|_2^2 = \inf_{X=X'} \text{trace}(CXC')$ s.t. $AX + XA' + BB' < 0$. So we can use LMI techniques to solve for the H_2 norm.

H_∞ norm computation is another example of LMI application. As we have seen section 3.1, based on Theorem 3.1 γ iteration is used to compute H_∞ norm, subjected to the corresponding Hamilton matrix has no eigenvalue on imaginary axis. We also can solve system H_∞ norm with LMI techniques: $\min_X \gamma^2$ subject to the LMI constraint (3.18) or (3.19) in Theorem 3.6.

Theorem 3.6 Continuous-time LTI system (A,B,C,D) is stable and the L_2 gain is less than γ if and only if there exists some symmetric matrix X, such that

$$\begin{aligned} \begin{bmatrix} A'X+XA & XB & C' \\ B'X & -\gamma^2 I & D' \\ C & D & -I \end{bmatrix} &< 0 \\ X &> 0 \end{aligned} \tag{3.18}$$

or equivalently there exists some symmetric matrix Y, such that

$$\begin{aligned} \begin{bmatrix} AY+YA' & B & YC' \\ B' & -I & D' \\ CY & D & -\gamma^2 I \end{bmatrix} &< 0 \\ Y &> 0 \end{aligned} \tag{3.19}$$

Proof: If A is Hurwitz, then

$$L_2 \text{ gain less than } \gamma \iff$$
$$0 > \|y\|_2^2 - \gamma^2\|u\|_2^2 = \int_0^\infty [(Cx+Du)'(Cx+Du) - \gamma^2 u'u]dt$$
$$= \int_0^\infty [\frac{d}{dt}(x'Xx) + (Cx+Du)'(Cx+Du) - \gamma^2 w'w]dt - \int_0^\infty \frac{d}{dt}(x'Xx)dt$$

Substituting $\dot{x} = Ax + Bu$, and completing the square, we can obtain

$$\begin{aligned} 0 &> \begin{bmatrix} A'X + XA + C'C & XB + C'D \\ B'X + D'C & D'D - \gamma^2 I \end{bmatrix} \\ &= \begin{bmatrix} A'X + XA & XB \\ B'X & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C' \\ D' \end{bmatrix} [C\ D] \end{aligned} \tag{3.20}$$

Using the Schur complement (in Section 2.2), we can get the LMI (3.18). And with Laypunov stability theory, we know $X > 0$ guarantees A is Hurwitz. Similarly we can obtain the LMI (3.19).

Remark: If we multiply both sides of inequality (3.20) by $\frac{1}{\gamma^2}$ or $\frac{1}{\gamma}$, with Schur complement we can get another two equivalent expressions of the LMI (3.18).

$$\begin{bmatrix} A'X + XA & XB & C' \\ B'X & -\gamma I & D' \\ C & D & -\gamma I \end{bmatrix} < 0, X > 0$$

or

$$\begin{bmatrix} A'X + XA & XB & C' \\ B'X & -I & D' \\ C & D & -\gamma^2 I \end{bmatrix} < 0, X > 0$$

Similarly we also can get two equivalent expressions of the LMI (3.19).

3.3.2 LMI-based H_∞ Control with Full State Feedback

Using Theorem 3.5, we can derive the LMI-based H_∞ synthesis approach with full state feedback. It is a direct optimization procedure, not like that in Section 3.2 where a series of suboptimal problems are solved.

Given an LTI system,

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \end{aligned} \tag{3.21}$$

with the full-state feedback $u = Fx$, the close loop would be

$$\left[\begin{array}{c|c} A + B_2 F & B_1 \\ \hline C_1 + D_{12} F & D_{11} \end{array}\right]$$

Substitute the closed-loop description into LMI (3.19), we can get the necessary and sufficient condition for H_∞ control with full-state feedback:

$$\begin{aligned} \begin{bmatrix} (A + B_2 F)Y + Y(A + B_2 F)' & B_1 & Y(C_1 + D_{12} F)' \\ B_1' & -I & D_{11}' \\ (C_1 + D_{12} F)Y & D_{11} & -\gamma^2 I \end{bmatrix} &< 0 \\ Y &> 0 \end{aligned} \tag{3.22}$$

In inequality (3.22), gain F and symmetric matrix Y are unknown. It doesn't look like a LMI problem. However, if we define $\hat{Y} = FY$, then the constraint is in LMI form with matrix variable Y, $\hat{Y}$ and a scalar variable γ^2. So the full-state feedback problem is a convex minimization problem, which can be solved efficiently with LMI techniques ([Gap94], [GNL95], [MOS98]).

Corollary 3.1 Optimal H_∞ problem with full state feedback is equivalent to the following linear objective minimization under LMI constraints:

$$\min_{Y, \hat{Y}, \gamma^2} \gamma^2$$

$$\text{s.t.} \quad \begin{bmatrix} AY + YA' + B_2 \hat{Y} + \hat{Y}' B_2' & B_1 & Y C_1' + \hat{Y}' D_{12}' \\ B_1' & -I & D_{11}' \\ C_1 Y + D_{12} \hat{Y} & D_{11} & -\gamma^2 I \end{bmatrix} < 0 \tag{3.23}$$

$$Y = Y' > 0$$

and the feeback gain is $F = \hat{Y} Y^{-1}$

3.3.3 LMI-based H_∞ Control with Dynamic Full-Order Output Feedback

The dynamic full-order output feedback H_∞ problem also can be solved concisely with LMI techniques.

Theorem 3.7 (Suboptimal H_∞) Given a constant $\gamma > 0$, and the LTI plant

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned}$$

there exists an output feedback controller $u = K_{sub}(s)y$, such that L_2 gain bound of the closed loop $w \to z$ $\|H_{zw}\|_\infty < \gamma$ if and only if there exist symmetric matrices R and S satisfying the following LMI constraints:

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'S + SA & SB_1 & C_1' \\ B_1'S & -\gamma I & D_{11}' \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0 \tag{3.24}$$

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} AR + RA' & RC_1' & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1' & D_{11}' & -\gamma I \end{bmatrix} \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0 \tag{3.25}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{3.26}$$

where N_R and N_s are the null space of $[B_2', D_{12}']$ and $[C_2, D_{21}]$, respectively. Moreover, there exist suboptimal controllers of lower order $k < n$ if and only if

$$\text{rank}(I - RS) \leq k \tag{3.27}$$

The proof of Theorem 3.7 follows from Theorem 3.6. Suppose

$$K_{sub}(s) := \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

A_K is an $k \times k$ matrix, then the closed loop $w \rightarrow z$ would be

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} & \begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D21 \end{bmatrix} \\ \hline [C_1 + D_{12} D_K C_2 \; D_{12} C_K] & D_{11} + D_{12} D_K D_{21} \end{array}\right]$$

Substituting it into Theorem 3.7, we can get

$$\begin{bmatrix} A_c' X_c + X_c A_c & X_c B_c & C_c' \\ B_c' X_c & -\gamma I & D_c' \\ C_c & D_c & -\gamma I \end{bmatrix} < 0 \tag{3.28}$$

for some matrix $X_c' = X_c > 0$. Partition X_c as

$$X_c := \begin{bmatrix} S & N \\ N' & * \end{bmatrix}, \; X_c^{-1} := \begin{bmatrix} R & M \\ M' & * \end{bmatrix} \tag{3.29}$$

where S and R are $n \times n$ matrices. Substituting the partition of X_c into (3.28), and eliminating the controller parameters A_K, B_K, C_K and D_K, we can obtain the LMI (3.24). Similarly we can get the LMI (3.25). Moreover, $X_c > 0$ and X_c is $k \times k$ dimension are equivalent to (3.27) and the LMI (3.26). (see reference [GaP94] for details).

Controller Reconstruction: The controller can be reconstructed with feasible pair (R, S) [GaA94]. Since

$$\begin{bmatrix} S & N \\ N' & * \end{bmatrix} \begin{bmatrix} R & M \\ M' & * \end{bmatrix} = I$$

we can get two full column rank $n \times k$-dimensional matrices M and N from $MN' = I - RS$, then a feasible X_c can be obtained as

$$X_c = \begin{bmatrix} R & I \\ M' & 0 \end{bmatrix}^{-1} \begin{bmatrix} I & S \\ 0 & N' \end{bmatrix}$$

. Once X_c is known, (3.28) becomes an LMI with respect to A_K, B_K, C_K and D_K, thus the controller can be reconstructed. More strict arguments in [GaA94] shows that the reconstruction procedure is always possible if pair (R, S) is feasible. Explicit formulas for reconstruction of the controller with pair (R, S) were investigated in [IwS94] and [Gah96]. In reference [EOA97], another simple LMI procedure to reconstruct the controller from the feasible pair (R, S) is proposed given some α stability margin.

Inequalities (3.24), (3.25)and (3.26) are in LMI form, so H_∞ suboptimal problem with **full-order output feedback** becomes a feasible LMI problem, and full-order H_∞ optimal control becomes linear objective (convex) minimization subject to LMI constraints. However, convexity is destroyed by adding constraint (3.27) on the controller order and the problem is much harder. There were some attempts with non-differential programming in reference [GaI94] and GaA94]. More satisfactory approaches were reported later [EOA97] [GSS98], which we discuss in next section.

3.4 Static Output Feedback and Reduced Order H_∞ Control

3.4.1 System Augmentation of Reduced Order Control

As we have already discussed in the chapter about H_2 optimal control, reduced-order control problems are readily transformed into static-output feedback problem, using system augmentation techniques.

Given an nth order plant model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{3.30}$$

and the kth order output controller ($k < n$) to be designed

$$\begin{aligned} \dot{x}_K &= A_K x_K + B_K y \\ u &= C_K x_K + D_K y \end{aligned} \tag{3.31}$$

where $x \in R^n$, $u \in R^{n_u}$, $y \in R^{n_y}$, $z \in R^{n_z}$ and $x_K \in R^k$.

Assume $\tilde{x} = \begin{bmatrix} x \\ x_K \end{bmatrix}$, $\tilde{u} = \begin{bmatrix} \dot{x}_K \\ u \end{bmatrix}$, and $\tilde{y} = \begin{bmatrix} x_K \\ y \end{bmatrix}$, then we can get an augmented plant

$$\left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{array}\right] := \left[\begin{array}{c|cc} \begin{bmatrix} A & 0 \\ 0 & 0_{k\times k} \end{bmatrix} & \begin{bmatrix} B_1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & B_2 \\ I_{k\times k} & 0 \end{bmatrix} \\ \hline [C_1 \ , \ 0] & D_{11} & [0 \ , \ D_{12}] \\ \begin{bmatrix} 0 & I_{k\times k} \\ C_2 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} & 0 \end{array}\right] \tag{3.32}$$

and the "static" output controller

$$\tilde{u} = \tilde{F}\tilde{y} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \tilde{y} \tag{3.33}$$

Therefore, the reduced-order control problem becomes a static output control problem.

Finally, note that, other than direct design for reduced order controller in the framework of static output feedback, model order reduction is a widely used techniques to get reduced order H_∞ controller, please refer to Section 2.6.

3.4.2 H_∞ Control with Static Output Feedback

Many problems with static output feedback remain open [SAD97]. In the past several years, there has been great process contributed by El Ghaoui [EOA96], Skelton [IwS94] [GrS96], Geromel [GSS98] *et al.*. In this section, we will extend some techniques proposed for static output stabilization problem to handle static H_∞ control efficeintly.

For the given plant model (3.30), with the static output feeback $u = Fy$ the closed loop is

$$\left[\begin{array}{c|c} A + B_2FC_2 & B_1 + B_2FD_{21} \\ \hline C_1 + D_{12}FC_2 & D_{11} + D_{12}FD_{21} \end{array}\right]$$

Using Theorem 3.6, we can get the necessary and sufficient condition for static sub-optimal H_∞ control:

$$\begin{bmatrix} (A + B_2FC_2)'S + S(A + B_2FC_2) & S(B_1 + B_2FD_{21}) & (C_1 + D_{12}FC_2)' \\ (B_1 + B_2FD_{21})'S & -\gamma I & (D_{11} + D_{12}FD_{21})' \\ C_1 + D_{12}FC_2 & D_{11} + D_{12}FD_{21} & -\gamma I \end{bmatrix} < 0 \tag{3.34}$$

$$S = S' > 0 \tag{3.35}$$

Begin with (3.34) and follow a procedure similar to the derivation of Theorem 3.7, or just by setting $k = 0$ in condition (3.27), we can get another necessary and sufficient condition, as stated in Theorem 3.8 [GaP94].

Theorem 3.8 Given a constant $\gamma > 0$ and the LTI plant (3.30), there exist a suboptimal H_∞ control with static output feedback gain F, such that the closed loop $||H_{zw}||_\infty < \gamma$ if and only if there exist symmetric matrices R and S satisfying the following constraints:

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'S + SA & SB_1 & C_1' \\ B_1'S & -\gamma I & D_{11}' \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0 \tag{3.36}$$

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} AR + RA' & RC_1' & B_1 \\ C_1R & -\gamma I & D_{11} \\ B_1' & D_{11}' & -\gamma I \end{bmatrix} \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0 \tag{3.37}$$

$$S = R^{-1} > 0 \tag{3.38}$$

where N_R and N_s are the null space of $[B_2', D_{12}']$ and $[C_2, D_{21}]$, respectively.

Remark: with some matrix operations, we can get two equivalent expressions of the LMI (3.36) and (3.37):

$$N_S' \begin{bmatrix} A'S + SA + \frac{1}{\gamma}C_1'C_1 & SB_1 + \frac{1}{\gamma}C_1'D_{11}' \\ B_1'S + \frac{1}{\gamma}D_{11}'C_1 & -\gamma I + \frac{1}{\gamma}D_{11}'D_{11} \end{bmatrix} N_S \tag{3.39}$$

$$N_R' \begin{bmatrix} AR + RA + \frac{1}{\gamma}B_1B_1' & RC_1' + \frac{1}{\gamma}B_1D_{11}' \\ C_1R + \frac{1}{\gamma}D_{11}B_1' & -\gamma I + \frac{1}{\gamma}D_{11}D_{11}' \end{bmatrix} N_R \tag{3.40}$$

However, the constraint (3.38) $S = R^{-1}$ destroys the convexity. To track this constraint is not trival. This type of constraint $S = R^{-1}$ also appears in the static output stability problem and there are several methods proposed, such as the alternating projection method [GrS95], min/max algorithm [GSS98] and the cone complementarity algorithm [EOA97]. As we mentioned in Chapter 2, the min/max algorithm [GSS98] is an excellent method to tack the static output stability problem. However, if it is extended to H_∞ problem, the feasibility of this procedure lost and convergence is not guaranteed, similar as to H_2 problem. Cone complementarity algorithm [MaP95] is proved extremely efficient for static output stability problem, and has also been used in certain robust control problem ([EOA97]). In the following we will make a more comprehensive extension. Recall the following Lemma 2.3 mentioned in Chapter 2:

Lemma: For any pair of symmetric matrices (X,Y), if $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$, then trace($XY$) $\geq n$, and the equality holds iff $YX = I$.

Thus we can rewrite constraint (3.38) as

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} \geq 0 \tag{3.41}$$

and trace(SR) $= n$.

then the H_∞ suboptimal problem can be tracked by solving:

$$\begin{aligned} &\min \ \text{trace}(RS) \\ &\text{s.t. } LMI \ (3.36), (3.37) \text{ and } (3.41) \\ &\quad \text{or } LMI \ (3.39), (3.40) \text{ and } (3.41) \end{aligned} \tag{3.42}$$

and there exists an H_∞ control with static output iff the minimum achieved is n.

This problem is a bilinear objective optimization with linear matrix inequality constraints. The sequential linearization method (described in [MaP95]) can be adopted to solve this type of problem:

Cone Complementarity Linearization Algorithm:

Step 1. Find a start point S_0 and R_0 satisfying LMI (3.36) (or LMI /(3.39)) and (3.37) (or LMI /(3.40)) respectively, set $k = 1$.

Step 2. Solve the LMI linear minimization (convex) problem:

$$\begin{aligned} \min f_k(R,S) = \ & \operatorname{trace}(R_{k-1}S + RS_{k-1}) \\ \text{s.t. } & LMI \ (3.36), (3.37) \text{ and } (3.41) \\ \text{or } & LMI \ (3.39), (3.40) \text{ and } (3.41) \end{aligned} \tag{3.43}$$

Step 3. If the stopping criterion is satisfied, exit; otherwise, set $(S_k, R_k) = \arg\min\ f_k(R,S)$, $k = k+1$ and go to step 2.

The above algorithm will generate a decreasing sequence f_k which is bounded from below, so it will converge to some value.

Theorem 3.9 The cone complementarity algorithm presents the following properties: (i) $2n \leq f_{k+1} \leq f_k$, (ii) $f(X^*, Y^*) = 2n$ iff $X^*Y^* = I$.

The above cone complementarity linearization algorithm is very easy to implement using the LMI toolbox. El Ghaoui etc ([EOA96] [EOA98]) have used this method, and showed it is extremely satisfactory for the α-stability problem with static output feedback or reduced order controller. They gave some intuitive explanation from the view of primal-and-dual, and named it the "cone complementarity linearization algorithm". El Ghaoui etc also used it for certain H_∞ suboptimal problems for the case $D_{11} = 0$ and $D_{21} = 0$. Our extension here is more general and requires less computation since we decrease the decision matrix number from four to two. Numerical comparison and extension to H_2 problem were reported in reference [OlG97].

Once we get the feasible pair (R, S), we can reconstruct the static gain F with the method in [GaA94] or [Gah96]. The method developed in the stabilization problem in [IwS94] also can be extended directly for H_∞ control reconstruction.

3.5 Decentralized H_∞ Control

In Section 3.4, we discussed the H_∞ control with lower-order and static output feedback, and solve them efficiently with cone complementarity linearization algorithm and LMI solver. However, if there are more constraints on the static gain, or not all sensor signals are available for each actuator, the above algorithm will fail, since we can't reconstruct the controller with pair (R,S). Static decentralized architecture is one such typical constraint of practical importance. Similar as Section 3.4, decentralized reduced order control can also be cast as a static decentralized problem with system augmentation. So in this section we will mainly focus on static decentralized control.

3.5.1 Static Decentralized H_∞ Problem and BMI

The static Decentralized H_∞ Problem: Given an LTI plant model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + \sum_{i=1}^{N} B_{2i} u_i \\ z &= C_1 x + D_{11} w + \sum_{i=1}^{N} D_{12i} u_i \\ y_i &= C_{2i} x + D_{21i} w, i = 1, 2, ..., N \end{aligned} \tag{3.44}$$

where y_i is the i-th measurement available for the i-th control vector u_i. Design the decentralized static output feedback controller $u_i = F_{Di} y_i$, $i = 1, 2, ..., N$, such that closed-loop system is stable and $\|H_{zw}\|_\infty < \gamma$.

Define $B_2 = [B_{21}, B_{22}, ...B_{2N}]$, $C_2' = [C_{21}', C_{22}', ...C_{2N}']$, $D_{12} = [D_{121}, D_{122}, ...B_{12N}]$, $D_{21}' = [D_{211}', D_{212}', ...D_{21N}']$,$u = [u_1', u_2', ...u_N']'$, and $y = [y_1', y_2', ...y_N']'$. The controller can be written as

$$u = F_d\ y = \begin{bmatrix} F_{D1} & & & \\ & F_{D2} & & \\ & & \ddots & \\ & & & F_{DN} \end{bmatrix} y$$

With Theorem 3.6, we can conclude that F_d is a decentralized static sub-optimal H_∞ controller if and only if there exists some symmetric matrix S such that:

$$\begin{bmatrix} (A+B_2F_dC_2)'S+S(A+B_2F_dC_2) & S(B_1+B_2F_dD_{21}) & (C_1+D_{12}F_dC_2)' \\ (B_1+B_2F_dD_{21})'S & -\gamma I & (D_{11}+D_{12}F_dD_{21})' \\ C_1+D_{12}F_dC_2 & D_{11}+D_{12}F_dD_{21} & -\gamma I \end{bmatrix} < 0 \tag{3.45}$$

$$S = S' > 0 \tag{3.46}$$

The condition (3.45) and (3.46) look similar as (3.34) and (3.35), but there is a great difference. since F_d is has block diagonal structure, we can't use elimination lemma to arrive at the equivalent conditions as in Theorem 3.8. So the decentralized static sub-optimal H_∞ problem is essentially a non-convex bilinear matrix problem (BMI).

BMI was popularized after a series of papers ([SGL94], [GTS94], [GSP94], [SGL96]) by Safonov *et al.*. Unfortunately BMI problems are generally NP-hard, not solvable in polynomial time. However, practical algorithms do exist for BMI problems, even although not efficient. LMI-based iteration ([ShC96], [CSM98]), nonlinear programming ([ImF99], [Lee99]), or homotopy methods ([IZF96], HHB99], [IZF99]) are used to search the local minima. Branch-and-bound methods ([GSP94], [Tas97], [KSK97], [BVB97], [SMP99], [TAN00], [TAN00]) are also used to solve for the global minima and various tricks were used to simplify the calculation of the upper or lower bound, but it still takes too much time even for a modest-size problem [VaB00]. In addition, some sufficient conditions for decentralized H_∞ control were also proposed in [GBP94] and [CrT99].

3.5.2 Algorithms for Decentralized H_∞

In the following, we will introduce some easily-implemented algorithms for decentralized H_∞ control: alternative minimization, iterative LMI, and homotopy. For nonlinear programming approach and brunch-and-bound method, please refer to the literature we previously mentioned.

Alternative Minimization Algorithm:

The most easy to implement algorithm for decentralized H_∞ is probably the alternative minimization:

Start with a stabilizing F_d, repeat OP1 and OP2 until γ can not decrease any more:

OP1: Fix F_d, search for $S = \arg\ \min_S \gamma$, subject to constraints (3.45) and (3.46);

OP2: Fix S, search for $F_d = \arg\ \min_{F_d} \gamma$, subject to constraint (3.45).

After the matrices F_d or S is fixed, OP1 and OP2 are LMI problems, which can be solved easily with LMI solver. The alternative minimization will generate a decreasing sequence of γ, and it works well in many practical problems. However, this algorithm might converge very slowly, and might even stop at a non-stationary point.

Iterative LMI Algorithms:

Shiau and Chow [ShC96] developed an iterative LMI approach for decentralized state feedback. They first designed a full state H_∞ controller (seen in Theorem 3.3), and parameterized the set of all decentralized state-feedback H_∞ controllers with parameters M and F_d. To solve for F_d, they used the equality $(X-M)'B_2B_2'(X-M) \geq 0$ to relax the BMI conditions and iteratively solve the LMI problem with fixed $X^{(k)} = M^{(k-1)}$.

Cao *et al.* [CSM98] used an approach similar to that of Shiau and Chow to relax BMI for decentralized output-feedback stabilization problem, and used it as a framework for the H_∞ problem. Their ideas are summarized as following:

1. Define

$$\bar{S} := \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \bar{A} := \begin{bmatrix} A & B_1 & 0 \\ 0 & \frac{\gamma}{2}I & 0 \\ C_1 & D_{11} & \frac{\gamma}{2}I \end{bmatrix}, \bar{B} := \begin{bmatrix} B_2 \\ 0 \\ D_12 \end{bmatrix}, \bar{C} := [C_2\ D_{21}\ 0]$$

then the sub-optimal condition (3.45) and (3.46) can be re-arranged as

$$\bar{S}(\bar{A} + \bar{B}F_d\bar{C}) + (\bar{A} + \bar{B}F_d\bar{C})'\bar{S} \tag{3.47}$$

which is exactly the same as a decentralized stabilization problem with the augmented plant $(\bar{A}, \bar{B}, \bar{C}, 0)$ and some structure on $\bar{S}$.

2. For the LTI plant $(A, B, C, 0)$, it is well known that there exist a static stabilizing decentralized controller $u = F_d\, y$ if and only if there exists some symmetric matrix $S > 0$ such that

$$S(A + BF_dC) + (A + F_dC)'S < 0 \tag{3.48}$$

Add an immaterial item $C'F_dF_dC$ to the above inequality, and it is shown that (3.48) holds if and only if

$$A'S + SA - SBB'S + (B'S + F_dC)'(B'S + F_dC) < 0 \tag{3.49}$$

.

3. Since $(X - S)'BB'(X - S) \geq 0$, we can get a sufficient condition for inequality (3.49)

$$A'S + SA - XBB'S - SBB'X + XBB'X + (B'S + F_dC)'(B'S + F_dC) < 0 \tag{3.50}$$

This condition become necessary if $X = S$ holds. Using the Schur complement, we can get

$$\begin{bmatrix} A'S + SA - XBB'S - SBB'X + XBB'X & (B'S + F_dC)' \\ B'S + F_dC & -I \end{bmatrix} < 0 \tag{3.51}$$

4. The algorithm for the decentralized stabilization problem is following. (This algorithm can be applied to decentralized H_∞ problem with the augmented plant $(\bar{A}, \bar{B}, \bar{C}, 0)$ and structured S).

Step 1. Solve the following Riccati equation for S with some $Q > 0$, then set X as S

$$A'S + SA - SBB'S + Q = 0$$

Step 2. Fix X, solve problems OP1 and OP2 with the LMI solver:

OP1: generalized eigenvalue problem

$$\min_{S>0, F_d, \alpha} \alpha$$

$$\text{s.t.} \quad \begin{bmatrix} A'S + SA - XBB'S - SBB'X + XBB'X - \alpha S & (B'S + F_dC)' \\ B'S + F_dC & -I \end{bmatrix} < 0$$

OP2: Fix the α as that achieved in OP1

$$\min_{S>0,F_d} trace(S)$$
$$\text{s.t.} \begin{bmatrix} A'S + SA - XBB'S - SBB'X + XBB'X - \alpha S & (B'S + F_d C)' \\ B'S + F_d C & -I \end{bmatrix} < 0$$

Step 3. If $\alpha < 0$, and F_d is a feasible decentralized gain, stop; else if $\|X - S\| < \delta$ (a pre-determined tolerance), there may be no feasible F_d, stop; otherwise update X as S achieved in OP2, and go to Step 2.

Problem OP1 is used to move the closed-loop poles from right to left, and in OP2 trace(S) is bounded from below since $S > 0$. Theoretically Cao's iterative LMI procedure will generate a decreasing sequence α, and a bounded sequence trace(S). So it should be convergent, although it might not converge to an acceptable solution.

Homotopy Algorithms

Hassibi *et al.* [HHB99] proposed a homotopy method to solve BMI problems in control. They linearize the BMI with a first-order perturbation approximation, and continuously make slight improvements of the controller performance with the LMI solver. They showed the effectiveness of the approach, but there are still no convergence guarantees.

Zhai *et al.* [ZIF01] [IZF96] proposed another homotopy method to deform the centralized gain into a decentralized controller. Write the left side of inequality (3.45) as $F(F_d, S)$, and define a matrix function $H(F_d, S, \lambda) = F(\ (1-\lambda)F_d + \lambda F_C, S)$, where F_C is a centralized output feedback gain, which can be found using the approaches in last section. Observe that $(1-\lambda)F_d + \lambda F_C = F_C$ if $\lambda = 0$, and $(1-\lambda)F_d + \lambda F_C = F_d$ if $\lambda = 1$. The idea of their approach is:

Step 1: Find a centralized suboptimal H_∞ gain F_C and the corresponding S, set $S_0 = S$

Step 2: Set $\lambda_k = k/2^M$, solve the following LMI feasible problems for $k = 1$ to 2^M,

OP1: search for F_{dk} with fixed S_{k-1} subject to $H(F_{dk}, S_{k-1}, \lambda_k) < 0$, OP2: search for S_k with fixed F_{dk} subject to $H(F_{dk}, S_k, \lambda_k) < 0$ and $S_k > 0$.

If this procedure is feasible for all λ_k, then we have obtained a suboptimal H_∞ controller F_d. If the procedure is infeasible for some selected λ_k, Zhai *et al.* will increase M or restart the procedure with another centralized F_C. Zhai's method is easy to implement, but similar to the alternative minimization there is no guarantee that an acceptable solution can be found with their algorithm.

As we know, the computation for a centralized suboptimal H_∞ gain F_C is also troublesome. So Zhai *et al.* further extends their approach by starting with a full-order centralized H_∞ controller, which can be obtained with the existing Matlab toolbox. They constructs some uncontrollable and unobservable (but stable) modes and deform it into a reduced-order decentralized controller. However, this will extremely increase the number of design variables in S.

We propose another homotopy approach here, which will only require the initial controller to be stabilizing. Represent the left side of inequality (3.45) as $F(F_d, S, \gamma)$ and let γ_g be the goal we are required to achieve.

Step1: Find a stabilizing F_{d0}, and the corresponding closed loop H_∞ norm γ_0,

Step 2: Set a trial step-size $\delta_\gamma = (\gamma_0 - \gamma_g)/N_m$, $\gamma_k = \gamma_0 - k\delta_\gamma$, $k = 1$ to N_m. For every γ_k, find a pair (F_{dk}, S_k) such that $F(F_{dk}, S_k, \gamma_k) < 0$ and $S_k > 0$ with the pair $(F_{d(k-1)}, S_{k-1})$

Wherein, for every k, the pair (F_{dk}, S_k) can be obtained by shifting the poles of $F(F_d, S, \gamma_k) < 0$ continuously from left to right: start with $F_{d(k-1)}$, alternatively fix F_d and S, minimize t, such that $F(F_d, S, \gamma_k) < tI$ and $S > 0$, until we arrive at $t < 0$.

Although this homotopy method has outer and inner LMI iterations, our (limited) experience showed that the computation efficiency is almost the same as alternative minimization. The reason might be that the inner LMI iteration does't require feasibility. For this reason, the initial γ_0 can be chosen less than the closed loop H_∞ norm with F_{d0} feedback. To accelerate convergence, we can use diminishing step-sizes of γ.

3.6 Application: Passive Mechanical System Design

Example: Five-Mass System

We apply the approaches of structured control to the design of the lumped-parameter mechanical structure. Figure 3-4 depicts a five mass system composing masses, springs and dashpots. We are given that $m1 = m2 = m3 = m4 = m5 = 1$, $k1 = k3 = k5 = 1$, and $c1 = c3 = c5 = 0$. Our goal is to choose $k2, c2, k4$ and $c4$ to minimize the H_∞ norm from $Fd1$ and $Fd2$ to the velocities of masses $m2$ and $m4$. It can be cast as a decentralized H_∞ problem with static output feedback. The plant order is 10, closed loop is 2 by 2, and the controller is a block diagonal matrix composed of two 2×1 blocks.

This example is borrowed from the literature [SMP99], in which a new branch and bound method is used and the achievable H_∞ norm is obtained between 2.775 and 3.083 after 6661 iterations nearly one and a quarter hours' computation on a Pentium II 400, while the traditional branch and bound can't get convergence after 12 hours of computation.

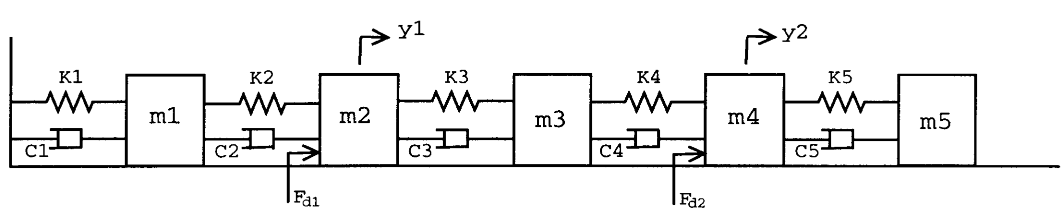

Figure 3-4: Diagram of the five-mass system.

With the alternative minimization algorithm, after 1070 LMI iterations with initial parameters

$$F_d = \begin{bmatrix} k2 & c2 & 0 & 0 \\ 0 & 0 & k4 & c4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

we get the H_∞ norm 3.044 and the optimal parameters

$$F_d = \begin{bmatrix} 0.7007 & 0.1032 & 0 & 0 \\ 0 & 0 & 0.0192 & 0.4890 \end{bmatrix}$$

With the homotopy algorithm we proposed, set initial γ_0 as 6, and step as -0.1, we can achieve H_∞ norm 3.10 after 899 LMI iterations.

With cone complementary algorithm, we obtained a centralized static output feedback gain

$$F_C = \begin{bmatrix} 0.5376 & 0.4285 & -0.4306 & 0.6011 \\ -0.2891 & 0.3104 & 0.2690 & 1.2730 \end{bmatrix}$$

(It converged very quickly for this plant model). With this F_C, we tried Zhai's homotopy method, for $\gamma = 3.10$ it converged after $2^{12} = 4096$ LMI iterations. (It couldn't produced a acceptable solution after $2^{11} = 2048$ LMI iterations.)

The singular value of the closed-loop system is shown in Figure 3-5, comparing with the case $k2 = k4 = 1$ without damping. In this example, alternative searching algorithm and pole shifting algorithm converge much faster than the branch and bound method. We also tried other initial parameters, or optimize three parameters rather than four, and it is found that both algorithms work pretty efficiently.

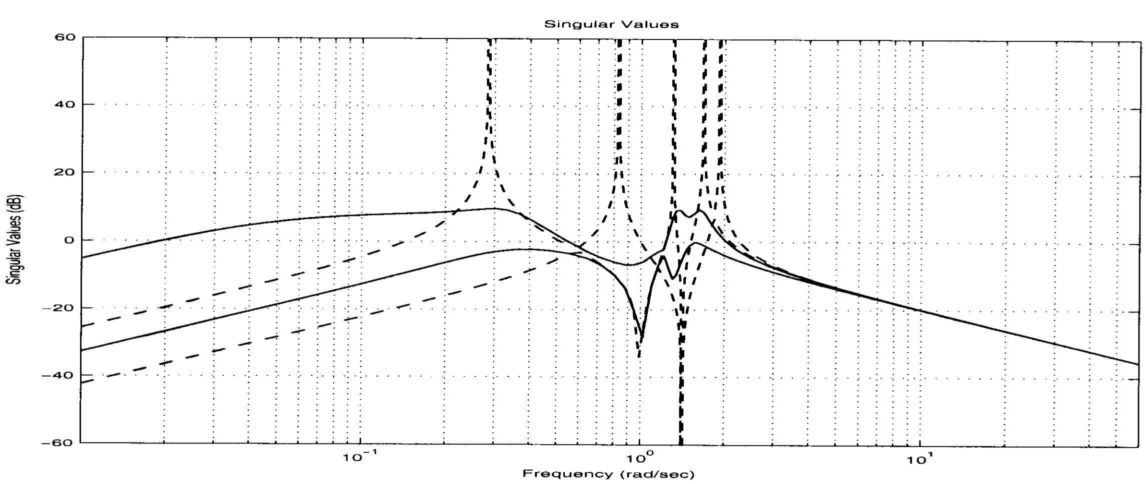

Figure 3-5: Closed-loop singular value of five-mass system.

In addition, we also applied the structured control techniques to TMD design. An SODF TMD can be taken as a centralized static output feedback problem. The cone complementary algorithm and alternative LMI converge very fast for this 4-order system. We can get an H_∞ optimal tuned ratio and damping. One example can be found in Section 5.6. We would like to point out that H_∞ design is not the same as Dan Hartog's analytic result. However Dan Hartog's design is quite close to that of H_∞, as found by Nishihara and Asami [NiA00]. The multi-degree-freedom TMD can be cast as a decentralized control and designed via H_∞ optimization.

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Chapter 4

Eigenvalue & Eigenstructure Treatment

Poles determine the rate of the system's response; right eigenvectors fix the modal shapes and left eigenvectors plays a determining role of the observability. Pole placement (the so-called inverse eigenvalue problems) is a well-known approach in dynamics and modern control since 1960s. Eigenstructure assignment, which can assign eigenvalues and eigenvectores simultaneously, has also been proposed in the past two decades. In this chapter, the important results and techniques of eigenstructure assignment are reviewed, including full-state feedback, static output feedback, constrained output feedback, and reginal pole placement. A new approach is proposed to treat poles of architecture constrained systems, so as to maximized the minimal damping. Practical examples are given to demonstrate the application to the design of passive mechanical systems. The performances of the closed-loop system produced with H_2, H_∞ and pole treatment are compared.

4.1 Modal Decomposition and Eigenstructure Treatment

4.1.1 Modal Decomposition

Consider the following n-dimensional system with initial condition:

$$\dot{x} = A_c x + B_c r,\ x(0) = x_0 \tag{4.1}$$

It is well-known that the zero-input response to initial condition is

$$x(t) = e^{A_c t} x_0$$

Suppose A_c has eigenvalues λ_i and right eigenvectors (eigendirections) v_i,

$$A_c v_i = \lambda_i v_i,\ i = 1, 2, ...n \tag{4.2}$$

If all eigenvalues are distinct, then

$$A_c = V^{-1} \Lambda V$$

where $V = [v_1, v_2, ...v_n]$, $\Lambda = \text{diag}\{[\lambda_1, \lambda_2, ...\lambda_n]\}$. Now we can write the response as:

$$\begin{aligned} x(t) &= e^{A_c t} x_0 = e^{V^{-1}\Lambda V t} x_0 = V^{-1} e^{\Lambda t} V x_0 \\ &= \sum_{i=1}^{n} v_i e^{\lambda_i t} w_i' x_0 \end{aligned} \tag{4.3}$$

where w_i is the it hrow of $W = V^{-1}$, which is known as the left eigenvector since $w_i A_c = \lambda_i w_i$.

Equation (4.3) is called the **modal decomposition** or **modal expansion** of the undriven system. With modal transformation, we can also obtain the modal decomposition for zero-state response with nonzero driven $r(t)$:

$$x(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} w_i' * B_c r(t) \tag{4.4}$$

where '*' means convolution.

The total response is the sum of the zero-input responses and zero-state responses. If the eigenvalues are not distinct, A_c might not be diagonalizable. If so we can use Jordan form for modal decomposition.

From Equation (4.3) and (4.4), we can see that

- The eigenvalues (poles) determine the response rate of the system; but poles (together with zeros) can't define a whole system (other than the single-input-single-output case);
- The right eigenvectors fix the modal shapes of the response;
- The left eigenvectors can influence the observability;
- The initial condition determines to which degree each mode will participate in the undriven response;
- The control matrix B_c determines to which degree the input will contribute in the zero-state response.

4.1.2 Eigenvalue & Eigenstructure Treatment

Because the importance of eigenvalues and eigenvectors, it is useful to change the closed-loop modes with the feedback so as to meet performance requirements. For examples, we may wish to assign poles precisely (or arbitrary close) to the preselected self-conjugate set $\{\lambda_i^d\}$, or assign the eigenvalues and eigenvectors (so-called eigenstructures) precisely to the self-conjugate scalar set $\{\lambda_i^d\}$ and corresponding vector set $\{v_i^d\}$, or place the poles in some ideal regions. We use the the term eigenvalue/eigenstructure treatment to refer to eigenvalue/eigenstructure assignment, pole regional placement, and the pole shifting.

Consider the plant model:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{4.5}$$

where $x \in R^n$, $u \in R^m$, and $y \in R^r$.

Our problem is to design the controller gain F, to meet the specification on the eigenvalues or eigenstructures of the closed-loop $A + BFC$. For general C, the above is a output feedback problem. Full-state feedback problem is the case $C = I$. If some entries of F are prescribed as zero, such as decentralized, this is architecture constrained problem.

Eigenvalue/Eigenstructure problems have been investigated extensively since the 1960s. Wonham [Won67] originally connected full-state pole placement with controllability. Davidson [Dav70] examined the pole assignment with static output feedback. Kimura [Kim75], Davidson and Wang [DaW75] independently proposed the same sufficient condition $m + r > n$ for output pole assignment, and this condition was extended by Wang [Wang92]. Various algorithms for pole placement have been proposed. Pole placement problem was extended to eigenstructures by recognition that multi-input systems can have a multiplicity of eigenvector associated with each eigenvalue. Moore [Moo76] described the freedom available to assign eigenvectors other than poles for full-state feedback. Srinathkumar [Sri76] investigated the freedom of eigenstructure assignment using output feedback. O'Reilly and his coworkers [FaO82] [RoO87] [FaO88] parameterized the eigenstructure assignment for full-state and output feedback. Liu *et al.* [LCT93] parameterized the decentralized eigenstructures assignment. Shapiro and his coworkers [ShC81] [ASC83] [SSA94] [PSS94] investigated the 'best achievable' eigenstructures via projection. Kautsky *et al.*. [KNV82] [KNV85] developed the robust eigenstructure assignment. Regional pole placement is also investigated with other objectives, such as H_∞ minimization [PaL94] [YeL95] [CGA99] or H_2 minimization [YaB92] [YAJ96] or both [BSU94] [FNM97] [CGP99]. Eignstructure assignment control toolbox was developed in 1994 by Liu and Patton. Eigenstructure assignment has been successfully applied in some flight control system. More information and application can be found in the survey paper [ASC83] [Spu90] [SSA94] [Whi95] [SAD97] and the book by Liu and Patton [LiP98].

In the following of this Chapter, we will survey some the important results and algorithms. Then proposed a subgradient-based approach to shift the poles so as to maximize the minimal damping for architecture-constrained systems.

4.2 Pole/Eigenvalue Assignment

In this section we will discuss the conditions and algorithms for the problems of eigenvalue assignment with full-state feedback, static output feedback and decentralized output feedback. Some other aspects, such as fixed modes, partial pole assignment, and robust pole assignment are also worthy to mention.

4.2.1 Eigenvalue Assignment with Full-State Feedback

Consider the full-state feedback problem: Given a self-conjugate complex scalar set $\{\lambda_i^d\}$, $i = 1, 2, ...n$, find a $m \times n$ real matrix F, such that the eigenvalue of $A + BF$ are precisely the set $\{\lambda_i^d\}$.

Theorem 4.1 There exists a real matrix F, such that

$$\det(\lambda I - A - BF) = \prod_{i=1}^{n}(\lambda - \lambda_i^d) \tag{4.6}$$

for an arbitrary self-conjugate complex scalar set $\{\lambda_i^d\}$, $i = 1, 2, ...n$ if and only if (A, B) is controllable: $\text{rank}\{[B, AB, A^2B, ...A^{n-1}B]\} = n$.

The proof can be seen in the lecture notes by Dahleh *et al.* [DDV99].

Below we will demonstrate the pole placement design for single-input case $B = b$. It is well-known that a nonsingular similarity transformation will not change the eigenvalues and controllability of the system. Since (A, b) is controllable, there exists a transform matrix T to change the pair (A, b) into the control canonical form [LiP98]:

$$\bar{A} = T^{-1}AT = \begin{bmatrix} -\bar{a}_1 & -\bar{a}_2 & \dots & -\bar{a}_{n-1} & -\bar{a}_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \tag{4.7}$$

$$\bar{b} = T^{-1}b = [1\ 0\ 0\ ...\ 0]' \tag{4.8}$$

And the characteristic polynomial is given by the first row of $\bar{A}$:

$$\det(\lambda I - A) = \det(\lambda I - \bar{A}) = \lambda^n + \bar{a}_1\lambda^{n-1} + \bar{a}_2\lambda^{n-2} + ... + \bar{a}_{n-1}\lambda + \bar{a}_n \quad (4.9)$$

Denote the full-state feedback $\bar{F}$ in the new coordinates as

$$\bar{F} = FT = [\bar{f}_1\ \bar{f}_2\ \bar{f}_3\ ...\ \bar{f}_n]$$

then the closed-loop matrix has the form:

$$\bar{A} + \bar{b}\bar{F} = T^{-1}(A + bF)T = \begin{bmatrix} -\bar{a}_1 + \bar{f}_1 & -\bar{a}_2 + \bar{f}_2 & ... & -\bar{a}_{n-1} + \bar{f}_{n-1} & -\bar{a}_n\bar{f}_n \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 1 & 0 \end{bmatrix} \quad (4.10)$$

and the closed-loop characteristic polynomial is given by

$$\begin{aligned} &\det(\lambda I - A - bF) = \det(\lambda I - \bar{A} - \bar{b}\bar{F}) \\ &= \lambda^n + (\bar{a}_1 - \bar{f}_1)\lambda^{n-1} + (\bar{a}_2 - \bar{f}_2)\lambda^{n-2} + ... + (\bar{a}_{n-1} - \bar{f}_{n-1})\lambda + (\bar{a}_n - \bar{f}_n) \end{aligned} \quad (4.11)$$

Rewrite Equation (4.6) as

$$\det(\lambda I - A - BF) = \prod_{i=1}^{n}(\lambda - \lambda_i^d) = \lambda^n + \bar{a}_1^d\lambda^{n-1} + \bar{a}_2^d\lambda^{n-2} + ... + \bar{a}_{n-1}^d\lambda + \bar{a}_n^d \quad (4.12)$$

Compare thing coefficients in (4.11) and (4.12), we have

$$\bar{f}_i = -\bar{a}_i^d - \bar{a}_i, i = 1, 2, ...n \quad (4.13)$$

And the full-state feedback gain in the original coordinate is obtained as

$$F = \bar{F}T^{-1} = [\bar{f}_1\ \bar{f}_2\ \bar{f}_3\ ...\ \bar{f}_n]T^{-1} \quad (4.14)$$

The matrix T which transforms the pair (A, b) in to the control canonical form $(\bar{A}, \bar{b})$ is given by

$$T = [b\ Ab\ A^2b\ ...\ A^{n-1}b][\bar{b}\ \bar{A}\bar{b}\ \bar{A}^2\bar{b}\ ...\ \bar{A}^{n-1}\bar{b}]^{-1} \quad (4.15)$$

The above procedure results in the following concise formula, which is commonly called Ackermann's formula

$$F = -[0\ 0\ ...\ 0\ 1][b\ Ab\ A^2b\ ...\ A^{n-1}b]^{-1}(A^n + \bar{a}_1^d A^{n-1} + \bar{a}_2^d A^{n-2} ... + \bar{a}_n^d I) \quad (4.16)$$

A similar procedure can be developed for multi-input case by introducing the MIMO control canonical form. For the MIMO case, the feedback matrix for pole assignment is not unique, and this additional freedom can be used to assign eigenvectors. We will discuss the details in the next section.

4.2.2 Eigenvalue Assignment with Static Output Feedback

Assignment conditions with output feedback

Pole placement with static output feedback has remained challenging for several decades. Given a minimal-realized (controllable and observable) n-order real-coefficient system $(A, B, C, 0)$ and a self-conjugate complex scalar set $\{\lambda_i^d\}$, $i = 1, 2, ...n$, find an $m \times n$ real matrix F such that the eigenvalue of $A + BFC$ are precisely (or arbitrary close to) the set $\{\lambda_i^d\}$. Usually we assume B has full column rank m and C has full raw rank r.

In 1975 Davidson and Wang [DaW75], and Kimura [Kim75] proved that a *sufficient* condition for arbitrary (self-conjugate) pole assignment with a real matrix F is that

$$m + r > n \quad (4.17)$$

Herman and Martin [HeM77] showed that if *complex* feedback is allowed a necessary and sufficient condition of static output pole assignment is

$$mr \geq n \quad (4.18)$$

Tarokh [Tar89] used the rank$(E) = n$ as the pole assignability criterion with real F, where E is $n \times mr$ matrix whose kth row is formed from the rows of $CA^{k-1}B$. But Carotenuto [CFM01] provided a counter example to show that Tarokh's assertion is only true for complex F, and for the case of real F it is false. Wang [Wan92] provided the best-known tight condition as below. An alternative concise proof can be found in [RSW95].

Theorem 4.2 If the degree of Grassmannian

$$d_{m,r} = \frac{1!2!...(r-1)!mp!}{m!(m+1)!...(m+p-1)!} \tag{4.19}$$

is odd, $mr \geq n$ is a necessary and sufficient condition of arbitrary pole assignability with real static output F; if $d_{m,r}$ is even, $mr > n$ is a sufficient condition.

The case $mr = n$ and $d_{m,r}$ even is left inclusive. Carotenuto [CFM01] showed that for arbitrary selected plant with $n = 4$, $m = 2$ and $r = 2$ ($d_{m,r}$ even), the chance of pole assignability is around 85%. Some new conditions can be found in [RoS98]. More related information can be found in the survey paper by Rosenthal and Wang [RoW97].

Unlike the above conditions for *arbitrary* pole placement, in [FaO88] a necessary and sufficient condition is used to check the assignability of certain preselected poles λ_i: rank[C adj$(\lambda_i I_n - A)B$] ≥ 1 for real eigenvalue λ_i, and rank[C adj$(\lambda_i I_n - A)B$, C adj$(\lambda_i^* I_n - A)B$] ≥ 2 for real eigenvalue λ_i. The proof can be found therein.

Algorithms for output feedback

Most available algorithms of output pole assignment required $m + r > n$ and only place some part of poles. We will see these algorithms in Section 4.3. Here we mainly focus on the algorithms to place whole set of poles.

Fahmy and O'Reilly [FaO88] proposed a multistage parametric approach to assign some eigenvalues and right eigenvectors first with a matrix F_1, and convert these eigenstructures into uncontrollable and unobservable modes in a particular n-order 'equivalent' system $(A + BF_1C, B\tilde{B}, \tilde{C}C, 0)$, then assign some other eigenvalues and left eigenvectors. This approach can be used to assign n self-conjugate poles if $m + r > n$. We will see their parameterization and protection method in next section.

After Wang [Wan92] provided the tighter sufficient condition $mr > n$ some new algorithms were proposed. Soylemez and Munro [SoM98] proposed an approach to place some poles successively with a single input, and to make these poles uncontrollable from other input so as to place all the poles. Alexandridis and Paraskevopoulos [AlP96] proposed a relatively easy approach to assign whole set of eigenvalues via two coupled Sylvester matrix equations. This algorithm is detailed below:

Suppose the set of the self-conjugate closed-loop poles $\{\lambda_i^d\}$, $i = 1, 2, ...n$ are distinct and don't coincide with the open-loop poles. Assume that the set can be partitioned into two self-conjugate sets $\{\lambda_i^d\}$, $i = 1, 2, ...r$ and $\{\lambda_i^d\}$, $i = r+1, r+2, ...n$. Write $\Lambda_r^d = \text{diag}\{[\lambda_1^d, ..., \lambda_r^d]\}$ and $\Lambda_{n-r}^d = \text{diag}\{[\lambda_{r+1}^d, ..., \lambda_n^d]\}$. Assuming v_i, $i = 1, 2, ...r$, is the ith eigenvector (to be determined) corresponding λ_i, $V_r = [v_1, ..., v_r]$, we obtain

$$(A + BFC)v_i = \lambda_i^d v_i, \;\; i = 1, 2, ..., r$$

or

$$(A + BFC)V_r = V_r \Lambda_r^d$$

Defining $\Psi_r = FCV_r$, we obtain the well-known Sylvester matrix equation

$$V_r \Lambda_r^d - AV_r = B\Psi_r \tag{4.20}$$

Since C is full row rank and Λ_r^d contains no poles of A, CV_r is invertible. So the output feedback is obtained as

$$F = \Psi_r (CV_r)^{-1} \tag{4.21}$$

Next we will see how to choose V_r so as to place the rest of the poles Λ_{n-r}^d. In [AlP96] it is shown that if F satisfies (4.20) and (4.21), the remaining $n-r$ closed-loop eigenvalues are the eigenvalues of the $(n-r) \times (n-r)$ matrix

$$A_{22} - LA_{12} \tag{4.22}$$

where

$$L = \bar{C} V_r (CV_r)^{-1} \tag{4.23}$$

and $A_{22} = \bar{C}AT_2$, $A_{12} = CAT_2$, $\bar{C}$ is any $(n-r) \times n$ matrix such that $[\bar{C}', \, C]$ is nonsingular, $[T_1, T_2] = [\bar{C}', \, C]^{-1}$. Further it is shown that the pair (A_{22}, A_{12}) is observable if and only if pair (A, C) is observable. So the remaining problem is to choose L so as to place the poles of $A_{22} - LA_{12}$ as Λ^d_{n-r} subject the constraint (4.23). Note that to determine the poles of $A_{22} - LA_{12}$ is equivalent to an observer pole placement problem, which is the duality of full-state feedback. Consequently, another Sylvester matrix equation is obtained as:

$$\Lambda^d_{n-r} W^T_{n-r} - W^T_{n-r} A_{22} = \Phi^T_{n-r} A_{12} \tag{4.24}$$

where $W_{n-r} = [w_{r+1}, ..., w_n]$ are the left eigenvectors of the $n - r$ order matrix $A_{22} - LA_{12}$, and $\Phi_{n-r} = [\xi_{r+1}, ..., \xi_n]$, $\xi_j = -L^T w_j$, $j = r+1, ..., n$. So

$$L = -W^{-1}_{n-r} \Phi^T_{n-r} \tag{4.25}$$

To simplify the solution of two Sylvester matrix equations (4.20) and (4.24) coupled with (4.22) and (4.23), [Alp96] used the result of [Dua93] to parameterize V_r, Ψ_r W_{n-r} and Φ_{n-r} as:

$$\Psi_r = [z_1, z_2, ..., z_r] \tag{4.26}$$

$$V_r = [S_1 z_1, S_2 z_2, ..., S_r z_r] \tag{4.27}$$

$$\Phi_{n-r} = [z_{r+1}, z_{r+2}, ..., z_n] \tag{4.28}$$

$$W_{n-r} = [R_{r+1} z_{r+1}, R_{r+2} z_{r+2}, ..., R_n z_n] \tag{4.29}$$

where $S_i = (\lambda^d_i I - A)^{-1} B$, z_i is a vector of length m of the form $[z_{i1}, z_{i2}, ..., z_{i(m-1)}, 1]^T$, for $i = 1, 2, ..., r$, $R_j = [CAT_2(\lambda^d_j I - \bar{C}AT_2)^{-1}]^T$, z_j is a vector of length r in the form $[z_{j1}, z_{j2}, ..., z_{j(r-1)}, 1]^T$, for $j = r+1, r+2, ..., n$. Further, the vectors z_i and z_j satisfy the algebraic conditions:

$$z_i^T M_{ij} z_j = 0 \text{ for } i = 1, 2, ..., r \text{ and } j = r+1, r+2, ..., n \tag{4.30}$$

where M_{ij} is an $m \times r$ matrix given by

$$M_{ij} = S_i^T(\bar{C}^T R_j + C^T) = \{C[I + AT_2(\lambda^d_j I - \bar{C}AT_2)^{-1})^{-1}\bar{C}](\lambda^d_i I - A)^{-1} B\}^T$$

Now we summarize the above pole placement procedure as the following. Solve the $r(n-r)$ equations (4.30) for the $r(m-1)$ free parameters in z_i and $(n-r)(r-1)$ free parameters in z_j, assemble Ψ_r and V_r as (4.26) and (4.27), then the output feedback gain F is computed using equation (4.21). For the existence of a solution, it is necessary that $r(m-1)+(n-r)(r-1) \geq r(n-r)$, i.e., $mr \geq n$; if $mr > n$ the remaining free parameters can be used for the eigenvectors; if $m+r > n$, the z_j can be chosen as a constant vector, and equations (4.30) become linear.

Leventides and Karcanias [LeK95] proposed a novel global asymptotic linearization method to place all the poles with static output feedback ($mr \geq n$), which is quite different from the state-space methods. Suppose the open-loop transfer function is $H(s) = C(sI-A)^{-1}B = N(s)D(s)^{-1}$, then the closed-loop polynomial may expressed by

$$p(s) = \det(I - FH(s)) \tag{4.31}$$

A degenerate point is a gain F which make the system not well-posed, i.e., $p(s) = 0$ for any s. They explored the properties of system degeneracy, and linearize the system around a degenerate gain. This method, however, might produce asymptotically infinite gains in the feedback.

4.2.3 Eigenvalue Assignment with Decentralized Feedback

Pole placement with architecture constrained (decentralized, etc.) static output feedback is still an open problem. Given a minimally-realized N-channel LTI plant model

$$\dot{x} = Ax + \sum_{i=1}^{N} B_i u_i$$
$$y_i = C_i x, \ \ i = 1, 2, ..., N \tag{4.32}$$

where $x \in R^n$, $u_i \in R_i^m$ and $y_i \in R_i^r$, design an N-channel decentralized feedback

$$u_i = F_i y_i, \ \ i = 1, 2, ..., N \tag{4.33}$$

to place n preselected self-conjugate poles.

Assignment conditions with decentralized feedback

If complex feedback is allowed, one *necessary* condition for arbitrary decentralized pole placement is known as

$$\sum_{i=1}^{N} m_i r_i \geq n \tag{4.34}$$

Unlike the centralized case, the above condition is not necessary for real feedback .

Tarokh [Tar89] used the rank of an $n \times \sum m_i r_i$ matrix E_D to check the sufficiency, where the kth row of E_D is formed from $CA^{k-1}B$ after removing the elements corresponding to the prescribed elements in constrained F. Lu *et al.* [LCT93] restated Tarokh's condition with a matrix of Kronecker products to check the possibility of partial pole assignability. However, Carotenuto [CFM01] showed that Tarokh's assertion is only true for complex decentralized feedback. Leventides and Karcanias [LeK98] independently obtained such a matrix E_D, and explored its relation with the gradients of the characteristic polynomial coefficients. They proved that

$$\text{rank}(E_D) = n \text{ and } \sum_{i=1}^{N} m_i r_i \geq n \tag{4.35}$$

are *sufficient* conditions for arbitrary pole assignment with decentralized *complex* static-output feedback.

Wang [Wan94] provided some sufficient conditions for arbitrary pole assignability with a decentralized *real* matrix:

$$dG(m_1, ...m_N, r_1...r_N) \text{ is odd and } \sum_{i=1}^{N} m_i r_i \geq n \tag{4.36}$$

$$\text{or} \qquad dG(m_1, ...m_N, r_1...r_N) \text{ is even, } \sum_{i=1}^{N} m_i r_i \geq n$$

$$\text{and } m_1 = m_2 = ... = m_N \text{ or } r_1 = r_2 = ... = r_N \tag{4.37}$$

where $dG(m_1, ...m_N, r_1...r_N)$ is the degree of product Grassmannian under Plucker-Segre embedding:

$$dG(m_1, ...m_N, r_1...r_N) = \frac{n_1! + ... + n_N!}{n_1!...n_N!} \prod_{i=1}^{N} m_i, \text{ where } n_i = \begin{pmatrix} m_i + r_i \\ r_i \end{pmatrix} - 1$$

Another sufficient condition for decentralized (real-matrix) pole assignment can be found in [LeK93].

Algorithms for decentralized feedback

Least-square-like approach ([ASC83] [PSS94]) and parametric decentralized method ([LCT93]) are also proposed for architecture-constrained partial eigenvalue assignment together with some eigenvectors. we will summarize the methods in the next section.

The reports we found to assign whole set of poles with static decentralized output feedback are [LRD85], [Tar89], [WiD92], and [LKL97]. [LRD85] and [Wi92] adopted continuation/homotopy method via Kronecker product and matrix. In [LKL97] homotopy method is also used with matrix E_D for the decentralized pole placement. Leventides *et al.* [LKL97] extended their early work in [LeK95] to asymptotically linearize the problem around a decentralized degenerate gain and place the poles.

In the author's view, one multistage nonlinear programming algorithm is also ready to implement: evaluate the eigenvalue sensitivity respect to the decentralized gain F, and so we can efficiently place the first complex conjugate eigenvalue pair (or real eigenvalue) with gradient-based methods, then make eigenvalue pair uncontrollable with the techniques in [FaO88], and assign the the rest of pole pair by pair (or one by one).

4.2.4 Other Research about Pole Placement

At the end of this section, it is also worthy to note other interesting aspects of pole placement research. Although mostly we would like to have no fixed modes, such that the system is arbitrarily pole-assignable. However, it is also possible to achieve the disturbance rejection via closed-loop fixed modes. Details can be found in the papers [MMD98] [DeM00]. In practice sometime only small number of poles are not ideal, so the methods for partial pole reassignment without changing the rest poles are also proposed in [DaS99] [DER00]. Note that in these two papers a second order description is used rather than the common first order sate-space description. On the contrast, most partial eigenvalue placement methods associated with eigenstructure assignment can't predict the other poles, even yield an unstable system. The worst reputation about pole assignment is the eigenvalue sensitivity. So robust pole assignment techniques are also presented by minimizing the eigenvalue sensitivity [KNV85], the norm or feedback gain [Var00], or by guaranteing some norm condition [ChL01]. We will discuss some details in next section.

4.3 Eigenstructure Assignment

As we stated in Section 4.1, eigenvalues can't uniquely define a system;eigenvectors also play an important role is the system response. So soon after eigenvalue placement was brought up, eigenstructure assignment was also proposed. In this section we will discuss four catalogs of algorithms: parametric method, protection method, projection method, and robust eigenstructure assignment. Full-state feedback, static output feedback and constrained (decentralized) feedback are all considered. The readers can also refer to the surveys in [ASC83] [Spu90] [Whi95] and the book [LiP98].

4.3.1 Freedom of Eigenstructure Assignment

As it well-known, eigenstructure assignment is an over-determined problem. The below two theorems originally contributed by Moore [Moo76], Kimura [Kim75], and Srinathkumar [Sri78] give the freedom of eigenstructure assignment with full-state and static-output feedback. The proof can also be found in the parametric algorithm later.

Theorem 4.3 Assume (A, B) is controllable, $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $\{\lambda_i\}_{i=1}^n$ is a self-conjugate set of distinct complex numbers. There exists a real $m \times n$ matrix F such that

$$(A + BF)v_i = \lambda_i v_i, \ i = 1, 2, ..., n \tag{4.38}$$

if and only if, for each i

a) vectors of $\{v_i\}_{i=1}^n$ are linearly independent set in C^n;

b) $v_i = v_j^*$ if $\lambda_i = \lambda_j^*$;

c) $v_i \in \text{span}\{N_{\lambda_i}\}$, where N_{λ_i} is the compatible partition of the null space of $[\lambda_i I - A \ B]$.

Also, if F exists and $\text{rank}(B) = m$, then F is unique.

Theorem 4.4 Given a controllable and observable plant $(A, B, C, 0)$, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $\text{rank}(B) = m$, $C \in R^{r\times n}$, and $\text{rank}(C) = r$. Then $\max(m, r)$ closed-loop eigenvalues can be assigned and $\max(m, r)$ eigenvectors (or left eigenvectors) can be partially assigned with $\min(m, r)$ entries in each vector arbitrary chosen using real static output feedback.

4.3.2 Parametric Method for Eigenstructure Assignment

Fahmy and O'Reilly ([FaO82] [FaO83] [Fa85]) developed a parametric method for eigenstructure assignment using **full-state feedback**.

Assume the closed-loop eigenvalues are distinct. With the feedback $u = Fx$, we have:

$$(A + BF)v_i = \lambda_i v_i, \;\; i = 1, 2, ..., n$$

Rearranging the above equation, we write

$$(\lambda_i I_n - A)v_i = BFv_i, \;\; i = 1, 2, ..., n$$

Define $z_i = Fv_i \in C^m$. If the closed-loop eigenvalues $\{\lambda_i\}_{i=1}^n$ are different from the open-loop eigenvalues, then $(\lambda_i I_n - A)$ is invertible, and

$$v_i = (\lambda_i I_n - A)^{-1} B z_i, \;\; i = 1, 2, ..., n \tag{4.39}$$

and the feedback gain is obtained as

$$F = [z_1, z_2, ...z_n][(\lambda_1 I_n - A)^{-1} B z_1, (\lambda_2 I_n - A)^{-1} B z_2, ..., (\lambda_n I_n - A)^{-1} B z_n]^{-1} \tag{4.40}$$

Equations (4.39) and (4.40) are the parametric expression of eigenstructure assignment with full-state feedback, in which the z_i are the free parameters used to select the appropriate eigenvector from m-dimensional subspace (so-called eigenspace) $\text{span}\{(\lambda_n I_n - A)^{-1} B\}$. When determining the z_i, we should ensure that $\{v_i\}_{i=1}^n$ be self-conjugate, which can be guaranteed if we partition the eigenstructure into real and imaginary parts. With preselected poles, the free z_i's parameters can be used to achieve other objectives, such as minimal norm [FaO83b]. Fahmy and O'Reilly also treated the more general cases [FaO85] where multiple eigenvalues or some closed-loop eigenvalues coincide with open-loop ones.

O'Reilly and his coworkers [RoO87] [FaO88] further extended the parametric approach to **static output feedback**. Given an nth order controllable and observable plant $(A, B, C, 0)$, $B \in R^{n\times m}$, $\text{rank}(B) = m$, $C \in R^{r\times n}$ $\text{rank}(C) = r$. Below is the approach to assign r eigenvalues and the corresponding right eigenvectors with r m-dimensional vector parameters. Let the open-loop characteristic polynomial be $\Delta_o(\lambda)$ and adjoint matrix be $\Psi(\lambda)$

$$\Delta_o(\lambda) = \det(\lambda I_n - A)$$

$$\Psi(\lambda) = \text{adj}(\lambda I_n - A)$$

The closed-loop characteristic polynomial can be written as

$$\det(\lambda I_n - A - BFC) = \det[\Delta_o(\lambda) I_m - FC\Psi(\lambda)B]/\Delta_o^{r-1}(\lambda) \qquad (4.41)$$

Assume the desired self-conjugate closed-loop eigenvalues $\{\lambda_i\}_{i=1}^r$ are distinct with each other and different from the open-loop eigenvalues. Then

$$\det[\Delta_o(\lambda_i) I_m - FC\Psi(\lambda_i)B] = 0, \ i = 1, 2, ..., r \qquad (4.42)$$

So for some non-null m-dimensional vector z_i, we have

$$FC\Psi(\lambda_i)Bz_i = \Delta_o(\lambda_i)z_i, \ i = 1, 2, ..., r$$

Therefore, under mild conditions:

a) $\det[C\Psi(\lambda_i)Bz_1, ..., C\Psi(\lambda_i)Bz_r] \neq 0$;

b) $z_i \in R^m$ if $\lambda_i \in R$; $z_i = z_j^* \in C^m$ if $\lambda_i = \lambda_j \in C$;

the partial-eigenstructure assignment problem can be parameterized as

$$v_i = \Psi(\lambda_i)Bz_i, \ i = 1, 2, ..., r \qquad (4.43)$$

$$F = [\Delta_o(\lambda_1)z_1, ..., \Delta_o(\lambda_r)z_r][C\Psi(\lambda_i)Bz_1, ..., C\Psi(\lambda_i)Bz_r]^{-1} \qquad (4.44)$$

where z_i are free parameter vectors.

In [FaO88] the case of less than r eigenvalues and right eigenvectors to be assigned is handled by partitioning the gain F and output matrix C. The procedure of assigning m eigenvalues and the associated right eigenvectors with m r-dimensional parametric vectors is just the dual of the above. Partial right eigenvectors and left eigenvectors can also be assigned successively with a protection modification [FaO88] using output feedback.

Lu *et al.* [LCT93] extended the parametric approach to partial-eigenstructure assignment using **decentralized feedback**, and obtained some nonlinear equations. For the case where all eigenvalues $\{\lambda_i\}_{i=1}^k$ are distinct from each other and are different from the open-loop poles, they concluded that the assignment of eigenvalues $\{\lambda_i\}_{i=1}^k$ and the corresponding eigenvectors can be parameterized with k m-dimensional vectors

$$v_i = (\lambda_i I_n - A)^{-1} B z_i, \ i = 1, 2, ..., k \tag{4.45}$$

$$\begin{aligned} F_j &= E_j[z_1, ..., z_k][C_j(\lambda_1 I_n - A)^{-1} B z_1, ... C_j(\lambda_k I_n - A)^{-1} B z_k]^+, \\ & \quad j = 1, 2, ..., N \end{aligned} \tag{4.46}$$

under the constraints:

a) $z_i \in R^m$ if $\lambda_i \in R$; $z_i = z_j^* \in C^m$ if $\lambda_i = \lambda_j \in C$;

b) the matrix $[C_j(\lambda_1 I_n - A)^{-1} B z_1, ... C_j(\lambda_k I_n - A)^{-1} B z_k] \in C^r_{j \times k}$ is full rank, $j = 1, 2, ..., N$;

c) $E_j[z_1, ..., z_k][C_j(\lambda_1 I_n - A)^{-1} B z_1, ... C_j(\lambda_k I_n - A)^{-1} B z_k]^\perp = 0, \ j = 1, 2, ..., N$, where E_j is an $m_j \times m$ matrix formed from the rows of $m \times m$ identity matrix $I_m = [E_1^T, E_2^T, ..., E_N^T]$, M^+ and $M^\perp$ are defined as

$$M^+ = \begin{cases} M^T(MM^T)^{-1}, & \text{if } M \in C^{p \times q}, \ p < q \\ (M^T M)^{-1} M^T, & \text{if } M \in C^{p \times q}, \ p \geq q \end{cases}$$

$$M^\perp = I_q - M^+ M$$

To finish the procedure, one can use any method to solve the nonlinear equations (c) for $m \times k$ parameters in z_i. So Lu's parametric approach is somewhat cumbersome.

4.3.3 Projection Method for Eigenstructure Assignment

Shapiro and his coworkers [ShC81] [ASC83] [SSA94] [PSS94] developed a projection method for eigenstructure assignment. They also proposed a least-squares-like approximation for architecture constrained feedback.

As stated in Theorem 4.3 and Theorem 4.4, only some elements of the eigenvector v_i can be selected arbitrarily. In general the desired eigenvector v_i^d will not reside in the eigenspace $\text{span}(N_{\lambda_i})$, where the columns of $[N'_{\lambda_i},\ \star]'$ form the basis for null space of $[\lambda_i I_n - A\ B]$. (If λ_i is not the eigenvalue of A and B is full column rank, N_{λ_i} can be chosen as $N_{\lambda_i} = (\lambda_i I_n - A)^{-1}B$.) The *best achievable* eigenvector v_i is the projection of v_i^d onto the eigenspace:

$$v_i = \arg \min_{v_i = N_{\lambda_i} z_i} \|v_i^d - v_i\|^2 \tag{4.47}$$

Solving this least square problem, the m-dimensional vector z_i and the best achievable eigenvector v_i can be obtained as

$$z_i = (N_{\lambda_i}^T N_{\lambda_i})^{-1} N_{\lambda_i}^T v_i^d \tag{4.48}$$

$$v_i = N_{\lambda_i} z_i = N_{\lambda_i} (N_{\lambda_i}^T N_{\lambda_i})^{-1} N_{\lambda_i}^T v_i^d \tag{4.49}$$

In many practical applications, complete specification of v_i^d is not required or known. Rather, the designer is interested in certain elements of the eigenvector. Projection methods were also proposed in [ASC83] for the *partial specifications of* v_i^d. Pick the k specified elements of v_i^d and arrange them as a k-dimensional vector $\bar{v}_i^d$. Also pick the rows of N_{λ_i} corresponding the specified elements of v_i^d and reorder them as $k \times m$ matrix $\bar{N}_{\lambda_i}$. Proceeding in the same manner as before, we can obtain the the m-dimensional vector z_i and the best achievable eigenvector v_i as

$$z_i = (\bar{N}_{\lambda_i}^T \bar{N}_{\lambda_i})^{-1} \bar{N}_{\lambda_i}^T \bar{v}_i^d \tag{4.50}$$

$$v_i = N_{\lambda_i} z_i = N_{\lambda_i} (\bar{N}_{\lambda_i}^T \bar{N}_{\lambda_i})^{-1} \bar{N}_{\lambda_i}^T \bar{v}_i^d \tag{4.51}$$

Once z_i and v_i are obtained, Equation (4.40) or (4.44) can be used to construct the full-state feedback controller $u = Fx$ and output feedback controller $u = Fy$, or we can use the procedure described in [ASC83]. To avoid ill-condition in computation, singular value decomposition is suggested [SSA94], which we will also see in the decentralized case below.

Shapiro and his coworkers ([ASC83] [PSS94]) also extended the projection method to eigenstructure assignment with architecture constrained gain, such as **decentralized control**. However, unlike the previous projection method which was able to assign the eigenvalues precisely and eigenvectors approximately, this approach can't even guarantee the eigenvalues. Assume $\{\lambda_i^d\}_{i=1}^k$ and $\{v_i^d\}_{i=1}^k$ are the desired self-conjugate eigenvalues and the corresponding desired eigenvectors, $k \leq n$. Then

$$(A + \sum_{j=1}^{N} B_j F_j C_j) v_i = \lambda_i v_i, \;\; i = 1, 2, ...k \tag{4.52}$$

Writing it in matrix form, we have

$$M\Lambda - AM = BFCM = B \begin{bmatrix} F_1 C_1 M \\ \vdots \\ F_N C_N M \end{bmatrix} \tag{4.53}$$

where Λ is the $k \times k$ diagonal matrix diag$\{[\lambda_1^d, ..., \lambda_k^d]\}$, $M = [v_1^d, ..., v_k^d] \in C^{n \times k}$, and F is the $m \times r$ decentralized (real) gain matrix diag$\{[F_1, F_2, ..., F_N]\}$. We can write the $n \times m$ matrix B and $r_j \times k$ matrix $C_j M$ in the form of singular value decomposition:

$$\begin{aligned} B &= [U_0 \; U_1] \begin{bmatrix} \Sigma V^T \\ 0 \end{bmatrix} = U_0 \Sigma V^T \\ C_j M &= [U_{j0} \; U_{j1}] \begin{bmatrix} \Sigma_j V_j^T \\ 0 \end{bmatrix} = U_{j0} \Sigma_j V_j^T, \text{ for } k \leq r_j \\ \text{or } C_j M &= [U_j \Sigma_j \; 0] \begin{bmatrix} V_{j0}^T \\ V_{j1}^T \end{bmatrix} = U_j \Sigma_j V_{j0}^T, \text{ for } k > r_j \end{aligned}$$

Thus

$$M\Lambda - AM = \begin{bmatrix} U_0 \Sigma V^T F_1 \hat{U}_1 \Sigma_1 \hat{V}_1^T \\ \vdots \\ U_0 \Sigma V^T F_N \hat{U}_N \Sigma_N \hat{V}_N^T \end{bmatrix} \tag{4.54}$$

where $\hat{U}_j = U_{j0}$ for $k \leq r_j$ or $U_j fork > r_j$, and $\hat{V}_j = V_j$ for $k \leq r_j$ or V_{j0} for $k > r_j$. Note that equation (4.54) is generally over-determined. Partition $M\Lambda - AM$ into N matrices with dimension $m_j \times k$, and denote them as $(M\Lambda - AM)_j$. Now we can obtain the projection-like solution

$$F_j = V\Sigma^{-1} U_0^T (M\Lambda - AM)_j \hat{V}_j \Sigma_j^{-1} \hat{U}_j^T, \; j = 1, 2, ...N \tag{4.55}$$

I call this approach as 'projection-like' method, because it is different from the the projection method mentioned before. It is not the least-square solution of $\min_{F_j} \|M\Lambda - (A + \sum_{j=1}^{N} B_j F_j C_j)M\|^2$, rather it is just the solution to $\min_{F_j} \|(M\Lambda - AM)_j - BF_j C_j M\|^2$, $j = 1, 2, ...N$. Thus this approximation can't guarantee the eigenvalues even when the desired eigenstructures are achievable. However, since the eigenstructure assignment is generally redundant, this concise projection-like method might be acceptable in some application.

4.3.4 Robust Eigenstructure Assignment

The fatal disadvantage of eigenstructure assignment is sensitivity. So robust eigenstructure assignment has been proposed to reduce the closed-loop eigenvalue sensitivity [CNK84] [KNV85] [MuP88] [PaL94]. Suppose v_i and w_i, $i = 1, 2, ..., n$, are the right and left eigenvectors of the closed-loop $A_c = A + BF$. It is known that the sensitivity of the eigenvalue λ_i depends on the condition number c_i:

$$\frac{d\lambda_i}{d\|A_c\|} \propto c_i = \frac{\|w_i\|_2 \|v_i\|_2}{|w_i^T v_i|} \tag{4.56}$$

so minimizing the condition number c_i will reduce the sensitivity of λ_i and increase the stability robustness of eigenstructure assignment. A bound upon the individual eigenvalue sensitivities is given in [KNV85] as

$$\max_{i=1,...n} c_i \leq \kappa(V) = \|V\|_2 \|V^{-1}\|_2 \tag{4.57}$$

where $\kappa(V)$ denotes the condition number of the right eigenvector matrix $[v_1, v_2, ..., v_n]$. Thus an iterative process to minimize $\kappa(V)$ is proposed [KNV85] to choose the eigenvector from achievable eigenspace and make it maximally orthogonal to the column space of $V_i = [v_1, ..., v_{i-1}, v_{i+1}, ...v_n]$. Performing the QR decomposition of V_i, we can obtain

$$V_i = [Q_i, y_i] \begin{bmatrix} R_i \\ 0 \end{bmatrix}$$

and the $n \times 1$ vector y_i is orthogonal to column space of V_i, then project y_i onto the achievable eigenspace, a new eigenvector v_i is obtained. The procedure is then repeated until no more improvement in the condition number $\kappa(V)$. The control gain F is obtained as Equation (4.40).

This technique was extended to output feedback by performing the QR decomposition of B and C [CNK84]. Liu and Patton [LiP98b] presented a parametric optimization method for low-sensitive and robust eigenstructure assignment with output feedback. Eigenstructure assignment with matrix-family modelling uncertainty [Dua92] [PSS94] was also proposed. In [ABG93] pole placement in a sector with uncertain an A matrix was examined with Laypunov-type approach. Chilali *et al.* [CGA99] investigated the robust pole placement in LMI regions under the plant uncertainty with H_∞ bounded. They used LMI tools to solve for the full-state or full-order controller.

4.3.5 Protection Method for Eigenstructure Assignment

Fletcher and Magni ([Fle81], [FlM87]) developed the protection method for eigenvalue assignment with full-state and static output feedback. This method is further developed in [Alp96] [SoM98]. Fahmy and O'Reilly [FaO88] extended it to eigenvector projection, so that it can also be used for eigenstructure assignment.

First one eigenvalue λ_i is selected and the parameter vector z_i is also chosen to pick the corresponding right eigenvector v_i from the eigenspace N_{λ_i}, $v_i = N_{\lambda_i} z_i$. It can be shown easily that for any m-dimensional vector t_i satisfying

$$t_i^T C z_i = 1$$

the below unity-rank gain matrix F assigns eigenvalue λ_i and the eigenvector v_i

$$F^{(i)} = v_i t_i^T \tag{4.58}$$

For a matrix $\bar{C}^{(i)}$ satisfying

$$\bar{C}^{(i)} C z_i = 0$$

the closed loop system with gain $F^{(i)}$ is $(A + BF^{(i)}C, B, \bar{C}^{(i)}C, 0)$. This augmented system has an unobservable eigenvalue λ_i. This means that the eigenvalue λ_i is protected. Note that the dimension of the output matrix $\bar{C}^{(i)}C$ is reduced to $(m-1) \times n$. Dually, we can make the eigenvalue λ_j uncontrollable in the augmented system $(A + BF^{(j)}C, B\bar{B}^{(j)}, C, 0)$. This procedure is repeated for another eigenvalue. The final gain matrix can be obtained by combining the separate gain matrices by superposition, passed through the protection matrices $\bar{C}^{(i)}$ and $\bar{B}^{(i)}$.

We should know, for an uncontrollable eigenvalue, we still can change the associated right eigenvector with feedback. Same is true for the left eigenvector associated an unobservable eigenvalue. In [FaO88] it is shown that the right (left) eigenvector associated with the unobservable (uncontrollable) eigenvalue is invariant with output feedback. Thus, in the augmented system $(A + BF^{(i)}C, B, \bar{C}^{(i)}C, 0)$, not only λ_i is protected (uncontrollable), the right eigenvector v_i is also protected (but w_i is not); similar result can be obtained for λ_j and w_j in $(A + BF^{(j)}C, B\bar{B}^{(j)}, C, 0)$. In [FaO88] p ($p \leq r$ eigenvalues and the associated right eigenvectors are assigned first, and with protection method, some other eigenvalues and the associated left eigenvectors are assigned. One potential problem is that when we protect some eigenvectors, the others might also be protected, and the assignable number decreases.

In recent years some methods were also developed to eigenvalue or eigenstructure assignment via properly chosen of the weighting matrices in LQR ([SZC90] [LuL95] [Sug98] [ChS99] [BoF99]). This is the so-called **inverse LQR problem**. Luo and Lan [LuL95] used that fact that the n closed-loop poles are among $2n$ Hamiltonian matrix eigenvalues, and developed a method for determining the weighting matrices of LQG to produce specified closed-loop eigenvalues. Choi and Seo [ChS98] examined the projection eignstruture assignment and the Riccati equation of full-state feedback

LQG, and concluded that they result in the same gain under some conditions. Sugimoto [Sug98] partitioned the right coprime factor as a two factors, one of which is used to place m poles exactly and he other is use to guarantee LQG optimality with full-state feedback by choosing some weighting Q. Genetic algorithms were used in [BoF99] to find the weighting Q and R so as to minimizing the eigenvalue sensitivity while placing poles in some internals. Kawaski and Shimemura [KaS83] proposed a procedure to choose weighting matrix Q of LQG so as to place the closed-loop pole in the $\pm\pi/4$ sector with full-state feedback, and this procedure is further simplified by Shieh *et al.* [SZC90] to place the pole in the $\pm\pi/2k$ sector ($k \geq 2$) through solving Lyapunov or Ricatti equations. We should mention that the inverse LQR problem is not limited to eigenvalue or eigenstructure assignment. Some other performance can also be achieved by weighting selection in quadratic optimal control, such as mixed H_2/H_∞ ([ZhS90]).

4.4 Regional Pole Placement

In practice, the exact positions of ideal poles are not known or necessary. Rather, as we do in classical control design, we would like to specify them in some region in the right half plane, so as to guarantee the some performance, such as rise time, settling time and overshot. So regional pole placement makes more sense sometimes. The shaded region in Figure 4-1 is ideal in most applications. To simplify the algorithms, different regions are utilized to approximate the region shown in Figure 4-1, such as vertical strips, disks, sectors, ellipses, or parabolic regions.

Anderson and Moore [AnM90] showed before 1970 that the poles of the closed-loop real matrix A_c reside in the left side of the vertical line $\mathrm{Re}(s) = -\alpha$ if and only if $A_C + \alpha I$ is stable. Or we say that there exists some $n \times n$ positive definite symmetric matrix P such that

$$(A_C + \alpha I)'P + P(A_C + \alpha I) < 0 \tag{4.59}$$

In the control literature, this is sometimes called α-stability.

Furtuta and Kim [FuK87] investigated circle-region pole placement of continuous

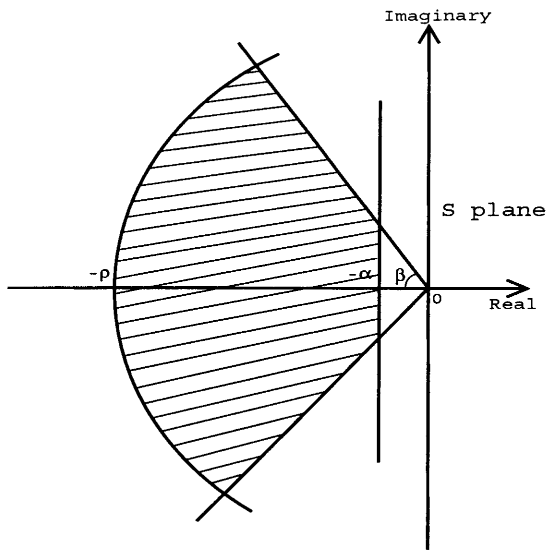

Figure 4-1: Pole regional constraints.

and discrete systems and concluded that the poles of a real matrix A_c reside in a disk $D(\alpha, \rho)$ with radius r and center at $-\alpha + 0j$ if and only if there exists some $n \times n$ positive definite matrix P such that

$$\alpha A_c' P + \alpha P A_c + A_c P A_c + (\alpha^2 - \rho^2) P < 0 \tag{4.60}$$

They also obtained a discrete Ricatti equation for full-state feedback, which is easy to understood with the fact that the poles of stable discrete systems is in unit circle region. This is called D-stability in the control literature. (In some literature, the term 'D-stability' is also used to general pole-region stability.)

Aderson *et al.* [ABJ75] also show that the poles of A_c reside in a sector $\pm\beta$ if and only if $\begin{bmatrix} A_c\sin\beta & A_c\cos\beta \\ -A_c\cos\beta & A_c\sin\beta \end{bmatrix}$ is stable, or we say there exist some $2n \times 2n$ positive definite matrix $\bar{P}$ such that

$$\begin{bmatrix} A_c\sin\beta & A_c\cos\beta \\ -A_c\cos\beta & A_c\sin\beta \end{bmatrix}' \bar{P} + \bar{P} \begin{bmatrix} A_c\sin\beta & A_c\cos\beta \\ -A_c\cos\beta & A_c\sin\beta \end{bmatrix} < 0 \tag{4.61}$$

And in [ChG96] it was shown that the condition (4.61) is equivalent to that there exists some $n \times n$ positive definite symmetric matrix P such that

$$\begin{bmatrix} (A_c P + P A_c')\sin\beta & (A_c P - P A')\cos\beta \\ (P A_c - A_c P)\cos\beta & (A_c P + P A_c')\sin\beta \end{bmatrix} < 0 \tag{4.62}$$

Similar description for Ellipse and Parabolic region can be found in the paper [BSU94]. Chilali and Gahinet [ChG96] generalized the above characterization using linear matrix inequality (LMI regions), including not only above typical regions, but also any convex polygonal regions symmetric with real axis. Further they showed that the intersection of any LMI regions can be characterized using the same matrix P without conservation.

Theorem 4.5 The real matrix A_c has all its eigenvalues in the LMI region

$$\{s | s \in C : [l_{ij} + m_{ij}s + m_{ij}s^*]_{i,j} < 0\}$$

if and only if there exist a symmetric $P > 0$ such that

$$[l_{ij}P + m_{ijA_c'P+m_{ij}}PA_c]_{i,j} < 0 \tag{4.63}$$

The region $S(\alpha, \rho, \beta)$ in Figure 4-1 is one example of the LMI regions. The poles of A_c reside in $S(\alpha, \rho, \beta)$ if and only if there exists an $n \times n$ symmetric matrix $P > 0$ such that

$$\begin{gathered} A_cP + PA_c' + 2\alpha P < 0 \\ \begin{bmatrix} -\rho P & A_cP \\ PA_c' & -\rho P \end{bmatrix} < 0 \\ \begin{bmatrix} (A_cP + PA_c')\sin\beta & (A_cP - PA')\cos\beta \\ (PA_c - A_cP)\cos\beta & (A_cP + PA_c')\sin\beta \end{bmatrix} < 0 \end{gathered} \tag{4.64}$$

and LMI techniques [GNL95] can be utilized to assign poles in the LMI regions with full-state feedback directly. The above equations can also be written in the form of the generalized Lyapunov equation [BSU94], so Lyapunov/Ricatti-based methods can be used for the case of full state feedback.

Keerthi and Phatak [KeP95] described a class of pole regions with equalities and described poles with equalities, then used homotopy method to find the architecture constrained static gain.

Another great advantage of regional pole placement is that we have more freedom to specify other performance, such as H_2/LQG and H_∞. This is multi-objective control associated with pole regional placement. The examples are, H_2/LQG optimization with regional pole placement ([HaB92] [Seh93] [SKK93] [Mis96] [YAJ96]), H_∞ control with regional pole placement ([PaL94] [YeL95] [ChG96]), and mixed H_2/H_∞ with regional pole placement [BSU94] [FND97] [CGP99]. More details will be discussed in next chapter.

4.5 Maximize the Minimal Damping via Pole Shifting

In this section a subgradient-based minimax method is adopted to shift the poles so as to maximize the minimal damping.

4.5.1 Introduction and Problem Statement

As we know, arbitrary eigenvalue assignment requires some conditions, and regional pole placement offers more flexibility if such requirements are not met. However, it still faces a similar problem: the preselected region might be unassignable. Also, the LMI characterization of the pole region is not convenient for structured control, such as static output feedback or decentralized feedback. In addition, it is easy to understand that traditional regional placement is conservative, since the transition performance of a system is dominated by some part of poles, mostly the low-frequency poles. Therefore, in this section we propose a subgradient-based approach to shift the critical poles into some sector region, and make the sector region as small as possible. The physical meaning is to maximize the minimal damping of the poles in some frequency range. This is extremely helpful for the structured control, where the controller structural constraints might limit the flexibility of pole treatment too much. Another advantage is that this approach can also handle unstable or marginal stable closed-loop system, which is impossible in the framework of H_2 or H_∞. Further, practical questions, such as nonnegative parameters and hysteric damping are considered.

The problem can be stated as:

$$\begin{aligned} & \max_{F\in\Omega}(\min_{i\in I}(\zeta_i(F))) \\ & I = \{i|\omega_l \leq \omega_i \leq \omega_h,\, \omega_i = \mathrm{Im}(|\mathrm{eig}(A + B_2FC_2)|)\} \end{aligned} \tag{4.65}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{r\times n}$, Ω is the set of real or complex matrices F with prescribed structure constraints (such as decentralized), and internal $[\omega_l, \omega_h]$ is the preselected frequency bandwidth.

This is a nonsmooth optimization problem, which have been examined extensively after Polyak(1969) and Dem'yanov (1974). As we stated in Chapter 2, gradient-based algorithms converge much faster than Simplex or Powell algorithms in constrained or unconstrained smooth optimization. However $\min_{i\in I}\xi_i(F)$ is a nonsmooth function, so we can't use the well-known conjugate gradient or BFGS algorithm. For nonsmooth optimization, the subgradient or ϵ-subgradient plays an important role similar to the gradient in smooth optimization. So in the following we will give some introduction to convexity, the subgradient, and non-smooth optimization, then adopt the minimax algorithm ([Dem74]) to solve the problem (4.65).

4.5.2 Convexity and Subgradient

Below are some concepts from nonsmooth optimization.

Convex Set: Let S be a subset of R^n, we say that S is convex if

$$\alpha x + (1-\alpha)y \in S,\ \forall x, y \in S, \forall\alpha \in [0,1] \tag{4.66}$$

A *convex hull* is a special convex set. Convex hull of X, denoted as $\mathrm{conv}(X)$, is the set of all convex combinations of the elements of X. In particular, if X consists of a finite number of vectors, $X = \{x_i\}_{i=1}^m$, then its convex hull is closed and can be expressed as

$$\mathrm{conv}\{x_i|i = 1, 2, ..., m\} = \left\{\sum_{i=1}^{m}\alpha_i x_i|\alpha_i \geq 0, i = 1, 2, ..., m, \sum_{i=1}^{m}\alpha_i = 1\right\} \tag{4.67}$$

Projection: Let S be a closed convex set, then for every x, there is a unique vector x^+ in S, such that

$$x^+ = \arg\min_{z\in S}\|x - z\| \tag{4.68}$$

x^+ is called the *projection* of x onto S. This definition is identical to

$$(x - x^+)'(y - x^+) \leq 0, \ \forall y \in S \tag{4.69}$$

With this definition, we can see the projection of a vector x onto a convex hull $\text{conv}\{x_i | i = 1, 2, ..., m\}$ is just a least square problem with convex constrain:

$$\begin{aligned} x^+ &= \arg \min_{\alpha_i, i=1,2,\ldots,m} \|\sum_{i=1}^{m} \alpha_i x_i - x\| \\ &\text{s.t.} \ \sum_{i=1}^{m} \alpha_i x_i = 1, \ \alpha_i \geq 0 \end{aligned} \tag{4.70}$$

Such problems can be solved using a standard code, such as the function `lsqlin` from *Matlab Optimization Toolbox.*

Convex Function: Let S be a convex set in R^n, a function $f : S \rightarrow R$ is said to be convex if

$$f[\alpha x + (1 - \alpha)y] \leq \alpha f(x) + (1 - \alpha)f(y), \ \forall x, y \in S, \forall \alpha \in [0, 1] \tag{4.71}$$

If $f(x)$ is convex on S, the $-f(x)$ is concave on S. If $f_i(x) : S \rightarrow R$ is convex, the index $i \in I$, and S is convex, then $\max_{i \in I} f(x)$ is convex, and $\min_{i \in I} f(x)$ is concave. One important property of a convex function are that local minima of convex function are global.

Subgradient: Given a convex function $f : R^n \rightarrow R$, a vector d is called a *subgradient* of f at x if

$$f(z) \geq f(x) + d'(z - x), \forall z \in R^n \tag{4.72}$$

The set of all subgradients of f at x, denoted by $\partial f(x)$, is called the **subdifferential** of f at x. It is a nonempty, convex and compact set. Figure 4-2 shows the subdifferential of a scalar function

$$f(x) = \max(-2x + 2, -0.5x + 1, x - 2)$$

It is easy to see that if $f(x)$ is differentiable at x_0, then $\partial f(x_0) = \{\nabla_x f(x_0)\}$. The the relation of the subdifferential $\partial f(x)$ and the *directional derivative* $f'(x; y)$ is:

$$f'(x; y) := \lim_{\alpha \searrow 0} \frac{f(x + \alpha y) - f(x)}{\alpha} = \max_{d \in \partial f(x)} y'd, \ \forall y \in R^n \tag{4.73}$$

From (4.73) it can be seen that x^* is the optimal point if and only if

$$0 \in \partial f(x^*) \tag{4.74}$$

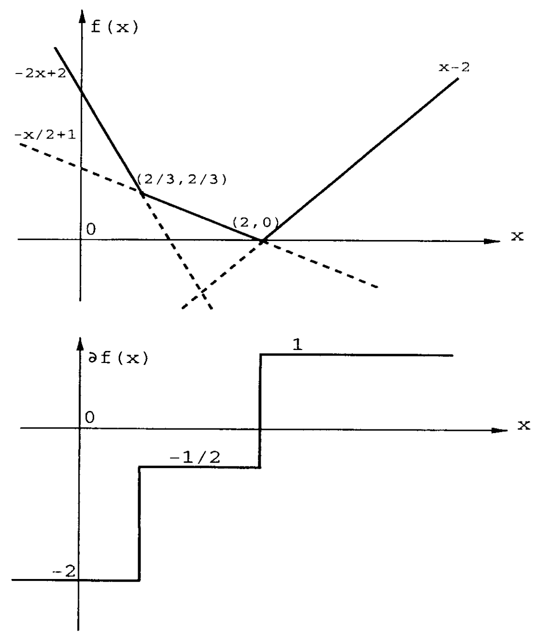

Figure 4-2: Example of subgradient

ε-Subgradient: Given a convex function $f : R^n \to R$, for a scalar $\varepsilon > 0$, we say that a vector d is an *ε-subgradient* of f at x if

$$f(z) + \varepsilon \geq f(x) + d'(z - x), \forall z \in \Re^n \tag{4.75}$$

The set of all ε-subgradients of f at x, denoted by $\partial_\varepsilon f(x)$, is called the **$\varepsilon$-subdifferential** of f at x. With this definition it can be shown that ϵ-subgradient has the following properties:

$$\inf_{\alpha>0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha} = \max_{d \in \partial_\epsilon f(x)} y'd \tag{4.76}$$

$$0 \in \partial_\epsilon f(x) \text{ if and only if } f(x) \leq f(x^*) + \epsilon \tag{4.77}$$

In [Dem74] Dem'yanov called such a point satisfying (4.77) as an ϵ-stationary point.

Similarly, we can define the subgradient and ϵ-subgradient for a concave function.

4.5.3 Nonsmooth Optimization and Minimax Theories

From (4.72) and (4.73), we can see that any subgradient is a descent direction for convex function $f : R^n \to R$. So *arbitrarily selected* subgradients can yield a descent sequence. A general convex function over a convex domain $M \in R^n$ has a similar property.

Key Lemma [Ber01]: Consider the minimization of a nonsmooth convex function $f(x)$ over a closed convex set $M \in R^n$ and the subgradient method:

$$x^{k+1} = [x^k - \alpha^k d^k]^+ \tag{4.78}$$

where d^k is an arbitrary subgradient $d^k \in \partial f(x^k)$, α^k is a positive scalar stepsize, and $[\cdot]^+$ denotes the projection onto the set M. Assume $\|d^k\| \leq \gamma, \forall k$. Then for all $y \in M$ and k,

$$\|x^{k+1} - y\|^2 \leq \|x^k - y\|^2 - 2\alpha^k[f(x^k) - f(y)] + (\alpha^k)^2\gamma^2 \tag{4.79}$$

The Key Lemma hints that if the stepsize α^k is small enough, the distance of x^{k+1} to x^* is improved, but the cost function might not improve. (If $M = R^n$, then the cost also improves.) Based on the Key Lemma, three typical stepsize rules are provided: constant stepsize, diminishing step size, or dynamic step size [AHK87] [Ber01]. For a small constant stepsize $\alpha^k = \alpha$, if

$$0 < \alpha < \frac{2(f(x^k) - f^*)}{\gamma^2}$$

the sequence will tend to a level set of f^* with maximal error $\alpha\gamma^2/2$. For a dimimin-ishing stepsize

$$\alpha^k \to 0, \ \sum_k \alpha^k = \infty$$

then $\lim_{k\to\infty} f(x^k) = f^*$. For a dynamic stepsize

$$\alpha^k = \frac{f(x^k) - f_k}{\gamma^2}$$

where f_k is an estimate of f^*. If $f_k = f^*$, it makes progress at every iteration; if $f_k < f^*$, it tends to oscillate around the optimum; if $f_k > f*$, it tends to a level set of f^* with maximal error $f_k - f^*$.

The above is subgradient-based optimization for nonsmooth convex functions. The interesting thing is that any subgradient can be used as a search direction, which is convenient for the case where the whole set subdifferential is hard to evaluate. However, for the nonsmooth convex function $\max_{i\in I} f_i(x)$ or concave function $\min_{i\in I} f_i(x)$, where I is a finite index set, it is easy to find the whole set subdifferential.

Theorem 4.6 (Danskin's Theorem, [Ber01]): If $f_i(x) R^n \rightarrow R$ is smooth for all $i \in$I, then the subdifferential of f(x) = $\max\limits_{i\in I}$ f$_i$(x) is

$$\partial f(x) = \text{conv}\{\nabla_x f_j(x) | j \in \bar{I}(x)\} \tag{4.80}$$

where $\bar{I}(x) = \{i | f_i(x) = \max\limits_{i\in I} f_i(x)\}$, and $\nabla_x f_j(x)$ is the gradient of f_j at x.

If $0 \in \partial f(x)$, then the optimal point of

$$\min_{x\in R^n} \max_{i\in I} f_i(x)$$

is obtained; otherwise, a special subgradient can be obtained by projection of 0 onto $\partial f(x)$: $\arg \min\limits_{d\in\partial f(x)} ||d(x)||$.

Dem'yanov [Dem74] showed that this special subgradient has some important property.

Theorem 4.7: The necessary condition of a continuous nonsmooth (not necessarily convex) function $f(x)$: $R^n \rightarrow R$ to attain a minimum at x^*–sufficient also if $f(x)$ is convex–is that $0 \in \partial f(\ x^*)$ if $0 \notin \partial f(x)$. Then the direction

$$- \arg \min_{d\in \partial f(x)} ||d(x)||$$

is the steepest descent direction.

Theorems 4.6 and 4.7 suggest an approach for standard minmax problems (or max-min problems): Start with an arbitrary initial point x_0, evaluate $\partial f(x_0)$, update x_0 in the direction of $- \arg \min\limits_{d\in\partial f(x_0)} ||d(x_0)||$ with one dimensional optimization, then repeat this procedure. One might expect the limit point of this descent sequence to be a minimal point of the nonsmooth function. However, due to the lack of smooth, other than the phenomena of zip-zag, the limit point of the sequence generated with above algorithm based on Theorem 4.6 may not even be a stationary point of $f(x)$, please see the example in [Dem74] p.74. Thus an ϵ-subgradient algorithm is suggested for minimax problems.

Theorem 4.8: Assume $\varepsilon > 0$, $I_\varepsilon = \{i | \max_{i\in I} f_i(x) - f_i(x) < \varepsilon\}$, and $\partial_\varepsilon f(x) = \text{conv}\{\nabla_x f_j(x) | j \in I_\varepsilon(x)\}$ is not empty. Then the searching direction $-\arg\min_{d\in\partial_\varepsilon f(x)} ||d(x)||$ and one-dimension minimizing step size will yield a sequence whose limit point is an ε-stationary point of $f(x)$, which is an approximation to a stationary point with the absolute error at most ε.

4.5.4 Minimax Pole-Shifting Algorithm

With the above background of nonsmooth optimization, let's come back to our problem (4.65): determine the structural constrained feedback gain F to maximize the minimal damping in a certain frequency bandwidth. Our algorithm is based on Theorems 4.6 and 4.8. To evaluate the gradient of jth mode damping $\nabla_F \xi_j(F)$, we need to introduce an eigenvalue sensitivity theorem [Cal86] [MaG97].

Eigenvalue Sensitivity: Given a real-coefficient dynamic system $\dot{x} = (A + BFC)x$, the sensitivity of the jth eigenvalue λ_j to changes in the klth element of F is

$$\frac{\partial \lambda_j}{\partial F_{kl}} = \frac{w_j' b_k c_l v_j}{w_j' v_j} \tag{4.81}$$

where v_j and w_j are the jth right and left eigenvector of $A+BFC$, respectively, b_k is the kth column of B, and c_l is the lth row of C.

Since A, B, C and F are real matrices, the eigenvalues of $A + BFC$ are symmetric with respect to the real axle in complex plane. We need only consider the poles and the associated damping in the second quadrant. The damping ratio

$$\zeta_j(F) = \frac{-Re(\lambda_j)}{|\lambda_j|}$$

so we can write the damping sensitivity with the chain rule:

$$\frac{\partial \zeta_j}{\partial F_{kl}} = (-|\lambda_j|^{-\frac{1}{2}} + \text{Re}(\lambda_j)^2 |\lambda_j|^{-\frac{3}{2}}) \text{Re}(\frac{\partial \lambda_j}{\partial F_{kl}}) + \text{Im}(\lambda_j)\text{Re}(\lambda_j)|\lambda_j|^{-\frac{3}{2}})\text{Re}(\frac{\partial \lambda_j}{\partial F_{kl}}) \tag{4.82}$$

Now we can set of the procedure as below:

Step1: Choose the initial parameters–a block diagonal matrix F.

Step2: Solve for steepest descent subgradient $drt(F)$:

Evaluate the eigenvalues and eigenvectors of $A + BFC$, find the set $I_\varepsilon(F) = \{i|\zeta_i(F) = \min_{j\in I} \zeta_j(F)\}$, and evaluate $\nabla_F \zeta_j(F)$ with respect to the free design variables in F for all $j \in I_\varepsilon(F)$ using (4.81) and (4.82). We obtain a convex set $\partial_\varepsilon \zeta(F)$. Solve the problem $\min\limits_{d\in\partial_\varepsilon\zeta(F)} ||d(F)||$ using Equation (4.70) and obtain the steepest descent subgradient $\mathrm{drt}(F) = -\arg \min\limits_{d\in\partial_\varepsilon\zeta(F)} ||d(F)||$. If $\mathrm{drt}(F) = 0$, stop; otherwise go to step 3.

Step3: One-dimensional minimum:

Search in the direction $\mathrm{drt}(F)$, get the stepsize a which maximizes the function $\min\limits_j \zeta_j(F + \alpha \cdot \mathrm{drt}(F))$, $j \in I$, and update F with $F + \alpha \cdot drt(F)$. Then go to step 2.

Remark 1: Generally, gradient-based methods are not finitely convergent. So typically we can stop computation when $||\mathrm{drt}(F)||$ or $\alpha||\mathrm{drt}(F)||$ becomes sufficient small, not necessarily $\mathrm{drt}(F) = 0$.

Remark 2: To make the approach more practical, we also can maximize the weighted minimal damping in a selected frequency range.

Remark 3: In practical design of passive mechanical systems, we would like to ensure that the parameters be nonnegative.In this case, we can replace F_{kl} with F_{kl}^2, and replace the right side of equation (4.81) by $2F_{kl}\frac{w_j' b_k c_l v_j}{w_j' v_j}$, then we can get all nonnegative parameters. More generally, if we would like to constrain some parameter F_{kl} to be in some internal $[r_1, r_2]$, we can specify F_{kl} with one parameter r:

$$0.5(r_1 + r_2) + 0.5(r_2 - r_1)\sin r$$

and replace the right side of equation (4.81) as

$$F_{kl} = 0.5(r_2 - r_1)\cos r \frac{w_j' b_k c_l v_j}{w_j' v_j}$$

Remark 4: For the systems with hysteretic damping, the matrices A, B and F are complex. There are seldom reports for design of this kind system, although it is important in practice. We can extend the eigenvalue sensitivity theorem to the case of complex coefficients:

$$\frac{\partial \lambda_j}{\partial \mathrm{Re}(F_{kl})} = \frac{w_j' b_k c_l v_j}{w_j' v_j}, \qquad \frac{\partial \lambda_j}{\partial \mathrm{Im}(F_{kl})} = i \frac{w_j' b_k c_l v_j}{w_j' v_j}$$

where i is $\sqrt{-1}$, and the other variables are the same as in Theorem 4.1. Then we can get the modal damping sensitivity $\frac{\partial \zeta_j}{\partial \mathrm{Re}(F_{kl})}$ and $\frac{\partial \zeta_j}{\partial \mathrm{Im}(F_{kl})}$ to changes in the real part and imaginary part of the free design parameters in F.

The above is the subgradient-based approach to shift some dominant poles into a sector region and make this region as small as possible. It's useful in the design for some performance with structure constrained feedback. This gradient-based approach can also be extended to general pole regional placement with structured feedback gain.

4.6 Application to the Design of Mechanical Systems

In this section we will give some application of the decentralized eigenstructure assignment, and the minimax pole-shifting method to the design of mechanical systems. First we apply above two approaches to real-coefficient static decentralized control (2DOF TMD). Then provide an example of a marginally-stable system with a complex matrix feedback (hysteretic damping). Finally we compare the three techniques of decentralized H_2, decentralized H_∞, and decentralized minimax in the example of the five-mass system introduced in Chapter 3.

4.6.1 Example 1, Decentralized Eigenstructure Assignment and Minimax

A two-DOF primary system that can translate in the x direction and rotate about the z axis as shown in Figure 4.6.1. Our task is to choose k_1, k_2, c_1, and c_2 to damp the two modes of the main system. As we mentioned in Chapter 1, the design of this mechanical system can be cast as a decentralized control problem, where the feedback

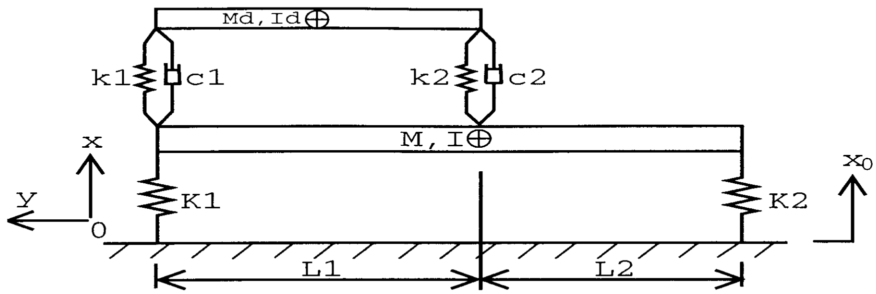

Figure 4-3: Sketch of the two-DOF system with tuned-mass damper (TMD): $L_1 = 0.25$ m, $L_2 = 0.2$ m, $M = 5$ kg, $I = 0.1$ kg·m^2, $K_1 = 50$ kN/m, $K_2 = 80$ kN/m, $M_d = 0.05M$, $I_d = 0.035I$

gain is

$$F = \begin{bmatrix} k_1 & c_1 & 0 & 0 \\ 0 & 0 & k_2 & c_2 \end{bmatrix}$$

One intuitive approach to design an MDOF TMD is to match the modal shapes and frequency of the damper (with the main mass fixed) and the main system (without damper), so that more energy will be dissipated in resonance. Thus we can propose one procedure as: (1) choose the ideal undamped eigenstructure of TMD; (2) calculate the equivalent mass ratios for each mode via total modal energy; (3) obtained the ideal damped eigenvalues by analogy with Den Hartog's formula for SDOF TMD; 4) assign the ideal damped eigenstructure. In the above procedure, Step (1) is important and depends on the designer's experience, Steps (2) and (3) is straightforward, and Step (4) is decentralized eigenstructure assignment. One possible method for Step (1) is to use the modal output of the main mass system at the point of the damper cg while setting the damper mass to zero. The available approaches for decentralized eigenstructure assignment are Lu's [LCT93] and Shapiro's [PSS94]. As we have seen, Lu's parametric method [LCT93] is cumbersome in computation, and Shapiro's projection-like approximation [PSS94] is not even in least square sense. Consider the over-determination of eigenstructure assignment, below we will directly assign the eigenstructure via nonlinear programming with Shapiro's approximation as the start point.

Suppose v_i^d and λ_i^d, $i = 1, 2, 3, 4$, are the ideal self-conjugate (normalized) eigenvectors and eigenvalues of the system with main mass fixed.

$$\min_{F} \sum_{i=1}^{4} [\alpha_i \|v_i - v_i^d\|^2 + \beta_i(\lambda_i/\lambda_i^d - 1)^2] \tag{4.83}$$
$$\text{s.t. } (A + BFC)v_i = \lambda_i v_i, i = 1, 2, 3, 4$$

where α_i and β_i are the weighting coefficients, and A, B, C is the system matrices with the main mass fixed. In this example the above procedure yield the closed-loop damping ratio 1.52%, 1.53%, 2.41%, and 2.58%. For brevity we omit the details.

In this mechanical problem, although we know intuitively that the eigenstructure of the damper should be close to that of the main mass, the true ideal eigenstructure is still unknown. And the proper choice of the weighting matrix in (4.83) requires several trials. So maximizing the minimal damping is the straightforward algorithm for this problem. With initial guess [500,50,0,0; 0,0,500,50] the minimax algorithm converges to

$$F = \begin{bmatrix} 6036 & 11.7 & 0 & 0 \\ 0 & 0 & 2678 & 5.9 \end{bmatrix}$$

and produces a system with the four modes:

Mode	Damping Ratio (%)	Frequency (Hz)
1	8.77	24.24
2	8.77	24.24
3	13.50	36.66
4	8.77	40.71

We can see that it is much better than the result of eigenstructure assignment. The bode plots of the transitions from ground vertical input x_0 to x and θ of the main mass Cg are shown in Figure 4-4.

4.6.2 Example 2, Eigenvalue Treatment for a Marginally-Stable System

Now consider planar vibration of the free-free tube beam shown in Figure 4-5. The bending stiffness EI is 1.636×10^5 Nm^2, the mass per unit length is 23.245 kg/m,

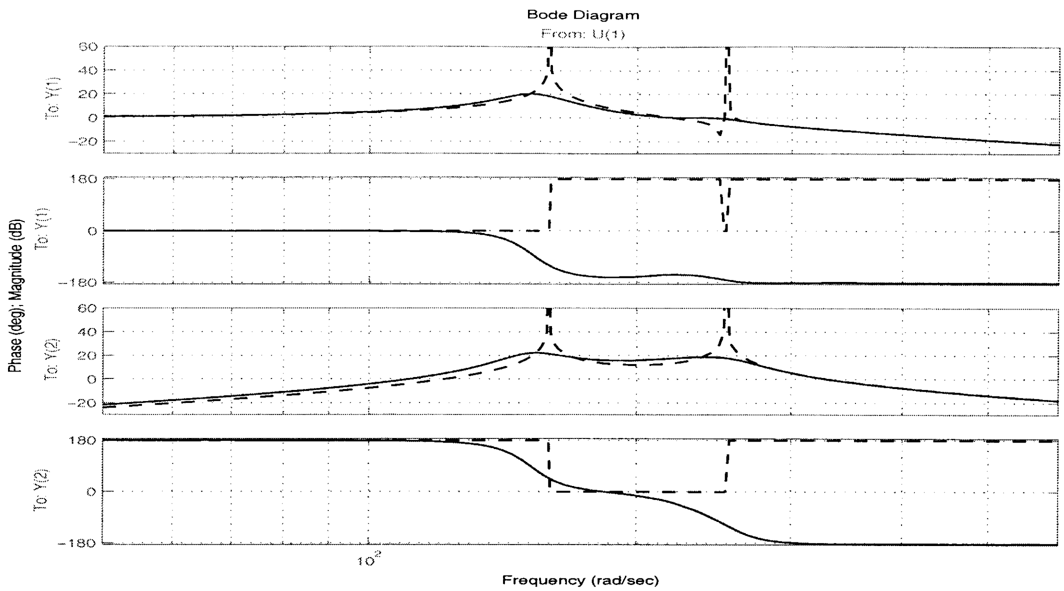

Figure 4-4: Bode plots of transmission from ground vertical input x_0 to x and θ of the main mass

and the rotational inertia is 6.997×10^{-2} kgm. We are required to damp the first three flexural modes. In the following we will provide three designs: traditional multiple SDOF TMDs, minimax multiple SDOF TMDs, and a multi-DOF TMD. The advantages of the minimax algorithm and multi-DOF TMD are highlighted. Complex feedback (hysteretic damping) is employed in this example.

The beam is a distributed parameter system, and we discretize it into twelve segments with thirteen nodes. Each of the node has three degrees of freedom of vibration in plane.

The fist setup is a 3DOF TMD, as shown in Figure 4-5. Attached to the end of beam is a small rigid block whose mass is 4% of that of the beam and whose lenght is 10% of the length of the beam. It is mounted with three flexures 55.4mm from the neural axis of the beam.

We cast the problem as an 82nd-order system with decentralized static output feedback. Using the minimax algorithm and eigenvalue sensitivity, we maximize the minimal damping ratios of first three modes with a weighting 1:1.1:1.6, and obtain the optimal parameters for both hysteretic and viscous 3DOF TMDs:

The corresponding damping ratios achieved by hysteretic and viscous 3DOF TMDs

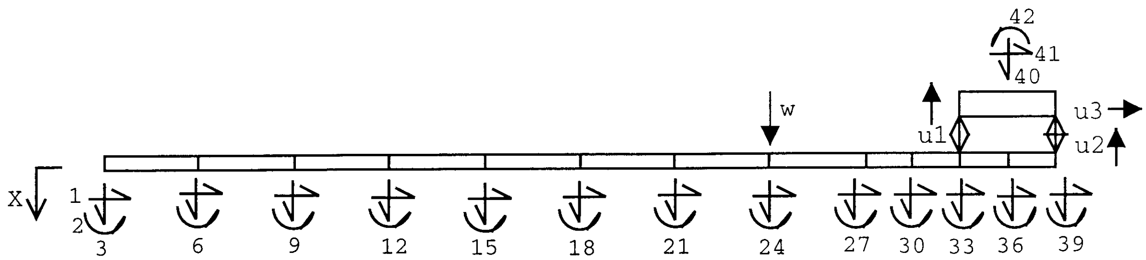

Figure 4-5: 3-DOF TMD for a 39-DOF discretized free-free beam.

Table 4.1: Optimal parameters of 3DOF TMD via minimax algorithm

i	Hysteretic 3DOF TMD Stiffness k_i (N/m) & Lost Factor η_i		Viscous 3DOF TMD Stiffness k_i (N/m) & Damping c_i (Ns/m)	
1	1.468×10^5	0.4420	1.5256×10^5	125.42
2	9.308×10^5	0.4187	9.8195×10^5	248.17
3	1.548×10^7	0.3066	1.5800×10^7	1556.6

are listed in Table 4.2: One typical frequency response (from disturbance force w to

Table 4.2: Damping ratio achieved by hysteretic and viscous 3DOF TMDs

mode	hysteretic		viscous	
1	10.67%	10.67%	10.81%	10.81%
2	9.70%	9.71%	9.82%	9.83%
3	6.67%	6.67%	6.80%	7.11%

the local position output in the x direction) is show in Figure 4-6. We can see that the performance with hysteretic and viscous TMDs are almost the same. Usually structural material has viscoelastic property, and can be characterized as hysteretic model; while most fluid dampers show ideal viscous property.

We note that Zhang et al [ZMH89] developed a two-DOF cantilever-type TMD to damp the first two modes of a railway wheel; they used a transfer function to get

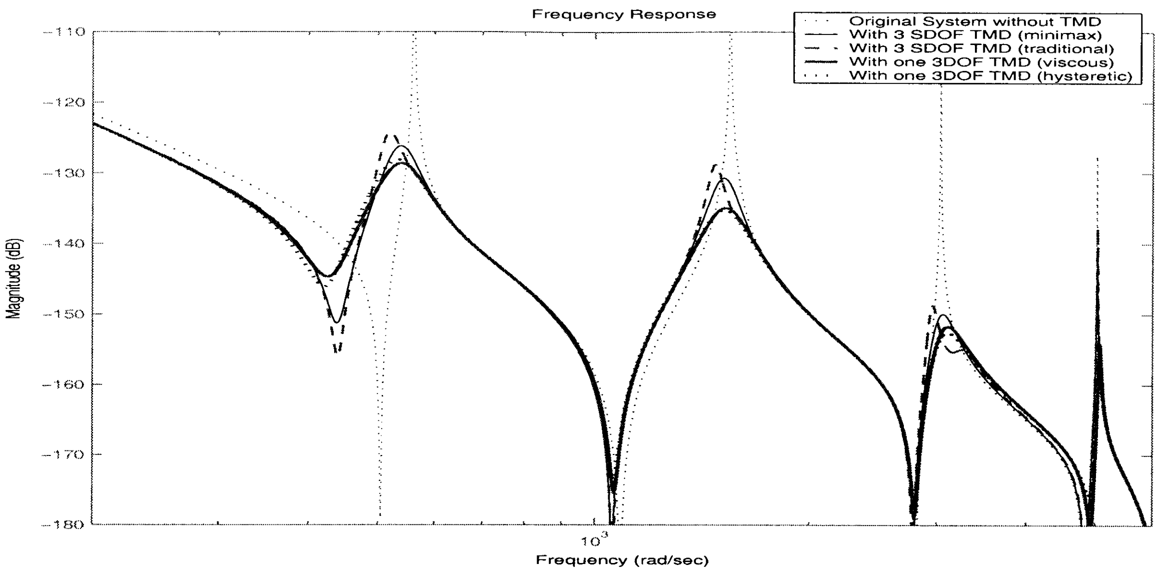

Figure 4-6: Frequency Responses of the TMD systems

an equivalent mass ratio, and obtained tuning parameters by analogy with an SDOF primary system. To the authors' knowledge, this is the only study where both the damper and main system have more than one coupled degree of freedom. Zhang's approach is hard to generalize; while the minimax optimization based on decentralized control make the design much easier and more general.

Currently, in practical design, multiple SDOF TMDs are used to damp more than one mode, and each SDOF TMD is independently tuned to a target mode by taking the main system as an equivalent SDOF system [SMT97]. As a comparison, we design three SDOF TMDs. The setup is shown in Figure 4-7, and each small mass is 1.33% of that of the beam.

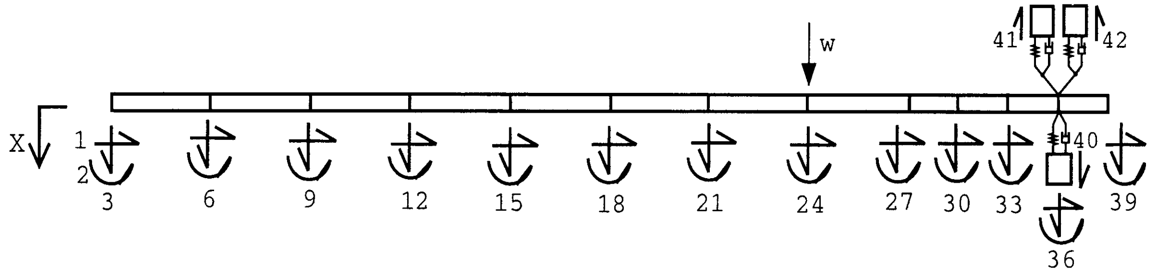

Figure 4-7: Three SDOF TMDs for a 39-DOF discretized free-free beam.

The procedure for traditional design is quite standard:

(1) Compute the equivalent SDOF main mass of the beam at the connection point j via the modal energy relation proposed in [SOY87] for the kth mode, $k = 1, 2, 3$:

$$M_{eq}^{(k)} x_j^2 = \sum_i M_i x_i^2$$

where x_j is the displacement of the connection point, M_i is the generalized mass (including mass or rotational inertia), and x_i is the displacement of i-th mass in the kth mode.

(2) Ignore the coupling of the other modes, and design the parameters for each mode based on the equivalent SDOF main mass with mass ratio $\mu^{(k)} = m_k / M_{eq}^{(k)}$, use the well-known SDOF TMD formula [Den47]:

$$f = \frac{1}{1+\mu}, \qquad \zeta = \sqrt{\frac{3\mu}{8(1+\mu)^3}} \tag{4.84}$$

where the tuning ratio f is defined as the ratio of the natural frequency of damper and that of the main mass, and ζ is the damping ratio of the damper.

From step (2), we can see that if the modes of main mass don't separate well at the connection points, this approach based on equivalent SDOF systems will not produce a good design. In this example, the equivalent mass ratios and the parameters obtained according to the above procedure are shown in Table 4.3.

Further we cast the system as a 82nd-order plant with decentralized feedback, and proceed to design via the minimax algorithm with the weighting 1:1.1:1.6. The optimal parameters obtained are also shown in Table 4.3.

The damping ratios achieved by the three designs are compared in Table 4.4:

From Table 4.4 and Figure 4-6, we can see that:

- The minimax algorithm yields TMDs with performance much better than the traditional design;

- MDOF TMDs have the capacity to damp more than one modes, and the performance is better than multiple SDOF TMDs. And decentralized control techniques provide a general and uniform framework for the design.

Table 4.3: Design of three SDOF TMDs to damp three modes

k	Undamped Modal Freq, (Hz)	Equivalent Mass $M_{eq}^{(k)}$, and $\mu^{(k)} = m_k/M_{eq}^{(k)}$	Traditional Design $k^{(k)}$ (N/m) $c^{(k)}$ (N · s/m)	Minimax Design $k^{(k)}$ (N/m) $c^{(k)}$ (N · s/m)
1	89.33	19.388Kg 2.924%	1.686×10^5 63.804	1.540×10^5 89.07
2	246.29	32.525Kg 1.743%	1.311×10^6 138.188	1.200×10^6 194.76
3	483.09	62.955Kg 0.900%	5.129×10^6 197.270	4.6250×10^6 323.78

Table 4.4: Comparison of the damping ratios achieved by three (viscous) design

Mode	Three SDOF TMDs Traditional Design	Three SDOF TMDs Minimax Design	One 3DOF TMD Minimax Design
1	3.38% 7.22%	6.78% 8.74%	10.81% 10.81%
2	2.44% 5.92%	6.17% 6.17%	9.83% 9.92%
3	1.52% 4.62%	4.38% 6.22%	6.79% 7.12%

Following similar procedure, we designed the parameters of two-DOF TMD to damp the first two nonzero modes of free-free beam, and implemented it in experiment. We designed spring-dashpot pairs in the form of flexures whose stiffness and damping are independently adjustable, as shown in Figure 4-9 and 4-9. In the experiment we hang the beam using latex tubing so as to approximate a free-free beam, and use an impact hammer and accelerometer to measure transfer functions. One typical transfer function (from force to acceleration at the end is shown in Figure 4-10, where we plot the predicted and measured responses with and without the damper. As one would expect, the system initially exhibits almost no damping, with $\zeta \approx 10^{-4}$ for each of the first three modes. With the damper installed and properly tuned, each

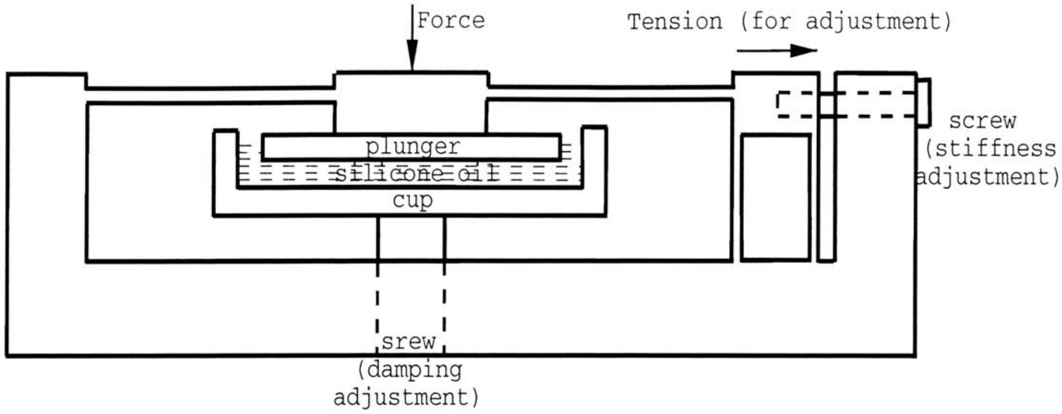

Figure 4-8: Flexure with adjustable stiffness and damping

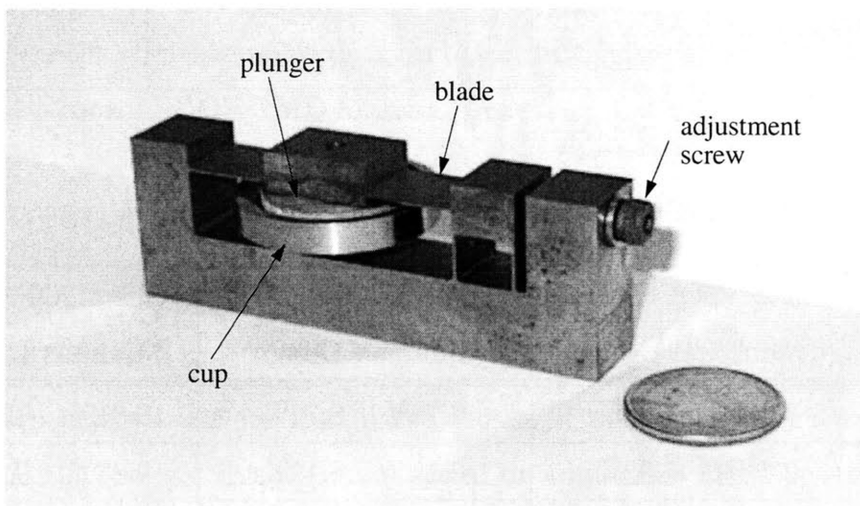

Figure 4-9: Photograph of a flexure with adjustable stiffness and damping

of the first two modes of the beam exhibit damping close to that predicted. More details can be found in the author's paper [ZuN02].

4.6.3 Case Study: Comparison of Decentralized H_2, H_∞, and Minimax

We will take the serial five-mass system discussed by Sipila *et al.* [SMP99] as an example, as shown in Figure 4-11.

In last Chapter 3 we gave the design of the system with H_∞. We further design the parameters k_2, c_2, k_4 and c_4 with decentralized H_2 and the minimax approach.

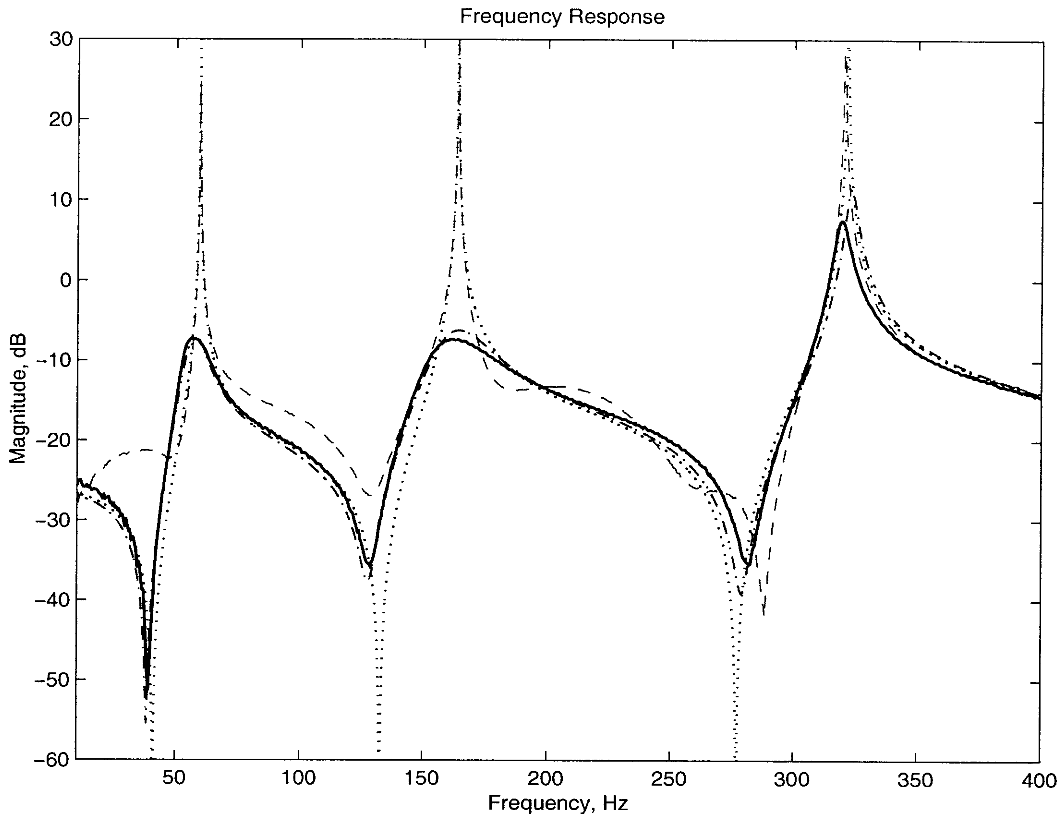

Figure 4-10: Force-to-acceleration frequency response at node 1 of the free-free beam: comparison of the predicted undamped response (dotted), measured undamped response (dashed), predicted damped response (dash-dot), and measured damped response (solid).

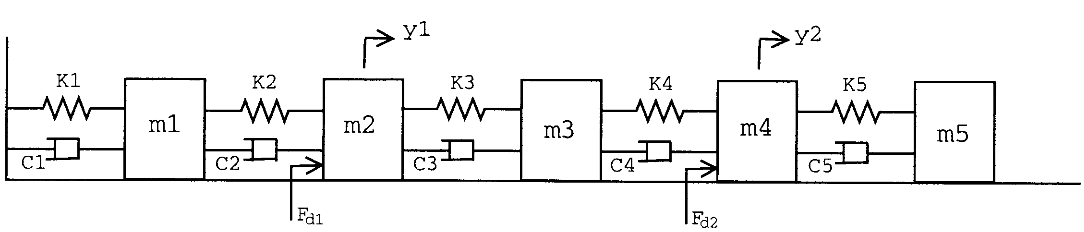

Figure 4-11: Diagram of the five-mass system.

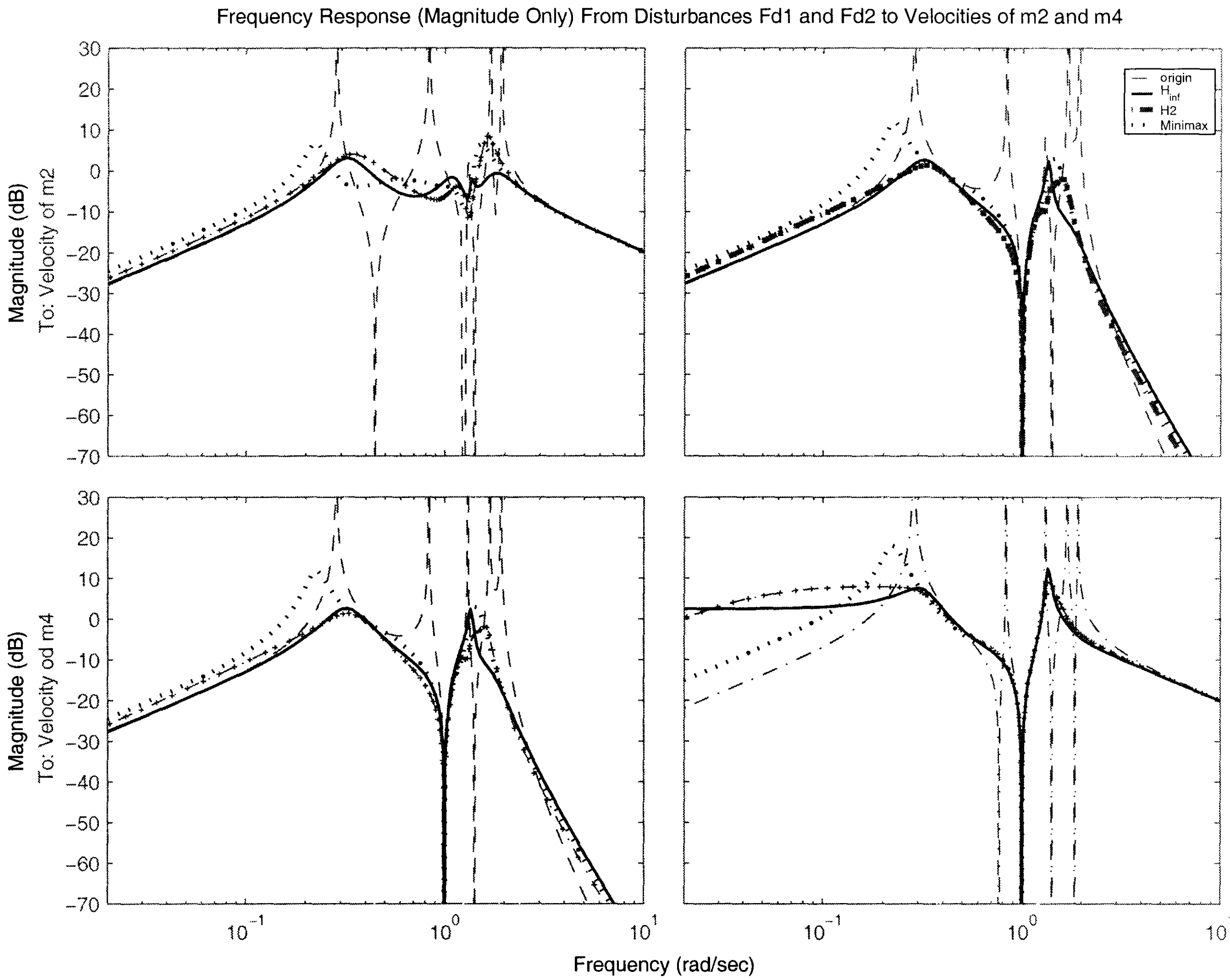

Figure 4-12: Frequency response of five-mass system.

Figure 4-12 shows the frequency response from each of the two inputs to each of the two outputs for each of the design methods. Figure 4-13 shows the impulse response of the closed-loop system designed with H_2, H_∞, and minimax approaches. Table 4.5 gives the resulting H_2, H_∞ norms obtained and minimal damping from the above three decentralized techniques

From Figures 4-12 and 4-13, we can see that the minimax design gives the fastest decay for each mode of excitation, but it takes no account of the system zeros and therefore produces a damping ratio of greater than 10 per cent in each mode. The H2

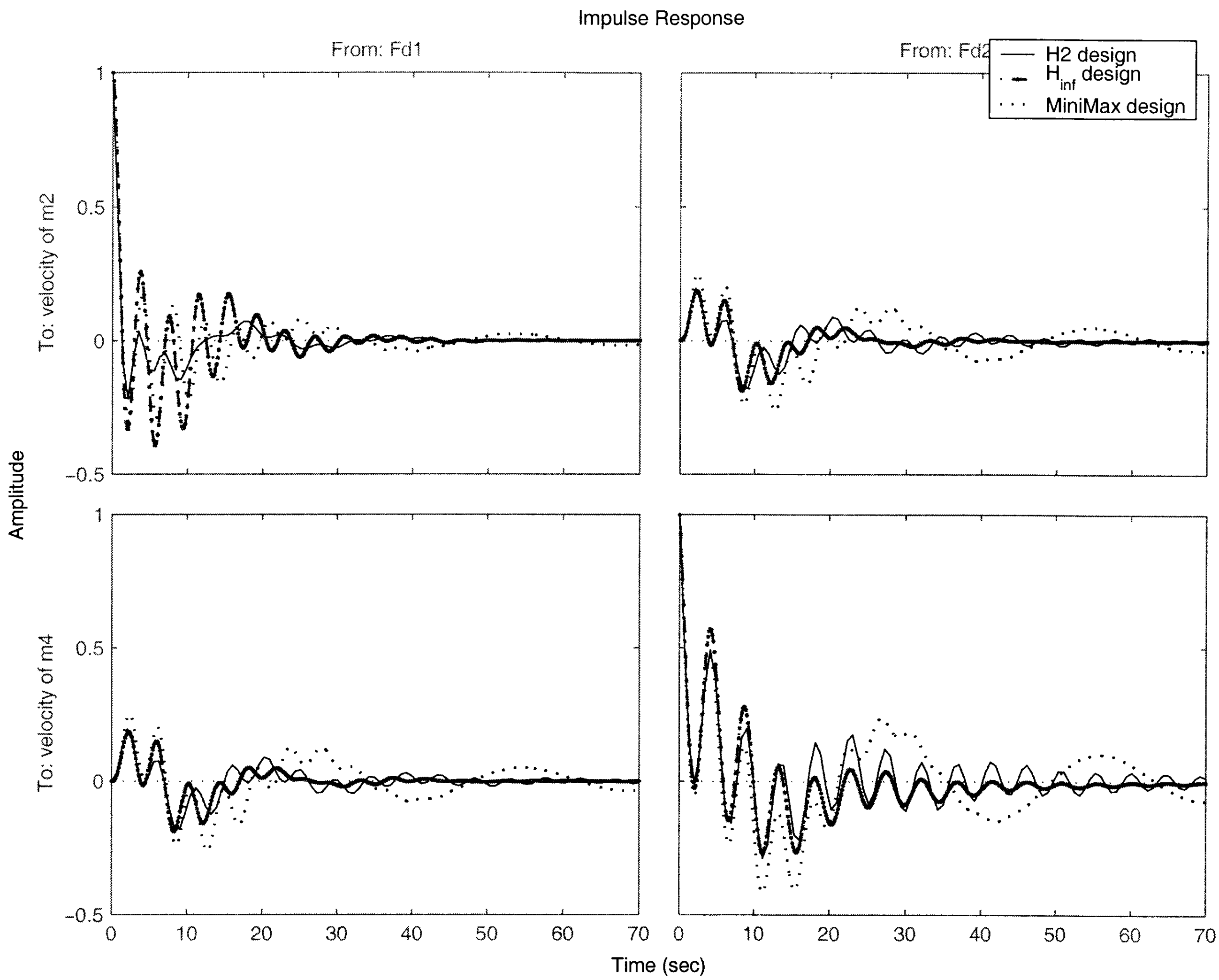

Figure 4-13: Impulse response of five-mass system.

Table 4.5: Results of the three optimization methods applied to the five-mass system

Method	H2 norm	H_∞ norm	minimal damping
H2	1.52	4.68	2.70%
H_∞	1.72	3.04	5.73%
MiniMax	2.058	10.02	10.9%

design has the lowest energy but disturbances diminish somewhat more slowly than in H_∞ design. Both the H2 and H_∞ design take advantage of the zeros of the system to minimize the effects of modes at certain positions; therefore they produce some lightly damped modes (with damping ratios as low as 2.7 per cent) as well as some overdamped modes.

Decentralized LQG/H2 optimization yields a very good design for suppression of white noise input, and the BFGS method is an efficient numerical algorithm for its solution. H_∞ optimization minimizes the amplitude of vibration under several sinusoidal frequency inputs since it minimizes the "peak" of the frequency response. And the minimax algorithm gives the parameters that ensure very good damping. The minimax algorithm can specify the design in some frequency range directly, while for H2 or H_∞ optimization we must use some dynamic shape filter as the weight to specify the performance in some frequency band or even might be impossible. (the TMD for free-free beam is such a example).

We should also note that there are some imperfect points for each method. It's not easy to choose the weights for LQG/H2 optimization (mostly we need to try). Static decentralized H_∞ optimization is NP-hard; there is no guarantee of convergence for the the LMI-based algorithms employed here, though they work well in practice. Like the steepest method for smooth optimization, the steepest-subgradient method doesn't converge very fast.

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Chapter 5

Multi-Objective Control

5.1 Introduction

As we have seen in Chapters 2, 3 and 4, H_2 or LQR, H_∞, and eigenvalue/eigenstructure treatment provide us three basic synthesis approaches. In the past decades, L_1 synthesis has also been investigated. H_2 design is based upon a stochastic noise disturbance model possessing a fixed covariance (power spectral density). Since it tries to minimize the energy of the impulse response of the closed loop—or system output variance with white noise input—H_2 design is particularly suited to meet some performance specification. H_∞ design is predicated on a deterministic disturbance model consisting of bounded power. It tries to minimize the worst-case attenuation regardless of the frequency. The H_∞ norm is also a natural tool to model plant (unstructured) uncertainty and thus H_∞ theory is suitable for practical robust design, in which H_2 with output feedback has been shown to be weak [BeH89]. (For structured uncertainties the H_∞ framework can be refined as μ-synthesis.) While H_∞ is concerned with the robust stability and frequency specification, it tells little about time-domain performance. Poles have an intuitive and direct connection with the transition characteristics of the closed loop, but pole placement might be too sensitive to parameter changes. L_1 optimal control can directly specify the signal magnitude in the time-domain, as well as the optimal rejection to the persistent bounded disturbance. But in the usual L_1 design, the controller order can be arbitrarily high [Hal00].

Obviously any practical design is a tradeoff between different and often conflicting objectives, such as robust stability, simultaneous rejection of disturbances with different characteristics (white noise, bounded energy or power, bonded magnitude), good tracking, finite capacity of actuators, and close-loop bandwidth. Therefore, in the past decade multi-objective control techniques have been developed, which might include two or more specifications on the system H_2 norm, H_∞ norm, L_1 induced norm, pole regional placement, or passivity, and so on. And various approaches has been used, such as linear and nonlinear programming, Riccati-based solutions, LMI convex optimization, and Youla parameterization.

Some pioneering work in this area are done by Bernsten and Haddad [BeH89] [HaB92], Dolye and Zhou ([DZB89] [ZGB94] [DZG94]), Khagoneckar ([KhR91] [SKK93], Stoorvogel [Sto93], Sznaier ([SHB95]), Chilali and Gahinet (ChG96], El Ghaoui [GhF96], and others. Brief survey can be found in [VrJ97]. Although multi-objective control has great potential in practical applications, most interesting problems in this area are still open, even for full-state or full-order feedback. There are also several reports attempting to treat static output feedback or even decentralized feedback.

In the following, we try to discuss the multi-objective control in the classification of H_2/H_∞, H_2/regional pole placement, H_∞ or H_2/H_∞/regional pole placement, and L_1 associated multi-objectives. Unstructured control (full-state feedback, full-order feedback) and structured control (static output feedback, lower-order control, and decentralized control) are covered. Other than a comprehensive survey, in this Chapter we extend the cone-complementary linearization algorithm [EOA97] to general multiobjective suboptimal control with static output feedback; we propose a new approach for decentralized H_2 with arbitrary pole regional constraints; and a tractable optimization problem is posed for decentralized H_2/H_∞/poles optimal control.

5.2 H_2/H_∞ Control

Mixed H_2/H_∞ has attracted great deal of attention. It is probably the most intuitive and direct way in many practical designs: to minimize the H_2 norm (optimal performance) under the constraint of H_∞ bound (robust stability).

Suppose the plant $G(s)$ is given as in Figure 5-1 with three sets of inputs and outputs,

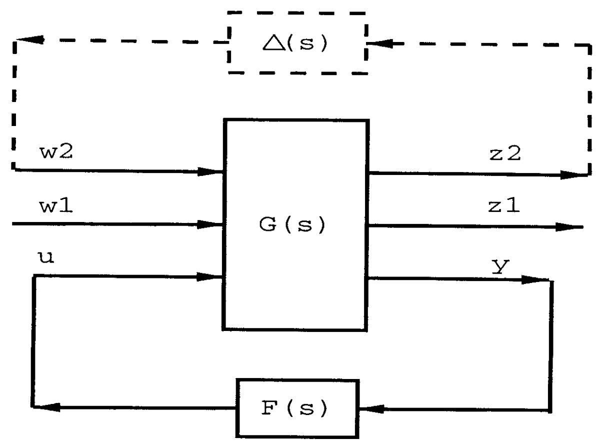

Figure 5-1: Multi-objective control configuration

where G(s) is given as

$$\dot{x} = Ax + B_1 w_1 + B_2 w_2 + B_3 u \tag{5.1}$$

$$z_1 = C_1 x + D_{11} w_1 + D_{12} w_2 + D_{13} u \tag{5.2}$$

$$z_2 = C_2 x + D_{21} w_1 + D_{22} w_2 + D_{23} u \tag{5.3}$$

$$y = C_3 x + D_{31} w_1 + D_{32} w_2 \tag{5.4}$$

and $x \in R^n$, $w_1 \in R^{p_1}$, $w_1 \in R^{p_2}$, $z_1 \in R^{q_1}$, $z_1 \in R^{q_2}$, $u \in R^m$, $y \in R^r$.

Without lose of generality, we have assumed $D_{33} = 0$. For the mixed H_2/H_∞ problem, the sets (z_1, w_1) are related to the H_2 performance, whereas $z_2, w_2)$ are related to the H_∞ requirement.

Various of approaches has been proposed for the mixed H_2/H_∞ problem for various configurations. (For example, some researchers set $w_1 = w_2$, and others set $z_1 = z_2$.) Below we will classify the approaches as the Bernsten-Haddad-type auxiliary cost optimization, LMI convex optimization, worst-case design, and Q-parameterization.

5.2.1 Bernsten-Haddad-Type Auxiliary Cost Optimization

Bernsten and Haddad [BeH89] did the pioneering work on mixed H_2/H_∞, where they addressed the synthesis of a strictly proper fixed-order controller to minimize an auxiliary cost which is the upper bound of the H_2 norm.

They considered the case $w_1 = w_2$, as in Figure 5-2,

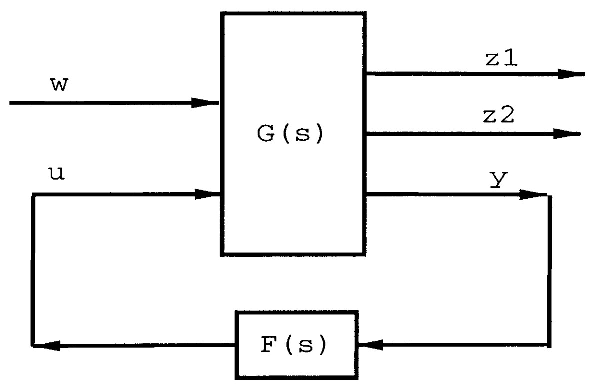

Figure 5-2: Mixed H_2/H_∞ control configuration I

$$G(s) := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{21} \\ C_2 & 0 & D_{22} \\ C_3 & D_{13} & 0 \end{array}\right]$$

where it is assumed D_{21} and D_{22} have full column ranks. Suppose the controller is kth order strictly-proper

$$K(s) := \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right]$$

then the closed-loop $w \longrightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C}_1 & \tilde{D}_1 \\ \tilde{C}_2 & \tilde{D}_2 \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} A & B_2C_k \\ B_kC_3 & A_k \end{bmatrix} & \begin{bmatrix} B_1 \\ B_kD_{13} \end{bmatrix} \\ \hline [C_1 \; D_{21}C_k] & 0 \\ [C_2 \; D_{22}C_k] & 0 \end{array}\right]$$

From Theorem 2.1, we know that the H_2 norm can be calculated as

$$||Hwz_1||_2^2 = J(A_k, B_k, C_k) = \text{trace}(\tilde{Q}\tilde{C}_1'\tilde{C}_1)$$

where $\tilde{Q}$ satisfies the Lyapunov equation

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}' + \tilde{B}_1\tilde{B}_1' = 0$$

Bernsten and Haddad [BeH89] showed that if there exists some nonnegative definite matrix Q satisfying the Ricatti equation

$$\tilde{A}Q + Q\tilde{A}' + \gamma^{-2}Q\tilde{C}_2'\tilde{C}_2Q + \tilde{B}_1\tilde{B}_1' = 0 \tag{5.5}$$

then $\tilde{Q} \leq Q$. Hence

$$||Hwz_1||_2^2 = \text{trace}(\tilde{Q}\tilde{C}_1'\tilde{C}_1) \leq \text{trace}(Q\tilde{C}_1'\tilde{C}_1) \tag{5.6}$$

using the controllability and observability Gramians, they further showed that

$$||Hwz_2||_\infty \leq \gamma \tag{5.7}$$

The Bernsten-Haddad-type auxiliary cost minimization, which will guarantee an upper bound on the H_2 norm $w \longrightarrow z_1$ subject to the requirement on the H_∞ norm $w \longrightarrow z_2$, is,

$$\min_{A_k, B_k, C_k, Q} \text{trace}(Q\tilde{C}_1'\tilde{C}_1), \text{ s.t. Equation}(5.5) \tag{5.8}$$

As a simplification, Bernsten and Haddad [BeH89] assumed $C_1'D_{21} = 0$, $B_1D_{13}' = 0$, and $C_2'D_{22} = 0$. With matrix Lagrange multipliers, they obtained four highly-coupled Ricatti equations, and the homotopy method was developed to solve the four coupled Ricatti equations numerically [GCW94]. (But it is computationally expensive). Khargoneckar and Rotea [KhR91] showed that the Bernsten-Haddad-type auxiliary cost minimization with full-state feedback can be handled with convex optimization, and the full-order output feedback problem can be chosen as a combination of the H_∞ state estimator and a full-state feedback gain for the mixed H_2/H_∞ synthesis problem of an auxiliary plant.

Doyle *et al.* [DZB89] [ZGB94] and Zhou *et al.* [ZGB94] investigated another system with two sets of inputs and one set of cost output, as shown in Figure 5-3,

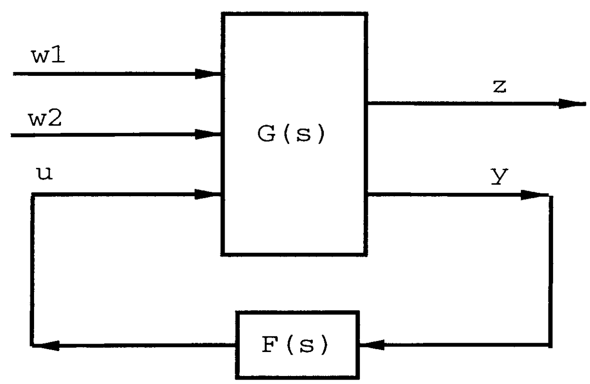

Figure 5-3: Mixed H_2/H_∞ control configuration II

where w_1 is a vector signal of bounded spectrum (white noise)

$$\|w_1\|_S := \sqrt{\|S_{w1}(jw)\|_\infty} = I$$

and w_2 is a vector signal of bounded power independent of w_1 or dependent causally on w_1

$$\|w_2\|_P := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace } S_{w_2}(jw)dw}$$

The objective is to design a stabilizing controller $K(s)$ such that

$$\sup_{w_2 \in P} \inf_{K(s)} \{\|z\|_P^2 - \gamma^2 \|w_2\|_P^2\}$$

They obtained necessary and sufficient conditions for the mixed H_2/H_∞ control. Yeh *et al.* [YBC92] proved that the results of Bernsten and Haddad and the results of Doyle and Zhou are in fact dual to each other.

In [VrJ97] the Bernsten-Haddad-type auxiliary cost minimization is extended to the case of $w_1 \neq w_2$ and $z_1 \neq z_2$, just by replacing $\tilde{C}_1$ in (5.5) by $\begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix}$.

Although Bernsten and Haddad only tried to design a strict proper dynamic controller, their idea of can be extended to more cases of (centralized) static output feedback. With the well-known system augmentation we introduced in Chapter 2, reduced-order control can also be handled as static output feedback. In fact this approach would be more concise than Bernsten and Haddad's approach, since it will only result in two coupled Ricatti equations, not four, even for the more general cases $C_1' D_{21} \neq 0$, $B_1 D_{13}' \neq 0$, and $C_2' D_{22} \neq 0$ or $w_1 \neq w_2$ and $z_1 \neq z_2$. In this way, the gradient-based approach used in Chapter 2 can also be applied to Bernsten-Haddad-type auxiliary cost H_2/H_∞ minimization with decentralized control. However, the more constrained, the more conservative is the design.

In addition, the entropy of H_{wz} ($\|H_{wz}\|_\infty < \gamma$) defined in [Mus89] [MGL91] [YaS97] for the case $w_1 \neq w_2$ and $z_1 \neq z_2$ also provides an upper bound of the H_2 norm:

$$\lim_{s_0 \to \infty} \{ \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln |\det[I - \gamma^2 H_{wz}(jw) H_{wz}'(jw)]| (\frac{s_0}{s_0 - jw})^2 dw \}$$

And it is proved that negative of the entropy equals the auxiliary cost defined by Bernsten and Haddad. For the full-state feedback, decoupled Ricatti equations are also obtained therein. Yaesh and Shaked [YaS97] investigated the static output-feedback entropy optimization with an H_∞ norm bound. They obtained a modified Riccati equation and a Lyapunov equation, and adopted a homotopy method to solve these two decoupled equations.

5.2.2 LMI Convex Optimization

Soon after LMI-based H_2 or H_∞ was proposed in [GaP94], [BEF94], [GNL95], the approach attracted the attention for its potential application in multi-objective control immediately ([ElF96], [SGC97], [Sch00], [Lei01]). As we know, a combination of several LMIs is still an LMI, so many specifications can be characterized as LMIs and be handled at a time. Currently most work only focuses on unstructured cases (full-state feedback or full-order feedback) and they only obtain a suboptimum with common LMI variables. More recently non-common LMI approach has also been proposed to obtain a less conservative design [FaF00] [EHS01]. Static H_2/H_∞ output feedback controllers are also examined in the past two years [Sch00] [Lei01].

Let's consider the full-state feedback for the general configuration (5-1) with plant model (5.1)-(5.3). The matrix D_{11} is assumed to be 0 to ensure a well-posed problem. Recall the LMI form in Chapter 2 (or see [GNP95]) for H_2 control: The closed loop with full-state feedback $\|H_{w_1 z_1}\|_2 < \gamma_2$ iff there exist some $X_2 \geq 0$ and a matrix T_2 such that

$$\begin{bmatrix} AX_2 + X_2A' + B_3T_2 + T_2'B_3' & B_1 \\ B_1' & -I \end{bmatrix} < 0 \tag{5.9}$$

$$\begin{bmatrix} S & C_1X_2 + D_{13}T_2 \\ (C_1X_2 + D_{13}T_2)' & X_2 \end{bmatrix} < 0 \tag{5.10}$$

$$\text{trace}(S) < \gamma_2^2 \tag{5.11}$$

Recalled the LMI form in Chapter 3 about H_∞ control: The closed loop with full-state feedback $\|H_{w_2 z_2}\|_\infty < \gamma_\infty$ iff there exist some positive definite matrix $X_\infty > 0$ and a matrix T_∞ such that

$$\begin{bmatrix} AX_\infty + X_\infty A' + B_3T_\infty + T_\infty' B_3' & B_2 & (C_2X_\infty + D_{23}T_\infty)' \\ B_2' & -I & D_{22}' \\ C_2X_\infty + D_{23}T_\infty & D_{22} & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{5.12}$$

The variables T_2 and T_∞ are introduced through the change of variables:

$$T_2 := FX_2, \; T_\infty := FX_\infty$$

The mixed H_2/H_∞ problem is stated as:

$$\min_{\gamma_\infty \leq \hat{\gamma}_\infty} \gamma_2 \tag{5.13}$$

and the general H_2/H_∞ multi-objective problem is stated as:

$$\min \alpha\gamma_2 + (1-\alpha)\gamma_\infty \tag{5.14}$$

where α is a given scalar in the interval [0,1]. These problems are nonconvex because of the constraint

$$F := T_\infty X_\infty^{-1} = T_2 X_2^{-1} \tag{5.15}$$

To avoid this difficulty, recently this nonconvex problem has been translated into a convex problem by imposing the extra (technical) constraints $X_2 = X_\infty$ and $T_2 = T_\infty$, and thus a suboptimum can be obtained easily with some reservation [GNL95].

For the full-order H_2/H_∞ problem, similar technical constraints were enforced, and a convex programming method was formulated. For details, please refer to [GNL95].

Leibritz [Lei01] examined the suboptimum of H_2/H_∞ with static output the suboptimal for the plant setup in Figure 5-3. However, in practice we have more interest in the configuration as in Figure 5-1, which enable us to achieve optimal performance in one channel while keeping the stability requirement through the other channel. Below we will extend El Ghaoui's cone complementary linerization algorithm [EAO97] to the mixed H_2/H_∞ problem for the configuration in Figure 5-1. The proposed approach can also be utilized to treat the reduced-order H_2/H_∞ problem with system augmentation.

Suppose the plant model is given by (5.1)-(5.3) and the measurement output is

$$y = C_3 x + D_{32} w_2$$

and we would like to design a static output feedback controller $u = Fy$ such that the H_2 norm of channel $w_1 \to z_1$ is less than γ_2 and the H_∞ norm of channel $w_2 \to z_2$ is less than γ_∞. Recall the single-objective H_2 γ-suboptimization problem:

$$\text{trace}(X_2 Y_2) = n \tag{5.16}$$

$$\text{trace}(B_1' X_2 B_1) < \gamma_2^2 \tag{5.17}$$

$$N_y' \begin{bmatrix} A'X_2 + X_2 A & C_1' \\ C_1 & -I \end{bmatrix} N_y < 0 \tag{5.18}$$

$$N_u' \begin{bmatrix} AY + Y_2 A' & Y_2 C_1' \\ C_1 Y_2 & -I \end{bmatrix} N_u < 0 \tag{5.19}$$

$$\begin{bmatrix} X_2 & I \\ I & Y_2 \end{bmatrix} > 0 \tag{5.20}$$

where N_y and N_u are, respectively, the orthogonal bases to the null space of $[C_3, 0]$ and $[B_3', D_{13}']$. Also recall the single-objective H_∞ γ-suboptimization problem:

$$\text{trace}(X_\infty Y_\infty) \;=\; n \tag{5.21}$$

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'X_\infty + X_\infty A & X_\infty B_2 & C_2' \\ B_2' X_\infty & -\gamma_\infty I & D_{22}' \\ C_2 & D_{22} & -\gamma_\infty I \end{bmatrix} \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} \;<\; 0 \tag{5.22}$$

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} AY_\infty + Y_\infty A' & Y_\infty C_2' & B_2 \\ C_2 Y_\infty & -\gamma_\infty I & D_{22} \\ B_2' & D_{22}' & -\gamma_\infty I \end{bmatrix} \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} \;<\; 0 \tag{5.23}$$

$$\begin{bmatrix} X_\infty & I \\ I & Y_\infty \end{bmatrix} \;>\; 0 \tag{5.24}$$

where N_R and N_s are, respectively, the null spaces of $[B_3', D_{23}']$ and $[C_3, D_{32}]$.

Now impose the technical constraints

$$X := X_2 = X_\infty, \text{ and } Y := Y_2 = Y_\infty$$

then we can use the cone complementary linerization algorithm [EAO97] to solve the bilinear objective minimization problem subject to LMI constraints:

$$\min_{X,Y} \text{trace}(XY), \text{ s.t. LMI } (5.17), (5.18), (5.19), (5.20), (5.22) \text{ and } (5.23) \tag{5.25}$$

If this minimization results in n, we obtain the pair (X, Y), and the static H_2/H_∞ controller can be reconstructed in the same way as the single-objective case. We haven't yet perfomed numerical experiments for this approach, it efficiency should be similar is to the cone complementary linerization algorithm. Like the full-state feedback case, the result of this approach is conservative, since we impose a common Lyapunov function for H_2 and H_∞.

Lee [Lee99] changed the H_∞ BMI feasible problem into a highly nonlinear minimization problem while keeping the controller structure (such as decentralization) inside. The conditional gradient method was used to solve the problem via iterative LMI. An example of mixed H_2/H_∞ control was also given therein.

To obtain a less conservative design, Shimomura and his coworkers [ShF00] [EHS01] investigated noncommon LMI approaches to the unstructured or structured controller design via iteration. The design is less conservative at the cost of computational complexity.

5.2.3 Worst-Case Design

Stoorvogel [Sto93], Steinbuch and Bosgra [StB94], Limebeer *et al.* [LAH94], and Chen *et al.* [CSZ98] considered the H_2 norm under the worst disturbance. For the configuration shown in Figure 5-1, the problem of worst-case design is:

$$\sup_{\|\Delta\|_\infty \leq 1} \min_{K(s)} \|H_{w_1 z 1}(K, \Delta)\|_2 \tag{5.26}$$

Stoorvogel [Sto93] analyzed the worst-case effect of a disturbance w_2 on the H_2 norm $w_1 \rightarrow z_1$ and obtained an upper bound (tight if w_1 is a scalar) dominated by an algebraic Riccati equation with an additional Lagrange multiplier. Nonlinear time-varying $\Delta(s)$ are also considered.

Steinbuch and Bosgra [StB94] assumed the worst-case $\Delta(s)$ is achieved as a finite-order "less bounded real" transfer function

$$\Delta^T(-s)\Delta(s) = I$$

which can be expressed by a rational "lossless positive real" transfer function $\Gamma(s)$, $\Delta(s) = [I - \Gamma(s)][(I + \Gamma(s)]^{-1}$. (This assumption is reasonable although not strictly proved, if $\Delta(s)$ is finite-order, causal, and LTI.) Then based on the definition of "lossless positive real" $\Gamma(s) + \Gamma^T(-s) = 0$, $\Gamma(s)$ is parameterized by three matrices A_t, B_t, and D_t: $B_t(sI - A_t)^{-1}B_t^T + D_t$, where $A_t + A_t^T = 0$ and $D_t + D_t^T = 0$. Thus worst-case design became a unconstrained optimization problem:

$$\max_{A_t, B_t, D_t} \min_{K(s)} \|H_{w_1 z_1}(u = K(s)y)\|_2 \tag{5.27}$$

Note that for a fixed (A_t, B_t, D_t), the problem $\min_{K(s)} \|H_{w_1 z_1}(u = K(s)y)\|_2$ becomes a standard H_2 optimization with a dynamic $(n + n_t)$-order controller, where n_t is the order of $\Delta(s)$. This observation can be used to simplify the gradient-based computation. The drawback of this method is that the controller order might be very high, and it is only limited to finite-order LTI $\Delta(s)$ (which may not be true in practice).

Limebeer *et al.* [LAH94] proposed the Nash game approach for the worst-case solution with full-state feedback for the case of $w_1 = w_2$ and $z_1 = z_2$. Chen *et al.* [CSZ98] extended this approach to the case of full-order output feedback and to the case of $w_1 \neq w_2$ and $z_1 \neq z_2$. The Nash game is a two-player nonzero sum game with two performance criteria. In these two papers the Nash game is used to characterize the mixed H_2/H_∞ problem by assigning one performance index to reflect the H_2 requirement and another one to reflect the H_∞ constraint. Three coupled Ricatti differential equations or algebraic equations are obtained to minimize the energy of the output z_1 under the worst-case disturbance w_2 applied to the system. Nonlinear $\Delta(s)$ are also considered.

5.2.4 Youla-Parameterization

Consider the configuration in Figure 5-1. It is known that the closed-loop $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \rightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is

$$\{G(s) = P_{wz}(s) + P_{uz}(s)K(s)[I - P_{uy}(s)K(s)]^{-1}P_{wy}(s) \mid K(s) \text{ stabilizing}\}$$

With the Youla parameterization Q, all achievable closed loop can be represented in the affine form in Q [HHB98]:

$$\{G(s) = W(s) - U(s)Q(s)V(s) \mid Q \in RH^{\infty}\} \tag{5.28}$$

where RH^{∞} denotes the function space with finite H_{∞} norm and analytic continuity for $R_e(s) > 0$, $H(s), U(s), V(s)$ are stable matrix transfer functions with appropriate dimensions.

Now the mixed H_2/H_{∞} problem become

$$\min_{Q \in RH^{\infty}} \|W_2(s) - U_2(s)Q(s)V_2(s)\|_2, \text{ s.t. } \|W_{\infty}(s) - U_{\infty}(s)Q(s)V_{\infty}(s)\|_{\infty} < \gamma_{\infty} \tag{5.29}$$

and the multi-objective H_2/H_{∞} problem become:

$$\min_{Q \in RH^{\infty}} \alpha\|W_2(s) - U_2(s)Q(s)V_2(s)\|_2 + (1 - \alpha\|W_{\infty}(s) - U_{\infty}(s)Q(s)V_{\infty}(s)\|_{\infty} \tag{5.30}$$

where α is given in [0,1].

Generally this is infinite dimension optimizational. Megretski [Meg94] showed has shown that the exact optimum of H_2/H_{∞} via Youla parameterization is generally infinite order. In the literature finite dimensional optimization is used to approximate it. Most reports of Youla-parameterization are for discrete systems ([Sch95] [HHB98] [Sch00] [DCR01]). The FIR (finite impulse response) structure of the Youla parameterization is used, and LMI convex optimization is formulated by fixing A_Q and B_Q in the state-space realization of $Q(s)$ $\left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]$ ([HHB98]). In [Sch900], Scherer considered a very special class of plant with static output feedback and formulated the H_2/H_{∞} control for this plant as an LMI problem, then used Youla parameterization to force the dynamic output feedback of more general plants into that class. In this approach, the size LMI is reduced. Sznaier *et al.* [SRB00] recast the continuous-time H_2/H_{∞} problem into discrete equivalent via bilinear transformation, and a ϵ-suboptimal solution was found by solving a sequence discrete truncated problems. Compared with other techniques, Youla parameterization algorithms reduces the conservatism, but the controller order usually increases significantly.

So far, we have surveyed the main techniques for mixed H_2/H_∞ or multi-objective H_2/H_∞ control. We see that no readily-to-implement approach for H_2/H_∞ control without much conservatism is available, even for full-state feedback. H_2/H_∞ control with static output feedback can also be obtained with LMI with some conservatism. Iterative LMI might be promising for decentralized H_2/H_∞, but a further investigation is needed.

5.3 Optimal H_2 Control with Regional Pole Placement

5.3.1 Introduction

As we have pointed out in Chapter 4, the poles determine the dynamic performance of the system to some extent, especially the transient response. Usually, exact pole placement is impractical and unimportant as long as the poles are in some region. Regional pole placement offers additional freedom to meet other specification. H_2 control with full-sate feedback can guarantee 60^o phase margin, but H_2 with output feedback might yield a system with very light damping or small decay rate. So H_2 control with regional pole placement is proposed to enhance the robustness of the closed loop. In general applications, the ideal pole region is shown in Figure 5-4. Strip, disk, or sector regions are used to approximate it. Elliptical or parabolic region are also addressed.

Consider a plant mode given by

$$\dot{x} = Ax + B_1 w + B_2 u \tag{5.31}$$

$$z = C_1 x + D_{11} w + D_{12} u \tag{5.32}$$

$$y = C_2 x \tag{5.33}$$

where D_{11} is assumed to be zero to ensure the H_2 problem is well posed. Design a controller $u = K(s)y$ such that the H_2 norm of the closed loop $w \to z$ is minimal and the poles are in some prescribed region.

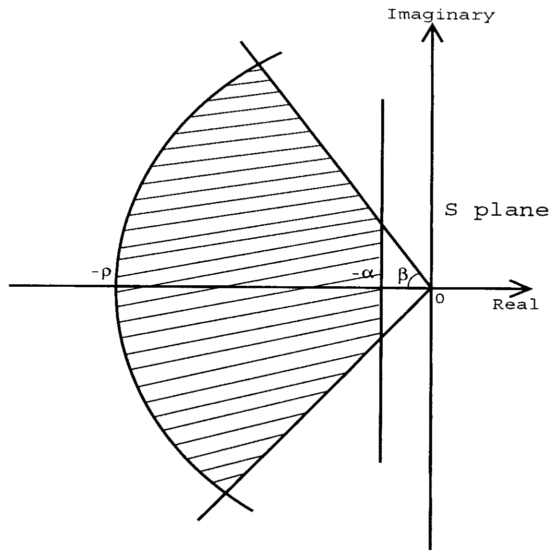

Figure 5-4: Pole regional constraints.

In the following section, Riccati-based approaches are surveyed for the auxiliary minimization problem defined by Haddad and Bernstein [HaB92]. LMI-based suboptimal approaches are discussed. A simple example is given to show that they are fairly conservative. In order to directly minimize the structured (or unstructured) H_2 norm with general pole regional constraints, we formulate a tractable optimization problem, and some approaches are hinted to solve it. We also present the method of multipliers for more general regional constraints, or constraints on partial poles only.

5.3.2 Auxiliary Function Minimization and LMI Suboptimization

Since dynamic feedback can be cast as a static output feedback problem, let's consider static output feedback back first. With $u = Fy$, the closed loop is

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} A + B_2FC_2 & B_1 \\ \hline C_1 + D_{12}FC_2 & 0 \end{array}\right]$$

Recall Theorem 2.1 again for the computation of H_2 norm:

$$\|H_{zw}\|_2^2 = \text{trace}(C_cQ_cC_c') \tag{5.34}$$

where Q_c is a nonnegative definite matrix satisfying

$$A_c Q_c + Q_c A_c + B_c B_c' = 0 \tag{5.35}$$

Also recall the Lyapunov equation description of the pole region. All poles are in the α-stability region ($R_e(s) < -\alpha$) iff there exists some positive definite Q [AnM90], such that

$$(A_c + \alpha I)Q + Q(A_c + \alpha I)' + V = 0 \tag{5.36}$$

where V is any positive definite matrix. All poles are in the the disk region $D(\alpha, \rho)$ ($\alpha \geq \rho > 0$) in the left half plane iff there exists some positive definite Q [FuK87], such that

$$(A_c + \alpha I)Q + Q(A_c + \alpha I)' + \frac{1}{\rho}(A_c + \alpha I)Q(A_c + \alpha I)' + V = 0 \tag{5.37}$$

where V is any positive definite matrix.

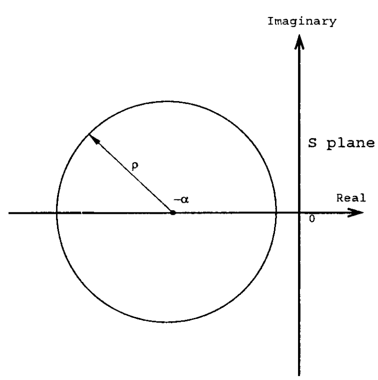

Figure 5-5: Disk pole region

Assume that B_c has full column rank. If there exist some matrix $Q \geq 0$ satisfying (5.36) or (5.37) in which the matrix V is taken as $B_c B_c'$, it can be shown ([HaB92]) that there exist some $Q_c \geq 0$ satisfying (5.35), and thus $\|H_{zw}\|_2^2 = \text{trace}(C_c Q_c C_c') \leq \text{trace}(C_c Q C_c')$ Therefore an auxiliary minimization is formulated as:

$$\min_F \text{trace}(C_c Q C_c')$$

$$\text{s.t. } (A_c + \alpha I)Q + Q(A_c + \alpha I)' + \frac{1}{\rho}(A_c + \alpha I)Q(A_c + \alpha I)' + B_c B_c' = 0 \tag{5.38}$$

$$\text{or } (A_c + \alpha I)Q + Q(A_c + \alpha I)' + B_c B_c' = 0 \tag{5.39}$$

Solving this auxiliary minimization problem, we obtain a controller which guarantees an upper bound on the H_2 norm with all closed loop poles placed in the α-stability region or disk region. For α-stability H_2 control, the design procedure is the same as the single-objective H_2 control if we replace A by $A - \alpha I$. For the case of disk-regional pole constraint, with Lagrange matrix multiplier method a discrete-time algebraic Riccati equation is obtained for the auxiliary minimization problem with full-state feedback. Two decoupled modified Riccati equations [HaB92] or two decoupled discrete-time algebraic Riccati equations [SKK93] are obtained for full-order feedback. For static output feedback, two coupled Riccati equations are obtained. The auxiliary cost minimization problem is usually solved with homotopy methods [GCW94], in which the controller structure can't be considered. For this reason, we extend the gradient-based method (discussed in Section 2.7 for single-objective H_2) to solve the auxiliary minimization problem (5.38) with decentralized F. However, the auxiliary minimization (5.38) is generally quite conservative. This can be seen from the simple example of an SDOF TMD with a prescribed disk region. (Please see that last section of this chapter.)

The sector region is also interesting, since it can specify the damping ratio directly. Consider the sector region in Figure 5-6. Recall that all poles reside in this sector

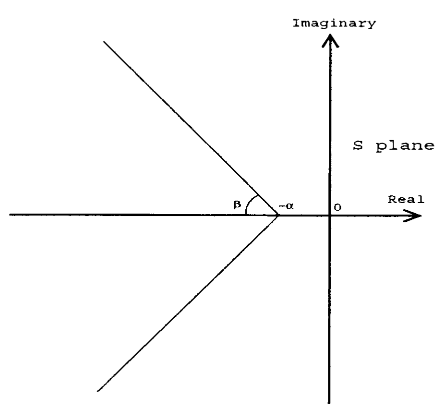

Figure 5-6: Sector region

region iff

$$\begin{bmatrix} (A_c - \alpha I)\sin\beta & (A_c - \alpha I)\cos\beta \\ -(A_c - \alpha I)\cos\beta & (A_c - \alpha I)\sin\beta \end{bmatrix}$$

is stable. In [HaB92] a $2n$-order system was augmented, and nth order dynamic feedback controller can be obtained by solving four nonlinearly coupled matrix equations by iteration. However, although all poles is placed in the sector region, the LQR cost of this design has no direct relation to the original H_2 norm requirement.

In Chapter 4, we have already seen that the α-stability region, disk region, sector region, etc., can be characterized with a matrix in LMI form. In [ChG96] more general convex regions are characterized in LMI form (so-called LMI regions). By imposing additional (technical) constraints similar to these discussed in previous section, H_2, H_∞, or H_2/H_∞ control with LMI pole regional constraint can be treated naturally by the promising LMI tools. For unstructured H_2 with regional pole placement, the LMI-based approach is more efficient and convenient than the Riccati-based approach, and it can be applied to more general regions, such as the sector region or the ideal region in Figure 5-4. We also can extend the LMI-based approach to static output feedback H_2 suboptimal control with pole regional constraints, by following the spirit of the extension for the mixed H_2/H_∞ control.

5.3.3 Direct Minimization

Since the previous auxiliary minimization may produce a quite conservative design, so direct minimization was also developed. Liu and Yedavalli [LiY93] consider full-sate H_2 optimization with the α-stability pole region constraint, and use the Lagrange multiplier method. Yuan *et al.* [YAJ96] investigated optimal H_2 control by static output feedback with pole placement within the sector region shown in Figure 5-6. They proposed a iteration which will converge to a stationary point if B_2 and C_1 have full rank. However, in practice, the sector region with prescribed damping and stability specifications (Figure 5-7) is suitable. So Sehitoglu [Seh93] approximated it with a hyperbola and translated into the left half plane of an associated $4n$-th order system. Unlike [HaB92], in this approach the H_2 norm can be minimized directly. But the numerical computation is not so efficient.

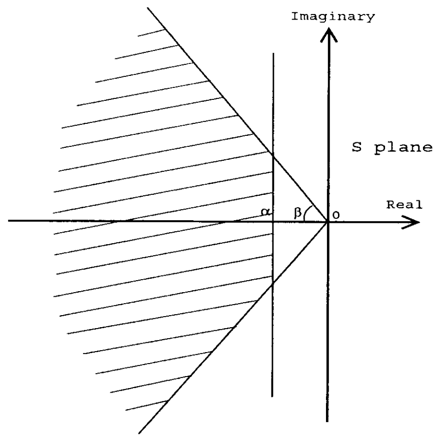

Figure 5-7: Sector region with prescribed damping and stability specifications

In the following, we will extended the techniques for single-objective structured H_2 in Chapter 2 to the H_2 control with pole placement in a general region. Suppose the prescribed pole region is the intersection region of the α-stability region, a disk $D(\alpha_2, \rho)$, and a sector $(\alpha_3, \pm\beta)$. This is reasonable in practice, and one special example (with $\alpha_2 = \alpha_3 = 0$) is the ideal pole region in Figure 5-4. (More general LMI regions can be extended in the same way.) This region is convex and can be characterized by three matrices $Q_1 > 0$, $Q_2 > 0$, and $Q_3 > 0$:

$$(A_c + \alpha I)Q_1 + Q_1(A_c + \alpha I)' + V_1 = 0 \tag{5.40}$$

$$(A_c + \alpha_2 I)Q_2 + Q_2(A_c + \alpha_2 I)' + \frac{1}{\rho}(A_c + \alpha_2 I)Q_2(A_c + \alpha_2 I)' + V_2 = 0 \tag{5.41}$$

$$\begin{bmatrix} (A_c - \alpha_3 I)\sin\beta & (A_c - \alpha_3 I)\cos\beta \\ -(A_c - \alpha_3 I)\cos\beta & (A_c - \alpha_3 I)\sin\beta \end{bmatrix} Q_3 + Q_3 \begin{bmatrix} (A_c - \alpha_3 I)\sin\beta & (A_c - \alpha_3 I)\cos\beta \\ -(A_c - \alpha_3 I)\cos\beta & (A_c - \alpha_3 I)\sin\beta \end{bmatrix}' + V_3 = 0 \tag{5.42}$$

where V_1, V_2, and V_3 are any given positive definite matrices of compatible dimension, and

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} A + B_2FC_2 & B_1 \\ \hline C_1 + D_{12}FC_2 & 0 \end{array}\right]$$

Therefore, we can formulate the direct minimization of the H_2 norm with pole placement in the prescribed region as a constrained optimization problem:

$$\begin{aligned} &\min_{Q_c \geq 0, Q_1 > 0, Q_2 > 0, Q_3 > 0, F} \text{trace}(C_c Q_c C_c') \\ &\text{s.t. } A_c Q_c + Q_c A_c + B_c B_c' = 0 \\ &\quad \text{and Equations } (5.40), (5.41), (5.42) \end{aligned} \tag{5.43}$$

In this problem, the feedback gain F is retained, thus we can consider structured control, such as decentralization. Note that for a fixed F, all the four equation constraints are in easy-to-be-solved Lyapunov or Riccati form, which is similar to single H_2 optimization. Thus the gradient-based algorithms, homotopy method, descent Anderson-Moore algorithm [ToM85], or iterative LMI (the constraints (5.40), (5.41), (5.42) can be replaced with LMI [ChG96]) can be used to minimized the H_2 norm exactly with pole regional constraint. Moreover, in the problem (5.43) we can also add the H_∞ constraint. Further investigation, especially of convergence properties, is required.

For disk region, Fischman *et al.* [FND97] developed an LMI-based approach to take into account the controller structure without introducing conservatism. We will discuss this approach later.

5.3.4 Method of Multipliers for Structured H_2 with Regional Pole Placement

In the previous two subsections, we discussed LMI suboptimal design and the auxiliary minimization of the H_2 upper bound. We also examined direct minimization, where all poles are placed in some typical region. However, we know that the performance of the system transient response is determined by the dominant poles. So it is not necessary to require all the poles to be within the region. Below we propose another direct minimization approach: the method of multipliers. It is able to specify the region for some dominant poles only. Compared with the methods in previous section, another advantage is that we can describe more general pole regional constraints. The computational efficiency is also comparable, since we only take the free variables in structured gain F as our design variables and the procedure is gradient-based.

The multiplier method combines the advantage of the Lagrange multiplier method and penalty function approach. It generates a sequence to approximate the Lagrange multipliers. Let us briefly introduce this method.

Theorem 5.1 ([Ber95]): If x^* is a local minimum of

$$\begin{aligned} &\min_x f(x) \\ &\text{s.t. } h(x) = 0, \; g(x) \leq 0 \end{aligned} \tag{5.44}$$

where $f : R^n \to R$, $h : R^n \to R^m$, and $g : R^n \to R^r$ are given smooth functions. Define the *augmented Lagrangain function* $L_c : R^n \times R^m \times R^r \to R$,

$$L_c(x, \lambda, \mu) = f(x) + \lambda' h(x) + \frac{c}{2}||h(x)||^2 + \sum_{j=1}^{r} \left\{ mu_j g_j^+(x, \mu, c) + \frac{c}{2}[g_j^+(x\mu, c)]^2 \right\} \tag{5.45}$$

where $g_j^+(x, \mu, c) = \max\{g_j(x), -\frac{\mu}{c}\}$. Then there exist (λ^*, μ^*) that minimize $L_c(x^*, \lambda^*, \mu^*)$ if c is large enough. In addition, (λ^*, μ^*) are the Lagrange multipliers if the original problem adopts Lagrangian multipliers.

Note that $L_c(x, \lambda, \mu)$ is differentiable. The augmented Lagrangain function is the combination of a penalty function and a Lagrangain function. A **Multiplier method** [Ber95] was proposed to avoid the fact that (λ^*, μ^*) is unknown: For a given c^k, update (λ, μ) as:

$$\begin{aligned} \lambda^{k+1} &= \lambda^k + c^k h(x^k) \\ \mu_j^{k+1} &= \mu_j^k + c^k g_j^+(x^k) \end{aligned} \tag{5.46}$$

where x^k is the (unconstrained) local minimum of $L_{c^k}(x, \lambda^k, \mu^k)$. Usually c^k is a gradually increasing sequence: $c^{k+1} = \beta c^k$, $\beta \in [5, 10]$. It was shown that this procedure converges to a local minimum of $f(x)$ with the constraint of $h(x)$ and $g(x)$, and

(λ^k, μ^k) converges to Lagrange multipliers (λ^*, μ^*). For details about this algorithm, please refer to [Ber95].

The specification of regional constraints on partial poles can be characterized with the eigenvalues of $A + B_2FC_2$. Suppose it is $g(A + B_2FC_2) \leq 0$. The structured H_2 problem with regional constraints on all or partial poles can be formulated as:

$$\begin{aligned} &\min J(F) = \|H_{zw}\|_2^2 = \text{trace}(B_1' Q_c B_1) \qquad (5.47) \\ &\text{s.t. } Q_c(A + B_2FC_2) + (A + B_2FC_2)'Q_c + (C_1 + D_{12}FC_2)'(C_1 + D_{12}FC_2) = 0 \\ &\quad g(A + B_2FC_2) \leq 0 \end{aligned}$$

Map the free design parameters in F to the vector x. In this problem, we can evaluate the conditional gradient of $\partial J/\partial F$ by solving the two associated Lyapunov equations similar as Section 2.7. So we can avoid introducing $h(x)$ and λ. The gradient of $g(A + B_2FC_2)$ with respect to F is easy to evaluate via the eigenvalue sensitivity formula in Section 4.5. Introducing the multipliers μ and c, we can use BFGS unconstrained optimization to solve the structured H_2 problem with pole regional constraints. Numerical examples will be given in the last section of this chapter.

Other than the above H_2 optimization with regional pole placement, H_2 control with exact eigenvalue/eigenstructure assignment [LiP98], by inverting LQR problems to determine the weighting matrix, have also been proposed recently. About this, we already give a brief review at the end of Section 4.3.

5.4 H_∞ or H_2/H_∞ Control with Regional Pole Placement

As a frequency domain method, H_∞ optimization can't guarantee the performance in the time domain. So H_∞ or H_2/H_∞ control with regional pole placement has also proposed recently [BSU94] [PaL94] [YeL95] [ChG96] [FND97] [CGA99] [ShF00] [EhS01].

5.4.1 LMI Suboptimization and Nonlinear Programming

The plant model is given in (5.31)–(5.33). Since the dynamic feedback can be cast as a static output feedback, let's consider the static output feedback first. The problem is to design some feedback gain $u = Fy$ such that the closed loop poles are placed in some prescribed region, and the H_∞ norm system $w \rightarrow z$ is minimal or less than some prescribed γ.

With $u = Fy$, the closed loop is $H_{zw}(s) = B_c(sI - A_c)^-1C_c + D_c$,

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} A + B_2FC_2 & B_1 \\ \hline C_1 + D_{12}FC_2 & D_{11} \end{array}\right]$$

Recall some results about the system H_∞ norm:

$$||H_{zw}||_\infty = \sup_w \sigma_{max}[H_{zw}(\omega)] \tag{5.48}$$

For a stable system, $||H_{zw}||_\infty < \gamma$ iff there exists a symmetrix matrix $X_\infty > 0$ such that

$$\begin{bmatrix} A_c'X_\infty + X_\infty A_c & X_\infty B_c & C_c' \\ B_c'X_\infty & -\gamma^2 I & D_C' \\ C_c & D_c & -I \end{bmatrix} < 0 \tag{5.49}$$

Consider the pole region in Figure 5-4 characterized in LMI form: there exists $X_D > 0$ such that

$$\begin{gathered} A_cX_D + X_DA_c' + 2\alpha X_D < 0 \\ \begin{bmatrix} -\rho X_D & A_cX_D \\ X_DA_c' & -\rho X_D \end{bmatrix} < 0 \\ \begin{bmatrix} (A_cX_D + X_DA_c')\text{sin}\beta & (A_cX_D - X_DA')\text{cos}\beta \\ (X_DA_c - A_cX_D)\text{cos}\beta & (A_cX_D + X_DA_c')\text{sin}\beta \end{bmatrix} < 0 \end{gathered} \tag{5.50}$$

With the additional (technical) constraint $X_D = X_\infty$ [ChG96], full-state feedback H_∞ with pole regional constraints can be naturally handled in the framework of LMI with some conservatism. Full-order output controller can also be handled in a similar way. It is also easy to understand that the full-state (or full-order) H_2/H_∞ control with pole regional constraints is also tractable with some conservatism by imposing a common Lyapunov function. Multi-objective control by centralized static (or reduced-order) output feedback can also be handled with some extension of cone complementary algorithm proposed in [EOA97].

Although the additional technique constraint makes the centralized multi-objective problem tractable and yields some acceptable suboptimal design, it can be fairly conservative in some systems. So direct minimization is also proposed. One example is non-common LMI iteration [EHS01]. Another example is successive approximation via nonlinear programming [YeK95]. Genetic algorithms are also used [PaL94] and have attracted more attention recently [CaC01]. Bambang *et al.* [BSU94] investigated the (centralized) static output feedback, and an iterative algorithm was proposed for a less conservative upper bound of the H_2 norm.

5.4.2 H_2/H_∞ with Pole Placement in a Disk Region

For H_2/H_∞ control with pole placement in a disk region $D(\rho, \rho)$, an attractive approach has been developed by Fischman *et al.* [FND97]. They formulated the constraints as bilinear equations and linear matrix inequalities without conservatism, and the control structure can also be handled therein. The plant in Figure 5-2 was examined by Fischman *et al.*. (Their idea is readily extended to the general plant shown in Figure 5-1.) Suppose the plant is given as:

$$\begin{aligned} \dot{x} &= Ax + B_1 w_1 + B_2 w_2 + B_3 u \\ z_1 &= C_1 x + D_{12} w_2 + D_{13} u \\ z_2 &= C_2 x + D_{21} w_1 + D_{22} w_2 + D_{23} u \\ y &= C_3 x \end{aligned} \tag{5.51}$$

and B_1 and C_3 are assumed to have full rank. With a structured static output controller $u = Fy$, the closed loop is

$$\left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & 0 & D_{c12} \\ C_{c2} & D_{c21} & D_{c22} \end{array}\right] := \left[\begin{array}{c|cc} A + B_3 F C_3 & B_1 & B_2 \\ \hline C_1 + D_{13} F C_3 & 0 & D_{12} \\ C_2 + D_{23} F C_3 & D_{21} & D_{22} \end{array}\right]$$

We would like to design a structured controller gain F such that the H_2 norm of the closed loop $w_1 \longrightarrow z_1$ is less than a given value γ_2, the H_∞ norm of $w_2 \longrightarrow z_2$ less than γ_∞, and the closed loop poles reside in the disk region $D(\alpha, \rho)$.

For the disk specification (5.37) and Schur complement (in Section 2.2), it can be obtained [GBA95] that the closed-loop poles residue in the (open) disk $D(\alpha, \rho)$ in left half plane iff there exist matrices (P_1, W_1, F) such that

$$\begin{bmatrix} P_1 & (A + \alpha I + B_3 F C_3)' \\ A + \alpha I + B_3 F C_3 & W_1 \end{bmatrix} > 0 \tag{5.52}$$

and $W_1 P_1 = \rho^2 I$.

For the system $(A_c, B_{c1}, C_{c1}, 0)$, from Section 2.1, we know that

$$\|H_{z_1 w_1}\|_2^2 = \inf\{\text{trace}(B_{c1}' P_2 B_{c_1}) : \; A_{c1}' P_2 + P_2 A_{c1} + C_{c1}' C_{c1} < 0\} \tag{5.53}$$

Add an immaterial item $\frac{1}{\rho_2^2} A_c' P_2 A_c$ to the inequality, the following conclusion can be drawn with some matrix operations. The H_2 norm of $w_1 \longrightarrow z_1$ is less than γ_2 iff there exist matrices (P_2, W_2, F) and a scalar ρ_2 such that

$$\gamma_2 - \text{trace}(B_{c1}' P_2 B_{c_1}) > 0 \tag{5.54}$$

$$\begin{bmatrix} P_2 & (A + \rho_2 I + B_3 F C_3)' & (C_1 + D_{13} F C_3)' \\ A + \rho_2 I + B_3 F C_3 & W_2 & 0 \\ (C_1 + D_{13} F C_3 & 0 & \rho_2 I \end{bmatrix} > 0 \tag{5.55}$$

and $W_2 P_2 = \rho_2^2 I$.

Fischman *et al.* [FND97] also obtained that the H_∞ norm of $w_2 \longrightarrow z_2$ is less than γ_∞ iff there exist matrices (P_3, W_3, F) and scalar ρ_3 such that

$$\begin{bmatrix} P_3 & (A + \rho_3 I + B_3 F C_3)' & P_3 B_2 + \frac{(C_2 + D_{23} F C_3)' D_{22}}{\gamma_\infty} & (C_2 + D_{23} F C_3)' \\ A + \rho_3 I + B_3 F C_3 & W_3 & 0 & 0 \\ B_2' P_3 + \frac{D_{22}'(C_2 + D_{23} F C_3)}{\gamma_\infty} & 0 & \rho_3(\gamma_\infty I - \frac{D_{22}' D_{22}}{\gamma_\infty}) & 0 \\ (C_2 + D_{23} F C_3 & 0 & 0 & \rho_3 I \end{bmatrix} > 0 \tag{5.56}$$

and $W_3 P_3 = \rho_3^2 I$.

With Lemma 2.3, we know that $W_iP_i = \rho_i I$ is equivalent to

$$\begin{bmatrix} P_i & \rho_i I \\ \rho_i I & W_i \end{bmatrix} \geq 0, \; i = 1, 2, 3 \tag{5.57}$$

and $\text{trace}(W_iP_i) - \rho_i^2 = 0$, for $i = 1, 2, 3$, and $\rho_1 = \rho$ (disk radius).

Therefore, the multi-objective control problem becomes

$$\min_{W_1,W_2,W_3,P_1,P_2,P_3,\rho_2,\rho_3} \text{trace}(W_1P_1 + W_2P_2 + W_3P_3) - \rho^2 - \rho_2^2 - \rho_3^3 \tag{5.58}$$
$$\text{s.t. } LMI(5.52), (5.54), (5.55), (5.56), \text{ and } (5.57)$$

A successive linearization procedure is proposed similar to that in [EOA97]. Considering the excellent performance of the algorithm in [EOA97], the computational efficiency of this successive linearization procedure should be satisfactory. But we worry that ρ_2 might be very large or converges to infinity, because if γ_2 is too tight the item $\frac{1}{\rho_2^2}A_c'P_2A_c$ only becomes "immaterial" for large enough ρ_2. This is a similar problem with ρ_3. More investigation is deserved, since the controller structure can be specified directly in this approach.

5.5 L_1 Norm (Peak to Peak Gain) and Associated Multi-Objective Control

In the previous Chapters of this thesis, we mainly concentrate on the H_2 norm (impulse $\rightarrow$ energy, or white noise $\rightarrow$ covariance), H_∞ norm ($L_2 \rightarrow L_2$ gain). In the past fifteen years L_1 (or l_1 for discrete-time) norm control techniques have also been developed and extended to multi-objective control thanks to the contribution of Dahleh, Sznaier, and others. In the following we will introduce some concepts about the L_1 norm and its computation methods, then survey some important investigations about the associated multi-objective control.

5.5.1 System L_1 Norm

Recall the definition of the signal norm. The L_1 norm of a vector valued signal $f(t)$ is

$$\|f\|_1 = \sum_i \int_0^\infty |f(t)|dt$$

And the L_∞-norm of a vector valued signal $f(t)$ is the maximal magnitude (peak) of the entries evaluated for all all time:

$$\|f\|_\infty = \max_i \sup_{t \geq 0} |f(t)|$$

For a causal system $H: w(t) \rightarrow z(t)$, the system L_1 norm is defined as the system induced $L_\infty \rightarrow L_\infty$ (peak $\rightarrow$ peak) gain:

$$\|H\|_1 := \sup_{\|w\|_\infty \leq 1 \text{ for } t \geq 0} \{\|z\|_\infty : x(0) = 0, \forall t \geq 0\} \tag{5.59}$$

If the system is causal and LTI $z(t) = h(t) * w(t)$, where $h(t)$ is an $r \times q$ impulse response matrix, the L_1 norm turns out to be

$$\|H\|_1 = \sup_{\|w\|_\infty \leq 1} \|h * w\|_\infty = \max_i \sum_{j=1}^{q} \|h_{ij}\|_1 \tag{5.60}$$

For a causal system, it can be proved that the system L_1 norm is an upper bound of the system L_p norm [SzB98]:

$$\|H\|_p \leq \|H\|_1, \quad p = 1, 2, \infty \tag{5.61}$$

From the definition (5.59), we can see that L_1 norm $\|H\|_1$ is the worst-case peak gain in time domain, while we already know in Chapter 3 that H_∞ norm is the worst-case peak gain in the frequency domain. Besides specifying the signal magnitude requirement in the time domain, the L_1 norm can also specify the rejection to persistent bounded disturbance, or describe the system robustness under uncertainty.

Unlike the H_2 or H_∞ problem whose optimal controller order is limited by the generalized plant order, in L_1 (or l_1 for discrete) control the order of the optimal solution can be arbitrarily high. Dahleh developed a synthesis approach via linear programming combined with duality theory, and more details can be found in the book [DaD95]. More recently, Kahammash [Kal00] proposed a new approach for l_1 optimization, referred as the scaled-Q method. There some computational burden is avoided and the controller recovery was made straightforward.

5.5.2 L_1 Norm Associated Multi-Objective Control

In the past several years, many L_1 (or l_1 associated multi-objective optimizations have been addressed, such as H_2/l_1 or L_1 ([ElD98] [AmA99]), L_1/H_2 ([ABS98]), l_1/H_∞ ([SzB98]), l_1/poles ([Hal00]), and more general multi-objective control [SGC97] [QKS01]. LMI tools are also used for H_2/L_1 or l_1/H_p sub-optimization ([SHB95] [SGC97] [BuS97]).

Elia and Dahleh [ElD98] generalized the *linear programming approach* for the standard l_1 problem to multi-objective control. They expressed a large class of specifications as linear equalities or inequalities in the unknowns. Then a linear primal problem and its dual are obtained:

$$\text{(primal)}\ \min_x c^T x \qquad \text{s.t.} \begin{cases} Ax = b \\ x_i \geq 0,\ i = 1, ...n \end{cases} \qquad\qquad \text{(dual)}\ \max_y y^T b \qquad \text{s.t. } y^T A \leq c^T$$

By examining the dual problem, it can be shown that the general multi-block problem, which has infinitely many variable and constraints, is partly finite dimensional. The parts which are still infinite dimensional are approximated by appropriate truncation of the original problem. The basic approximation methods ([DaB95] [VrJ97]) are

1. Finitely many variables (FMV): provide a suboptimal polynomial feasible solution by constraining the number of (primal) variable to be finite;

2. Finitely many equations (FME): provide a superoptimal infeasible solution by including only a finite number of (primal) equality constraints;

3. Delay augmentation (DA): provides both a suboptimal and a superoptimal solution by embedding the problem into a one-block problem through the aug-

mentation of the operators U and V with delays, where U and V are within the Youla parameterization $H_{zw} = W + UQV$. This method is used more often then FME/FMV since it doesn't necessarily suffer from order-inflation when inputs and outputs are (re)ordered properly.

Sznaier and his coworkers ([ABS98] [AmS99]) extended Elia and Dahleh's work [ElD98] to continuous-time systems. They showed that the mixed H_2/L_1 and L_1/H_2 problems lead to non-rational solutions, even when the original plant is rational. And they showed that the optimal cost can be approximated arbitrarily closely by rational controllers that can be synthesized by solving an auxiliary discrete-time problem, obtained by Euler approximation $s = (z-1)/\tau$ of the plant, together with an additional interpolation constraint.

Sznair and Bu [SzB98] investigated the mixed l_1/H_∞ problem, and obtained the rational ϵ-optimal controller by solving a finite dimensional *convex optimization* problem together with an unconstrained H_∞ problem.

$$\begin{aligned} \inf_{Q(z)\in RH^\infty} \|H_{z_1 w_1}(z)\|_1 &= \inf_{Q(z)\in RH^\infty} \|W_1(z) + U_1(z)Q(z)V_1(z)\|_1 \\ \text{s.t. } \|H_{w_2 z_2}(z)\|_\infty &= \|W_2(z) + U_2(z)Q(z)V_2(z)\|_\infty \leq \gamma \end{aligned} \tag{5.62}$$

To avoid the potential failure of convergence by sampling on the unit circle, they used $RH^{\infty,\delta}$ to replace RH^∞, and thus a sequence of modified problems were solved and proved to converge to an optimum.

$$\|H(z)\|_{\infty,\delta} = \sup_{0\leq w\leq\pi} \sigma_{\max}[H(\delta e^{jw})]$$

They also showed that contrary to the H_2/H_∞ problem [Meg94], the l_1/H_∞ problem admits an optimal solution in l_1. Continuous-time L_1/H_∞ ϵ-optimal controller was synthesized by solving a discrete-time l_1/H_∞ problem obtained by Euler approximation. The most severe limitation is that a very high order controller might be produced and model reduction is necessary.

The scaled-Q method proposed by Khammash [Kha00] has also been used in $l_1/H_2/H_\infty$ problem. l_1 suboptimal control with exact pole placement was also reported [Hal00]

LMI techniques have also been used for L_1 (l_1) sub-optimization [NAP94], and extended to L_1 (l_1) associated multi-objective control with full state or full-order feedback [SHB94] [SGC97] [BuS97]. Given a closed-loop system (A_c, B_c, C_c, D_c), Nagpal and Abedor [NAP94] [ANP96] showed that L_1 norm is bounded from above by λ

$$||H||_1 < \lambda$$

where λ satisfies the following matrix inequalities with matrix parameter P and scalar parameters α and μ:

$$\begin{aligned} \alpha &> 0 \\ \begin{bmatrix} A_c'P + PA_c + \alpha P & PB_c \\ B_c'P & -\mu I \end{bmatrix} &< 0 \\ \begin{bmatrix} \alpha P & 0 & C_c' \\ 0 & (\lambda - \mu)I & D_c' \\ C_c & D_c & \lambda I \end{bmatrix} &> 0 \end{aligned} \tag{5.63}$$

Note that because of the item αP, the inequalities (5.63) are in LMI form only for fixed α, hence a linear search over $\alpha > 0$ is required. It also should be clear that that the upper bound obtained in such a way might be fairly conservative, especially for a lightly-damped closed-loop system. By imposing a common matrix P, the above (5.63) is readily combined with other specifications in multi-objective control synthesis via full-state or full-order feedback ([SHB95] [SGC97]).

No structured controller synthesis has been seen for L_1 associated multi-objective control.

We also notice that recently generalized H_2 performance (energy $\rightarrow$ peak) [Rot93]) and general quadratic constraints [WaW01] are also formulated in LMI form, and have also been taken account of in multi-objective control [SGC97]. Some other objectives, such as direct time specification [QKS01], or passivity requirement [SGC97], were also discussed.

5.6 Applications: Design of Passive Mechanical Systems

The ground vehicle (suspension and steering) is one typical application of multi-objective control ([TCZ98] [YoC99] [WaW01] [ChG01]). Other applications can be found in aerospace systems [PMF00], power systems [CML00] and others. In the following we will give some examples to show the application to passive mechanical system design via structured H_2 control with pole regional placement and other methods.

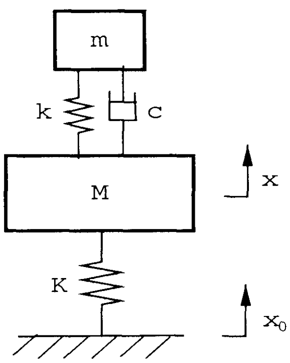

Figure 5-8: Single-Degree-of-Freedom Tuned Mass Damper

We apply the multi-objective control to multiple and single DOF (degree-of-freedom) tuned-mass dampers (TMD). The SDOF TMD system is shown in the Figure 5-8. This is a 4-th order system with structured feedback. (Although this is a static output problem, all the techniques we mention below can be used for decentralized cases). Suppose M and K are normalized as one, and the mass ratio m/M is 10%. We employ five design methods: (1) the gradient-based H_2 optimization with control structure constraints (presented in Chapter 2); (2) the alternative LMI minimization of H_∞ design in Chapter 3; (3) the minimax method proposed in Chapter 4; (4) H_2 sub-optimization via auxiliary cost (defined by Harddad and Bernsten [HaB92]) with pole placement in a disk region $D(10, 9.99)$ (solved by the gradient-based method); (5) optimal H_2 with pole placement in a sector (10.5% damping required) via the multiplier method we proposed in Section 5.3.

Table 5.1: Results of the five methods applied to the design of SDOF TMD

Method	tuning ratio & damping ratio		closed loop model dampings (%)		H2 norm	H_∞ norm
Den Hartog	0.90909	0.18464	9.607	9.607	1.7890	4.5902
# 1	0.93154	0.15254	7.207	8.649	1.7681	5.2368
# 2	0.90927	0.18309	9.518	9.533	1.7877	4.5900
# 3	0.90909	0.30151	15.811	15.811	1.9607	6.4141
# 4	0.9245	0.2092	10.1	11.7	1.811*	5.044
# 5	0.9091	0.2016	10.5	10.5	1.805	4.653

$*$ note: the auxiliary function minimization yielded an upper bound 2.5434

The design results are shown in Table 5.1 compared with Den Hartog's design. From the table we can see that the Harddad–Bernsten auxiliary cost (upper bound of H_2) (method 4) is quite conservative. Moreover, minimization of the auxiliary cost might make the real H_2 norm worse, which can be seen from Figure 5-9, where the iteration begins with the result produced with minmax. The method of multipliers (method 5) provides a better way to find the optimal H_2 norm with pole placement in some region efficiently. The data in Table 5.1 also show that Den Hartog's fixed-points method (Equ 4.84) produces a system which is very close to that of H_∞ design.

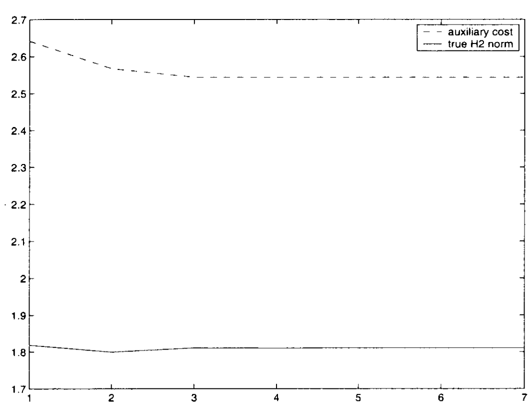

Figure 5-9: Auxiliary cost decreases but H_2 norm becomes worse

The Bode plots of the closed-loops produced with method (1), (2), (3), and (5) are shown in Figure 5-10, and impulse response are shown in Figure 5-11. We can see that optimal H_2 with regional pole placement can produce a system with very good performance. All of the five methods converge well in this simple example. Generally, the computational efficiency of gradient-based methods for structured H_2 optimization are very good, while the subgradient-based minimax converges slowly to the optimum. And there are no convergence guarantees for alternative LMI minimization of H_∞. The efficiency of multiplier methods is fine.

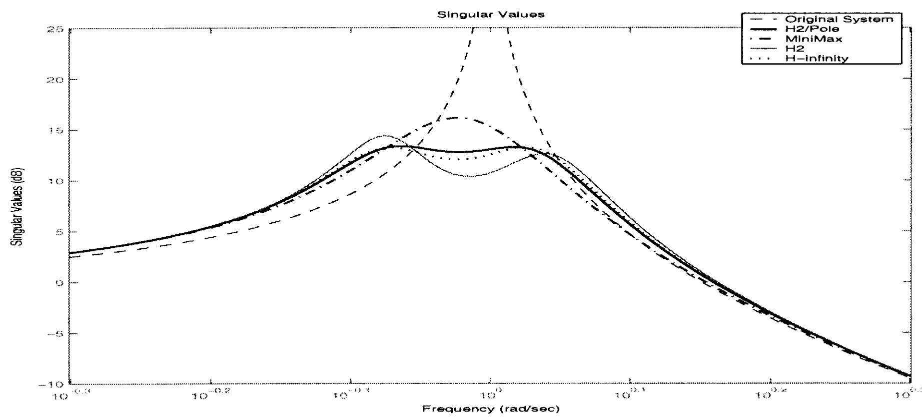

Figure 5-10: Magnitude frequency response of closed loops via different designs

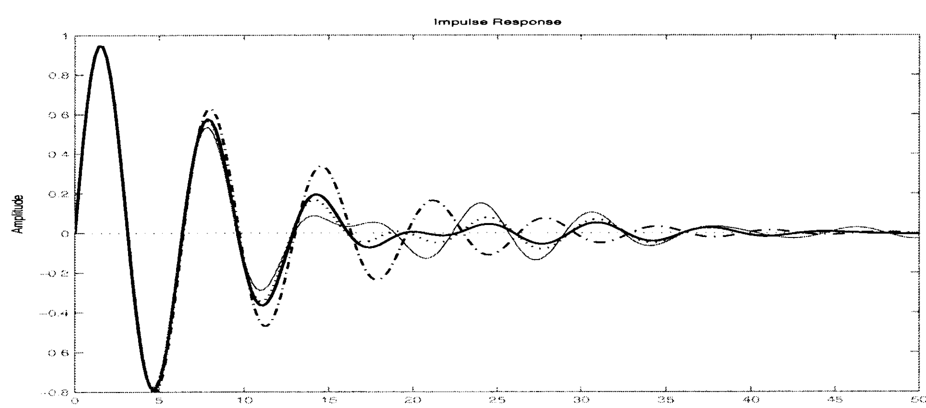

Figure 5-11: Impulse response of closed loops via different designs

Figure 5-12 shows the results of a passive 3DOF TMD designed with decentralized H_2, minimax, and decentralized H_2/pole placement in a sector region (via the method of multipliers). We can see that for this plant H_2/pole region yielded a system with better performance than pure H_2 or minimax pole shifting.

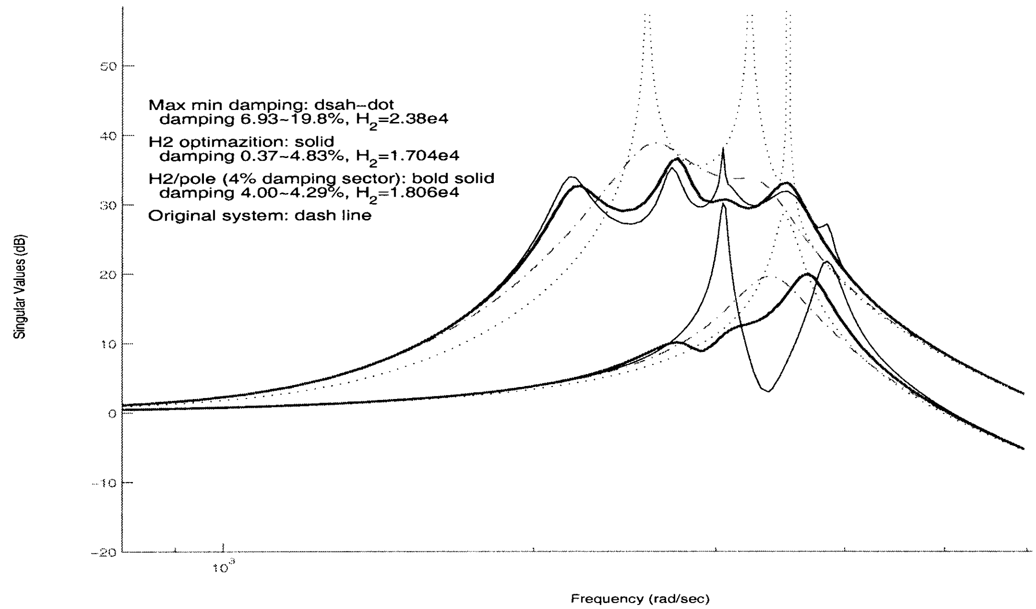

Figure 5-12: Sigma values of 3DOF TMD via minimax, H_2, and H_2/pole

5.7 References

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Chapter 6

Conclusion

6.1 Summary

In this thesis, the parameter design of passive mechanical systems is cast as a state-space control problem with controller structure constraints. A wide class of control techniques are surveyed and investigated, including H_2 control, H_∞ control, eigenvalue and eigenstructure treatment, L_1 control, and multi-objective optimization. Structured control—static output feedback, fixed-order control, and decentralized control—are discussed, as well as unstructured cases—full state feedback and full-order control. Riccati-based methods, LMI, homotopy methods, and nonlinear programming are used.

The system H_2 norm is the energy of the impulse response at zero initial states, or the asymptotic value of system output variance with unit-energy white noise input. It is the 2 norm of the LTI transfer matrix, which can be evaluated by solving a Lyapunov equation. The unstructured (full-state or full-order) H_2 controller can be synthesized via one or two decoupled Riccati equations, or via the LMI techniques. Model-order reduction is a useful approach to get a lower-order suboptimal controller from the full-order design. Projective control and Kosut's suboptimization are concise (but possibly unstabilizing) approximation for structured control. Gradient-based nonlinear programming methods (steepest descent, conjugate gradient, FBGS, etc.) are efficient for decentralized H_2 optimization, as well as static output feedback or reduced-order controller. Homotopy method is also a good approach for optimal H_2 with structure constraints. H_2/LQR design is suitable to meet performance specifications under white noise disturbances.

The system H_∞ norm is the L_2 to L_2 gain, or we say the worst-case peak-to-peak gain in the frequency domain. For LTI systems it is the maximal sigma value of the transfer matrix evaluated at all frequencies. The H_∞ norm provides a natural way to model system uncertainty and is therefore suitable for robust control. Full-state feedback H_∞ controller can be solved by a Riccati equation, while full-order H_∞ optimization yields two coupled Riccati equations. Both can be obtained efficiently with the LMI tools. The cone complementary linearization method is extremely efficient for centralized static output or fixed-order H_∞ control. Decentralized H_∞ results in a BMI problem, and is still open. LMI-based iterations and homotopy methods have been proposed for it.

The so-called L_1 norm is the L_∞ to L_∞ gain, or the worst case peak-to-peak gain in the time domain. The L_1 norm is suited to meet some time-domain specifications, or to reject persistent disturbances. Linear programming with duality, and convex optimization have been developed for the synthesis. However, unlike the optimal H_2 or H_∞ design, the optimal L_1 controllers usually have a very high order, which might be much larger than the generalized plant. LMI-based suboptimal L_1 control has also been developed.

Eigenvalues determine the response rate of a system; right eigenvalues fix the modal shapes and left eigenvalues influence the observability. Under some conditions, eigenvalues can be assigned arbitrarily. Eigenstructure assignment is essentially overdetermined. Parameterization, projection, and protection methods have be proposed for eigenvalue and eigenstructure assignment with full-state feedback, static output feedback, and decentralized feedback. Robust eigenvalue and eigenstructure assignment has also been developed. Regional pole placement has been proposed together with other objectives. With controller structural constraint, it might be impossible to place pole exactly, but maximizing the minimal damping (or shifting the poles to some region) might be ideal in many applications.

Each of the above techniques has its own advantages and disadvantages. Any real design in practice must meet more than one specification. Multi-objective control has been brought up recently, such as H_2/H_∞, H_2/poles, H_∞/poles, H_2/H_∞/poles, L_1/H_2, L_1/H_∞. Auxiliary cost optimization yields some coupled Riccati equations for centralized control. By imposing an additional technical constraint, unstructured multi-objective suboptimal control can be obtained via LMI optimization. Youla parameterization, linear programming, and nonlinear programming approaches have been used. The exact optimal solution of multi-objective control is still a hard problem, even for full state feedback. The decentralized optimal H_2 control with regional pole placement seems to be tractable with nonlinear programming methods. Decentralized H_∞ associated multi-objective control is open.

Decentralized H_2, decentralized H_∞, minimax pole shifting, and decentralized H_2 with regional pole placement have been used in the design of passive mechanical systems. Some numerical examples and experimental results are given.

6.2 Conclusions and Contributions

From the the survey of related literature, we see that unstructured H_2, H_∞, and eigenvalue (and eigenstructure somehow) assignment have been solved. Efficient algorithms for H_2 and H_∞ optimization with static output or reduced-order feedback have also been developed. Static output eigenvalue and eigenstructure assignment are also clear. However, decentralized problems are generally open. Approaches have been proposed for the decentralized H_2 and work well in practice, but convergency requires some strong conditions or are generally dependent on the initial guess. No efficient method is known for the decentralized H_∞ problem. Decentralized pole placement also needs further investigation. Minimax pole shifting has been proposed. Optimal multi-objective control remains an open problem, even for full-state feedback. Centralized suboptimal multi-objective control (H_2, H_∞, eigenvalues) can be obtained with LMI or Riccati-based approach. Decentralized multi-objective control is still under way.

Our main contributions herein are highlighted here:

1. We cast the parameter design of general passive mechanical systems as a decentralized control problem, and a wide range of decentralized control techniques have been used to treat it according different specification requirements.

2. A comprehensive survey of structured and unstructured control has been presented. Topics include H_2, H_∞, eigenvalue and eigenstructure, multi-objective optimization, as well as L_1 control. Some future research directions are suggested.

3. In structured H_2 optimization, more practical constraints are considered. For instance, the parameters are nonnegative, symmetric, or reside in certain ranges. We propose easy-to-implement modifications to handle such constraints.

4. We extend the cone-complementary algorithm to more general static output-feedback H_∞ control, and also multi-objective (H_2/H_∞/poles) control with static output feedback.

5. We present a homotopy method to solve for decentralized H_∞ control, and its efficiency is shown by examples.

6. We use the subgradient-based minimax algorithm and eigenvalue sensitivity to maximize the minimal damping; hysterically damped systems (complex-coefficient matrix feedback) and marginally stable systems are also considered.

7. We adopt the method of multipliers to solve the problem of optimal structured H_2 control with general pole regional constraints. And a tractable optimization problem is also posed for decentralized multi-objective control.

8. Successful application to passive system design. An MDOF (multi-degree-of-freedom) TMD (tuned mass damper) has been designed and built. To the author's knowledge, in the 70 years of TMD research this is the first time that six modes are damped with one passive TMD.

6.3 Future Work

Since many topics that we covered are still open, there is a lot of challenging work we can do in the near future.

LMI is a well-known technique for full-state and full-order control. Recently LMI has been extended to the problems of stabilization, H_2 and H_∞ optimization with centralized static output, and achieved extremely satisfactory computational efficiency. Decentralized control usually yields a BMI problem, which has been proved to be NP hard. Decentralized stabilization with identical controller or controller gain bound has been proved to be NP hard. Hopefully some efficient and ready-to-implement LMI-based algorithm for decentralized (suboptimal) control might be developed, especially for decentralized H_∞. And it deserves further investigation.

Gradient-based algorithms and other similar iterations have been shown to be efficient in solving for the local minimal structured H_2 controller. The convergence needs more extension. And it might also be possible use nonlinear programming methods for H_∞ and multi-objective control. To find the global optimum, it might help to combine the gradient-based methods and genetic algorithms.

To achieve performance requirements under the system uncertainties and disturbance, we might turn to the (structured and unstructured) multi-objective control. Although some problems have been proved to be almost "hopeless" (such as pure mixed H_2/H_∞), it is still worthy for further investigation considering its potential application in practice. This area, especially for the exact optimal solution, is far from the end, and the future direction is not so clear. Linear and nonlinear programming, LMI, and Q parameterization are examined currently.

To make the application to the parameter design of passive mechanical systems more practical, we need to investigate the approaches for robust performance. Because there are a lot of uncertainties in real systems; and the passive parameters are much harder to adjust than the gains in control application.