On Representations of Quantum Groups and Cherednik Algebras

by

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Abstract

In the first part of the thesis, we study quantum groups associated to a semisimple Lie algebra $\mathfrak{g}$. The classical Chevalley theorem states that for $\mathfrak{h}$ a Cartan subalgebra and W the Weyl group of $\mathfrak{g}$, the restriction of $\mathfrak{g}$-invariant polynomials on $\mathfrak{g}$ to $\mathfrak{h}$ is an isomorphism onto the W-invariant polynomials on $\mathfrak{h}$, $\mathrm{Res}\colon \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^{W}$. A recent generalization of [36] to the case when the target space $\mathbb{C}$ of the polynomial maps is replaced by a finite-dimensional representation V of $\mathfrak{g}$ shows that the restriction map $\mathrm{Res}\colon (\mathbb{C}[\mathfrak{g}] \otimes V)^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}] \otimes V$ is injective, and that the image can be described by three simple conditions. We further generalize this to the case when a semisimple Lie algebra $\mathfrak{g}$ is replaced by a quantum group. We provide the setting for the generalization, prove that the restriction map $\mathrm{Res}\colon (O_q(G) \otimes V)^{U_q(\mathfrak{g})} \to O(H) \otimes V$ is injective and describe the image.

In the second part we study rational Cherednik algebras $H_{1,c}(W, \mathfrak{h})$ over the field of complex numbers, associated to a finite reflection group W and its reflection representation $\mathfrak{h}$. We calculate the characters of all irreducible representations in category $\mathcal{O}$ of the rational Cherednik algebra for W the exceptional Coxeter group H_3 and for W the complex reflection group G_{12}. In particular, we determine which of the irreducible representations are finite-dimensional, and compute their characters.

In the third part, we study rational Cherednik algebras $H_{t,c}(W, \mathfrak{h})$ over the field of finite characteristic p. We first prove several general results about category $\mathcal{O}$, and then focus on rational Cherednik algebras associated to the general and special linear group over a finite field of the same characteristic as the underlying algebraically closed field. We calculate the characters of irreducible representations with trivial lowest weight of the rational Cherednik algebra associated to $GL_n(\mathbb{F}_{p^r})$ and $SL_n(\mathbb{F}_{p^r})$, and characters of all irreducible representations of the rational Cherednik algebra associated to $GL_2(\mathbb{F}_p)$.

Thesis Supervisor: Pavel Etingof
Title: Professor of Mathematics

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Finally, to Michael, for always being here, I want to thank in the poetic words of Mike Slackenerny:

"I'm loving you to pieces
To you I dedicate my thesis."

Contents

Chapter 1

Introduction

1.1 Chevalley restriction theorem for quantum groups

The first part (Chapter 2) of this thesis generalizes the classical Chevalley restriction theorem about restrictions of invariant polynomial maps on Lie algebras to the case when the polynomial maps are vector-valued, with values in a representation V, and the Lie algebra is replaced by a quantum group.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ its Cartan subalgebra, W its Weyl group, G the connected simply connected algebraic group associated to $\mathfrak{g}$, and H the maximal torus of G corresponding to $\mathfrak{h}$.

We can consider the space $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ of polynomial functions from $\mathfrak{g}$ to $\mathbb{C}$, invariant with respect to the coadjoint action of $\mathfrak{g}$. Such functions can be restricted to polynomial functions on $\mathfrak{h}$. The classical Chevalley restriction theorem (the graded version of the Harish-Chandra isomorphism) states that the restriction map $\mathrm{Res}\colon \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]$ is injective, and that the image is $\mathbb{C}[\mathfrak{h}]^W$, the space of polynomial functions $\mathfrak{h} \to \mathbb{C}$ invariant under the action of W.

There is also a version of this isomorphism for quantum groups, see [46].

Recently Khoroshkin, Nazarov and Vinberg [36] generalized this result to the case when the target space of polynomial maps is V, a finite-dimensional representation of $\mathfrak{g}$. Let E_i be the Chevalley generators of $\mathfrak{g}$ associated to positive simple roots α_i. Use $X.v$ to denote the action of $X \in \mathfrak{g}$ on $v \in V$, and similarly $X.f(y)$ to denote the action of $X \in \mathfrak{g}$ on the value $f(y)$ of a function $f\colon \mathfrak{h} \to V$ (so, if we think of f as an element of $\mathbb{C}[\mathfrak{h}] \otimes V$, then $E_i.f = (1 \otimes E_i)f$). [36] showed:

Theorem 2.1.1. *The map* $\mathrm{Res}\colon (\mathbb{C}[\mathfrak{g}] \otimes V)^G \to \mathbb{C}[\mathfrak{h}] \otimes V$ *is injective. Its image consists of those functions* $f \in \mathbb{C}[\mathfrak{h}] \otimes V$ *that satisfy:*

1. $f \in \mathbb{C}[\mathfrak{h}] \otimes V[0]$;
2. f *is* W*-equivariant;*
3. *for every simple root* $\alpha_i \in \Pi$ *and every* $n \in \mathbb{N}$*, the polynomial* $E_i^n.f$ *is divisible by* α_i^n.

We want to generalize this theorem to quantum groups. This is more convenient to do in the setting of algebraic groups, to which [36] theorem generalizes naturally and with an almost identical proof.

Consider the Hopf algebra $O(G)$ of polynomial functions on the group G. It comes with a natural restriction map to $O(H)$, which can be extended to the case when the target space is V, giving $O(G) \otimes V \to O(H) \otimes V$. We are interested in what this restriction map does to equivariant functions $(O(G) \otimes V)^G$, namely the ones that satisfy $f(gxg^{-1}) = g.f(x)$ for all $x, g \in G$. The result of [36] modified to the setting of algebraic groups is:

Theorem 2.1.5. *The map* $\mathrm{Res}\colon (O(G) \otimes V)^G \to O(H) \otimes V$ *is injective. Its image consists of those functions* $f \in O(H) \otimes V$ *that satisfy:*

1. $f \in O(H) \otimes V[0]$;
2. f *is* W*-equivariant;*
3. *for every simple root* α_i *and every* $n \in \mathbb{N}$*, the polynomial* $E_i^n.f$ *is divisible by* $(1 - e^{\alpha_i})^n$.

This setting is more convenient for generalization to quantum groups for the following reason. By the Peter-Weyl theorem, $O(G) \cong \bigoplus_L L^* \otimes L$, where the direct sum is taken over isomorphism classes of finite-dimensional irreducible representations of

G (equivalently: over dominant integral weights λ - in that case $L = L_\lambda$), and L^* denotes the dual representation. In this setting $O(H) \cong \oplus_\mu \mathbb{C}_\mu^* \otimes \mathbb{C}_\mu$, where for any integral weight μ, $\mathbb{C}_\mu$ is the one dimensional representation on which H acts by the character e^μ. With these isomorphisms, the restriction map $\mathrm{Res}\colon O(G) \to O(H)$ is easy to describe and corresponds to decomposing the irreducible representation L into its weight spaces (see discussion after theorem 2.1.5 for details).

By analogy with the classical case, in the quantum case we define $O_q(G) = \bigoplus_L {}^*L \otimes L$, with ${}^*L$ being the left dual of L (see Section 7.3.3), and the sum again being over all dominant integral weights λ, with $L = L_\lambda$ the irreducible representation of $U_q(\mathfrak{g})$ with highest weight λ. As all such representations have integral weights, there is again a natural restriction map to $O(H)$. For V a finite-dimensional representation of $U_q(\mathfrak{g})$, we then consider the restriction map

$$(O_q(G) \otimes V)^{U_q(\mathfrak{g})} \to O(H) \otimes V.$$

Let E_i denote the standard generator of $U_q(\mathfrak{g})$ associated to α_i, and let $q_i = q^{d_i} = q^{<\alpha_i, \alpha_i>/2}$. The main result of this part of the thesis is:

Theorem 2.2.4. *The map* $\mathrm{Res}\colon (O_q(G) \otimes V)^{U_q(\mathfrak{g})} \to O(H) \otimes V$ *is injective. Its image consists of those functions* $f \in O(H) \otimes V$ *that satisfy:*

1. $f \in O(H) \otimes V[0]$;
2. f *is invariant under the (unshifted) action of the dynamical Weyl group (see Section 2.3)*
3. *for every simple root* α_i *and every* $n \in \mathbb{N}$*, the polynomial* $E_i^n.f$ *is divisible by*

$$(1 - q_i^2 e^{\alpha_i})(1 - q_i^4 e^{\alpha_i}) \dots (1 - q_i^{2n} e^{\alpha_i}).$$

Obviously, this statement is a direct generalization of the one for $q = 1$ case. Checking that restrictions to $O(H) \otimes V$ satisfy properties (1) and (3) is a direct computation; checking (2) requires more tools. The most involved part of the proof is checking that every function in $O(H) \otimes V$ that satisfies $(1)-(3)$ is a restriction of an element of $(O_q(G) \otimes V)^{U_q(\mathfrak{g})}$. The proof of the analogous statement in [36] uses some basic geometric observations which are not available in the quantum case (these observations follow from $O(G)$ being an algebra of polynomial functions on the algebraic variety G). This is the reason their proof cannot be directly generalized.

Instead, the space of invariants can be rewritten in another way, namely as

$$(O_q(G) \otimes V)^{U_q(\mathfrak{g})} = \bigoplus_L ({}^*L \otimes L \otimes V)^{U_q(\mathfrak{g})} \cong \bigoplus_L \mathrm{Hom}_{U_q(\mathfrak{g})}(L, L \otimes V).$$

This natural isomorphism composed with the restriction map above reformulates the problem in terms of traces of intertwining operators $L \to L \otimes V$, as it turns out that $\Phi \in \mathrm{Hom}_{U_q(\mathfrak{g})}(L, L \otimes V)$ maps to the function on H given by $x \mapsto \mathrm{Tr}|_L(\Phi \circ x)$. Such functions have been extensively studied in recent years, among others by Etingof and Varchenko in [27], [26], and satisfy a number of remarkable symmetry properties and difference equations. Reframing the problem in terms of trace functions enables us to draw from those results to prove Theorem 2.2.4.

This work is available in [2].

1.2 Rational Cherednik algebras and their representations

A rational Cherednik algebra $H_{t,c}(W, \mathfrak{h})$ is a certain associative, noncommutative, infinite-dimensional algebra over an algebraically closed field $\Bbbk$, associated to a finite reflection group W, its reflection representation $\mathfrak{h}$ and a collection of parameters t and c. Parameter t is an element of the field $\Bbbk$, and c is a collection of elements of $\Bbbk$ parametrized by the conjugacy classes of reflections in W. The algebra $H_{t,c}(W, \mathfrak{h})$ is a deformation of the semidirect product of the group algebra $\Bbbk[W]$ and the symmetric algebra $S(\mathfrak{h} \oplus \mathfrak{h}^*)$, and $H_{0,0}(W, \mathfrak{h}) \cong \Bbbk[W] \ltimes S(\mathfrak{h} \oplus \mathfrak{h}^*)$.

In case W is a Weyl group, these algebras are rational degenerations of double affine Hecke algebras, which were defined by Cherednik [14] and used to prove Macdonald conjectures. They can also be thought of in relation to completely integrable systems as algebras encoding the structure of Dunkl operators (considered by Dunkl in [18] and Dunkl and Opdam in [20]), and Calogero-Moser systems [21], or as a special case of symplectic reflection algebras of Etingof and Ginzburg [23]. Such algebras and their representation theory have been intensively studied in the last fifteen years.

A natural class of representations to consider is the category $\mathcal{O}$ (sometimes called $\mathcal{O}_{t,c}$ or $\mathcal{O}_{t,c}(W,\mathfrak{h})$). One can define standard or Verma modules $M_{t,c}(\tau)$, which are certain lowest weight modules parametrized by the set of irreducible representations τ or the group W. The algebra $H_{t,c}(W,\mathfrak{h})$ and the modules $M_{t,c}(\tau)$ are graded. There exists a contravariant form B on Verma modules, such that its kernel on $M_{t,c}(\tau)$ is the maximal proper graded submodule of $M_{t,c}(\tau)$. The quotient of $M_{t,c}(\tau)$ by this kernel is irreducible, and we call it $L_{t,c}(\tau)$. Category $\mathcal{O}$ is defined in such a way that all irreducible objects in it (up to, possibly, grading shifts) are of the form $L_{t,c}(\tau)$.

An open question in representation theory of rational Cherednik algebras is to describe the modules $L_{t,c}(\tau)$ (for example, by calculating their characters, finding the dimensions of finite-dimensional ones, finding the composition series of Verma modules in terms of irreducible modules, or even describing the values of t,c and sets of τ for which such representations are finite-dimensional).

1.3 Rational Cherednik algebras over $\mathbb{C}$

There is a difference in the behavior of the irreducible quotients $L_{t,c}(\tau)$ and consequently in the definition of category $\mathcal{O}$ between cases when the ground field $\Bbbk$ has characteristic 0 and characteristic $p > 0$. The case that has been studied the most is when $\Bbbk = \mathbb{C}$. In that case, the category $\mathcal{O}$ is semisimple for generic values of parameters t and c, and as a consequence $M_{t,c}(\tau)$ are irreducible and equal to $L_{t,c}(\tau)$ for all τ.

More precisely, parameter t can be rescaled to allow us to assume $t = 0$ or $t = 1$. For $t = 0$, the algebra $H_{t,c}(W,\mathfrak{h})$ has a large center, so $M_{t,c}(\tau)$ has a large submodule and the category $\mathcal{O}$ is never semisimple. For $t = 1$, there exists a KZ functor (defined by Ginzburg, Guay, Opdam and Rouquier in [30]) from category $\mathcal{O}$ to the category of representations of the Hecke algebra $\mathcal{H}_q(W)$, with the parameter q depending on c. If c is a constant, then $q = e^{2\pi ic}$. Using this functor one can prove that the $\mathbb{C}$-algebra $\mathcal{H}_q(W)$ is semisimple if and only if $\mathcal{O}_{1,c}$ is semisimple. This, along with the information about semisimplicity conditions for Hecke algebras from [13], [15], [29] and [38], resolves the questions about $L_{1,c}(\tau)$ for generic values of parameter c (namely, $L_{1,c}(\tau) = M_{1,c}(\tau)$ for generic c, and the KZ functor enables us to find conditions on c for which this is true). However, it gives no information about what happens at $t = 1$ and special values of c. The structure of category $\mathcal{O}$ can be quite complicated there, and no general description of the irreducible modules, or even information about which ones are finite-dimensional, is known.

Partial information known includes: for $(W, \mathfrak{h})$ of type A, Berest, Etingof and Ginzburg [8] calculate the character formulas for all the finite-dimensional $L_{1,c}(\tau)$. Also for type A, Rouquier [41] calculates all the characters for c not a half integer, and conjectures that the analogous formulas hold for c a half integer. For dihedral groups, Chmutova [16] computes the characters of irreducible modules in category $\mathcal{O}$. Varagnolo and Vasserot [47] answer the question of when is the representation $L_{1,c}(\tau)$ finite-dimensional for W a Weyl group, c a constant, and τ a trivial representation of W. A generalization of this is a recent result by Etingof [22], which gives an answer for any finite Coxeter group W, trivial τ, and any value of the parameter c.

All these results are standard, and described in Chapter 3.

1.4 Rational Cherednik algebras associated to reflection groups H_3 and G_{12}

We tackle the problem of describing the modules $L_{1,c}(\tau)$ for all values of c and τ for $\Bbbk = \mathbb{C}$, and for W the exceptional Coxeter group H_3 (Chapter 4) and the complex reflection group G_{12} in the Shephard-Todd notation (Chapter 5). Both these groups have only one conjugacy class of reflections, so c is a single complex parameter.

The strategy is similar for both $W = H_3$ and $W = G_{12}$. We first use KZ_c functor to determine the set of parameters c for which category $\mathcal{O}_{1,c}$ is not semisimple. This gives a description of $L_{1,c}(\tau)$ (namely, $L_{1,c}(\tau) = M_{1,c}(\tau)$) for all but countably many rational values of c. For $W = H_3$, these are rational numbers of the form $c = m/d$, whose denominator d divides a degree of a basic invariant of H_3, so $d \in \{2, 3, 5, 6, 10\}$. For $W = G_{12}$, we use the CHEVIE package of the computer algebra software GAP [38] to find the semisimplicity conditions on the Hecke algebra, and translating them to the rational Cherednik algebra using the KZ functor, we get that $\mathcal{O}_{1,c}$ is semisimple unless $c = m/12$, $m \in \mathbb{Z}$, $m \equiv 1, 3, 4, 5, 6, 7, 8, 9, 11 (\mathrm{mod}\, 12)$.

Next, we use a series of equivalences of categories to reduce the set of pairs (c, τ) for which we need to calculate the characters to a small finite set. There are equivalences between $\mathcal{O}_{1,c}$ and $\mathcal{O}_{1,fc}$ for any character f of the group, coming from the isomorphism between $H_{1,c}(W, \mathfrak{h})$ and $H_{1,fc}(W, \mathfrak{h})$. By defining f to be a signum character (taking value -1 on all simple reflections), we can assume $c > 0$. Next, there are equivalences of categories between category $\mathcal{O}_{1/d}(W, \mathfrak{h})$ and category $\mathcal{O}_{r/d}(W, \mathfrak{h})$, in the case $d \neq 2$ (defined by Rouquier in [40]), and finally between category $\mathcal{O}_c(W, \mathfrak{h})$ and category $\mathcal{O}_{c+1}(W, \mathfrak{h})$ for $c >> 0$ (defined by Berest, Etingof and Ginzburg in [8]). It is known how these functors act on the standard and irreducible modules, and consequently how the characters transform under them.

All this allows us to reduce the possible values of c that we need to consider for each group to a very small set; $c \in \{1/10, 1/6, 1/5, 1/3, 1/2, 3/2\}$ for $W = H_3$ and $c \in \{1/12, 1/4, 1/3, 1/2\}$ for G_{12}. For those values, we use a variety of algebraic, combinatorial and computational methods, including representation theory of finite groups, induction and restriction functors for rational Cherednik algebras, and explicit calculation of the contravariant form B. This form can be calculated inductively on the graded pieces on $M_{1,c}(\tau)$, and in cases where we cannot resolve the structure of $L_{1,c}(\tau)$ in any other way, we use MAGMA algebra software [11] to calculate this form and its kernel explicitly.

The work about representations of $H_{1,c}(H_3, \mathfrak{h})$ is joint with Arjun Puranik and available in [6], and the work about representations of $H_{1,c}(G_{12}, \mathfrak{h})$ is joint with

Christopher Policastro and available in [5]. The main results of this part are Theorems 4.2.1 and 5.2.1.

1.5 Rational Cherednik algebras in positive characteristic

When $\Bbbk$ is an algebraically closed field of positive characteristic p, the rational Cherednik algebra $H_{t,c}(W, \mathfrak{h})$ has a large center. Consequently, the modules $M_{t,c}(\tau)$ always have a large submodule, and we can define *baby Verma modules* $N_{t,c}(\tau)$ as quotients of Verma modules by this submodule. These modules are graded, the form B descends to them, and irreducible modules can be alternatively realized as quotients of baby Verma modules by the kernel of the induced form B, which is the maximal proper submodule. The baby Verma modules are always finite-dimensional.

In finite characteristic, we define category $\mathcal{O}$ to be the category of finite-dimensional graded modules. It contains irreducible modules and baby Verma modules, but not Verma modules. All the irreducible objects in it, up to grading shifts, are of the form $L_{t,c}(\tau)$. The kernel of B on $M_{t,c}(\tau)$ is the maximal proper graded submodule, and the kernel of B on $N_{t,c}(\tau)$ is the maximal proper submodule.

Every graded piece of the graded module $M_{t,c}(\tau)$ and all its submodules and quotients is a representation of W. We define characters which reflect this information. We then prove that for $t = 1$ and generic c characters are of specific form, depending on the structure of a certain *reduced module*. We also give an upper bound for the dimension of irreducible modules for $t = 1$.

These definitions and general observations about category $\mathcal{O}$ for rational Cherednik algebras in finite characteristic are new, and given in Chapter 6.

1.6 Rational Cherednik algebras associated to general and special linear group over a finite field

Chapter 7 calculates the character formulas for the irreducible representations $L_{t,c}(\tau)$ for all values of (t, c), for τ = triv the trivial representation of W, and for $W = GL_n(\mathbb{F}_q)$ and $W = SL_n(\mathbb{F}_q)$ for all $n \geq 2$, $q = p^r$, and $\mathbb{F}_q$ the finite field of characteristic p with $q = p^r$ elements. The main theorems are 7.2.1, 7.2.8, 7.3.6 and 7.3.11. The descriptions of these characters is somewhat surprising: for fixed t and for n and p large enough, the characters don't depend on the value of c, and the form B diagonalizes in the appropriate basis. These phenomena never occur in characteristic zero.

Chapter 8 calculates the characters of representations $L_{t,c}(\tau)$ for generic c and all τ for the case when W is the group $GL_2(\mathbb{F}_p)$. The main result is 8.2.1.

The methods of chapters 7 and 8 are direct combinatorial, group theoretic and representation theoretic computations. We used MAGMA in the initial stages of the project, to calculate low rank examples and form conjectures, but we don't reference those computations in proofs. The hope is that these detailed computations of the structure of category $\mathcal{O}$ for specific classes of groups, alongside with those from Chapters 4 and 5, can serve as steps in the direction of fuller understanding of rational Cherednik algebras associated to any group W.

The work about rational Cherednik algebras in positive characteristic is joint with Harrison Chen and available in [4], [3].

Chapter 2

Chevalley Restriction Theorem for Vector-valued Functions on Quantum Groups

2.1 The generalized Chevalley restriction theorem in the classical case

Through this chapter, let $C = (a_{ij})$ be a Cartan matrix of finite type of size r, and $(\mathfrak{h}, \mathfrak{h}^*, \Pi, \Pi^\vee)$ its realization. This means that $\mathfrak{h}$ is an r-dimensional vector space over $\mathbb{C}$ with a basis $\Pi^\vee = \{h_1, \dots h_r\}$, $\mathfrak{h}^*$ its dual space with a basis of simple positive roots $\Pi = \{\alpha_1, \dots \alpha_r\}$, and $\alpha_i(h_j) = a_{ji}$. The matrix C is symmetrizable, so we let d_i be the minimal positive integers that satisfy $d_i a_{ij} = d_j a_{ji}$. Define a symmetric bilinear form on $\mathfrak{h}^*$ by $\langle \alpha_i, \alpha_j \rangle = d_i a_{ij}$ and on $\mathfrak{h}$ by $\langle h_i, h_j \rangle = d_i^{-1} a_{ji}$. Both of these forms induce the same isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ by $\alpha_i \leftrightarrow d_i h_i$. Let H be a complex torus of rank r, so that the Lie algebra of H is $\mathfrak{h}$, and let $\exp\colon \mathfrak{h} \to H$ be the exponential map, such that its kernel is $\mathbb{Z}$-spanned by $2\pi i h_j$ (in other words, exp realizes H as a quotient of $\mathfrak{h}$ by the lattice $\mathbb{Z}2\pi i \Pi^\vee$). We will write elements of H by $x = \exp(h) = e^h$, $h \in \mathfrak{h}$, and characters on the group accordingly, meaning $e^\alpha\colon H \to \mathbb{C}$ is a character corresponding to $\alpha \in \mathfrak{h}^*$, such that $e^\alpha(e^h) = e^{\alpha(h)}$. Let W be the Weyl group associated to this data.

Let P be the weight lattice (set of all $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) \in \mathbb{Z}\ \forall i$), and P_+ the set of dominant integral weights ($\lambda \in P$ such that $\lambda(h_i) \in \mathbb{N}_0\ \forall i$).

Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over $\mathbb{C}$ with a Cartan matrix C and Cartan subalgebra $\mathfrak{h}$. Let G be the connected simply connected complex algebraic group with Lie algebra $\mathfrak{g}$, maximal torus H, and $\exp\colon \mathfrak{g} \to G$ the exponential map that restricts to $\exp\colon \mathfrak{h} \to H$. For each root α let $\mathfrak{g}_\alpha$ be the appropriate root space, and for every simple root α_i let $E_i \in \mathfrak{g}_{\alpha_i}, F_i \in \mathfrak{g}_{-\alpha_i}$ denote the Chevalley generators of $\mathfrak{g}$; these satisfy $[E_i, F_j] = \delta_{ij} h_i$ and for every i determine a copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

For every $\lambda \in P$ let M_λ be the Verma module with highest weight λ, generated by a distinguished highest weight vector m_λ. For every dominant integral $\lambda \in P_+$, the module M_λ has an irreducible finite-dimensional quotient that we call L_λ. Call the image of m_λ in it l_λ. For any finite-dimensional $\mathfrak{g}$-module V and any $\nu \in P$, set $V[\nu] = \{v \in V | h.v = \nu(h)v\ \forall h \in \mathfrak{h}\}$, the weight space of V of weight ν.

Because G is simply connected, finite-dimensional representation theory of G and $\mathfrak{g}$ is the same. In particular, for any finite-dimensional V with an action of G and action of $\mathfrak{g}$ derived from it, the set of invariants is the same: $\{v \in V | g.v = v\ \forall g \in G\} = V^G = V^{\mathfrak{g}} = \{v \in V | X.v = 0\ \forall X \in \mathfrak{g}\}$. Because of this, in this section we will be passing from G representations to $\mathfrak{g}$ representations and back without comments.

Consider the set $\mathbb{C}[\mathfrak{g}]$ of all polynomial functions on $\mathfrak{g}$. The group G acts on it by the coadjoint action: for $f \in \mathbb{C}[\mathfrak{g}], X \in \mathfrak{g}, g \in G$, $(gf)(X) = f(\mathrm{Ad}(g^{-1})X)$. Let V be any finite-dimensional G and $\mathfrak{g}$ representation; we will write both actions with a dot: $g.v$ and $X.v$. Consider the space $\mathbb{C}[\mathfrak{g}] \otimes V$ of polynomial functions on $\mathfrak{g}$ with values in V. Let G act on this space diagonally on both tensor factors. This means that $g \in G$ maps $f \in \mathbb{C}[\mathfrak{g}] \otimes V$ to a polynomial function on $\mathfrak{g}$ given by $X \mapsto g.f(\mathrm{Ad}(g^{-1})X)$. Call the set of invariants with respect to this diagonal action $(\mathbb{C}[\mathfrak{g}] \otimes V)^G$; these are functions f that satisfy $g.f(X) = f(\mathrm{Ad}(g)X)$ for all $g \in G, X \in \mathfrak{g}$.

There is an obvious restriction map $\mathrm{Res}\colon \mathbb{C}[\mathfrak{g}] \otimes V \to \mathbb{C}[\mathfrak{h}] \otimes V$, and $\mathrm{Res}\colon (\mathbb{C}[\mathfrak{g}] \otimes V)^G \to \mathbb{C}[\mathfrak{h}] \otimes V$. The graded version of the main result of [36] (Theorem 2) describes this latter map.

Theorem 2.1.1. *The map* $\mathrm{Res}\colon (\mathbb{C}[\mathfrak{g}] \otimes V)^G \to \mathbb{C}[\mathfrak{h}] \otimes V$ *is injective. Its image* *consists of those functions* $f \in \mathbb{C}[\mathfrak{h}] \otimes V$ *that satisfy:*

1. $f \in \mathbb{C}[\mathfrak{h}] \otimes V[0]$;
2. f *is* W*-equivariant;*
3. *for every simple root* $\alpha_i \in \Pi$ *and every* $n \in \mathbb{N}$*, the polynomial* $E_i^n.f$ *is divisible by* α_i^n.

We recall the proof of [36]. We first need a technical lemma about algebraic geometry.

Lemma 2.1.2. *Let* X *be a smooth connected complex algebraic variety,* f *a rational function on* X*, and* Z *a divisor in* X *such that* f *is regular on* $X \setminus Z$*. Assume that for a generic point* z *of* Z *there exists a regular map* $c_z \colon \mathbf{D} \to X$ *from a formal disk* $\mathbf{D}$ *to* X*, such that* $c_z(0) = z$*,* c_z *does not factor through* Z *(so the limit* $\lim_{t\to 0} f(c_z(t))$ *is well-defined), and this limit is finite (equivalently,* $f(c_z(t)) \in \mathbb{C}[[t]]$*). Then* f *is regular at a generic point of* Z*, and hence it is a regular function on* X*.*

Proof. The singular set of the rational function f is a finite union of irreducible divisors, and it is by assumption contained in the divisor Z. So, it is enough to show that f is regular at a generic point of Z to see that f is regular on X.

This is a local problem. By localizing to an open subset of X, we may assume without loss of generality that X is affine and Z is irreducible. One may also assume that Z is given by a polynomial equation $\{Q = 0\}$, for some regular function Q on X such that $dQ \neq 0$ for a generic point of Z.

Since f is a rational function, there exists the smallest integer $m \geq 0$ such that the function $P = fQ^m$ is regular at a generic point of Z.

For a generic point $z \in Z$, using that $\lim_{t\to 0} f(c_z(t))$ is finite, we get

$$P(z) = \lim_{t\to 0} P(c_z(t)) = \lim_{t\to 0} f(c_z(t)) \cdot Q^m(c_z(t)) =$$

$$= \lim_{t\to 0} f(c_z(t)) \cdot \lim_{t\to 0} Q^m(c_z(t)) = \lim_{t\to 0} f(c_z(t)) \cdot 0^m.$$

If $m > 0$, then this implies that $P(z) = 0$ for a generic point z, so P/Q is regular on Z and we can replace P by P/Q and m by $m - 1 \geq 0$, contrary to our choice of m as minimal. So, $m = 0$ and $f = P$ is regular at a generic point of Z.

□

Remark 2.1.3. *Note that the existence of c_z for only one specific point z does not guarantee that f is regular at it. To prove that f is regular at one point z, one would need to show that the limit is finite when approaching z from any direction, not just along c_z. However, the assumption of the lemma is that a function c_z exists for many points of Z at once. In that case, as we showed, f is regular at all points of Z, and hence the limit of f is indeed finite when approaching any point of Z from any direction.*

Remark 2.1.4. *We will first apply this lemma in the proof of theorem 2.1.1 for $X = \mathfrak{g}$, where $c_z(t)$ can be chosen to be linear functions, and then in the proof of 2.1.5 for $X = G$ an algebraic group, where $c_z(t)$ can be chosen to be multiplication by an appropriate element of G.*

We now recall the proof of Theorem 2.1.1.

Proof. Let us first show that the conditions 1)-3) are necessary. Let $f \in (\mathbb{C}[\mathfrak{g}] \otimes V)^G$, and let us abuse notation and write f for $\mathrm{Res} f$.

1) is necessary: For any $x \in H$, $h \in \mathfrak{h}$, we have

$$x.f(h) = f(\mathrm{Ad}(x)h) = f(h).$$

From this is follows that $f(h) \in V[0]$.

2) is necessary: For $N_G(H)$ the normalizer of H in G and $Z_G(H)$ the centralizer of H in G, $Z_G(H) = H$, we have $W = N_G(H)/Z_G(H) = N_G(H)/H$. The space $V[0]$ is the $e^0 = 1$-eigenspace of H, so $N_G(H)$ preserves it and H fixes it pointwise; therefore W acts on it. Because $N_G(H) \subseteq G$, the functions we get are W- equivariant, meaning:

$$f(wh) = w.f(h).$$

3) is necessary: For f a polynomial function on $\mathfrak{g}$ and $X, Y \in \mathfrak{g}$, let

$$\frac{\partial f}{\partial X}(Y) = \frac{\mathrm{d}}{\mathrm{d}t} f(Y + tX)|_{t=0}$$

be the directional derivative. We have the usual Taylor series expansion for the function of one complex variable $t \mapsto f(Y + tX)$ given by

$$f(Y + tX) = \sum_{n \geq 0} \frac{1}{n!} t^n \frac{\partial^n f}{\partial X^n}(Y).$$

Let us write down the invariance condition of f with respect to $\exp(tE_i) \in G$. For $h \in \mathfrak{h}$, and $t \in \mathbb{C}$, we have:

$$\exp(tE_i).f(h) = f(\mathrm{Ad}(\exp(tE_i))h)$$

Expand both sides into a power series in t to get

$$\begin{aligned} \sum_{n \geq 0} \frac{1}{n!} t^n E_i^n.f(h) &= f(\exp(\mathrm{ad}(tE_i))h) \\ &= f(h - t\alpha_i(h)E_i) \\ &= \sum_{n \geq 0} \frac{1}{n!} (-1)^n t^n \alpha_i(h)^n \frac{\partial^n f}{\partial E_i^n}(h) \end{aligned}$$

Looking at the corresponding terms in the power series, we get

$$E_i^n.f(h) = (-1)^n \alpha_i(h)^n \frac{\partial^n f}{\partial E_i^n}(h),$$

which is divisible by α_i^n.

The map Res *is injective:* The set of elements in $\mathfrak{g}$ that are $\mathrm{Ad}(G)$-conjugate to an element of $\mathfrak{h}$ is dense in $\mathfrak{g}$. So, if two G-equivariant polynomial functions on $\mathfrak{g}$ match on $\mathfrak{h}$, they match on its dense G-orbit, so they are the same.

1)-3) are sufficient: So far we have seen that the image of Res is contained in the set of all functions satisfying $1) - 3)$. To see that all functions satisfying 1)-3) are restrictions of equivariant functions on $\mathfrak{g}$, let f be a polynomial function on $\mathfrak{h}$ satisfying 1)-3) and let us try extending it to $\mathfrak{g}$.

Call elements of $\mathfrak{h}$ that are not fixed by any nontrivial element of the Weyl group *regular*, and call the set of all such elements $\mathfrak{h}_{reg}$. It is a complement of finitely many hyperplanes of the form $\mathrm{Ker}\alpha$ in $\mathfrak{h}$, for α a root. Call elements of $\mathfrak{g}$ that are $\mathrm{Ad}(G)$-conjugate to an element of $\mathfrak{h}_{reg}$ *regular semisimple*, and the set of all such elements $\mathfrak{g}_{rs}$. This is an algebraic variety, open and dense in $\mathfrak{g}$. More precisely, there exists a polynomial in $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$, called the discriminant, such that the set $\mathfrak{g}_{rs}$ is the complement of its zero set. The restriction of this polynomial to $\mathfrak{h}$ is the product of all roots of $\mathfrak{g}$.

We can extend the function f to elements of $\mathfrak{g}_{rs}$ by defining $f(\mathrm{Ad}(g)h) = g.f(h)$. More precisely, consider the diagram

$$\begin{array}{ccc} G \times \mathfrak{h}_{reg} & \xrightarrow{\overline{f}} & V \\ \downarrow{\scriptstyle a} & & \uparrow \\ \mathfrak{g}_{rs} & \xrightarrow{i} & \mathfrak{g} \end{array}$$

Here, $a(g,h) = \mathrm{Ad}(g)h$, and i is the inclusion. The map a is surjective and $\mathfrak{g}_{rs}$ is dense in $\mathfrak{g}$, so $i \circ a$ is dominant. It is compatible with the map $\overline{f}(g,h) = g.f(h)$. Indeed, if $i(a(g_1,h_1)) = i(a(g_2,h_2))$, then $h_1 = \mathrm{Ad}(g_1^{-1}g_2)h_2$, so $g_1^{-1}g_2$ is in the normalizer of $\mathfrak{h}$ in G, and therefore a representative of an element of W. Using that $f\colon \mathfrak{h} \to V$ is W-invariant, we get

$$\overline{f}(g_1,h_1) = g_1.f(h_1) = g_1.f(\mathrm{Ad}(g_1^{-1}g_2)h_2) = g_1g_1^{-1}g_2.f(h_2) = \overline{f}(g_2,h_2).$$

Therefore, there is a well-defined rational function $\mathfrak{g} \to V$ that makes the above diagram commute. We claim this is the required extension of $f\colon \mathfrak{h} \to V$ to a G-invariant function on $\mathfrak{g}$.

The restriction of this function to $\mathfrak{h}$ is f, so we abuse notation and call this rational function on $\mathfrak{g}$ by the same name f. By construction, it is G-invariant on $\mathfrak{g}_{rs}$ which is dense in $\mathfrak{g}$, so it is G-invariant on the maximal domain in $\mathfrak{g}$ where it is regular. To prove this is the required function in $(\mathbb{C}[\mathfrak{g}] \otimes V)^G$, we just need to show that it is a regular function on $\mathfrak{g}$.

As it is regular on $\mathfrak{g}_{rs}$, we need to show it is regular on the divisor $D = \mathfrak{g} \setminus \mathfrak{g}_{rs}$, and for that we will use lemma 2.1.2. The assumptions of this lemma refer to a generic point of the divisor D. The set of elements whose semisimple part is conjugate to an element of $\mathfrak{h}$ which is contained in only one hyperplane of the form Kerα is Zariski dense in D. More precisely, the irreducible components of D are

$$D_\alpha = \{\text{elements of } \mathfrak{g} \text{ whose semisimple part is conjugate to an element of Ker}\alpha\},$$

for α a representative of a W-conjugacy class of roots. Therefore we choose all representatives α to be simple roots. Then the set D_{α_i} is equal to

$$D_{\alpha_i} = \{\text{elements whose semisimple part is conjugate to an element of Ker}\alpha_i\},$$

and contains a Zariski dense subset

$$D'_{\alpha_i} = \mathrm{Ad}(G)\{h + E_i | h \in \mathrm{Ker}\alpha_i, h \notin \mathrm{Ker}\beta\ \forall \text{ root } \beta \neq \pm\alpha_i\}.$$

We will check the assumptions of Lemma 2.1.2 on any element $z = \mathrm{Ad}(g)(h+E_i)$ the set D'_{α_i} for any α_i, $g \in G, h \in \mathrm{Ker}\alpha_i$. We construct a function c_z, such that $\lim_{t\to 0} f(c_z(t))$ is finite. Pick $y \in \mathfrak{h}$ such that $\alpha_i(y) = 1$ and define $c_z(t) = \mathrm{Ad}(g)(h + ty + E_i)$. Clearly $c_z(0) = z$. The element $h + ty + E_i$ is conjugate, via $\exp(t^{-1}E_i)$, to

$$\mathrm{Ad}(\exp(t^{-1}E_i))(h + ty + E_i) = h + ty,$$

which is in $\mathfrak{h}_{reg}$ for small $t \neq 0$, so f is well-defined there and c_z doesn't factor through the divisor D. Now calculate the limit, using that f is $G-$invariant:

$$\begin{aligned}
\lim_{t\to 0} f(c_z(t)) &= \lim_{t\to 0} f(\mathrm{Ad}(g)(h+ty+E_i)) \\
&= \lim_{t\to 0} g.f(h+ty+E_i) \\
&= \lim_{t\to 0} g.f(\mathrm{Ad}(\exp(-t^{-1}E_i))(h+ty)) \\
&= \lim_{t\to 0} g.\exp(-t^{-1}E_i).f(h+ty) \\
&= g.\lim_{t\to 0}\sum_{n\geq 0}\frac{1}{n!}(-t)^{-n}E_i^n.f(h+ty).
\end{aligned}$$

This sum is finite because f takes values in V, a finite-dimensional representation on which E_i is nilpotent. $h+ty$ is in $\mathfrak{h}$, and by 3) every term $E_i^n.f(h+ty)$ is divisible by $\alpha_i(h+ty)^n = t^n$. So we can exchange limit and sum, and all of the summands are finite when we let $t \to 0$.

Using lemma 2.1.2, we conclude that f is regular on $\mathfrak{h}$, as required.

□

This theorem can be restated terms of polynomial functions on the group G, with an almost identical proof. Let $O(G)$ be the algebra of polynomial functions on the algebraic group G, and $O(H)$ the algebra of polynomial functions on the subgroup H. There is again the obvious restriction map (quotient of algebras) that we will call $\mathrm{Res}\colon O(G) \to O(H)$. Let Res also denote the tensor product of this map with the identity map on a representation, $\mathrm{Res}\colon O(G)\otimes V \to O(H)\otimes V$. There is also a natural G-action on this tensor product, by acting on the first tensor factor by dual of conjugation in the group, and on the second by a given action on V. The invariants are then functions that satisfy:

$$g.f(x) = f(gxg^{-1}) \ \forall g, x \in G,$$

and the analogous theorem is:

Theorem 2.1.5. *The map* $\mathrm{Res}\colon (O(G)\otimes V)^G \to O(H)\otimes V$ *is injective. Its image* *consists of those functions $f \in O(H) \otimes V$ that satisfy:*

1. $f \in O(H) \otimes V[0]$;

2. f *is W-equivariant;*

3. *for every simple root α_i and every $n \in \mathbb{N}$, the polynomial $E_i^n.f$ is divisible by $(1 - e^{\alpha_i})^n$.*

Proof. Essentially, this proof is the same as proof of Theorem 2.1.1. Necessity of conditions 1) and 2) follows directly. To check condition 3), calculate for $h \in \mathfrak{h}, t \in \mathbb{C}$:

$$\begin{aligned} \exp(-h)\exp(tE_i)\exp(h) &= \exp(Ad(\exp(-h))tE_i) \\ &= \exp(\exp(ad(-h))tE_i) \\ &= \exp(e^{-\alpha_i(h)}tE_i) \end{aligned}$$

It follows that

$$\exp(tE_i)\exp(h)\exp(-tE_i) = \exp(h)\exp((e^{-\alpha(h)} - 1)tE_i).$$

Now

$$\begin{aligned} \sum_{n\geq 0} \frac{1}{n!} t^n E_i^n.f(\exp h) &= \exp(tE_i).f(\exp(h)) \\ &= f(\exp(tE_i)\exp(h)\exp(-tE_i)) \\ &= f(\exp(h)\exp((e^{-\alpha_i(h)} - 1)tE_i)) \\ &= \sum_{n\geq 0} \frac{1}{n!}(e^{-\alpha_i(h)} - 1)^n t^n R_{E_i}^n f(\exp(h)) \\ &= \sum_{n\geq 0} \frac{1}{n!} e^{-n\alpha_i(h)}(1 - e^{\alpha_i(h)})^n t^n R_{E_i}^n f(\exp(h)) \end{aligned}$$

Here $R_{E_i} f$ denotes the derivative of f with respect to left invariant vector field E_i. It is a polynomial function on G. It follows that

$$E_i^n.f(\exp h) = e^{-n\alpha_i(h)}(1 - e^{\alpha_i(h)})^n R_{E_i}^n f(\exp(h)),$$

and it is divisible by $(1 - e^{\alpha_i(h)})^n$.

If the function $f \in O(H) \otimes V$ that satisfies 1)-3) can be extended to a G-equivariant function on G, this can be done in a unique way, because the set of elements conjugate to an element of H is dense in G.

To see it always extends, just as in Theorem 2.1.1, we first extend it to the set of regular semisimple elements of G, and then use lemma 2.1.2.

Every element of $g \in G$ has a decomposition $g = g_s g_u$, where g_s is semisimple, g_u is unipotent and $g_s g_u = g_u g_s$. Every semisimple element is contained in some maximal torus. All maximal tori are conjugate. A semisimple element of G is called *regular* if there is only one such torus containing g_s, equal to the identity component of the centralizer $Z_G(g_s)$. The set of regular elements G_{rs} of G is open dense in G. (See [10]).

We can extend $f\colon H \to V$ to a G-invariant polynomial function on the set of all regular semisimple elements of G, by using that such an element is conjugate to an element of the fixed torus H, and that two elements of H are G-conjugate if and only if they are W-conjugate. Because the set of regular semisimple elements is open dense in G, we can consider f to be a rational function $G \to V$, regular except maybe on the set $G \setminus G_{rs}$. We will use lemma 2.1.2 to show that it is in fact regular everywhere.

We have $G \setminus G_{rs} = \cup_{\alpha,m} D_{\alpha,m}$, where α is an arbitrary root and m an arbitrary integer, and

$$D_{\alpha,m} = \{\text{elements whose semisimple part is conjugate to}$$

$$\text{some} \exp(h) \in H, h \in \mathfrak{h}, \text{ such that } \alpha(h) = 2\pi i m\}.$$

Some of these sets coincide (for example, if they are labeled by W- conjugate roots and appropriately chosen integers). For the purposes of applying lemma 2.1.2, we will use the following dense subset of $D_{\alpha_i,m}$:

$$D'_{\alpha_i,m} = \{G - \text{conjugates of} \exp(h)\exp(E_i) \in H,$$

$h \in \mathfrak{h}$ such that $\alpha_i(h) = 2\pi i m$, and $e^{\beta(h)} \neq 1 \ \forall$ root $\beta \neq \pm\alpha_i\}$.

Let $z = g \cdot \exp(h)\exp(E_i) \cdot g^{-1}$ be an arbitrary element of $D'_{\alpha_i,m}$. Pick $y \in \mathfrak{h}$ with $\alpha_i(y) = 1$. Then

$$[y, E_i] = E_i \quad [h, E_i] = 2i\pi m E_i,$$

so for $t \in \mathbb{C}$

$$\exp(ty)E_i = e^t E_i \exp(ty) \quad \exp(h)E_i = E_i \exp(h).$$

From this it follows

$$\exp(\frac{-1}{e^{-t}-1}E_i)\exp(h+ty)\exp(E_i)\exp(\frac{1}{e^{-t}-1}E_i) = \exp(h+ty),$$

in other words $\exp(h+ty)\exp(E_i)$ is conjugate to a regular element of H. We define

$$c_z(t) = g \cdot \exp(h+ty)\exp(E_i) \cdot g^{-1}$$

and calculate, as before:

$$\begin{aligned}
\lim_{t\to 0} f(c_z(t)) &= \lim_{t\to 0} f(g \cdot \exp(h+ty)\exp(E_i) \cdot g^{-1}) \\
&= \lim_{t\to 0} g.f(\exp(h+ty)\exp(E_i)) \\
&= \lim_{t\to 0} g.f(\exp(\frac{1}{e^{-t}-1}E_i)\exp(h+ty)\exp(\frac{-1}{e^{-t}-1}E_i)) \\
&= \lim_{t\to 0} g.\exp(\frac{1}{e^{-t}-1}E_i).f(\exp(h+ty)) \\
&= g.\lim_{t\to 0}\sum_{n\geq 0}\frac{1}{n!}\frac{1}{(e^{-t}-1)^n}E_i^n.f(\exp(h+ty)).
\end{aligned}$$

This sum is finite, and every $E_i^n.f(\exp(h+ty))$ is by assumption 3) divisible by

$$(1-e^{\alpha_i})^n(\exp(h+ty)) = (1-e^{\alpha_i(h+ty)})^n = (1-e^{2i\pi m+t})^n = (1-e^t)^n.$$

Since $\lim_{t\to 0}(\frac{1-e^t}{e^{-t}-1})^n = \lim_{t\to 0} e^{nt} = 1$, we see that the limit of every summand is finite. So, $\lim_{t\to 0} f(c_z(t))$ is finite, and by lemma 2.1.2, f is regular at the generic point of $G \setminus G_{rs}$, and hence it is regular everywhere. □

The main reason for reformulating Theorem 2.1.1 in terms of Theorem 2.1.5 is that the latter allows generalization to quantum groups. Namely, use the Peter-Weyl theorem to write

$$O(G) \cong \bigoplus_{L \in \widehat{G}} L^* \otimes L.$$

Here the sum is over $\widehat{G}$, the set of irreducible finite-dimensional representations L of G; equivalently, it is over all dominant integral weights $\mu \in P_+$, with $L = L_\mu$. The module L^* is the dual space of L, with the natural G action $g\varphi = \varphi \circ g^{-1}$. The isomorphism $A\colon \bigoplus_{L \in \widehat{G}} L^* \otimes L \to O(G)$ is determined by sending $\varphi \otimes l \in L^* \otimes L$ to a function on G given by $x \mapsto A(\varphi \otimes l)(x) = \varphi(xl)$. It is a matrix coefficient of L, and therefore a polynomial function on G. If we put the natural action of G on every tensor product $L^* \otimes L$, meaning letting $g \in G$ act by $g \otimes g$, then A is an isomorphism of G representations: $(Ag(\varphi \otimes l))(x) = (A(\varphi \circ g^{-1}) \otimes (gl))(x) = \varphi(g^{-1}xgl) = A(\varphi \otimes l)(gxg^{-1})$.

The action we had on $O(G) \otimes V$ was the natural action on the tensor product, so $A \otimes id_V\colon \bigoplus_{L \in \widehat{G}} L^* \otimes L \otimes V \to O(G) \otimes V$ is also an isomorphism of representations. There is also a natural isomorphism $B\colon L^* \otimes L \otimes V \to \mathrm{Hom}_{\mathbb{C}}(L, L \otimes V)$, by $B(\varphi \otimes l \otimes v)(l') = \varphi(l')l \otimes v, l' \in L$. The map B is a G-isomorphism with respect to the following G-action on $\mathrm{Hom}_{\mathbb{C}}(L, L \otimes V)$: for $\Phi \in \mathrm{Hom}_{\mathbb{C}}(L, L \otimes V)$, $l \in L$, $g \in G$, $(g\Phi)(l) = (g \otimes g).(\Phi(g^{-1}l))$. Notice that the invariants with respect to this action are exactly the G-intertwining operators $L \to L \otimes V$. It is also interesting to note that the composite map $(A \otimes id) \circ B^{-1}\colon \mathrm{Hom}_{\mathbb{C}}(L, L \otimes V) \to O(G) \otimes V$ is the trace map; more precisely, for a basis l_i of L, dual basis φ_i of L^*, and $\Phi \in \mathrm{Hom}_{\mathbb{C}}(L, L \otimes V)$, its image $(A \otimes id)(B(\Phi))$ is a polynomial on G given by

$$x \mapsto \mathrm{Tr}|_L(\Phi \circ x) = \sum_i (\varphi_i \otimes id)\Phi(xl_i) \in V.$$

To summarize, we have the following diagram:

$$\begin{array}{ccccc}
O(G)\otimes V & \xleftarrow{A\otimes id} & \bigoplus_{\mu\in P_+} L_\mu^*\otimes L_\mu\otimes V & \xrightarrow{B} & \bigoplus_{\mu\in P_+}\mathrm{Hom}_{\mathbb{C}}(L_\mu, L_\mu\otimes V) \\
\downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\
O(H)\otimes V & \xleftarrow{A\otimes id} & \bigoplus_{\nu\in P}\mathbb{C}_\nu^*\otimes \mathbb{C}_\nu\otimes V & \xrightarrow{B} & \bigoplus_{\nu\in P}\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\nu, \mathbb{C}_\nu\otimes V)
\end{array}$$

where $\mathbb{C}_\nu$ denotes a one dimensional representation of H on which H acts by a character ν. The isomorphisms A and B for H are completely analogous to those for G. All Res maps are naturally defined. Res: $O(G)\to O(H)$ is restricting a polynomial map to a subvariety. Res: $\bigoplus_{\mu\in P_+} L_\mu^*\otimes L_\mu \to \bigoplus_{\nu\in P}\mathbb{C}_\nu^*\otimes\mathbb{C}_\nu$ corresponds to decomposing representations L_μ and L_μ^* into their H-isotypic components $L_{\mu,\nu}$ and $L_{\mu,\nu}^*$, then annihilating all parts that are not diagonal, i.e. parts of the form $L_{\mu,\nu}^*\otimes L_{\mu,\eta}, \nu\neq\eta$, and finally taking a trace $L_{\mu,\nu}^*\otimes L_{\mu,\nu}\to\mathbb{C}_\nu^*\otimes\mathbb{C}_\nu$. In other words, for $\varphi_\nu\otimes v_\nu\in\mathbb{C}_\nu^*\otimes\mathbb{C}_\nu$ a fixed basis with $\varphi_\nu(v_\nu)=1$, and for $\varphi\in L_{\mu,\nu}^*$, $v\in L_{\mu,\nu}$, $v'\in L_{\mu,\nu'}$ with $\nu'\neq\nu$, the map is $\varphi\otimes v\mapsto\varphi(v)\varphi_\nu\otimes v_\nu$, and $\varphi\otimes v'\mapsto 0$.

This Res is defined to make the left square in the diagram commute. Analogously, the rightmost Res corresponds to viewing the homomorphisms as maps of H-representations, decomposing and forgetting the nondiagonal parts. The right square in the diagram also commutes.

Theorem 2.1.5 can now be restated as follows:

Corollary 2.1.6. *For every $\mu\in P_+$, let $(\Phi_{\mu,j})_j$ be a basis of the space of intertwining operators $\mathrm{Hom}_G(L_\mu, L_\mu\otimes V)$. For every such operator $\Phi_{\mu,j}$ define its trace function to be $\Psi_{\mu,j}\in O(H)\otimes V$, given by $\Psi_{\mu,j}(x)=\mathrm{Tr}|_{L_\mu}(\Phi_{\mu,j}\circ x)$. Then the set of all $\Psi_{\mu,j}$ is a basis of the space of functions in $O(H)\otimes V$ that satisfy 1)-3) from the statement of Theorem 2.1.5.*

This is the form of the theorem that we will prove in the quantum case. Now let us illustrate this form of the theorem with a simple example where we can write everything explicitly.

Example 2.1.7. *Let $\mathfrak{g}=\mathfrak{sl}_2$, $G=SL_2$. The rank of $\mathfrak{g}$ is 1, so identify $\mathfrak{h}^*$ with $\mathbb{C}$ by $z\mapsto z\frac{\alpha}{2}$ for α the positive root. Then the dominant weights are identified with* *nonnegative integers. Let* $V = L_2$ *be the three dimensional irreducible representation with highest weight* 2*. Pick a basis* v_{-2}, v_0, v_2 *of weight vectors for it, so that* $v_i \in V[i]$*, and* $F.v_2 = v_0, F^2.v_2 = v_{-2}$*. Pick an analogous basis for any* L_μ*, by* $F^i.l_\mu, i = 0, \dots \mu$*, and a dual basis to it* φ_i*,* $\varphi_i(F^j.l_\mu) = \delta_{ij}$*.*

Let $\mu \in \mathbb{N}_0$ *be arbitrary. Let us first describe all intertwining operators*

$$\Phi \in \mathrm{Hom}_{SL_2}(L_\mu, L_\mu \otimes V) = \mathrm{Hom}_{\mathfrak{sl}_2}(L_\mu, L_\mu \otimes V).$$

The map Φ *is determined by* $\Phi(l_\mu)$*, which needs to be a singular vector in* $L_\mu \otimes V$ *of total weight* μ*. So,*

$$\Phi(l_\mu) = c_0 l_\mu \otimes v_0 + c_1 F.l_\mu \otimes v_2.$$

The condition that this needs to be a singular vector in $L_\mu \otimes V$ *gives a recursion on the coefficients* c_i*. In general (for any* $\mathfrak{g}$ *and any* V*), if* $c_0 = 0$ *then* $\Phi = 0$ *(see [27], or Lemma 2.3.1 below). Scaling so that* $c_0 = 1$ *in this example we get*

$$\Phi(l_\mu) = l_\mu \otimes v_0 - \frac{2}{\mu} F.l_\mu \otimes v_2.$$

The dimension of the space of $\mathfrak{g}$*-intertwiners* $L_\mu \to L_\mu \otimes V$ *is* 1*, except when* $\mu = 0$*, when it is* 0*. This also illustrates the general case, when for generic* μ *the spaces* $\mathrm{Hom}_\mathfrak{g}(L_\mu, L_\mu \otimes V)$ *and* $V[0]$ *are isomorphic. The isomorphism sends* $v \in V[0]$ *to the* $\mathfrak{g}$*-intertwining operator* Φ *determined by* $\Phi(l_\mu) = l_\mu \otimes v +$ *terms with first factor of lower weight (again, see [27] or Lemma 2.3.1 below).*

Set $h = h_1$*. Any element of* $\mathfrak{h}$ *is of the form* $zh, z \in \mathbb{C}$*, and* $\alpha(h) = 2$*. The trace* *function Ψ on $H = \exp \mathfrak{h}$ is then, for $z \in \mathbb{C}$*

$$\begin{aligned}
\Psi(e^{zh}) &= \mathrm{Tr}_{L_\mu}(\Phi \circ e^{zh}) = \sum_{i=0}^{\mu} (\varphi_i \otimes id)(\Phi(e^{zh}(F^i.l_\mu)) \\
&= \sum_{i=0}^{\mu} e^{(\mu-i\alpha)(zh)}(\varphi_i \otimes id)(F^i.(l_\mu \otimes v_0 - \frac{2}{\mu}F.l_\mu \otimes v_2)) \\
&= \sum_{i=0}^{\mu} e^{(\mu-2i)z}(1-\frac{2}{\mu}i)v_0 \\
&= \sum_{\mu-2i\geq 0} \frac{\mu-2i}{\mu}(e^{z\cdot(\mu-2i)} - e^{-z\cdot(\mu-2i)})v_0
\end{aligned}$$

In this notation, $O(H) = \mathbb{C}[e^z, e^{-z}] = span\{e^{zh} \mapsto e^{nz}, n \in \mathbb{Z}\}$. As we vary $\mu \in \mathbb{N}$ and allow linear combinations of such trace functions, we can obviously get all the functions in $O(H) \otimes V$ of the form $f(e^{zh}) = \sum_n a_n e^{nz} v_0$ that satisfy $a_n = -a_{-n} \forall n \in \mathbb{N}_0$. On the other hand, a function $f(e^{zh}) = \sum_{n,i} a_{n,i} e^{nz} v_i$ in $O(H) \otimes V$ that satisfies 1)-3) must have:

1. *$a_{n,i} = 0$ unless $i = 0$, so $f(e^{zh}) = \sum_n a_n e^{nz} v_0$;*

2. *the Weyl group invariance: the Weyl group of $\mathfrak{sl}_2$ is $\mathbb{Z}_2$, and the nontrivial element acts on the 0 weight subspace of L_{2m} by $(-1)^m$; so in our case $f(e^{-zh}) = -f(e^{zh})$, which means $a_n = -a_{-n}$;*

3. *$E.f(e^z h) = \sum_n a_n e^{nz} E.v_0 = \sum_{n>0} a_n(e^{nz} - e^{-nz}) \cdot 2v_2$; every term $e^{nz} - e^{-nz}$ is divisible by $(1-e^\alpha)(e^{zh}) = 1 - e^{2z}$ in the ring $O(H)$ (and this condition is trivial in this case; see remark 2.4.10).*

Here we can directly see these are the same spaces of functions, as claimed by the theorem.

2.2 The generalized Chevalley isomorphism in the quantum case

We keep the notation from Section 2.1: $C = (a_{ij})$ is a Cartan matrix of finite type, $(\mathfrak{h}, \mathfrak{h}^*, \Pi = \{\alpha_1, \ldots \alpha_r\}, \Pi^\vee = \{h_1, \ldots h_r\})$ is its realization, d_i the symmetrizing integers, $\langle \cdot, \cdot \rangle$ the form identifying $\mathfrak{h}$ and $\mathfrak{h}^*$, H the complex torus with the map $\exp\colon \mathfrak{h} \to H$ whose kernel is $2i\pi$ times the dual weight lattice, W the Weyl group, P the weight lattice and P_+ the set of dominant integral weights.

Let $q \in \mathbb{C}^\times$ not a root of unity. Pick $t \in \mathbb{C}$ such that $e^t = q$. For $x \in \mathbb{C}$, define $q^x = e^{tx}$. For $h \in \mathfrak{h}$, define $q^h = e^{th} \in H$, and for $\lambda \in \mathfrak{h}^*$ use the identification $\mathfrak{h}^* \cong \mathfrak{h}$ to define $q^\lambda = e^{t\lambda} \in H$. For a function $e^\nu \in O(H), \nu \in P$, we now have

$$e^\nu(q^h) = e^\nu(e^{th}) = e^{t\nu(h)} = q^{\nu(h)},$$

$$e^\nu(q^\lambda) = e^\nu(e^{t\lambda}) = e^{t\langle \nu, \lambda \rangle} = q^{\langle \nu, \lambda \rangle}.$$

To this data one may associate a quantum group $U_q(\mathfrak{g})$, and its representations. First define quantum integers as $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$, quantum factorials as $[m]_q! = [m]_q \cdot [m-1]_q \cdot \ldots [1]_q$, and $q_i = q^{d_i} = q^{\langle \alpha_i, \alpha_i \rangle/2}$. As an associative algebra, $U_q(\mathfrak{g})$ is given by generators $E_1, \ldots E_r, F_1, \ldots F_r$, and $q^h, h \in \mathfrak{h}$ (here, q^h is a formal symbol for a generator, meant to suggest how this element will act on weight spaces), with relations

$$q^h q^{h'} = q^{h+h'} \qquad [E_i, F_j] = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}$$
$$q^h E_i q^{-h} = q^{\alpha_i(h)} E_i \qquad q^h F_i q^{-h} = q^{-\alpha_i(h)} E_i,$$

with $q_i^{h_i} = q^{d_i h_i}$, and Serre relations

$$\sum_{k=0}^{1-a_{ij}} \frac{(-1)^k}{[k]_{q_i}! [1 - a_{ij} - k]_{q_i}!} E_i^{1-a_{ij}-k} E_j E_i^k = 0$$

$$\sum_{k=0}^{1-a_{ij}} \frac{(-1)^k}{[k]_{q_i}![1-a_{ij}-k]_{q_i}!} F_i^{1-a_{ij}-k} F_j F_i^k = 0.$$

$U_q(\mathfrak{g})$ is a Hopf algebra, with the coproduct Δ, counit ε, and the antipode S given on the generators by

$$\begin{array}{ccc} \Delta(q^h) = q^h \otimes q^h & \Delta(E_i) = E_i \otimes q_i^{h_i} + 1 \otimes E_i & \Delta(F_i) = F_i \otimes 1 + q_i^{-h_i} \otimes F_i \\ \varepsilon(q^h) = 1 & \varepsilon(E_i) = 0 & \varepsilon(F_i) = 0 \\ S(q^h) = q^{-h} & S(E_i) = -E_i q_i^{-h_i} & S(F_i) = -q_i^{h_i} F_i \end{array}$$

Representations of $U_q(\mathfrak{g})$ that we are going to consider are going to be in category $\mathcal{O}$ and of type I. This means that a representation is a vector space V with an algebra homomorphism $U_q(\mathfrak{g}) \to End(V)$, such that the weight spaces $V[\nu] = \{v \in V | q^h.v = q^{\nu(h)}v \ \forall h \in \mathfrak{h}\}, \nu \in P$, are all finite-dimensional, $V = \bigoplus_{\nu \in P} V[\nu]$, and all weights appearing with nonzero weight space will be contained in a union of finitely many cones of the form $\{\nu - \sum_i n_i\alpha_i,\ n_i \in \mathbb{N}_0\}$ in P. Moreover, we will only be interested in finite-dimensional representations and Verma modules, defined below.

As $U_q(\mathfrak{g})$ is a Hopf algebra, its representations form tensor category, as an element $X \in U_q(\mathfrak{g})$ acts on a tensor product of representations by $\Delta(X)$. We can also define duals of representations. For a finite-dimensional $U_q(\mathfrak{g})$ module V, define its left dual ${}^*V$ to be the space of functionals on V together with the $U_q(\mathfrak{g})$ action $(X\varphi)(v) = \varphi(S^{-1}(X)v)$. Left dual space ${}^*V$ comes with natural isomorphisms ${}^*V \otimes U \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, {}^*V \otimes U) \cong \mathrm{Hom}_{\mathcal{C}}(V, U)$ for every module U. In the classical case of $U(\mathfrak{g})$, we have $S = S^{-1}$, as $S(X) = -X$ for $X \in \mathfrak{g}$, so left dual modules for quantum groups are one of two possible generalizations of the notion of dual module for enveloping algebras. The other one is the right dual module, defined using S instead of S^{-1}.

For any $\mu \in P$, let M_μ denote the Verma module with highest weight μ. It is a module generated over $U_q(\mathfrak{g})$ by a distinguished singular vector m_μ, with relations $E_i m_\mu = 0$, $q^h m_\mu = q^{\mu(h)} m_\mu$. If $\mu \in P_+$, then M_μ has a finite-dimensional irreducible quotient; call it L_μ, and call the image of m_μ in it l_μ. As in the classical case, the finite-dimensional irreducible representations we are interested in are labeled by integral dominant weights. We will mainly be interested in them, and occasionally use an auxiliary Verma module.

Remark 2.2.1. *Note that the symbol* q^h *denotes both the element* $\exp(th)$ *of the group* H *and the generator of* $U_q(\mathfrak{g})$. *This makes sense because on any representation* V *in category* $\mathcal{O}$, *these elements diagonalize with the same weight spaces, and act on such a weight space* $V[\nu]$ *with the same eigenvalues:* $q^h \in U_q(\mathfrak{g})$ *acts by* $q^{\nu(h)}$, $e^{th} \in H$ *acts by* $e^{\nu(th)}$, *and* $q^{\nu(h)} = e^{t\nu(h)} = e^{\nu(th)}$.

In other words, there exists a group homomorphism from the multiplicative group of all elements of the form $q^h \in U_q(\mathfrak{g})$ *to* H *given by* $q^h \mapsto \exp(th)$. *It is surjective, its kernel is the set of all* $q^h \in U_q(\mathfrak{g}), h \in 2\pi i\mathbb{Z}\Pi^\vee/t$, *and any representation in category* $\mathcal{O}$ *factors through this homomorphism.*

Remark 2.2.2. *Another way to define the setup we need is to avoid defining the quantum group* $U_q(\mathfrak{g})$ *altogether, and to instead just define its category* $\mathcal{O}$ *of representations. Namely, we define objects in the category to be* P*-graded vector spaces* V *with graded pieces* $V[\nu], \nu \in P$, *together with operators* E_i, F_i, *such that:*

- *all* $V[\nu]$ *are finite-dimensional:*
- *the set of all* ν *with* $V[\nu] \neq 0$ *is contained in a union of finitely many cones of the form* $\{\lambda - \sum_i n_i\alpha_i | n_i \in \mathbb{N}_0\}$;
- $E_i \colon V[\nu] \to V[\nu + \alpha_i]$, $F_i \colon V[\nu] \to V[\nu - \alpha_i]$;
- E_i, F_i *satisfy Serre relations;*
- $[E_i, F_j]|_{V[\nu]} = \delta_{ij} \frac{q_i^{\nu(h_i)} - q_i^{-\nu(h_i)}}{q_i - q_i^{-1}} id|_{V[\nu]}$.

Morphisms in the category are morphisms of graded vector spaces that commute with the operators E_i, F_i. *Tensor structure of the category can be defined by similar formulas.*

It is obvious that these two definitions of category $\mathcal{O}$ *are equivalent. The first one is the usual definition of an algebra and its category of representations. The advantages* *of the second one are that it avoids the ambiguity of defining $U_q(\mathfrak{g})$, allows for a very clear restriction functor from this category to the category of representations of H, avoids the representations of $U_q(\mathfrak{g})$ which are not of type I, and is a more direct generalization of the the category of representations of $U(\mathfrak{g})$ that we considered, because just replacing q by 1 and $[m]_q$ by m in all formulas gives exactly the category of representations of $U(\mathfrak{g})$ we considered.*

A practical consequence of the last remark is that there is a functor from the category of $U_q(\mathfrak{g})$ representations considered above to the category of representations of the torus H, given by letting $x \in H$ act on the space $V[\mu]$ by $e^{\mu}(x)\mathrm{id}_{V[\mu]}$. In light of Remark 2.2.1, it corresponds to restricting the representation of $U_q(\mathfrak{g})$ to the subalgebra generated by all the q^h.

Inspired by the classical case, define functions on a quantum group to be

$$O_q(G) = \bigoplus_{\mu \in P_+} {}^*L_\mu \otimes L_\mu.$$

This is a $U_q(\mathfrak{g})$ module with the usual action of $U_q(\mathfrak{g})$ on ${}^*L_\mu \otimes L_\mu$.

Remark 2.2.3. *See [37] for a discussion on various equivalent definitions of quantized algebras of functions on a Lie group. In particular, in Chapter 3, Proposition 2.1.2 it is shown that one can define $O_q(G)$ as $\bigoplus_\mu L_\mu \otimes L_\mu^*$. This definition is equivalent to ours, as there is an isomorphism $L_\mu^* \to L_{\mu^*}$ whose dual is an isomorphism $L_\mu \to {}^*L_{\mu^*}$. Here μ^* is a dual weight to μ, and can be calculated as $\mu^* = -w_0\mu$ for w_0 the longest element of W. The map $\mu \mapsto \mu^*$ is an involution on the set of dominant integral weights.*

This will be the setting for the rest of the chapter. Also, let V will be a finite-dimensional representation of type I in category $\mathcal{O}$, that is a direct sum of finitely many L_μ. As we are interested in describing restrictions of functions with values in V, which corresponds to taking tensor products with V, all the statements and conditions will behave nicely with respect to decomposing V into direct sums. This means that the restriction theorems will hold for V if and only if they will hold for every direct summand of V. As a consequence, we can at any point assume V is irreducible, and all the conclusions we make will hold for any V that is a direct sum of (possibly infinitely many) irreducible finite-dimensional modules.

As in the classical case, we have the following diagram:

$$\begin{array}{c} O_q(G) \otimes V = \bigoplus_{\mu \in P_+} {}^*L_\mu \otimes L_\mu \otimes V \cong \bigoplus_{\mu \in P_+} \mathrm{Hom}_{\mathbb{C}}(L_\mu, L_\mu \otimes V) \\ \Big\downarrow \mathrm{Res} \\ O(H) \otimes V \cong \bigoplus_{\nu \in P} \mathbb{C}_\nu^* \otimes \mathbb{C}_\nu \otimes V \cong \bigoplus_{\nu \in P} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\nu, \mathbb{C}_\nu \otimes V) \end{array}$$

Here we are using that on H, $S = S^{-1}$ so ${}^*\mathbb{C}_\nu = \mathbb{C}_\nu^*$. Let us list all the maps and all the actions of $U_q(\mathfrak{g})$ on these spaces; checking that all maps are isomorphisms of $U_q(\mathfrak{g})$ modules is then a direct computation.

- The map Res, as in the classical case, corresponds to decomposing the representation $L = L_\mu$ into weight spaces, making an H-representation out of each weight space by defining $x|_{L[\nu]} = e^\nu(x)\mathrm{id}_{L[\nu]}$, annihilating the nondiagonal part and taking the trace. As in Remark 2.2.2, this corresponds to understanding a representation L of a quantum algebra $U_q(\mathfrak{g})$ as an H-representation given by weight decomposition together with the operators $E_i \colon L[\nu] \to L[\nu + \alpha_i]$ and $F_i \colon L[\nu] \to L[\nu - \alpha_i]$. Alternatively, it corresponds to understanding H as a multiplicative subgroup of $U_q(\mathfrak{g})$ like in Remark 2.2.1.

- The maps ${}^*L_\mu \otimes L_\mu \otimes V \to \mathrm{Hom}_{\mathbb{C}}(L_\mu, L_\mu \otimes V)$ and $\mathbb{C}_\nu^* \otimes \mathbb{C}_\nu \otimes V \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\nu, \mathbb{C}_\nu \otimes V)$ are natural, as they were in the classical case:

$$\varphi \otimes l \otimes v \mapsto \big(l' \mapsto \varphi(l') l \otimes v\big)$$

  for $l, l' \in L_\mu, \varphi \in {}^*L_\mu, v \in V$, or for $l, l' \in \mathbb{C}_\nu, \varphi \in \mathbb{C}_\nu^*, v \in V$.

- The map $\mathbb{C}_\nu^* \otimes \mathbb{C}_\nu \otimes V \to O(H) \otimes V$ is, as in the classical case, given by

$$\varphi \otimes l \otimes v \mapsto \big(x \mapsto \varphi(xl) v\big)$$

  for $\varphi \in \mathbb{C}_\nu^*, l \in \mathbb{C}_\nu, v \in V, x \in H$.

- $U_q(\mathfrak{g})$ action on ${}^*L_\mu \otimes L_\mu \otimes V$ is the usual one on a triple tensor product, with $X \in U_q(\mathfrak{g})$ acting by $\Delta^2(X) = (\Delta \otimes id) \circ \Delta(X) = X_{(1)} \otimes X_{(2)} \otimes X_{(3)}$ in Sweedler's notation.

- The $U_q(\mathfrak{g})$-action on $\mathrm{Hom}_{\mathbb{C}}(L_\mu, L_\mu \otimes V)$ is as follows: for $X \in U_q(\mathfrak{g})$, $l \in L_\mu$, $\Phi \in \mathrm{Hom}_{\mathbb{C}}(L_\mu, L_\mu \otimes V)$,

$$(X\Phi)(l) = \Delta(X_{(2)}).\Phi(S^{-1}(X_{(1)})l) = (X_{(2)} \otimes X_{(3)}).\Phi(S^{-1}(X_{(1)})l).$$

We are interested in the $U_q(\mathfrak{g})$ invariants in $O_q(G) \otimes V$ and their restrictions to $O(H) \otimes V$. The above action of $U_q(\mathfrak{g})$ on $\mathrm{Hom}_{\mathbb{C}}(L_\mu, L_\mu \otimes V)$ is the usual one, so the space of invariants is exactly

$$\mathrm{Hom}_{\mathbb{C}}(L_\mu, L_\mu \otimes V)^{U_q(\mathfrak{g})} = \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu \otimes V).$$

We can again write explicitly the composite map $\mathrm{Hom}_{\mathbb{C}}(L_\mu, L_\mu \otimes V) \to O(H) \otimes V$; it is the trace map. It maps $\Phi \in \mathrm{Hom}_{\mathbb{C}}(L_\mu, L_\mu \otimes V)$ to the polynomial function $\Psi \colon H \to V$ given by:

$$\Psi(x) = Tr|_{L_\mu}(\Phi \circ x) \in V.$$

By further abuse of notation, we will call this map Res, as well as its restriction to the space of invariants.

We can now state the main theorem, analogous to Theorem 2.1.5.

Theorem 2.2.4. *The map* $\mathrm{Res}\colon (O_q(G) \otimes V)^{U_q(\mathfrak{g})} \to O(H) \otimes V$ *is injective. Its image consists of those functions* $f \in O(H) \otimes V$ *that satisfy:*

1. $f \in O(H) \otimes V[0]$;

2. *f is invariant under the (unshifted) action of the dynamical Weyl group (see Section 2.3)*

3. *for every simple root α_i and every $n \in \mathbb{N}$, the polynomial $E_i^n.f$ is divisible by*

$$(1 - q_i^2 e^{\alpha_i})(1 - q_i^4 e^{\alpha_i}) \dots (1 - q_i^{2n} e^{\alpha_i}).$$

The proof of the theorem will be given in Section 2.4. We will review the definition and some properties of the dynamical Weyl group in Section 2.3.

2.3 Intertwining operators, dynamical Weyl group and trace functions

Let us first fix some conventions for the rank one case $U_q(\mathfrak{sl}_2)$. In that situation, the Cartan matrix is $C = [2]$ and $\mathfrak{h}$ is one dimensional. As $r = 1$, we will use the notation $h = h_1$ and $\alpha = \alpha_1$; we have $\alpha(h) = 2$. Then we can identify $\mathfrak{h}^*$ with $\mathbb{C}$ by $n \leftrightarrow n\frac{\alpha}{2}$. Integral weights P are thus identified with integers $\mathbb{Z}$ and dominant integral ones P_+ with nonnegative integers $\mathbb{N}_0$.

Next, let us describe the notion of expectation value for general C and $U_q(\mathfrak{g})$. Let V be a finite-dimensional representation and ν a weight of V. Any operator $\Phi \in \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\nu} \otimes V)$ is of the form $\Phi(m_\lambda) = m_{\lambda-\nu} \otimes v + \text{l.o.t.}$, where l.o.t. denotes the lower order terms, meaning terms with first coordinate in a lower weight space. Obviously, $v \in V[\nu]$. Define the expectation value of Φ to be $\langle\Phi\rangle = v$. That means that if $\varphi_{\lambda-\nu}$ denotes an element of the (algebraic) dual of $M_{\lambda-\nu}$ that is 1 on $m_{\lambda-\nu}$ and 0 on all other weight spaces of $M_{\lambda-\nu}$, then $\langle\Phi\rangle = (\varphi_{\lambda-\nu} \otimes id)(\Phi(m_\lambda)) \in V[\nu]$. An analogous map exists for the situation when Verma modules are replaced by irreducible modules, and we will also write it as $\langle\cdot\rangle : \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_{\lambda-\nu} \otimes V) \to V[\nu]$.

A morphism of $U_q(\mathfrak{g})$ modules $M_\lambda \to M_\mu \otimes V$ or $L_\lambda \to L_\mu \otimes V$ is clearly determined by the image of the highest weight vector, but for generic λ even more is true: it is determined by the first term of it, more precisely by the expectation value. The precise statement is in the following lemma:

Lemma 2.3.1. *1. For generic λ and for λ integral dominant with sufficiently large coordinates $\lambda(h_i)$, the expectation value maps $\langle\cdot\rangle$ define isomorphisms*

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\nu} \otimes V) \cong V[\nu] \cong \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_{\lambda-\nu} \otimes V).$$

2. *For $\nu = 0$ and λ dominant integral, the image of the injective map*

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_\lambda \otimes V) \to V[0]$$

is

$$\{v \in V[0] \mid E_i^{\lambda(h_i)+1} v = 0, \; i = 1, \dots r\}.$$

3. *For $U_q(sl_2)$, $V = L_{2m}$, $\nu = 0$ and λ dominant integral, the expectation value map*

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_\lambda \otimes V) \to V[0]$$

is an isomorphism if and only if $\lambda \notin \{0, 1, \dots m-1\}$. If $\lambda \in \{0, 1, \dots m-1\}$, then $\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_\lambda \otimes V) = 0$.

Proof. 1. For λ generic or integral dominant with sufficiently large coordinates we have the following diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\nu} \otimes V) & \longrightarrow & \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, L_{\lambda-\nu} \otimes V) & \xrightarrow{\simeq} & \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_{\lambda-\nu} \otimes V) \\ & \searrow & & \swarrow & \\ & & V[\nu] & & \end{array}$$

The map $\mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\nu} \otimes V) \to \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, L_{\lambda-\nu} \otimes V)$ is the composition with the projection map $M_{\lambda-\nu} \otimes V \to L_{\lambda-\nu} \otimes V$, and it is defined for any λ. In general, it is not injective.

The map $\mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, L_{\lambda-\nu} \otimes V) \to \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_{\lambda-\nu} \otimes V)$ is defined when all homomorphisms $M_\lambda \to L_{\lambda-\nu} \otimes V$ factor through L_λ. In particular, this happens if λ is generic (in which case $M_\lambda = L_\lambda$ and the map is the identity), or when $\lambda - \nu$ is dominant integral (in which case $L_{\lambda-\nu} \otimes V$ is finite-dimensional, so every map $M_\lambda \to L_{\lambda-\nu} \otimes V$ factors through the finite-dimensional L_λ). In both of these cases, the map is an isomorphism.

Both maps to $V[\nu]$ are the expectation value maps.

Let us show that $\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_{\lambda-\nu} \otimes V) \to V[\nu]$ is injective. Pick a basis

v_i of weight vectors for V. Let $\Phi \neq 0 \in \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_{\lambda-\nu} \otimes V)$. Consider $\Phi(l_\lambda) = \sum_i l_i \otimes v_i$ for some $l_i \in L_{\lambda-\nu}$. Because $\Phi(l_\lambda)$ and all v_i are weight vectors and v_i are a basis, all the l_i are weight vectors as well. Pick $l_{i_0} \neq 0$ with a highest weight among all nonzero l_i. Because $\Phi(l_\lambda)$ is singular and l_{i_0} has highest weight, l_{i_0} is a singular vector in $L_{\lambda-\nu}$. Thus, $l_{i_0} = c \cdot l_{\lambda-\nu}$ for some $c \neq 0 \in \mathbb{C}$, and $\langle \Phi \rangle = c \cdot v_{i_0} \neq 0$, so the expectation value map is injective.

Lemma 1 in [27] states that for λ generic, and in particular integral dominant with sufficiently large coordinates, the expectation value map

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\nu} \otimes V) \to V[\nu]$$

is an isomorphism. The proof is straightforward, by noticing that the conditions on this map being an isomorphism are that a certain set of linear equations has a unique solution. It is a general argument of the type we used in Example 2.1.7 for $\mathfrak{sl}_2$.

As the diagram from the beginning of the proof commutes, whenever the map

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\nu} \otimes V) \to V[\nu]$$

is an isomorphism, the map $\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_{\lambda-\nu} \otimes V) \to V[\nu]$ is surjective and therefore also an isomorphism.

2. The map is injective due to proof of part (1). This proof also shows that with the assumptions of (2), namely $\nu = 0$ and λ dominant integral,

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_\lambda \otimes V) \cong \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, L_\lambda \otimes V).$$

The Verma module M_λ is induced to $U_q(\mathfrak{g})$ from the subalgebra $U_q(\mathfrak{b}_+)$, generated by all q^h and E_i; the $U_q(\mathfrak{b}_+)$ module we are inducing from is the one dimensional module $\mathbb{C}_\lambda$, with q^h acting on it by $q^{\lambda(h)}\mathrm{id}$ and E_i acting on it by

0. So,

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, L_\lambda \otimes V) \cong$$

$$\cong \mathrm{Hom}_{U_q(\mathfrak{b}_+)}(\mathbb{C}_\lambda, L_\lambda \otimes V) \cong \mathrm{Hom}_{U_q(\mathfrak{b}_+)}(\mathbb{C}_\lambda \otimes L_\lambda^*, V).$$

L_λ^* is a lowest weight module with the lowest weight $-\lambda$. We can define the lowest weight analogue of Verma module $M_{-\lambda}^-$, which is induced from the module $\mathbb{C}_{-\lambda}$ over the subalgebra generated by all q^h to the algebra $U_q(\mathfrak{b}_+)$; so as a vector space it is isomorphic to the subalgebra $U_q(\mathfrak{n}_+)$ generated by all the E_i. Call its lowest weight vector $\phi_{-\lambda}$. The module L_λ^* is then known to be the quotient of $M_{-\lambda}^-$ by relations $E_i^{\lambda(h_i)+1}\phi_{-\lambda} = 0$.

Because of that, any $U_q(\mathfrak{b}_+)$ map $\mathbb{C}_\lambda \otimes L_\lambda^* \to V$ is determined by the image of the lowest weight vector $1 \otimes \phi_{-\lambda}$ in V. This must be a vector $v \in V$ of weight $\lambda - \lambda = 0$, such that $E_i^{\lambda(h_i)+1}.v = 0$. It is clear that any such vector will define a $U_q(\mathfrak{b}_+)$ intertwining operator $\mathbb{C}_\lambda \otimes L_\lambda^* \to V$.

The only thing left to notice is that under the isomorphism

$$\mathrm{Hom}_{U_q(\mathfrak{b}_+)}(\mathbb{C}_\lambda \otimes L_\lambda^*, V) \cong \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_\lambda \otimes V),$$

the vector v from above corresponds to the expectation value of an intertwining operator $L_\lambda \to L_\lambda \otimes V$.

3. This follows directly from 2). $V[0]$ is one dimensional, so either the injective map

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_\lambda \otimes V) \to V[0]$$

is an isomorphism or the space $\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_\lambda \otimes V)$ is zero. As $d = 1$, and after the identification $\mathfrak{h}^* \cong \mathbb{C}$ we have $\langle \lambda, \alpha \rangle = \lambda \frac{\langle \alpha, \alpha \rangle}{2} = \lambda$, part 2) of the lemma tells us that the image of the expectation value map is the set of $v \in V[0]$ such that $E^{\lambda+1}.v = 0$. The maps $E\colon V[2i] \to V[2i+2]$ are injective for $i \neq m$, and $E^{\lambda+1}.v \in V[2\lambda+2]$, we conclude that the image of the map is zero unless $\lambda + 1 > m$, that is if $0 \leq \lambda \leq m-1$. If $\lambda \geq m$, the set of such v that $E^{\lambda+1}.v = 0$

is the entire $V[0]$, so the injective map is an isomorphism.

This ends the proof, but it is interesting to note that the last case of $\lambda \in \{0, \ldots m-1\}$ is exactly when the commutative diagram from the beginning of this proof fails to be a commutative diagram of isomorphisms: $\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda, L_\lambda \otimes V) = 0 = \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, L_\lambda \otimes V)$; the spaces $V[0]$ and $\mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_\lambda \otimes V) \cong \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{-\lambda-2} \otimes V)$ are one dimensional, but the map between them is 0.

□

Remark 2.3.2. *Another way to prove (3) is to calculate explicitly the conditions on a vector in $L_\lambda \otimes V$ to be a singular vector of weight λ, and get a set of linear equations that have a solution if and only if λ is in the above set. This is done in the first part of Theorem 7.1. in [26].*

Following the notation in [27], for those λ for which the expectation value map

$$\langle \cdot \rangle : \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\nu} \otimes V) \to V[\nu]$$

is an isomorphism, let $v \mapsto \Phi_\lambda^v$ be the inverse map; i.e. Φ_λ^v is an intertwining operator such that $\langle \Phi_\lambda^v \rangle = v$. For the same situation, let $\overline{\Phi}_\lambda^v$ be the intertwiner $L_\lambda \to L_{\lambda-\nu} \otimes V$ with $\langle \overline{\Phi}_\lambda^v \rangle = v$.

The Weyl group W is generated by simple reflections s_i associated to simple roots α_i. Let $\rho \in P \subseteq \mathfrak{h}^*$ be a weight such that $\rho(h_i) = 1 \,\forall i$. Let the dot $w \cdot \lambda = w(\lambda+\rho) - \rho$ denote the shifted action of the Weyl group on $\mathfrak{h}^*$. The dynamical Weyl group of V is a collection of operator valued functions $A_{w,V}(\lambda)$ labeled by $w \in W$, rational in $q^\lambda, \lambda \in \mathfrak{h}$, with $A_{w,V}(\lambda) \colon V[\nu] \to V[w \cdot \nu]$. To define these operators, we first need a bit more notation and results from [26]. Let $w = s_{i_1} \ldots s_{i_l}$ be a reduced decomposition of $w \in W$. Let $\lambda \in P_+$, and let $\alpha^l = \alpha_{i_l}$, $\alpha^j = (s_{i_l} \ldots s_{i_{j+1}})\alpha_{i_j}$, $j = 1, \ldots l-1$. Let $n_j = 2 \langle \lambda + \rho, \alpha^j \rangle / \langle \alpha^j, \alpha^j \rangle$. These are positive integers. Let $d^j = d_{i_j}$ be the symmetrizing numbers defined before. The following is Lemma 2 from [27]:

Lemma 2.3.3. *For $\lambda \in P_+$, the set of pairs $(n_1, d^1), \ldots (n_l, d^l)$ and the product*

$F_{i_1}^{n_1} \dots F_{i_l}^{n_l}$ *don't depend on the reduced decomposition of* $w \in W$*. Hence, the vector*

$$m_{w\cdot\lambda}^{\lambda} = \frac{F_{i_1}^{n_1} \dots F_{i_l}^{n_l}}{[n_1]_{q^{d_1}} \dots [n_l]_{q^{d_l}}} m_\lambda \in M_\lambda$$

is well-defined. It is a singular vector of weight $w \cdot \lambda$.

We will use Proposition 15 and Corollary 16 from [27] to define the dynamical Weyl group action.

Definition 2.3.4. *Let* $v \in V[\nu]$, $w \in W$, $\lambda \in P_+$ *with large enough coordinates compared with* ν*. We have*

$$\Phi_\lambda^v(m_\lambda) = m_{\lambda-\nu} \otimes v + l.o.t..$$

Define $A_{w,V}(\lambda)v \in V[w \cdot \nu]$ *by*

$$\Phi_\lambda^v(m_{w\cdot\lambda}^{\lambda}) = m_{w\cdot(\lambda-\nu)}^{\lambda-\nu} \otimes A_{w,V}(\lambda)v + l.o.t.$$

(The proof that this is well-defined, i.e. that the vector $\Phi_\lambda^v(m_{w\cdot\lambda}^{\lambda})$ *is of that form, is in [27]).*

The operators $A_{w,V}(\lambda)$*, defined for* λ *dominant integral with large enough coordinates, depend rationally on* q^λ *(in the sense that their coefficients in any basis are rational functions of* $q^{\lambda(h_i)}$*), so they can be uniquely extended to rational functions of* q^λ*, for* $\lambda \in \mathfrak{h}^*$.

The operators $A_{w,V}(\lambda)$ do not, in general, define a representation of the Weyl group. However, we have a weaker result below (Lemma 17 and Corollary 29 from [27]). Let l be the length function on the Weyl group W, defined to be the length of the shortest reduced expression.

Proposition 2.3.5. *1. If* $w_1, w_2 \in W$ *such that* $l(w_1w_2) = l(w_1) + l(w_2)$*, then*

$$A_{w_1w_2,V}(\lambda) = A_{w_1,V}(w_2 \cdot \lambda) A_{w_2,V}(\lambda).$$

2. Restrictions of operators $A_{w,V}(\lambda)$ to $V[0]$ satisfy

$$A_{w_1w_2,V}(\lambda) = A_{w_1,V}(w_2 \cdot \lambda)A_{w_2,V}(\lambda)$$

without any requirements on the length of $w_i \in W$.

For $U_q(\mathfrak{sl}_2)$ and V a simple finite-dimensional module, $V[0]$ is either 0 (if $V = L_{2m+1}$) or one dimensional (if $V = L_{2m}$). In the latter case, the operators $A_V(\lambda)$ restricted to $V[0]$ are just rational functions of q^λ times the identity operator on $V[0]$. We can calculate them explicitly:

Lemma 2.3.6. *For $U_q(\mathfrak{sl}_2)$, $V = L_{2m}$, and s the nontrivial element of the Weyl group $W = \mathbb{Z}_2$,*

$$A_{s,V}(\lambda) = (-1)^m \prod_{j=1}^{m} \frac{[\lambda+1+j]_q}{[\lambda+1-j]_q} \mathrm{id}_{V[0]}.$$

Proof. Follows directly from Corollary 8 (iii) and Proposition 12 in [27]. □

One can now define two actions of the dynamical Weyl group on rational functions of q^λ with values in $V[0]$:

Definition 2.3.7. *1. The shifted action is given by*

$$(w \circ f)(\lambda) = A_{w,V}(w^{-1} \cdot \lambda) f(w^{-1} \cdot \lambda).$$

2. Define $\mathcal{A}_{w,V}(\lambda) = A_{w,V}(-\lambda - \rho)$.

3. The unshifted action is given by

$$(w * f)(\lambda) = \mathcal{A}_{w,V}(w^{-1}\lambda) f(w^{-1}\lambda).$$

Corollary 2.3.8. *Restricted to $V[0]$, the operators $\mathcal{A}_{w,V}(\lambda)\colon V[0] \to V[0]$ satisfy*

$$\mathcal{A}_{w_1w_2,V}(\lambda) = \mathcal{A}_{w_1,V}(w_2\lambda)\mathcal{A}_{w_2,V}(\lambda).$$

Remark 2.3.9. *In general, the shifted and the unshifted action are defined for rational functions with values in V. Because of Proposition 2.3.5, in that case they don't define a representation of the Weyl group W, but define an action of a braid group of W. However, we will need them only for functions with values in $V[0]$, where both actions define a representation of W (again due to Proposition 2.3.5).*

The statement of the main theorem, 2.2.4, refers to the unshifted dynamical action from this definition. Here one must remember that we can use the form $\langle\cdot,\cdot\rangle$ to identify $\mathfrak{h} \cong \mathfrak{h}^*$, so this definition of functions on $\mathfrak{h}^*$ can be applied to functions on $\mathfrak{h}$. With that identification, the part of the theorem "$f \in O(H) \otimes V$ invariant under the unshifted action of dynamical Weyl group" means that for every $w \in W$, $\lambda \in \mathfrak{h}^*$,

$$f(q^{2w\lambda}) = \mathcal{A}_{w,V}(\lambda) f(q^{2\lambda}).$$

To prove the dynamical Weyl group invariance, we need to invoke several more definitions and results form [27] and [26].

Remember that for μ large dominant and $v \in V[0]$ we defined

$$\Phi^v_\mu \in \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\mu, M_\mu \otimes V)$$

such that $\langle \Phi^v_\mu \rangle = v$, and analogously $\overline{\Phi}^v_\mu \in \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu \otimes V)$ such that $\langle \overline{\Phi}^v_\mu \rangle = v$. We also defined their trace functions. To introduce notation of [27], for $\lambda \in \mathfrak{h}^*$, define

$$\Psi^v(\lambda, \mu) = \mathrm{Tr}|_{M_\mu}(\Phi^v_\mu q^{2\lambda}) \in V[0].$$

The functions we are interested in are

$$\Psi^v_\mu(\lambda) = \mathrm{Tr}|_{L_\mu}(\overline{\Phi}^v_\mu q^{2\lambda}) \in V[0].$$

The paper [27] also uses generating functions for these trace functions. Pick a basis v_i of $V[0]$ and let $v_i^* \in V^*[0]$ be the dual basis. Then define the generating functions as

$$\Psi_V(\lambda,\mu) = \sum_i \Psi^{v_i}(\lambda,\mu)\otimes v_i^* \in V[0]\otimes V^*[0] \cong \mathrm{Hom}_{\mathbb{C}}(V[0],V[0])$$

$$\Psi_V^{\mu}(\lambda) = \sum_i \Psi_{\mu}^{v_i}(\lambda)\otimes v_i^* \in V[0]\otimes V^*[0].$$

We are interested in functions of the type $f(q^{2\lambda}) = \Psi_\mu^v(\lambda)$. More results are available about functions $\Psi^v(\lambda,\mu)$. Fortunately, there is a theorem allowing us to translate results of one type to another, analogous to Weyl character formula and proved as Proposition 42 in [27]:

Proposition 2.3.10. $\Psi_\mu^v(\lambda) = \sum_{w\in W}(-1)^w \Psi^v(\lambda, w\cdot\mu)A_{w,V}(\mu)$.

Let $\delta_q(\lambda)$ be the Weyl denominator $\delta_q(\lambda) = \sum_{w\in W}(-1)^w q^{2\langle\lambda, w\rho\rangle}$. It satisfies

Lemma 2.3.11. $\delta_q(w\lambda) = (-1)^w\delta_q(\lambda)$.

Proof. It follows directly from the W-invariance of the form $\langle\cdot,\cdot\rangle$. □

For finite-dimensional $U_q(\mathfrak{g})$ modules U, V, define the fusion matrix $J_{UV}(\lambda)\colon U\otimes V \to U\otimes V$ as follows. For generic λ and $v\in V[\mu], u\in U[\nu]$, it is an operator such that

$$(\Phi^u_{\lambda-\mu}\otimes 1)\circ \Phi^v_\lambda = \Phi_\lambda^{J_{UV}(\lambda)(u\otimes v)}.$$

It is a rational function of q^λ, and an invertible operator (see [27], Section 2.6).

If $J_{U,{}^*U}(\lambda) = \sum_i c_i\otimes c_i'$, with $c_i\in \mathrm{End}(U), c_i'\in\mathrm{End}({}^*U)$, define $Q_U(\lambda) = \sum_i (c_i')^* c_i \in \mathrm{End}(U)$ (see [26]). Use these to define the renormalized trace functions

$$F_V(\lambda,\mu) = \delta_q(\lambda)\Psi_V(\lambda, -\mu-\rho)Q_V^{-1}(-\mu-\rho).$$

These satisfy (see Proposition 45 in [27]):

Proposition 2.3.12. $F_V(\lambda,\mu) = \big(\mathcal{A}_{w,V}(w^{-1}\lambda)\otimes\mathcal{A}_{w,V^*}(w^{-1}\mu)\big)F_V(w^{-1}\lambda, w^{-1}\mu)$.

These operators appear in many formulas because they transform the action of operators $A_{w,V}(\lambda)$ on the space V and its duals. One of these, a special case of Proposition 20 in [27], is the following proposition:

Proposition 2.3.13. *When restricted to* $V[0]$,

$$A_{w,V^*}(\lambda)^* = Q_V(\lambda)A_{w,V}(\lambda)^{-1}Q_V(w\cdot\lambda)^{-1}.$$

2.4 Proof of Theorem 2.2.4

As we identified $(O_q(G)\otimes V)^{U_q(\mathfrak{g})} \cong \bigoplus_{\mu\in P_+}\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu\otimes V)$ and are thus interested in the map $\mathrm{Res}\colon \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu\otimes V)\to O(H)\otimes V$, all the claims can be stated and proved in this language of traces of intertwining operators. The main Theorem 2.2.4 can be restated in this language as the following theorem, analogous to Corollary 2.1.6.

Theorem 2.4.1. *For any intertwining operator* $\Phi\in \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu\otimes V)$ *define its weighted trace as a function* $\Psi\in O(H)\otimes V$ *given by* $\Psi(x) = \mathrm{Tr}_{L_\mu}(\Phi\circ x)$. *Then the map*

$$\mathrm{Res}\colon \bigoplus_{\mu\in P_+}\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu\otimes V)\to O(H)\otimes V$$

given by $\mathrm{Res}\Phi = \Psi$ *is injective, and its image consists of all the functions* $f\in O(H)\otimes V$ *that satisfy*

1. $f\in O(H)\otimes V[0]$;

2. f *is invariant under the (unshifted) action of the dynamical Weyl group, meaning that for all* $w\in W$, $\lambda\in\mathfrak{h}^*$,

$$f(q^{2w\lambda}) = \mathcal{A}_{w,V}(\lambda)f(q^{2\lambda});$$

3. *for every* $\alpha_i\in\Pi$ *and every* $n\in\mathbb{N}$, *the polynomial* $E_i^n.f$ *is divisible by*

$$(1-q_i^2e^{\alpha_i})(1-q_i^4e^{\alpha_i})\dots(1-q_i^{2n}e^{\alpha_i}).$$

Lemma 2.4.2. *Trace functions* $\Psi = \mathrm{Res}\Phi$ *satisfy 1), i.e.* $\Psi\in O(H)\otimes V[0]$.

Proof. Let $\Phi \in \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu \otimes V)$. We can assume we are calculating the trace of Φ using a basis of weight vectors in L. The image of every weight vector l in L_μ under Φ is going to be a weight vector of the same weight, so when we write it as a sum of elementary tensors and pick the elementary tensor whose first component is l, the second component is going to have weight 0. □

Lemma 2.4.3. *Trace functions* $\Psi = \mathrm{Res}\Phi$ *satisfy 2), i.e. for every* $w \in W$, $\lambda \in \mathfrak{h}^*$,

$$\Psi(q^{2w\lambda}) = \mathcal{A}_{w,V}(\lambda)\Psi(q^{2\lambda}).$$

Proof. Using the definition of renormalized trace functions from section 2.3, Proposition 2.3.12, definition of shifted and unshifted action of dynamical Weyl group, Proposition 2.3.13, defintion of renormalized trace functions again, and finally Lemma 2.3.11, we get

$$\begin{aligned}
&\Psi_V(\lambda,\mu) = \\
&= \delta_q(\lambda)^{-1}F_V(\lambda,-\mu-\rho)Q_V(\mu) \\
&= \delta_q(\lambda)^{-1}\mathcal{A}_{w,V}(w^{-1}\lambda)F_V(w^{-1}\lambda, w^{-1}(-\mu-\rho))\mathcal{A}_{w,V^*}(w^{-1}(-\mu-\rho))^*Q_V(\mu) \\
&= \delta_q(\lambda)^{-1}A_{w,V}(-w^{-1}\lambda-\rho)F_V(w^{-1}\lambda, w^{-1}(-\mu-\rho))A_{w,V^*}(w^{-1}\cdot\mu)^*Q_V(\mu) \\
&= \delta_q(\lambda)^{-1}A_{w,V}(-w^{-1}\lambda-\rho)F_V(w^{-1}\lambda, -w^{-1}(\mu+\rho))Q_V(w^{-1}\cdot\mu)A_{w,V}(w^{-1}\cdot\mu)^{-1} \\
&= \delta_q(\lambda)^{-1}\delta_q(w^{-1}\lambda)A_{w,V}(-w^{-1}\lambda-\rho)\Psi_V(w^{-1}\lambda, w^{-1}(\mu+\rho)-\rho)\cdot \\
&\qquad\qquad \cdot Q_V(w^{-1}(\mu+\rho)-\rho)^{-1}Q_V(w^{-1}\cdot\mu)A_{w,V}(w^{-1}\cdot\mu)^{-1} \\
&= (-1)^w A_{w,V}(-w^{-1}\lambda-\rho)\Psi_V(w^{-1}\lambda, w^{-1}\cdot\mu)A_{w,V}(w^{-1}\cdot\mu)^{-1}.
\end{aligned}$$

As we are interested in traces of intertwining operators on irreducible modules and not on Verma modules, we use Poposition 2.3.10 to translate the above identity to those functions:

$$\begin{aligned}
\Psi^v_\mu(\lambda) &= \sum_{w\in W}(-1)^w\Psi_V(\lambda, w\cdot\mu)A_{w,V}(\mu)\\
&= \sum_{w\in W}(-1)^w(-1)^w A_{w,V}(-w^{-1}\lambda-\rho)\Psi_V(w^{-1}\lambda, w^{-1}\cdot(w\cdot\mu))\cdot\\
&\qquad \cdot A_{w,V}(w^{-1}(w\cdot\mu))^{-1}A_{w,V}(\mu)\\
&= \sum_{w\in W} A_{w,V}(-w^{-1}\lambda-\rho)\Psi_V(w^{-1}\lambda,\mu)\\
&= \sum_{w\in W} \mathcal{A}_{w,V}(w^{-1}\lambda)\Psi_V(w^{-1}\lambda,\mu).
\end{aligned}$$

Finally, we use this and Corollary 2.3.8 to conclude that for any $w'\in W$,

$$\begin{aligned}
\mathcal{A}_{w',V}(\lambda)\Psi^v_\mu(\lambda) &= \sum_{w\in W}\mathcal{A}_{w',V}(\lambda)\mathcal{A}_{w,V}(w^{-1}\lambda)\Psi_V(w^{-1}\lambda,\mu)\\
&= \sum_{w\in W}\mathcal{A}_{w'w,V}(w^{-1}\lambda)\Psi_V(w^{-1}\lambda,\mu)\\
&= \sum_{w\in W}\mathcal{A}_{w,V}(w^{-1}w'\lambda)\Psi_V(w^{-1}w'\lambda,\mu)\\
&= \Psi^v_\mu(w'\lambda)
\end{aligned}$$

as required. This proves the lemma with the above convention

$$\Psi(q^{2\lambda}) = \mathrm{Tr}|_{L_\mu}(\Phi\circ q^{2\lambda}) = \Psi^v_\mu(\lambda).$$

Notice that we used the fact that Φ^v_μ span the space $\bigoplus_\mu \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu\otimes V)$, so it is enough to prove the invariance for Ψ^v_μ. □

Lemma 2.4.4. *Trace functions Ψ satisfy 3), i.e. for every $i=1,\ldots r$ and every $n\in\mathbb{N}$, the polynomial $E_i^n.\Psi$ is divisible by $(1-q_i^2e^{\alpha_i})(1-q_i^4e^{\alpha_i})\ldots(1-q_i^{2n}e^{\alpha_i})$.*

Proof. If $\Phi\in\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu\otimes V)$, we can define its trace function not only as a function on H, but on the entire $U_q(\mathfrak{g})$, by $f(X)=\mathrm{Tr}|_{L_\mu}(\Phi\circ X)$. The restriction of f to the subalgebra generated by all the q^h is the trace function Ψ as in the claim of the lemma.

This defines a map from $\mathrm{Hom}_{\mathbb{C}}(L, L\otimes V) \cong {}^*L\otimes L\otimes V$ to linear functions from $U_q(\mathfrak{g})$ to V. We can make $U_q(\mathfrak{g})$ act on the algebraic dual of $U_q(\mathfrak{g})$ tensored with V in a way to make the above defined trace map a morphism of $U_q(\mathfrak{g})$ modules. The easiest way to do that is to remember that the compatible definition of action of $Y \in U_q(\mathfrak{g})$ on ${}^*L\otimes L\otimes V$ was by $Y_{(1)}\otimes Y_{(2)}\otimes Y_{(3)}$, and to notice that the map from ${}^*L\otimes L\otimes V$ to the linear functions $f\colon U_q(\mathfrak{g}) \to V$ corresponding to the one we just defined above is $\varphi\otimes l\otimes v \mapsto (X \mapsto \varphi(Xl)v)$. From this it is clear that the action of $Y \in U_q(\mathfrak{g})$ on the space of linear functions $f\colon U_q(\mathfrak{g}) \to V$ is

$$(Yf)(X) = Y_{(3)}.f(S^{-1}(Y_{(1)})XY_{(2)}).$$

If the function $f\colon U_q(\mathfrak{g}) \to V$ was defined as a trace $f(X) = \mathrm{Tr}(\Phi\circ X)$ of an intertwining operator Φ, then it is invariant with respect to the above action, and hence satisfies

$$\varepsilon(Y)f(X) = Y_{(3)}.f(S^{-1}(Y_{(1)})XY_{(2)}).$$

Specializing this identity to $Y = q^h$, for which $\varepsilon(q^h) = 1$, $S^{-1}(q^h) = q^{-h}$, and $\Delta^2 q^h = q^h\otimes q^h\otimes q^h$, we get that f satisfies

$$q^h.f(X) = f(q^hXq^{-h}).$$

Specializing it to $Y = E_i$ instead, for which $\varepsilon(E_i) = 0$, $S^{-1}(E_i) = -q_i^{-h_i}E_i$ and

$$\Delta^2 E_i = E_i\otimes q_i^{h_i}\otimes q_i^{h_i} + 1\otimes E_i\otimes q_i^{h_i} + 1\otimes 1\otimes E_i,$$

we get that f also satisfies

$$0 = q_i^{h_i}.f(-q_i^{-h_i}E_iXq_i^{h_i}) + q_i^{h_i}.f(XE_i) + E_i.f(X)$$

so

$$E_i.f(X) = f(E_iX) - f(q_i^{h_i}XE_iq_i^{-h_i}).$$

Using this formula, we will now prove by induction on n that

$$E_i^n.f(q^h) = (1 - q_i^2 q^{\alpha_i(h)}) \cdot (1 - q_i^4 q^{\alpha_i(h)}) \ldots (1 - q_i^{2n} q^{\alpha_i(h)}) f(E_i^n q^h).$$

For $n = 0$ the claim is trivial. Assume that it is true for $n - 1$ and calculate

$$\begin{aligned}
&E_i^n.f(q^h) = \\
&= (1 - q_i^2 q^{\alpha_i(h)}) \ldots (1 - q_i^{2(n-1)} q^{\alpha_i(h)}) E_i.f(E_i^{n-1} q^h) \\
&= (1 - q_i^2 q^{\alpha_i(h)}) \ldots (1 - q_i^{2(n-1)} q^{\alpha_i(h)}) \big(f(E_i^n q^h) - f(q_i^{h_i} E_i^{n-1} q^h E_i q_i^{-h_i})\big) \\
&= (1 - q_i^2 q^{\alpha_i(h)}) \ldots (1 - q_i^{2(n-1)} q^{\alpha_i(h)}) \big(f(E_i^n q^h) - q_i^{2n} q^{\alpha_i(h)} f(E_i^n q^h)\big) \\
&= (1 - q_i^2 q^{\alpha_i(h)}) \ldots (1 - q_i^{2n} q^{\alpha_i(h)}) f(E_i^n q^h).
\end{aligned}$$

This ends the induction and proves the lemma. □

Remark 2.4.5. *[37], Chapter 3, Definition 1.2.1, defines $O_q(G)$ as the Hopf subalgebra of the space of linear functionals on $U_q(\mathfrak{g})$ generated by the matrix elements of the finite-dimensional representations of type I. The map from the beginning of the above proof, associating to $\varphi \otimes l \in {}^*L_\lambda \otimes L_\lambda$ the functional on $U_q(\mathfrak{g})$ given by $X \mapsto \varphi(Xl)$, is exactly the isomorphism from $\oplus_\lambda {}^*L_\lambda \otimes L_\lambda$ to this subalgebra of linear functionals, establishing the equivalence of these two definitions.*

Lemma 2.4.6. *The restriction map $(O_q(G) \otimes V)^{U_q(\mathfrak{g})} \to O(H) \otimes V$ is injective.*

Proof. We will prove that the map

$$\bigoplus_{\mu \in P_+} \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu \otimes V) \to O(H) \otimes V$$

associating to the intertwining operator Φ its weighted trace $\Psi(x) = \mathrm{Tr}|_{L_\mu}(\Phi \circ x)$ is injective.

Let

$$\Phi = \sum_\mu \Phi_\mu^{v_\mu}$$

be an element of $\bigoplus_{\mu\in P_+}\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\mu, L_\mu\otimes V)$ whose weighted trace is zero. Notice that

$$\mathrm{Tr}(\Phi_\mu^{v_\mu}\circ x)=\sum_\nu u_{\mu,\nu}e^\nu(x)$$

for some $u_{\mu,\nu}\in V[0]$, with $u_{\mu,\nu}=0$ unless ν is a weight of L_μ, and with $u_{\mu,\mu}=v_\mu$. The fact that Φ maps to zero can be written as

$$\sum_\mu\sum_\nu u_{\mu,\nu}e^\nu=0.$$

Assume $\Phi\neq 0$. Then one can pick $\mu=\mu_0$ so that $\Phi_\mu^{v_\mu}$ is nonzero and μ_0 is a highest weight with that property. Using the fact that e^ν are linearly independent, the coefficient with e^{μ_0} in the above equation is

$$0=u_{\mu_0,\mu_0}=v_{\mu_0}.$$

But then $0=\Psi_{\mu_0}^{v_{\mu_0}}$, contrary to the choice of μ_0. So, $\Phi=0$ and the map is injective. □

Lemma 2.4.7. *Theorem 2.4.1 holds for* $U_q(\mathfrak{sl}_2)$.

Proof. As stated before, we are using identifications $\mathbb{C}\cong\mathfrak{h}^*$, $z\mapsto z\frac{\alpha}{2}$. Let us use a slightly different convention for $\mathfrak{h}$: it is also one dimensional, so we can write any element of it as zh, for $h=h_1$ the standard generator of $\mathfrak{h}$ and $z\in\mathbb{C}$. The space $V[0]$ is one dimensional, so pick any $v_0\neq 0\in V[0]$ and identify $V[0]\cong\mathbb{C}$ by it. The polynomial functions in $O(H)\otimes V[0]$ we talk about are, with all these identifications, $\mathbb{C}[q^z,q^{-z}]$, spanned by functions $e^{n\alpha/2}(q^{zh})=q^{nz}$, $n\in\mathbb{Z}$. With all these conventions, $zh\leftrightarrow z\alpha=2z\frac{\alpha}{2}$, so and the above definitions of trace functions give $\Psi(q^{zh})=\Psi_\mu^v(z\frac{\alpha}{2})$. This is a good convention because $z\alpha/2\leftrightarrow z$. The dynamical Weyl group invariance, with all these identifications, has the form

$$\mathcal{A}_{s,V}(z)\Psi(q^{zh})=\Psi(q^{-zh}).$$

We are proving that two subspaces of polynomial functions $O(H)\otimes V$ are equal:

the space of traces of intertwining operators and the space of functions satisfying 1)-3) from the statement of Theorem 2.4.1. Lemmas 2.4.2, 2.4.3 and 2.4.4 show that the space of traces of intertwining operators is contained in the space of functions satisfying 1)-3). We will now prove this lemma by proving that these two spaces of functions are of the same size (more accurately, as they are infinite-dimensional, that there is a filtration on $O(H)$ such that dimensions match on every filterered piece; the filtration we use will be the obvious filtration by degree of a polynomial).

As stated in Lemma 2.3.1, the space of intertwining operators $L_\mu \to L_\mu \otimes V$ for $V = L_{2m}$ is zero if $\mu = 0, 1, \dots m-1$, and is one dimensional if $\mu \in \mathbb{N}, \mu \geq m$. The trace of such an operator is a Laurent polynomial $\Psi(x) = \mathrm{Tr}|_{L_\mu}(\Phi \circ x) = \sum_\nu a_\nu e^\nu(x)$, with all the ν that appear being weights of L_μ. So, it is a Laurent polynomial of maximal (positive and negative) degree μ. Using Lemma 2.4.6 that allows us to calculate the dimension of the space of trace functions by calculating the dimension of the appropriate space of intertwining operators, we can conclude that for any large enough positive integer N, the space of trace functions of maximal (positive) degrees less or equal to N has dimension $N - m + 1$.

Now, let us calculate the dimension of the space of functions that satisfy 1)-3) and have degree $\leq N$. It is enough to show that it has dimension less or equal to $N - m + 1$; from this it will follow that it has exactly this dimension and that the two spaces are equal.

Let f be such a function. Condition 1) of $\mathrm{Im} f \in V[0]$ means we can regard f as an element of $\mathbb{C}[q^z, q^{-z}]$ after taking into account $V[0] \cong \mathbb{C}$. So, f is of the form $f(x) = \sum_n a_n e^{n\alpha/2}(x)$, with only finitely many $n \in \mathbb{Z}, |n| \leq N$ appearing.

Condition 3) is about $E^n.f$ being divisible by a certain function. We are in a $U_q(\mathfrak{sl}_2)$ module V, which has 1 dimensional weight spaces $V[0], V[2], \dots V[2m]$, with $E \colon V[2i] \to V[2i+2]$ being injective for $i = 0, \dots m-1$, and being zero for $i = m$. The functions $E^n.f$ are of the form rational function times a basis vector for $V[2n]$. For $n > m$ this is zero, so it is divisible by anything. For $n = 1, \dots m$, the rational function in $E^n.f$ is, up to a multiplicative constant, equal to f. Condition 3) in this case says that f is divisible in the ring $\mathbb{C}[e^z, e^{-z}]$ by

$$(1-q^{2+2z})(1-q^{4+2z})\dots(1-q^{2m+2z}) =$$

$$= (-1)^m \cdot q^{m(m+1)/2} \cdot q^{mz} \cdot (q^{z+1}-q^{-z-1})(q^{z+2}-q^{-z-2})\dots(q^{z+m}-q^{-z-m}).$$

This is equivalent to saying it is divisible by

$$(q^{z+1}-q^{-z-1})(q^{z+2}-q^{-z-2})\dots(q^{z+m}-q^{-z-m}).$$

Condition 2) says that f is invariant under the unshifted action of the dynamical Weyl group. As the Weyl group in this case has only one nontrivial element, call it s, this really means f is invariant under the action of the operator $\mathcal{A}_{s,V}$, which was explicitly calculated in 2.3.6. We know s acts on $\mathfrak{h}$ by -1, so this means

$$\begin{aligned} f(q^{-z}) &= \mathcal{A}_{s,V}(z) f(q^z) \\ &= A_{s,V}(-z-1) f(q^z) \\ &= (-1)^m \prod_{j=1}^{m} \frac{[-z+j]_q}{[-z-j]_q} \cdot f(q^z) \\ &= \prod_{j=1}^{m} \frac{q^{j-z}-q^{-j+z}}{q^{j+z}-q^{-j-z}} \cdot f(q^z) \end{aligned}$$

Thus, the function

$$g(q^z) = \frac{f(q^z)}{(q^{z+1}-q^{-z-1})(q^{z+2}-q^{-z-2})\dots(q^{z+m}-q^{-z-m})}$$

is W-equivariant, in the sense $g(q^z) = g(q^{-z})$. The function g is also a Laurent polynomial, because by condition 3) above, f is divisible in $\mathbb{C}[q^z, q^{-z}]$ by the denominator of g. If f was of degree $\leq N$, then g is of degree $\leq N-m$. Invariance under W and limitations on the maximal degree mean that g is of the form

$$g(q^z) = \sum_{n=0}^{N-m} b_n (q^{zn} + q^{-zn}).$$

The space of all such functions has dimension $N-m+1$; so, the space of all possible f that satisfy 1)-3) and have degree less or equal to N also has dimension less or equal to $N-m+1$.

This proves the lemma. $\square$

To finish the proof of Theorem 2.4.1 we need to show that any function $f \in O(H)\otimes V$ satisfying 1)-3) from the statement can be written as a linear combination of trace functions. For this, write f as a sum of characters of H,

$$f(x) = \sum_{\mu\in P} v_\mu e^\mu(x),$$

with $v_\mu \in V[0]$.

For any fixed $i = 1,\dots r$ one can decompose f as follows:

$$f = \sum_{\beta\in P_+/\mathbb{Z}\alpha_i} f_\beta, \qquad f_\beta = \sum_{\mu\in\beta} v_\mu e^\mu.$$

Lemma 2.4.8. *Every f_β is a sum of trace functions for the subalgebra $U_{q_i}(\mathfrak{sl}_2)$ of $U_q(\mathfrak{g})$ generated by $E_i, F_i, q^{zh_i}, z\in\mathbb{C}$.*

Proof. Due to Lemma 2.4.7, it is enough to prove they satisfy 1)-3) for $U_{q_i}(\mathfrak{sl}_2)$.

Condition 1) is clear.

Condition 2) is about dynamical Weyl group invariance. The operators $A_{s_i,V}(\mu)\colon V[\nu]\to V[s_i\nu]$ for $U_q(\mathfrak{g})$ and $A_{s_i,V}(\mu(h_i))\colon V[\nu]\to V[s_i\nu]$ for $U_{q_i}(\mathfrak{sl}_2)$ coincide. Using this, the fact that $\mathcal{A}_{s_i,V}(\lambda) = A_{s_i,V}(-\lambda-\rho)$, and the condition that f is invariant under the action of s_i, meaning

$$\sum_\beta f_\beta(q^{2s_i\lambda}) = \mathcal{A}_{s_i,V}(\lambda)\sum_\beta f_\beta(q^{2\lambda}),$$

we get

$$\sum_\beta f_\beta(q^{2s_i\lambda}) = \sum_\beta A_{s_i,V'}(-d_i^{-1}(\langle\alpha_i,\lambda\rangle+1))f_\beta(q^{2\lambda}).$$

Now decompose both of these functions into their β parts as we did with f. Call the left hand side function l and the right hand side r. Using $\langle \mu, s_i\lambda\rangle = \langle s_i\mu, \lambda\rangle$ and the fact that $\mu \in \beta$ implies $s_i\mu \in \beta$, we get $l_\beta(q^\lambda) = f_\beta(q^{2s_i\lambda})$. On the right hand side, we know that $A_{s_i,V}(-d_i^{-1}(\langle \alpha_i, \lambda\rangle + 1))$ maps $V[\nu]$ to $V[s_i\nu]$, so $r_\beta(q^\lambda) = A_{s_i,V}(-d_i^{-1}(\langle \alpha_i, \lambda\rangle + 1))f_\beta(q^{2\lambda})$. Thus we have

$$f_\beta(q^{2s_i\lambda}) = A_{s_i,V}(-d_i^{-1}(\langle \alpha_i, \lambda\rangle + 1))f_\beta(q^{2\lambda}).$$

Remembering the identification $\mathfrak{h} \cong \mathfrak{h}^*$ and restricting this to $2\lambda = zd_i^{-1}\alpha_i$, which corresponds to zh_i we get

$$f_\beta(q^{-zh_i}) = A_{s_i,V'}(d_i^{-1}(-z-1))f_\beta(e^{zh_i}).$$

This is exactly the dynamical Weyl group invariance for $U_q(\mathfrak{sl}_2)$, as the function f is defined in terms of powers of q, and the operator $A_{s_i,V}$, which was defined for $U_{q_i}(\mathfrak{sl}_2)$, in terms of $q_i = q^{d_i}$. Replacing q with q_i we get the required dynamical Weyl group invariance for $U_{q_i}(\mathfrak{sl}_2)$.

Condition 3): we know that $E_i^n.f$ is divisible by $(1-q_i^2e^{\alpha_i})\dots(1-q_i^{2n}e^{\alpha_i})$, call the quotient $g \in O(H)$ and write

$$E_i^n.\sum_\beta f_\beta = (1-q_i^2e^{\alpha_i})\dots(1-q_i^{2n}e^{\alpha_i}) \cdot \sum_\beta g_\beta.$$

Decompose both sides into their β parts to get

$$E_i^n.f_\beta = (1-q_i^2e^{\alpha_i})\dots(1-q_i^{2n}e^{\alpha_i})g_\beta.$$

Replacing q by q_i in all the functions we get the required statement for $U_{q_i}(\mathfrak{sl}_2)$.

□

For f as above, $f = \sum_{\mu\in P} v_\mu e^\mu$, let $D(f) = \{\mu \in P | v_\mu \neq 0\}$, and let $C(f)$ be the convex hull of $D(f)$ in the Euclidean space $\mathfrak{h}_\mathbb{R}^*$. Then define the weight diagram of f to be the set $\mathbf{WD}(f) = C(f) \cap P$. We will prove the theorem by induction on the size of the set $\mathbf{WD}(f)$.

Lemma 2.4.9. *If f satisfies 1)-3), then $D(f)$, and consequently $\mathbf{WD}(f)$, is W-invariant.*

Proof. First note that it follows directly from the proof of Lemma 2.4.7 that this is true for $U_q(\mathfrak{sl}_2)$, as

$$f(q^{zh}) = g(q^{zh}) \cdot (q^{z+1} - q^{-z-1}) \dots (q^{z+m} - q^{-z-m})$$

with g being W-equivariant, so the set of powers of q^z that appear in f is symmetric around 0. This means $D(f)$, and therefore also $C(f)$ and $\mathbf{WD}(f)$, is W-invariant.

Next, write $f = \sum f_\beta$ as before, for $\beta \in P_+/\mathbb{Z}\alpha_i$. The simple reflection s_i preserves the equivalence classes β. So, $s_i D(f) = D(f)$ if and only if $s_i D(f_\beta) = D(f_\beta)$.

From the previous lemma, f_β is the sum of trace functions, so from the comment at the beginning of this proof, $D(f_\beta)$ is invariant under the action of the Weyl group of $U_{q_i}(\mathfrak{sl}_2)$. The nontrivial element of this Weyl group is the reflection with respect to the only simple root for $U_{q_i}(\mathfrak{sl}_2)$, which is α_i. So, every $D(f_\beta) \subseteq P$ is preserved under s_i, hence $s_i D(f) = D(f)$ and $s_i \mathbf{WD}(f) = \mathbf{WD}(f)$.

Geometrically in the lattice P, the argument in the last paragraph corresponds to decomposing $D(f)$ into sets $D(f_\beta)$, so that every $D(f_\beta)$ consists of points of $D(f)$ that lie on one of the parallel lines in $\mathfrak{h}^*$, passing through β, in the direction of α_i. Then we note that s_i preserves each of these lines, and that every such line is symmetric with respect to the hyperplane through 0 orthogonal to α_i. This is exactly the reflection hyperplane of s_i, so $D(f)$ is symmetric with respect to this hyperplane and preserved by s_i.

Of course, once we proved $D(f)$ and $\mathbf{WD}(f)$ are preserved by all the simple reflections s_i, we immediately conclude that they are preserved by the entire group W generated by all the s_i.

□

Proof of Theorem 2.4.1. Let us prove Theorem 2.4.1 by induction on the size of the finite set $\mathbf{WD}(f)$.

If the set $\mathbf{WD}(f)$ is empty, $f = 0$ and there is nothing to prove.

Otherwise, assume we have proved the theorem for all functions whose weight diagram has fewer elements than $\mathbf{WD}(f)$.

Pick $\mu \in \mathbf{WD}(f)$ an extremal point, meaning a point μ such that it is not in the convex hull of $\mathbf{WD}(f)\backslash\{\mu\}$. Such a point exists, as $\mathbf{WD}(f)$ is a finite set. Moreover, such a point μ is in $D(f)$. To see that, notice that $\mu \in C(f)$ means that either $\mu \in D(f)$, or $\mu = \sum_i t_i\mu_i$ for some $t_i \geq 0, \sum_i t_i = 1$ and some $\mu_i \in D(f), \mu_i \neq \mu$. In the latter case $\mu_i \in D(f)\backslash\{\mu\} \subseteq \mathbf{WD}(f)\backslash\{\mu\}$, so $\mu = \sum_i t_i\mu_i$ is in the convex hull of $\mathbf{WD}(f)\backslash\{\mu\}$, contrary to the choice of μ.

As $D(f)$ and $\mathbf{WD}(f)$ are W-invariant, we can assume without loss of generality that μ is a dominant weight. Finally, for any $i = 1, \dots r$, the weight $\mu + \alpha_i$ is not in $\mathbf{WD}(f)$. To see that, consider two cases: either $\langle \mu, \alpha_i \rangle \neq 0$ or $\langle \mu, \alpha_i \rangle = 0$. If $\langle \mu, \alpha_i \rangle \neq 0$, then $s_i\mu = \mu - 2\frac{\langle \mu, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \in \mathbf{WD}(f)\backslash\{\mu\}, \mu + \alpha_i \in \mathbf{WD}(f)\backslash\{\mu\}$ implies that μ is in the convex hull of $\mathbf{WD}(f)\backslash\{\mu\}$, contrary to the choice of μ. If $\langle \mu, \alpha_i \rangle = 0$, then the same can be concluded from $\mu + \alpha_i \in \mathbf{WD}(f)\backslash\{\mu\}, s_i(\mu + \alpha_i) = \mu - \alpha_i \in \mathbf{WD}(f)\backslash\{\mu\}$.

Let us now restrict f to $U_{q_i}(\mathfrak{sl}_2)$ and decompose into f_β as before. Then Lemma 2.4.8 tells us that all f_β, and in particular the f_β such that $\mu \in \beta$, are traces of intertwining operators for $U_{q_i}(\mathfrak{sl}_2)$. Lemma 2.3.1, 2) then implies that $E_i^{d_i^{-1}\langle \mu, \alpha_i \rangle + 1} v_\mu = 0$.

Since this statement is valid for every $i = 1 \dots r$, the same Lemma 2.3.1, 2) implies that there is an intertwining operator $\bar{\Phi}_\mu^{v_\mu} : L_\mu \to L_\mu \otimes V$. Its trace function $\Psi_\mu^{v_\mu}$ has $\mathbf{WD}(\Psi_\mu^{v_\mu})$ equal to the convex hull of the set of weights of L_μ, i.e. equal to the convex hull of the W orbit of μ. This is contained in the set $\mathbf{WD}(f)$. So, the function $f - \Psi_\mu^{v_\mu}$ satisfies 1)-3), has $\mathbf{WD}(f - \Psi_\mu^{v_\mu})$ contained in $\mathbf{WD}(f)$, and has the coefficient of e^μ equal to $v_\mu - v_\mu = 0$. This means $D(f - \Psi_\mu^{v_\mu})$ is a subset of $\mathbf{WD}(f)$ which does not contain μ, so it's convex hull doesn't contain μ, and so $\mathbf{WD}(f - \Psi_\mu^{v_\mu})$ is a proper subset of $\mathbf{WD}(f)$.

By induction assumption we can now express $f - \Psi_\mu^{v_\mu}$ as a linear combination of trace functions. So, we can express f as a linear combination of trace functions. This proves the theorem.

□

Remark 2.4.10. *It is explained in [36] how theorem 2.1.1 reduces when V is small enough in the appropriate sense. First, if V is a trivial representation of $\mathfrak{g}$, then conditions* (1) *and* (3) *of theorem 2.1.1 are automatically satisfied, so the statement becomes the Chevalley isomorphism theorem $\mathbb{C}[\mathfrak{g}]^G \tilde{\to} \mathbb{C}[\mathfrak{h}]^W$. The second special case is when V is small in the sense of [12], meaning that for every root α, 2α is not a weight of V. In that case, any function $f \in \mathbb{C}[\mathfrak{h}] \otimes V$ that satisfies conditions* (1) *and* (2) *automatically satisfies* (3) *as well. To see that, first note that $E_i^n.f = 0$ for all $n \geq 2$. Next, for any vector $v \in V[0]$, either $E_i.v = 0$, or $E_i.v \neq 0, E_i^2.v = 0$ and v generates an $(\mathfrak{sl}_2)_i$ representation isomorphic to either the three dimensional irreducible representation L_2, or the direct sum of the trivial representation L_0 with L_2. Then condition* (1) *means that f is a sum of functions of the form f_1v_1 and functions of the form f_2v_2, for some $f_{1,2} \in \mathbb{C}[\mathfrak{h}]$, some v_1 which generate a trivial $(\mathfrak{sl}_2)_i$ representation and some v_2 in the zero weight space of some three dimensional $(\mathfrak{sl}_2)_i$ representation. Condition* (2) *implies that f_1 is an even function and f_2 an odd one with respect to the action of the element $s_i \in W$ corresponding to $(\mathfrak{sl}_2)_i$, so $E_i.f(h) = f_2(h)E_i.v_2$ is divisible by α_i. This reduces the theorem 2.1.1 to theorem 1 in [12].*

In the context of quantum groups, the same analysis applies to theorem 2.2.4. If V is trivial, then conditions (1) *and* (3) *are satisfied, and condition* (2) *reduces to $f(q^{w\lambda}) = f(q^\lambda)$ because $\mathcal{A}_{w,V}(\lambda) = id$ (by lemma 2.3.6). If V is small in the sense of [12], then for any copy of $U_q(\mathfrak{sl}_2)_i$, the only representations of it that contain a nonzero vector in $V[0]$ are direct sums of trivial and three dimensional irreducible representations. We again conclude that any function $f \in O(H) \otimes V$ which satisfies* (1) *and* (2) *must be a sum of functions of the form f_1v_1 and f_2v_2, with v_1 in some copy of L_0 and v_2 in some copy of L_2. If f also satisfies* (2)*, then by the proof of lemma 2.4.7, f_2 is of the form $f_2(q^z) = g(q^z) \cdot (q^{z+1} - q^{-z-1})$, so $E_i.f_2$ is divisible by $(1 - q_i^2 e^{\alpha_i})$. All the other parts of condition* (3) *are satisfied trivially, as $E_i.v_1 = 0$ and $E_i^2.v_2 = 0$. So, in the case V is small, condition* (3) *is unnecessary, and theorem 2.2.4 reduces to a quantum version of theorem 1 from [12].*

Chapter 3

Rational Cherednik Algebras

3.1 Definitions and basic properties

3.1.1 Reflection groups

For the remainder of the thesis we study rational Cherednik algebras over an algebraically closed field $\Bbbk$. We first present definitions and properties which do not depend on characteristic of $\Bbbk$, then focus on $\Bbbk = \mathbb{C}$ for the rest of this chapter and for Chapters 4 and 5, and then let $\Bbbk$ have finite characteristic in Chapters 6,7 and 8.

The definitions and statements in this chapter are standard and can, unless otherwise stated, be found in [21] or [24].

Let W be a finite group, and $\mathfrak{h}$ its n-dimensional faithful representation over $\Bbbk$. We say that an element $s \in W$ is a *reflection* if $\mathrm{rank}(1-s)|_{\mathfrak{h}} = 1$. We say that W is a *reflection group* if it is generated by the set S of reflections in W. We call $\mathfrak{h}$ its *reflection representation.*

Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$, and $(\cdot,\cdot)\colon \mathfrak{h}^*\times\mathfrak{h} \to \Bbbk$, $(\cdot,\cdot)\colon \mathfrak{h}\times\mathfrak{h}^* \to \Bbbk$ the canonical pairings. The group W acts naturally on $\mathfrak{h}^*$ by the dual representation, and that an element $s \in W$ is a reflection on $\mathfrak{h}$ if and only if it is a reflection on $\mathfrak{h}^*$.

Lemma 3.1.1. *For any reflection $s \in S$, let $\alpha_s^\vee \in \mathfrak{h}$ and $\alpha_s \in \mathfrak{h}^*$ be such that s acts on all $x \in \mathfrak{h}^*$ by*

$$s.x = x - (\alpha_s^\vee, x)\alpha_s.$$

Then the action of s on $y \in \mathfrak{h}$ is

$$s.y = y + \frac{(y, \alpha_s)}{1 - (\alpha_s, \alpha_s^\vee)} \alpha_s^\vee.$$

The element α_s is a basis of $\mathrm{Im}(1-s)|_{\mathfrak{h}^*}$*, and $\alpha_s^\vee$ is a basis of* $\mathrm{Im}(1-s)|_{\mathfrak{h}}$*. Let $\lambda_s = 1 - (\alpha_s, \alpha_s^\vee)$. If $\lambda_s \neq 1$, then the reflection s is diagonalizable on $\mathfrak{h}^*$ with eigenvalues $\lambda_s, 1 \ldots 1$ and on $\mathfrak{h}$ with eigenvalues $\lambda_s^{-1}, 1 \ldots 1$. If $\lambda_s = 1$, then s is not diagonalizable, has a generalized eignevalue* 1 *and* $n-1$ *eigenvalues* 1*. Such reflections of finite order are unipotent and only exist in finite characteristic.*

The reflection s is uniquely determined by $\alpha_s \otimes \alpha_s^\vee \in \mathfrak{h}^* \otimes \mathfrak{h}$. The vectors α_s and $\alpha_s^\vee$ are determined by s up to mutual rescaling.

Proof. The rank of $1 - s$ on $\mathfrak{h}^*$ is 1, so there exist $\alpha_s^\vee$ and α_s, unique up to mutual rescaling, such that $(1-s)|_{\mathfrak{h}^*} = (\alpha_s^\vee, \cdot)\alpha_s$. The formula for the action of s on $\mathfrak{h}$ follows from this and the definition of dual representation, which is

$$(s.y, s.x) = (y, x).$$

It is clear from these formulas that $\alpha_s^\vee$ and α_s are basis vectors for the image of $1-s$ on $\mathfrak{h}$ and $\mathfrak{h}^*$, and that they are (generalized) eigenvectors for s on $\mathfrak{h}$ and $\mathfrak{h}^*$ with (generalized) eigenvalue λ_s^{-1} and λ_s. All vectors in $\mathfrak{h}$ which are in the kernel of $(\alpha_s, \cdot)$ are eigenvectors of s with eigenvalue 1, and all vectors in $\mathfrak{h}^*$ which are in the kernel of $(\alpha_s^\vee, \cdot)$ are eigenvectors of s with eigenvalue 1. If $\lambda = 1$, then for any $x \in \mathfrak{h}^*$,

$$s^n.x = x - n(\alpha_s^\vee, x)\alpha_s.$$

From this it follows that $(1-s)^2 = 0$, so $1 - s$ is nilpotent and s is unipotent. If the order of s s finite (and we consider only finite reflection groups), then some power of s is the identity, and it follows that $n(\alpha_s^\vee, x)\alpha_s$ for some n and all v, so $\Bbbk$ has finite characteristic. □

Occasionally, in literature where only semisimple reflections are considered, a different normalization of α_s and $\alpha_s^\vee$ is chosen, such that $(\alpha_s, \alpha_s^\vee) = 2$. This brings about an additional factor of $\frac{2}{1-\lambda_s}$ in some formulas.

Remark 3.1.2. *A large class of reflection groups is formed by Weyl groups, and more generally Coxeter groups. These have the property that all reflections s have non-unit eigenvalue λ_s equal to -1, and that they preserve an inner product on $\mathfrak{h}$. Using this inner product, $\mathfrak{h}$ and $\mathfrak{h}^*$ can be identified. Sometimes in literature only s satisfying this extra property are called reflections, and only the groups generated by them, and thus subgroups of the orthogonal group of $\mathfrak{h}$, are called reflection groups. In that notation, the names for the more general elements and groups we defined are* complex refections *(or* unitary reflections, *or* pseudoreflections*) and* complex reflection groups.

3.1.2 Definition of rational Cherednik Algebras

Let $t \in \Bbbk$, and let $c\colon S \to \Bbbk$ be a conjugation invariant function on the set of reflections. Denoting it $s \mapsto c_s$, ti satisfies $c_s = c_{gsg^{-1}}$ for all $s \in S$, $g \in W$. For any vector space V, let SV and TV denote the symmetric and tensor algebra on V, and S^iV and T^iV the i-th homogeneous part of it.

Definition 3.1.3. *The rational Cherednik algebra $H_{t,c}(W, \mathfrak{h})$ is the quotient of the semidirect product $\Bbbk[W] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the ideal generated by relations*

$$[x, x'] = 0,\ [y, y'] = 0,\ [y, x] = (y, x)t - \sum_{s \in S} c_s((1-s).x, y)s,$$

for all $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$.

Note that for $g \in W$ and $y \in \mathfrak{h}$, we use notation gy for multiplication in the algebra, and $g.y$ for the action from the representation; because of the semidirect product they are related by $gyg^{-1} = g.y$.

Parameters t and c can be simultaneously rescaled, in the sense that $H_{t,c}(W, \mathfrak{h}) \cong H_{at,ac}(W, \mathfrak{h})$ for any $a \in \Bbbk^\times$ (the isomorphism sends generators $x \in \mathfrak{h}^*$ to tx and fixes $\mathfrak{h}$ and G). This implies that it is enough to study two cases with respect to t, $t = 0$ and $t = 1$. We are mostly interested in the $t = 1$ case.

An analogue of the PBW theorem holds for $H_{t,c}(W,\mathfrak{h})$. Let $\{x_1,\dots,x_n\}$ be any basis of $\mathfrak{h}^*$ and $\{y_1,\dots,y_n\}$ any basis of $\mathfrak{h}$. The proof of the following theorem, not depending on the characteristic, can be found in [33].

Theorem 3.1.4. *Let $(x_1,\dots,x_n)$ be a basis of $\mathfrak{h}^*$ and $(y_1,\dots,y_n)$ a basis of $\mathfrak{h}$. The elements of the form*

$$x_1^{a_1}\dots x_n^{a_n} g y_1^{b_1}\dots y_n^{b_n},$$

for $g\in W$, $a_i,b_i\in\mathbb{Z}_{\geq 0}$, form a basis of $H_{t,c}(W,\mathfrak{h})$.

From now on, let $\{y_1,\dots,y_n\}$ and $\{x_1,\dots,x_n\}$ be fixed dual bases of $\mathfrak{h}$ and $\mathfrak{h}^*$.

Lemma 3.1.5. *Let $f\colon W\to\Bbbk$ be a group character. Then $c\cdot f\colon S\to\Bbbk$ is conjugation invariant, and the algebras $H_{t,c}(W,\mathfrak{h})$ and $H_{t,c\cdot f}(W,\mathfrak{h})$ are isomorphic.*

Proof. The isomorphism is defined to be the identity on the generators from $\mathfrak{h}$ and $\mathfrak{h}^*$, and to send $g\in W$ to $f(g)\cdot g$. □

3.1.3 Verma modules and Dunkl operators

A natural class of representations of rational Cherednik algebra $H_{t,c}(W,\mathfrak{h})$ are standard or Verma modules. Let τ be an irreducible representation of W. Define a $\Bbbk[W]\ltimes S\mathfrak{h}\subset H_{1,c}(W,\mathfrak{h})$ action on τ by requiring that $\mathfrak{h}$ acts on it by zero. The *standard* or *Verma module* $M_{t,c}(\tau)$ (sometimes denoted $M_{t,c}(W,\mathfrak{h},\tau)$) is the induced module

$$M_{t,c}(\tau)=\mathrm{Ind}_{\mathbf{k}[W]\ltimes S\mathfrak{h}}^{H_{t,c}(W,\mathfrak{h})}\tau=H_{t,c}(W,\mathfrak{h})\otimes_{\mathbf{k}[W]\ltimes S\mathfrak{h}}\tau.$$

As induced modules, Verma modules satisfy the following universal mapping property:

Lemma 3.1.6. *Let M be an $H_{t,c}(W,\mathfrak{h})$-module. Let $\tau\subset M$ be a W-submodule on which $\mathfrak{h}\subseteq H_{t,c}(G,\mathfrak{h})$ acts as zero. Then there is a unique homomorphism $\phi\colon M_{t,c}(\tau)\to M$ of $H_{t,c}(W,\mathfrak{h})$-modules such that $\phi|_\tau$ is the identity.*

By the PBW theorem,

$$M_{t,c}(\tau)\cong S\mathfrak{h}^*\otimes\tau$$

as $\Bbbk$-vector spaces. Through this identification, the action of the generators of $H_{t,c}(G,\mathfrak{h})$ can be explicitly written as follows. Let $f\otimes v \in S\mathfrak{h}^*\otimes\tau \cong M_{t,c}(\tau)$, $x\in\mathfrak{h}^*$, $y\in\mathfrak{h}$ and $g\in W$. Then

$$x.(f\otimes v) = (xf)\otimes v,$$

$$g.(f\otimes v) = g.f\otimes g.v$$

$$y.(f\otimes v) = t\partial_y(f)\otimes v - \sum_{s\in S} c_s\frac{(y,\alpha_s)}{\alpha_s}(1-s).f\otimes s.v.$$

The operators

$$D_y = t\partial_y\otimes 1 - \sum_{s\in S} c_s\frac{(y,\alpha_s)}{\alpha_s}(1-s)\otimes s$$

are called *Dunkl operators.*

Remark 3.1.7. *This differs from the more usual definition of the Dunkl operator by a factor of* $\frac{2}{1-\lambda_s}$ *in the coefficient of* $s\in S$. *The reason for the different convention is explained in the comment after Lemma 3.1.1.*

Define a grading on $H_{t,c}(W,\mathfrak{h})$ by letting $x\in\mathfrak{h}^*$ have degree 1, $y\in\mathfrak{h}$ have degree -1, and $g\in G$ have degree 0. We will denote by a subscript $+$ the positive degree elements of a graded module.

We can define a grading on $M_{t,c}(\tau)\cong S\mathfrak{h}^*\otimes\tau$ by letting the i-th graded piece be $S^i\mathfrak{h}^*\otimes\tau$. It is clear from the above formulas that this grading is compatible with the grading on the algebra $H_{t,c}(W,\mathfrak{h})$, and so are any shifts of it (gradings on $M_{t,c}(\tau)$ defined by $M_{t,c}(\tau)_{i+h} = S^i\mathfrak{h}^*\otimes\tau$ for any grading shift h).

We say a homogeneous element $v\in M_{t,c}(\tau)$ is *singular* if $D_y v = 0$ for all $y\in\mathfrak{h}$. They are interesting because singular vectors of positive degree generate a proper $H_{t,c}(W,\mathfrak{h})$ submodule. If v is singular and generates an irreducible W representation, then this $H_{t,c}(W,\mathfrak{h})$ submodule is isomorphic to a quotient of $M_{t,c}(Gv)$.

We will often use the following simple observation:

Lemma 3.1.8. *If* $M=\bigoplus_i M_i$ *is a graded* $H_{t,c}(W,\mathfrak{h})$*-module isomorphic to a Verma module or any of its submodules or quotients, then for every* i *the map* $\mathfrak{h}\otimes M_{i+1}\to M_i$

defined as

$$y \otimes m \mapsto D_y(m)$$

is a homomorphism of group representations.

Even thought this claim is very simple, it is useful in finding singular vectors, as the subspace σ of M_{i+1} consists of singular vectors if and only if $\mathfrak{h} \otimes \sigma$ maps to zero under the above map. This enables us to make use Schur's lemma in this situation.

3.1.4 Contravariant form and modules $L_{t,c}(\tau)$

There is an analogue of Shapovalov form on Verma modules. First, for any graded $H_{t,c}(W, \mathfrak{h})$-module $M = \bigoplus_i M_i$ with finite-dimensional graded pieces M_i, define its restricted dual $M^\dagger$ to be the $\bigoplus_i M_i^*$. It is a left module for the opposite algebra $H_{t,c}(W, \mathfrak{h})^{opp}$ of $H_{t,c}(G, \mathfrak{h})$. Let $\bar{c}\colon S \to \Bbbk$ be the function $\bar{c}(s) = c(s^{-1})$. There is a natural isomorphism $H_{t,c}(W, \mathfrak{h})^{opp} \to H_{t,\bar{c}}(W, \mathfrak{h}^*)$ which is the identity on $\mathfrak{h}$ and $\mathfrak{h}^*$, and sends $g \mapsto g^{-1}$ for $g \in W$, making $M^\dagger$ into an $H_{t,\bar{c}}(W, \mathfrak{h}^*)$-module.

Definition 3.1.9. *Let τ be an an irreducible finite-dimensional representation of W, $M_{t,c}(W, \mathfrak{h}, \tau)$ the Verma module for $H_{t,c}(W, \mathfrak{h})$ with lowest weight τ, and $M_{t,\bar{c}}(W, \mathfrak{h}^*, \tau^*)$ the Verma module for $H_{t,\bar{c}}(W, \mathfrak{h}^*)$ with lowest weight τ^*. By Lemma 3.1.6, there is a unique homomorphism $\phi\colon M_{t,c}(W, \mathfrak{h}, \tau) \to M_{t,\bar{c}}(W, \mathfrak{h}^*, \tau^*)^\dagger$ which is the identity in the lowest graded piece τ. By adjointness, it is equivalent to the* contravariant form *pairing*

$$B\colon M_{t,c}(W, \mathfrak{h}, \tau) \times M_{t,\bar{c}}(W, \mathfrak{h}^*, \tau^*) \to \Bbbk.$$

Proposition 3.1.10. *The contravariant form B satisfies the following properties.*

a) It is W-invariant: for $g \in W$, $f \in M_{t,c}(\tau)$, $h \in M_{t,\bar{c}}(\tau^*)$, $B(g.f, g.h) = B(f, h)$.

b) For $x \in \mathfrak{h}^*$, $f \in M_{t,c}(\tau)$, and $h \in M_{t,\bar{c}}(\tau^*)$, $B(xf, h) = B(f, D_x(h))$.

c) For $y \in \mathfrak{h}$, $f \in M_{t,c}(\tau)$, and $h \in M_{t,\bar{c}}(\tau^*)$, $B(f, yh) = B(D_y(f), h)$.

d) The form is zero on elements in different degrees: if $f \in M_{t,c}(\tau)_i$ and $h \in M_{t,\bar{c}}(\tau^*)_j$, $i \neq j$, then $B(f, h) = 0$.

e) The form is the canonical pairing of τ and τ^* in the zeroth degree: for $v \in \tau = M_{t,c}(\tau)_0$, $f \in \tau^* = M_{t,\bar{c}}(\tau^*)_0$, $B(v,f) = (v,f)$.

As B respects the grading of $M_{t,c}(\tau)$ and $M_{t,\bar{c}}(\tau^*)$, we can think of it as a collection of bilinear forms on finite-dimensional graded pieces. Let B_i be the restriction of B to the $M_{t,c}(\tau)_i \otimes M_{t,\bar{c}}(\tau^*)_i$. By definition, $\mathrm{Ker} B = \mathrm{Ker}\phi$, which it is a submodule of $M_{t,c}(G,\mathfrak{h},\tau)$. Singular vectors of positive degree in $M_{t,c}(\tau)$ are in $\mathrm{Ker}B$, and so are the submodules generated by them.

Definition 3.1.11. *For τ an irreducible G-representation, define the $H_{t,c}(G,\mathfrak{h})$ representation $L_{t,c}(\tau) = L_{t,c}(G,\mathfrak{h},\tau)$ as the quotient $M_{t,c}(G,\mathfrak{h},\tau)/\mathrm{Ker}(B)$.*

$L_{t,c}(\tau)$ are graded modules. We are going to show that they are irreducible, and define category $\mathcal{O}$ in such a way that they will be the only irreducible modules in it (up to grading shifts). The proof will be different in characteristic zero and characteristic p, as is the behavior of these modules. In characteristic zero and for generic t and c, the module $M_{t,c}(\tau)$ is irreducible and hence equal to $L_{t,c}(\tau)$. In characteristic p, or in characteristic zero and for $t = 0$, this never happens. On the contrary, all $L_{t,c}(\tau)$ are finite-dimensional, and $M_{t,c}(\tau)$ always have a large submodule. Because of that, definitions of category $\mathcal{O}$ differ, and in characteristic p we sometimes prefer using *baby Verma modules* defined in Chapter 6 instead of Verma modules.

3.1.5 Grading

For any graded vector space M, we use the notation M_i for the i-th graded piece, and $M[j]$ for the same vector space with the grading shifted by j, meaning $M[j]_i = M_{i+j}$. It is explained above that every Verma module $M_{t,c}(\tau)$ is a graded module with $M_{t,c}(\tau)_i = S^i\mathfrak{h}^* \otimes \tau$. Then for any j, $M_{t,c}(\tau)[j]$ is also a graded module. However, some gradings shifts are more natural then others.

If characteristic of $\Bbbk$ is different then 2, consider the following element of $H_{t,c}(W,\mathfrak{h})$:

$$\mathbf{h} = \sum_i x_i y_i + \frac{\dim \mathfrak{h}}{2} - \sum_{s\in S} c_s s.$$

For any conjugacy class C of reflections in W, the element $\sum_{s\in C} s$ is central in the group algebra $\Bbbk[W]$. From this it follows that $\mathbf{h}$ acts by constant on every irreducible representation of W. It is also easy to see that $\mathbf{h}$ does not depend on the choice of the dual bases $(x_i)_i$ and $(y_i)_i$, and that it satisfies:

$$[\mathbf{h}, x] = tx, \quad [\mathbf{h}, y] = -ty, \quad [\mathbf{h}, g] = 0$$

for $x \in \mathfrak{h}^*, y \in \mathfrak{h}, g \in W$.

Let $h_c(\tau) \in \Bbbk$ be the constant by which $\mathbf{h}$ acts on τ. Then its action on the i-th graded pieces of Verma module $M_{t,c}(\tau)$ is by a constant $h_c(\tau) + ti$. From this it follows:

- If $t = 0$, then $\mathbf{h}$ acts on $M_{0,c}(\tau)$ and all its submodules and quotients by the constant $h_c(\tau)$.

- In particular, if there is a subspace of singular vectors isomorphic to a W-representation σ in $M_{0,c}(\tau)_i$, then the action of $\mathbf{h}$ on it is by $h_c(\tau)$ (as it is a subrepresentation of $M_{0,c}(\tau)$) and by $h_c(\sigma)$ (as it is a quotient of $M_{0,c}(\sigma)$); hence,
$$h_c(\tau) = h_c(\sigma).$$

- If $t = 1$ and $\operatorname{char}\Bbbk = 0$, then $\mathbf{h}$ diagonalizes on $M_{1,c}(\tau)$ with eigenvalues of the form $h_c(\tau) + \mathbb{Z}_{\geq 0}$. These eigenvalues define a natural $\mathbb{Z}$-grading on $M_{1,c}(\tau)$, which is compatible with the grading on the algebra $H_{1,c}(W, \mathfrak{h})$, and differs from the grading defined in the previous section by a $h_c(\tau)$-shift.

- In particular, for $t = 1$ and $\operatorname{char}\Bbbk = 0$, if there is a subspace of singular vectors isomorphic to a W-representation σ in $M_{1,c}(\tau)_i$, then
$$h_c(\tau) + i = h_c(\sigma).$$

- If $t = 1$ and $\operatorname{char}\Bbbk = p$, then $\mathbf{h}$ diagonalizes on $M_{1,c}(\tau)$, and its eigenvalues define a natural $\mathbb{Z}/p\mathbb{Z}$-grading.

- In particular, for $t = 1$ and $\operatorname{char} \Bbbk = p$, if there is a subspace of singular vectors isomorphic to a W-representation σ in $M_{1,c}(\tau)_i$, then

$$h_c(\tau) + i = h_c(\sigma) \pmod{p}.$$

If $\operatorname{char} \Bbbk = 2$, we can use $\mathbf{h}' = \sum_i x_i y_i - \sum_{s\in S} c_s s$ instead, and analogous statements hold. The reason for the convention to use the $\frac{\dim \mathfrak{h}}{2}$ constant shift in the definition of $\mathbf{h}$ is the following special case.

Assume that $\Bbbk = \mathbb{C}$ and that $\mathfrak{h}$ has an inner product which W respects, so that $W \subseteq O(\mathfrak{h})$. As $\mathbf{h}$ does not depend on the basis, let $(y_i)_i$ be an orthonormal basis of $\mathfrak{h}$ and $(x_i)_i$ the dual basis of $\mathfrak{h}^*$. The following elements of $H_{1,c}(W, \mathfrak{h})$

$$\mathbf{E} = \frac{1}{2}\sum_i x_i^2, \quad \mathbf{F} = -\frac{1}{2}\sum_i y_i^2$$

then satisfy

$$[\mathbf{h}, \mathbf{E}] = 2\mathbf{E}, \quad [\mathbf{h}, \mathbf{F}] = -2\mathbf{F}, \quad [\mathbf{E}, \mathbf{F}] = \mathbf{h},$$

so they form a copy of $\mathfrak{sl}_2(\mathbb{C})$ in $H_{1,c}(W, \mathfrak{h})$. This is useful for finding finite-dimensional representations of $H_{1,c}(W, \mathfrak{h})$: if L is a finite-dimensional representation, graded by the eigenvalues of $\mathbf{h}$, then it is in particular a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. So, $h_c(\tau)$ is a lowest $\mathfrak{sl}_2(\mathbb{C})$ weight and thus a negative integer, and dimensions of graded pieces of L are palindromic: $\dim L_i = \dim L_{-i}$. Integrating this representation to a representation of the group $SL_2(\mathbb{C})$ and using that $\mathfrak{sl}_2(\mathbb{C})$ commutes with W, we get that the graded pieces of L satisfy $L_i \cong L_{-i}$ as W-representations. All this information provides constriants to values of c for which finite-dimensional representations can exist.

3.2 Representations of $H_{1,c}(W,\mathfrak{h})$ over $\Bbbk = \mathbb{C}$

3.2.1 Category $\mathcal{O}$

As explained above, the definitions of category $\mathcal{O}$ are different for characteristic 0 and characteristic p. For the rest of this chapter, we let $\Bbbk = \mathbb{C}$ and $t = 1$.

Definition 3.2.1. *Category $\mathcal{O}$ (also called $\mathcal{O}_{1,c}(W,\mathfrak{h})$ or $\mathcal{O}_{1,c}$) is a category whose objects are $H_{1,c}(W,\mathfrak{h})$-modules which are finitely generated over $S\mathfrak{h}^*$ and locally nilpotent over $S\mathfrak{h}$. It is a full subcategory of $H_{1,c}(W,\mathfrak{h})$-*Mod*, closed under subquotients and extensions.*

It is clear that Verma modules $M_{1,c}(\tau)$ and their quotients $L_{1,c}(\tau)$ are in $\mathcal{O}$. The element $\mathbf{h}$ acts locally finitely on $\mathcal{O}$, and diagonalizes on $M_{1,c}(\tau)$ with eigenvalues $h_c(\tau) + \mathbb{Z}_{\geq 0}$. We will always consider objects from category $\mathcal{O}$ with this natural grading.

3.2.2 Irreducible representations in category $\mathcal{O}$

As announced before, we can describe all irreducible objects in $\mathcal{O}$.

Proposition 3.2.2. *All $L_{1,c}(\tau)$ are irreducible $H_{1,c}(W,\mathfrak{h})$-modules. All irreducible objects in $\mathcal{O}$ are of the form $L_{1,c}(\tau)$ for some irreducible W-representaton τ.*

For completeness, we sketch the proof.

Proof. The grading element $\mathbf{h}$ diagonalizes on all proper submodules of $M_{t,c}(\tau)$, and has eigenvalues which are all $> h_c(\tau)$. As a consequence, the sum $J_{1,c}(\tau)$ of all proper submodules does not contain $\tau \cong M_{1,c}(\tau)_0$, and so it is a proper submodule. The quotient $K_c(\tau) = M_{1,c}(\tau)/J_{1,c}(\tau)$ is irreducible.

The proof that $J_{1,c}(\tau)$ is the kernel of the contravariant form B and that consequently $K_c(\tau) = L_{1,c}(\tau)$ is completely analogous to the proof of the similar claim about Shapovalov form in Lie theory. Namely, the contravariant form was defined by the map

$$\phi\colon M_{t,c}(W,\mathfrak{h},\tau) \to M_{t,\bar{c}}(W,\mathfrak{h}^*,\tau^*)^\dagger,$$

which factors trough

$$K_{\bar{c}}(\tau^*)^\dagger \hookrightarrow M_{t,\bar{c}}(W, \mathfrak{h}^*, \tau^*)^\dagger.$$

As $K_c(\tau)$ is irreducible, there exists an isomorphism $K_c(\tau) \to K_{\bar{c}}(\tau^*)^\dagger$, which maps the lowest graded piece τ of $K_c(\tau)$ to the lowest graded piece τ of $K_{\bar{c}}(\tau^*)^\dagger$ by the identity map. The uniqueness of the map ϕ implies that, up to a constant, it is equal to the composition

$$M_{1,c}(\tau) \twoheadrightarrow K_c(\tau) \tilde{\to} K_{\bar{c}}(\tau^*)^\dagger \hookrightarrow M_{t,\bar{c}}(W, \mathfrak{h}^*, \tau^*)^\dagger.$$

As a consequence,

$$\mathrm{Ker} B = \mathrm{Ker}\phi = J_{1,c}(\tau),$$

and

$$K_c(\tau) = L_{1,c}(\tau)$$

is irreducible.

Now let L be any irreducible $H_{1,c}(W, \mathfrak{h})$-module. It has a lowest $\mathbf{h}$-eigenspace, and let τ be an irreducible W-subrepresentation of it. Then there is a map $M_{1,c}(\tau) \to L$ which is the identity on τ. Its image has to be L, as it is irreducible. So, $L \cong L_{t,c}(\tau)$. □

3.2.3 Characters

We define characters so that they contain all the information about graded pieces of a module, seen as a representation of W.

Let N be an object of $\mathcal{O}$. It decomposes as $N = \oplus_\alpha N_\alpha$, for N_α a generalized eigenspace of $\mathbf{h}$ with eigenvalue α. Let $K(W)$ be the Grothendieck group of finite-dimensional W representations. We define two versions of character. For a formal variable z and $[N_\alpha]$ a representative of N_α in the Grothendieck group $K(W)$, χ is a

$K(W)$-valued formal power series

$$\chi_N(z) = \sum_\alpha [N_\alpha] z^\alpha.$$

We also define ch as a function of $g \in W$ and z,

$$\mathrm{ch}_N(z, g) = \sum_\alpha \mathrm{Tr}|_{N_\alpha}(g) z^\alpha.$$

Over $\Bbbk = \mathbb{C}$, these two characters are equivalent, and we mostly work with ch.

It is easy to see that

$$\mathrm{ch}_{M_{1,c}(\tau)}(z, g) = \sum_{i \in \mathbb{Z}_{\geq 0}} \mathrm{Tr}|_{S^i\mathfrak{h}^* \otimes \tau}(g) z^{h_c(\tau)+i} = \frac{z^{h_c(\tau)} \mathrm{Tr}|_\tau(g)}{\det|_{\mathfrak{h}^*}(1 - zg)}.$$

3.2.4 Grothendieck Group

Let $K(\mathcal{O})$ be the Grothendieck group of the category $\mathcal{O} = \mathcal{O}_{1,c}(W, \mathfrak{h})$. All modules in $\mathcal{O}$ have finite length (this follows from them being finitely generated over $S\mathfrak{h}^*$, and $\mathbf{h}$ acting on them locally finitely). All irreducible modules in $\mathcal{O}$ are of the form $L_{1,c}(\tau)$, so the set $[L_{1,c}(\tau)]$, for all irreducible representations τ of W, is a $\mathbb{Z}$-basis of $K(\mathcal{O})$.

In particular, there exist $\hat{n}_{\tau,\sigma} \in \mathbb{Z}_{\geq 0}$ such that

$$[M_{1,c}(\tau)] = \sum_\sigma \hat{n}_{\tau,\sigma} [L_{1,c}(\sigma)].$$

The grading on all modules is given by the action of $\mathbf{h}$, so these integers satisfy:

- $\hat{n}_{\tau,\tau} = 1$;

- if $\tau \neq \sigma$, and $\hat{n}_{\tau,\sigma} \neq 0$, then $h_c(\sigma) - h_c(\tau) \in \mathbb{Z}_{>0}$.

As a consequence, in the appropriate ordering on the equivalence classes of irreducible representations of W, the matrix $[\hat{n}_{\tau,\sigma}]$ is upper triangular with 1 on the diagonal, and hence invertible. So, there exist $n_{\tau,\sigma} \in \mathbb{Z}$ such that

$$[L_{1,c}(\tau)] = \sum_{\sigma} n_{\tau,\sigma}[M_{1,c}(\sigma)].$$

If the integers $n_{\tau,\sigma}$ are known, then one can easily compute the characters of $L_{1,c}(\tau)$, as

$$\chi_{L_{1,c}(\tau)}(z) = \sum_{\sigma} n_{\tau,\sigma}\chi_{M_{1,c}(\sigma)}(z) = \sum_{\sigma} n_{\tau,\sigma} \sum_{i\geq 0}[S^i\mathfrak{h}^* \otimes \sigma]z^{h_c(\sigma)+i}$$

and

$$\mathrm{ch}_{L_{1,c}(\tau)}(z,g) = \sum_{\sigma} n_{\tau,\sigma}\mathrm{ch}_{M_{1,c}(\sigma)}(z,g) = \sum_{\sigma} n_{\tau,\sigma}\frac{z^{h_c(\sigma)}\mathrm{Tr}|_{\sigma}(g)}{\det|_{\mathfrak{h}^*}(1-zg)}.$$

The numbers $\hat{n}_{\tau,\sigma}$ are easy to compute from $n_{\tau,\sigma}$, so this gives the composition factors of every $M_{1,c}(\tau)$ as well.

Chapters 4 and 5 are dedicated to calculating $n_{\tau,\sigma}$ for all τ, σ, for all $H_{1,c}(W, \mathfrak{h})$, and for W of the type H_3 and G_{12}.

3.2.5 A useful lemma

We will use the following lemma several times.

Lemma 3.2.3. *Let $\sigma \subset \mathfrak{h}^* \otimes \tau \subset M_{1,c}(\tau)$ be an irreducible subrepresentation of W. The elements of $\mathfrak{h}$ act on σ by zero if and only if $h_c(\sigma) - h_c(\tau) = 1$.*

Proof. This lemma can be found in [25] as Lemma 3.5. The proof uses the observation from Lemma 3.1.8, which in this case states that applying the Dunkl operator to elements in the graded piece $S^1\mathfrak{h}^* \otimes \tau$ of $M_{1,c}(\tau)$, $y \otimes v \mapsto D_y(v)$, is a homomorphism $\mathfrak{h} \otimes \mathfrak{h}^* \otimes \tau \to \tau$. By adjointness, this can be thought of as an endomorphism

$$\mathfrak{h}^* \otimes \tau \to \mathfrak{h}^* \otimes \tau,$$

which is then explicitly calculated to be

$$\mathrm{id} - \sum_{s\in S} c_s(1-s)\otimes s.$$

This operator acts on $\sigma \subset \mathfrak{h}^* \otimes \tau$ by a constant $1 + h_c(\tau) - h_c(\sigma)$, proving the lemma. □

3.2.6 Support of a module

We will use the main result from [22].

Just for this subsection, let W be a finite Coxeter group, and assume $s \mapsto c_s$ is a constant function on the set of reflections, and $\tau = \mathrm{triv}$ the trivial representation of W. We are interested in the growth of $L_{1,c}(\mathrm{triv})$.

The paper [19] (as well as the discussion in 3.2.8) shows that the set of $c \in \mathbb{C}$ for which $L_{1,c}(\mathrm{triv}) \neq M_{1,c}(\mathrm{triv})$ is equal to

$$\bigcup_{m\geq 0}\bigcup_{i=1}^{n}\bigcup_{j=1}^{d_j-1}\{m+\frac{j}{d_i}\}.$$

Here $d_i = d_i(W)$ are degrees of basic invariants of W, defined (see Chevalley-Shephard-Todd theorem) to be integers such that the ring of invariants $\mathbb{C}[\mathfrak{h}]^W$ is a polynomial ring generated by homogeneous elements of degrees $d_1, \dots d_n$.

The module $L_{1,c}(\mathrm{triv})$ is a module over the commutative algebra $\mathbb{C}[\mathfrak{h}] \cong S\mathfrak{h}^* \subseteq H_{1,c}(W,\mathfrak{h})$. Its support is defined to be the set of all $a \in \mathfrak{h} = \mathrm{Spec}\mathbb{C}[\mathfrak{h}]$ such that the localization of $L_{1,c}(\mathrm{triv})$ at the maximal ideal a is nonzero, and is equal to the set of common zeroes of the $\mathbb{C}[\mathfrak{h}]$ ideal $J_{1,c}(\mathrm{triv})$.

When c is not of the form $m + \frac{j}{d_i}$, then $L_{1,c}(\mathrm{triv}) = M_{1,c}(\mathrm{triv})$, which is identified with $\mathbb{C}[\mathfrak{h}]$ as a $\mathbb{C}[\mathfrak{h}]$-module, $J_{1,c}(\mathrm{triv}) = 0$, and the support is the whole $\mathfrak{h}$.

When c is of of the form $m + \frac{j}{d_i}$, the support is calculated in [22]. Let $w \mapsto l(w)$ denote the length function on the Coxeter group W, P_W the Poincaré polynomial of

W, defined as

$$P_W(q) = \sum_{w \in W} q^{l(w)} = \prod_i \frac{1-q^{d_i}}{1-q},$$

and $W_a = \mathrm{Stab}_W a$ the stabilizer of $a \in \mathfrak{h}$ in the group W (which is also known to be a Coxeter group). Theorem 3.1. in [22] states:

Theorem 3.2.4. *A point $a \in \mathfrak{h}$ belongs to the support of $L_{1,c}(\mathrm{triv})$ if and only if*

$$\frac{P_W}{P_{W_a}}(e^{2\pi i c}) \neq 0,$$

in other words, if and only if

$$\#\{i | d \text{ divides } d_i(W)\} = \#\{i | d \text{ divides } d_i(W_a)\}.$$

As the support of $L_{1,c}(\mathrm{triv})$ is the subvariety of $\mathfrak{h}$ determined by the ideal $J_{1,c}(\mathrm{triv})$, its dimension is equal to the degree of the pole at $t=1$ of the Hilbert series of $L_{1,c}(\mathrm{triv})$ with respect to the usual grading on $S\mathfrak{h}^*$. This grading and the grading by $\mathbf{h}$ action differ by a constant $h_c(\mathrm{triv})$, so Hilbert series defined using these two gradings differ by factor of $z^{h_c(\mathrm{triv})}$, and have the same order of pole at $t=1$.

On the other hand, this Hilbert series is equal to the character $\mathrm{ch}_{L_{1,c}(\mathrm{triv})}(z,g)$ evaluated at $g = 1 \in W$. This will help us determine the coefficients in the character formulas for $L_{1,c}(\mathrm{triv})$.

In particular, $L_{1,c}(\mathrm{triv})$ is finite-dimensional if and only if there is no pole, meaning if the support is zero dimensional and equal to $\{0\}$.

3.2.7 Parabolic induction and restriction functors

A subgroup W' of a reflection group W is called parabolic if it is of the form $\mathrm{Stab}_W a$ for some $a \in \mathfrak{h}$. In that case, W' is a reflection group, with reflection representation $\mathfrak{h}' = \mathfrak{h}/\mathfrak{h}^{W'}$, where $\mathfrak{h}^{W'}$ is the subspace of W'-invariant elements.

Let c' be the restriction of c to the reflections in W'. Induction and restriction functors, introduced in [9], give a way of relating modules for $H_{1,c}(W,\mathfrak{h})$ and of

$H_{1,c'}(W', \mathfrak{h}/\mathfrak{h}^{W'})$. We omit the details of their construction and give only the properties we will use.

Proposition 3.2.5. *There exist induction and restriction functors*

$$\mathrm{Res}_a \colon \mathcal{O}_{1,c}(W, \mathfrak{h}) \to \mathcal{O}_{1,c'}(W', \mathfrak{h}')$$

$$\mathrm{Ind}_a \colon \mathcal{O}_{1,c'}(W', \mathfrak{h}') \to \mathcal{O}_{1,c}(W, \mathfrak{h})$$

associated to $a \in \mathfrak{h}$ such that $W' = \mathrm{Stab}_W a$. These functors are exact. The following formulas hold for generic c, and on the level of Grothendieck group for every c:

$$\mathrm{Res}_a(M_{1,c}(W, \mathfrak{h}, \tau)) = \bigoplus_{\sigma \in \mathrm{Irred}(W')} \dim(\mathrm{Hom}(\sigma, \tau|_{W'})) M_{1,c'}(W', \mathfrak{h}', \sigma),$$

$$\mathrm{Ind}_a(M_{1,c'}(W', \mathfrak{h}', \sigma)) = \bigoplus_{\tau \in \mathrm{Irred}(W)} \dim(\mathrm{Hom}(\sigma, \tau|_{W'})) M_{1,c}(W, \mathfrak{h}, \tau).$$

3.2.8 Iwahori-Hecke algebras, KZ functor and semisimplicity

Category $\mathcal{O}_{1,c}(W, \mathfrak{h})$ is semisimple for generic values of parameter c. We are interested in describing the irreducible objects $L_{1,c}(\tau)$ in cases when it is not semisimple. The first step is to determine for which values of c this happens.

For this section, assume $c \in \mathbb{C}$ is a constant.

Proposition 3.2.6. *If W is a Coxeter group and c a constant, then $\mathcal{O}_{1,c}(W, \mathfrak{h})$ is semisimple unless $c \in \mathbb{Q}$, and writing $c = \frac{m}{d}$ as a reduced fraction, d divides a degree of a basic invariant of W.*

This statement can be found in [9], and is a combination of results from [30] and [19].

For a reflection group W, let $H_s = \mathrm{Ker}(1-s)|_{\mathfrak{h}}$ be the reflection plane associated to the reflection s, and let $\mathfrak{h}_{reg}$ be the complement of all the reflection planes in $\mathfrak{h}$. The *braid group* BW is then defined as the fundamental group of the quotient space $\pi_1(\mathfrak{h}_{reg}/W)$. It is generated by elements T_H for all reflection hyperplanes H, where T_H is defined as a small circle around the image of $H = H_s$ in the quotient space. It is known that the group W is the quotient of BW by relations of the type $T_H^{m_s} = 1$, where $m_H \in \mathbb{Z}_{>0}$ is the (maximal) order of the reflection s in W defining the hyperplane $H = H_s$. For a constant $q \in \mathbb{C}$, define the Iwahori-Hecke algebra $\mathcal{H}_q(W)$ as the quotient of $\mathbb{C}[BW]$ by the relations

$$(T_H - 1)\prod_{j=1}^{m_H-1}(T_H - e^{2\pi ij/m_H}q).$$

(A version of $\mathcal{H}_q(W)$ with q a collection of conjugation invariant parameters is available, but we will not use it). Clearly, if $q = 1$, the $H_q(W) = \mathbb{C}W$.

We also define the generic Hecke algebra as an algebra with the same generators and relations as above, but over $\mathbb{Z}[q, q^{-1}]$ for q an indeterminate. For the rest of this section, we assume that W is such that the generic Hecke algebra is a free $\mathbb{Z}[q, q^{-1}]$-module of rank $|W|$. This is known for cases when W is a Coxeter group or a simple complex reflection group different than $G_{12,\dots,19}$, $G_{24,\dots,27}$, G_{29} and G_{31-34}, and conjectured in other cases. All the results quoted in this section depend on this assumption. We will apply the results of these section in Chapter 5 to the case $W = G_{12}$, for which the assumption is known to be satisfied.

Example 3.2.7. *The Coxeter group $W = H_3$ has a presentation by generators and relations*

$$\left\langle s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = 1, s_1s_3 = s_3s_1, s_1s_2s_1 = s_2s_1s_2, s_2s_3s_2s_3s_2 = s_3s_2s_3s_2s_3\right\rangle.$$

Its braid group BW is

$$\langle T_1, T_2, T_3 | T_1T_3 = T_3T_1, T_1T_2T_1 = T_2T_1T_2, T_2T_3T_2T_3T_2 = T_3T_2T_3T_2T_3\rangle.$$

The Hecke algebra $\mathcal{H}_q(W)$ is a $\mathbb{C}$-algebra with generators T_1, T_2, T_3 and relations

$$T_1T_3 = T_3T_1, T_1T_2T_1 = T_2T_1T_2, T_2T_3T_2T_3T_2 = T_3T_2T_3T_2T_3$$

$$(T_i - 1)(T_i + q) = 0 \quad i = 1, 2, 3.$$

Example 3.2.8. *The complex reflection group $W = G_{12}$ has a presentation by generators and relations*

$$\left\langle s_e, s_f, s_g | s_e^2 = s_f^2 = s_g^2 = 1, (s_e s_f s_g)^4 = (s_g s_e s_f)^4 = (s_f s_g s_e)^4 \right\rangle,$$

its braid group BW is

$$\left\langle T_e, T_f, T_g \,|\, (T_e T_f T_g)^4 = (T_g T_e T_f)^4 = (T_f T_g T_e)^4 \right\rangle,$$

and its Hecke algebra $\mathcal{H}_q(W)$ is the quotient of the group algebra of the braid group of W by the extra relations

$$(T_e - 1)(T_e + q) = 0, \;\; (T_f - 1)(T_f + q) = 0, \;\; (T_g - 1)(T_g + q) = 0.$$

For M a module in $\mathcal{O}$, let M_{reg} be its localization at $\mathfrak{h}_{reg}$. Let $\mathcal{O}^{tor}$ be the subcategory of $\mathcal{O}$ consisting of those modules for which the localization $M_{reg} = 0$.

Theorem 3.2.9. *There exists a functor*

$$KZ \colon \mathcal{O}_{1,c}(W, \mathfrak{h}) \to \mathcal{H}_q(W) - mod$$

associating to every module in category $\mathcal{O}$ for $H_{1,c}(W, \mathfrak{h})$ a module for the Hecke algebra $\mathcal{H}_q(W)$, where q is a constant depending on c. If the order of the generating reflections of W is 2, then $q = e^{2\pi i c}$.

The functor KZ induces an equivalence of categories

$$\mathrm{KZ} \colon \mathcal{O}_{1,c}/\mathcal{O}_{1,c}^{tor} \to \mathcal{H}_q(W)\text{-}\mathrm{Mod}.$$

The definition of KZ and the proof of this theorem first appeared in [30]. It is useful to us due to the following corollary:

Corollary 3.2.10. *The* $\mathbb{C}$ *algebra* $\mathcal{H}_q(W)$ *is semisimple if and only if* $\mathcal{O}_c$ *is semisimple.*

Proof. $\mathcal{H}_q(W)$ is semisimple if and only if the number of irreducible representations is equal to the number $|\widehat{W}|$ of irreducible representations of W. The number of simple objects in $\mathcal{O}$ is always equal to $|\widehat{W}|$. Thus, if either $\mathcal{O}$ or $\mathcal{H}_q(W)$ is semisimple, then $\mathcal{O}_{1,c}^{tor} = 0$, KZ is an equivalence KZ: $\mathcal{O}_{1,c} \to \mathcal{H}_q(W)$- Mod, and both $\mathcal{O}$ and $\mathcal{H}_q(W)$ are semisimple. □

The following result is a combination of [13], [15] and [29].

Proposition 3.2.11. *The Hecke algebra* $\mathcal{H}_q(W)$ *for* $q = v^2$ *is semisimple if and only if there exists a symmetrizing form on it, which is if and only is all* Schur elements s_τ, *for* $\tau \in \widehat{W}$, *satisfy*

$$s_\tau(v) \neq 0.$$

Schur elements are a family of polynomials, labeled by $\widehat{W}$. They have been calculated, and can be accessed for various groups W via the CHEVIE package of the algebra software GAP, see [38].

We will use the combination of these results:

Corollary 3.2.12. *For constant* $c > 0$, *category* $\mathcal{O}_{1,c}(W, \mathfrak{h})$ *is semisimple if and only if for all* $\tau \in \widehat{W}$, *for* $v = e^{\pi i c}$,

$$s_\tau(v) \neq 0.$$

3.2.9 Scaling functors

For the set of constant c such that $\mathcal{O}_{1,c}$ is not semisimple, there are some equivalences of categories that enable us to consider a subset of all such c. The following are results from [41] (see Theorem 5.5.) and [32] (sections 2.6. and 2.16).

Theorem 3.2.13. *Assume that* g *is an element of the Galois group* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, *and* $c \in \mathbb{C}$, $c \notin \frac{1}{2} + \mathbb{Z}$, $r \in \mathbb{R}_{>0}$ *such that*

$$g(e^{2\pi i c}) = e^{2\pi i r c}.$$

Then there exists an equivalence of categories

$$\Phi_{c,rc} \colon \mathcal{O}_{1,c} \to \mathcal{O}_{1,rc}$$

and a permutation $\varphi_{c,rc}$ *of* $\widehat{W}$ *such that*

$$\Phi_{c,rc}(M_{1,c}(\tau)) = M_{1,rc}(\varphi_{c,rc}(\tau))$$

$$\Phi_{c,rc}(L_{1,c}(\tau)) = L_{1,rc}(\varphi_{c,rc}(\tau)).$$

We use this to transfer information about characters of irreducible representations in $\mathcal{O}_{1,1/d}$ to $\mathcal{O}_{1,r/d}$, for $r \in \mathbb{Z}_{>0}$ relatively prime to d.

An effective way of calculating the permutation $\varphi_{c,rc}$ is given in [32] by formula (11). Let $c = 1/d$, $d, r \in \mathbb{Z}_{>0}$, and let $g \in \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/2d})/\mathbb{Q})$ such that

$$g(e^{2\pi i/d}) = e^{2\pi i r/d}.$$

Let

$$\eta = e^{-2\pi i r/2d} \cdot g(e^{2\pi i/2d}) \in \mathbb{C}.$$

Then the irreducible characters χ' of the Hecke algebra $\mathcal{H}_q(W)$ are related by

$$\chi'_{\varphi_{1/d,r/d}(\tau)}(w)(q) = (g\chi'_\tau(w))(\eta q).$$

After evaluating at $q = 1$ to get irreducible characters of W, this becomes:

- If $\eta = 1$, then $\chi'_{\varphi_{1/d,r/d}(\tau)} = g\chi'_\tau$, so $\varphi_{1/d,r/d}$ is a permutation given by the action of g on the characters of W.
- If $\eta = -1$, then the character formula becomes $\chi'_{\varphi_{1/d,r/d}(\tau)}(w)(1) = g\chi'_\tau(w)(-1)$. The left hand side can be interpreted as a group character, while the right hand side needs to be transformed. The permutation $\varphi_{1/d,r/d}$ is the composition of the permutation given by the action of g on the characters of W and the permutation resulting from the transformation, which can be read off from [39].

While there might be several choices for g, the permutation φ doesn't depend on them.

3.2.10 Shift functors

Scaling functors described above are known to be equivalences of categories for $d \neq 2$ and conjectured to be equivalences for $d = 2$. In absence of a proof, we use shift functors for equivalences between various parameters of the form $c = r/2$. For references on shift functors, see [8].

Let W be a reflection group generated by reflections of order two. We call $c \in \mathbb{C}$ *spherical* if every module in $\mathcal{O}_{1,c}$, seen as a W-representation, has a nontrivial W-invariant. Let $\mathcal{O}^+_{1,c}$ be the subcategory of modules in $\mathcal{O}_{1,c}$ that contain no notrivial invariants (W-subrepresentations isomorphic to the trivial representation), and $\mathcal{O}^-_{1,c}$ the subcategory of modules in $\mathcal{O}_{1,c}$ that contain no anti-invariants (W-subrepresentations isomorphic to the signum representation).

Theorem 3.2.14. *For $c = r/2$, $r \in \mathbb{Z}_{\geq 0}$, there exists an equivalence of categories*

$$\Phi_{c,c+1}\colon \mathcal{O}_{1,c}/\mathcal{O}^+_{1,c} \to \mathcal{O}_{1,c+1}/\mathcal{O}^-_{1,c+1}.$$

If c is spherical, then $\Phi_{c,c+1}$ an equivalence of categories between $\mathcal{O}_{1,c}$ and $\mathcal{O}_{1,c+1}$. In that case $c+1$ is spherical as well, and

$$\Phi_{c,c+1}(M_{1,c}(\tau)) = M_{1,c+1}(\varphi_{c,c+1}(\tau))$$

$$\Phi_{c,c+1}(L_{1,c}(\tau)) = L_{1,c+1}(\varphi_{c,c+1}(\tau)).$$

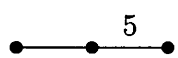
As a consequence, if $c = r/2$ is positive and spherical, then category $\mathcal{O}_{1,r/2}$ is equivalent to all $\mathcal{O}_{1,m+r/2}$ for $m \in \mathbb{Z}_{\geq 0}$, and it is possible to calculate character formulas in $\mathcal{O}_{1,m+r/2}$ using the permutation $\varphi_{c,c+1}$ and character formulas in $\mathcal{O}_{1,r/2}$.

Chapter 4

Representations of Rational Cherednik Algebras Associated to the Coxeter Group H_3

4.1 The group H_3

We will study the rational Cherednik algebra associated to $W = H_3$, the exceptional Coxeter group with the Coxeter graph



Its presentation with generators and relations is

$$\left\langle s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = 1, s_1s_3 = s_3s_1, s_1s_2s_1 = s_2s_1s_2, s_2s_3s_2s_3s_2 = s_3s_2s_3s_2s_3 \right\rangle.$$

It is the group of symmetries of the regular icosahedron, with Coxeter generators s_1, s_2, s_3 corresponding to the reflections along the planes with respective angles of $\pi/2$, $\pi/3$ and $\pi/5$ with each other. The space $\mathfrak{h}$ is the complexification of the 3 dimensional real space which realizes H_3 as such symmetry group. H_3 preserves the standard inner product product, so $H_3 \subset O(\mathfrak{h})$, and this inner product induces an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$.

The group H_3 has 120 elements and is isomorphic to $\mathbb{Z}_2 \times A_5$. Here $\mathbb{Z}_2$ is a cyclic group of order 2 containing the identity and the central symmetry of the icosahedron, and A_5 is a group of even permutations of the set of 5 elements, in this case the 5 tetrahedra that are formed by centers of the faces of the icosahedron. We will write elements of H_3 as elements of $\mathbb{Z}_2 \times A_5$, using notation $\pm$ for elements of $\mathbb{Z}_2$ and cyclic notation for elements of $A_5 \subset S_5$. One instance of an isomorphism $H_3 \to \mathbb{Z}_2 \times A_5$ is

$$s_1 \mapsto -(12)(34), \quad s_2 \mapsto -(15)(34), \quad s_3 \mapsto -(13)(24).$$

Let us write down the character table of H_3 in this realization. The group $\mathbb{Z}_2$ has two one-dimensional irreducible representations, the trivial one and the signum one. The group A_5 has five irreducible representations: the trivial one, that we will call $\mathbf{1}$; a three dimensional one called $\mathbf{3}$, that realizes it as rotations of an icosahedron; another three dimensional one, called $\widetilde{\mathbf{3}}$, obtained from $\mathbf{3}$ by twisting by conjugation with the element $(12) \in S_5$ (S_5 is the symmetric group, and conjugating by (12) in S_5 preserves $A_5 \subset S_5$); a four dimensional representation $\mathbf{4}$ (the permutation representation of A_5 obtained from it acting on the 5 tetrahedra is reducible; it has a 1 dimensional trivial subrepresentation and $\mathbf{4}$ as irreducible components); and a 5 dimensional $\mathbf{5}$, that is an irreducible subrepresentation of a 6 dimensional representation arising from the fact that A_5 permutes the 6 great diagonals of the icosahedron.

Every irreducible representation of $H_3 \cong \mathbb{Z}_2 \times A_5$ is a tensor product of an irreducible representation of $\mathbb{Z}_2$ and an irreducible representation of A_5. Let us denote the tensor product of a representation triv of $\mathbb{Z}_2$ and a representation τ of A_5 bt τ_+, and the tensor product of the signum representation of $\mathbb{Z}_2$ and a representation τ of A_5 bt τ_-. In this notation, $\mathfrak{h} \cong \mathfrak{h}^* \cong \mathbf{3}_-$.

The character table of H_3, is Table 4.1. For references about H_3 and its representations see [34], [28].

There is one conjugacy class of reflections in H_3, with a representative $-(12)(34)$ and 15 reflections in it. The conjugation invariant function $c\colon S \to \mathbb{C}$ is a complex constant.

	Id	-Id	(123)	-(123)	(12)(34)	-(12)(34)	(12345)	-(12345)	(13245)	-(13245)
#	1	1	20	20	15	15	12	12	12	12
$\mathbf{1}_+$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_-$	1	-1	1	-1	1	-1	1	-1	1	-1
$\mathbf{3}_+$	3	3	0	0	-1	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$\mathbf{3}_-$	3	-3	0	0	-1	1	$\frac{1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
$\widetilde{\mathbf{3}}_+$	3	3	0	0	-1	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$\widetilde{\mathbf{3}}_-$	3	-3	0	0	-1	1	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
$\mathbf{4}_+$	4	4	1	1	0	0	-1	-1	-1	-1
$\mathbf{4}_-$	4	-4	1	-1	0	0	-1	1	-1	1
$\mathbf{5}_+$	5	5	-1	-1	1	1	0	0	0	0
$\mathbf{5}_-$	5	-5	-1	1	1	-1	0	0	0	0

Table 4.1: The character table for $H_3 \cong \mathbb{Z}_2 \times A_5$

We can also easily calculate the action of the central element $\sum_{s\in S} s$ on any representation. For example, in $\mathbf{5}_-$, it is a constant on a 5 dimensional space, whose trace is $\mathrm{tr} = \sum_{s\in S} \mathrm{tr} s = -15$, so it is $-15/5 = 3$. Doing this calculation for every irreducible representation τ, we get Table 4.2.

$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\widetilde{\mathbf{3}}_+$	$\widetilde{\mathbf{3}}_-$	$\mathbf{4}_+$	$\mathbf{4}_-$	$\mathbf{5}_+$	$\mathbf{5}_-$
15	-15	-5	5	-5	5	0	0	3	-3

Table 4.2: The action of the central element $\sum_{s\in S} s \in H_3$ on all τ

Table 4.2 now enables us to calculate the action of $\mathbf{h}$ on any lowest weight τ, as $h_c(\tau) = \frac{3}{2} - c\sum_{s\in S} s|_\tau$.

4.2 Main theorem

Theorem 4.2.1. *For the Coxeter group H_3, its reflection representation $\mathfrak{h}$, c any complex number, and τ an irreducible representation of H_3, the expression in the Grothendieck group $K(\mathcal{O}_{1,c})$ for the irreducible module $L_{1,c}(\tau)$ in terms of standard modules $M_{1,c}(\tau)$ is as below. Any module $L_{1,c}(\tau)$ for which we do not explicitly write its dimension is infinite-dimensional. We leave out the index $(1,c)$ in $L_{1,c}(\tau)$ and $M_{1,c}(\tau)$ whenever it is clear from the context, and write $L(\tau)$ instead of $[L_{1,c}(\tau)]$. Here $r \in \mathbb{N}, d \in \{2,3,5,6,10\}$, and all fractions r/d are reduced.*

- *If c is not of the form $c = r/d$ or $c = -r/d$, then for all τ,*

$$L_{1,c}(\tau) = M_{1,c}(\tau).$$

- *If $c = -r/d$, then the formulas for $L_{1,c}(\tau)$ in terms of $M_{1,c}(\sigma)$ follow from formulas for $L_{1,-c}(\mathbf{1}_- \otimes \tau)$ in terms of $M_{1,c}(\mathbf{1}_- \otimes \sigma)$, which are given below. More precisely, if*

$$[L_{1,r/d}(\tau)] = \sum_\sigma n_{\tau,\sigma}[M_{1,r/d}(\sigma)]$$

then

$$[L_{1,-r/d}(\mathbf{1}_- \otimes \tau)] = \sum_\sigma n_{\tau,\sigma}[M_{1,-r/d}(\mathbf{1}_- \otimes \sigma)].$$

Consequently, $L_{1,c}(\tau)$ is finite-dimensional if and only if $L_{1,-c}(\mathbf{1}_- \otimes \tau)$ is.

- $\mathbf{c = r/10, r \neq 3, 7 \,(\mathrm{mod}\, 10)}$

$$\begin{aligned}
L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{3}_-) + M(\mathbf{3}_+) - M(\mathbf{1}_-) \\
L(\mathbf{3}_+) &= M(\mathbf{3}_+) - M(\mathbf{1}_-) \\
L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{3}_+) + M(\mathbf{1}_-)
\end{aligned}$$

Every $L_{1,r/10}(\mathbf{1}_+)$ is finite-dimensional, with $\dim L_{1,r/10}(\mathbf{1}_+) = r^3$ and

$$\mathrm{ch}_{L_{1,r/10}(\mathbf{1}_+)}(z, g) = \frac{\det_{\mathfrak{h}^*}(1 - z^r g)}{\det_{\mathfrak{h}^*}(1 - zg)}.$$

- $\mathbf{c = r/10, r = 3, 7 \,(\mathrm{mod}\, 10)}$

$$\begin{aligned}
L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\tilde{\mathbf{3}}_-) + M(\tilde{\mathbf{3}}_+) - M(\mathbf{1}_-) \\
L(\tilde{\mathbf{3}}_+) &= M(\tilde{\mathbf{3}}_+) - M(\mathbf{1}_-) \\
L(\tilde{\mathbf{3}}_-) &= M(\tilde{\mathbf{3}}_-) - M(\tilde{\mathbf{3}}_+) + M(\mathbf{1}_-)
\end{aligned}$$

Every $L_{1,r/10}(\mathbf{1}_+)$ is finite-dimensional, with $\dim L_{1,r/10}(\mathbf{1}_+) = r^3$.

- $\mathbf{c = r/6}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\mathbf{1}_-) \\ L(\mathbf{5}_+) &= M(\mathbf{5}_+) - M(\mathbf{5}_-) + M(\mathbf{1}_-) \\ L(\mathbf{5}_-) &= M(\mathbf{5}_-) - M(\mathbf{1}_-) \end{aligned}$$

Every $L_{1,r/6}(\mathbf{1}_+)$ *is finite-dimensional, with* $\dim L_{1,r/6}(\mathbf{1}_+) = 5r^3$ *and*

$$\mathrm{ch}_{L_{1,r/6}(\mathbf{1}_+)}(z,g) = \frac{\det_{\mathfrak{h}^*}(1-z^r g)}{\det_{\mathfrak{h}^*}(1-zg)} \cdot \big(\mathrm{Tr}_{\mathbf{1}_+}(g) z^{-r} + \mathrm{Tr}_{\mathbf{3}_-}(g) + \mathrm{Tr}_{\mathbf{1}_+}(g) z^r\big).$$

- $\mathbf{c = r/5, r = 1, 9} \pmod{\mathbf{10}}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{4}_-) + M(\widetilde{\mathbf{3}}_+) \\ L(\widetilde{\mathbf{3}}_-) &= M(\widetilde{\mathbf{3}}_-) - M(\mathbf{4}_+) + M(\mathbf{1}_-) \\ L(\mathbf{4}_+) &= M(\mathbf{4}_+) - M(\mathbf{1}_-) \\ L(\mathbf{4}_-) &= M(\mathbf{4}_-) - M(\widetilde{\mathbf{3}}_+) \end{aligned}$$

- $\mathbf{c = r/5, r = 2, 8} \pmod{\mathbf{10}}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{4}_+) + M(\mathbf{3}_+) \\ L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{4}_-) + M(\mathbf{1}_-) \\ L(\mathbf{4}_-) &= M(\mathbf{4}_-) - M(\mathbf{1}_-) \\ L(\mathbf{4}_+) &= M(\mathbf{4}_+) - M(\mathbf{3}_+) \end{aligned}$$

- $\mathbf{c = r/5, r = 3, 7\,(\text{mod}\,10)}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{4}_-) + M(\mathbf{3}_+) \\ L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{4}_+) + M(\mathbf{1}_-) \\ L(\mathbf{4}_+) &= M(\mathbf{4}_+) - M(\mathbf{1}_-) \\ L(\mathbf{4}_-) &= M(\mathbf{4}_-) - M(\mathbf{3}_+) \end{aligned}$$

- $\mathbf{c = r/5, r = 4, 6\,(\text{mod}\,10)}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{4}_+) + M(\widetilde{\mathbf{3}}_+) \\ L(\widetilde{\mathbf{3}}_-) &= M(\widetilde{\mathbf{3}}_-) - M(\mathbf{4}_-) + M(\mathbf{1}_-) \\ L(\mathbf{4}_-) &= M(\mathbf{4}_-) - M(\mathbf{1}_-) \\ L(\mathbf{4}_+) &= M(\mathbf{4}_+) - M(\widetilde{\mathbf{3}}_+) \end{aligned}$$

- $\mathbf{c = r/3, r}$ ***odd***

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{5}_+) + M(\mathbf{4}_-) \\ L(\mathbf{4}_+) &= M(\mathbf{4}_+) - M(\mathbf{5}_-) + M(\mathbf{1}_-) \\ L(\mathbf{5}_-) &= M(\mathbf{5}_-) - M(\mathbf{1}_-) \\ L(\mathbf{5}_+) &= M(\mathbf{5}_+) - M(\mathbf{4}_-) \end{aligned}$$

- $\mathbf{c = r/3, r}$ ***even***

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{5}_+) + M(\mathbf{4}_+) \\ L(\mathbf{4}_-) &= M(\mathbf{4}_-) - M(\mathbf{5}_-) + M(\mathbf{1}_-) \\ L(\mathbf{5}_-) &= M(\mathbf{5}_-) - M(\mathbf{1}_-) \\ L(\mathbf{5}_+) &= M(\mathbf{5}_+) - M(\mathbf{4}_+) \end{aligned}$$

- $\mathbf{c = r/2}$

$$\begin{aligned}
L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{3}_-) - M(\widetilde{\mathbf{3}}_-) + M(\mathbf{5}_+) - M(\mathbf{5}_-) + \\
&\quad + M(\mathbf{3}_+) + M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-) \\
L(\mathbf{3}_+) &= M(\mathbf{3}_+) - M(\mathbf{1}_-) \\
L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\mathbf{3}_+) \\
L(\widetilde{\mathbf{3}}_+) &= M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-) \\
L(\widetilde{\mathbf{3}}_-) &= M(\widetilde{\mathbf{3}}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\widetilde{\mathbf{3}}_+) \\
L(\mathbf{5}_+) &= M(\mathbf{5}_+) - 2 \cdot M(\mathbf{5}_-) + M(\mathbf{3}_+) + M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-). \\
L(\mathbf{5}_-) &= M(\mathbf{5}_-) - M(\mathbf{3}_+) - M(\widetilde{\mathbf{3}}_+) + M(\mathbf{1}_-)
\end{aligned}$$

For every r, three of these modules are finite-dimensional, with $\dim L_{1,r/2}(\mathbf{1}_+) = 115r^3$, $\dim L_{1,r/2}(\mathbf{3}_-) = 10r^3$, *and* $\dim L_{1,r/2}(\widetilde{\mathbf{3}}_-) = 10r^3$, *and*

$$\mathrm{ch}_{L_{1,r/2}(\mathbf{3}_-)}(z,g) = \frac{\det_{\mathfrak{h}^*}(1-z^r g)}{\det_{\mathfrak{h}^*}(1-zg)} \cdot \left(\mathrm{Tr}_{\mathbf{3}_-}(g)z^{-r} + \mathrm{Tr}_{\mathbf{1}_+}(g) + \mathrm{Tr}_{\mathbf{3}_+}(g) + \mathrm{Tr}_{\mathbf{3}_-}(g)z^r\right),$$

$$\mathrm{ch}_{L_{1,r/2}(\widetilde{\mathbf{3}}_-)}(z,g) = \frac{\det_{\mathfrak{h}^*}(1-z^r g)}{\det_{\mathfrak{h}^*}(1-zg)} \cdot \left(\mathrm{Tr}_{\widetilde{\mathbf{3}}_-}(g)z^{-r} + \mathrm{Tr}_{\mathbf{4}_+}(g) + \mathrm{Tr}_{\widetilde{\mathbf{3}}_-}(g)z^r\right).$$

Proof. By Proposition 3.2.6, the only values of c for which $\mathcal{O}_{1,c}$ is not semisimple are of the form $c = \pm r/d$, with $r, d \in \mathbb{Z}_{>0}$, and d dividing the order of a basic invariant of H_3. Basic invariants of H_3 have degrees $2, 6, 10$, so $d \in \{2, 3, 5, 6, 10\}$.

By Lemma 3.1.5 applied to the group character $f = \mathbf{1}_-$ which sends all simple reflections to -1, the algebras $H_{1,c}(H_3, \mathfrak{h})$ and $H_{1,-c}(H_3, \mathfrak{h})$ are isomorphic. Twisting by this isomorphism makes an irreducible representation $L_{1,-c}(\tau)$ into an irreducible representation of $H_{1,c}(H_3, \mathfrak{h})$ with the lowest weight $\mathbf{1}_- \otimes \tau$, and similarly for Verma modules. Using this, it is enough to calculate Grothendieck group expressions for $c > 0$ and they can be transformed into those for $c < 0$ as is described in the statement of the theorem.

We calculate the Grothendieck group expressions for $c = 1/d, d \in \{10, 6, 5, 3\}$ and for $c = 1/2, 3/2$ directly in Theorems 4.4.1, 4.5.1, 4.6.1, 4.7.1, 4.8.1 and 4.9.1. The

Grothendieck group expressions for all other $c = r/d > 0$ follow from equivalences of categories $\mathcal{O}_{1,1/d} \to \mathcal{O}_{1,r/d}$ described in section 3.2.9 and $\mathcal{O}_{1,c} \to \mathcal{O}_{1,c+1}$ described in section 3.2.10. To use the latter one, we check in Lemma 4.9.2 that $c = 3/2$ is indeed aspherical for H_3. To use both, we explicitly calculate the permutation $\varphi_{c,c'}$ described in section 3.2.9 in Lemma 4.3.1. □

As the characters of Verma modules $M_{1,c}(\sigma)$ are known (see section 6.2), the characters of all irreducible modules $M_{1,c}(\tau)$ can be calculated from the above theorem. For completeness, we also calculate the Grothendieck group expressions of $M_{1,c}(\tau)$ in terms of $L_{1,c}(\tau)$ in the next corollary; in other words, calculate the composition series of standard modules.

Corollary 4.2.2. *For the Coxeter group H_3, its reflection representation $\mathfrak{h}$, c any complex number, and τ an irreducible representation of H_3, the expression in the Grothendieck group $K(\mathcal{O}_{1,c})$ for the standard module $M_{1,c}(\tau)$ in terms of irreducible modules $L_{1,c}(\tau)$ is as below. We leave out the index c in $L_{1,c}(\tau)$ and $M_{1,c}(\tau)$ whenever it is clear from the context. Here $r \in \mathbb{N}, d \in \{2,3,5,6,10\}$, and all fractions r/d are reduced.*

- *If c is not of the form $c = r/d$ or $c = -r/d$, then for all τ,*

$$M_{1,c}(\tau) = L_{1,c}(\tau).$$

- *If $c = -r/d$, then the formulas for $M_{1,c}(\tau)$ in terms of $L_{1,c}(\sigma)$ follow from formulas for $M_{1,-c}(\mathbf{1}_- \otimes \tau)$ in terms of $L_{1,-c}(\mathbf{1}_- \otimes \sigma)$, which are given below. More precisely, if*

$$M_{1,r/d}(\tau) = \sum_{\sigma} \hat{n}_{\tau,\sigma} L_{1,r/d}(\sigma)$$

then

$$M_{1,-r/d}(\mathbf{1}_- \otimes \tau) = \sum_{\sigma} \hat{n}_{\tau,\sigma} L_{1,-r/d}(\mathbf{1}_- \otimes \sigma).$$

- $\mathbf{c = r/10, r \neq 3, 7 \,(\mathrm{mod}\, 10)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{3}_-) \\ M(\mathbf{3}_+) &= L(\mathbf{3}_+) + L(\mathbf{1}_-) \\ M(\mathbf{3}_-) &= L(\mathbf{3}_-) + L(\mathbf{3}_+) \end{aligned}$$

- $\mathbf{c = r/10, r = 3, 7 \,(\mathrm{mod}\, 10)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\tilde{\mathbf{3}}_-) \\ M(\tilde{\mathbf{3}}_+) &= L(\tilde{\mathbf{3}}_+) + L(\mathbf{1}_-) \\ M(\tilde{\mathbf{3}}_-) &= L(\tilde{\mathbf{3}}_-) + L(\tilde{\mathbf{3}}_+) \end{aligned}$$

- $\mathbf{c = r/6}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{5}_+) \\ M(\mathbf{5}_+) &= L(\mathbf{5}_+) + L(\mathbf{5}_-) \\ M(\mathbf{5}_-) &= L(\mathbf{5}_-) + L(\mathbf{1}_-) \end{aligned}$$

- $\mathbf{c = r/5, r = 1, 9 \,(\mathrm{mod}\, 10)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{4}_-) \\ M(\widetilde{\mathbf{3}}_-) &= L(\widetilde{\mathbf{3}}_-) + L(\mathbf{4}_+) \\ M(\mathbf{4}_+) &= L(\mathbf{4}_+) + L(\mathbf{1}_-) \\ M(\mathbf{4}_-) &= L(\mathbf{4}_-) + L(\widetilde{\mathbf{3}}_+) \end{aligned}$$

- $\mathbf{c = r/5, r = 2, 8 \,(\mathrm{mod}\, 10)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{4}_+) \\ M(\mathbf{3}_-) &= L(\mathbf{3}_-) + L(\mathbf{4}_-) \\ M(\mathbf{4}_-) &= L(\mathbf{4}_-) + L(\mathbf{1}_-) \\ M(\mathbf{4}_+) &= L(\mathbf{4}_+) + L(\mathbf{3}_+) \end{aligned}$$

- $\mathbf{c = r/5, r = 3, 7 \,(\mathrm{mod}\, 10)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{4}_-) \\ M(\mathbf{3}_-) &= L(\mathbf{3}_-) + L(\mathbf{4}_+) \\ M(\mathbf{4}_+) &= L(\mathbf{4}_+) + L(\mathbf{1}_-) \\ M(\mathbf{4}_-) &= L(\mathbf{4}_-) + L(\mathbf{3}_+) \end{aligned}$$

- $\mathbf{c = r/5, r = 4, 6 \,(\mathrm{mod}\, 10)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{4}_+) \\ M(\widetilde{\mathbf{3}}_-) &= L(\widetilde{\mathbf{3}}_-) + L(\mathbf{4}_-) \\ M(\mathbf{4}_-) &= L(\mathbf{4}_-) + L(\mathbf{1}_-) \\ M(\mathbf{4}_+) &= L(\mathbf{4}_+) + L(\widetilde{\mathbf{3}}_+) \end{aligned}$$

- $\mathbf{c = r/3, r}$ ***odd***

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{5}_+) \\ M(\mathbf{4}_+) &= L(\mathbf{4}_+) + L(\mathbf{5}_-) \\ M(\mathbf{5}_-) &= L(\mathbf{5}_-) + L(\mathbf{1}_-) \\ M(\mathbf{5}_+) &= L(\mathbf{5}_+) + L(\mathbf{4}_-) \end{aligned}$$

- $\mathbf{c = r/3, r}$ ***even***

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{5}_+) \\ M(\mathbf{4}_-) &= L(\mathbf{4}_-) + L(\mathbf{5}_-) \\ M(\mathbf{5}_-) &= L(\mathbf{5}_-) + L(\mathbf{1}_-) \\ M(\mathbf{5}_+) &= L(\mathbf{5}_+) + L(\mathbf{4}_+) \end{aligned}$$

- $\mathbf{c = r/2}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{3}_-) + L(\widetilde{\mathbf{3}}_-) + L(\mathbf{5}_+) + L(\mathbf{5}_-) + L(\mathbf{1}_-) \\ M(\mathbf{3}_+) &= L(\mathbf{3}_+) + L(\mathbf{1}_-) \\ M(\mathbf{3}_-) &= L(\mathbf{3}_-) + L(\mathbf{5}_+) + L(\mathbf{5}_-) + L(\mathbf{3}_+) + L(\mathbf{1}_-) \\ M(\widetilde{\mathbf{3}}_+) &= L(\widetilde{\mathbf{3}}_+) + L(\mathbf{1}_-) \\ M(\widetilde{\mathbf{3}}_-) &= L(\widetilde{\mathbf{3}}_-) + L(\mathbf{5}_+) + L(\mathbf{5}_-) + L(\widetilde{\mathbf{3}}_+) + L(\mathbf{1}_-) \\ M(\mathbf{5}_+) &= L(\mathbf{5}_+) + 2 \cdot L(\mathbf{5}_-) + L(\mathbf{3}_+) + L(\widetilde{\mathbf{3}}_+) + L(\mathbf{1}_-). \\ M(\mathbf{5}_-) &= L(\mathbf{5}_-) + L(\mathbf{3}_+) + L(\widetilde{\mathbf{3}}_+) + L(\mathbf{1}_-) \end{aligned}$$

4.3 Two auxiliary lemmas

Lemma 4.3.1. *The permutation $\varphi = \varphi_{1/d,r/d}$, for $d = 2,3,5,6,10$ and $r \in \mathbb{Z}_{>0}$ relatively prime to d, is given by:*

- $d = 2$, $r = 1,3 \pmod 4$, $\varphi = \mathrm{id}$
- $d = 3$, $r = 1,5 \pmod 6$, $\varphi = \mathrm{id}$
- $d = 3$, $r = 2,4 \pmod 6$, $\varphi = (\mathbf{4}_+, \mathbf{4}_-)$
- $d = 5$, $r = 1,9 \pmod{10}$, $\varphi = \mathrm{id}$
- $d = 5$, $r = 2,8 \pmod{10}$, $\varphi = (\mathbf{3}_-, \widetilde{\mathbf{3}}_-)(\mathbf{3}_+, \widetilde{\mathbf{3}}_+)(\mathbf{4}_+, \mathbf{4}_-)$
- $d = 5$, $r = 3,7 \pmod{10}$, $\varphi = (\mathbf{3}_-, \widetilde{\mathbf{3}}_-)(\mathbf{3}_+, \widetilde{\mathbf{3}}_+)$

- $d = 5$, $r = 4, 6 \pmod{12}$, $\varphi = (\mathbf{4}_+, \mathbf{4}_-)$
- $d = 6$, $r = 1, 11 \pmod{12}$, $\varphi = \mathrm{id}$
- $d = 6$, $r = 5, 7 \pmod{12}$, $\varphi = (\mathbf{4}_+, \mathbf{4}_-)$
- $d = 10$, $r = 1, 9 \pmod{20}$, $\varphi = \mathrm{id}$
- $d = 10$, $r = 3, 7 \pmod{20}$, $\varphi = (\mathbf{3}_-, \widetilde{\mathbf{3}}_-)(\mathbf{3}_+, \widetilde{\mathbf{3}}_+)(\mathbf{4}_+, \mathbf{4}_-)$
- $d = 10$, $r = 11, 19 \pmod{20}$, $\varphi = (\mathbf{4}_+, \mathbf{4}_-)$
- $d = 10$, $r = 13, 17 \pmod{20}$, $\varphi = (\mathbf{3}_-, \widetilde{\mathbf{3}}_-)(\mathbf{3}_+, \widetilde{\mathbf{3}}_+)$

Proof. Proof is a direct computation, following the algorithm explained in section 4.3.1. The algorithm consisted of setting $\xi = e^{\pi i/d}$, and finding an element $g \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ such that $g(\xi^2) = \xi^{2r}$. The permutation $\varphi_{1/d,r/d}$ was then the composition of the permutation realizing the action of g on characters of irreducible representations, and of an additional permutation in case $\eta = g(\xi)/\xi^r$ is equal to -1. In case $W = H_3$, [39] calculates that this additional permutation is the transposition $(\mathbf{4}_+, \mathbf{4}_-)$.

From this it is clear that the permutations associated to r and to $r + 2d$ are the same. When calculating the permutation associated to $r + d$, we can use the same Galois element g as for r, and η will differ by a factor of -1. So, the permutation for $r + d$ is a composition of the permutation for r and the transposition $(\mathbf{4}_+, \mathbf{4}_-)$.

Let us list g and η for $d = 2, 3, 5, 6, 10$ and $1 \leq r < d$ relatively prime to d.

- If $r = 1$, then $g = 1$, $\eta = 1$ and $\varphi = \mathrm{id}$.
- $d = 3, r = 2$, $g =$ complex conjugation, $\eta = -1$, $\varphi = (\mathbf{4}_+, \mathbf{4}_-)$
- $d = 6, r = 5$, $g =$ complex conjugation, $\eta = -1$, $\varphi = (\mathbf{4}_+, \mathbf{4}_-)$
- $d = 5, r = 2$, $g(\xi) = \xi^7$, $g(\sqrt{5}) = -\sqrt{5}$, $\eta = -1$, $\varphi = (\mathbf{3}_-, \widetilde{\mathbf{3}}_-)(\mathbf{3}_+, \widetilde{\mathbf{3}}_+)(\mathbf{4}_+, \mathbf{4}_-)$
- $d = 5, r = 3$, $g(\xi) = \xi^3$, $g(\sqrt{5}) = -\sqrt{5}$, $\eta = 1$, $\varphi = (\mathbf{3}_-, \widetilde{\mathbf{3}}_-)(\mathbf{3}_+, \widetilde{\mathbf{3}}_+)$

- $d = 5, r = 4$, $g(\xi) = \xi^9$, $g(\sqrt{5}) = \sqrt{5}$, $\eta = -1$, $\varphi = (\mathbf{4}_+, \mathbf{4}_-)$
- $d = 10, r = 3$, $g(\xi) = \xi^{13}$, $g(\sqrt{5}) = -\sqrt{5}$, $\eta = -1$, $\varphi = (\mathbf{3}_-, \widetilde{\mathbf{3}}_-)(\mathbf{3}_+, \widetilde{\mathbf{3}}_+)(\mathbf{4}_+, \mathbf{4}_-)$
- $d = 10, r = 7$, $g(\xi) = \xi^{17}$, $g(\sqrt{5}) = -\sqrt{5}$, $\eta = -1$, $\varphi = (\mathbf{3}_-, \widetilde{\mathbf{3}}_-)(\mathbf{3}_+, \widetilde{\mathbf{3}}_+)(\mathbf{4}_+, \mathbf{4}_-)$
- $d = 10, r = 9$, $g(\xi) = \xi^9$, $g(\sqrt{5}) = \sqrt{5}$, $\eta = 1$, $\varphi = \mathrm{id}$.

□

The following lemma is a slightly stronger version of the discussion from section 3.2.4, applied to $W = H_3$.

Lemma 4.3.2. *The integers $\hat{n}_{\tau,\sigma}$ and $n_{\tau,\sigma}$ encoding the changes of $\mathbb{Z}$-bases of $K(\mathcal{O})$ between $M_{1,c}(\tau)$ and $L_{1,c}(\sigma)$ satisfy: if $\hat{n}_{\tau,\sigma} \neq 0$ or $n_{\tau,\sigma} \neq 0$, then*

- $h_c(\sigma) - h_c(\tau) = i \in \mathbb{Z}_{>0}$
- *if $-\mathrm{Id} \in H_3$ acts of both τ and σ by the same constant (either 1 or -1), then i is even; otherwise i is odd.*

Proof. If $\hat{n}_{\tau,\sigma} \neq 0$, then $L_{1,c}(\sigma)$ is a composition factor of $M_{1,c}(\tau)$, and σ is a W-subrepresentation of some degree graded piece $\mathrm{S}^i\mathfrak{h}^* \otimes \tau \subset M_{1,c}(\tau)$. The action of the grading element $\mathbf{h}$ on this σ is thus

$$h_c(\sigma) = h_c(\tau) + i.$$

The action of the central element $-\mathrm{Id}$ on every irreducible representation is by 1 or -1 (the notation for representations of H_3 is chosen so that this is signaled by the subscript of the representation name). The action on $\mathfrak{h}^* \cong \mathbf{3}_-$ is by -1, so the action on $S^i\mathfrak{h}^* \otimes \tau$ differs from the one on τ by a factor of $(-1)^i$.

The inverse $n_{\tau,\sigma}$ of the block-diagonal, upper-triangular matrix $\hat{n}_{\tau,\sigma}$ has the same properties.

□

4.4 Calculations for $c = 1/10$

This section contains the explicit calculations in $\mathcal{O}_{1,1/10}$. We omit the subscript $c = 1/10$ in in $L_{1,c}(\tau)$ this section, as it is clear from the context, and write $L(\tau)$ instead of $[L_{1,c}(\tau)] \in K(\mathcal{O})$ for better readability.

Theorem 4.4.1. *Irreducible representations in category $\mathcal{O}_{1/10}(H_3, \mathfrak{h})$ have the following descriptions in the Grothendieck group:*

$$\begin{aligned}
L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{3}_-) + M(\mathbf{3}_+) - M(\mathbf{1}_-) \\
L(\mathbf{1}_-) &= M(\mathbf{1}_-) \\
L(\mathbf{3}_+) &= M(\mathbf{3}_+) - M(\mathbf{1}_-) \\
L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{3}_+) + M(\mathbf{1}_-) \\
L(\widetilde{\mathbf{3}}_+) &= M(\widetilde{\mathbf{3}}_+) \\
L(\widetilde{\mathbf{3}}_-) &= M(\widetilde{\mathbf{3}}_-) \\
L(\mathbf{4}_+) &= M(\mathbf{4}_+) \\
L(\mathbf{4}_-) &= M(\mathbf{4}_-) \\
L(\mathbf{5}_+) &= M(\mathbf{5}_+) \\
L(\mathbf{5}_-) &= M(\mathbf{5}_-)
\end{aligned}$$

Among these representations only $L(\mathbf{1}_+)$ is finite-dimensional, with $\mathrm{ch}_{L(\mathbf{1}_+)}(z, g) = 1$.

The rest of this chapter is the proof of this theorem. Let us first calculate the constants $h_{1/10}(\tau) = \frac{3}{2} - \frac{1}{10}\sum_{s\in S} s|_\tau$ (see Table 4.3).

$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\widetilde{\mathbf{3}}_+$	$\widetilde{\mathbf{3}}_-$	$\mathbf{4}_+$	$\mathbf{4}_-$	$\mathbf{5}_+$	$\mathbf{5}_-$
0	3	2	1	2	1	3/2	3/2	6/5	9/5

Table 4.3: $h_{1/10}(\tau)$

Using lemma 4.3.2 we immediately conclude:

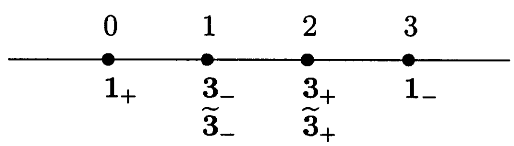
$$L(\mathbf{4}_+) = M(\mathbf{4}_+)$$

$$L(\mathbf{4}_-) = M(\mathbf{4}_-)$$

$$L(\mathbf{5}_+) = M(\mathbf{5}_+)$$

$$L(\mathbf{5}_-) = M(\mathbf{5}_-)$$

Mark the lowest weights of other modules on the real line as



This picture represents Lemma 4.3.2 graphically, meaning that $n_{\tau,\sigma}$ can be nonzero only if both τ and σ are represented on the line, with σ to the right of τ. From this we can also immediately conclude that

$$L(\mathbf{1}_-) = M(\mathbf{1}_-).$$

To calculate character formulas for $L(\mathbf{3}_+)$ and $L(\widetilde{\mathbf{3}}_+)$, we will use Lemma 3.2.3. First calculate the decomposition into irreducible H_3 representations of $\mathfrak{h}^* \otimes \mathbf{3}_+$ and $\mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_+$. A computation with characters of finite group H_3 (see [28] and Table 4.1) gives

$$\mathfrak{h}^* \otimes \mathbf{3}_+ = \mathbf{3}_- \otimes \mathbf{3}_+ \cong \mathbf{1}_- \oplus \mathbf{5}_- \oplus \mathbf{3}_-$$

and

$$\mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_+ = \mathbf{3}_- \otimes \widetilde{\mathbf{3}}_+ \cong \mathbf{4}_- \oplus \mathbf{5}_-.$$

Lemma 3.2.3 now implies that the subrepresentation $\sigma = \mathbf{1}_- \subseteq \mathfrak{h}^* \otimes \mathbf{3}_+$ consists of singular vectors, and hence that $M(\mathbf{1}_-)$ is a subrepresentation of $M(\mathbf{3}_+)$. It is the maximal proper subrepresentation, and it follows that $L(\mathbf{3}_+) \cong M(\mathbf{3}_+)/M(\mathbf{1}_-)$, so in the Grothendieck group

$$L(\mathbf{3}_+) = M(\mathbf{3}_+) - M(\mathbf{1}_-).$$

On the other hand, decomposition of $\mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_+$ does not have $\mathbf{1}_-$ as a subrepresentation, so

$$L(\widetilde{\mathbf{3}}_+) = M(\widetilde{\mathbf{3}}_+).$$

Next, using the decomposition $\mathrm{S}^2\mathfrak{h}^* \cong \mathbf{1}_+ \oplus \mathbf{5}_+$, let us decompose two more H_3 representations:

$$\mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_- \cong \mathbf{4}_+ \oplus \mathbf{5}_+$$

$$\mathrm{S}^2\mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_- \cong (\mathbf{1}_+ \oplus \mathbf{5}_+) \otimes \widetilde{\mathbf{3}}_- \cong 2\widetilde{\mathbf{3}}_- \oplus \mathbf{3}_- \oplus \mathbf{4}_- \oplus \mathbf{5}_-.$$

Because neither $\mathbf{3}_+$ nor $\widetilde{\mathbf{3}}_+$ appear as subrepresentations of $\mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_-$, nor does $\mathbf{1}_-$ appear in the decomposition of $\mathrm{S}^2\mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_-$, the module $M(\widetilde{\mathbf{3}}_-)$ must be simple:

$$L(\widetilde{\mathbf{3}}_-) = M(\widetilde{\mathbf{3}}_-).$$

Corresponding decompositions for $\mathbf{3}_-$ are

$$\mathfrak{h}^* \otimes \mathbf{3}_- \cong \mathbf{1}_+ \oplus \mathbf{3}_+ \oplus \mathbf{5}_+$$

$$\mathrm{S}^2\mathfrak{h}^* \otimes \mathbf{3}_- \cong 2 \cdot \mathbf{3}_- \oplus \widetilde{\mathbf{3}}_- \oplus \mathbf{4}_- \oplus \mathbf{5}_-.$$

From the first of these formulas and using lemma 3.2.3 we can now conclude that $\mathbf{3}_+ \subseteq S^2\mathfrak{h}^* \otimes \mathbf{3}_- \subseteq M(\mathbf{3}_-)$ consists of singular vectors, so it generates a $H_{1,c}(H_3\mathfrak{h})$ subrepresentation. $\mathbf{1}_-$ does not appear in the decomposition of $\mathrm{S}^2\mathfrak{h}^* \otimes \mathbf{3}_-$, so the subrepresentation generated by $\mathbf{3}_+$ is the whole $J(\mathbf{3}_-)$. Looking at the computations for $L(\mathbf{3}_+)$ we see that the only lowest weight representations with lowest weight $\mathbf{3}_+$ are $M(\mathbf{3}_+)$ and $L(\mathbf{3}_+) = M(\mathbf{3}_+) - M(\mathbf{1}_-)$. Thus in Grothendieck group, $L(\mathbf{3}_-) = M(\mathbf{3}_-) - M(\mathbf{3}_+) + n_{\mathbf{3}_-,\mathbf{1}_-}M(\mathbf{1}_-)$, for $n_{\mathbf{3}_-,\mathbf{1}_-} = 0$ or $n_{\mathbf{3}_-,\mathbf{1}_-} = 1$. To see which one of these it is, notice that $\mathbf{1}_-$ does not appear as an H_3 subrepresentation in $\mathrm{S}^2\mathfrak{h}^* \otimes \mathbf{3}_- \subseteq M(\mathbf{3}_-)$, but it does in $\mathrm{S}^1\mathfrak{h}^* \otimes \mathbf{3}_+ \subseteq M(\mathbf{3}_+)$. That means that $M(\mathbf{3}_+)$ cannot be a submodule of $M(\mathbf{3}_-)$; so $J(\mathbf{3}_-) = L(\mathbf{3}_+)$, and the expression in Grothendieck group we wanted is

$$L(\mathbf{3}_-) = M(\mathbf{3}_-) - M(\mathbf{3}_+) + M(\mathbf{1}_-).$$

None of the modules considered so far in this chapter is finite-dimensional. An easy way to see that is to consider them as $\mathfrak{sl}_2$ representations. The lowest occuring weights are then given by Table 4.3. Every finite-dimensional $\mathfrak{sl}_2$ representation will have integral weights, with the lowest one being less or equal then 0. As none of the lowest weights of these modules is a nonpositive integer, they are not finite-dimensional.

This reasoning does not apply to the one module left to describe, $L(\mathbf{1}_+)$. Its lowest $\mathbf{h}$ weight is 0, so it could be finite-dimensional in case it was a trivial one dimensional module. That is exactly what happens: it is easy to see by direct calculation that setting $x = 0, y = 0, w = 1, x \in \mathfrak{h}^*, y \in \mathfrak{h}, w \in H_3$ defines an action of $H_{1,c}(H_3, \mathfrak{h})$ on $\mathbb{C}$. So, there is a trivial module at $c = 1/10$, whose lowest weight is $\mathbf{1}_+$, and it has to be $L(\mathbf{1}_+)$. This computation appears in [7], Prop 2.1.

The character of $L(\mathbf{1}_+)$ is naturally $\mathbf{1}$; to express it in terms of characters of $M(\sigma)$, we count the dimensions of $\mathbf{h}$ weight spaces. Clearly the copy of $\mathbf{3}_- \subseteq \mathfrak{h}^* \otimes \mathbf{1}_+ M(\mathbf{1}_+)$ consists of singular vectors, spanning either $M(\mathbf{3}_-)$ or $L(\mathbf{3}_-)$. To see which one it is, look at the next $\mathbf{h}$ weight space, where $\dim S^2\mathfrak{h}^* \otimes \mathbf{3}_-) = 9 > 6 = \dim S^2\mathfrak{h}^* \otimes \mathbf{1}_+$. So, the submodule with the lowest weight in $\mathbf{h}$ weight space 1 is $L(\mathbf{3}_-)$. The dimensions of all higher $\mathbf{h}$ weight spaces of $M(\mathbf{1}_+)/L(\mathbf{3}_-)$ are 0, so $J(\mathbf{1}_+) = L(\mathbf{3}_-)$ and

$$L(\mathbf{1}_+) = M(\mathbf{1}_+) - L(\mathbf{3}_-) = M(\mathbf{1}_+) - M(\mathbf{3}_-) + M(\mathbf{3}_+) - M(\mathbf{1}_-).$$

4.5 Calculations for $c = 1/6$

Theorem 4.5.1. *Irreducible representations in category $\mathcal{O}_{1/6}(H_3, \mathfrak{h})$ have the following descriptions in the Grothendieck group:*

$$\begin{aligned}
L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\mathbf{1}_-) \\
L(\mathbf{1}_-) &= M(\mathbf{1}_-) \\
L(\mathbf{3}_+) &= M(\mathbf{3}_+) \\
L(\mathbf{3}_-) &= M(\mathbf{3}_-) \\
L(\widetilde{\mathbf{3}}_+) &= M(\widetilde{\mathbf{3}}_+) \\
L(\widetilde{\mathbf{3}}_-) &= M(\widetilde{\mathbf{3}}_-) \\
L(\mathbf{4}_+) &= M(\mathbf{4}_+) \\
L(\mathbf{4}_-) &= M(\mathbf{4}_-) \\
L(\mathbf{5}_+) &= M(\mathbf{5}_+) - M(\mathbf{5}_-) + M(\mathbf{1}_-) \\
L(\mathbf{5}_-) &= M(\mathbf{5}_-) - M(\mathbf{1}_-)
\end{aligned}$$

Only $L(\mathbf{1}_+)$ among these representations is finite-dimensional, with character

$$\mathrm{ch}_{L(\mathbf{1}_+)}(z, g) = \mathrm{Tr}_{\mathbf{1}_+}(g) z^{-1} + \mathrm{Tr}_{\mathbf{3}_-}(g) + \mathrm{Tr}_{\mathbf{1}_+}(g) z.$$

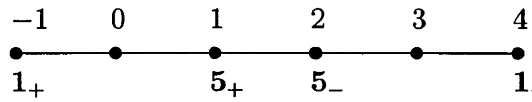
Let us again first calculate the constants $h_{1/6}(\tau) = \frac{3}{2} - \frac{1}{6}\sum_{s\in S} s|_\tau$ (see Table 4.4).

$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\widetilde{\mathbf{3}}_+$	$\widetilde{\mathbf{3}}_-$	$\mathbf{4}_+$	$\mathbf{4}_-$	$\mathbf{5}_+$	$\mathbf{5}_-$
-1	4	7/3	2/3	7/3	2/3	3/2	3/2	1	2

Table 4.4: $h_{1/6}(\tau)$

We immediately conclude that $M(\mathbf{3}_+)$, $M(\mathbf{3}_-)$, $M(\widetilde{\mathbf{3}}_+)$, $M(\widetilde{\mathbf{3}}_-)$, $M(\mathbf{4}_+)$ and $M(\mathbf{4}_-)$ are simple.

The remaining modules have lowest weights represented in the following picture:



So, $M(\mathbf{1}_-)$ is also simple.

Calculate

$$S^2\mathfrak{h}^* \otimes \mathbf{5}_- \cong \mathbf{1}_- \oplus \mathbf{3}_- \oplus \widetilde{\mathbf{3}}_- \oplus 2\cdot\mathbf{4}_- \oplus 3\cdot\mathbf{5}_-$$

From this we can conclude that $L(\mathbf{5}_-) = M(\mathbf{5}_-) - n_{\mathbf{5}_-,\mathbf{1}_-}\cdot M(\mathbf{1}_-)$, with $n_{\mathbf{5}_-,\mathbf{1}_-} \in \{0,1\}$. It is possible to deduce $n_{\mathbf{5}_-,\mathbf{1}_-}$ from the rank of the contravariant form B restricted to $S^4\mathfrak{h}^* \otimes \mathbf{5}_- \subseteq M(\mathbf{5}_-)$, but we will use a less direct argument here.

Let us focus on $\mathbf{5}_+$ for a while. We notice that

$$\mathfrak{h}^* \otimes \mathbf{5}_+ \cong \mathbf{3}_- \oplus \widetilde{\mathbf{3}}_- \oplus \mathbf{4}_- \oplus \mathbf{5}_-,$$

so by Lemma 3.2.3, the H_3 subrepresentation $\mathbf{5}_-$ consists of lowest weight vectors. We know from the previous paragraph that if $n_{\mathbf{5}_-,\mathbf{1}_-} = 0$, then there is just one representation with lowest weight $\mathbf{5}_-$, that is $M(\mathbf{5}_-)$, and if $n_{\mathbf{5}_-,\mathbf{1}_-} = 1$ there are two, namely the standard one $M(\mathbf{5}_-)$ and the irreducible one $M(\mathbf{5}_-) - M(\mathbf{1}_-)$. The module $M(\mathbf{5}_+)$ can also have a b-dimensional space of singular vectors in $S^4\mathfrak{h}^* \otimes \mathbf{5}_+ \subseteq M(\mathbf{5}_+)$, $b \in \mathbb{N}_0$. So, the expression for $L(\mathbf{5}_+)$ is either

$$L(\mathbf{5}_+) = M(\mathbf{5}_+) - M(\mathbf{5}_-) - bM(\mathbf{1}_-),$$

or

$$L(\mathbf{5}_+) = M(\mathbf{5}_+) - M(\mathbf{5}_-) + (n_{\mathbf{5}_-,\mathbf{1}_-} - b)M(\mathbf{1}_-).$$

Now use the decompositions

$$S^3\mathfrak{h}^* \otimes \mathbf{5}_+ \cong 3\cdot\mathbf{3}_- \oplus 3\cdot\widetilde{\mathbf{3}}_- \oplus 3\cdot\mathbf{4}_- \oplus 4\cdot\mathbf{5}_-$$

$$S^2\mathfrak{h}^* \otimes \mathbf{5}_- \cong \mathbf{1}_- \oplus \mathbf{3}_- \oplus \widetilde{\mathbf{3}}_- \oplus 2\cdot\mathbf{4}_- \oplus 3\cdot\mathbf{5}_-$$

to deduce that the graded piece of $L(\mathbf{5}_+)$ with $\mathbf{h}$ weight 4 has, in the Grothendieck group of H_3 representations, one of the following two decompositions:

$$\begin{aligned} L(\mathbf{5}_+)_4 &= M(\mathbf{5}_+)_4 - M(\mathbf{5}_-)_4 - b\cdot M(\mathbf{1}_-)_4 \\ &= \mathrm{S}^3\mathfrak{h}^*\otimes \mathbf{5}_+ - \mathrm{S}^2\mathfrak{h}^*\otimes \mathbf{5}_- - b\cdot \mathbf{1}_- \\ &= 3\cdot \mathbf{3}_- + 3\cdot \widetilde{\mathbf{3}}_- + 3\cdot \mathbf{4}_- + 4\cdot \mathbf{5}_- - \mathbf{1}_- - \mathbf{3}_- - \widetilde{\mathbf{3}}_- - 2\cdot \mathbf{4}_- - 3\cdot \mathbf{5}_- - b\cdot \mathbf{1}_- \\ &= (-1-b)\cdot \mathbf{1}_- + 2\cdot \mathbf{3}_- + 2\cdot \widetilde{\mathbf{3}}_- + \mathbf{4}_- + \mathbf{5}_-, \end{aligned}$$

or

$$\begin{aligned} L(\mathbf{5}_+)_4 &= M(\mathbf{5}_+)_4 - M(\mathbf{5}_-)_4 + (a-b)\cdot M(\mathbf{1}_-)_4 \\ &= (n_{\mathbf{5}_-,\mathbf{1}_-} - 1 - b)\cdot \mathbf{1}_- + 2\cdot \mathbf{3}_- + 2\cdot \widetilde{\mathbf{3}}_- + \mathbf{4}_- + \mathbf{5}_-. \end{aligned}$$

These are decompositions of an actual representation of H_3, so all the coefficients need to be nonnegative integers. $-1-b$ can never be more than -1, so the correct decomposition is the second one, $n_{\mathbf{5}_-,\mathbf{1}_-}=1, b=0$, and the correct formulas for both irreducible modules are

$$L(\mathbf{5}_-) = M(\mathbf{5}_-) - M(\mathbf{1}_-)$$

$$L(\mathbf{5}_+) = M(\mathbf{5}_+) - M(\mathbf{5}_-) + M(\mathbf{1}_-).$$

To describe the module $L(\mathbf{1}_+)$ we use Theorem 3.2.4. It says that its support, when viewed as a $\mathbb{C}[\mathfrak{h}]$ module, is the set of $a\in\mathfrak{h}$ such that $\#\{i|6 \text{ divides } d_i(W)\} = \#\{i|6 \text{ divides } d_i(W_a)\}$. Degrees of H_3 are $2, 6, 10$, so the size of that set is 1. Maximal parabolic subgroups W_a of H_3 are Coxeter groups obtained by deleting a node from the Coxeter graph of H_3, so they are $A_1\times A_1$, $I_2(5)$ and A_2. The degrees of their basic invariants are: $d_1(A_1)=2, d_1(I_2(5))=2, d_2(I_2(5))=5, d_1(A_2)=2, d_2(A_2)=3$. Since 6 does not divide any of them, theorem implies that support of $L(\mathbf{1}_+)$ is just $0\in\mathfrak{h}$. Thus, its Hilbert-Poincaré series does not have a pole at t=1, so it is a polynomial, and $L(\mathbf{1}_+)$ is finite-dimensional.

We know that the expression $L(\mathbf{1}_+)$ in the Grothendieck group is of the form

$$L(\mathbf{1}_+) = M(\mathbf{1}_+) + n_{\mathbf{1}_+,\mathbf{5}_+}M(\mathbf{5}_+) + n_{\mathbf{1}_+,\mathbf{5}_-}M(\mathbf{5}_-) + n_{\mathbf{1}_+,\mathbf{1}_-}M(\mathbf{1}_-).$$

The characters of these representations relate in the same way. Substituting the known expression for character of $M_{1,c}(\tau)$ and evaluating at $g = 1$, we get that

$$\mathrm{ch}_{L(\mathbf{1}_+)}(\mathrm{Id}, z) = \frac{z^{-1}}{(1-z)^3} + n_{\mathbf{1}_+,\mathbf{5}_+} \cdot \frac{5z}{(1-z)^3} + n_{\mathbf{1}_+,\mathbf{5}_-} \cdot \frac{5z^2}{(1-z)^3} + n_{\mathbf{1}_+,\mathbf{1}_-} \cdot \frac{z^4}{(1-z)^3}$$

must be regular at $z = 1$, i.e. that

$$z^{-1} + n_{\mathbf{1}_+,\mathbf{5}_+} \cdot 5z + n_{\mathbf{1}_+,\mathbf{5}_-} \cdot 5z^2 + n_{\mathbf{1}_+,\mathbf{1}_-} \cdot z^4$$

must vanish to order 3 at $z = 1$. Solving this system we get that the only case when this happens is $n_{\mathbf{1}_+,\mathbf{5}_+} = n_{\mathbf{1}_+,\mathbf{1}_-} = -1, n_{\mathbf{1}_+,\mathbf{5}_-} = 1$, so the Grothendieck group expression is, as claimed,

$$L(\mathbf{1}_+) = M(\mathbf{1}_+) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\mathbf{1}_-).$$

It is now an easy computation of H_3 characters in $\mathrm{S}^2\mathfrak{h}^* \otimes \mathbf{1}_+$ to see that the character is equal to

$$\mathrm{Tr}_{\mathbf{1}_+}(g)z^{-1} + \mathrm{Tr}_{\mathbf{3}_-}(g) + \mathrm{Tr}_{\mathbf{1}_+}(g)z.$$

Looking at lowest $\mathbf{h}$ weights again, we see that no module other then $L(\mathbf{1}_+)$ can be finite-dimensional, which completes the proof.

4.6 Calculations for $c = 1/5$

Theorem 4.6.1. *Irreducible representations in category $\mathcal{O}_{1/5}(H_3, \mathfrak{h})$ have the following descriptions in the Grothendieck group:*

$$\begin{aligned}
L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{4}_-) + M(\widetilde{\mathbf{3}}_+)\\
L(\mathbf{1}_-) &= M(\mathbf{1}_-)\\
L(\mathbf{3}_+) &= M(\mathbf{3}_+)\\
L(\mathbf{3}_-) &= M(\mathbf{3}_-)\\
L(\widetilde{\mathbf{3}}_+) &= M(\widetilde{\mathbf{3}}_+)\\
L(\widetilde{\mathbf{3}}_-) &= M(\widetilde{\mathbf{3}}_-) - M(\mathbf{4}_+) + M(\mathbf{1}_-)\\
L(\mathbf{4}_+) &= M(\mathbf{4}_+) - M(\mathbf{1}_-)\\
L(\mathbf{4}_-) &= M(\mathbf{4}_-) - M(\widetilde{\mathbf{3}}_+)\\
L(\mathbf{5}_+) &= M(\mathbf{5}_+)\\
L(\mathbf{5}_-) &= M(\mathbf{5}_-)
\end{aligned}$$

None of these representations is finite-dimensional.

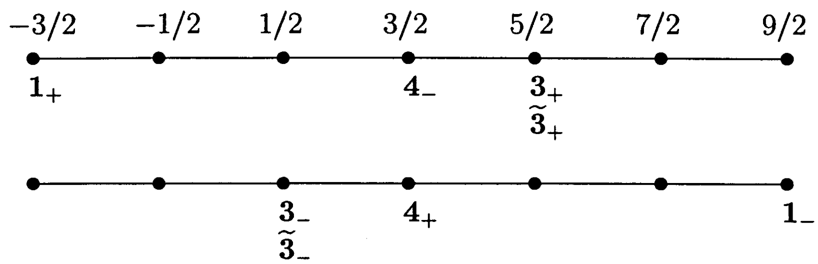
In this case, $h_{1/5}(\tau) = \frac{3}{2} - \frac{1}{5}\sum_{s\in S} s|_\tau$ are as follows (see Table 4.5):

$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\widetilde{\mathbf{3}}_+$	$\widetilde{\mathbf{3}}_-$	$\mathbf{4}_+$	$\mathbf{4}_-$	$\mathbf{5}_+$	$\mathbf{5}_-$
-3/2	9/2	5/2	1/2	5/2	1/2	3/2	3/2	9/10	21/10

Table 4.5: $h_{1/5}(\tau)$

An observation we can immediately make by restricting the representations to the $\mathfrak{sl}_2$ subalgebra is that there are no finite-dimensional modules at $c = 1/5$, because those would have integral weights. We can also immediately say that $M(\mathbf{5}_+)$ and $M(\mathbf{5}_-)$ are simple.

Taking into consideration Lemma 4.3.2, draw the remaining 8 representations schematically as



This picture means that $n_{\tau,\sigma}$ can only be nonzero if τ and σ are on the same line, and σ is to the right of τ. The fact that there are now two lines takes into account the second part of Lemma 4.3.2, meaning the action of a central element $-\mathrm{Id} \in H_3$.

From this we conclude that modules $M(\mathbf{3}_+)$, $M(\widetilde{\mathbf{3}}_+)$ and $M(\mathbf{1}_-)$ are also simple.

To describe $L(\mathbf{4}_-)$, it is enough to calculate

$$\mathfrak{h}^* \otimes \mathbf{4}_- \cong \widetilde{\mathbf{3}}_+ \oplus \mathbf{4}_+ \oplus \mathbf{5}_+,$$

and use Lemma 3.2.3 to conclude

$$L(\mathbf{4}_-) = M(\mathbf{4}_-) - M(\widetilde{\mathbf{3}}_+).$$

To describe $L(\mathbf{1}_+)$, use Theorem 3.2.4 again. The denominator of 1/5 divides just one of the degrees of basic invariants of H_3, namely 10. Thus, the support of this module is the set of all $a \in \mathfrak{h}$ whose stabilizer contains $I_2(5)$, which is a 1-dimensional set (union of lines). That means that the character of $L(\mathbf{1}_+)$, which is of the form

$$\mathrm{ch}_{L(\mathbf{1}_+)} = \mathrm{ch}_{M(\mathbf{1}_+)} + n_{\mathbf{1}_+,\mathbf{4}_-} \cdot \mathrm{ch}_{L(\mathbf{4}_-)} + n_{\mathbf{1}_+,\mathbf{3}_+} \cdot \mathrm{ch}_{L(\mathbf{3}_+)} + n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+} \cdot \mathrm{ch}_{L(\widetilde{\mathbf{3}}_+)},$$

has a pole of order 1 at $z = 1$, i.e. that the function

$$z^{-3/2} + 4n_{\mathbf{1}_+,\mathbf{4}_-} t^{3/2} + 3n_{\mathbf{1}_+,\mathbf{3}_+} z^{5/2} + 3n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+} z^{5/2}$$

vanishes at $z = 1$ to order 2. This translates into: $n_{\mathbf{1}_+,\mathbf{4}_-} = -1$, $n_{\mathbf{1}_+,\mathbf{3}_+} + n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+} = 1$. This means that there is a 4 dimensional set of singular vectors in $M(\mathbf{1}_+)_{3/2}$; using the fact that $\mathfrak{h}^* \otimes \mathbf{4}_- \cong \widetilde{\mathbf{3}}_+ \oplus \mathbf{4}_+ \oplus \mathbf{5}_+$, we conclude they span a copy of $L(\mathbf{4}_-)$, so

$n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+} = 1$, $n_{\mathbf{1}_+,\mathbf{3}_+} = 0$, and the Grothendieck group expression is:

$$L(\mathbf{1}_+) = M(\mathbf{1}_+) - M(\mathbf{4}_-) + M(\widetilde{\mathbf{3}}_+).$$

To calculate the characters of $L(\mathbf{4}_+)$, $L(\mathbf{3}_-)$ and $L(\widetilde{\mathbf{3}}_-)$, let us do more computations of characters of H_3. Namely, we use their characters to see that the multiplicity of $\mathbf{1}_-$ in $\mathrm{S}^3\mathfrak{h}^* \otimes \mathbf{4}_+$ is 1, the multiplicity of $\mathbf{1}_-$ in both $\mathrm{S}^4\mathfrak{h}^* \otimes \mathbf{3}_-$ and $\mathrm{S}^4\mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_-$ is 0, and that $\mathbf{4}_+$ appears with multiplicity 1 the decomposition of $\mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_-$ and not at all in the decomposition of $\mathfrak{h}^* \otimes \mathbf{3}_-$. From this we can conclude:

$$L(\mathbf{3}_-) = M(\mathbf{3}_-)$$

$$L(\mathbf{4}_+) = M(\mathbf{4}_+) - M(\mathbf{1}_-)$$

$$L(\widetilde{\mathbf{3}}_-) = M(\widetilde{\mathbf{3}}_-) - M(\mathbf{4}_+) + M(\mathbf{1}_-).$$

4.7 Calculations for $c = 1/3$

Theorem 4.7.1. *Irreducible representations in category $\mathcal{O}_{1/3}(H_3, \mathfrak{h})$ have the following descriptions in the Grothendieck group:*

$$\begin{aligned}
L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{5}_+) + M(\mathbf{4}_-) \\
L(\mathbf{1}_-) &= M(\mathbf{1}_-) \\
L(\mathbf{3}_+) &= M(\mathbf{3}_+) \\
L(\mathbf{3}_-) &= M(\mathbf{3}_-) \\
L(\widetilde{\mathbf{3}}_+) &= M(\widetilde{\mathbf{3}}_+) \\
L(\widetilde{\mathbf{3}}_-) &= M(\widetilde{\mathbf{3}}_-)
\end{aligned}$$

$$
\begin{aligned}
L(\mathbf{4}_-) &= M(\mathbf{4}_-)\\
L(\mathbf{4}_+) &= M(\mathbf{4}_+) - M(\mathbf{5}_-) + M(\mathbf{1}_-)\\
L(\mathbf{5}_-) &= M(\mathbf{5}_-) - M(\mathbf{1}_-)\\
L(\mathbf{5}_+) &= M(\mathbf{5}_+) - M(\mathbf{4}_-)
\end{aligned}
$$

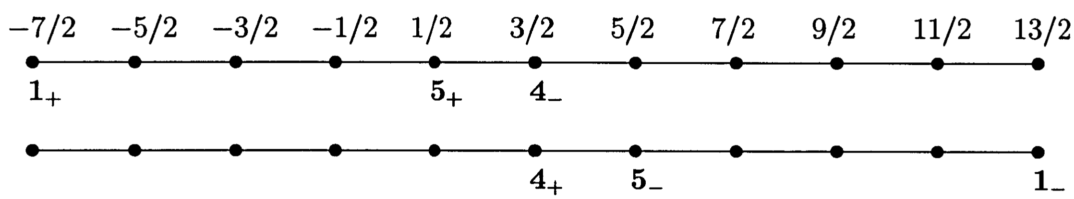
None of these representations is finite-dimensional.

The constants $h_{1/3}(\tau) = \frac{3}{2} - \frac{1}{3}\sum_{s\in S} s|_\tau$ are as follows (see Table 4.6):

$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\widetilde{\mathbf{3}}_+$	$\widetilde{\mathbf{3}}_-$	$\mathbf{4}_+$	$\mathbf{4}_-$	$\mathbf{5}_+$	$\mathbf{5}_-$
-7/2	13/2	19/6	-1/6	19/6	-1/6	3/2	3/2	1/2	5/2

Table 4.6: $h_{1/3}(\tau)$

Thus, $M(\mathbf{3}_+)$, $M(\mathbf{3}_-)$, $M(\widetilde{\mathbf{3}}_+)$ and $M(\widetilde{\mathbf{3}}_-)$ are simple. The other standard modules fall apart into two families:



So $M(\mathbf{4}_-)$ and $M(\mathbf{1}_-)$ are simple too. To describe $L(\mathbf{5}_+)$, calculate

$$\mathfrak{h} \otimes \mathbf{5}_+ \cong \mathbf{3}_- \oplus \widetilde{\mathbf{3}}_- \oplus \mathbf{4}_- \oplus \mathbf{5}_-,$$

and conclude using Lemma 3.2.3

$$L(\mathbf{5}_+) = M(\mathbf{5}_+) - M(\mathbf{4}_-).$$

Next, let us describe $L(\mathbf{1}_+)$. Again, the number of degrees of basic invariants that 3 divides is 1 (namely, $d_2 = 6$). The support is the set of all $a \in \mathfrak{h}$ that have stabilizer containing A_2, which is a union of lines. So, the character of $L(\mathbf{1}_+)$ evaluated at

$g = \mathrm{Id}$ has a pole of order 1 and the function

$$z^{-7/2} + n_{\mathbf{1}_+,\mathbf{5}_+} \cdot 5z^{1/2} + n_{\mathbf{1}_+,\mathbf{4}_-} \cdot 4z^{3/2}$$

has a zero of order 2 at $z = 1$. Writing out this condition gives $n_{\mathbf{1}_+,\mathbf{5}_+} = -1$, $n_{\mathbf{1}_+,\mathbf{4}_-} = 1$, so the expression for $L(\mathbf{1}_+)$ is:

$$L(\mathbf{1}_+) = M(\mathbf{1}_+) - M(\mathbf{5}_+) + M(\mathbf{4}_-).$$

Next, we want to describe the structure of $L(\mathbf{5}_-)$. As explained before, all the standard modules $M_{1,c}(\tau)$ have a contravariant bilinear form B on them, whose kernel is $J_{1,c}(\tau)$. The form respects the grading of $M_{1,c}(\tau)$, in the sense that graded pieces of $M_{1,c}(\tau)$ are orthogonal to each other. Let the restriction of the form B to $\mathrm{S}^k\mathfrak{h}^* \otimes \tau$ be called B_k. It is easy to compute B_k recursively on k using MAGMA algebra software. If

$$L(\mathbf{5}_-) = M(\mathbf{5}_-) - a \cdot M(\mathbf{1}_-),$$

then the rank of the form B_4 on $M(\mathbf{5}_-)$ is

$$\dim L(\mathbf{5}_-)_{13/2} = \dim M(\mathbf{5}_-)_{13/2} - a \cdot \dim M(\mathbf{1}_-)_{13/2} = 75 - a.$$

Calculating the rank of the same B_4 in MAGMA, we get that it is 74; hence, $a = 1$ and

$$L(\mathbf{5}_-) = M(\mathbf{5}_-) - M(\mathbf{1}_-).$$

To do $L(\mathbf{4}_+)$, notice that the multiplicity of $\mathbf{5}_-$ in $\mathrm{S}^1\mathfrak{h}^* \otimes \mathbf{4}_+$ is 1, that the multiplicity of $\mathbf{1}_-$ in $\mathrm{S}^5\mathfrak{h}^* \otimes \mathbf{4}_+$ is 1, and that the multiplicity of $\mathbf{1}_-$ in $\mathrm{S}^4\mathfrak{h}^* \otimes \mathbf{5}_-$ is 2. So, writing out the condition that the multiplicity of $\mathbf{1}_-$ in $L(\mathbf{4}_+)$ must be nonnegative, we get that the expression for it is

$$L(\mathbf{4}_+) = M(\mathbf{4}_+) - M(\mathbf{5}_-) + M(\mathbf{1}_-).$$

4.8 Calculations for $c = 1/2$

Theorem 4.8.1. *Irreducible representations in category $\mathcal{O}_{1/2}(H_3, \mathfrak{h})$ have the following descriptions in the Grothendieck group:*

$$
\begin{aligned}
L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{3}_-) - M(\widetilde{\mathbf{3}}_-) + M(\mathbf{5}_+) - M(\mathbf{5}_-) + M(\mathbf{3}_+) + M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-) \\
L(\mathbf{1}_-) &= M(\mathbf{1}_-) \\
L(\mathbf{3}_+) &= M(\mathbf{3}_+) - M(\mathbf{1}_-) \\
L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\mathbf{3}_+) \\
L(\widetilde{\mathbf{3}}_+) &= M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-) \\
L(\widetilde{\mathbf{3}}_-) &= M(\widetilde{\mathbf{3}}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\widetilde{\mathbf{3}}_+) \\
L(\mathbf{4}_+) &= M(\mathbf{4}_+) \\
L(\mathbf{4}_-) &= M(\mathbf{4}_-) \\
L(\mathbf{5}_+) &= M(\mathbf{5}_+) - 2 \cdot M(\mathbf{5}_-) + M(\mathbf{3}_+) + M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-). \\
L(\mathbf{5}_-) &= M(\mathbf{5}_-) - M(\mathbf{3}_+) - M(\widetilde{\mathbf{3}}_+) + M(\mathbf{1}_-)
\end{aligned}
$$

The following of these representations are finite-dimensional: $L(\mathbf{1}_+)$ (dim $= 115$), $L(\mathbf{3}_-)$ (with $\mathrm{ch}_{L(\mathbf{3}_-)}(z, g) = \mathrm{Tr}_{\mathbf{3}_-}(g)z^{-1} + \mathrm{Tr}_{\mathbf{1}_+}(g) + \mathrm{Tr}_{\mathbf{3}_+}(g) + \mathrm{Tr}_{\mathbf{3}_-}(g)z$) and $L(\widetilde{\mathbf{3}}_-)$ (with $\mathrm{ch}_{L(\widetilde{\mathbf{3}}_-)}(g)z = \mathrm{Tr}_{\widetilde{\mathbf{3}}_-}(g)z^{-1} + \mathrm{Tr}_{\mathbf{4}_+}(g) + \mathrm{Tr}_{\widetilde{\mathbf{3}}_-}(g)z$).

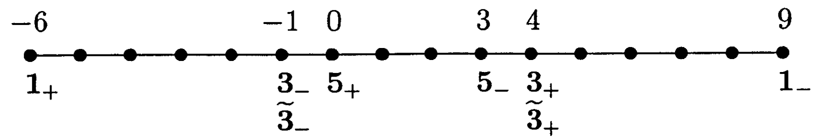
In this case, $h_{1/2}(\tau) = \frac{3}{2} - \frac{1}{2}\sum_{s \in S} s|_\tau$ are (see Table 4.7):

$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\widetilde{\mathbf{3}}_+$	$\widetilde{\mathbf{3}}_-$	$\mathbf{4}_+$	$\mathbf{4}_-$	$\mathbf{5}_+$	$\mathbf{5}_-$
-6	9	4	-1	4	-1	3/2	3/2	0	3

Table 4.7: $h_{1/2}(\tau)$

So, $M(\mathbf{4}_-)$ and $M(\mathbf{4}_+)$ are simple.

Graphic representation of Lemma 4.3.2 is now



Again, $M(\mathbf{1}_-)$ is simple.

Let us first analyze $L(\mathbf{3}_-)$ and $L(\widetilde{\mathbf{3}}_-)$. In both cases

$$\dim \operatorname{Hom}(\mathbf{5}_+, \mathfrak{h}^* \otimes \mathbf{3}_-) = \dim \operatorname{Hom}(\mathbf{5}_+, \mathfrak{h}^* \otimes \widetilde{\mathbf{3}}_-) = 1,$$

so by Lemma 3.2.3 there is a 5-dimensional set of lowest weight vectors at $\mathbf{h}$ weight 0. The dimension of both of these modules at $\mathbf{h}$ weight 2 is

$$3 \cdot \binom{3+2}{2} - 5 \cdot \binom{2+2}{2} = 0.$$

This means both these modules are finite-dimensional, and we can immediately determine their characters by decomposing weight spaces at $\mathbf{h}$ weights 0 and 1 into H_3 irreducible representations. They are:

$$\operatorname{ch}_{L(\mathbf{3}_-)}(z, g) = \operatorname{Tr}_{\mathbf{3}_-}(g) z^{-1} + \operatorname{Tr}_{\mathbf{1}_+}(g) + \operatorname{Tr}_{\mathbf{3}_+}(g) + \operatorname{Tr}_{\mathbf{3}_-}(g) z$$

$$\operatorname{ch}_{L(\widetilde{\mathbf{3}}_-)}(g, z) = \operatorname{Tr}_{\widetilde{\mathbf{3}}_-}(g) z^{-1} + \operatorname{Tr}_{\mathbf{4}_+}(g) + \operatorname{Tr}_{\widetilde{\mathbf{3}}_-}(g) z.$$

To express them in terms of characters of standard modules, write them in the Grothendieck group as

$$L(\mathbf{3}_-) = M(\mathbf{3}_-) - M(\mathbf{5}_+) + n_{\mathbf{3}_-,\mathbf{5}_-} M(\mathbf{5}_-) + n_{\mathbf{3}_-,\mathbf{3}_+} M(\mathbf{3}_+) + n_{\mathbf{3}_-,\widetilde{\mathbf{3}}_+} M(\widetilde{\mathbf{3}}_-) + n_{\mathbf{3}_-,\mathbf{1}_-} M(\mathbf{1}_-)$$

$$L(\widetilde{\mathbf{3}}_-) = M(\widetilde{\mathbf{3}}_-) - M(\mathbf{5}_+) + n_{\widetilde{\mathbf{3}}_-,\mathbf{5}_-} M(\mathbf{5}_-) + n_{\widetilde{\mathbf{3}}_-,\mathbf{3}_+} M(\mathbf{3}_+) + n_{\widetilde{\mathbf{3}}_-,\widetilde{\mathbf{3}}_+} M(\widetilde{\mathbf{3}}_-) + n_{\widetilde{\mathbf{3}}_-,\mathbf{1}_-} M(\mathbf{1}_-).$$

Then write the condition that dimensions of all $\mathbf{h}$ weight spaces above 2 must be 0 (it is enough to write the equations for weights 3, 4 and 9). This produces linear equations in $n_{\tau,\sigma}$ with solutions: $n_{\mathbf{3}_-,\mathbf{5}_-} = n_{\widetilde{\mathbf{3}}_-,\mathbf{5}_-} = 1$, $n_{\mathbf{3}_-,\mathbf{3}_+} + n_{\mathbf{3}_-,\widetilde{\mathbf{3}}_+} = n_{\widetilde{\mathbf{3}}_-,\mathbf{3}_+} + n_{\widetilde{\mathbf{3}}_-,\widetilde{\mathbf{3}}_+} = -1, n_{\mathbf{3}_-,\mathbf{1}_-} = n_{\widetilde{\mathbf{3}}_-,\mathbf{1}_-} = 0$. Finally, writing the H_3 character of $M(\mathbf{3}_-)_4 - M(\mathbf{5}_+)_4 +$

$M(\mathbf{5}_-)_4$ we conclude it is isomorphic to $\mathbf{3}_+$, whereas $M(\widetilde{\mathbf{3}}_-)_4 - M(\mathbf{5}_+)_4 + M(\mathbf{5}_-)_4 \cong \widetilde{\mathbf{3}}_+$, so the required expressions are:

$$L(\mathbf{3}_-) = M(\mathbf{3}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\mathbf{3}_+)$$

$$L(\widetilde{\mathbf{3}}_-) = M(\widetilde{\mathbf{3}}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\widetilde{\mathbf{3}}_+)$$

Remark 4.8.2. *Values of c for which modules with zero set of W invariants exist are called aspherical. The module $L_{1/2}(\widetilde{\mathbf{3}}_-)$ we just described has no H_3 invariants and so shows that $c = 1/2$ is an aspherical value for $(H_3, \mathfrak{h})$. In particular, $\Phi_{1/2,3/2}$ from section 3.2.10 is not an equivalence of categories.*

We use MAGMA to calculate the rank of the form B_5 on $M(\mathbf{3}_+)$ and on $M(\widetilde{\mathbf{3}}_+)$ and in both cases get 62. This means there is a $3 \cdot \binom{7}{2} - 62 = 1$ dimensional kernel, and that

$$L(\mathbf{3}_+) = M(\mathbf{3}_+) - M(\mathbf{1}_-),$$

$$L(\widetilde{\mathbf{3}}_+) = M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-).$$

To analyze $L(\mathbf{1}_+)$, note that the number of degrees of basic invariants of H_3, that 2 divides is 3 (all the degrees 2, 6 and 10 are even). This is bigger then the number of even basic invariants of any parabolic subgroup of H_3 except H_3 itself, so the support of $L(\mathbf{1}_+)$ is the set of elements of $\mathfrak{h}$ fixed by the entire H_3, i.e. just a zero dimensional set consisting only of the origin. That means that the module $L(\mathbf{1}_+)$ is finite-dimensional.

Remark 4.8.3. *Notice that the previous argument depended only on the denominator of $c = 1/2$; it actually proves that $L_{1,r/2}(\mathbf{1}_+)$ is finite-dimensional for all odd $r \geq 0$.*

Now we use MAGMA [11] to calculate the rank of the form B restricted to $M(\mathbf{1}_+)_{-1} = S^5\mathfrak{h}^* \otimes \mathbf{1}_+$. This is 21 dimensional space, and the rank of the form is 15. Since both $\mathbf{3}_-$ and $\widetilde{\mathbf{3}}_-$ appear in the decomposition of $S^5\mathfrak{h}^*$ into H_3 subrepresentations, and each of them with multiplicity 2, we need some more calculations to see how this 6 dimensional space of singular vectors looks. To do that, again use

MAGMA to compute the H_3 character on the 6 dimensional kernel of B on $M(\mathbf{1}_+)_{-1}$. This computation shows that the kernel is $\mathbf{3}_- \oplus \widetilde{\mathbf{3}}_-$, so $n_{\mathbf{1}_+,\mathbf{3}_-} = n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_-} = 1$.

Because $L(\mathbf{1}_+)$ is finite-dimensional and has an $\mathfrak{sl}_2$ representation structure, we know that $\dim L(\mathbf{1}_+)_j = \dim L(\mathbf{1}_+)_{-j}$ for every integer j. This gives us a system of linear equations whose only solution yields the following expression for the irreducible module:

$$L(\mathbf{1}_+) = M(\mathbf{1}_+) - M(\mathbf{3}_-) - M(\widetilde{\mathbf{3}}_-) + M(\mathbf{5}_+) - M(\mathbf{5}_-) +$$

$$+ n_{\mathbf{1}_+,\mathbf{3}_+} M(\mathbf{3}_+) + n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+} M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-)$$

with $n_{\mathbf{1}_+,\mathbf{3}_+} + n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+} = 2$.

To calculate $n_{\mathbf{1}_+,\mathbf{3}_+}, n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+}$ we make the following observation. As the copy of $\mathfrak{sl}_2$ in $H_{1,c}(H_3, \mathfrak{h})$ commutes with H_3, for any $H_{1,c}(H_3, \mathfrak{h})$ module M and any irreducible representation τ of H_3 we can put the $\mathfrak{sl}_2$ module structure on $\mathrm{Hom}_{H_3}(\tau, M)$ by letting $\mathfrak{sl}_2$ act on the value. If $M = L(\mathbf{1}_+)$, then this module is finite-dimensional, so dimensions of weight spaces are symmetric around 0. In other words, $\dim \mathrm{Hom}_{H_3}(\tau, L(\mathbf{1}_+)_{-j}) = \dim \mathrm{Hom}_{H_3}(\tau, L(\mathbf{1}_+)_j)$. Doing this computation for $\tau = \mathbf{3}_+$ and $j = 4$ gives us that this dimension is $\dim \mathrm{Hom}_{H_3}(\mathbf{3}_+, S^2\mathfrak{h}^*) = 0$.

Representations of A_5 and H_3 are defined over the field $\mathbb{Q}[\sqrt{5}]$, which is a field extension of $\mathbb{Q}$ of degree 2. The Galois action of $\mathbb{Z}_2$ corresponding to this extension is $\sqrt{5} \mapsto -\sqrt{5}$. It acts on all characters, and it is clear from the character table 4.1 that the action of the Galois group on the character of a representation V of H_3 is trivial if and only if

$$\dim \mathrm{Hom}_{H_3}(\mathbf{3}_-, V) + \dim \mathrm{Hom}_{H_3}(\mathbf{3}_+, V) = \dim \mathrm{Hom}_{H_3}(\widetilde{\mathbf{3}}_-, V) + \dim \mathrm{Hom}_{H_3}(\widetilde{\mathbf{3}}_+, V),$$

in other words, if, seen as a representation of A_5 and decomposed into irreducible subrepresentations, V has the same multiplicity of $\mathbf{3}$ and $\widetilde{\mathbf{3}}$.

Calculation of the H_3 characters for $L(\mathbf{1}_+)_{-4}$ and $L(\mathbf{1}_+)_4 = S^{10}\mathfrak{h}^* \otimes \mathbf{1}_+ - S^5\mathfrak{h}^* \otimes (\mathbf{3}_+ \oplus \widetilde{\mathbf{3}}_+) + S^4\mathfrak{h}^* \otimes \mathbf{5}_+ - S^1\mathfrak{h}^* \otimes \mathbf{5}_- + n_{\mathbf{1}_+,\mathbf{3}_+}\mathbf{3}_+ + n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+}\widetilde{\mathbf{3}}_+$ (an elementary computation of H_3 characters, though a tedious one) show the character of $L(\mathbf{1}_+)_{-4}$ is invariant under the above Galois action, and that the character of $L(\mathbf{1}_+)_4$ (which is the same) is invariant if and only if $n_{\mathbf{1}_+,\mathbf{3}_+} = n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+}$. So, they both have to be 1, and the character is, as claimed in the theorem,

$$L(\mathbf{1}_+) = M(\mathbf{1}_+) - M(\mathbf{3}_-) - M(\widetilde{\mathbf{3}}_-) + M(\mathbf{5}_+) - M(\mathbf{5}_-) + M(\mathbf{3}_+) + M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-).$$

4.8.1 Cherednik algebra $H_{1/2}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathfrak{h}')$ and calculation of $L(\mathbf{5}_-)$

We will calculate $L(\mathbf{5}_-)$ using the induction functor from section 3.2.7. To do that, let us first describe the algebra we will be inducing from.

A way to get a maximal parabolic subgroups of Coxeter groups is to remove one vertex from the Coxeter graph, which corresponds to removing one generator. In this case, let us remove the middle vertex of the H_3 graph, and thus get a disconnected graph

of $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the isomorphism $H_3 \cong \mathbb{Z}_2 \times A_5$, we can take the Coxeter generators of H_3 to be $s_1 = -(12)(34), s_2 = -(15)(34), s_3 = -(13)(24)$. Then the generators of $W' = \mathbb{Z}_2 \times \mathbb{Z}_2$ are s_1, s_3. Let us write the character table of $\mathbb{Z}_2 \times \mathbb{Z}_2$, with the main purpose of introducing notation and names of representations: see Table 4.8.

	Id	$-(12)(34)$	$-(13)(24)$	$(14)(23)$
$\mathbf{1}_{++}$	1	1	1	1
$\mathbf{1}_{+-}$	1	1	-1	-1
$\mathbf{1}_{-+}$	1	-1	1	-1
$\mathbf{1}_{--}$	1	-1	-1	1

Table 4.8: Character table for $\mathbb{Z}_2 \times \mathbb{Z}_2$

Working out the irreducible modules $L_{1/2}(\tau), \tau \in \widehat{W'}$ is really easy in this case. They have the lowest weights given in table 4.9. So using only Lemma 3.2.3 and the

$\mathbf{1}_{++}$	$\mathbf{1}_{+-}$	$\mathbf{1}_{-+}$	$\mathbf{1}_{--}$
0	1	1	2

Table 4.9: $h_{1/2}(\tau), \tau$ irreducible representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$

fact $\mathfrak{h}' \cong \mathbf{1}_{-+} \oplus \mathbf{1}_{+-}$, we get that the module

$$L_{1/2}(\mathbf{1}_{++}) = M_{1/2}(\mathbf{1}_{++}) - M_{1/2}(\mathbf{1}_{+-}) - M_{1/2}(\mathbf{1}_{-+}) + M_{1/2}(\mathbf{1}_{--})$$

is a one dimensional representation of $H_{1/2}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathfrak{h}')$.

Let b be any point whose stabilizer is this copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$. We are going to apply the induction functor Ind_b to the one dimensional module $L_{1/2}(\mathbf{1}_{++})$. Before we do that, let us decompose all the representations of H_3 into representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$:

	Id	$-(12)(34)$	$-(13)(24)$	$(14)(23)$	$\cong$
$\mathbf{1}_+$	1	1	1	1	$\mathbf{1}_{++}$
$\mathbf{1}_-$	1	-1	-1	1	$\mathbf{1}_{--}$
$\mathbf{3}_+$	3	-1	-1	-1	$\mathbf{1}_{-+} \oplus \mathbf{1}_{+-} \oplus \mathbf{1}_{--}$
$\mathbf{3}_-$	3	1	1	-1	$\mathbf{1}_{-+} \oplus \mathbf{1}_{+-} \oplus \mathbf{1}_{++}$
$\widetilde{\mathbf{3}}_-$	3	-1	-1	-1	$\mathbf{1}_{-+} \oplus \mathbf{1}_{+-} \oplus \mathbf{1}_{--}$
$\widetilde{\mathbf{3}}_-$	1	1	1	-1	$\mathbf{1}_{-+} \oplus \mathbf{1}_{+-} \oplus \mathbf{1}_{++}$
$\mathbf{4}_+$	4	0	0	0	$\mathbf{1}_{--} \oplus \mathbf{1}_{-+} \oplus \mathbf{1}_{+-} \oplus \mathbf{1}_{++}$
$\mathbf{4}_-$	4	0	0	0	$\mathbf{1}_{--} \oplus \mathbf{1}_{-+} \oplus \mathbf{1}_{+-} \oplus \mathbf{1}_{++}$
$\mathbf{5}_+$	5	1	1	1	$\mathbf{1}_{--} \oplus \mathbf{1}_{-+} \oplus \mathbf{1}_{+-} \oplus 2 \cdot \mathbf{1}_{++}$
$\mathbf{5}_-$	5	-1	-1	-1	$2 \cdot \mathbf{1}_{--} \oplus \mathbf{1}_{-+} \oplus \mathbf{1}_{+-} \oplus \mathbf{1}_{++}$

Table 4.10: Decomposition of irreducible representations of H_3 as representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$

So, using Proposition 3.2.5, the expression in the Grothendieck group of $\mathcal{O}_{1/2}(H_3, \mathfrak{h})$ for the induced module $\mathrm{Ind}_b(L_{1/2}(\mathbf{1}_{++}))$ is

$$\mathrm{Ind}_b(L_{1/2}(\mathbf{1}_{++})) =$$

$$= \mathrm{Ind}_b(M_{1/2}(\mathbf{1}_{++})) - \mathrm{Ind}_b(M_{1/2}(\mathbf{1}_{-+})) - \mathrm{Ind}_b(M_{1/2}(\mathbf{1}_{+-})) + \mathrm{Ind}_b(L_{1/2}(\mathbf{1}_{--}))$$

$$= M(\mathbf{1}_+) - M(\mathbf{3}_+) - M(\widetilde{\mathbf{3}}_+) + M(\mathbf{5}_-) + M(\mathbf{5}_+) - M(\mathbf{3}_-) - M(\widetilde{\mathbf{3}}_-) + M(\mathbf{1}_-).$$

This means there is a module in $\mathcal{O}_{1/2}(H_3, \mathfrak{h})$ with this expression in the Grothendieck group. Its composition series must contain an irreducible module containing $M(\mathbf{1}_+)$ in its Grothendieck group expression, and there is only one such. Subtracting the known Grothendieck group expression of $L(\mathbf{1}_+)$ from the one for $\mathrm{Ind}_b(L_{1/2}(\mathbf{1}_{++}))$, we get that there must exist a module with Grothendieck group expression

$$2\big(M(\mathbf{5}_-) - M(\mathbf{3}_+) - M(\widetilde{\mathbf{3}}_+) + M(\mathbf{1}_-)\big).$$

Now, Lemma 3.2.3 and the decomposition of $\mathfrak{h}^* \otimes \mathbf{5}_-$ into irreducible subrepresentations imply that the irreducible module $L(\mathbf{5}_-)$ is of the form

$$L(\mathbf{5}_-) = M(\mathbf{5}_-) - M(\mathbf{3}_+) - M(\widetilde{\mathbf{3}}_+) + a \cdot M(\mathbf{1}_-),$$

with $a \in \mathbb{Z}$. There are 3 copies of $\mathbf{1}_-$ in $M(\mathbf{5}_-)_9$, 2 copies of $\mathbf{1}_-$ in $M(\mathbf{3}_+)_9$ and 2 copies of $\mathbf{1}_-$ in $M(\widetilde{\mathbf{3}}_+)_9$, so $3 - 2 - 2 + a \geq 0$ and $a \geq 1$.

Subtracting two times this expression from the above expression for the module we concluded must exist, we get that there also must be a module with Grothendieck group expression

$$2(1-a)M(\mathbf{1}_-),$$

i.e. that $a \leq 1$, so $a = 1$. This proves that the expression for the irreducible module we wanted is

$$L(\mathbf{5}_-) = M(\mathbf{5}_-) - M(\mathbf{3}_+) - M(\widetilde{\mathbf{3}}_+) + M(\mathbf{1}_-).$$

4.8.2 Cherednik algebra $H_{1/2}(S_3, \mathfrak{h}')$ and calculation of $L(\mathbf{5}_+)$

We start by doing the MAGMA computation of rank of B in degrees 3 and 4 we get that it is 40 and 51, so the Grothendieck group expression is of the form

$$L(\mathbf{5}_+) = M(\mathbf{5}_+) - 2 \cdot M(\mathbf{5}_-) + n_{\mathbf{5}_-,\mathbf{3}_+} M(\mathbf{3}_+) + n_{\mathbf{5}_-,\widetilde{\mathbf{3}}_+} M(\widetilde{\mathbf{3}}_+) + n_{\mathbf{5}_-,\mathbf{1}_-} M(\mathbf{1}_-),$$

with $n_{\mathbf{5}_-,\widetilde{\mathbf{3}}_+} + n_{\mathbf{5}_-,\mathbf{3}_+} = 2$. Looking at the dimension of $L(\mathbf{5}_+)_k$, which is a quadratic polynomial in k with leading term $\frac{1}{2}(1+n_{\mathbf{5}_-,\mathbf{1}_-})k^2$, and writing the condition that it is ≥ 0 for large k, we conclude $n_{\mathbf{5}_-,\mathbf{1}_-} \geq -1$.

To finish the analysis, we need to look at another Cherednik algebra associated to a parabolic subgroup of H_3, like in the last section. This time, remove the rightmost vertex in the Coxeter graph, to get a group $W' = S_3$ generated by $s_1 = -(12)(34)$, $s_2 = -(15)(34)$. Its character table is very well known:

	Id	s_1	(125)
#	1	3	2
$\mathbf{1}_+$	1	1	1
$\mathbf{1}_-$	1	-1	1
$\mathbf{2}$	2	0	-1

Table 4.11: Character table for S_3

Working out the irreducible modules $L_{1/2}(\tau), \tau \in \widehat{W'}$ is again really easy. The lowest weights are

$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}$
-1/2	5/2	1

Table 4.12: $h_{1/2}(\tau), \tau$ irreducible representation of S_3

The denominator of 1/2 is a degree of a basic invariant of S_3, so the category $\mathcal{O}_{1/2}(S_3, \mathfrak{h}')$ is not semisimple. So, $M_{1/2}(\mathbf{1}_+)$ is not simple (as the other two are). Looking at the possible options and decomposing $\mathrm{S}^2\mathfrak{h}' \otimes \mathbf{1}_+ = \mathrm{S}^2\mathbf{2} = \mathbf{2} \oplus \mathbf{1}_-$, we conclude $L_{1/2}(\mathbf{1}_+) = M_{1/2}(\mathbf{1}_+) - M_{1/2}(\mathbf{1}_-)$.

Now let b be any point with a stabilizer W' and apply Ind_b to $L_{1/2}(\mathbf{1}_+)$. In the same way as before (using decompositions of H_3 representations into S_3 irreducible components, and applying Lemma 3.2.5), we get that there is a module in $\mathcal{O}_{1,c}(H_3, \mathfrak{h})$ with the Grothendieck group description

$$M(\mathbf{1}_+) + M(\mathbf{3}_-) + M(\widetilde{\mathbf{3}}_-) + M(\mathbf{5}_+) - M(\mathbf{5}_-) - M(\mathbf{3}_+) - M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-).$$

Subtracting $L(\mathbf{1}_+)$, which has to be in its composition series, from it, and doing the same thing for $L(\mathbf{3}_-)$ and $L(\widetilde{\mathbf{3}}_-)$, we see that there is a module in $\mathcal{O}_{1,c}(H_3, \mathfrak{h})$ with the Grothendieck group description

$$4(M(\mathbf{5}_+) - M(\mathbf{5}_-)).$$

$L(\mathbf{5}_+)$ must appear as a factor in the composition series of this module 4 times. So, subtract $4 \cdot L(\mathbf{5}_+)$ from it to get that there exists a module with expression

$$4 \cdot M(\mathbf{5}_-) - 4 \cdot n_{\mathbf{5}_-,\mathbf{3}_+} M(\mathbf{3}_+) - 4 \cdot n_{\mathbf{5}_-,\widetilde{\mathbf{3}}_+} M(\widetilde{\mathbf{3}}_+) - 4 \cdot n_{\mathbf{5}_-,\mathbf{1}_-} M(\mathbf{1}_-).$$

Subtracting the known expression for $4 \cdot L(\mathbf{5}_-)$, we get that there must be a module with expression:

$$4 \cdot (1 - n_{\mathbf{5}_-,\mathbf{3}_+}) M(\mathbf{3}_+) + 4 \cdot (1 - n_{\mathbf{5}_-,\widetilde{\mathbf{3}}_+}) M(\widetilde{\mathbf{3}}_+) - 4 \cdot (1 + n_{\mathbf{5}_-,\mathbf{1}_-}) M(\mathbf{1}_-).$$

This implies $1 - n_{\mathbf{5}_-,\mathbf{3}_+} \geq 0$ and $1 - n_{\mathbf{5}_-,\widetilde{\mathbf{3}}_+} \geq 0$, which together with $n_{\mathbf{5}_-,\mathbf{3}_+} + n_{\mathbf{5}_-,\widetilde{\mathbf{3}}_+} = 2$ means $n_{\mathbf{5}_-,\mathbf{3}_+} = n_{\mathbf{5}_-,\widetilde{\mathbf{3}}_+} = 1$. The last module then becomes

$$-4 \cdot (1 + n_{\mathbf{5}_-,\mathbf{1}_-}) M(\mathbf{1}_-),$$

so $1 + n_{\mathbf{5}_-,\mathbf{1}_-} \leq 0$, which means $n_{\mathbf{5}_-,\mathbf{1}_-} = -1$. Therefore, we have

$$L(\mathbf{5}_+) = M(\mathbf{5}_+) - 2 \cdot M(\mathbf{5}_-) + M(\mathbf{3}_+) + M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-).$$

Remark 4.8.4. *As explained before, for $c = r/d$, $d \geq 3$, we will use scaling functors $\Phi_{1/d,r/d} : \mathcal{O}_{1,1/d} \to \mathcal{O}_{1,r/d}$ from section 3.2.9 to get the descriptions of all modules $L_{1,r/d}(\tau)$. Scaling functors are only conjectured to be equivalences of categories for half integers, so instead of them for $c = r/2$ we use shift functors $\Phi_{c,c+1}$ from section 3.2.9 to show equivalence $\mathcal{O}_{1,(r-2)/2} \to \mathcal{O}_{1,r/2}$. The functor $\Phi_{c,c+1}$ is an equivalence of categories if c is aspherical. For H_3, $c = 1/2$ is not aspherical, as the module $L_{1/2}(\widetilde{\mathbf{3}}_-)$ contains no H_3 invariants. So, we do the analogous calculation for $c = 3/2$ and then* *check if 3/2 is aspherical.*

4.9 Calculations for $c = 3/2$

Theorem 4.9.1. *Irreducible representations in category $\mathcal{O}_{3/2}(H_3, \mathfrak{h})$ have the following descriptions in the Grothendieck group:*

$$\begin{aligned}
L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{3}_-) - M(\widetilde{\mathbf{3}}_-) + M(\mathbf{5}_+) - M(\mathbf{5}_-) + M(\mathbf{3}_+) + M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-)\\
L(\mathbf{1}_-) &= M(\mathbf{1}_-)\\
L(\mathbf{3}_+) &= M(\mathbf{3}_+) - M(\mathbf{1}_-)\\
L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\mathbf{3}_+)\\
L(\widetilde{\mathbf{3}}_+) &= M_{1/2}(\widetilde{\mathbf{3}}_+) - M_{1/2}(\mathbf{1}_-)\\
L(\widetilde{\mathbf{3}}_-) &= M(\widetilde{\mathbf{3}}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) - M(\widetilde{\mathbf{3}}_+)\\
L(\mathbf{4}_+) &= M(\mathbf{4}_+)\\
L(\mathbf{4}_-) &= M(\mathbf{4}_-)\\
L(\mathbf{5}_+) &= M(\mathbf{5}_+) - 2\cdot M(\mathbf{5}_-) + M(\mathbf{3}_+) + M(\widetilde{\mathbf{3}}_+) - M(\mathbf{1}_-)\\
L(\mathbf{5}_-) &= M(\mathbf{5}_-) - M(\mathbf{3}_+) - M(\widetilde{\mathbf{3}}_+) + M(\mathbf{1}_-)
\end{aligned}$$

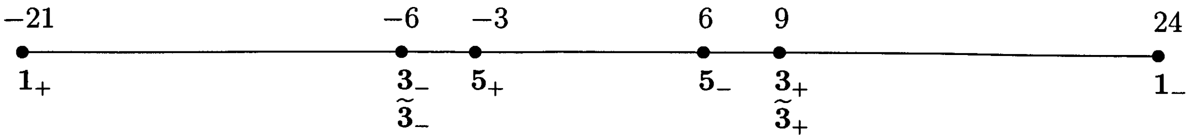
Three of these representations are finite-dimensional: $L(\mathbf{1}_+)$, $L(\mathbf{3}_-)$ and $L(\widetilde{\mathbf{3}}_-)$.

In this case, $h_{3/2}(\tau) = \frac{3}{2} - \frac{1}{2}\sum_{s\in S} s|_\tau$ are (see Table 4.7):

$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\widetilde{\mathbf{3}}_+$	$\widetilde{\mathbf{3}}_-$	$\mathbf{4}_+$	$\mathbf{4}_-$	$\mathbf{5}_+$	$\mathbf{5}_-$
-21	24	9	-6	9	-6	3/2	3/2	-3	6

Table 4.13: $h_{3/2}(\tau)$

Graphic representation of Lemma 4.3.2 is:



The functor $\Phi_{1/2,3/2}$ is an equivalence of $\mathcal{O}_{1/2}/\mathcal{O}^+_{1/2}$ and $\mathcal{O}_{3/2}/\mathcal{O}^-_{3/2}$, we can conclude that the modules $L(\tau) = L_{3/2}(\tau)$ for $\tau \in \{\mathbf{4}_+, \mathbf{4}_-, \mathbf{5}_+, \mathbf{5}_-, \mathbf{3}_+, \widetilde{\mathbf{3}}_+, \mathbf{1}_-\}$ have the Grothendieck group expressions analogous to those in $c = 1/2$ case.

The argument from the previous chapter shows that $L(\mathbf{1}_+)$ is finite-dimensional. Calculating the rank of the form B_{15} and the character on the kernel lets us conclude that $J(\mathbf{1}_+)_{-6} \cong \mathbf{3}_- \oplus \widetilde{\mathbf{3}}_-$. Solving the system of equations $\dim L(\mathbf{1}_+)_k = \dim L(\mathbf{1}_+)[-k]$ in $n_{\mathbf{1}_+,\sigma}$ (it is enough to do so for $k = 3, 6, 9, 24$) gives all the coefficients of the Grothendieck group, except $n_{\mathbf{1}_+,\mathbf{3}_+}$ and $n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+}$, for which we can only conclude that their sum is 2. Then we look at the H_3 characters on spaces $L(\mathbf{1}_+)_9$ and $L(\mathbf{1}_+)_{-9}$; the condition that they must be invariant under the Galois group action $\sqrt{5} \mapsto -\sqrt{5}$ implies that $n_{\mathbf{1}_+,\mathbf{3}_+} = n_{\mathbf{1}_+,\widetilde{\mathbf{3}}_+}$. This gives us the desired formula for $L(\mathbf{1}_+)$ in terms of $M(\sigma)$.

To describe $L(\mathbf{3}_-)$ and $L(\widetilde{\mathbf{3}}_+)$, we first use MAGMA to compute the rank of the form B on the $\mathbf{h}$ weight space -3. In both cases we get that there is a 5 dimensional kernel. Writing out the dimensions of graded pieces we can again conclude that these modules are finite-dimensional, with the Grothendieck group expressions

$$L(\mathbf{3}_-) = M(\mathbf{3}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) + n_{\mathbf{3}_-,\mathbf{3}_+} M(\mathbf{3}_+) + n_{\mathbf{3}_-,\widetilde{\mathbf{3}}_+} M(\widetilde{\mathbf{3}}_+)$$

$$L(\widetilde{\mathbf{3}}_-) = M(\widetilde{\mathbf{3}}_-) - M(\mathbf{5}_+) + M(\mathbf{5}_-) + n_{\widetilde{\mathbf{3}}_-,\mathbf{3}_+} M(\mathbf{3}_+) + n_{\widetilde{\mathbf{3}}_-,\widetilde{\mathbf{3}}_+} M(\widetilde{\mathbf{3}}_+),$$

with $n_{\mathbf{3}_-,\mathbf{3}_+} + n_{\mathbf{3}_-,\widetilde{\mathbf{3}}_+} = n_{\widetilde{\mathbf{3}}_-,\mathbf{3}_+} + n_{\widetilde{\mathbf{3}}_-,\widetilde{\mathbf{3}}_+} = -1$. Finally, looking at the trace of an element (12345) on $L(\mathbf{3}_-)$ and $L(\widetilde{\mathbf{3}}_-)$, which of course needs to be 0, we can conclude that all $n_{\tau,\sigma}$ are as in the statement of the theorem.

By inspection, all modules in $\mathcal{O}_{1,3/2}$ contain an H_3-invariant. We conclude:

Lemma 4.9.2. *Functors $\Phi_{c,c+1}\colon \mathcal{O}_{1,c} \to \mathcal{O}_{c+1}$ are equivalences of categories for $c = r/2$, r odd, $r \geq 3$.*

This lemma allows us to derive formulas for Grothendieck group expressions of $L_{1,c}(\tau)$ in terms of $M_{1,c}(\tau)$ for all $c = r/2, r > 3$. It is used in the proof of Theorem 4.2.1.

Chapter 5

Representations of Rational Cherednik Algebras Associated to the Complex Reflection Group G_{12}

5.1 The Group G_{12}

In this chapter, we focus on rational Cherednik algebras associated to the complex reflection group G_{12} in the Shephard-Todd classification [42]. It is a group of order 48, given by generators and relations (see [13]) as

$$G_{12} = \left\langle e, f, g \mid e^2 = f^2 = g^2 = 1, (efg)^4 = (fge)^4 = (gef)^4 \right\rangle.$$

Alternatively, it can be realized as $\mathrm{GL}_2(\mathbb{F}_3)$ or as a nonsplit central extension of S_4 by $\mathbb{Z}_2$. More precisely, there is a short exact sequence of groups

$$1 \to \mathbb{Z}_2 \to G_{12} \to S_4 \to 1$$

with the map $\mathbb{Z}_2 \to G_{12}$ given by $-1 \mapsto (efg)^4$ and the map $G_{12} \to S_4$ given by $e \mapsto (12), f \mapsto (34), g \mapsto (23)$. Its reflection representation $\mathfrak{h}$ is two dimensional. For

$\zeta = e^{\pi i/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, the reflection action is given by

$$e \mapsto \frac{1}{2}\begin{pmatrix} \zeta^3-\zeta & -\zeta^3+\zeta \\ -\zeta^3+\zeta & -\zeta^3+\zeta \end{pmatrix} \qquad f \mapsto \frac{1}{2}\begin{pmatrix} \zeta^3-\zeta & \zeta^3-\zeta \\ \zeta^3-\zeta & -\zeta^3+\zeta \end{pmatrix} \qquad g \mapsto \begin{pmatrix} 0 & -\zeta \\ \zeta^3 & 0 \end{pmatrix}.$$

The representatives of conjugacy classes in G_{12} are $\{id, (efg)^4, e, eg, ef, fg, efg, egf\}$. It contains 12 reflections, all in the same conjugacy class with a representative e. The character table of G_{12} is given in table 5.1.

	1	$(efg)^4$	e	eg	ef	fg	efg	egf
size	1	1	12	8	6	8	6	6
order	1	2	2	3	4	6	8	8
$\mathbf{1}_+$	1	1	1	1	1	1	1	1
$\mathbf{1}_-$	1	1	-1	1	1	1	-1	-1
$\mathbf{2}$	2	2	0	-1	2	-1	0	0
$\mathbf{2}_+$	2	-2	0	-1	0	1	$\sqrt{-2}$	$-\sqrt{-2}$
$\mathbf{2}_-$	2	-2	0	-1	0	1	$-\sqrt{-2}$	$\sqrt{-2}$
$\mathbf{3}_+$	3	3	1	0	-1	0	-1	-1
$\mathbf{3}_-$	3	3	-1	0	-1	0	1	1
$\mathbf{4}$	4	-4	0	1	0	-1	0	0

Table 5.1: Character table of G_{12}

As stated in the character table, G_{12} has two one-dimensional representations: the trivial one we call $\mathbf{1}_+$ and the signum one we call $\mathbf{1}_-$. The reflection representation is written as $\mathfrak{h} \cong \mathbf{2}_+$ and its dual is $\mathfrak{h}^* \cong \mathbf{2}_-$. Projection to S_4 gives, in addition to both one dimensional representations, another three representations $\mathbf{2}$, $\mathbf{3}_+$ and $\mathbf{3}_-$. Finally, there is a four dimensional representation that can be realized as $\mathbf{4} \cong \mathbf{2}\otimes\mathbf{2}_+ \cong \mathbf{2}\otimes\mathbf{2}_-$. It should be noted that the names of the representations are chosen to encode the dimension and the result of tensoring with the signum representation; for example, $\mathbf{2}_- \otimes \mathbf{1}_- \cong \mathbf{2}_+$ and $\mathbf{2} \otimes \mathbf{1}_- \cong \mathbf{2}$.

5.2 Main theorem

In this section we list the main results, which describe the structure of category $\mathcal{O}_{1,c}$ for an arbitrary complex parameter c. We will write all fractions r/d reduced, and assume $r, d > 0$. Whenever c and the fact we are working in the Grothendieck group are clear from context, we write $L(\tau), M(\tau)$ instead of $[L_{1,c}(\tau)]$, $[M_{1,c}(\tau)]$ for readability.

Theorem 5.2.1. *1. If c is not of the form*

$$c = m/12,\ m \in \mathbb{Z},\ m \equiv 1,3,4,5,6,7,8,9,11 (\mathrm{mod}\, 12),$$

then $\mathcal{O}_{1,c}(G_{12}, \mathfrak{h})$ is semisimple and $M_{1,c}(\tau) = L_{1,c}(\tau)$ for all τ.

2. If

$$[L_{1,r/d}(\tau)] = \sum_{\sigma} n_{\tau,\sigma}[M_{1,r/d}(\sigma)],$$

then

$$[L_{1,-r/d}(\mathbf{1}_- \otimes \tau)] = \sum_{\sigma} n_{\tau,\sigma}[M_{1,-r/d}(\mathbf{1}_- \otimes \sigma)].$$

In particular, $\dim(L_{1,r/d}(\tau)) < \infty$ *iff* $\dim(L_{1,-r/d}(\mathbf{1}_- \otimes \tau)) < \infty$*, and the character formulas for $c = -r/d$ are easily derived from those for $c = r/d$.*

3. The expressions for $[L_{1,r/d}(\tau)]$ for $r > 0$ and $d \in \{2,3,4,12\}$ are given below. We include characters for those $L_{1,r/d}(\tau)$ that are finite-dimensional. For all pairs (c, τ) that don't appear on the list, $M_{1,c}(\tau) = L_{1,c}(\tau)$.

- $\mathbf{d = 12, r \equiv 1, 11, 17, 19 (\mathrm{mod}\, 24)}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{2}_-) + M(\mathbf{1}_-) \\ L(\mathbf{2}_-) &= M(\mathbf{2}_-) - M(\mathbf{1}_-) \end{aligned}$$

$L_{r/12}(\mathbf{1}_+)$ has dimension r^2, and character

$$\mathrm{ch}_{L_{1,r/12}(\mathbf{1}_+)}(z, g) = \frac{\det_{\mathfrak{h}^*}(1 - z^r g)}{\det_{\mathfrak{h}^*}(1 - zg)} \cdot \chi_{\mathbf{1}_+}(g) \cdot z^{1-r}\ .$$

- $\mathbf{d = 12, r \equiv 5, 7, 13, 23 (\mathrm{mod}\, 24)}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{2}_+) + M(\mathbf{1}_-) \\ L(\mathbf{2}_+) &= M(\mathbf{2}_+) - M(\mathbf{1}_-) \end{aligned}$$

$L_{r/12}(\mathbf{1}_+)$ has dimension r^2, and character

$$\mathrm{ch}_{L_{1,r/12}(\mathbf{1}_+)}(z,g) = \frac{\det_{\mathfrak{h}}(1 - z^r g)}{\det_{\mathfrak{h}^*}(1 - zg)} \cdot \chi_{\mathbf{1}_+}(g) \cdot z^{1-r} \,.$$

- $\mathbf{d = 4, r \equiv 1, 3 (\mathrm{mod}\, 8)}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{3}_+) + M(\mathbf{2}_+) \\ L(\mathbf{2}_+) &= M(\mathbf{2}_+) - M(\mathbf{3}_-) + M(\mathbf{1}_-) \\ L(\mathbf{3}_+) &= M(\mathbf{3}_+) - M(\mathbf{2}_+) - M(\mathbf{4}) + M(\mathbf{3}_-) \\ L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{1}_-) \\ L(\mathbf{4}) &= M(\mathbf{4}) - M(\mathbf{3}_-) \end{aligned}$$

$L_{1,r/4}(\mathbf{1}_+)$ has dimension $3r^2$, and character

$$\mathrm{ch}_{L_{1,r/4}(\mathbf{1}_+)}(z,g) = \frac{\det_{\mathfrak{h}^*}(1 - z^r g)}{\det_{\mathfrak{h}^*}(1 - zg)} \cdot (\chi_{\mathbf{1}_+}(g) \cdot z^{1-3r} + \chi_{\mathbf{2}_-}(g) \cdot z^{1-2r}) \,.$$

$L_{1,r/4}(\mathbf{2}_+)$ has dimension $3r^2$, and character

$$\mathrm{ch}_{L_{1,r/4}(\mathbf{2}_+)}(z,g) = \frac{\det_{\mathfrak{h}^*}(1 - z^r g)}{\det_{\mathfrak{h}^*}(1 - zg)} \cdot (\chi_{\mathbf{2}_+}(g) \cdot z + \chi_{\mathbf{1}_+}(g) \cdot z^{1+r}) \,.$$

$L_{1,r/4}(\mathbf{3}_+)$ has dimension $3r^2$, and character

$$\mathrm{ch}_{L_{1,r/4}(\mathbf{3}_+)}(z,g) = \frac{\det_{\mathfrak{h}^*}(1 - z^r g)}{\det_{\mathfrak{h}^*}(1 - zg)} \cdot \chi_{\mathbf{3}_+}(g) \cdot z^{1-r} \,.$$

- $\mathbf{d=4, r\equiv 5,7 (\mathrm{mod}\, 8)}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{3}_+) + M(\mathbf{2}_-) \\ L(\mathbf{2}_-) &= M(\mathbf{2}_-) - M(\mathbf{3}_-) + M(\mathbf{1}_-) \\ L(\mathbf{3}_+) &= M(\mathbf{3}_+) - M(\mathbf{2}_-) - M(\mathbf{4}) + M(\mathbf{3}_-) \\ L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{1}_-) \\ L(\mathbf{4}) &= M(\mathbf{4}) - M(\mathbf{3}_-) \end{aligned}$$

$L_{1,r/4}(\mathbf{1}_+)$ has dimension $3r^2$, and character

$$\mathrm{ch}_{L_{1,r/4}(\mathbf{1}_+)}(z,g) = \frac{\det_{\mathfrak{h}}(1-z^r g)}{\det_{\mathfrak{h}^*}(1-zg)} \cdot (\chi_{\mathbf{1}_+}(g)\cdot z^{1-3r} + \chi_{\mathbf{2}_+}(g)\cdot z^{1-2r})\,.$$

$L_{1,r/4}(\mathbf{2}_-)$ has dimension $3r^2$, and character

$$\mathrm{ch}_{L_{1,r/4}(\mathbf{2}_-)}(z,g) = \frac{\det_{\mathfrak{h}}(1-z^r g)}{\det_{\mathfrak{h}^*}(1-zg)} \cdot (\chi_{\mathbf{2}_-}(g)\cdot z + \chi_{\mathbf{1}_+}(g)\cdot z^{1+r})\,.$$

$L_{1,r/4}(\mathbf{3}_+)$ has dimension $3r^2$, and character

$$\mathrm{ch}_{L_{1,r/4}(\mathbf{3}_+)}(z,g) = \frac{\det_{\mathfrak{h}}(1-z^r g)}{\det_{\mathfrak{h}^*}(1-zg)} \cdot \chi_{\mathbf{3}_+}(g)\cdot z^{1-r}\,.$$

- $\mathbf{d=3, r\equiv 1,2 (\mathrm{mod}\, 3)}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{2}) + M(\mathbf{1}_-) \\ L(\mathbf{2}) &= M(\mathbf{2}) - M(\mathbf{1}_-) \end{aligned}$$

$L_{1,r/3}(\mathbf{1}_+)$ has dimension $16r^2$, and character

$$\mathrm{ch}_{L_{1,r/3}(\mathbf{1}_+)}(z,g) = \frac{\det_{\mathfrak{h}^*}(1-z^r g)}{\det_{\mathfrak{h}^*}(1-zg)} \cdot (\chi_{\mathbf{1}_+}(g)\cdot z^{1-4r} + \chi_{\mathbf{2}_-}(g)\cdot z^{1-3r} + \chi_{\mathbf{3}_+}(g)\cdot z^{1-2r} +$$

$$+\chi_{\mathbf{4}}(g)\cdot z^{1-r} + \chi_{\mathbf{3}_+}(g)\cdot z + \chi_{\mathbf{2}_+}(g)\cdot z^{1+r} + \chi_{\mathbf{1}_+}(g)\cdot z^{1+2r})\,.$$

- $\mathbf{d=2, r\equiv 1 (\mathrm{mod}\, 2)}$

$$\begin{aligned} L(\mathbf{1}_+) &= M(\mathbf{1}_+) - M(\mathbf{3}_+) + M(\mathbf{2}) \\ L(\mathbf{2}) &= M(\mathbf{2}) - M(\mathbf{3}_-) + M(\mathbf{1}_-) \\ L(\mathbf{3}_+) &= M(\mathbf{3}_+) - M(\mathbf{2}) - M(\mathbf{1}_-) \\ L(\mathbf{3}_-) &= M(\mathbf{3}_-) - M(\mathbf{1}_-) \end{aligned}$$

$L_{1,r/2}(\mathbf{1}_+)$ *has dimension* $12r^2$*, and character*

$$\mathrm{ch}_{L_{1,r/2}(\mathbf{1}_+)}(z,g) = \frac{\det_{\mathfrak{h}^*}(1-z^r g)}{\det_{\mathfrak{h}^*}(1-zg)} \cdot (\chi_{\mathbf{1}_+}(g)\cdot z^{1-6r} + \chi_{\mathbf{2}_-}(g)\cdot z^{1-5r} +$$

$$+\chi_{\mathbf{3}_+}(g)\cdot z^{1-4r} + \chi_{\mathbf{4}}\,(g)\cdot z^{1-3r} + \chi_{\mathbf{2}}\,(g)\cdot z^{1-2r})\,.$$

$L_{1,r/2}(\mathbf{2}\,)$ *has dimension* $12r^2$*, and character*

$$\mathrm{ch}_{L_{1,r/2}(\mathbf{2}\,)}(z,g) = \frac{\det_{\mathfrak{h}^*}(1-z^r g)}{\det_{\mathfrak{h}^*}(1-zg)} \cdot (\chi_{\mathbf{2}}\,(g)\cdot z + \chi_{\mathbf{4}}\,(g)\cdot z^{1-r} +$$

$$+\chi_{\mathbf{3}_+}(g)\cdot z^{1-2r} + \chi_{\mathbf{2}_+}(g)\cdot z^{1-3r} + \chi_{\mathbf{1}_+}(g)\cdot z^{1-4r})\,.$$

Proof. The proof of the first statement, about the values of c for which $\mathcal{O}_{1,c}(G_{12},\mathfrak{h})$ is not semisimple, is obtained using Corollary 3.2.12 and inspecting the table A.3 of Schur polynomials, which was calculated using the CHEVIE packet of the algebra software GAP [38].

The statement (2), about translating character formulas from $c<0$ to $c>0$, comes from twisting the representations using the isomorphism $H_{1,c}(G_{12},\mathfrak{h}) \cong H_{1,-c}(G_{12},\mathfrak{h})$ from Lemma 3.1.5.

We calculate the characters of $L_{1,c}(\tau)$ for $c = 1/12, 1/4, 1/3, 1/2$ explicitly in sections 5.4, 5.5, 5.6 and 5.7. The characters for $c = r/2, r/3, r/4, r/12$ for $r>1$ then follow from these results using the scaling and shift functors $\Phi_{1/d,r/d}$ from sections 3.2.9 and 3.2.10. We calculate the permutation $\varphi_{1/d,r/d}$ in Lemma 5.3.1.

□

Using this result, the expressions in the Grothendieck group for $[M_{1,r/d}(\tau)]$ in terms of $[L_{1,r/d}(\sigma)]$ are immediate, and we include them for the sake of thoroughness.

Corollary 5.2.2. *If*

$$[M_{1,r/d}(\tau)] = \sum_{\sigma} \hat{n}_{\tau,\sigma}[L_{1,r/d}(\sigma)],$$

then

$$[M_{1,-r/d}(\mathbf{1}_- \otimes \tau)] = \sum_{\sigma} \hat{n}_{\tau,\sigma}[L_{1,-r/d}(\mathbf{1}_- \otimes \sigma)] \, .$$

The Grothendieck group expressions for standard modules in terms of irreducible modules are given below for all $c = r/d > 0$. For pairs (c, τ) that don't appear on the list, $M_{1,c}(\tau) = L_{1,c}(\tau)$.

- $\mathbf{d = 12, r \equiv 1, 11, 17, 19 (\mathrm{mod}\, 24)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{2}_-) \\ M(\mathbf{2}_-) &= L(\mathbf{2}_-) + L(\mathbf{1}_-) \end{aligned}$$

- $\mathbf{d = 12, r \equiv 5, 7, 13, 23 (\mathrm{mod}\, 24)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{2}_+) \\ M(\mathbf{2}_+) &= L(\mathbf{2}_+) + L(\mathbf{1}_-) \end{aligned}$$

- $\mathbf{d = 4, r \equiv 1, 3 (\mathrm{mod}\, 8)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{3}_+) + L(\mathbf{4}) \\ M(\mathbf{2}_+) &= L(\mathbf{2}_+) + L(\mathbf{3}_-) \\ M(\mathbf{3}_+) &= L(\mathbf{3}_+) + L(\mathbf{2}_+) + L(\mathbf{4}) + L(\mathbf{3}_-) \\ M(\mathbf{3}_-) &= L(\mathbf{3}_-) + L(\mathbf{1}_-) \\ M(\mathbf{4}) &= L(\mathbf{4}) + L(\mathbf{3}_-) + L(\mathbf{1}_-) \end{aligned}$$

- $\mathbf{d=4, r \equiv 5, 7 (\mathrm{mod}\, 8)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{3}_+) + L(\mathbf{4}) \\ M(\mathbf{2}_-) &= L(\mathbf{2}_-) + L(\mathbf{3}_-) \\ M(\mathbf{3}_+) &= L(\mathbf{3}_+) + L(\mathbf{2}_-) + L(\mathbf{4}) + M(\mathbf{3}_-) \\ M(\mathbf{3}_-) &= L(\mathbf{3}_-) + L(\mathbf{1}_-) \\ M(\mathbf{4}) &= L(\mathbf{4}) + L(\mathbf{3}_-) + L(\mathbf{1}_-) \end{aligned}$$

- $\mathbf{d=3, r \equiv 1, 2 (\mathrm{mod}\, 3)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{2}) \\ M(\mathbf{2}) &= L(\mathbf{2}) + L(\mathbf{1}_-) \end{aligned}$$

- $\mathbf{d=2, r \equiv 1 (\mathrm{mod}\, 2)}$

$$\begin{aligned} M(\mathbf{1}_+) &= L(\mathbf{1}_+) + L(\mathbf{3}_+) + L(\mathbf{1}_-) \\ M(\mathbf{2}) &= L(\mathbf{2}) + L(\mathbf{3}_-) \\ M(\mathbf{3}_+) &= L(\mathbf{3}_+) + L(\mathbf{2}) + L(\mathbf{3}_-) + L(\mathbf{1}_-) \\ M(\mathbf{3}_-) &= L(\mathbf{3}_-) + L(\mathbf{1}_-) \end{aligned}$$

It is a direct calculation to derive the set of aspherical values for G_{12} from this data. As the description of such values for any complex reflection group is an interesting open question, we include the list here.

Corollary 5.2.3. *The set of aspherical values of for G_{12} is*

$$\Sigma(G_{12}, \mathfrak{h}) = \left\{ \frac{1}{4}, \frac{-1}{2}, \frac{-1}{3}, \frac{-2}{3}, \frac{-1}{4}, \frac{-3}{4}, \frac{-5}{4}, \frac{-1}{12}, \frac{-5}{12}, \frac{-7}{12}, \frac{-11}{12}, \right\}.$$

Proof. It follows from Theorem 5.2.1. It is obviously enough to inspect the irreducible modules and check for which c there exist $L_{1,c}(\tau)$ with no nontrivial G_{12}-invariants.

According to Theorem 4.1 in [9], a module in $\mathcal{O}_{1,c}(W,\mathfrak{h})$ which has no nontrivial W-invariants is either finite-dimensional or induced from a module in $\mathcal{O}_{1,c}(W',\mathfrak{h}')$, where W' is a parabolic subgroup of W, and c is aspherical for W'. It is easy to see that the only nontrivial parabolic subgroups of G_{12} are isomorphic to $\mathbb{Z}_2$, and that the only aspherical value for $\mathbb{Z}_2$ is $c=-1/2$. Indeed, the modules $L_{-1/2}(\mathbf{1}_-), L_{-1/2}(\mathbf{2}), L_{-1/2}(\mathbf{3}_-)$ have no W-invariants, and $-1/2$ is aspherical for G_{12}.

We are left with the task of examining the finite-dimensional modules for invariants. First assume $c>0$. It can be read off from the character formulas for the finite-dimensional modules from 5.2.1 that all the $L_{1,r/d}(\tau)$, for $0<r/d<1$, contain an invariant, except $L_{r/4}(\mathbf{3}_+)$. Checking values $r/4$ one by one we see that $c=1/4$ is indeed aspherical (with $L_{1/4}(\mathbf{3}_+)$ containing no invariant), $c=3/4$ and $c=5/4$ are spherical ($L_{3/4}(\mathbf{3}_+)$ and $L_{3/4}(\mathbf{3}_+)$ contain $\mathbf{1}_+\subset \mathbf{S^2}\mathfrak{h}^*\otimes\mathbf{3}_+$). Finally, from the properties of shift functors at the end of section 3.2.10 it follows that if c is aspherical, then every module in $\mathcal{O}_{c+1}$ contains both a W-invariant and a W-antiinvariant, so in particular $c+1$ is spherical too, and conclude that the only positive aspherical value for G_{12} is $1/4$.

Next, it is clear from the equivalence $\mathcal{O}_{1,c}\to\mathcal{O}_{1,-c}$ which is realized by twisting by the isomorphism $H_{1,c}(W,\mathfrak{h})\to H_{1,-c}(W,\mathfrak{h})$ that for $c<0$ the module $L_{1,c}(\tau)$ has no W-invariants if and only if the corresponding module $L_{-c}(\mathbf{1}_-\otimes\tau)$ has no W-antiinvariants. A similar computation as above, case by case, finishes the proof of the corollary. □

Finally, this data shows that, as expected,Theorem 3.2.4 (cited from [22]) about supports of representations $L_{1,c}(\mathrm{triv})$ can not be trivially extended from Coxeter groups to complex reflection groups. A corollary (3.3 in [22]) of Theorem 3.2.4 is that for W a Coxeter group, $c=n/m\in\mathbb{Q}-\mathbb{Z}$, $W'\subset W$ a parabolic subgroup, and $d_i(W)$ degrees of a reflection group, the module $\dim L_{1,c}(\mathrm{triv})$ is finite-dimensional if and only if

$$\#\{i:m\mid d_i(W)\}>\#\{i:m\mid d_i(W')\}$$

The degrees of G_{12} are $d_1(G_{12}) = 6$ and $d_2(G_{12}) = 8$ (see [13]). Moreover, G_{12} has a maximal parabolic subgroup isomorphic to $\mathbb{Z}_2$, with the only degree being 2. By Theorem 5.2.1, $L_{1,1/12}(\mathrm{triv})$ is finite-dimensional. However, $12 \nmid 6$, $12 \nmid 8$, and $12 \nmid 2$, so this does not hold for G_{12}.

5.3 Preliminary calculations

5.3.1 The permutation φ

Proposition 5.3.1. *For $d \in \{2, 3, 4, 12\}$ and $r \in \mathbb{Z}_{>0}$ relatively prime to d, the permutation $\varphi = \varphi_{1/d,r/d}$ realizing the equivalence of categories $\Phi_{1/d,r/d}$ is as follows:*

- *If $d = 2$, then $\varphi = \mathrm{id}$;*
- *If $d = 3$, $r \equiv 1, 2 (\mathrm{mod}\, 6)$, then $\varphi = \mathrm{id}$;*
- *If $d = 3$, $r \equiv 4, 5 (\mathrm{mod}\, 6)$, then $\varphi = (\mathbf{2}_+ \mathbf{2}_-)$;*
- *If $d = 4$, $r \equiv 1, 3 (\mathrm{mod}\, 8)$ then $\varphi = \mathrm{id}$;*
- *If $d = 4$, $r \equiv 5, 7 (\mathrm{mod}\, 8)$ then $\varphi = (\mathbf{2}_+ \mathbf{2}_-)$;*
- *If $d = 12$, $r \equiv 1, 11, 17, 19 (\mathrm{mod}\, 24)$ then $\varphi = \mathrm{id}$;*
- *If $d = 12$, $r \equiv 5, 7, 13, 23 (\mathrm{mod}\, 24)$ then $\varphi = (\mathbf{2}_+ \mathbf{2}_-)$;*

Proof. The calculation is explained in section 3.2.9, and analogous to the one in Lemma 4.3.1. As there, it is enough to calculate $\varphi_{1/d,r/d}$ for $d = 2, 3, 4, 12$ and $1 \leq r < 2d$. The permutation for r differs from the one for $r + d$ by a transposition $(\mathbf{2}_+ \mathbf{2}_-)$, so it is enough to calculate them for $1 \leq r < d$. For $r = 1$, $\varphi = \mathrm{id}$. We list the values of the Galois group element g and η for all the remaining cases.

- $d = 2$, $\varphi = \mathrm{id}$;
- $d = 3$, $r = 2$, $g =$ complex conjugation, $\eta = -1$, $\varphi = \mathrm{id}$;

- $d = 4$, $r = 3$, $g =$ complex conjugation, $\eta = -1$, $\varphi = \mathrm{id}$;
- $d = 12$, $r = 5$, $\xi = e^{\pi i/12}$ $g(\xi) = \xi^{-7}$, $\eta = -1$, $\varphi = (\mathbf{2}_+\mathbf{2}_-)$;
- $d = 12$, $r = 7$, $\xi = e^{\pi i/12}$ $g(\xi) = \xi^{7}$, $\eta = 1$, $\varphi = (\mathbf{2}_+\mathbf{2}_-)$;
- $d = 12$, $r = 11$, $\xi = e^{\pi i/12}$ $g(\xi) = \xi^{-1}$, $\eta = -1$, $\varphi = \mathrm{id}$.

□

We use this permutation to get formulas for transforming characters of irreducible modules: if

$$[L_{1,1/d}(\tau)] = \sum_\sigma n_{\tau,\sigma}[M_{1,1/d}(\sigma)],$$

then

$$[L_{1,r/d}(\varphi_{1/d,r/d}(\tau))] = \sum_\sigma n_{\tau,\sigma}[M_{1,r/d}(\varphi_{1/d,r/d}(\sigma))].$$

Using this and the formula for the character of $M_{1,c}(\tau)$, we get:

Corollary 5.3.2. *Let $\gamma_{1/d,r/d} \in \mathrm{Gal}(\mathbb{Q}(\sqrt{-2})/\mathbb{Q})$ be the identity or complex conjugation, depending on whether $\varphi_{1/d,r/d}$ is identity or transposition $(\mathbf{2}_+, \mathbf{2}_-)$. Then*

$$\mathrm{ch}_{L_{1,r/d}(\tau)}(z,g) = \frac{\det_{\varphi_{1/d,r/d}(\mathfrak{h}^*)}(1 - z^r g)}{\det_{\mathfrak{h}^*}(1 - zg)} \cdot z^{1-r} \cdot \gamma_{1/d,r/d}(\mathrm{ch}_{L_{1,1/d}(\tau)}(g, z^r)).$$

In particular, $L_{1,1/d}(\tau)$ is finite-dimensional iff $L_{1,r/d}(\varphi_{1/d,r/d}(\tau))$ is, with

$$\dim(L_{1,r/d}(\varphi_{1/d,r/d}(\tau))) = r^2 \cdot \dim(L_{1,1/d}(\tau)).$$

5.3.2 Lowest h-weights

The constants $h_c(\tau)$ by which the grading element

$$\mathbf{h} = \sum_i x_i y_i + \frac{\dim \mathfrak{h}}{2} - c \sum_{s \in S} s$$

acts on lowest weights of modules in $\mathcal{O}_{1,c}$ is easy to calculate directly, and given in Table 5.2.

τ	1_+	1_-	2	2_+	2_-	3_+	3_-	4
$\sum_{s\in S} s \mid_\tau$	12	-12	0	0	0	4	-4	0
$h_c(\tau)$	1-12c	1+12c	1	1	1	1-4c	1+4c	1

Table 5.2: Lowest **h**-weights for G_{12}

5.3.3 A-matrix

The following lemma appears in [45], and provides a simple yet computationally very effective necessary condition on the structure constants $n_{\tau,\sigma}$ for the module $L_{1,c}(\tau)$ to be finite-dimensional.

Fix τ and c, and suppose the representation $L_{1,c}(\tau)$ is finite-dimensional. Write $[L_{1,c}(\tau)] = \sum_\sigma n_{\tau,\sigma}[M_{1,c}(\sigma)]$. For $g \in W$, consider $\det_{\mathfrak{h}^*}(1 - gz)$ as a polynomial (or a polynomial times a fixed fractional power of z) in z, and let $z_{g,1}, z_{g,2}$ be its roots. Clearly, $g \mapsto \{z_{g,1}, z_{g,2}\}$ is constant on conjugacy classes.

Define a matrix A, with columns indexed by $\widehat{W}$ and rows indexed by the ordered pairs (g, i), for g a conjugacy class in W and $i = 1, 2$, by setting the element in the row labeled by (g, i) and column labeled by σ to be equal to $z_{g,i}^{h_c(\sigma)}\mathrm{Tr}_\sigma(g)$.

Lemma 5.3.3. *If $L_{1,c}(\tau)$ is finite-dimensional, then the column vector $[n_{\tau,\sigma}]_{\sigma\in\widehat{W}}$ is a nullvector of A.*

Proof. Since $L_{1,c}(\tau)$ is finite-dimensional, it follows that its character is a finite sum of powers of z. On the other hand, it can be written as

$$\mathrm{ch}_{L_{1,c}(\tau)}(z, g) = \frac{\sum_\sigma n_{\tau,\sigma} \cdot \mathrm{Tr}_\sigma(g) \cdot z^{h_c(\sigma)}}{\det_{\mathfrak{h}^*}(1 - zg)}.$$

For every $z_{g,i}$, the denominator vanishes; since the character doesn't have a pole, the numerator must vanish as well. This gives the desired claim. □

The matrix A is easy to compute and gives a strong condition on $n_{\tau,\sigma}$. We compute it using MAGMA, for variable c. The matrix and basis vectors for its null-spaces at $c = 1/12, 1/4, 1/3, 1/2$ are given in tables A.1 and A.2 of the Appendix.

5.4 Calculations for $c = 1/12$

Evaluating the expressions in table 5.2 at $c = 1/12$, we get the lowest $\mathbf{h}$-weights in table 5.3.

τ	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}$	$\mathbf{2}_+$	$\mathbf{2}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\mathbf{4}$
$h_{1/12}(\tau)$	0	2	1	1	1	2/3	4/3	1

Table 5.3: Lowest $\mathbf{h}$-weights for $c = 1/12$

We use comments from section 3.2.4, saying that the necessary condition for $n_{\tau,\sigma}$ to be nonzero is that $h_{1/12}(\sigma) - h_{1/12}(\tau)$ is a positive integer. From this immediately follows that

$$M_{1,1/12}(\mathbf{1}_-) = L_{1,1/12}(\mathbf{1}_-),$$

$$M_{1,1/12}(\mathbf{3}_+) = L_{1,1/12}(\mathbf{3}_+),$$

$$M_{1,1/12}(\mathbf{3}_-) = L_{1,1/12}(\mathbf{3}_-).$$

By the same condition, the only $M_{1,1/12}(\sigma)$ that can appear in the Grothendieck group expressions for $L_{1,1/12}(\tau)$ for $\tau = \mathbf{2},\mathbf{2}_+,\mathbf{2}_-,\mathbf{4}$ is $\sigma = \mathbf{1}_-$. In those cases the difference $h_c(\sigma) - h_c(\tau)$ is 1, so the question is equivalent to checking whether there exists a subspace isomorphic to σ in $\mathfrak{h}^* \otimes \tau$ which consists of singular vectors. We use Lemma 3.2.3 which states that when $h_c(\sigma) - h_c(\tau) = 1$, any such subrepresentation σ will consist of singular vectors. Decomposing these group representations into irreducible subrepresentations, we get

$$S^1\mathfrak{h}^* \otimes \mathbf{2} \cong \mathbf{4} \qquad S^1\mathfrak{h}^* \otimes \mathbf{2}_+ \cong \mathbf{1}_+ \oplus \mathbf{3}_-$$

$$S^1\mathfrak{h}^* \otimes \mathbf{2}_- \cong \mathbf{1}_- \oplus \mathbf{3}_+ \qquad S^1\mathfrak{h}^* \otimes \mathbf{4} \cong \mathbf{2} \oplus \mathbf{3}_+ \oplus \mathbf{3}_-.$$

From this it follows

$$M_{1,1/12}(\mathbf{2}) = L_{1,1/12}(\mathbf{2}),$$

$$M_{1,1/12}(\mathbf{2}_+) = L_{1,1/12}(\mathbf{2}_+),$$

$$M_{1,1/12}(\mathbf{4}) = L_{1,1/12}(\mathbf{4}),$$

$$[L_{1,1/12}(\mathbf{2}_-)] = [M_{1,1/12}(\mathbf{2}_-)] - [M_{1,1/12}(\mathbf{1}_-)] .$$

Similarly, we calculate the decomposition of $S^1\mathfrak{h}^*\otimes\mathbf{1}_+ M_{1,1/12}(\mathbf{1}_+)$; it is $S^1\mathfrak{h}^*\otimes\mathbf{1}_+ = \mathbf{2}_-$. By lemma 3.2.3, the entire graded piece $S^1\mathfrak{h}^* \otimes \mathbf{1}_+ = M_{1,1/12}(\mathbf{1}_+)_1$ consists of singular vectors, and $L_{1,1/12}(\mathbf{1}_+)$ is the trivial one dimensional representation. To express it in terms of Verma modules in the Grothendieck group, we use that $\mathbf{2}_- \subseteq S^1\mathfrak{h}^* \otimes \mathbf{1}_+$ generates a Cherednik algebra subrepresentation of $M_{1,1/12}(\mathbf{1}_+)$. This subrepresentation is isomorphic to a quotient of $M_{1,1/12}(\mathbf{2}_-)$, which we saw was either $M_{1,1/12}(\mathbf{2}_-)$ or $L_{1,1/12}(\mathbf{2}_-)$. Comparing dimensions of $M_{1,1/12}(\mathbf{1}_+)$ and $M_{1,1/12}(\mathbf{2}_-)$ in the $\mathbf{h}$-eigenspace 2, we see that $\dim(M_{1,1/12}(\mathbf{2}_-)_2) = \dim(\mathfrak{h}^* \otimes \mathbf{2}_-) = 4$, while $\dim(M_{1,1/12}(\mathbf{1}_+)_2) = \dim(S^2\mathfrak{h}^* \otimes \mathbf{1}_+) = 3$, so $M_{1,1/12}(\mathbf{2}_-)$ cannot be a submodule of $M_{1,1/12}(\mathbf{1}_+)$. Dimensions of $L_{1,1/12}(\mathbf{2}_-)$ and $M_{1,1/12}(\mathbf{1}_+)$ match in all higher degrees, so we conclude $J_{1,1/12}(\mathbf{1}_+) = L_{1,1/12}(\mathbf{2}_-)$ and

$$[L_{1,1/12}(\mathbf{1}_+)] = [M_{1,1/12}(\mathbf{1}_+)] - [L_{1,1/12}(\mathbf{2}_-)] = [M_{1,1/12}(\mathbf{1}_+)] - [M_{1,1/12}(\mathbf{2}_-)] + [M_{1,1/12}(\mathbf{1}_-)].$$

This module is one dimensional, and is the only finite-dimensional irreducible module at $c = 1/12$. Its character corresponds to vector $e_{1/12}$ in the table A.2 in the appendix.

5.5 Calculations for $c = 1/4$

Using table 5.2, we get lowest weights for $c = 1/4$ in table 5.4.

τ	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}$	$\mathbf{2}_+$	$\mathbf{2}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\mathbf{4}$
$h_{1/12}(\tau)$	-2	4	1	1	1	0	2	1

Table 5.4: Lowest $\mathbf{h}$-weights for $c = 1/4$

It immediately follows that

$$M_{1,1/4}(\mathbf{1}_-) = L_{1,1/4}(\mathbf{1}_-).$$

We will need the following decompositions:

$$S^2\mathfrak{h}^* \otimes \mathbf{3}_+ = \mathbf{1}_+ \oplus \mathbf{2} \ \oplus \mathbf{3}_+ \oplus \mathbf{3}_-, \quad S^4\mathfrak{h}^* \otimes \mathbf{3}_+ = \mathbf{1}_+ \oplus \mathbf{2} \ \oplus 2 \cdot \mathbf{3}_+ \oplus 2 \cdot \mathbf{3}_-$$

$$S^1\mathfrak{h}^* \otimes \mathbf{2} \ = \mathbf{4}\,, \quad S^3\mathfrak{h}^* \otimes \mathbf{2} \ = \mathbf{2}_+ \oplus \mathbf{2}_- \oplus \cdot\mathbf{4}$$

$$S^1\mathfrak{h}^* \otimes \mathbf{2}_+ = \mathbf{1}_+ \oplus \mathbf{3}_-, \quad S^3\mathfrak{h}^* \otimes \mathbf{2}_+ = \mathbf{2} \ \oplus \mathbf{3}_+ \oplus \mathbf{3}_-$$

$$S^1\mathfrak{h}^* \otimes \mathbf{2}_- = \mathbf{1}_- \oplus \mathbf{3}_+, \quad S^3\mathfrak{h}^* \otimes \mathbf{2}_- = \mathbf{2} \ \oplus \mathbf{3}_+ \oplus \mathbf{3}_-$$

$$S^1\mathfrak{h}^* \otimes \mathbf{3}_+ = \mathbf{2}_+ \oplus \mathbf{4}$$

$$S^1\mathfrak{h}^* \otimes \mathbf{4} \ = \mathbf{2} \ \oplus \mathbf{3}_+ \oplus \mathbf{3}_-, \quad S^3\mathfrak{h}^* \otimes \mathbf{4} \ = \mathbf{1}_+ \oplus \mathbf{1}_- \oplus \mathbf{2} \ \oplus 2 \cdot \mathbf{3}_+ \oplus 2 \cdot \mathbf{3}_-$$

So

$$M_{1,1/4}(\mathbf{2}\ ) = L_{1,1/4}(\mathbf{2}\ )$$

$$M_{1,1/4}(\mathbf{2}_-) = L_{1,1/4}(\mathbf{2}_-)$$

are simple, because $\mathbf{3}_-$ and $\mathbf{1}_+$ don't appear in the right weight space.

We will use the expression for $L_{1,1/4}(\mathbf{2}_+)$ to study $L_{1,1/4}(\mathbf{3}_-)$. First of all, lemma 3.2.3 implies that $M_{1,1/4}(\mathbf{2}_+)$ contains a quotient of $M_{1,1/4}(\mathbf{3}_-)$. Comparing dimensions in $\mathbf{h}$-weight space 4, we see that this module cannot be $M_{1,1/4}(\mathbf{3}_-)$. So,

$$[L_{1,1/4}(\mathbf{3}_-)] = [M_{1,1/4}(\mathbf{3}_-)] - [M_{1,1/4}(\mathbf{1}_-)]$$

$$[L_{1,1/4}(\mathbf{2}_+)] = [M_{1,1/4}(\mathbf{2}_+)] - [M_{1,1/4}(\mathbf{3}_-)] + [M_{1,1/4}(\mathbf{1}_-)].$$

The latter is finite-dimensional.

To analyze $M_{1,1/4}(\mathbf{4}\,)$, first use lemma 3.2.3 to conclude its Grothendieck group expression is

$$[L_{1,1/4}(\mathbf{4}\,)] = [M_{1,1/4}(\mathbf{4}\,)] - [M_{1,1/4}(\mathbf{3}_-)] + n_{\mathbf{4}\,\mathbf{1}_-}[M_{1,1/4}(\mathbf{1}_-)].$$

To find $n_{\mathbf{4}\,,\mathbf{1}_-}$, we calculate the rank of the form B in $\mathbf{h}$-weight space 4 in MAGMA. We find it is 7. Since $\dim_3(M_{1,1/4}(\mathbf{4}\,)) = 16$, and $\dim_2(M_{1,1/4}(\mathbf{3}_-)) = 9$, there exist two possibilities. Either $M_{1,1/4}(\mathbf{1}_-)$ appears as a submodule of $M_{1,1/4}(\mathbf{4}\,)$ and $M_{1,1/4}(\mathbf{3}_-)$, or as a submodule of neither. While our methods cannot distinguish the two cases, in both we can conclude

$$[L_{1,1/4}(\mathbf{4}\,)] = [M_{1,1/4}(\mathbf{4}\,)] - [M_{1,1/4}(\mathbf{3}_-)]\,.$$

By lemma 3.2.3, we know that

$$[L_{1,1/4}(\mathbf{3}_+)] = [M_{1,1/4}(\mathbf{3}_+)] - [L_{1,1/4}(\mathbf{2}_+)] - [L_{1,1/4}(\mathbf{4}_;)] \pm \text{modules with lowest weight} > 1.$$

Looking at dimensions in $\mathbf{h}$-weight space 1, we see that $L_{1,1/4}(\mathbf{3}_+)$ is finite-dimensional. Writing down the condition that its character must be a polynomial, which we do by saying that it needs to be a linear combination of vectors in table A.2, we get

$$[L_{1,1/4}(\mathbf{3}_+)] = [M_{1,1/4}(\mathbf{3}_+)] - [M_{1,1/4}(\mathbf{2}_+)] - [M_{1,1/4}(\mathbf{4}\,)] + [M_{1,1/4}(\mathbf{3}_-)],$$

which corresponds to $-e^3_{1/4}$.

Using MAGMA, we find that the form B restricted to $M_{1/4}(\mathbf{1}_+)_0$ is zero. Therefore $M_{1,1/4}(\mathbf{1}_+)$ is finite-dimensional. Referring to table A.2 again, it is easily seen that the only possible linear combination is $e^1_{1/4} + e^3_{1/4}$, giving

$$[L_{1,1/4}(\mathbf{1}_+)] = [M_{1,1/4}(\mathbf{1}_+)] - [M_{1,1/4}(\mathbf{3}_+)] + [M_{1,1/4}(\mathbf{2}_+)]\,.$$

5.6 Calculations for $c = 1/3$

First we calculate the lowest weights $h_c(\tau)$, see table 5.5.

τ	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}$	$\mathbf{2}_+$	$\mathbf{2}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\mathbf{4}$
$h_{1/3}(\tau)$	-3	5	1	1	1	-1/3	7/3	1

Table 5.5: Lowest **h**-weights for $c = 1/3$

Since $h_{1/3}(\mathbf{1}_-) > h_{1/3}(\tau)$ for all $\tau \neq \mathbf{1}_-$, it follows that

$$M_{1,1/3}(\mathbf{1}_-) = L_{1,1/3}(\mathbf{1}_-).$$

SImilarly, $h_{1/3}(\mathbf{3}_\pm) - h_{1/3}(\tau) \notin \mathbb{Z}$ for all $\tau \neq \mathbf{3}_\pm$, so

$$M_{1,1/3}(\mathbf{3}_+) = L_{1,1/3}(\mathbf{3}_+)$$

$$M_{1,1/3}(\mathbf{3}_-) = L_{1,1/3}(\mathbf{3}_-).$$

If $M_{1,1/3}(\mathbf{1}_-)$ is contained in $M_{1,1/3}(\tau)$ as a submodule for $\tau = \mathbf{2}, \mathbf{2}_+, \mathbf{2}_-, \mathbf{4}$, then $\mathbf{1}_- \subseteq \mathbf{M}_{\mathbf{1},\mathbf{1/3}}(\tau)_\mathbf{5} = \mathbf{S^4\mathfrak{h}^*} \otimes \tau$. Decomposing $S^4\mathfrak{h}^* \otimes \tau$ in all of these cases, we get:

τ	$\mathbf{2}$	$\mathbf{2}_+$	$\mathbf{2}_-$	$\mathbf{4}$
$S^4\mathfrak{h}^* \otimes \tau$	$\mathbf{1}_+ \oplus \mathbf{1}_- \oplus \mathbf{2} \oplus \mathbf{3}_+ \oplus \mathbf{3}_-$	$\mathbf{2}_- \oplus 2 \cdot \mathbf{4}$	$\mathbf{2}_+ \oplus 2 \cdot \mathbf{4}$	$2 \cdot \mathbf{2}_+ \oplus 2 \cdot \mathbf{2}_- \oplus 2 \cdot \mathbf{4}$

Hence

$$M_{1,1/3}(\mathbf{2}_+) = L_{1,1/3}(\mathbf{2}_+)$$

$$M_{1,1/3}(\mathbf{2}_-) = L_{1,1/3}(\mathbf{2}_-)$$

$$M_{1,1/3}(\mathbf{4}) = L_{1,1/3}(\mathbf{4})$$

We see that $S^4\mathfrak{h}^* \otimes \mathbf{2}$ contains a copy of $\mathbf{1}_-$. Let us first analyze $L_{1,1/3}(\mathbf{1}_+)$ and return to the description of $L_{1,1/3}(\mathbf{2})$ after that.

$S^4\mathfrak{h}^* \otimes \mathbf{1}_+ = \mathbf{2} \oplus \mathbf{3}_+$, so it follows that $M_{1,1/3}(\mathbf{1}_+)$ can contain a quotient of $M_{1,1/3}(\mathbf{2})$. Using MAGMA, we compute the rank of the contravariant form B on

$M_{1,1/3}(\mathbf{1}_+)_1 = S^4\mathfrak{h}^* \otimes \mathbf{1}_+$. It is 3 while the dimension of this graded piece is 5, so $M_{1,1/3}(\mathbf{1}_+)$ does contain a quotient of $M_{1,1/3}(\mathbf{2}\,)$. Calculating $\dim(M_{1,1/3}(\mathbf{2}\,)_5) = 10$, while $\dim(M_{1,1/3}(\mathbf{1}_+)_5) = 9$. So we conclude that the module $M_{1,1/3}(\mathbf{2})$ cannot be simple. Combining this with the above analysis we conclude

$$[L_{1,1/3}(\mathbf{2}\,)] = [M_{1,1/3}(\mathbf{2}\,)] - [M_{1,1/3}(\mathbf{1}_-)]$$

$$[L_{1,1/3}(\mathbf{1}_+)] = [M_{1,1/3}(\mathbf{1}_+)] - [M_{1,1/3}(\mathbf{2}\,)] + [M_{1,1/3}(\mathbf{1}_-)].$$

5.7 Calculations for $c = 1/2$

Using the expressions in Table 5.2, we find the following lowest weights for $c = 1/2$.

τ	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}$	$\mathbf{2}_+$	$\mathbf{2}_-$	$\mathbf{3}_+$	$\mathbf{3}_-$	$\mathbf{4}$
$h_{1/12}(\tau)$	-5	7	1	1	1	-1	3	1

Table 5.6: Lowest $\mathbf{h}$-weights for $c = 1/2$

It immediately follows that

$$M_{1,1/2}(\mathbf{1}_-) = L_{1,1/2}(\mathbf{1}_-).$$

We will need the following decompositions:

$$S^4\mathfrak{h}^* \otimes \mathbf{3}_+ = \mathbf{1}_+ \oplus \mathbf{2} \oplus 2\cdot\mathbf{3}_+ \oplus 2\cdot\mathbf{3}_-, \quad S^8\mathfrak{h}^* \otimes \mathbf{3}_+ = \mathbf{1}_+ \oplus \mathbf{1}_- \oplus 2\cdot\mathbf{2} \oplus 4\cdot\mathbf{3}_+ \oplus 3\cdot\mathbf{3}_-$$

τ	$\mathbf{2}$	$\mathbf{2}_+$	$\mathbf{2}_-$	$\mathbf{3}_+$	$\mathbf{4}$
$S^2\mathfrak{h}^* \otimes \tau$	$\mathbf{3}_+ \oplus \mathbf{3}_-$	$\mathbf{2}_- \oplus \mathbf{4}$	$\mathbf{2}_+ \oplus \mathbf{4}$	$\mathbf{1}_+ \oplus \mathbf{2} \oplus \mathbf{3}_+ \oplus \mathbf{3}_-$	$\mathbf{2}_+ \oplus \mathbf{2}_- \oplus 2\cdot\mathbf{4}$
$S^6\mathfrak{h}^* \otimes \tau$	$\mathbf{2}_+ \oplus 2\cdot\mathbf{3}_+ \oplus 2\cdot\mathbf{3}_-$	$2\cdot\mathbf{2}_+ \oplus \mathbf{2}_- \oplus 2\cdot\mathbf{4}$	$\mathbf{2}_+ \oplus 2\cdot\mathbf{2}_- \oplus 2\cdot\mathbf{4}$		$2\cdot\mathbf{2}_+ \oplus 2\cdot\mathbf{2}_- \oplus 5\cdot\mathbf{4}$

So

$$M_{1,1/2}(\mathbf{2}_+) = L_{1,1/2}(\mathbf{2}_+)$$

$$M_{1,1/2}(\mathbf{2}_-) = L_{1,1/2}(\mathbf{2}_-)$$

$$M_{1,1/2}(\mathbf{4}\ ) = L_{1,1/2}(\mathbf{4}\ )$$

are simple because $\mathbf{3}_-$ and $\mathbf{1}_-$ do not appear in the appropriate graded pieces.

Completely analogous to previous cases, we calculate the rank of B on the 6 dimensional space $M_{1,1/2}(\mathbf{2}\ )_3$ and get that it is 3. Comparing dimensions of $M_{1,1/2}(\mathbf{2}\ )_7$ and $M_{1,1/2}(\mathbf{3}_-)_7$ (they are 14 and 15), we get

$$[L_{1,1/2}(\mathbf{2}\ )] = [M_{1,1/2}(\mathbf{2}\ )] - [M_{1,1/2}(\mathbf{3}_-)] + [M_{1,1/2}(\mathbf{1}_-)]$$

$$[L_{1,1/2}(\mathbf{3}_-)] = [M_{1,1/2}(\mathbf{3}_-)] - [M_{1,1/2}(\mathbf{1}_-)]\ .$$

Before considering $L_{1,1/2}(\mathbf{3}_+)$, let us determine $L_{1,1/2}(\mathbf{1}_+)$. Using MAGMA, we find that the rank of B on $M_{1/4}(\mathbf{1}_+)_{-1}$ is 2, while the space is 5 dimensional. So $M_{1,1/2}(\mathbf{1}_+)$ contains a quotient of $M_{1,1/2}(\mathbf{3}_+)$ starting at the $\mathbf{h}$-space -1, and is hence finite-dimensional. Since $\dim(M_{1,1/2}(\mathbf{1}_+)_1) = 7$, and $\dim(M_{1,1/2}(\mathbf{3}_+)_1) = 9$, it follows that $M_{1,1/2}(\mathbf{3}_+)$ is not simple, but contains a set of singular vectors isomorphic to $\mathbf{2}$ in $\mathbf{h}$-space 1. Using decompositions $S^2\mathfrak{h}^* \otimes \mathbf{3}_+ = \mathbf{1}_+ \oplus \mathbf{2}\ \oplus \mathbf{3}_+ \oplus \mathbf{3}_-$ and $S^6\mathfrak{h}^* \otimes \mathbf{1}_+ = \mathbf{1}_+ \oplus \mathbf{3}_+ \oplus \mathbf{3}_-$ and table A.2, we obtain

$$[L_{1,1/2}(\mathbf{1}_+)] = [M_{1,1/2}(\mathbf{1}_+)] - [M_{1,1/2}(\mathbf{3}_+)] + [M_{1,1/2}(\mathbf{2}\ )]$$

corresponding to $e^1_{1/2}$.

We know that

$$[L_{1,1/2}(\mathbf{3}_+)] = [M_{1,1/2}(\mathbf{3}_+)] - [L_{1/2}(\mathbf{2})] \pm \text{modules with lowest weights} > 1.$$

Using MAGMA, we find that B on $M_{1,1/2}(\mathbf{3}_+)_3$ has rank 9 and conclude

$$[L_{1,1/2}(\mathbf{3}_+)] = [M_{1,1/2}(\mathbf{3}_+)] - [M_{1,1/2}(\mathbf{2}\ )] + X \cdot [M_{1,1/2}(\mathbf{1}_-)]$$

for some integer X.

We will determine X using induction functors. Consider a point $a \neq 0$ on a reflection hyperplane. We know that the isotropy group of a is isomorphic to $\mathbb{Z}_2$. Let $\epsilon_+, \epsilon_- \in \widehat{\mathbb{Z}_2}$ be the trivial and sign representations. It is easily seen that the irreducible representations of $H_{1/2}(\mathbb{Z}/2\mathbb{Z}, \epsilon_-)$ have Grothendieck group expressions

$$[L_{1,1/2}(\epsilon_-)] = [M_{1,1/2}(\epsilon_-)]$$

$$[L_{1,1/2}(\epsilon_+)] = [M_{1,1/2}(\epsilon_+)] - [M_{1,1/2}(\epsilon_-)].$$

We now use proposition 3.2.5 to deduce

$$[\mathrm{Ind}_a(L_{1,1/2}(\epsilon_+))] = [M_{1,1/2}(\mathbf{1}_+)] - [M_{1,1/2}(\mathbf{1}_-)] + [M_{1,1/2}(\mathbf{3}_+)] - [M_{1,1/2}(\mathbf{3}_-)]\ .$$

Using the expressions already determined, this can be rewritten as

$$[\mathrm{Ind}_a(L_{1,1/2}(\epsilon_-))] = [L_{1,1/2}(\mathbf{1}_+)] + [L_{1,1/2}(\mathbf{2}\ )] + 2\cdot[L_{1,1/2}(\mathbf{3}_+)] - (2X+2)\cdot[L_{1,1/2}(\mathbf{1}_-)]\ .$$

Therefore $-(2X+2) \geq 0$ and $X \leq -1$.

Looking at the multiplicity of $\mathbf{1}_-$ in $S^8\mathfrak{h}^* \otimes \mathbf{3}_+$ (it is 1), in $S^6\mathfrak{h}^* \otimes \mathbf{2}$ (it is 0), it follows that $\mathbf{1}_-$ appears $1-0+X \geq 0$ times in $L_{1,1/2}(\mathbf{3}_+)_7$. This implies $X = -1$. So

$$[L_{1,1/2}(\mathbf{3}_+)] = [M_{1,1/2}(\mathbf{3}_+)] - [M_{1,1/2}(\mathbf{2}\ )] - [M_{1,1/2}(\mathbf{1}_-)].$$

Finally, we check that all modules $L_{1/2}(\tau)$ have a nontrivial G_{12}-invariant. This means $1/2$ is spherical for G_{12} and that shift functors $\Phi_{1/2,3/2}$, $\Phi_{3/2,5/2}$, etc. are all equivalences. So, the description of $\mathcal{O}_{1,1/2}$ can be used to describe $\mathcal{O}_{r/2}$ for all positive r.

Chapter 6

Rational Cherednik Algebras over Fields of Finite Characteristic

For the remaining three chapters, let $\Bbbk$ be an algebraically closed field of positive characteristic p. As before, W is a reflection group with a reflection representation $\mathfrak{h}$ over $\Bbbk$, and $\mathfrak{h}^*$ the dual representation. We continue the study of the rational Cherednik algebra $H_{t,c}(W,\mathfrak{h})$ over $\Bbbk$ from Chapter 3.

6.1 Baby Verma modules $N_{t,c}(\tau)$ and irreducible modules $L_{t,c}(\tau)$

As explained in Chapter 3, the main difference to characteristic zero is that $H_{t,c}(W,\mathfrak{h})$ has a large center, and all Verma modules $M_{t,c}(\tau)$ have large submodules. We adept the definition of category $\mathcal{O}$ to this situation.

6.1.1 Baby Verma modules

Let $(S\mathfrak{h}^*)^W$ be the subspace of W-invariants in $S\mathfrak{h}^*$, and $((S\mathfrak{h}^*)^W)_+$ the subspace of W-invariants in $S\mathfrak{h}^*$ of positive degree. At $t=1$, the subspace $((S\mathfrak{h}^*)^W)_+^p$ of p-th powers of elements from $((S\mathfrak{h}^*)^W)_+$, is central in $H_{1,c}(W,\mathfrak{h})$. As a consequence, $((S\mathfrak{h}^*)^W)_+^p M_{1,c}(\tau)$ is a proper submodule of $M_{1,c}(\tau)$.

Definition 6.1.1. *The* baby Verma module *for the algebra* $H_{1,c}(W,\mathfrak{h})$ *is the quotient*

$$N_{1,c}(\tau) = N_{1,c}(W,\mathfrak{h},\tau) = M_{1,c}(\tau)/((S\mathfrak{h}^*)^W)_+^p M_{1,c}(\tau).$$

Since $(S\mathfrak{h}^*)^W$ is graded, $N_{1,c}(\tau)$ is a graded module. The subspace $((S\mathfrak{h}^*)^W)_+^p M_{1,c}(\tau)$ is contained in Ker$B$. To see this, let $Z \in ((S\mathfrak{h}^*)^W)_+^p$ be an arbitrary homogeneous element of positive degree m, $v \in \tau$ and $y \in \mathfrak{h}$ arbitrary. Then $D_y(Z\otimes v) = (yZ).v = (Zy).v = Z.(y.v) = 0$, so $Z \otimes v$ is singular and therefore in Ker$B$.

Because of this, the form B descends to the $N_{1,c}(\tau)$, and $L_{1,c}(\tau)$ can be alternatively realized as the quotient of $N_{1,c}(\tau)$ by the kernel of the induced form.

To define baby Verma modules at $t = 0$, we use that $((S\mathfrak{h}^*)^W)_+$ is central in $H_{0,c}(W,\mathfrak{h})$, so $((S\mathfrak{h}^*)^W)_+ M_{0,c}(\tau)$ is a proper submodule of $M_{0,c}(\tau)$.

Definition 6.1.2. *The* baby Verma module *for the algebra* $H_{0,c}(W,\mathfrak{h})$ *is the quotient*

$$N_{0,c}(\tau) = N_{0,c}(W,\mathfrak{h},\tau) = M_{0,c}(\tau)/((S\mathfrak{h}^*)^W)_+ M_{0,c}(\tau).$$

By the same arguments as above, it is graded, the form B descends to it, and $L_{0,c}(\tau)$ can be alternatively realized as a quotient of $L_{0,c}(\tau)$ by the kernel of B.

Next, we turn to basic properties of modules $L_{t,c}(\tau)$ and $N_{t,c}(\tau)$. We will need the following lemma, which is a consequence of the Hilbert-Noether Theorem and can be found in [43] as Corollary 2.3.2.

Lemma 6.1.3. *For any finite group W, field F, and a finite-dimensional $F[W]$-module $\mathfrak{h}$, the algebra of invariants $(S\mathfrak{h})^W$ is finitely generated over F, and $S\mathfrak{h}$ is a finite integral extension of $(S\mathfrak{h})^W$.*

The following proposition is unique to fields of positive characteristic.

Proposition 6.1.4. *All $N_{t,c}(\tau)$, and thus $L_{t,c}(\tau)$, are finite-dimensional.*

Proof. The Hilbert series of a baby Verma module is defined as

$$\mathrm{Hilb}_{N_{1,c}(\tau)}(z) = \sum_i \dim N_{1,c}(\tau)_i z^i, \quad \mathrm{Hilb}_{N_{0,c}(\tau)}(z) = \sum_i \dim N_{0,c}(\tau)_i z^i.$$

The series at $t = 0$ and $t = 1$ are related by

$$\mathrm{Hilb}_{N_{1,c}(\tau)}(z) = \left(\frac{1-z^p}{1-z}\right)^n \mathrm{Hilb}_{N_{0,c}(\tau)}(z^p).$$

Because of this formula, and because $L_{t,c}(\tau)$ is a quotient of $N_{t,c}(\tau)$, it is enough to prove the proposition for $N_{0,c}(\tau)$.

Representation τ is finite-dimensional, so $M_{0,c}(\tau) = S\mathfrak{h}^* \otimes \tau$ is a finite module over $S\mathfrak{h}^*$. By Lemma 6.1.3, $S\mathfrak{h}^*$ is a finite module over $(S\mathfrak{h}^*)^W$.

For any commutative ring R, maximal ideal $\mathfrak{m}$, and finite R-module M, $M/\mathfrak{m}M$ is a finite-dimensional vector space over $R/\mathfrak{m}$. Applying this to $\mathfrak{m} = ((S\mathfrak{h}^*)^W)_+$, $R = ((S\mathfrak{h}^*)^W)$, and $M = M_{0,c}(\tau)$, it follows that $N_{0,c}(\tau)$ is finite-dimensional over $\Bbbk$. □

6.1.2 Modules $L_{t,c}(\tau)$

As in characteristic zero, $L_{t,c}(\tau)$ is irreducible. When proving this in characteristic zero, we used the fact that there is a natural grading on Verma modules given by the action of $\mathbf{h}$, so all submodules are graded, and as none of them contains anything from the lowest graded part τ, their sum does not contain anything in τ either and it is a proper submodule. In characteristic p this fails, as $\mathbf{h}$ only induces a natural $\mathbb{Z}/p\mathbb{Z}$-grading. There exist submodules of $M_{t,c}(\tau)$ which are not $\mathbb{Z}$-graded.

Example 6.1.5. *For any $f \in (S\mathfrak{h}^*)^W$, the subspace $S\mathfrak{h}^*(1+f^p)\otimes\tau$ is a proper submodule. The sum of all submodules of $M_{t,c}(\tau)$ of this form equals $M_{t,c}(\tau)$, so the sum of all proper submodules of $M_{t,c}(\tau)$ is the whole $M_{t,c}(\tau)$.*

However, the situation is better if we consider only graded submodules, or if we let baby Verma modules take over the role of Verma modules. This is explained more precisely by the following results.

To show irreducibility of $L_{t,c}(\tau)$, we will need the following form of Nakayama's lemma. Recall that the *Jacobson radical* of a commutative ring R, denoted $\mathrm{rad}(R)$, is the maximal ideal that annihilates all simple modules, or equivalently, the intersection of all maximal ideals. Also recall that $\Bbbk[[x_1, \ldots, x_n]]$ is local, so $\text{rad}(\Bbbk[[x_1, \ldots, x_n]]) = \langle x_1, \ldots, x_n \rangle$.

Lemma 6.1.6 (Nakayama). *Let R be a commutative ring, $I \subset rad(R)$ an ideal, and M a finitely generated R-module. Let $m_1, \ldots, m_n \in M$ be such that their projections generate M/IM over R/I. Then, $m_1, \ldots, m_n$ generate M over R.*

Lemma 6.1.7. *Let $L_{t,c}(\tau)_+$ be the positively graded part of $L_{t,c}(\tau)$. If $v_1, \ldots, v_m \in L_{t,c}(\tau)$ are such that their projections $\bar{v}_1, \ldots, \bar{v}_m \in L_{t,c}(\tau)/L_{t,c}(\tau)_+ \cong \tau$ span τ over $\Bbbk$, then $v_1, \ldots, v_m$ generate $L_{t,c}(\tau)$ as an $S\mathfrak{h}^*$ module.*

Proof. This is a direct application of Nakayama's lemma, with $R = \Bbbk[[x_1, \ldots, x_n]]$, $M = L_{t,c}(\tau)$, and $I = \langle x_1, \ldots, x_n \rangle = \text{rad}(R)$. By Lemma 6.1.6, $v_1, \ldots, v_m$ generate $L_{t,c}(\tau)$ as a $\Bbbk[[x_1, \ldots, x_n]]$-module. Since $L_{t,c}(\tau)$ is finite-dimensional, an infinite power series really acts on M as a finite polynomial. ☐

Proposition 6.1.8. *$L_{t,c}(\tau)$ is irreducible for every c and τ.*

Proof. Let f be any nonzero element of $L_{t,c}(\tau)$. We claim that it generates the entire $L_{t,c}(\tau)$ as an $H_{t,c}(W, \mathfrak{h})$ module.

If the projection $\bar{f}$ of f to $L_{t,c}(\tau)_0 \cong \tau$ is nonzero, then the set of W-translates of $\bar{f}$ spans the irreducible representation τ, so by Lemma 6.1.7 the set of W-translates of f generates $L_{t,c}(\tau)$ as an $S\mathfrak{h}^*$ module, and f generates $L_{t,c}(\tau)$ as an $H_{t,c}(W, \mathfrak{h})$-module.

If the projection of f to $L_{t,c}(\tau)_0 \cong \tau$ is zero, write $f = f_1 + \cdots + f_d$, with $f_i \in L_{t,c}(\tau)_i$. The form B is nondegenerate on $L_{t,c}(\tau)$, so $f \notin \text{Ker}B$; it respects the grading so there is some $r > 0$ such that $f_r \notin \text{Ker}(B)$. The form is bilinear, so there exists a monomial $y_1^{a_1} \cdots y_n^{a_n} \in S^r\mathfrak{h}$ and $v \in \tau^*$ such that $B(f_r, y_1^{a_1} \cdots y_n^{a_n} v) \neq 0$. By contravariance of B, and writing D_i for D_{y_i}, this is equal to $0 \neq B(D_1^{a_1} \cdots D_n^{a_n} f_r, v) = B(D_1^{a_1} \cdots D_n^{a_n} f, v)$. So, $D_1^{a_1} \cdots D_n^{a_n} f$ is a nonzero element of $L_{t,c}(\tau)$, with a nonzero projection to $L_{t,c}(\tau)_0 \cong \tau$. By the previous reasoning, $D_1^{a_1} \cdots D_n^{a_n} f$ generates $L_{t,c}(\tau)$ as an $H_{t,c}(W, \mathfrak{h})$-module, and thus f generates $L_{t,c}(\tau)$ as an $H_{t,c}(W, \mathfrak{h})$-module. ☐

Corollary 6.1.9. *The Verma module $M_{t,c}(\tau)$ has a unique maximal graded submodule.*

Proof. Consider the sum of all graded submodules. None of these submodules have elements in $L_{t,c}(\tau)_0 \cong \tau$, since such elements generate the entire module, so their sum is a proper submodule. □

Corollary 6.1.10. *The baby Verma module has a unique maximal submodule.*

Proof. Let N be any proper submodule, and $f \in N$ arbitrary nonzero element. Write $f = f_0 + \cdots + f_d$, with f_i in the i-th graded piece. Baby Verma modules are finite-dimensional, N is a proper submodule, so a similar argument as in Proposition 6.1.8 implies that $f_0 = 0$. Thus, any proper submodule has zero projection to the zeroth graded piece, and so the sum of all proper submodules is still proper. □

The unique maximal graded submodule of $M_{t,c}(\tau)$ descends to the unique maximal submodule of $N_{t,c}(\tau)$. We will denote this unique (graded) maximal submodule by $\bar{J}_{t,c}(\tau)$. The following corollary follows by irreducibility of $M_{t,c}(\tau)/\mathrm{Ker}(B)$.

Corollary 6.1.11. *The kernel of B is $J_{t,c}(\tau)$.*

Thus,

$$L_{t,c}(\tau) = M_{t,c}(\tau)/J_{t,c}(\tau) = M_{t,c}(\tau)/\mathrm{Ker}(B) \cong N_{t,c}(\tau)/\mathrm{Ker}(B) = N_{t,c}(\tau)/\bar{J}_{t,c}(\tau).$$

6.1.3 Category $\mathcal{O}$

We now define category $\mathcal{O}$ of $H_{t,c}(W,\mathfrak{h})$ modules. The definition, which is somewhat different than in characteristic zero, is justified by Example 6.1.5 and Proposition 6.1.13.

Definition 6.1.12. *The category $\mathcal{O}_{t,c}(W,\mathfrak{h})$ is the category of $\mathbb{Z}$-graded $H_{t,c}(W,\mathfrak{h})$-modules which are finite-dimensional over $\Bbbk$.*

We usually write $\mathcal{O}_{t,c}$ or $\mathcal{O}$ instead of $\mathcal{O}_{t,c}(W,\mathfrak{h})$, when it is clear what the arguments are.

The grading element $\mathbf{h}$ (or $\mathbf{h}'$) does not induce the natural $\mathbb{Z}$ grading on objects in category $\mathcal{O}$ as it does in characteristic zero. We allow all $\mathbb{Z}$ grading shifts.

Proposition 6.1.13. *For every irreducible $L \in \mathcal{O}_{t,c}(W, \mathfrak{h})$, there is a unique irreducible W-representation τ and $i \in \mathbb{Z}$ such that $L \cong L_{t,c}(W, \mathfrak{h}, \tau)[i]$.*

Proof. Let $L \in \mathcal{O}_{t,c}$ be any irreducible module in category $\mathcal{O}$. It is graded and finite-dimensional, so there must be a lowest graded piece L_i. Without loss of generality, we can shift indices so that the lowest graded piece is in degree zero. Further, if the degree zero part L_0, which is a W-representation, is reducible, then the proper W-subrepresentation of L_0 generates a proper $H_{t,c}(W, \mathfrak{h})$-subrepresentation of L. So, $L_0 \cong \tau$ for some irreducible W-representation τ. By Proposition 3.1.6, there exists a nonzero graded homomorphism $\phi \colon M_{t,c}(\tau) \to L$. Since L is irreducible, this homomorphism is surjective, and L is isomorphic to $M_{t,c}(\tau)/\mathrm{Ker}(\phi)$. Since $J_{t,c}(\tau)$ is the unique maximal graded submodule, $\mathrm{Ker}(\phi) = J_{t,c}(\tau)$ and the result follows. Uniqueness follows from the fact that $L_{t,c}(\tau)_0 \cong \tau$. □

6.2 Characters

6.2.1 Definition and basic properties

The two definitions of characters are the same as in characteristic zero, in section 6.2. In the following sections, we mostly use χ instead of ch.

Definition 6.2.1. *Let $K(W)$ be the Grothendieck group of the category of finite-dimensional representations of W over $\Bbbk$. For $M = \oplus_i M_i$ any graded $H_{t,c}(W, \mathfrak{h})$ module with finite-dimensional graded pieces, define its character to be the power series in formal variables z, z^{-1} with coefficients in $K(W)$*

$$\chi_M(z) = \sum_i [M_i] z^i,$$

or the following function of $g \in W$

$$\mathrm{ch}_M(z) = \sum_i \mathrm{Tr}|_{M_i}(g) z^i,$$

and define its Hilbert series as

$$\mathrm{Hilb}_M(z) = \sum_i \dim(M_i) z^i.$$

If M is in category $\mathcal{O}$, it is finite-dimensional and its character is in $K(W)[z, z^{-1}]$.

The character of $M_{t,c}(\tau)$ is

$$\chi_{M_{t,c}(\tau)}(z) = \sum_{i \geq 0} [S^i \mathfrak{h}^* \otimes \tau] z^i,$$

and its Hilbert series is

$$\mathrm{Hilb}_{M_{t,c}(\tau)}(z) = \frac{\dim(\tau)}{(1-z)^n}.$$

The character of $N_{t,c}(\tau)$ depends on whether $t = 0$ or $t \neq 0$; they are related by

$$\chi_{N_{1,c}(\tau)}(z) = \chi_{N_{0,c}(\tau)}(z^p) \cdot \left(\frac{1-z^p}{1-z}\right)^n.$$

If W is a reflection group for which the algebra of invariants $(S\mathfrak{h}^*)^W$ is a polynomial algebra with homogeneous generators of degrees $d_1, \ldots d_n$, then the characters of baby Verma modules are:

$$\chi_{N_{0,c}(\tau)}(z) = \chi_{M_{0,c}(\tau)}(z)(1 - z^{d_1})(1 - z^{d_2}) \ldots (1 - z^{d_n}),$$

$$\chi_{N_{1,c}(\tau)}(z) = \chi_{M_{1,c}(\tau)}(z)(1 - z^{pd_1})(1 - z^{pd_2}) \ldots (1 - z^{pd_n}).$$

The main focus of the next chapters is describing these modules for particular series of groups W, in terms of their characters, or through describing the generators for the maximal proper submodules $J_{t,c}(\tau)$, or through describing the composition series of baby Verma modules and Verma modules.

It is clear from the definition that

$$\mathrm{Hilb}_{L_{t,c}(\tau)}(z) = \sum_{i=0}^{\infty} \mathrm{rank}(B_i) z^i.$$

The matrices B_i and their ranks can be calculated in many examples using algebra software. We used MAGMA [11], and did these calculations for small examples in order to form conjectures which became chapters 7 and 8.

6.2.2 Characters of $L_{t,c}(\tau)$ at generic value of parameter c

By definition, the i-th graded piece of $L_{t,c}(\tau)$ is, as a representation of W, equal to the quotient of $S^i\mathfrak{h}^* \otimes \tau$ by the kernel of B_i. Let us fix t and consider $c = (c_s)_s$ as variables; B_i depends on them polynomially. Let $\Bbbk^{|conj|}$ be the space of functions from the finite set of conjugacy classes in W to $\Bbbk$, and think of it as the space of all possible parameters c.

Let d be the dimension of $S^i\mathfrak{h}^* \otimes \tau$ and let r be the rank of B_i, seen as an operator over $\Bbbk[c]$. For c outside of finitely many hypersurfaces in $\Bbbk^{|conj|}$, the rank of B_i evaluated at c is equal to r, and the kernel of B_i is some $(d-r)$-dimensional representation of W, depending on c. All these representations have the same composition series. (To see that, let $V(c)$ be a flat family of W-representations, for example $\mathrm{Ker} B_i$ for generic c. Let for σ_i be all irreducible W-representations and π_i their projective covers. Then the number $[V(c) : \sigma_i]$ of times σ_i appears as a composition factor in $V(c)$ is equal to the dimension of $\mathrm{Hom}(\pi_i, V(c))$. So, for generic c it is the same, and for special c it might be bigger. But $\sum_i [V(c) : \sigma_i] \dim(\sigma_i) = \dim V(c)$ is constant, so $[V(c) : \sigma_i]$ does not depend on c, and all $V(c)$ have the same composition series. They might however not be isomorphic, because they might be different extensions of their irreducible composition factors.)

The map $c \mapsto \mathrm{Ker}(B_i) = J_{t,c}(\tau)_i$, defined on the open complement of hypersurfaces in $\Bbbk^{|conj|}$, can be thought of as a rational function from $\Bbbk^{|conj|}$ to the Grassmanian of $(d-r)$-dimensional subspaces of $S^i\mathfrak{h}^* \otimes \tau$.

For c in some finite family of hypersurfaces in the parameter space $\Bbbk^{|conj|}$, the rank of B_i evaluated at c is smaller than r, and the dimension of the kernel $J_{t,c}(\tau)_i$ is larger then $d-r$. We want to use the above rational function to define a subspace $J_{t,0}(\tau)'_i \subseteq J_{t,0}(\tau)_i$ at $c = 0$, which has similar properties to those $J_{t,0}(\tau)_i$ would have if $c = 0$ was a generic point.

If $c=0$ is generic and rank of B_i at $c=0$ is d, let $J_{t,0}(\tau)'_i = J_{t,0}(\tau)_i$. Otherwise, pick a line in the parameter space $\Bbbk^{|conj|}$ which does not completely lie in one of the hypersurfaces, and which passes through 0. The composition of the inclusion of this line to $\Bbbk^{|conj|}$ and the rational map from $\Bbbk^{|conj|}$ to the Grassmanian is then a rational map from the punctured line to a projective space, and such a map can always be extended to a regular map on the whole line. This associates to $c=0$ a vector space $J_{t,0}(\tau)'_i$. It generally depends on the choice of a line in the parameter space, and it always has the following properties:

- $\dim(J_{t,0}(\tau)')_i = r-d$;
- $J_{t,0}(\tau)'_i \subseteq \mathrm{Ker} B_i = J_{t,0}(\tau)_i$;
- $J_{t,0}(\tau)'_i$ is W-invariant;
- $J_{t,0}(\tau)'_i$ has the same composition series as $J_{t,c}(\tau)_i$ for generic c.

By making consistent choices for all i (for example, by choosing the same line in the parameter space for all i), one can ensure an extra property:

- $J_{t,0}(\tau)' = \oplus_i J_{t,0}(\tau)'_i$ is a $H_{t,0}(W,\mathfrak{h})$ subrepresentation of $M_{t,0}(\tau)$.

So, this produces a subrepresentation $J'_{t,0}(\tau)$ at $c=0$ such that the quotient $M_{t,0}(\tau)/J_{t,0}(\tau)'$ behaves like $L_{t,c}(\tau)$ at generic c, even when $c=0$ is not generic. In particular, $M_{t,0}(\tau)/J_{t,0}(\tau)'$ and $L_{t,c}(\tau)$ at generic c have the same character.

Example 6.2.2. *For $W=GL_2(\mathbb{F}_2)$, $\tau=\mathrm{triv}$, the form B restricted to $M_{1,c}(\mathrm{triv})_4 \cong S^4\mathfrak{h}^*$ has a matrix, written here in the ordered basis $(x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4)$:*

$$B_4 = \begin{pmatrix} c^2(c+1) & c^2(c+1) & c^2(c+1) & c^2(c+1) & 0 \\ c^2(c+1) & c(c+1) & 0 & 0 & c^2(c+1) \\ c^2(c+1) & 0 & 0 & 0 & c^2(c+1) \\ c^2(c+1) & 0 & 0 & c(c+1) & c^2(c+1) \\ 0 & c^2(c+1) & c^2(c+1) & c^2(c+1) & c^2(c+1) \end{pmatrix}.$$

When $c \neq 0,1$, this matrix has rank 4, and a one-dimensional kernel $J_{1,c}(\mathrm{triv})_4$ spanned by $x_1^4 + x_1^2x_2^2 + x_2^4$. For $c = 0$, the matrix is zero and $J_{1,0}(\mathrm{triv})_4$ is the whole $S^4\mathfrak{h}^*$. The above procedure defines $J_{1,0}(\mathrm{triv})_4'$ to be $\Bbbk(x_1^4 + x_1^2x_2^2 + x_2^4)$.

We will now draw conclusions about the character of $L_{t,c}(\tau)$ for generic c using information about $M_{t,0}(\tau)/J'_{t,0}(\tau)$.

Lemma 6.2.3. *Let M be a free finitely-generated graded $S\mathfrak{h}^*$-module with free generators $b_1, \ldots, b_m$, and N a graded submodule of M. For $f \in S\mathfrak{h}^*$, $y \in \mathfrak{h}$, define $\partial_y f b_i = (\partial_y f)b_i$. If N is stable under ∂_y for all $y \in \mathfrak{h}$, then it is generated by elements of the form $\sum f_i^p b_i$ for some $f_i \in S\mathfrak{h}^*$.*

Proof. First, assume there is only one generator, so $M \cong S\mathfrak{h}^*$ as left $S\mathfrak{h}^*$ modules. Let $N' = \{f^p \mid f \in S\mathfrak{h}^*\} \cap N$. We claim that $S\mathfrak{h}^*N' = N$.

Clearly, $S\mathfrak{h}^*N' \subset N$. To show that $N \subset S\mathfrak{h}^*N'$, we need to show that any $f \in N$ can be written as a sum of elements of the form $h(x_1, \ldots, x_n)f'(x_1^p, \ldots, x_n^p)$, for some $h \in S\mathfrak{h}^*$ and $f'(x_1^p, \ldots, x_n^p) \in N$.

As N is graded, assume f is homogeneous of degree d. Write it as

$$f = \sum_{i=0}^{p-1} x_1^i f_i(x_1^p, x_2, \ldots, x_n).$$

The space N is stable under all partial derivatives, so for each $j = 0, \ldots, p-1$,

$$x_1^j \partial_1^j f = \sum_{i=1}^{p-1} i(i-1)\ldots(i-j+1)x_1^i f_i(x_1^p, x_2, \ldots, x_n)$$

is in N. The coefficient $i(i-1)\ldots(i-j+1)$ is zero for $i < j$ and is nonzero for $i = j$, so the matrix $[i(i-1)\ldots(i-j+1)]_{i,j}$ is invertible, implying that $x_1^i f_i(x_1^p, x_2, \ldots, x_n)$ is in N for all i, and therefore (after applying ∂_1^i), also $f_i(x_1^p, x_2, \ldots, x_n) \in N$.

Applying the same argument on each f_i for $x_2, \ldots, x_n$, it follows that f is of desired form.

The claim for $M \cong \oplus S\mathfrak{h}^* b_i$ follows directly from the one for $S\mathfrak{h}^*$. □

Let $S^{(p)}\mathfrak{h}^*$ be the quotient of $S\mathfrak{h}^*$ by the ideal generated by $x_1^p, \ldots x_n^p$.

Proposition 6.2.4. *The character of $L_{1,c}(\tau)$, for generic value of c, is of the form*

$$\chi_{L_{1,c}(\tau)}(z) = \chi_{S^{(p)}\mathfrak{h}^*}(z)H(z^p)$$

for $H \in K_0[z]$ character of some graded W-representation. In particular, the Hilbert series of $L_{1,c}(\tau)$ is of the form

$$\mathrm{Hilb}_{L_{1,c}(\tau)}(z) = \left(\frac{1-z^p}{1-z}\right)^n \cdot h(z^p),$$

for h a polynomial with nonnegative integer coefficients.

Proof. As commented above, the character of $L_{1,c}(\tau) = M_{1,c}(\tau)/J_{1,c}(\tau)$ is the same for all c outside of finitely many hypersurfaces, and it is equal to the character of $M_{1,0}(\tau)/J_{1,0}(\tau)'$. At these values of parameter, $t = 1$ and $c = 0$, Dunkl operators are particularly simple, and equal to partial derivatives: $D_y = \partial_y$. By the previous lemma, $J_{1,0}(\tau)'$ is generated by p-th powers. Let $f_i(x_1^p, \dots x_n^p) \otimes v_i$, for some $f_i \in S\mathfrak{h}^*$, $v_i \in \tau$, be these generators.

Define J^* to be the $(S\mathfrak{h}^*)^p$-module generated by $f_i(x_1^p, \dots x_n^p) \otimes v_i$. Let the *reduced module* $R_{t,c}(\tau)$ be the $\Bbbk[W] \ltimes S\mathfrak{h}^*$-module defined as the quotient of $S\mathfrak{h}^* \otimes \tau$ by the ideal generated by $f_i(x_1, \dots x_n) \otimes v_i$. Call its character (in the sense of Definition 6.2.1) the *reduced character* of $L_{t,c}(\tau)$, and let $H(z) \in K(W)[z]$ denote it.

Consider the multiplication map

$$\mu\colon S^{(p)}\mathfrak{h}^* \otimes ((S\mathfrak{h}^*)^p \otimes \tau)/J^* \to S\mathfrak{h}^* \otimes \tau/J_{1,0}(\tau)'.$$

It is an isomorphism of graded W-representations, so it preserves characters. From this it follows that for generic c,

$$\chi_{L_{1,c}(\tau)}(z) = \chi_{M_{1,0}(\tau)/J_{1,0}(\tau)'}(z) = \chi_{S^{(p)}\mathfrak{h}^*}(z)H(z^p).$$

□

By inspecting the proof and using that c is nongeneric on a union of finitely many hypersurfaces, one can strengthen the claim of the proposition as follows: for any hyperplane P passing through the origin in the space of functions from the conjugacy classes of W to $\Bbbk$, there exists a function $H_P(z) \in K(W)[z]$ such that, for c generic in P, the character of $L_{1,c}(\tau)$ is of the form $\chi_{S^{(p)}\mathfrak{h}^*}(z)H_P(z^p)$.

Corollary 6.2.5. *Let $t \neq 0$ and c be generic. The module $J_{t,c}(\tau)$ is generated under $S\mathfrak{h}^*$ by homogeneous elements in degrees divisible by p. The images of such elements of degree mp in the quotient*

$$(J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau))_{mp} = J_{t,c}(\tau)_{mp}/\mathfrak{h}^* J_{t,c}(\tau)_{mp-1} \subseteq S^{mp}\mathfrak{h}^* \otimes \tau/\mathfrak{h}^* J_{t,c}(\tau)_{mp-1}$$

form a subrepresentation of $S^{mp}\mathfrak{h}^* \otimes \tau/\mathfrak{h}^* J_{t,c}(\tau)_{mp-1}$ whose composition factors are a submultiset of composition factors of $(S^m\mathfrak{h}^*)^p \otimes \tau/(\mathfrak{h}^* J'_{t,0}(\tau)_{mp-1} \cap (S^m\mathfrak{h}^*)^p \otimes \tau)$.

Any such generator in degree mp is a singular vector in the quotient of $M_{t,c}(\tau)$ by the $S\mathfrak{h}^*$-submodule generated by all such generators from smaller degrees.

Proof. For representations σ and σ' of G, let us write $\sigma \preccurlyeq \sigma'$ if the multiset of composition factors of σ is a subset of the multiset of composition factors of σ'. If σ and σ' are graded G representations, we write $\sigma \preccurlyeq \sigma'$ if $\sigma_i \preccurlyeq \sigma'_i$ for all i. If $\sigma \preccurlyeq \sigma'$ and $\sigma' \preccurlyeq \sigma$, then $[\sigma] = [\sigma']$ in the Grothendieck group.

Let us first prove:

$$J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau) \preccurlyeq J'_{t,0}(\tau)/\mathfrak{h}^* J'_{t,0}(\tau) \preccurlyeq (S\mathfrak{h}^*)^p \otimes \tau/(\mathfrak{h}^* J'_{t,0}(\tau) \cap (S\mathfrak{h}^*)^p \otimes \tau).$$

As $J_{t,c}(\tau)$ is a deformation of $J_{t,0}(\tau)'$, for any degree $i > 0$ we have, in the Grothendieck group:

$$[J_{t,0}(\tau)'_i] = [J_{t,c}(\tau)_i]$$

$$[J_{t,0}(\tau)'_{i-1}] = [J_{t,c}(\tau)_{i-1}]$$

$$\mathfrak{h}^* J_{t,0}(\tau)'_{i-1} \preccurlyeq \mathfrak{h}^* J_{t,c}(\tau)'_{i-1}$$

so

$$J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau) \preccurlyeq J'_{t,0}(\tau)/\mathfrak{h}^* J'_{t,0}(\tau).$$

The statement $J'_{t,0}(\tau)/\mathfrak{h}^*J'_{t,0}(\tau) \preccurlyeq (S\mathfrak{h}^*)^p \otimes \tau/(\mathfrak{h}^*J'_{t,0}(\tau) \cap (S\mathfrak{h}^*)^p \otimes \tau)$ follows from $J'_{t,0}(\tau)$ being generated under $S\mathfrak{h}^*$ by p-th powers (see Lemma 6.2.3).

The module $J_{t,c}(\tau)$ is generated under $S\mathfrak{h}^*$ by elements which have nonzero projection to $J_{t,c}(\tau)/\mathfrak{h}^*J_{t,c}(\tau)$. Because of the above sequence of $\preccurlyeq$, such elements only exist in degrees divisible by p, and their images in $J_{t,c}(\tau)/\mathfrak{h}^*J_{t,c}(\tau) \subset S\mathfrak{h}^* \otimes \tau/\mathfrak{h}^*J_{t,c}(\tau)$ form a group representation which is $\preccurlyeq (S\mathfrak{h}^*)^p \otimes \tau/(\mathfrak{h}^*J'_{t,0}(\tau) \cap (S\mathfrak{h}^*)^p \otimes \tau)$.

For every $v \in J_{t,c}(\tau)_{mp}$ and every $y \in \mathfrak{h}$, $D_y(v) \in J_{t,c}(\tau)_{mp-1}$. So, if v is not in $\mathfrak{h}^*J_{t,c}(\tau)_{mp-1}$, then its projection is a nonzero vector in $J_{t,c}(\tau)/\mathfrak{h}^*J_{t,c}(\tau)$ with a property that $D_y(v)$ is zero in $J_{t,c}(\tau)/\mathfrak{h}^*J_{t,c}(\tau)$, in other words a singular vector. □

6.2.3 A dimension estimate for $L_{1,c}(\tau)$

Lemma 6.2.6. *Any irreducible $H_{1,c}(W,\mathfrak{h})$-representation has dimension less than or equal to $p^n|W|$.*

Proof. We begin with a definition, which will only be used in this proof. Let A be an algebra. A *polynomial identity* is a nonzero, noncommutative polynomial $f(x_1,\ldots,x_r)$ such that $f(a_1,\ldots,a_r) = 0$ for all $a_1,\ldots,a_r \in A$. Given an algebraically closed field $\Bbbk$, a *polynomial identity algebra*, or *PI algebra* is a $\Bbbk$-algebra A that satisfies a polynomial identity. We say a PI algebra has degree r if it satisfies the polynomial identity $s_{2r} = \sum_{\sigma\in S_{2r}} \mathrm{sgn}(\sigma) \prod_{i=1}^{2r} x_{\sigma(i)}$.

Our first claim is that $H_{1,c}(W,\mathfrak{h})$ is a PI algebra. By Proposition V.5.4 in [1], A is a PI algebra if and only if every localization of A is also a PI algebra. By the localization lemma (Proposition 3.12. in [24]), $H^{\mathrm{loc}}_{1,c}(W,\mathfrak{h}) \cong H^{\mathrm{loc}}_{1,0}(W,\mathfrak{h})$. Thus, it suffices to show that $H_{1,0}(W,\mathfrak{h})$ is a PI algebra.

Let Z be the center of $H_{1,0}(W,\mathfrak{h})$. It is easy to see that $Z = ((S\mathfrak{h})^p)^W \oplus ((S\mathfrak{h}^*)^p)^W$. Z is commutative, so we can consider $A' = \mathrm{Frac}(Z) \otimes_Z H_{1,0}(W,\mathfrak{h})$, which is an algebra over the field $\mathrm{Frac}(Z)$. By Theorem V.8.1 in [1], this is a *central simple algebra*, i.e. an algebra that is finite-dimensional, simple, and whose center is exactly its field of coefficients. By the Artin-Wedderburn theorem, a central simple algebra is isomorphic to the matrix algebra over a division ring. Thus, A' is isomorphic to a matrix algebra over some division ring. We would like to determine that dimension of A'. We will write $\Bbbk[x_1^p, \ldots, y_n^p]$ as shorthand for $\Bbbk[x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p]$ and likewise for $\Bbbk[x_1, \ldots, y_n]$. It is clear that

$$\dim_{\mathrm{Frac}(Z)} A' = \dim_{\mathrm{Frac}(\Bbbk[x_1^p,\ldots,y_n^p]^W)} \mathrm{Frac}(\Bbbk[x_1^p, \ldots, y_n^p]^W) \otimes_{\Bbbk[x_1^p,\ldots,y_n^p]^W} H_{1,0}(W, \mathfrak{h}) =$$

$$= \dim_{\mathrm{Frac}(\Bbbk[x_1^p,\ldots,y_n^p]^W)} \mathrm{Frac}(\Bbbk[x_1^p, \ldots, y_n^p]^W) \otimes_{\Bbbk[x_1^p,\ldots,y_n^p]^W}$$

$$\otimes_{\Bbbk[x_1^p,\ldots,y_n^p]^W} \Bbbk[x_1^p, \ldots, y_n^p] \otimes_{\Bbbk[x_1^p,\ldots,y_n^p]} (W \ltimes \Bbbk[x_1, \ldots, y_n]).$$

We claim that if W acts faithfully on a vector space V, then,

$$\dim_{\mathrm{Frac}(\Bbbk[x_1^p,\ldots,y_n^p]^W)} \mathrm{Frac}(\Bbbk[V]^W) \otimes_{\Bbbk[V]^W} \Bbbk[V] = |W|.$$

Choose some $f \in \Bbbk[V]$ such that the $W.f$ are distinct. It is clear that $1, f, \ldots, f^{|W|-1}$ is a basis. Thus, the left tensor product has dimension $|W|$ over $\mathrm{Frac}(Z)$. The right tensor product has dimension $p^{2n}|W|$. Thus, $\dim_{\mathrm{Frac}(Z)}(A') = (|W|p^n)^2$. By Corollary V.8.4 in [1], an $r \times r$ matrix algebra satisfies s_{2r}, so A' is a PI algebra of degree $p^n|W|$. Since A' is a localization of $H_{1,0}(W, \mathfrak{h})$, $H_{1,0}(W, \mathfrak{h})$ is a PI algebra of degree $p^n|W|$, as desired.

Thus, $H_{1,c}(W, \mathfrak{h})$ is a PI algebra of degree $p^n|W|$. By Proposition V.6.1(ii) in [1], an irreducible representation of a PI algebra of degree d must have dimension less than or equal to d, and the result follows. □

Corollary 6.2.7. *Let h be the reduced Hilbert series of $L_{1,c}(\tau)$ for generic c. Then $L_{1,c}(\tau)$ has dimension $h(1)p^n$, and $1 \leq h(1) \leq |W|$.*

6.2.4 Some observations, questions and remarks

Remark 6.2.8. *In many examples we considered, in particular whenever $W = GL_n(\mathbb{F}_q)$ or $W = SL_n(\mathbb{F}_q)$ and $\tau = \mathrm{triv}$, $h(1)$ is equal to 1 or to $|W|$. In many other cases, it divides $|W|$. However, this is not always true. For $W = GL_2(\mathbb{F}_p)$, $\tau = S^{p-2}\mathfrak{h}$, the order of the group is $(p^2-1)(p^2-p)$, and the reduced Hilbert series is*

$p+(p-1)z+pz^2$. *So,* $h(1)=3p-2$, *which does not always divide* $(p^2-1)(p^2-p)$ *(for example, when* $p=3$*).*

Question 6.2.9. *For* $h_1(z)$ *the reduced Hilbert series of* $L_{1,c}(\tau)$ *and* $h_0(z)$ *the Hilbert series of* $L_{1,c}(\tau)$*, does the inequality*

$$h_0 \leq h_1$$

hold coefficient by coefficient?

There is computational data supporting the positive answer. In many examples, particularly for $W=GL_n(F_q)$ and $SL_n(\mathbb{F}_p)$, the equality $h_0=h_1$ holds. An example when strict inequality is achieved is $W=SL_2(\mathbb{F}_3)$, $\tau=$ triv: the reduced Hilbert series is $h_1(z)=(1+z+z^2+z^3)(1+z+z^2+z^3+z^4+z^5)$, and the Hilbert series of $L_{0,c}(\tau)$ is $h_0(z)=1$.

Recall that a finite-dimensional $\mathbb{Z}_+$ graded algebra $A=\oplus_i A_i$ is *Frobenius* if the top degree A_d is one dimensional, and multiplication $A_i\otimes A_{d-i}\to A_d$ is a nondegenerate pairing. As a consequence, the Hilbert series of A is a palindromic polynomial.

The irreducible module $L_{t,c}(\mathrm{triv})$ is a quotient of $M_{t,c}(\mathrm{triv})\cong S\mathfrak{h}^*$ by an $H_{t,c}(W,\mathfrak{h})$ submodule $J_{t,c}(\mathrm{triv})$, which is in particular an $S\mathfrak{h}^*$ submodule. So, we can consider it as a quotient of the algebra $S\mathfrak{h}^*$ by the left ideal $J_{t,c}(\mathrm{triv})$, and therefore as a finite-dimensional graded commutative algebra.

Proposition 6.2.10. *Assume that* t,c,τ *are such that the top graded piece of* $L_{t,c}(\mathrm{triv})$ *is one dimensional. Then* $L_{t,c}(\mathrm{triv})$ *is Frobenius.*

Proof. Let us first prove: a finite-dimensional graded commutative algebra $A=\oplus_{i=0}^d A_i$ is Frobenius if and only if the kernel on A of multiplication by $A_+=\oplus_{i>0}A_i$ is one dimensional. One implication is clear: if A is Frobenius, the kernel is the one dimensional space A_d. For the other, assume the kernel on A of multiplication by A_+ is one dimensional. The top nontrivial graded piece A_d is always contained in it, so A_d is one dimensional and equal to the kernel. Now assume there exists a nonzero element $a_n\in A_n$ such that multiplication by a_n, seen as a map $A_{d-n}\to A_d$, is zero, and let $0 < n < d$ be the maximal index for which such an a_n exists. As a_n isn't in the kernel of the multiplication by A_+, there exists some $b \in A_+$ such that $a_n b \neq 0$. We can assume without loss of generality that b is homogeneous, $b \in A_m$, $0 < m < d-n$. Then $a_n b \in A_{n+m}$, with $n < n+m < d$, is a nonzero element such that multiplication by it, seen as a map $A_{d-n-m} \to A_d$, is zero, contrary to the choice of n as the largest such index.

Now assume that $A = L_{t,c}(\mathrm{triv})$ has one dimensional top degree. Let $0 \neq f$ be in the kernel of multiplication by A_+. As the kernel is graded, assume without loss of generality that f is homogeneous. Then $xf = 0 \in L_{t,c}(\mathrm{triv})$ for all $x \in \mathfrak{h}^*$, so x is a *highest weight vector.* Under the action of $H_{t,c}(W, \mathfrak{h})$, f generates a subrepresentation of $L_{t,c}(\mathrm{triv})$ for which the highest graded piece consists of W-translates of x. As $L_{t,c}(\mathrm{triv})$ is irreducible, this subrepresentation has to be the entire $L_{t,c}(\mathrm{triv})$, and f is in the top degree, which is by assumption one dimensional. □

Remark 6.2.11. *In many instances we observed, the algebra $L_{t,c}(\mathrm{triv})$ is Frobenius for generic c and has palindromic Hilbert series. However, this is not true in general: let $\Bbbk = \overline{\mathbb{F}_3}$, $W = S_5$ the symmetric group on five letters, $\mathfrak{h}$ the four dimensional reflection representation $\{(z_1, \ldots z_5) \in \Bbbk^5 | z_1 + \ldots + z_5 = 0\}$ with the action $s.z_i = z_{s(i)}$, and $\tau = \mathrm{triv}$. Then the Hilbert series of $L_{0,c}(\mathrm{triv})$ is*

$$(1+z)(1+z+z^2)(1+2z+3z^2+4z^3).$$

We thank Sheela Devadas and Steven Sam for pointing out this counterexample to us.

6.3 A lemma about finite fields

We finish this section with a lemma which we frequently use in computations in the next chapters.

Lemma 6.3.1. *Let $q = p^r$ be a prime power. Let $f \in \Bbbk[x_1, x_2, \ldots, x_n]$ be a polynomial* *in n variables, for which there exists a variable x_i such that*

$$\deg_{x_i}(f) < q-1.$$

Then

$$\sum_{x_1,\dots x_n\in\mathbb{F}_q} f(x_1,\dots x_n) = 0.$$

Proof. It is enough to prove the claim for all monomials f for which $\deg_{x_n}(f) < q-1$.

First, if $n=1$, then $f = x_1^m$ for some $m < q-1$. Let $S_m := \sum_{i\in\mathbb{F}_q} i^m$. For every $j \in \mathbb{F}_q$, $j^m S_m = \sum_{i\in\mathbb{F}_q}(ij)^m = S_m$, which is equivalent to $(1-j^m)S_m = 0$ for all $j \neq 0$. As $m < q-1$, there exists some j such that $1 - j^m \neq 0$, and so $S_m = 0$ and the claim is true for polynomials in one variable.

If $n > 1$ and $f = f' \cdot x_n^m$, with f' a polynomial in $x_1, \dots x_{n-1}$ and $m < q-1$, then

$$\sum_{x_1,\dots x_n\in\mathbb{F}_q} f(x_1,\dots x_n) = \sum_{x_1,\dots,x_{n-1}\in\mathbb{F}_q} f'(x_1,\dots x_{n-1}) \cdot \sum_{x_n\in\mathbb{F}_q} x_n^m =$$

$$= \sum_{(x_1,\dots,x_{n-1})\in\mathcal{M}_{n-1}} f'(x_1,\dots x_{n-1}) \cdot S_m = 0.$$

□

Remark 6.3.2. *In particular, the assumptions of the lemma are satisfied by all f such that* $\deg(f) < n(q-1)$.

Chapter 7

Representations of Rational Cherednik Algebras Associated to Groups $GL_n(\mathbb{F}_{p^r})$ and $SL_n(\mathbb{F}_{p^r})$

Let $q = p^r$ be a power of characteristic p of the ground field $\Bbbk$, and let $\mathbb{F}_q \subset \Bbbk$ be a finite field with q elements. In this chapter we study the case when $W = GL_n(\mathbb{F}_q)$ or $W = GL_n(\mathbb{F}_q)$, and $\mathfrak{h} = \Bbbk^n$ is the tautological representation. Let $\mathfrak{h}_{\mathbb{F}} = \mathbb{F}_q^n$ be the $\mathbb{F}_q$-form of it.

7.1 $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$ as reflection groups

7.1.1 Reflections in $GL_n(\mathbb{F}_q)$

Let us start by fixing some more notation and discussing conjugacy classes in $GL_n(\mathbb{F}_q)$.

For $\lambda \in \mathbb{F}_q, \lambda \neq 0$, let d_λ be the following elements of $GL_n(\mathbb{F}_q)$:

$$\text{for } \lambda \neq 1,\ d_\lambda = \begin{bmatrix} \lambda^{-1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad d_1 = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

$GL_n(\mathbb{F}_q)$ can be though of as a subset of $GL_n(\mathbb{k}) = GL(\mathfrak{h})$. The representation $\mathfrak{h}$ is tautological, $(y_i)_{1\leq i\leq n}$ is a tautological basis of $\mathfrak{h} = \mathbb{k}^n$, and $(x_i)_{1\leq i\leq n}$ is the dual basis of $\mathfrak{h}^*$. This means that the matrix of $g \in GL_n(\mathbb{F}_q)$ action on $\mathfrak{h}$ in basis $(y_i)_i$ is g, and the matrix ofits action on $\mathfrak{h}^*$ in basis $(x_i)_i$ is $(g^{-1})^t$. All d_λ are reflections, with eigenvalues λ^{-1} on $\mathfrak{h}$ and λ on $\mathfrak{h}^*$.

An element $s \in W$ is called a *semisimple reflection* if it is semisimple as an element of $GL(\mathfrak{h})$ and it is a reflection; such elements are conjugate in $GL(\mathfrak{h})$ to some d_λ with $\lambda \neq 1$. An element of $s \in W$ is called a *unipotent reflection* if it is unipotent as an element of $GL(\mathfrak{h})$ and it is a reflection; such an element is conjugate in $GL(\mathfrak{h})$ to d_1, and has the property that $s^p = 1$. Note that in characteristic zero, a unipotent reflection generates an infinite subgroup, so considering unipotent reflections is unique to working in positive characteristic.

Lemma 7.1.1. *Reflections in $GL_n(\mathbb{F}_q)$ are elements that are conjugate in $GL_n(\mathbb{F}_q)$ to one of the d_λ, $\lambda \in \mathbb{F}_q^\times$. The group $GL_n(\mathbb{F}_q)$ is generated by reflections. There are $q-1$ conjugacy classes of reflections in $GL_n(\mathbb{F}_q)$, with representatives d_λ. Each semisimple conjugacy class consists of $\frac{(q^n-1)q^{n-1}}{q-1}$ reflections. The unipotent conjugacy class (elements conjugate to d_1) consists of $\frac{(q^n-1)(q^{n-1}-1)}{(q-1)}$ reflections.*

Proof. The semisimple conjugacy class associated to an eigenvalue $\lambda \neq 1$ consists of conjugates of d_λ. Its centralizer is $GL_1(\mathbb{F}_q) \times GL_{n-1}(\mathbb{F}_q) \subseteq GL_n(\mathbb{F}_q)$, so the number of reflections in this conjugacy class is $|GL_n(\mathbb{F}_q)|/|GL_1(\mathbb{F}_q) \times GL_{n-1}(\mathbb{F}_q)| = \frac{(q^n-1)q^{n-1}}{q-1}$.

The unipotent conjugacy class is the orbit of d_1, which is centralized by any element of the form:

$$\begin{bmatrix} a & b & \cdots & c \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & * & * \\ 0 & f & * & * \end{bmatrix}$$

Here, $a \neq 0$, the rest of the variables are arbitrary, and the bottom right $(n-1)\times(n-2)$ submatrix is invertible. The order of the centralizer is $q^{2n-3}(q-1)|GL_{n-2}(\mathbb{F}_p)|$, and the order of the conjugacy class is $\frac{(q^n-1)(q^{n-1}-1)}{(q-1)}$. □

Let C_λ be the conjugacy class of reflections containing d_λ. Let c_λ denote the value of the function c on elements of C_λ.

The following lemma is a strengthening of Lemma 3.1.1 for the case $W = GL_n(\mathbb{F}_q)$.

Lemma 7.1.2. *There exists a bijection between the set of reflections in $GL_n(\mathbb{F}_q)$ and the set of all vectors $\alpha \otimes \alpha^\vee \neq 0$ in $\mathfrak{h}^*_{\mathbb{F}} \otimes \mathfrak{h}_{\mathbb{F}}$ such that $(\alpha, \alpha^\vee) \neq 1$. The reflection s corresponding to $\alpha \otimes \alpha^\vee$ acts:*

$$\text{on } \mathfrak{h}^* \text{ by} \quad s.x = x - (\alpha^\vee, x)\alpha$$
$$\text{on } \mathfrak{h} \text{ by} \quad s.y = y + \frac{(y,\alpha)}{1-(\alpha,\alpha^\vee)}\alpha^\vee.$$

Such a reflection s is semisimple with nonunit eigenvalue $\lambda = 1 - (\alpha^\vee, \alpha)$ on $\mathfrak{h}^*$ if $(\alpha, \alpha^\vee) \neq 0$, and unipotent if $(\alpha, \alpha^\vee) = 0$.

Example 7.1.3. *If $n = 2$, the parametrization of C_λ by $\alpha \otimes \alpha^\vee \in \mathfrak{h}^*_{\mathbb{F}} \otimes \mathfrak{h}_{\mathbb{F}}$ described in Lemma 7.1.2 is as follows:*

$$\lambda \neq 1: \quad C_\lambda \leftrightarrow \{\begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} 1-\lambda-bd \\ d \end{bmatrix} | b, d \in \mathbb{F}_q\} \cup \{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 1-\lambda \end{bmatrix} | a \in \mathbb{F}_q\}$$

$$\lambda = 1: \quad C_1 \leftrightarrow \{\begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} -bd \\ d \end{bmatrix} | b, d \in \mathbb{F}_q, d \neq 0\} \cup \{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 0 \end{bmatrix} | a \in \mathbb{F}_q, a \neq 0\}.$$

7.1.2 Invariants and characters of baby Verma modules for $W = GL_n(\mathbb{F}_q)$ and $W = SL_n(\mathbb{F}_q)$

Recall that the baby Verma module $N_{t,c}(\tau)$ is a quotient of $M_{t,c}(\tau) \cong S\mathfrak{h}^* \otimes \tau$ by $(S\mathfrak{h}^*)^W_+ \otimes \tau$ if $t = 0$ and by $((S\mathfrak{h}^*)^W)^p_+ \otimes \tau$ if $t \neq 0$. For all groups for which we know the Hilbert series of the space of invariants $(S\mathfrak{h}^*)^W$, we can calculate the character of baby Verma modules easily. An especially nice case is when $(S\mathfrak{h}^*)^W$ is a polynomial algebra generated by algebraically independent elements of homogeneous degrees $d_1, \dots d_n$. In that case, such elements are called *fundamental invariants*, and d_i are called *degrees* of W.

In [17], Dickson shows that $GL_n(\mathbb{F}_q)$ is such a group; the result for $SL_n(\mathbb{F}_q)$ follows easily and is explained in [35]. Let us recall the construction of invariants and the calculation of their degrees before calculating the characters of baby Verma modules.

As before, let $x_1, \dots, x_n$ be the tautological basis for $\mathfrak{h}^*_{\mathbb{F}}$. For an ordered n-tuple of nonnegative integers $e_1, \dots, e_n$, let $[e_1, \dots, e_n] \in S\mathfrak{h}^*$ be the determinant of the matrix whose entry in the i-th row and $j-th$ column is $x_j^{q^{e_i}}$. The action of W on $\mathfrak{h}^*$ is dual to the tautological action, so the matrix of $g \in GL_n(\mathbb{F}_q)$ in the basis $(x_i)_i$ is $(g^{-1})^t$. Taking determinants is a multiplicative map, so a direct calculation gives that for $g \in GL_n(\mathbb{F}_q)$,

$$g.[e_1, \dots, e_n] = (\det(g))^{-1}[e_1, \dots, e_n].$$

Define

$$L_n := [n-1, n-2, \dots, 1, 0],$$

$$Q_i := [n, n-1, \dots, i+1, i-1, \dots, 1, 0]/L_n, \quad i = 1, \dots n-1,$$

and

$$Q_0 = L_n^{q-1}.$$

The paper [17] shows that $[n, n-1, \dots, i+1, i-1, \dots, 1, 0]$ is divisible by L_n, and so Q_i are indeed in $S\mathfrak{h}^*$. From the observation that all $[e_1, \dots, e_n]$ transform as

$(\det g)^{-1}$ under the action of $g \in GL_n(\mathbb{F}_q)$, it follows that $Q_i, i = 0, \ldots, n-1$ are invariants in $S\mathfrak{h}^*$. The main theorem in [17] states a stronger claim:

Theorem 7.1.4. *Polynomials Q_i, $i = 0, \ldots, n-1$, form a fundamental system of invariants for $GL_n(\mathbb{F}_q)$ in $S\mathfrak{h}^*$, i.e. they are algebraically independent and generate the subalgebra of invariants.*

A comment in Section 3 of [35] gives us the following corollary.

Corollary 7.1.5. *Polynomials L_n and Q_i for $i = 1, \ldots, n-1$ form a fundamental system of invariants for $SL_n(\mathbb{F}_q)$.*

The degrees of these invariants are:

$$\deg L_n = 1 + q + \ldots + q^{n-1}$$

$$\deg Q_0 = (q-1)\deg L_n = (q-1)(1 + q + \ldots + q^{n-1}) = q^n - 1$$

$$\deg Q_i = (1 + q + \ldots + q^n - q^i) - (1 + q + \ldots + q^{n-1}) = q^n - q^i.$$

From this, we can calculate the characters of baby Verma modules for these groups.

Corollary 7.1.6. *For $W = GL_n(\mathbb{F}_q)$, the characters of baby Verma modules are*

$$\chi_{N_{0,c}(\tau)}(z) = \chi_{M_{0,c}(\tau)}(z) \prod_{n=0}^{n-1} (1 - z^{q^n - q^i}),$$

$$\chi_{N_{t,c}(\tau)}(z) = \chi_{M_{t,c}(\tau)}(z) \prod_{n=0}^{n-1} (1 - z^{p(q^n - q^i)}), \quad t \neq 0.$$

For $W = SL_n(\mathbb{F}_q)$, the characters of baby Verma modules are

$$\chi_{N_{0,c}(\tau)}(z) = \chi_{M_{0,c}(\tau)}(z)(1 - z^{1+q+\ldots+q^{n-1}}) \prod_{n=1}^{n-1} (1 - z^{q^n - q^i}),$$

$$\chi_{N_{t,c}(\tau)}(z) = \chi_{M_{t,c}(\tau)}(z)(1 - z^{p(1+q+\ldots+q^{n-1})}) \prod_{n=1}^{n-1} (1 - z^{p(q^n - q^i)}), \quad t \neq 0.$$

7.2 Description of $L_{t,c}(\text{triv})$ for $GL_n(\mathbb{F}_{p^r})$

7.2.1 Irreducible modules with trivial lowest weight for $GL_n(\mathbb{F}_q)$ at $t=0$

Theorem 7.2.1. *The characters of the irreducible modules $L_{0,c}(\text{triv})$ for the rational Cherednik algebra $H_{0,c}(GL_n(\mathbb{F}_q),\mathfrak{h})$ are:*

(q,n)	c	$\chi_{L_{0,c}(\text{triv})}(z)$	$\text{Hilb}_{L_{0,c}(\text{triv})}(z)$
$(q,n)\neq(2,2)$	*any*	$[\text{triv}]$	1
$(2,2)$	0	$[\text{triv}]$	1
$(2,2)$	$c\neq 0$	$[\text{triv}]+[\mathfrak{h}^*]z+([S^2\mathfrak{h}^*]-[\text{triv}])z^2+$ $+([S^3\mathfrak{h}^*]-[\mathfrak{h}^*]-[\text{triv}])z^3$	$1+2z+2z^2+z^3$

Proof. We claim that when $(n,q)\neq(2,2)$, all the vectors $x\in\mathfrak{h}^*\otimes\text{triv}\cong M_{0,c}(\text{triv})_1$ are singular. To see that, remember that the Dunkl operator associated to $y\in\mathfrak{h}$ is

$$D_y=t\partial_y\otimes 1-\sum_{s\in S}c_s\frac{(y,\alpha_s)}{\alpha_s}(1-s)\otimes s,$$

which for $t=0$ and $\tau=\text{triv}$ becomes

$$D_y=-\sum_{s\in S}c_s\frac{(y,\alpha_s)}{\alpha_s}(1-s).$$

To see that $D_y(x)=0$ for all $x\in\mathfrak{h}^*,y\in\mathfrak{h}$ for all values of parameter c, let us calculate the coefficient of c_λ in $D_y(x)$. Using parametrization of conjugacy classes from Proposition 7.1.2, this coefficient is equal to

$$-\sum_{s\in C_\lambda}\frac{(y,\alpha_s)}{\alpha_s}(1-s).x=-\sum_{\substack{\alpha\otimes\alpha^\vee\neq 0\\(\alpha,\alpha^\vee)=1-\lambda}}(y,\alpha)(\alpha^\vee,x)=-\sum_{\alpha}(\alpha,y)\left(x,\sum_{(\alpha,\alpha^\vee)=1-\lambda}\alpha^\vee\right).$$

We claim that for fixed $\alpha\in\mathfrak{h}^*$, the sum $\sum_{(\alpha,\alpha^\vee)=1-\lambda}\alpha^\vee$ is zero. Fix α and let us change the coordinates so that α is the first element of some new ordered basis.

Write the sum $\sum_{(\alpha,\alpha^\vee)=1-\lambda}\alpha^\vee$ in the dual of this basis. The set of all nonzero $\alpha^\vee$ such that $(\alpha,\alpha^\vee)=1-\lambda$ written in these new coordinates then consists of all $\alpha^\vee = (1-\lambda, a_2,\dots,a_n)\neq 0$, for $a_i\in\mathbb{F}_q$. If $\lambda\neq 1$, the sum of all such $\alpha^\vee$ is the sum over al $a_2,\dots,a_n\in\mathbb{F}_q$, so the sum is zero. If $\lambda=1$, the first coordinate is always zero, so the sum is over all $a_i\in\mathbb{F}_q$ which are not all simultaneously 0. However, adding $(0,\dots,0)$ doesn't change the sum, which is then equal to

$$\sum_{a_2,\dots,a_n\in\mathbb{F}_q}(0,a_2,\dots a_n)=(0,q^{n-2}\sum_{a\in\mathbb{F}_q}a,\dots,q^{n-2}\sum_{a\in\mathbb{F}_q}a).$$

This is equal to 0, as claimed, unless $n=2$ and $q=2$.

In case $(n,q)=(2,2)$, direct computation of matrices of the bilincar form B imply the claim. More precisely, if $c=0$ then all $D_y=0$, the form B is zero in degree one, and the module $L_{0,0}(\text{triv})$ is one dimensional. If $c\neq 0$, then the only vectors in the kernels of matrices B_i are the invariants in degrees 2 and 3 and all their $S\mathfrak{h}^*$ multiples. This implies that $L_{0,c}(\text{triv}) = N_{0,c}(\text{triv})$, and gives the character formula from the statement of the theorem.

□

7.2.2 Irreducible modules with trivial lowest weight for $GL_n(\mathbb{F}_q)$ at $t\neq 0$

For the rest of this section we assume $t=1$. As t and c can be simultaneously rescaled, the results we obtain for $t=1$ hold (after rescaling c by $1/t$) for all $t\neq 0$.

Proposition 7.2.2. *Suppose $(q,n)\neq(2,2)$. For any $x\in\mathfrak{h}^*$, the vector x^p is singular in $M_{1,c}(\text{triv})$.*

Proof. Method of proof is an explicit calculation analogous to the proof of Theorem

7.2.1. By definition,

$$\begin{aligned} D_y(x^p) &= \partial_y(x^p) - \sum_{s\in S} c_s(\alpha_s, y)\frac{1}{\alpha_s}(1-s).x^p \\ &= 0 - \sum_{s\in S} c_s(\alpha_s, y)\frac{1}{\alpha_s}((1-s).x)^p \end{aligned}$$

We will show that for every conjugacy class C_λ of reflections, the sum

$$\sum_{s\in C_\lambda}(\alpha_s, y)\frac{1}{\alpha_s}((1-s).x)^p$$

vanishes. By Proposition 7.1.2, this is equal to

$$\sum_{\substack{\alpha\otimes\alpha^\vee\neq 0\\ (\alpha,\alpha^\vee)=1-\lambda}}(\alpha, y)\frac{1}{\alpha}((\alpha^\vee, x)\alpha)^p = \sum_{\alpha\neq 0}(\alpha, y)\alpha^{p-1}\sum_{\substack{\alpha^\vee\neq 0\\ (\alpha,\alpha^\vee)=1-\lambda}}(\alpha^\vee, x)^p$$

It is enough to fix α and show that the inner sum over $\alpha^\vee$ is zero. After fixing α, let us change the basis so that α is the first element of the new ordered basis. The set of all $\alpha^\vee \neq 0$ such that $(\alpha, \alpha^\vee) = 1-\lambda$, written in the dual of this new basis, is $A := \{((1-\lambda), a_2, \ldots, a_n) \neq 0 \mid a_2, \ldots, a_n \in \mathbb{F}_q\}$. For a fixed x, the expression

$$\sum_{\alpha^\vee\in A}(x, \alpha_s^\vee)^p$$

is a sum over all possible values of $n-1$ variables $a_2, \ldots, a_n$ of a polynomial of degree p. By Lemma 6.3.1, this is zero if $p < (n-1)(q-1)$. This is only violated when $n = 2$, $p = q$. In that case $(\alpha_s^\vee, x)^p = (\alpha_s^\vee, x)$ for all $x \in \mathfrak{h}_{\mathbb{F}}^*$, so the expression is equal to the sum over all possible values of one variable a_2 of a polynomial of degree 1; this sum is again by Lemma 6.3.1 equal to zero whenever $1 < p-1$. So, the expression is equal to zero, as desired, whenever $(n, q) \neq (2, 2)$. □

When $(n, q) = (2, 2)$, the claim of the lemma is not true, and the irreducible module with trivial lowest weight is bigger. We will settle the case $(n, q) = (2, 2)$

separately by explicit calculations.

As explained before, studying $L_{1,c}(\mathrm{triv})$ is the same as studying the contravariant form B on $M_{1,c}(\mathrm{triv})$. The following proposition tells us that the set of singular vectors from the previous proposition is large, in the sense that the quotient of $M_{1,c}(\mathrm{triv})$ by the submodule generated by them is already finite-dimensional.

Corollary 7.2.3. *Suppose $(q,n) \neq (2,2)$. Then, the form $B_k = 0$ and $L_{1,c}(\mathrm{triv})_k = 0$ for all $k \geq np - n + 1$.*

Proof. Any degree $n(p-1)+1$ monomial must have one of the n basis elements, say x_i, raised to at least the p-th power by the pigeonhole principle. Then, $B(x_1^{a_1} \dots x_n^{a_n}, y) = B(x_i^p, y') = 0$, where y' is the result of the Dunkl operator being applied to y successively. □

We write matrices of the form B_k in the monomial basis in x_i for $S\mathfrak{h}^*$ and y_i for $S\mathfrak{h}$, both in lexicographical order. In the case of $GL_n(\mathbb{F}_q)$ these matrices are surprisingly simple.

Proposition 7.2.4. *Suppose $(q,n) \neq (2,2)$. Then the matrices of B_k are diagonal for all k.*

Proof. We will use invariance of B with respect to W to show that for $(a_1, \dots, a_n)$, $(b_1, \dots, b_n) \in \mathbb{Z}_{\geq 0}^n$, such that $\sum a_i = \sum b_i = k$, if $B(x_1^{a_1} \dots x_n^{a_n}, y_1^{b_1} \dots y_n^{b_n}) \neq 0$ then $(a_1, \dots, a_n) = (b_1, \dots, b_n)$. This means the matrices B_k, written in monomial basis in lexicographical order, are diagonal.

Let $g \in W$ be a diagonal matrix with entries $\lambda_1, \dots, \lambda_n \in \mathbb{F}_q^\times$, so that $g.y_i = \lambda_i y_i$ and $g.x_i = \lambda_i^{-1} x_i$. Then for any $(a_1, \dots, a_n)$, $(b_1, \dots, b_n) \in \mathbb{Z}_{\geq 0}^n$, such that $\sum a_i = \sum b_i = k$, we have

$$\begin{aligned} B(x_1^{a_1} \dots x_n^{a_n}, y_1^{b_1} \dots y_n^{b_n}) &= B(g.(x_1^{a_1} \dots x_n^{a_n}), g.(y_1^{b_1} \dots y_n^{b_n})) \\ &= B((\lambda_1^{-1} x_1)^{a_1} \dots (\lambda_n^{-1} x_n)^{a_n}, (\lambda_1 y_1)^{b_1} \dots (\lambda_n y_n)^{b_n}) \\ &= \lambda_1^{b_1 - a_1} \dots \lambda_n^{b_n - a_n} B(x_1^{a_1} \dots x_n^{a_n}, y_1^{b_1} \dots y_n^{b_n}). \end{aligned}$$

So, whenever $B(x_1^{a_1} \dots x_n^{a_n}, y_1^{b_1} \dots y_n^{b_n}) \neq 0$, it follows that $\lambda_1^{b_1-a_1} \dots \lambda_n^{b_n-a_n} = 1$ for all $\lambda_1, \dots, \lambda_n \in \mathbb{F}_q^\times$. Fix i. Set all λ_j where $j \neq i$ to be equal to 1, and set λ_i to be a multiplicative generator of $\mathbb{F}_q^\times$. Then, necessarily, $b_i - a_i \equiv 0 \pmod{q-1}$.

If $q > p$ and $b_i - a_i \equiv 0 \pmod{q-1}$, then either $a_i = b_i$ for all i, or there exists an index i for which $b_i - a_i \equiv 0 \pmod{q-1}$, $a_i \neq b_i$. In the second case, either a_i or b_i is greater or equal to p, so by the previous proposition, $B(x_1^{a_1} \dots x_n^{a_n}, y_1^{b_1} \dots y_n^{b_n}) = 0$. This finishes the proof if $q > p$.

Now assume $q = p$, and $B(x_1^{a_1} \dots x_n^{a_n}, y_1^{b_1} \dots y_n^{b_n}) \neq 0$, so $a_i \equiv b_i \pmod{p-1}$ for all i, and $a_i, b_i < p$ for all i. Then either all $a_i = b_i$ as claimed, or, there exists an index i such that $\{a_i, b_i\} = \{0, p-1\}$. Assume without loss of generality that $i = 1$, $a_1 = 0$, $b_1 = p-1$. Using that $\sum a_j = \sum b_j = k$, there exists another index, which we can assume without loss of generality to be 2, such that $a_2 = p-1$, $b_2 = 0$. Now we are claiming that for any f monomial in $x_3, \dots, x_n$, any f' monomial in $y_3, \dots, y_n$,

$$B(x_2^{p-1} f, y_1^{p-1} f') = 0.$$

We will be working only with indices 1 and 2 and choosing $g \in W$ which fixes all others, so assume without loss of generality that $n = 2$, $f = f' = 1$. We use the invariance of B with respect to $d_1 \in W$:

$$\begin{aligned} B(x_1^{p-1}, y_1^{p-1}) = B(d_1.(x_1^{p-1}), d_1.(y_1^{p-1})) =& B((x_1 - x_2)^{p-1}, y_1^{p-1}) \\ =& B((x_1^{p-1} + \dots + x_2^{p-1}), y_1^{p-1}) \\ =& B(x_1^{p-1}, y_1^{p-1}) + 0 + B(x_2^{p-1}, y_1^{p-1}) \end{aligned}$$

where the terms in the middle are all zero, as their exponents do not differ by a multiple of $p-1$. Thus, $B(x_2^{p-1}, y_1^{p-1}) = 0$ as desired. □

Elements of $W \subseteq H_{t,c}(W, \mathfrak{h})$ have degree 0 and preserve the graded pieces. So, every graded piece is a finite-dimensional representation of W. This makes the following lemma useful.

Lemma 7.2.5. *Suppose $(q,n) \neq (2,2)$. Then, for every i, $S^i\mathfrak{h}^*/(\langle x_1^p, \dots, x_n^p\rangle \cap S^i\mathfrak{h}^*)$ is either zero or an irreducible W-representation.*

Proof. Suppose i is such that $S^i\mathfrak{h}^*/(\langle x_1^p, \dots, x_n^p\rangle \cap S^i\mathfrak{h}^*)$ is nonzero, and let $V \neq 0$ be a submodule of it. Let $\mathcal{M}$ be the set of all n-tuples of integers $(m_1, \dots, m_n)$ such that $0 \leq m_i < p$ and $\sum_{r=1}^n m_r = i$. For $m \in \mathcal{M}$, we will denote $x_1^{m_1} \cdots x_n^{m_n}$ by x^m; these monomials are a basis of $S^i\mathfrak{h}^*/(\langle x_1^p, \dots, x_n^p\rangle \cap S^i\mathfrak{h}^*)$. We say that $m, m' \in \mathcal{M}$ re congruent modulo $q-1$, and write $m \equiv m' \pmod{q-1}$, if for every i the integers m_i and m_i' are congruent modulo $q-1$.

Claim 1 If $v = \sum_{m \in \mathcal{M}} a_m x^m \in V$, and $m^{(0)}$ is such that $a_{m^{(0)}} \neq 0$, then

$$\sum_{m \equiv m^{(0)} \pmod{q-1}} a_m x^m \in V.$$

To prove this, let $g_\lambda^{(j)} \in GL_n(\mathbb{F}_q)$ be the diagonal matrix which has λ^{-1} on the j-th place and 1 everywhere else on the diagonal. It acts on $\mathfrak{h}^*$ by $x_j \mapsto \lambda x_j$, $x_l \mapsto x_l$ for $j \neq l$, and on $S\mathfrak{h}^*$ by $x^m \mapsto \lambda^{m_i} x^m$. Fix j, and let $\lambda \in \mathbb{F}_q$. This can be thought of as a linear system of equations whose matrix is the Vandermonde matrix $[\lambda^j]_{\lambda \in \mathbb{F}_q^\times, 0 \leq j \leq q-2}$. It is invertible, which allows us to express $\sum_{m \equiv m' \pmod{q-1}} a_m x^m$ as a linear combination of $g_\lambda^{(j)}.v$. This proves it is in V.

Claim 2 V contains some monomial x^m.

If $q > p$, then the nonzero element of V from Claim 1, $\sum_{m \equiv m^{(0)} \pmod{q-1}} a_m x^m$, is a monomial because of the $m_j < p$ condition.

For the rest of the proof of Claim 2, assume $q = p$. Let $\sum_{m \equiv m^{(0)} \pmod{p-1}} a_m x^m \in V$. If it has only one term, then it is a multiple of a monomial and we are done. If it has at least two terms, with multi-indices $m^{(0)}$ and $m^{(1)}$, then $m^{(1)} \equiv m^{(0)}$ and $\sum_j m_j^{(0)} = \sum_j m_j^{(1)} = i$ mean there exist two indices, for which we assume without loss of generality to be 1 and 2, such that $m_1^{(0)} = p-1$, $m_2^{(0)} = 0$, $m_1^{(1)} = 0$, $m_2^{(1)} = p-1$. The vector v can then be written as

$$v = x_1^{p-1} f_1(x_3, \dots x_n) + x_2^{p-1} f_2(x_3, \dots x_n)$$

for some polynomials f_1, f_2 not depending on x_1, x_2. The group element d_1 fixes $x_2, \ldots x_n$ and maps $x_1 \mapsto x_1 - x_2$, so

$$d_1.v = (x_1^{p-1} + x_1^{p-2}x_2 + \ldots + x_2^{p-1})f_1(x_3, \ldots x_n) + x_2^{p-1}f_2(x_3, \ldots x_n) \in V.$$

Applying Claim 1 on this element, we get that

$$(x_1^{p-1} + x_2^{p-1})f_1(x_3, \ldots x_n) + x_2^{p-1}f_2(x_3, \ldots x_n) \in V,$$

and subtracting v from this element we get

$$x_2^{p-1}f_1(x_3, \ldots x_n) \in V.$$

Thus, starting from v, we produced an element of V which is a linear combination of monomials x^m whose exponents $m = (m_1, m_2, \ldots m_n)$ all have the same (m_1, m_2). Repeating this, we get a monomial in V.

Claim 3 $V = S^i\mathfrak{h}^*/(\langle x_1^p, \ldots, x_n^p \rangle \cap S^i\mathfrak{h}^*)$.

By using group elements which permute the variables, and Claim 2, it is enough to see that if $m \in \mathcal{M}$ with $m_1 > 0$ and $x^m \in V$, then for $m' = (m_1-1, m_2+1, m_3, \ldots m_n)$, $x^{m'} \in V$ as well. This is seen by applying d_1 to x^m and using Claim 1.

□

Proposition 7.2.6. *Suppose $(q, n) \neq (2, 2)$. Then, in each degree i, the diagonal elements of the matrix of B_i are constant multiples of the same polynomial in c_s.*

Proof. By previous lemmas, every B_i is a diagonal matrix, with diagonal entries polynomials f_m in c parametrized by n-tuples of integers $m = (m_1, \ldots, m_n)$ such that $\sum m_j = i$. The kernel of B at specific c is spanned by all monomials x^m for which $f_m(c) = 0$. As the kernel is a submodule of $S^i\mathfrak{h}^*$ containing $< x_1^p, \ldots, x_n^p > \cap S^i\mathfrak{h}^*$, by the previous lemma it can either be $< x_1^p, \ldots, x_n^p > \cap S^i\mathfrak{h}^*$ or the whole $S^i\mathfrak{h}^*$. In other words, all polynomials f_m where one of m_j is $\geq p$ are identically zero, and all others have the same roots, so they are constant multiples of the same polynomial in c_s. □

Next, we will find these polynomials f_m in each degree and calculate their zeroes.

Proposition 7.2.7. *Suppose $(q,n) \neq (2,2)$. Then,*

a) If $n = 2$ and $q = p$, then B_i depends on the c_s for $p-1 \leq i \leq 2p-1$. The diagonal entries of B_i are $\Bbbk$-multiples of $c_1 + \ldots + c_{p-1} - 1$ for $i \geq p-1$ and $\Bbbk$-multiples of $(c_1 + \ldots + c_{p-1} - 1)(c_1 + 2c_2 + \ldots + (p-1)c_{p-1} + 1)$ for $i \geq p$.

b) If $n = 3$ and $q = 2$, then B_i depends on the c_s for $i = 2,3$, and the diagonal entries are $\Bbbk$-multiples of $c_1 + 1$.

c) Otherwise, the form B doesn't depend on c.

Proof. The matrices B_i are diagonal, with all diagonal entries being constant multiples of the same polynomial. Our strategy is to compute one nonzero diagonal entry.

First, we will show claim c). It is sufficient to show that the Dunkl operators on the quotient $M_{1,c}(\tau)/\mathrm{Ker}B$ are independent of c. As in the proof of Proposition 7.2.2, we compute the part of the Dunkl operator associated to the conjugacy class C_λ with eigenvalue λ, and claim that for any monomial $f \in S^i\mathfrak{h}^*$, the part of $D_y(f)$ which is the coefficient of c_λ,

$$-\sum_{s \in C_\lambda} (y, \alpha_s) \frac{1}{\alpha_s}(1-s).f,$$

is in $\mathrm{Ker}(B)$.

Using Proposition 7.1.2, we can write this sum over nonzero $\alpha \in \mathfrak{h}^*$ and $\alpha^\vee \in \mathfrak{h}$, such that $(\alpha, \alpha^\vee) = 1-\lambda$. Writing it as consecutive sums over α and then over $\alpha^\vee$, it is enough to show that the inner sum, over all $\alpha^\vee$ such that $(\alpha, \alpha^\vee) = 1-\lambda$, is contained in $\langle x_1^p, \ldots, x_n^p \rangle$. As in the previous calculations, we fix α, and change basis of $\mathfrak{h}^*$ to $x'_1, \ldots, x'_n$ so that $x'_1 = \alpha$. Let the dual basis of $\mathfrak{h}$ be $y'_1, \ldots, y'_n$. The inner sum, with vectors written in the basis y'_i, is then over $\alpha^\vee \in A_\lambda := \{((1-\lambda), a_2, \ldots, a_n) \neq 0 \mid a_2, \ldots, a_n \in \mathbb{F}_q\}$. By Proposition 7.1.2, the reflection s corresponding to $\alpha \otimes \alpha^\vee$, $\alpha = (1, 0, \ldots, 0)$, $\alpha^\vee = ((1-\lambda), a_2, \ldots, a_n)$, acts on $\mathfrak{h}^*$ as

$$s.x'_1 = \lambda x'_1$$

$$s.x_i' = x_i' - a_i x_1', \;\; i > 1$$

In addition to fixing α, let us also factor the constant $-(y,\alpha)$ out. The inner sum for fixed α is then $\sum_{\alpha^\vee \in A_\lambda} \frac{1}{\alpha}(1-s).f$. The set A_λ is parametrized by $(a_2,\dots,a_n) \in \mathbb{F}_q^{n-1}$ if $\lambda \neq 1$, and by $(a_2,\dots,a_n) \neq 0 \in \mathbb{F}_q^{n-1}$ if $\lambda = 1$. However, if $\lambda = 1$, the summand corresponding to $(a_2,\dots,a_n) = 0$ is 0, so we can assume the sum is over all $(a_2,\dots,a_n) \in \mathbb{F}_q^{n-1}$ in both cases.

The inner sum we are calculating is equal to

$$\sum_{(a_2,\dots,a_n)\in\mathbb{F}_q^{n-1}} \frac{1}{x_1'}\Big(f(x_1',\dots,x_n') - f(\lambda x_1', x_2' - a_2x_1', \dots, x_n' - a_n x_1')\Big). \qquad (\star)$$

It is enough to calculate it for f of the form $f = x_1'^{b_1}\dots x_n'^{b_n}$, $b_i < p$. For such f,

$$(\star) = \sum_{(a_2,\dots,a_n)\in\mathbb{F}_q^{n-1}} \frac{-1}{x_1'}\Big((\lambda x_1')^{b_1}(x_2' - a_2x_1')^{b_2}\dots(x_n' - a_nx_1')^{b_n} - x_1'^{b_1}\dots x_n'^{b_n}\Big) =$$

$$= \sum_{(a_2,\dots,a_n)\in\mathbb{F}_q^{n-1}} \sum_{i_2,\dots i_n} -\binom{b_2}{i_2}\dots\binom{b_n}{i_n}\lambda^{b_1}(-a_2)^{i_2}\dots(-a_n)^{i_n} x_1'^{b_1+i_2+\dots i_n-1}x_2'^{b_2-i_2}\dots x_n'^{b_n-i_n}$$

the last sum being over all $0 \le i_j \le b_j$ such that not all i_j are 0 at the same time. The coefficient of each monomial in x_i can be seen as a monomial of degree $i_2 + \dots i_n$ in variables a_i, so when we sum it over all $(a_2,\dots a_n) \in \mathbb{F}_q^{n-1}$ to get the sum $(\star)$, we can use Lemma 6.3.1 to conclude $(\star)$ is 0 whenever the degree of all polynomials appearing is small enough, more precisely, whenever there exists an index j such that $i_j < q-1$. As $i_j \le b_j < p$ for all j, this only fails when $p = q$ and $b_2 = b_3 = \dots = b_n = p-1$. In other words, this proves c) whenever $q \neq p$.

Now assume $q = p$. By the above argument, the only monomials f for which $(\star)$ is not yet known to be zero are the ones of the form $f = x_1'^b x_2'^{p-1}\cdots x_n'^{p-1}$. For such f, the sum $(\star)$ is by the above argument equal to

$$(-1)^{(p-1)(n-1)+1} \sum_{(a_2,\dots,a_n)\in\mathbb{F}_q^{n-1}} \lambda^b (a_2)^{p-1}\dots(a_n)^{p-1} x_1'^{b+(p-1)(n-1)-1}$$

While this is not 0, the monomial $x_1'^{b+(p-1)(n-1)}$ is by Lemma 7.2.2 in Ker$B$ whenever it has degree at least p, meaning whenever

$$b+(p-1)(n-1)-1 \geq p.$$

If $n=2$, this condition is $b \geq 2$, which is not satisfied only when $b=0,1$. Thus, for $n=2$, $q=p$, the diagonal matrices B_i don't depend on c in degrees $i<p-1$, their entries are multiples of some polynomial in c in degree $p-1$, and some other polynomial (divisible by the first one) in degrees p and higher. Finally, by corollary 7.2.3, all matrices B_i become zero at degrees $2p-1$ and higher.

For $n=3$, this condition is $b+2(p-1)-1 \geq p$, which is equivalent to $p+b \geq 3$. This will not be satisfied only if $p=2$ and $b=0$. So, for $n=3$, $p=2$, the matrices B_0 and B_1 don't depend on c, the diagonal entries of B_i are multiples of the same polynomial in c for $i=2,3$, and $B_4=0$. For $n=3$ and all other p, the matrices B_i don't depend on c.

For $n>3$, the inequality $b+(p-1)(n-1)-1 \geq p$ is always satisfied, so there is never a dependence of matrices B_i on the parameters c.

This finishes the proof of (c) and describes the cases in a) and b) for which there might be dependence on parameters c. To finish the proof, it remains to find specific polynomials in cases: (a) $n=2, q=p$, degrees $p-1$ and p and; (b)$n=3, q=2$, degrees 2 and 3.

Next, we prove (a). Let $n=2$ and $q=p$. We need to compute one nonzero entry of the matrix B_{p-1} and one nonzero entry of B_p.

To compute the polynomial in degree $p-1$, by Proposition 7.2.6, it suffices to compute $B(x_1^{p-1}, y_1^{p-1}) = B(D_{y_1}(x_1^{p-1}), y_1^{p-2})$. For that, by Proposition 7.2.4, it suffices to show that the coefficient of x_1^{p-2} in $D_{y_1}(x_1^{p-1})$ is a constant multiple of $c_1+\ldots+c_{p-1}-1$. Compute

$$D_{y_1}(x_1^{p-1}) = \partial_{y_1}(x_1^{p-1}) - \sum_C c_s \sum_{s\in C} (\alpha_s, y_1)\frac{1}{\alpha_s}(x_1^{p-1}-(s.x_1)^{p-1}).$$

The coefficient of x_1^{p-2} in $\partial_{y_1}(x_1^{p-1}) = (p-1)x_1^{p-2}$ is $p-1=-1$, so it suffices to show that for each conjugacy class C the coefficient of x_1^{p-2} in

$$\sum_{s\in C}(\alpha_s, y_1)\frac{1}{\alpha_s}((s.x_1)^{p-1} - x_1^{p-1})$$

is 1. Using $\binom{p-1}{i} = (-1)^i$, we can write this as

$$\sum_{s\in C}(\alpha, y_1)\frac{1}{\alpha_s}((x_1 - (x_1, \alpha_s^\vee)\alpha_s)^{p-1} - x_1^{p-1}) =$$

$$= \sum_{s\in C}(\alpha_s, y_1)\sum_{i=1}^{p-1}(\alpha_s^\vee, x_1)^i \alpha_s^{i-1} x_1^{p-1-i}.$$

The bases x_i and y_i are dual, so $\alpha_s = (\alpha_s, y_1)x_1 + (\alpha_s, y_2)x_2$, and the coefficient of x_1^{i-1} in α_s^{i-1} is $(\alpha_s, y_1)^{i-1}$. Thus, the coefficient of x_1^{p-2} in the above sum is

$$\sum_{s\in C}(y_1, \alpha_s)\sum_{i=1}^{p-1}(\alpha_s^\vee, x_1)^i (y_1, \alpha_s)^{i-1}$$

which can be written as

$$\sum_{s\in C}\left(((\alpha_s^\vee, x_1)(\alpha_s, y_1) - 1)^{p-1} - 1\right).$$

Each term $((\alpha_s^\vee, x_1)(\alpha_s, y_1) - 1)^{p-1} - 1$ is nonzero if and only if $(\alpha_s^\vee, x_1)(\alpha_s, y_1) = 1$, in which case it is -1. There are $p-1$ choices of the first coordinates of $\alpha_s, \alpha_s^\vee$ that make their product 1. The product of the second coordinate must now be $(\alpha, \alpha^\vee) - 1$. This is nonzero, so there are $p-1$ choices for the second coordinates. Hence, the sum is $(p-1)^2(-1) = -1$, as desired. Note that this term will appear as a multiplicative factor in higher degrees, since the matrices of B are defined inductively. This proves the claim for degree $p-1$.

Now let us consider degree p, with $n=2$ and $q=p$. We calculate

$$B(x_1^{p-1}x_2, y_1^{p-1}y_2) = B(D_{y_2}(x_1^{p-1}x_2), y_1^{p-1}),$$

which is equal to the product of the coefficient of x_1^{p-1} in $D_{y_2}(x_1^{p-1}x_2)$ and $B(x_1^{p-1}, y_1^{p-1})$. We proved that $B(x_1^{p-1}, y_1^{p-1})$ is a constant multiple of $c_1 + \ldots + c_{p-1} - 1$, so we are now calculating the coefficient of x_1^{p-1} in

$$\begin{aligned} D_{y_2}(x_1^{p-1}x_2) &= \partial_{y_2}(x_1^{p-1}x_2) - \sum_\lambda c_\lambda \sum_{s \in C\lambda} (\alpha_s, y_2)\frac{1}{\alpha_s}(1-s).(x_1^{p-1}x_2) \\ &= x_1^{p-1} - \sum_\lambda c_\lambda \sum_{s \in C_\lambda} (\alpha_s, y_2)\frac{1}{\alpha_s}(x_1^{p-1}x_2 - (s.x_1)^{p-1}(s.x_2)). \end{aligned}$$

We now use parametrization of conjugacy classes C_λ by $\alpha \otimes \alpha^\vee$ from Lemma 7.1.2 and Example 7.1.3:

$$\lambda \neq 1: \quad C_\lambda \leftrightarrow \{\begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} 1-\lambda-bd \\ d \end{bmatrix} | b, d \in \mathbb{F}_p\} \cup \{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 1-\lambda \end{bmatrix} | a \in \mathbb{F}_p\}$$

$$\lambda = 1: \quad C_1 \leftrightarrow \{\begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} -bd \\ d \end{bmatrix} | b, d \in \mathbb{F}_p, d \neq 0\} \cup \{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 0 \end{bmatrix} | a \in \mathbb{F}_p, a \neq 0\}.$$

We are calculating the coefficient of x_1^{p-1} in

$$x_1^{p-1} - \sum_\lambda c_\lambda \sum_{\alpha \otimes \alpha^\vee \leftrightarrow C_\lambda} (\alpha, y_2)\frac{1}{\alpha}\left(x_1^{p-1}x_2 - (x_1 - (\alpha^\vee, x_1)\alpha)^{p-1}(x_2 - (\alpha^\vee, x_2)\alpha)\right) =$$

$$= x_1^{p-1} - \sum_\lambda c_\lambda \sum_{\alpha \otimes \alpha^\vee \leftrightarrow C_\lambda} (\alpha, y_2)\frac{1}{\alpha}\left(x_1^{p-1}x_2 - (x_1 - (\alpha^\vee, x_1)\alpha)^{p-1}x_2\right) -$$

$$- \sum_\lambda c_\lambda \sum_{\alpha \otimes \alpha^\vee \leftrightarrow C_\lambda} (\alpha, y_2)\frac{1}{\alpha}\left((x_1 - (\alpha^\vee, x_1)\alpha)^{p-1}(\alpha^\vee, x_2)\alpha\right).$$

The term $x_1^{p-1} - (x_1 - (\alpha^\vee, x_1)\alpha)^{p-1}$ is divisible by α, so

$$\frac{1}{\alpha}\left(x_1^{p-1}x_2 - (x_1 - (\alpha^\vee, x_1)\alpha)^{p-1}(x_2 - (\alpha^\vee, x_2)\alpha)\right),$$

written in a monomial basis in x_1 and x_2, is divisible by x_2. These terms can be disregarded when calculating the coefficient of x_1^{p-1} in the above sum.

Let $\alpha_b = x_1 + bx_2$. We are calculating the coefficient of x_1^{p-1} in

$$x_1^{p-1} - \sum_\lambda c_\lambda \sum_{\alpha\otimes\alpha^\vee \leftrightarrow C_\lambda} (\alpha, y_2)\frac{1}{\alpha}\left((x_1 - (\alpha^\vee, x_1)\alpha)^{p-1}(\alpha^\vee, x_2)\alpha\right) =$$

$$= x_1^{p-1} - \sum_\lambda c_\lambda \Big(\sum_{b,d} b\frac{1}{\alpha_b}\left((x_1 - (1-\lambda-bd)\alpha_b)^{p-1} d\alpha_b\right) + \sum_a \frac{1}{x_2}\left((x_1 - ax_2)^{p-1}(1-\lambda)x_2\right)\Big)$$

$$= x_1^{p-1} - \sum_\lambda c_\lambda \Big(\sum_{b,d} bd \sum_{i=0}^{p-1} (1-\lambda-bd)^i x_1^{p-1-i}(x_1+bx_2)^i + (1-\lambda)\sum_a (x_1 - ax_2)^{p-1}\Big)$$

Here, the sum over is over all $b, d \in \mathbb{F}_p$ and over all $a \in \mathbb{F}_p$ if $\lambda \neq 1$, and over $a, b, d \in \mathbb{F}_p$, $d, a \neq 0$ if $\lambda = 1$. The coefficient of x_1^{p-1} is:

$$1 - \sum_\lambda c_\lambda \Big(\sum_{b,d} bd \sum_{i=0}^{p-1} (1-\lambda-bd)^i + (1-\lambda)\sum_a 1\Big) =$$

$$1 - \sum_{\lambda\neq 1} c_\lambda \Big(\sum_{b,d} (bd)^{p-1}(-\lambda)\Big) - c_1\Big(\sum_{b,d}(bd)^{p-1}\Big) = 1 + \sum_\lambda \lambda c_\lambda.$$

This ends the proof of (a).

To prove (b), $n = 3$, $q = 2$, we computed the matrices B_i explicitly. □

The combination of these results and the explicit computations in case $(n, q) = (2, 2)$ gives us the main theorem of this section:

Theorem 7.2.8. *Let $\Bbbk$ be an algebraically closed field of characteristic p. Let $W = GL_n(\mathbb{F}_q)$ for $q = p^r$ and $n \geq 2$. The following is a complete classification of characters of $L_{1,c}(\mathrm{triv})$ for all values of c:*

(q,n)	c	$\chi_{L_{1,c}(\mathrm{triv})}(z)$	$\mathrm{Hilb}_{L_{1,c}(\mathrm{triv})}(z)$
$(q,n)\neq(2,2)$	*generic*	$\sum_{i\geq 0}([S^i\mathfrak{h}^*/\langle x_1^p,\dots,x_n^p\rangle\cap S^i\mathfrak{h}^*])z^i$	$\left(\frac{1-z^p}{1-z}\right)^n$
$(2,2)$	*generic*	$(\sum_{i\geq 0}[S^i\mathfrak{h}^*]\,z^i)\,(1-z^4)(1-z^6)$	$\frac{(1-z^4)(1-z^6)}{(1-z)^2}$
$(2,3)$	$c=1$	$[\mathrm{triv}]+[\mathfrak{h}^*]\,z$	$1+3z$
$(p,2),p\neq 2$	$\sum_{i=1}^{p-1}c_i=1$	$\sum_{i=0}^{p-2}[S^i\mathfrak{h}^*]\,z^i$	$\sum_{i=0}^{p-2}(i+1)z^i$
$(p,2),p\neq 2$	$\sum_{i=1}^{p-1}c_i\neq 1,\sum_{i=1}^{p-1}ic_i=-1$	$\sum_{i=0}^{p-1}[S^i\mathfrak{h}^*]\,z^i$	$\sum_{i=0}^{p-1}(i+1)z^i$
$(2,2)$	$c=1$	$[\mathrm{triv}]$	1
$(2,2)$	$c=0$	$[\mathrm{triv}]+[\mathfrak{h}^*]\,z+[\mathrm{triv}]\,z^2$	$1+2z+z^2$

Proof. For $(n,q)=(2,2)$, the matrices B_i of the form B can be computed explicitly, and one can see that they are 0 starting in degree 1 when $c=1$ and starting in degree 3 when $c=0$. For all other c, they are full rank on $N_{1,c}(\mathrm{triv})$, so $L_{1,c}(\mathrm{triv})=N_{1,c}(\mathrm{triv})$ is the quotient of the Verma module by squares of the invariants, which are in degrees 4 and 6.

For $(q,n)\neq(2,2)$ and generic c, we saw in Proposition 7.2.2 that $J_0'(\mathrm{triv})$ contains $\langle x_1^p,\dots,x_n^p\rangle$, so by Proposition 6.2.4 the reduced module $R_{1,c}(\mathrm{triv})$ is a quotient of the trivial module. From this it follows that for generic c, $J_{1,c}(\mathrm{triv})=\langle x_1^p,\dots,x_n^p\rangle$ and that the character of $L_{1,c}(\mathrm{triv})$ for generic c and $(q,n)\neq(2,2)$ is as stated above.

Characters for special c are computed by looking at roots of polynomials on the diagonal in B_i, and are computed directly from Proposition 7.2.7. □

Remark 7.2.9. *Notice that for n,p,r large enough, the character doesn't depend on c at all. This never happens in characteristic zero.*

Remark 7.2.10. *Notice that the claims from Remarks 6.2.8 and 6.2.11 and Question 6.2.9 hold in case of $W=GL_n(\mathbb{F}_q)$. Namely, by observing the characters one can see that all $L_{t,c}(\mathrm{triv})$ for generic c have one dimensional top degree and are thus Frobenius; that for h_1 the reduced Hilbert series of $L_{1,c}(\mathrm{triv})$ at generic c, $h_1(1)$ is either $|W|$ (in case of $(q,n)=(2,2)$, when they are both 6) or $h_1(1)=1$ (in all other cases), and that for h_0 the Hilbert series of $L_{0,c}(\mathrm{triv})$ at generic c, the equality $h_0=h_1$ always holds.*

7.3 Description of $L_{t,c}(\text{triv})$ for $SL_n(\mathbb{F}_{p^r})$

In this section we explore category $\mathcal{O}$ for the rational Cherednik algebra associated to the special linear group over a finite field. We start with some preliminary facts about relations between rational Cherednik algebras associated to some group and to its normal subgroup, and by looking more carefully into conjugacy classes of reflections in $SL_n(\mathbb{F}_{p^r})$.

7.3.1 Normal subgroups of reflection groups

Let $W \subseteq GL(\mathfrak{h})$ be any reflection group, and assume $N \subset W$ is a normal subgroup with a property that two reflections in N are conjugate in N if and only if they are conjugate in W. Let c be a $\Bbbk$-valued conjugation invariant function on reflections of N, and extend c to all reflections in W by defining it to be zero on reflections which are not in N. Then one can consider the rational Cherednik algebra $H_{t,c}(N,\mathfrak{h})$ as a subalgebra of $H_{t,c}(W,\mathfrak{h})$; it has fewer generators and the same relations.

Let τ be an irreducible representation of W, and assume it is irreducible as a representation $\tau|_N$ of N. Consider two representations of $H_{t,c}(N,\mathfrak{h})$: the irreducible representation $L_{t,c}(\tau|_N) = L_{t,c}(N,\mathfrak{h},\tau|_N)$, and the irreducible representation $L_{t,c}(\tau) = L_{1,c}(W,\mathfrak{h},\tau)$ of $H_{t,c}(W,\mathfrak{h})$ restricted to $H_{t,c}(N,\mathfrak{h})$.

Lemma 7.3.1. *As representations of $H_{1,c}(N,\mathfrak{h})$, $L_{1,c}(\tau|_N) \cong L_{1,c}(\tau)|_{H_{1,c}(N,\mathfrak{h})}$.*

Proof. The reflections in N are a subset of reflections in W. Because N is normal in W, every conjugacy class in W is either contained in N or doesn't intersect it. By the assumption, two reflections in N which are conjugate in W are also conjugate in N, so conjugacy classes in N are a subset of conjugacy classes in W.

The Verma modules $M_{t,c}(W,\mathfrak{h},\tau)$ and $M_{t,c}(N,\mathfrak{h},\tau|_N)$ don't invoke the group in their definition, and are naturally isomorphic as $H_{t,c}(N,\mathfrak{h})$ representations. The modules $L_{t,c}(\tau)$ and $L_{t,c}(\tau|_N)$ are their quotients by the kernel of the contravariant form, which is controlled by Dunkl operators. Because of the discussion of conjugacy classes in N and W and because of the definition of c, the Dunkl operators are the same for $H_{t,c}(N,\mathfrak{h})$ and $H_{t,c}(W,\mathfrak{h})$. □

One could define Verma modules, baby Verma modules and their quotients by the kernel of the contravariant form (chosen so that it is nondegenerate on lowest weights) even in cases when the lowest weight is not irreducible as a representation of the reflection group. In that case, a lemma analogous to the above one would be that the composition series of τ as a representation of N is the same as the composition series of $L_{t,c}(\tau)$ as a representation of $H_{t,c}(N,\mathfrak{h})$. We will not need this here.

7.3.2 Conjugacy classes of reflections in $SL_n(\mathbb{F}_q)$

In this section we will study $W = SL_n(\mathbb{F}_q)$ for $q = p^r$ is a prime power and $n > 1$. As before, let $\mathfrak{h} = \mathbb{k}^n$, $\mathbb{k} = \overline{\mathbb{F}_p}$, and τ be the trivial representation. Further, let Q be the set of nonzero squares in $\mathbb{F}_q$ and R be the set of non-squares.

All reflections in $SL_n(\mathbb{F}_q)$ are unipotent, conjugate in $GL_n(\mathbb{F}_q)$ to d_1. It is easy to see that $SL_n(\mathbb{F}_q)$ is generated by them. $SL_n(\mathbb{F}_q)$ is a normal subgroup of $GL_n(F_q)$ and it contains all the reflections from the unipotent conjugacy class in $GL_n(\mathbb{F}_q)$. However, the second condition from the above discussion, that two reflections are conjugate in $GL_n(\mathbb{F}_q)$ if and only if they are conjugate in $SL_n(\mathbb{F}_q)$, is not satisfied for all n, q. For example, in $SL_2(\mathbb{F}_3)$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is not conjugate to } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Proposition 7.3.2. *Let $q = p^r$ be a prime power. If $n \geq 3$, or $p = 2$, then two reflections are conjugate in $SL_n(\mathbb{F}_q)$ if and only if they are conjugate in $GL_n(\mathbb{F}_q)$, and there is one conjugacy class of reflections in $SL_n(\mathbb{F}_q)$. Otherwise, there are two conjugacy classes of unipotent reflections in $SL_n(\mathbb{F}_q)$.*

Proof. Every reflection $s \in SL_n(\mathbb{F}_q)$ is a unipotent reflection in $GL_n(\mathbb{F}_q)$, so there exists $g \in GL_n(\mathbb{F}_q)$ so that $s = gd_1g^{-1}$. To conclude that s and d_1 are conjugate in $SL_n(\mathbb{F}_q)$, it is enough to find some h in the centralizer $Z(d_1)$ of d_1 such that $gh \in SL_n(\mathbb{F}_q)$. For that, it is enough to find an element of $Z(d_1)$ of arbitrary nonzero determinant.

The general form of an element of $Z(d_1)$ is

$$h = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{11} & 0 & \cdots & 0 \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Its determinant is

$$a_{11}^2 \cdot \det \begin{bmatrix} a_{33} & \ldots & a_{3n} \\ \vdots & & \vdots \\ a_{n3} & \ldots & a_{nn} \end{bmatrix}$$

If $n \geq 3$, determinant of the $(n-2) \times (n-2)$ matrix in the bottom right corner can be any nonzero element of $\mathbb{F}_q$ (as this matrix is a arbitrary element of $GL_{n-2}(\mathbb{F}_q)$), so there exists $h \in Z(d_1)$ of an arbitrary determinant and the claim is proved.

If $n = 2$, the centralizer of d_1 in $GL_2(\mathbb{F}_q)$ consists of matrices of the form

$$h = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix},$$

whose determinants are of the form $a^2, a \in \mathbb{F}_q$.

If $p = 2$, then any nonzero element of $\mathbb{F}_{p^r}$ has a square root (as the map $a \mapsto a^2$ is injective and hence surjective on $\mathbb{F}_q^\times$), and there exist elements in $Z(d_1)$ of arbitrary determinant. So, for $SL_2(\mathbb{F}_{2^r})$ contains only one conjugacy class of reflections.

Finally, assume $n = 2$, $p \neq 2$ and let us show there are two conjugacy classes of reflections in $SL_2(\mathbb{F}_q)$, namely

$$C_Q = \{gd_1g^{-1} | g \in GL_2(\mathbb{F}_q), \det(g) \textit{ is a square}\}$$

and

$$C_R = \{gd_1g^{-1} | g \in GL_2(\mathbb{F}_q), \det(g) \textit{ is not a square}\}$$

Two reflections are conjugate in $SL_2(\mathbb{F}_q)$ if and only if they are conjugate in $GL_2(\mathbb{F}_q)$ by some element whose determinant is in Q. From this it is clear that any two elements in C_Q are conjugate in $SL_2(\mathbb{F}_q)$.

Next, sets Q and R both have $(q-1)/2$ elements, and satisfy $R \cdot R = Q$ and $R^{-1} = R$. From this it follows that any two elements of C_R are conjugate in $SL_2(\mathbb{F}_q)$: for $g_1 d_1 g_1^{-1}$ and $g_2 d_1 g_2^{-1}$ both in C_R, we can write $g_1 d_1 g_1^{-1} = (g_1 g_2^{-1})(g_2 d_1 g_2^{-1})(g_1 g_2^{-1})^{-1}$ to see that they are indeed conjugate via an element $g_1 g_2^{-1}$ whose determinant is a square.

Finally, assuming that some element $g d_1 g^{-1} \in C_R$ is conjugate in $SL_2(\mathbb{F}_q)$ to some element of C_Q means it is equal to $g' d_1 g'^{-1}$ with $g' \in SL_2(\mathbb{F}_q)$, so an $gg'^{-1} \in Z(d_1)$, and has the determinant equal to $\det g$. But $\det g$ is by assumption not a square, whereas every element of $Z(d_1)$ has determinant of the form a^2. So, C_Q and C_R are really separate conjugacy classes in $SL_2(\mathbb{F}_q)$. □

Let us continue using the same notation for conjugacy classes of reflections in $GL_n(\mathbb{F}_q)$, C_λ, $\lambda = 1, \ldots, q-1$. If $n \geq 3$ or $p = 2$, then $SL_n(\mathbb{F}_q)$ has only one conjugacy class, equal to C_1. Let us call it C, and the constant associated to it by c. If $n = 2$ and $p \neq 2$, there are two conjugacy classes C_R and C_Q, with associated parameters c_R and c_Q, and they satisfy $C_R \cup C_Q = C_1$. In the situation of only one conjugacy class we will use Lemma 7.3.1 to transfer character formulas for rational Cherednik algebras associated to $GL_n(\mathbb{F}_q)$ to character formulas for rational Cherednik algebras associated to $SL_n(\mathbb{F}_q)$. In the situation of two conjugacy classes, we will have to do more computations to get character formulas. Let us first look more closely into the situation of two conjugacy classes.

Lemma 7.3.3. *Let $n = 2$, $q = p^r$, and $p \neq 2$. Let $\gamma \in R$ be an arbitrary non-square in $\mathbb{F}_q$. Let s be a reflection in $SL_2(\mathbb{F}_q)$. Then, s and $\gamma s - (\gamma - 1)$ are in different conjugacy classes.*

Proof. The proof follows from Proposition 7.3.2. The map $F_\gamma : s \mapsto \gamma s - (\gamma - 1)$ maps reflections to reflections, and its inverse is $F_{\gamma^{-1}}$. So, it is enough to show it maps $s \in C_Q$ to an element of C_R.

Let d_γ as before denote the diagonal matrix with diagonal entries $\gamma^{-1}, 1, 1, \ldots, 1$. For $s = d_1$, we have $F_\gamma(d_1) = d_\gamma^{-1} d_1 d_\gamma \in C_R$. If s is a conjugate in $SL_2(\mathbb{F}_q)$ to d_1, say $s = h d_1 h^{-1}$, then $F_\gamma(s) = h d_\gamma^{-1} d_1 d_\gamma h^{-1} = (h d_\gamma^{-1} h^{-1}) s (h d_\gamma^{-1} h^{-1})^{-1} \in C_R$. □

The following lemma is useful in computations, and is a stronger version of Lemma 6.3.1.

Lemma 7.3.4. *If $d \equiv 0 \pmod{q-1}$, then $\sum_{i \in Q} i^d = \sum_{i \in R} i^d = \frac{q-1}{2}$. If $d \equiv \frac{q-1}{2} \pmod{q-1}$, then $\sum_{i \in Q} i^d = -\sum_{i \in R} i^d = \frac{q-1}{2}$. Otherwise, $\sum_{i \in Q} i^d = \sum_{i \in R} i^d = 0$.*

Proof. For this proof, let $S_Q = \sum_{i \in Q} i^d$ and $S_R = \sum_{i \in R} i^d$. Suppose $d \equiv 0 \pmod{q-1}$. Then, $i^d = 1$ for all nonzero $i \in \mathbb{F}_q$, and $S_R = S_Q = |Q| = |R| = \frac{q-1}{2}$. Suppose $d \equiv \frac{q-1}{2} \pmod{q-1}$. In that case, if $i \in Q$, $i^d = 1$, and if $i \in R$, then $i^d = -1$. Thus, $S_Q = -S_R = \frac{q-1}{2}$. Suppose neither holds. For any $a \in Q$ it is easy to see that $aR = R$ and $aQ = Q$, so $S_R = a^d S_R$ and $S_Q = a^d S_Q$. If a is a multiplicative generator of the cyclic multiplicative group $\mathbb{F}_q^\times$, then 1 and a^d are different elements of Q, so $(1 - a^d) S_Q = (1 - a^d) S_R = 0$ implies that $S_R = S_Q = 0$. □

Next, we parametrize reflections in each conjugacy class. Remember the notation from Proposition 7.1.2: unipotent reflections are identified with all $\alpha \otimes \alpha^\vee \in \mathfrak{h}^* \otimes \mathfrak{h}$ such that $(\alpha, \alpha^\vee) = 0$, in such a way that the action of a reflection s on $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$ is

$$s.x = x - (\alpha^\vee, x)\alpha$$

$$s.y = y + (\alpha, y)\alpha^\vee.$$

Lemma 7.3.5. *Conjugacy classes C_Q and C_R of reflections in $SL_2(\mathbb{F}_{p^r})$, $p > 2$, are parametrized through $\alpha \otimes \alpha^\vee$ as*

$$C_Q = \left\{ \gamma \begin{bmatrix} 1 \\ a \end{bmatrix} \otimes \begin{bmatrix} a \\ -1 \end{bmatrix} \mid a \in \mathbb{F}_q,\ \gamma \in Q \right\} \bigcup \left\{ \gamma \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid \gamma \in Q \right\},$$

$$C_R = \left\{ \gamma \begin{bmatrix} 1 \\ a \end{bmatrix} \otimes \begin{bmatrix} a \\ -1 \end{bmatrix} \mid a \in \mathbb{F}_q,\ \gamma \in R \right\} \bigcup \left\{ \gamma \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid \gamma \in R \right\}.$$

Proof. The proof is straightforward. The reflection d_1 is identified with $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Conjugating it by $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{F}_q)$ gives us a reflection gd_1g^{-1} identified with

$$\frac{1}{\det g} \begin{bmatrix} -c \\ a \end{bmatrix} \otimes \begin{bmatrix} a \\ c \end{bmatrix}.$$

As g can always be scaled by an element of the centralizer of d_1 so that either $c = -1$ or $c = 0$ and $a = 1$, and that gd_1g^{-1} is in C_Q or C_R depending whether $\det g$ is in Q or R, the description follows. □

7.3.3 Description of $L_{0,c}(\text{triv})$ for $SL_n(\mathbb{F}_{p^r})$

Theorem 7.3.6. *Characters of the irreducible modules* $L_{0,c}(\text{triv})$ *for the rational Cherednik algebra* $H_{0,c}(SL_n(\mathbb{F}_q,\mathfrak{h}))$ *are:*

(q,n)	c	$\chi_{L_{0,c}(\text{triv})}(z)$	$\text{Hilb}_{L_{0,c}(\text{triv})}(z)$
$(q,n) \neq (2,2)$	*any*	$[\text{triv}]$	1
$(2,2)$	0	$[\text{triv}]$	1
$(2,2)$	$c \neq 0$	$[\text{triv}] + [\mathfrak{h}^*]z + ([S^2\mathfrak{h}^*] - [\text{triv}])z^2 +$ $+([S^3\mathfrak{h}^*] - [\mathfrak{h}^*] - [\text{triv}])z^3$	$1 + 2z + 2z^2 + z^3$

Proof. $SL_n(\mathbb{F}_q)$ is a normal subgroup of $GL_n(F_q)$, and for $n \geq 3$ or $p = 2$, by Proposition 7.3.2 it satisfies the conditions of Lemma 7.3.1. Thus, in those cases, irreducible representations of $H_{t,c}(SL_n(\mathbb{F}_q),\mathfrak{h})$ have the same characters as irreducible representations of $H_{t,c}(GL_n(\mathbb{F}_q),\mathfrak{h})$, where c is extended to conjugacy classes of reflections in $GL_n(\mathbb{F}_q) - SL_n(\mathbb{F}_q)$ by zero. So, in those cases we can deduce character formulas for $L_{0,c}(\text{triv})$ from Theorem 7.2.1.

The remaining case is $SL_2(\mathbb{F}_q)$, $p \neq 2$, for which we claim that $D_y(x) = 0$ for all x and y, and so the character is trivial. We have

$$D_y(x) = -c_R \sum_{\alpha\otimes\alpha^\vee \in C_R} (y,\alpha)(x,\alpha^\vee) - c_Q \sum_{\alpha\otimes\alpha^\vee \in C_Q} (y,\alpha)(x,\alpha^\vee),$$

so it is enough to show that for $T=Q$ or $T=R$,

$$\sum_{\alpha\otimes\alpha^\vee\in C_T}\alpha\otimes\alpha^\vee$$

is zero.

We know from the proof of Theorem 7.2.1 that $\sum_{\alpha\otimes\alpha^\vee\in C_{R\cup S}}\alpha\otimes\alpha^\vee=0$, so it is enough to prove that the sum over C_Q is zero. For this, let us calculate, using parametrization from Lemma 7.3.5:

$$\sum_{\alpha\otimes\alpha^\vee\in C_Q}\alpha\otimes\alpha^\vee = \sum_{\gamma\in Q}\sum_{a\in\mathbb{F}_q}\gamma(x_1+ax_2)\otimes(ay_1-y_2)+\sum_{\gamma\in Q}\gamma x_2\otimes y_1$$

By Lemma 7.3.4, if $q\neq 3$, then $\sum_{\gamma\in Q}\gamma=0$ and the whole sum is zero, as claimed. For $q=3$, $Q=\{1\}$, so the sum is equal to

$$\sum_{a\in\mathbb{F}_3}\left(-x_1\otimes y_2-ax_2\otimes y_2+ax_1\otimes y_1+a^2x_2\otimes y_1\right)+x_2\otimes y_1=$$

$$-x_2\otimes y_1+x_2\otimes y_1=0.$$

□

7.3.4 Description of $L_{1,c}(\mathrm{triv})$ for $SL_n(\mathbb{F}_{p^r})$ if $n\geq 3$ or $p=2$

As explained above and demonstrated in case of $t=0$, we can get character formulas for $H_{1,c}(SL_n(\mathbb{F}_q),\mathfrak{h})$ directly from the ones for $H_{1,c}(GL_n(\mathbb{F}_q),\mathfrak{h})$ when $n\geq 3$ or $p=2$. The following is a corollary of Lemma 7.3.1, Proposition 7.3.2 and results from Section 7.2, most notably 7.2.8.

Corollary 7.3.7. *Let $n\geq 3$ or $p=2$. Consider the rational Cherednik algebra $H_{1,c}(SL_n(\mathbb{F}_q),\mathfrak{h})$, its representations $M_{1,c}(\mathrm{triv})$, the contravariant form B on it, and the irreducible quotient $L_{1,c}(\mathrm{triv})$. Then all the results we proved for the group $GL_n(\mathbb{F}_q)$ hold also for $SL_n(\mathbb{F}_q)$. Specifically,* *a)* $D_y(x^p) = 0$ *in* $M_{1,c}(\mathrm{triv})$.

b) *The form* B *on* $M_{1,c}(\mathrm{triv})$ *is zero in degrees* $np - n + 1$ *and higher.*

c) *The matrices of the form* B *on* $M_{1,c}(\mathrm{triv})$ *in lexicographically ordered monomial bases are diagonal in all degrees.*

d) *All diagonal elements of the matrix of the form* B_i *on any graded piece* $M_{1,c}(\mathrm{triv})_i$ *are* $\Bbbk$*-multiples of the same polynomial in* c.

e) *If* $(q, n) \neq (2, 3)$, $(q, n) \neq (2, 2)$ *the matrices* B_i *of the form on* $M_{1,c}(\mathrm{triv})_i$ *don't depend on* c.

f) *If* $(q, n) = (2, 3)$, *only the matrices* B_2 *and* B_3 *depend on* c. *Their nonzero diagonal coefficients are constant multiples of* $c + 1$.

g) *If* $(q, n) = (2, 2)$, *then* $GL_n(\mathbb{F}_q) = SL_n(\mathbb{F}_q)$ *so the character formulas are the same.*

h) *The character formulas for representation* $L_{1,c}(\mathrm{triv})$ *of the rational Cherednik algebra* $H_{1,c}(SL_n(\mathbb{F}_q), \mathfrak{h})$ *are the same as for the rational Cherednik algebra* $H_{1,c}(GL_n(\mathbb{F}_q), \mathfrak{h})$, *with* c *extended to all classes of reflections in* $GL_n(\mathbb{F}_q)$ *which are not in* $SL_n(\mathbb{F}_q)$ *by zero.*

7.3.5 Description of $L_{1,c}(\mathrm{triv})$ for $SL_n(\mathbb{F}_{p^r})$ if $n = 2$ and $p > 2$

As in the case of $t = 0$, we need to study the case $n = 2$ and $p > 2$, when there are two conjugacy classes of reflections in $SL_n(\mathbb{F}_q)$, separately. The case $q = 3$ is the most complicated and we solve it by calculating the matrices of the form B explicitly. The following results address the remaining cases.

Proposition 7.3.8. *Let* $n = 2$, $q = p^r$ *for* p *an odd prime, and* $q \neq 3$. *In the Verma module* $M_{1,c}(\mathrm{triv})$ *for* $H_{1,c}(SL_2(\mathbb{F}_q))$, *all the vectors* $x^p, x \in \mathfrak{h}^*$, *are singular.*

Proof. We need to show that for every conjugacy class C_T, for $T = R$ or $T = Q$, and any x, y, the coefficient in $D_y(x^p)$ of c_T is zero. This coefficient is equal to

$$\sum_{\alpha\otimes\alpha^\vee\in C_T} (y,\alpha)(x,\alpha^\vee)^p\alpha^{p-1}.$$

Again, as $C_R \cup C_Q = C_1$ is a conjugacy class of reflections in $GL_2(\mathbb{F}_q)$, and the result holds there, it is enough to show this for C_Q.

As in the proof of Proposition 7.2.2, we claim that after writing it as a double sum, with the outer sum being over α and the inner over $\alpha^\vee$, the inner sum is already zero for any α. Fix α and change coordinates, so that we assume without loss of generality that $\alpha = x_1$. Then the inner sum is

$$\sum_{\substack{\alpha^\vee=\gamma y_2\\ \gamma\in Q}} (x,\alpha^\vee)^p = (y_2,x)^p \sum_{\gamma\in Q}\gamma^p.$$

Using Lemma 7.3.4, this is zero unless $p \equiv 0, \frac{q-1}{2} \pmod{q-1}$. However, this only happens in cases we excluded: $q = 2$ and $q = 3$. □

Next, we prove that acting by Dunkl operator produces elements of $M_{1,c}(\mathrm{triv})$ of a specific form.

Lemma 7.3.9. *Let $n = 2$, $q = p^r$ for p an odd prime, and $q \neq 3$, and consider the Verma module $M_{1,c}(\mathrm{triv})$ for $H_{1,c}(SL_n(\mathbb{F}_q))$. For any $f \in M_{1,c}(\mathrm{triv})/\langle x_i^p\rangle \cong S\mathfrak{h}^*/\langle x_i^p\rangle$ and any $y \in \mathfrak{h}$, there exists $h \in S\mathfrak{h}^*$ such that, as elements of $M_{1,c}(\mathrm{triv})/\langle x_i^p\rangle$,*

$$D_y(f) = \partial_y f + c_Q\cdot h + c_R\cdot h$$

Proof. The Dunkl operator action in case of $SL_2(\mathbb{F}_q)$ is

$$D_y(f) = \partial_y(f) - \sum_{T\in\{Q,R\}} c_T \sum_{s\in C_T} \frac{(\alpha_s,y)}{\alpha_s}(1-s).f$$

The strategy is to compute the sum $\sum_{s\in C_T}\frac{(\alpha_s,y)}{\alpha_s}(1-s).(f)$ parallel for $T = Q, R$, disregarding all terms that don't depend on the choice of T (these contribute equally to the coefficient of c_Q and c_R), and any elements of the ideal $\langle x_i^p \rangle$. We will use:

$$\sum_{a\in\mathbb{F}_q} a^m = 0 \quad \text{unless} \quad m \equiv 0 \pmod{q-1}, m \neq 0 \tag{7.3.1}$$

$$\sum_{\gamma\in Q} \gamma^m = \sum_{\gamma\in R} \gamma^m \quad \text{unless} \quad m \equiv \frac{q-1}{2} \pmod{q-1} \tag{7.3.2}$$

$$x_i^p = 0 \quad \text{in} \quad M_{1,c}(\text{triv})/\langle x_i^p \rangle \tag{7.3.3}$$

and the parametrization of conjugacy classes from 7.3.5:

$$C_T = \left\{ \gamma \begin{bmatrix} 1 \\ a \end{bmatrix} \otimes \begin{bmatrix} a \\ -1 \end{bmatrix} \mid a \in \mathbb{F}_q,\ \gamma \in T \right\} \bigcup \left\{ \gamma \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid \gamma \in T \right\}. \tag{7.3.4}$$

The rest of the proof is this computation.

We will do it for $D_y(f)$ for $y = y_1$ and $f = x_1^u x_2^v$. The general statement follows from this case by symmetry and linearity. We can assume $u, v \leq p-1$.

We claim that the sum

$$\sum_{s\in C_T} \frac{(\alpha_s, y_1)}{\alpha_s}(x_1^u x_2^v - (s.x_1)^u (s.x_2)^v) \tag{$\star$}$$

doesn't depend on T.

Reflections s corresponding to elements of the form $\gamma \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ satisfy $(\alpha_s, y_1) = (\gamma x_2, y_1) = 0$, so they don't contribute to the sum.

For s of the form $\gamma \begin{bmatrix} 1 \\ a \end{bmatrix} \otimes \begin{bmatrix} a \\ -1 \end{bmatrix}$, let us write the action on $x_i \in \mathfrak{h}^*$ explicitly, using notation $\alpha'_a = x_1 + a x_2$. The explicit action is

$$\begin{aligned} s.x_1 &= x_1 - a\gamma\alpha'_a \\ s.x_2 &= x_2 + \gamma\alpha'_a. \end{aligned}$$

Substituting this into $(\star)$ we get

$$(\star) = \sum_{\gamma\in T}\sum_{a\in\mathbb{F}_q}\frac{(\gamma\alpha'_a,y_1)}{\gamma\alpha'_a}(x_1^u x_1^v-(x_1-a\gamma\alpha'_a)^u(x_2+\gamma\alpha'_a)^v)=$$

$$=\sum_{\gamma\in T}\sum_{a\in\mathbb{F}_q}\sum_{i=0}^{u}\sum_{\substack{j=0\\(i,j)\neq(0,0)}}^{v}\frac{1}{\alpha'_a}(-1)\binom{u}{i}\binom{v}{j}x_1^{u-i}(-1)^i a^i\gamma^i\alpha'^i_a x_2^{v-j}\gamma^j\alpha'^j_a=$$

$$=\sum_{\gamma\in T}\sum_{a\in\mathbb{F}_q}\sum_{i=0}^{u}\sum_{\substack{j=0\\(i,j)\neq(0,0)}}^{v}\binom{u}{i}\binom{v}{j}(-1)^{i+1}a^i\gamma^{i+j}x_1^{u-i}x_2^{v-j}(x_1+ax_2)^{i+j-1}=$$

$$=\sum_{\gamma\in T}\sum_{a\in\mathbb{F}_q}\sum_{i=0}^{u}\sum_{\substack{j=0\\(i,j)\neq(0,0)}}^{v}\sum_{k=0}^{i+j-1}\binom{u}{i}\binom{v}{j}\binom{i+j-1}{k}(-1)^{i+1}a^{i+k}\gamma^{i+j}x_1^{u+j-1-k}x_2^{v-j+k}$$

Let us evaluate the sum $\sum_{a\in\mathbb{F}_q}a^{i+k}$. By (7.3.1), this is only nonzero if $i+k$ is divisible by $q-1$. We know that

$$i+k\leq 2i+j-1\leq 2u+v-1\leq 3(p-1)-1<3(p-1)\leq 3(q-1),$$

so let us consider three different cases: $i+k=0$, $i+k=q-1$ and $i+k=2(q-1)$.

CASE 1: $i+k=0$. In that case, $\sum_{a\in\mathbb{F}_q}a^0=0$, so this doesn't contribute either.

CASE 2: $i+k=q-1$. After substituting $\sum_{a\in\mathbb{F}_q}a^{q-1}=-1$, $k=q-1-i$, and after that $m=i+j$, we get that part of $(\star)$ corresponding to this case equals

$$\sum_{\gamma\in T}\sum_{i=0}^{u}\sum_{\substack{j=0\\(i,j)\neq(0,0)}}^{v}\binom{u}{i}\binom{v}{j}\binom{i+j-1}{q-1-i}(-1)^i\gamma^{i+j}x_1^{u+j+i-q}x_2^{v-j+q-1-i}=$$

$$=\sum_{\gamma\in T}\sum_{m=1}^{u+v}\sum_{j=0}^{v}\binom{u}{m-j}\binom{v}{j}\binom{m-1}{q-1-m+j}(-1)^{m-j}\gamma^m x_1^{u+m-q}x_2^{v-m+q-1}$$

Now we use (7.3.2) to describe $\sum_{\gamma\in T}\gamma^m$ and disregard all terms except $m\equiv\frac{q-1}{2}$ (mod $q-1$) (these terms we disregarded contribute to the coefficient h). There are again few cases, as

$$m \leq u+v \leq 2(p-1) < \frac{5}{2}(p-1) \leq \frac{5(q-1)}{2},$$

so we consider separately $m = \frac{q-1}{2}$ and $m = \frac{3(q-1)}{2}$.

CASE 2.1: $m = \frac{q-1}{2}$. One of the binomial coefficients in the expression is $\binom{m-1}{q-1-m+j}$, and we claim it is always zero for this choice of m. This is because

$$q-1-m+j = \frac{q}{2} - \frac{1}{2} + 2j \geq \frac{q-1}{2} > \frac{q-1}{2} - 1 = m-1.$$

In other words, case 2.1. never actually appears in the sum.

CASE 2.2: $m = \frac{3(q-1)}{2}$. We will show that this part of the sum is zero. First,

$$\frac{3(q-1)}{2} = m \leq u+v \leq 2(p-1)$$

implies that this can only happen when $p = q$. Next, because we $x_1^p = 0$ in the quotient, the only terms that can be nonzero are the ones with the power of x_1 less than p, so

$$u+m-q \leq p-1$$

which means

$$u \leq p-1+q-m = \frac{p+1}{2}.$$

Next, the term $\binom{u}{m-j}$ is zero unless

$$j \geq m-u \geq \frac{3(p-1)}{2} - \frac{p+1}{2} = p-2.$$

Since

$$j \leq v \leq p-1,$$

it follows that $j \in \{p-2, p-1\}$. In both those cases, the binomial coefficient $\binom{m-1}{q-1-m+j}$ is 0, as the numerator has a factor p and the denominator doesn't.

CASE 3: $i+k=2(q-1)$. The part of the sum $(\star)$ corresponding to this case is

$$\sum_{\gamma\in T}\sum_{i=0}^{u}\sum_{\substack{j=0\\(i,j)\neq(0,0)}}^{v}\binom{u}{i}\binom{v}{j}\binom{i+j-1}{2q-2-i}(-1)^i\gamma^{i+j}x_1^{u+j-1-2q+2+i}x_2^{v-j+2q-2-i}=$$

$$\sum_{\gamma\in T}\sum_{m=1}^{u+v}\sum_{j=0}^{v}\binom{u}{m-j}\binom{v}{j}\binom{m-1}{2q-2-m+j}(-1)^{m-j}\gamma^m x_1^{u+m-1-2q+2}x_2^{v-m+2q-2}$$

The powers of x_2 in this sum and the original power of x_1 are both $\leq p-1$, so

$$p-1\geq v-m+2(q-1)$$

$$p-1\geq u\geq m-v\geq 2(q-1)-(p-1)\geq p-1.$$

From the last string of inequalities, $u=p-1$, $m=u+v$ and $p=q$. The above sum then becomes

$$\sum_{\gamma\in T}\sum_{j=0}^{v}\binom{p-1}{p-1+v-j}\binom{v}{j}\binom{p+v-2}{p-1-v+j}(-1)^{p-1+v-j}\gamma^{p-1+v}x_1^{v-1}x_2^{p-1}$$

As $j\leq v$, the first binomial coefficient in this sum is zero unless $j=v$, producing

$$\sum_{\gamma\in T}\binom{p+v-2}{p-1}\gamma^{p-1+v}x_1^{v-1}x_2^{p-1}.$$

The sum $\sum_{\gamma\in T}\gamma^{p-1+v}=\sum_{\gamma\in T}\gamma^v$ only depends on T if $v\equiv\frac{p-1}{2}\pmod{p-1}$, which only happens if $v=\frac{p-1}{2}$. In that case, $\binom{p+v-2}{p-1}=\binom{p+v-2}{p-1}=0$, as the numerator is divisible by p. □

We can use the previous proposition to transfer the results we had about $GL_2(\mathbb{F}_q)$ to $SL_2(\mathbb{F}_q)$, as in the previous chapter. Namely, the structure of irreducible modules for $H_{1,c}(SL_2(\mathbb{F}_q),\mathfrak{h})$, where c takes value c_Q on C_Q and c_R on C_R, is determined by Dunkl operators. By the previous proposition,

$$\sum_{s\in C_Q}\frac{(\alpha_s,y)}{\alpha_s}(1-s)=\sum_{s\in C_R}\frac{(\alpha_s,y)}{\alpha_s}(1-s),$$

so the Dunkl operator is equal to

$$\begin{aligned} D_y &= \partial_y - \sum_{s\in C_Q} c_Q \frac{(\alpha_s, y)}{\alpha_s}(1-s) - \sum_{s\in C_R} c_R \frac{(\alpha_s, y)}{\alpha_s}(1-s) \\ &= \partial_y - \frac{c_R + c_Q}{2} \sum_{s\in C_R\cup C_Q} \frac{(\alpha_s, y)}{\alpha_s}(1-s) \end{aligned}$$

In $GL_2(\mathbb{F}_q)$, the union $C_Q \cup C_R$ is one conjugacy class C_1 (unipotent reflections). Define the function c on all reflections in $GL_2(\mathbb{F}_q)$ by letting it be $c_1 = \frac{c_R+c_Q}{2}$ on all unipotent reflections, and $c_\lambda = 0$ on all semisimple reflections. Then the Dunkl operators controlling the structure of $L_{1,c}(\mathrm{triv})$ for $H_{1,c}(GL_2(\mathbb{F}_q), \mathfrak{h})$ are

$$\begin{aligned} D_y &= \partial_y - \sum_{\lambda=1}^{q} \sum_{s\in C_\lambda} c_\lambda \frac{(\alpha_s, y)}{\alpha_s}(1-s) \\ &= \partial_y - \sum_{s\in C_1} c_1 \frac{(\alpha_s, y)}{\alpha_s}(1-s), \end{aligned}$$

which is exactly the same as the Dunkl operator for $H_{1,c}(SL_2(\mathbb{F}_q), \mathfrak{h})$. From this we get:

Corollary 7.3.10. *Let $n = 2$, $q = p^r$ for p an odd prime, and $q \neq 3$, and consider the Verma module $L_{1,c}(\mathrm{triv})$ for $H_{1,c}(SL_n(\mathbb{F}_q))$. All the results we proved for the rational Cherednik algebra associated to $GL_n(\mathbb{F}_q)$ hold for $SL_n(\mathbb{F}_q)$. Namely,*

a) $D_y(x^p) = 0$ in $M_{1,c}(\mathrm{triv})$.

b) The form B on $M_{1,c}(\mathrm{triv})$ is zero in degrees $2p-1$ and higher.

c) The form B on $M_{1,c}(\mathrm{triv})$ is diagonal in all degrees.

d) All diagonal elements of the matrix of the form B_i on any graded piece $M_{1,c}(\mathrm{triv})_i$ are $\Bbbk$-multiples if a single polynomial in c.

e) If $q = p^r$ with $r > 1$, then B doesn't depend on c.

f) If $q = p$, the matrices of B_i on $M_{1,c}(\text{triv})_i$ are constant for $i = 0, \dots, p-2$, constant multiples of $c_Q + c_R - 2$ for $i = p-1$, and constant multiples of $(c_Q + c_R - 2)(c_Q + c_R + 2)$ for $i = p, \dots, 2p-2$.

Putting together the previous Corollary, Corollary 7.3.7, explicit computations for the rational Cherednik algebra associated to $SL_2(\mathbb{F}_3)$, and noticing that $SL_2(\mathbb{F}_2) = GL_2(\mathbb{F}_2)$, we get the main theorem of this section.

Theorem 7.3.11. *Let p be a prime, $q = p^r$ and $n \geq 2$. The characters of $L_{1,c}(\text{triv})$ for the rational Cherednik algebra $H_{1,c}(SL_n(\mathbb{F}_q), \mathfrak{h})$ over an algebraically closed field of characteristic p are as follows:*

(q,n)	c	$\chi_{L_{1,c}(\text{triv})}(z)$	$\text{Hilb}_{L_{1,c}(\text{triv})}(z)$
$(q,n) \neq (2,2),(3,2)$	*generic*	$\sum_{i\geq 0}([S^i\mathfrak{h}^* / \langle x_1^p, \dots, x_n^p\rangle \cap S^i\mathfrak{h}^*])z^i$	$\left(\frac{1-z^p}{1-z}\right)^n$
$(3,2)$	*generic*	$\left(\sum_{i\geq 0}[S^i\mathfrak{h}^*]\, z^i\right)(1-z^{12})(1-z^{18})$	$\frac{(1-z^{12})(1-z^{18})}{(1-z)^2}$
$(2,2)$	*generic*	$\left(\sum_{i\geq 0}[S^i\mathfrak{h}^*]\, z^i\right)(1-z^4)(1-z^6)$	$\frac{(1-z^4)(1-z^6)}{(1-z)^2}$
$(2,2)$	$c=1$	$[\text{triv}]$	1
$(2,2)$	$c=0$	$[\text{triv}] + [\mathfrak{h}^*]\,z + [\text{triv}]\,z^2$	$1+2z+z^2$
$(2,3)$	$c=1$	$[\text{triv}] + [\mathfrak{h}^*]\,z$	$1+3z$
$(p,2), p\neq 2,3$	$c_Q + c_R = 2$	$\sum_{i=0}^{p-2}[S^i\mathfrak{h}^*]\,z^i$	$\sum_{i=0}^{p-2}(i+1)z^i$
$(p,2), p\neq 2,3$	$c_Q + c_R = -2$	$\sum_{i=0}^{p-1}[S^i\mathfrak{h}^*]\,z^i$	$\sum_{i=0}^{p-1}(i+1)z^i$

We omit the characters for $(q,n) = (3,2)$ and special c as there are too many cases to concisely list.

Remark 7.3.12. *All $L_{t,c}(\text{triv})$ for generic c have one dimensional top degree and are Frobenius. For h_1 the reduced Hilbert series of $L_{1,c}(\text{triv})$ at generic c, $h_1(1)$ is either $|W|$ or 1. For h_0 the Hilbert series of $L_{0,c}(\text{triv})$ at generic c, the inequality $h_0 \leq h_1$ term by term always holds, but not the equality: for $SL_2(\mathbb{F}_3)$, $h_0(z) = 1$, and $h_1(z) = \frac{(1-z^4)(1-z^6)}{(1-z)^2}$.*

Chapter 8

Representations of Rational Cherednik Algebras Associated to the Group $GL_2(\mathbb{F}_p)$

8.1 The group $GL_2(\mathbb{F}_p)$

8.1.1 Reflections in $GL_2(\mathbb{F}_p)$

In this chapter we will be considering the rational Cherednik algebra associated to the group $W = GL_2(\mathbb{F}_p)$, for $\mathbb{F}_p \subseteq \Bbbk$ the finite field of p elements, and p an odd prime. This group has $(p^2-1)(p^2-p)$ elements, and so p divides the order of the group and its category of representations is not semisimple. Let $\mathfrak{h}_{\mathbb{F}} = \mathbb{F}_p^2$, $\mathfrak{h} = \Bbbk^2$, $\mathfrak{h}^*$, $\mathfrak{h}_{\mathbb{F}}^*$, x_1, x_2, y_1, y_2, C_λ, c_λ, d_λ be as in the previous chapter. Let us repeat the result of Lemma 7.1.2 and Example 7.1.3 in this case, which state that the conjugacy classes in $GL_2(\mathbb{F}_p)$ are C_λ, $\lambda \in \mathbb{F}_p^\times$, and are parametrized by $\alpha \otimes \alpha^\vee \in \mathfrak{h}_{\mathbb{F}}^* \otimes \mathfrak{h}_{\mathbb{F}}$ as

$$\lambda \neq 1: \quad C_\lambda \leftrightarrow \{ \begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} 1-\lambda-bd \\ d \end{bmatrix} | b,d \in \mathbb{F}_p\} \cup \{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 1-\lambda \end{bmatrix} | a \in \mathbb{F}_p\}$$

$$C_1 \leftrightarrow \{\begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} -bd \\ d \end{bmatrix} | b, d \in \mathbb{F}_p, d \neq 0\} \cup \{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 0 \end{bmatrix} | a \in \mathbb{F}_p, a \neq 0\}.$$

In this parametrization, the reflection s corresponding to $\alpha \otimes \alpha^\vee$ acts:

$$\text{on } \mathfrak{h}^* \text{ by} \quad s.x = x - (\alpha^\vee, x)\alpha$$

$$\text{on } \mathfrak{h} \text{ by} \quad s.y = y + \frac{(y, \alpha)}{1 - (\alpha, \alpha^\vee)}\alpha^\vee.$$

8.1.2 Invariants and reduced characters

Specializing the results of section 8.1.2 to $W = GL_2(\mathbb{F}_p)$, we see that the space of invariants $(S\mathfrak{h}^*)^W$ is a polynomial algebra generated by

$$Q_0 = L^{p-1} == (x_1^p x_2 - x_1 x_2^p)^{p-1} = \sum_{i=0}^{p-1} x_1^{(p-1)(p-i)} x_2^{(p-1)(i+1)}$$

$$Q_1 = \frac{[2,0]}{L} = \frac{[2,0]}{[1,0]} = \frac{x_1^{p^2} x_2 - x_1 x_2^{p^2}}{x_1^p x_2 - x_1 x_2^p} = \sum_{i=0}^{p} x_1^{(p-1)(p-i)} x_2^{(p-1)i}.$$

with degrees $\deg(Q_i) = p^2 - p^i$. Alternatively, the L can be described as a product of all the linear polynomials of the form $x_1 + ax_2$ and x_2, Q_0 as the product of all the nonzero linear polynomials $ax_1 + bx_2$, and Q_1 as the product of all the irreducible monic quadratic polynomials $x_1^2 + ax_1x_2 + bx_2^2$.

The characters and Hilbert series of baby Verma modules are

$$\chi_{N_{0,c}(\tau)}(z) = \chi_{M_{0,c}(\tau)}(z)(1 - z^{(p^2-1)})(1 - z^{(p^2-p)}),$$

$$\mathrm{Hilb}_{N_{0,c}(\tau)}(z) = \dim(\tau)\frac{(1 - z^{(p^2-1)})(1 - z^{(p^2-p)})}{(1-z)^2}.$$

$$\chi_{N_{1,c}(\tau)}(z) = \chi_{M_{t,c}(\tau)}(z)(1 - z^{p(p^2-1)})(1 - z^{p(p^2-p)}),$$

$$\mathrm{Hilb}_{N_{1,c}(\tau)}(z) = \dim(\tau)\frac{(1 - z^{p(p^2-1)})(1 - z^{p(p^2-p)})}{(1-z)^2}.$$

Let us recall from Proposition 6.2.4 that for $t \neq 0$ and generic c, the character of

$L_{t,c}(\tau)$ is of the form

$$\chi_{L_{t,c}(\tau)}(z) = \chi_{S^{(p)}\mathfrak{h}^*}(z)H(z^p),$$
$$\mathrm{Hilb}_{L_{t,c}(\tau)}(z) = \left(\frac{1-z^p}{1-z}\right)^n \cdot h(z^p).$$

In case of $GL_2(\mathbb{F}_p)$, the character of $S^{(p)}\mathfrak{h}^*$ is

$$\chi_{S^{(p)}\mathfrak{h}^*}(z) = \chi_{S\mathfrak{h}^*}(z) - 2\chi_{S\mathfrak{h}^*}(z)z^p + \chi_{S\mathfrak{h}^*}(z)z^{2p}$$

with the Hilbert series

$$\mathrm{Hilb}_{S^{(p)}\mathfrak{h}^*}(z) = \left(\frac{1-z^p}{1-z}\right)^2.$$

8.1.3 Representations of $GL_2(\mathbb{F}_p)$

Most of the results from this section can be found in [31], or proved directly.

Proposition 8.1.1. *All irreducible representations of $GL_2(\mathbb{F}_p)$ over $\Bbbk$ are of the form*

$$S^i\mathfrak{h} \otimes (det)^j,$$

for $i = 0, 1, \ldots p-1$, $j = 0, \ldots p-2$.

Proof. From the paper of Steinberg [44] it follows that all the irreducible representations of $SL_2(\mathbb{F}_p)$ are of the form $S^i\mathfrak{h}$. Any irreducible representation of $GL_2(\mathbb{F}_p)$ stays irreducible when restricted to $SL_2(\mathbb{F}_p)$. The group $GL_2(\mathbb{F}_p)$ is generated by $SL_2(\mathbb{F}_p)$ and the subgroup of elements of the form $\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$, which act by a character, say $\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \mapsto \lambda^j$ on the one-dimensional subrepresentation y_1^i. So, any irreducible representation of $GL_2(\mathbb{F}_p)$ is of the form $S^i\mathfrak{h} \otimes (det)^j$. □

Lemma 8.1.2. *As representations of $GL_2(\mathbb{F}_p)$,*

$$\mathfrak{h}^* \cong \mathfrak{h} \otimes (det)^{-1},$$

and the isomorphism is $x_1 \mapsto -y_2$, $x_2 \mapsto y_1$.

Proof. The matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{F}_p)$ acts on $\mathfrak{h}$, in basis y_1, y_2, in a tautological way as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It acts on $\mathfrak{h}^*$, in a basis x_1, x_2, as

$$(g^{-1})^t = \frac{1}{\det}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \frac{1}{\det}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

□

Next, we will need to know how the higher symmetric powers and tensor products of such representations decompose into irreducible components.

Proposition 8.1.3. *For any* $i, j > 0$*, there is a short exact sequence of* $GL_2(\mathbb{F}_p)$*-representations*

$$0 \to S^{i-1}\mathfrak{h} \otimes S^{j-1}\mathfrak{h} \otimes det \to S^i\mathfrak{h} \otimes S^j\mathfrak{h} \to S^{i+j}\mathfrak{h} \to 0.$$

The first map is $f \otimes g \mapsto (y_1 \otimes y_2 - y_2 \otimes y_1) \cdot f \otimes g$*, and the second map is* $f \otimes g \to f \cdot g$.

Proof. One can see directly that both maps are indeed $GL_2(\mathbb{F}_p)$-representation maps and that they compose to 0. Multiplication of factors $S^i\mathfrak{h} \otimes S^j\mathfrak{h} \to S^{i+j}\mathfrak{h} \to 0$ is surjective, multiplication by $(y_1 \otimes y_2 - y_2 \otimes y_1)$ is injective, and the dimensions agree: $i \cdot j + (i + j + 1) = (i + 1)(j + 1)$, so this is really a short exact sequence of group representations. □

Proposition 8.1.4. *Let* $0 \leq j < p$*, and* $n \geq 0$*. There is a short exact sequence of* $GL_2(\mathbb{F}_p)$*-representations:*

$$0 \to S^j\mathfrak{h} \otimes S^n\mathfrak{h} \to S^{j+pn}\mathfrak{h} \to S^{p-j-2}\mathfrak{h} \otimes S^{n-1}\mathfrak{h} \otimes \det^{j+1} \to 0.$$

Here we use the convention $S^i\mathfrak{h} = 0$ *if* $i < 0$. *The first map is*

$$\alpha(y_1^a y_2^b \otimes y_1^c y_2^d) = y_1^{a+cp} y_2^{b+dp}$$

and the second, for $0 \leq a, b < p$, *is*

$$\beta(y_1^{a+pc} y_2^{b+pd}) = \begin{cases} \binom{a}{j+1} \cdot y_1^{a-j-1} y_2^{b-j-1} \otimes y_1^c y_2^d\,,\ a+b = p+j \\ 0\,,\ \textit{otherwise} \end{cases}$$

Proof. Both α and β send monomials to monomials. Every monomial in $S^{j+pn}\mathfrak{h}$ can be written as $y_1^{a+cp} y_2^{b+dp}$ with $0 \leq a, b < p$. Then either $a + b = j$ and the monomial is in the image of α and the kernel of β, or $a + b = j + p$, so $y_1^{a+cp} y_2^{b+dp}$ is not in the image of α and $\beta(y_1^{a+cp} y_2^{b+dp}) \neq 0$. It is clear that α is injective and β surjective, so the sequence is really a short exact sequence of vector spaces.

It remains to see that both maps commute with $GL_2(\mathbb{F}_p)$-action. The map α can be written as a composition of raising all monomials in the second tensor factor to the p-th power and multiplication of tensor factors, both of which are $GL_2(\mathbb{F}_p)$-maps. To see that β is a $GL_2(\mathbb{F}_p)$-map, let $y_1^{a+cp} y_2^{b+dp} \in S^{j+pn}\mathfrak{h}$, with $0 \leq a, b < n$, and assume first that $a + b = j, c + d = n$. Then $\beta(y_1^{a+cp} y_2^{b+dp}) = 0$, and for any $g \in GL_2(\mathbb{F}_p)$, $\beta(g.(y_1^{a+cp} y_2^{b+dp})) = 0$.

Next, consider $y_1^{a+cp} y_2^{b+dp} \in S^{j+pn}\mathfrak{h}$, with $0 \leq a, b < n$, $a+b = p+j, c+d = n-1$, and let us show that applying β to this element commutes with the group action. For this it is enough to see that β commutes with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $e \in \mathbb{F}_p$, which generate $GL_2(\mathbb{F}_p)$.

Using $\det A = -1$, we get

$$\begin{aligned}
\beta(A.(y_1^{a+cp}y_2^{b+dp})) &= \beta(y_1^{b+dp}y_2^{a+cp}) \\
&= \binom{b}{j+1} y_1^{b-j-1}y_2^{a-j-1} \otimes y_1^d y_2^c \\
A.(\beta(y_1^{a+cp}y_2^{b+dp})) &= A.(\binom{a}{j+1} y_1^{a-j-1}y_2^{b-j-1} \otimes y_1^c y_2^d) \\
&= (-1)^{j+1}\binom{a}{j+1} y_1^{b-j-1}y_2^{a-j-1} \otimes y_1^d y_2^c.
\end{aligned}$$

However, these are the same because

$$(-1)^{j+1}\binom{a}{j+1} = \binom{j-a}{j+1} = \binom{b}{j+1}$$

so $\beta A = A\beta$.

Next, using $\det B = e$, we get

$$\beta(B.(y_1^{a+cp}y_2^{b+dp})) = \binom{a}{j+1} e^{a+c} y_1^{a-j-1}y_2^{b-j-1} \otimes y_1^c y_2^d = B.(\beta(y_1^{a+cp}y_2^{b+dp})).$$

Finally,

$$\begin{aligned}
\beta(C.(y_1^{a+cp}y_2^{b+dp})) &= \beta(\sum_{i=0}^{b}\sum_{l=0}^{d}\binom{b}{i}\binom{d}{l} y_1^{a+i}y_2^{b-i}y_1^{(c+l)p}y_2^{(d-l)p}) \\
&= \sum_{i=0}^{b-j-1}\sum_{l=0}^{d}\binom{b}{i}\binom{d}{l}\binom{a+i}{j+1} y_1^{a+i-j-1}y_2^{b-i-j-1} \otimes y_1^{c+l}y_2^{d-l} \\
C.(\beta(y_1^{a+cp}y_2^{b+dp})) &= C.(\binom{a}{j+1} y_1^{a-j-1}y_2^{b-j-1} \otimes y_1^c y_2^d) \\
&= \sum_{i=0}^{b-j-1}\sum_{l=0}^{d}\binom{b-j-1}{i}\binom{d}{l}\binom{a}{j+1} y_1^{a+i-j-1}y_2^{b-i-j-1} \otimes y_1^{c+l}y_2^{d-l}
\end{aligned}$$

So, the claim that $\beta C = C\beta$ is equivalent to showing that

$$\binom{b}{i}\binom{a+i}{j+1} = \binom{b-j-1}{i}\binom{a}{j+1}.$$

Using $a+b=p+j$ and $\binom{A}{i}=(-1)^i\binom{-A+i-1}{i}$, this is equivalent to

$$(-1)^i\binom{a+i-j-1}{i}\binom{a+i}{j+1}=(-1)^i\binom{a+i}{i}\binom{a}{j+1},$$

which is true because both left and right hand side are equal to

$$(-1)^i\frac{(a+i)(a+i-1)\dots(a-j)}{i!(j+1)!}.$$

□

8.2 Category $\mathcal{O}$ for the rational Cherednik algebra $H_{t,c}(GL_2(\mathbb{F}_p),\mathfrak{h})$

The main theorem of this chapter is:

Theorem 8.2.1. *Up to a grading shift, any irreducible representations in category $\mathcal{O}$ for the rational Cherednik algebra $H_{t,c}(GL_2(\mathbb{F}_p),\mathfrak{h})$ is isomorphic to $L_{t,c}(S^i\mathfrak{h}\otimes(\det)^j)$ for some $0\le i\le p-1$ and $0\le j\le p-2$. The characters and Hilbert series of these representations are as follows.*

For $t=0$:

- *If $0\le i\le p-3$,*

$$\chi_{L_{0,c}(S^i\mathfrak{h}\otimes(\det)^j)}(z)=[S^i\mathfrak{h}\otimes(\det)^j],$$

$$\mathrm{Hilb}_{L_{0,c}(S^i\mathfrak{h}\otimes(\det)^j)}(z)=i+1.$$

- *If $i=p-2$*

$$\chi_{L_{0,c}(S^i\mathfrak{h}\otimes(\det)^j)}(z)=[S^{p-2}\mathfrak{h}\otimes(\det)^j]+[S^{p-1}\mathfrak{h}\otimes(\det)^{j-1}]z+[S^{p-2}\mathfrak{h}\otimes(\det)^{j-1}]z^2,$$

$$\mathrm{Hilb}_{L_{0,c}(S^{p-2}\mathfrak{h}\otimes(\det)^j)}(z)=(p-1)+pz+(p-1)z^2.$$

- *If $i=p-1$*

$$\chi_{L_{0,c}(S^{p-1}\mathfrak{h}\otimes(\det)^j)}(z)=\chi_{M_{0,c}(S^{p-1}\mathfrak{h}\otimes(\det)^j)}(z)(1-z^{p-1})(1-z^{p^2-1}),$$

$$\mathrm{Hilb}_{L_{0,c}(S^{p-1}\mathfrak{h}\otimes(\det)^j)}(z)=p\frac{(1-z^{p-1})(1-z^{p^2-1})}{(1-z)^2}.$$

For $t=1$,

$$\chi_{L_{1,c}(\tau)}(z)=H_{L_{1,c}(\tau)}(z^p)\chi_{S^{(p)}\mathfrak{h}^*}(z),$$

$$\mathrm{Hilb}_{L_{1,c}(\tau)}(z)=h_{L_{1,c}(\tau)}(z^p)\left(\frac{1-z^p}{1-z}\right)^2,$$

where

$$\chi_{S^{(p)}\mathfrak{h}^*}(z)=\chi_{S\mathfrak{h}^*}(z)-2\chi_{S\mathfrak{h}^*}(z)z^p+\chi_{S\mathfrak{h}^*}(z)z^{2p}$$

and the reduced character and Hilbert series is:

- *If $0\leq i\leq p-3$,*

$$H_{L_{1,c}(S^i\mathfrak{h}\otimes(\det)^j)}(z)=[S^i\mathfrak{h}\otimes(\det)^j],$$

$$h_{L_{1,c}(S^i\mathfrak{h}\otimes(\det)^j)}(z)=i+1.$$

- *If $i=p-2$*

$$H_{L_{1,c}(S^{p-2}\mathfrak{h}\otimes(\det)^j)}(z)=[S^{p-2}\mathfrak{h}\otimes(\det)^j]+[S^{p-1}\mathfrak{h}\otimes(\det)^{j-1}]z+[S^{p-2}\mathfrak{h}\otimes(\det)^{j-1}]z^2,$$

$$h_{L_{1,c}(S^{p-2}\mathfrak{h}\otimes(\det)^j)}(z)=(p-1)+pz+(p-1)z^2.$$

- *If $i=p-1$*

$$H_{L_{1,c}(S^{p-1}\mathfrak{h}\otimes(\det)^j)}(z)=\chi_{M_{0,c}(S^{p-1}\mathfrak{h}\otimes(\det)^j)}(z)(1-z^{p-1})(1-z^{p^2-1}),$$

$$h_{L_{1,c}(S^{p-1}\mathfrak{h}\otimes(\det)^j)}(z)=p\frac{(1-z^{p-1})(1-z^{p^2-1})}{(1-z)^2}.$$

Proof. Lemma 8.2.7 shows that all the formulas for the characters of $L_{t,c}(S^i\mathfrak{h}\otimes(\det)^j)$ follow from the ones for $j=0$. Those are proved for $0\leq i<p-2$ in Propositions

8.3.1 (for $t = 0$) and 8.3.2 (for $t = 1$); for $i = p - 2$ in Propositions 8.4.1 (for $t = 0$) and 8.4.5 (for $t = 1$), and for $i = p - 1$ in Propositions 8.5.5 and 8.5.6 (for $t = 0$) Propositions 8.5.8 and 8.5.9 (for $t = 1$). □

Remark 8.2.2. *For $G = GL_2(\mathbb{F}_p)$, one can notice from the previous theorem that the reduced character for $L_{1,c}(\tau)$ is equal to the character of $L_{0,c}(\tau)$ for all τ. So, the answer to question 6.2.9 is affirmative in this case.*

8.2.1 Blocks

We are going to use the grading element of $H_{t,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$:

$$\mathbf{h} = \sum_{i=1,2} x_i y_i + 1 - \sum_{s \in S} c_s s.$$

Lemma 8.2.3. *For $t = 0$, the element $\mathbf{h}$ is central and acts by a constant*

$$h_c(\tau) h_c(\tau) = 1 - \sum_{s \in S} c_s s|_\tau$$

on $M_{0,c}(\tau)$. For $t = 1$, $\mathbf{h}$ acts by $h_c(\tau)$ on $M_{1,c}(\tau)_0$ and by $h_c(\tau) + i \in \Bbbk$ on $M_{1,c}(\tau)_i$.

It gives a $\mathbb{Z}_p$-grading on representations. A consequence of this is:

Lemma 8.2.4. *For $t = 0, 1$, if $L_{t,c}(\sigma)[m]$ is a composition factor of $M_{t,c}(\tau)$ or $N_{t,c}(\tau)$, then*

$$h_c(\sigma) - h_c(\tau) = t \cdot m \in \mathbb{Z}_p.$$

In particular, this happens if M is any quotient of $M_{t,c}(\tau)$, and $\sigma \subseteq M_m$ is a $GL_2(\mathbb{F}_p)$ subrepresentation consisting of singular vectors.

For $t = 1$ and generic c, the Hilbert series is of the form $(\frac{1-z^p}{1-z})^2 h(z^p)$, so the only composition factor in $M_{1,c}(\tau)$ and $N_{1,c}(\tau)$ are of the form $L_{1,c}(\sigma)[mp]$. Hence, for $t = 0$ or for $t = 1$ and generic c, the above condition reduces to

$$h_c(\sigma) = h_c(\tau),$$

and separates representations into blocks.

The constants $h_c(\tau)$ are easy to calculate directly.

Lemma 8.2.5. *For $GL_2(\mathbb{F}_p)$ and conjugacy class C_λ of reflections in it, the action of central elements $\sum_{s\in C_\lambda} s$ on symmetric powers of the reflection representation is:*

$$\text{For } \lambda \neq 1, \quad \sum_{s\in C_\lambda} s|_{S^i\mathfrak{h}} = \begin{cases} 0\,, \; i<p-1 \\ 1\,, \; i=p-1 \end{cases}$$

$$\text{For } \lambda = 1, \quad \sum_{s\in C_1} s|_{S^i\mathfrak{h}} = \begin{cases} -1\,, \; i<p-1 \\ 0\,, \; i=p-1 \end{cases}$$

So, for $\tau = S^i\mathfrak{h}\otimes \det^j$, the action of $\mathbf{h}$ *on the lowest weight $\tau \subseteq M_{t,c}(\tau)$ is by the constant*

$$h_c(\tau) = 1 + \begin{cases} c_1\,, \; i<p-1 \\ -\sum_{\lambda\neq 0,1}\lambda^j c_\lambda\,, \; i=p-1 \end{cases}$$

Proof. We use the parametrization of conjugacy classes from Lemma 7.1.2 and Example 7.1.3 .

As $\sum_{s\in C_\lambda} s$ is central in the group algebra and acts on $S^i\mathfrak{h}$ as a constant, it is enough to compute $\sum_{s\in C_\lambda} s.y_1^i$. As we are computing it, we may disregard all terms of the type $y_1^{i-j}y_2^j$ for $j>0$, as we know these sum up to zero. We use Lemma 6.3.1 several times.

For $\lambda \neq 1$, the action of $\sum_{s\in C_\lambda} s$ on y_1^i is by a constant:

$$\sum_{b,d\in\mathbb{F}_p}\left(1+\frac{1}{\lambda}\cdot 1\cdot(1-\lambda-bd)\right)^i + \sum_{a\in\mathbb{F}_p}\left(1+\frac{1}{\lambda}\cdot 0\right)^i =$$

$$= \frac{1}{\lambda^i}\sum_{b,d\in\mathbb{F}_p}(1-bd)^i = \frac{-1}{\lambda^i}\sum_{m\in\mathbb{F}_p} m^i = -\sum_{m\in\mathbb{F}_p} m^i = \begin{cases} 0\,, \; i<p-1 \\ 1\,, \; i=p-1 \end{cases}$$

For $\lambda = 1$, a similar computation yields:

$$\sum_{\substack{b,d\in\mathbb{F}_p\\ d\neq 0}} (1-bd)^i + \sum_{\substack{a\in\mathbb{F}_p\\ a\neq 0}} 1^i = (p-1)\left(\sum_{m\in\mathbb{F}_p} m^i + 1\right) = \begin{cases} -1\,,\; i<p-1 \\ 0\,,\; i=p-1. \end{cases}$$

The formulas for the action of $\mathbf{h}$ on irreducible representations $S^i\mathfrak{h}\otimes\det^j$ is now computed from this directly. ☐

Corollary 8.2.6. *For generic c, the representations of the form $L_{t,c}(S^{p-1}\mathfrak{h}\otimes\det^j)$ form blocks of size one, meaning that the only irreducible representations that appear as composition factors in any representation with lowest weight $S^{p-1}\mathfrak{h}\otimes\det^j$ are isomorphic, up to grading shifts, to $L_{t,c}(S^{p-1}\mathfrak{h}\otimes\det^j)$.*

8.2.2 Dependence of the character of $L_{t,c}(S^i\mathfrak{h}\otimes(\det)^j)$ on j

Lemma 8.2.7. *The algebras $H_{t,c}(G,\mathfrak{h})$ and $H_{t,c\cdot\det}(G,\mathfrak{h})$ are isomorphic. Therefore, for generic c and any irreducible representation τ of G, the Hilbert series of $L_{t,c}(\tau\otimes(\det)^j)$ doesn't depend on j, and their characters are related by*

$$\chi_{L_{t,c}(\tau\otimes\det)} = \chi_{L_{t,c}(\tau)}\cdot[\det].$$

Proof. The existence of the isomorphism follows directly from Lemma 3.1.5, for the group character $f = \det$. Twisting by this isomorphism makes a representation $L_{t,c\cdot\det}(\tau)$ of $H_{t,c\cdot\det}(G,\mathfrak{h})$ into a representation $L_{t,c}(\tau\otimes\det)$ of $H_{t,c}(G,\mathfrak{h})$. So, picking c is such that both c and $c\cdot\det$ are generic parameters, the Hilbert series of $L_{t,c}(\tau)$ and $L_{t,c}(\tau\otimes\det)$ are the same, and that their characters satisfy

$$\chi_{L_{t,c}(\tau\otimes\det)} = \chi_{L_{t,c\cdot\det}(\tau)}\cdot[\det] = \chi_{L_{t,c}(\tau)}\cdot[\det]$$

(here, multiplication is in the Grothendieck ring and corresponds to taking tensor products of representations). ☐

Note this is false for special values of parameter c; more specifically, it shows that

c is special for $\tau \otimes \det$ if and only if $c \cdot \det$ is special for τ.

Because of this lemma, for $G = GL_2(\mathbb{F}_p)$ it is enough to calculate the characters of $L_{t,c}(S^i\mathfrak{h})$ for generic c.

8.3 Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = 0 \dots p-3$

8.3.1 Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = 0 \dots p-3$ and $t = 0$

Proposition 8.3.1. *For $i = 0, \dots p-3$, $t = 0$ and all c, the space $M_{0,c}(S^i\mathfrak{h})_1$ consists of singular vectors. So, the character of $L_{0,c}(S^i\mathfrak{h})$ is*

$$\chi_{L_{0,c}(S^i\mathfrak{h})}(z) = [S^i\mathfrak{h}], \quad \mathrm{Hilb}_{L_{0,c}(S^i\mathfrak{h})}(z) = i + 1.$$

Proof. The space $M_{0,c}(S^i\mathfrak{h})_1$ is isomorphic to $\mathfrak{h}^* \otimes S^i\mathfrak{h}$ as a $GL_2(\mathbb{F}_p)$- representation. To show that it consists of singular vectors, we will show that for any $x \in \mathfrak{h}^*$, any $y \in h$, and any $f \in S^i\mathfrak{h}$,

$$D_y(x \otimes f) = t(y, x) - \sum_{s \in S} c_s(y, \alpha_s)\frac{(1-s).x}{\alpha_s} \otimes s.f$$

is zero. As $t = 0$ and we are claiming this holds for all c, it is equivalent to showing that, for any conjugacy class C_λ of reflections,

$$\sum_{s \in C_\lambda} (y, \alpha_s)\frac{(1-s).x}{\alpha_s} \otimes s.f$$

is zero.

Using Lemma 7.1.2, this sum is equal to

$$\sum_{\substack{\alpha\otimes\alpha^\vee \neq 0 \\ (\alpha,\alpha^\vee)=1-\lambda}} (y, \alpha)(x, \alpha^\vee) \otimes s.f.$$

We now use Example 7.1.3, which parametrizes all $\alpha\otimes\alpha^\vee$ such that $(\alpha, \alpha^\vee) = 1-\lambda$ as nonzero vectors in $\{\begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} 1-\lambda-bd \\ d \end{bmatrix} | b,d \in \mathbb{F}_p\} \cup \{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 1-\lambda \end{bmatrix} | a \in \mathbb{F}_p\}$. We write the above sum as a sum over $a, b, d \in \mathbb{F}_p$ which produce nonzero elements of this set. First note that if α or $\alpha^\vee$ are zero, they don't contribute to the sum, so we can sum over all $a, b, d \in \mathbb{F}_p$. Next, note that $s.f$ is a polynomial of degree i in a and in d, so the summand $(y,\alpha)(x,\alpha^\vee)\otimes s.f$ is a polynomial in degree $1+i \le p-2 < p-1$ in d and in a. By Lemma 6.3.1, this means the sum is zero, as claimed. □

8.3.2 Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = 0 \dots p-3$ and $t = 1$

A very similar computation gives the analogous answer in case $t = 1$.

Proposition 8.3.2. *For $i = 0, \dots p-3$, $t = 1$ and all c, all the vectors of the form $x^p \otimes v \in S^p\mathfrak{h}^* \otimes S^i\mathfrak{h}$ are singular. For generic c these vectors generate $J_{1,c}(S^i\mathfrak{h})$, and the character of the irreducible module $L_{1,c}(S^i\mathfrak{h})$ is*

$$\chi_{L_{1,c}(S^i\mathfrak{h})}(z) = \chi_{S^{(p)}\mathfrak{h}^*}(z) \cdot [S^i\mathfrak{h}],$$

its Hilbert series is

$$\mathrm{Hilb}_{L_{1,c}(S^i\mathfrak{h})}(z) = (i+1)\left(\frac{1-z^p}{1-z}\right)^2,$$

and its reduced character and Hilbert series are

$$H(z) = [S^i\mathfrak{h}], \qquad h(z) = i+1.$$

Proof. The proof is very similar to the proof of the previous proposition. To show that all vectors of the form $x^p \otimes v \in S^p\mathfrak{h}^* \otimes S^i\mathfrak{h} \cong M_{1,c}(S^i\mathfrak{h})_p$ are singular, we need to show that the

$$D_y(x^p \otimes f) = \partial_y x^p - \sum_{s\in S} c_s (y,\alpha_s)\frac{(x,\alpha_s^\vee)^p \alpha_s^p}{\alpha_s} \otimes s.f =$$

$$= -\sum_\lambda c_\lambda \sum_\alpha (y,\alpha)\alpha^{p-1} \sum_{\alpha^\vee} (x,\alpha^\vee)^p \otimes (s.f)$$

is zero. Again use Lemma 7.1.2 and Example 7.1.3 to write this as a sum over all $a, b, d \in \mathbb{F}_p$ parametrizing $\alpha \otimes \alpha^\vee$. For $x \in \mathfrak{h}_{\mathbb{F}}^* \subseteq \mathfrak{h}^*$, $(x, \alpha^\vee)$ is in $\mathbb{F}_p \subseteq \Bbbk$, and $(x, \alpha^\vee)^p = (x, \alpha^\vee)$. Using this, the inner sum over $\alpha^\vee$ again becomes a sum over all $d \in \mathbb{F}_p$ or over all $a \in \mathbb{F}_p$ of a polynomial $(x, \alpha^\vee) \otimes (s.f)$ of degree $1 + i < p - 1$ in d or a, so the sum is zero by Lemma 6.3.1.

To see these vectors generate $J_{1,c}(S^i\mathfrak{h})$ at generic c, we use Proposition 6.2.4, by which the character of $L_{1,c}(S^i\mathfrak{h})$ is of the form

$$\chi_{L_{t,c}(\tau)}(z) = \chi_{S^{(p)}\mathfrak{h}^*}(z)H(z^p).$$

The character of the quotient of $M_{1,c}(S^i\mathfrak{h})$ by the singular vectors found in this lemma is

$$\chi_{L_{t,c}(\tau)}(z) = \chi_{S^{(p)}\mathfrak{h}^*}(z)[S^i\mathfrak{h}],$$

and so the graded $GL_2(\mathbb{F}_p)$ representation with the character $H(z)$ is a quotient of the irreducible representation concentrated in one degree $[S^i\mathfrak{h}]$. As it is nonzero, there is no other choice then $H(z) = [S^i\mathfrak{h}]$, so the character of $L_{1,c}(S^i\mathfrak{h})$ is as claimed, and the maximal graded submodule $J_{1,c}(S^i\mathfrak{h})$ is generated by singular vectors of the form $x^p \otimes f$. □

Note this proposition says nothing about the character at special values of c; we can only conclude that for some special values of c, the modules $L_{1,c}(S^i\mathfrak{h})$ are smaller then the above described modules for generic c. The vectors $x^p \otimes f$ are still singular, but for particular values of c the character does not have to be of the form $\chi_{S^{(p)}\mathfrak{h}^*}(z)H(z^p)$, so there could be other singular vectors in degrees $1, 2, \ldots p - 1$.

In the appropriately chosen Grothendieck ring, for $d_1 = p^2 - p$, $d_2 = p^2 - 1$, and using

$$\chi_{N_{1,c}(\tau)} = \chi_{M_{1,c}(\tau)}(1 - z^{pd_1})(1 - z^{pd_2}),$$

we have

$$L_{1,c}(S^i\mathfrak{h}) = M_{1,c}(S^i\mathfrak{h}) - M_{1,c}(\mathfrak{h}^* \otimes S^i\mathfrak{h})[p] + M_{1,c}(\det \otimes S^i\mathfrak{h})[2p]$$

and

$$\chi_{L_{1,c}(S^i\mathfrak{h})} = \frac{\chi_{N_{1,c}(S^i\mathfrak{h})}(z) - \chi_{N_{1,c}(\mathfrak{h}^*\otimes S^i\mathfrak{h})}(z)z^p + \chi_{N_{1,c}(\det\otimes S^i\mathfrak{h})}(z)z^{2p}}{(1-z^{pd_1})(1-z^{pd_2})}.$$

8.4 Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = p-2$

8.4.1 Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = p-2$ and $t = 0$

Proposition 8.4.1. *The character of $L_{0,c}(S^{p-2}\mathfrak{h})$ is*

$$[S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h}\otimes(\det)^{-1}]z + [S^{p-2}\mathfrak{h}\otimes(\det)^{-1}]z^2,$$

and its Hilbert series is

$$(p-1) + pz + (p-1)z^2.$$

Proof. We will prove this in a series of lemmas. Let us outline the proof here, and define several auxiliary modules, used only in this subsection.

The character of the Verma module $M_{0,c}(S^{p-2}\mathfrak{h})$ is

$$\chi_{M_{0,c}(S^{p-2}\mathfrak{h})}(z) = \sum_{j\geq 0}[S^j\mathfrak{h}^*\otimes S^{p-2}\mathfrak{h}]z^j.$$

Lemma 8.4.2 shows that the space of singular vectors in $M_{0,c}(S^{p-2}\mathfrak{h})_1$ is isomorphic to $S^{p-3}\mathfrak{h}$, and consequently that $J_{0,c}(S^{p-2}\mathfrak{h})_1 \cong S^{p-3}\mathfrak{h}$. We define M^1 to be the quotient of the Verma module $M_{0,c}(S^{p-2}\mathfrak{h})$ by the submodule generated by these vectors.

The character of M^1 begins as

$$\chi_{M^1}(z) = [S^{p-2}\mathfrak{h}] + ([\mathfrak{h}^*\otimes S^{p-2}\mathfrak{h}] - [S^{p-3}\mathfrak{h}])z + ([S^2\mathfrak{h}^*\otimes S^{p-2}\mathfrak{h}] - [\mathfrak{h}^*\otimes S^{p-3}\mathfrak{h}])z^2 +$$

$$+([S^3\mathfrak{h}^*\otimes S^{p-2}\mathfrak{h}] - [S^2\mathfrak{h}^*\otimes S^{p-3}\mathfrak{h}])z^3 + \ldots,$$

which is, using Lemmas 8.1.3 and 8.1.4, equal to

$$\chi_{M^1}(z) = [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h}\otimes(\det)^{-1}]z + [S^p\mathfrak{h}\otimes(\det)^{-2}]z^2 + [S^{p+1}\mathfrak{h}\otimes(\det)^{-3}]z^3 + \ldots.$$

The module M^1 has the property that its zero-th and first graded piece are equal to those of the irreducible module $L_{0,c}(\tau)$.

However, M^1 is not irreducible. Lemma 8.4.3 shows that the space of singular vectors in $M^1_2 \cong S^p\mathfrak{h} \otimes (\det)^{-2}$ is isomorphic to $\mathfrak{h} \otimes (\det)^{-2}$. This subspace is thus also in $J_{0,c}(S^{p-2}\mathfrak{h})$. Define M^2 as the quotient of M^1 by the submodule generated by these vectors. M^2 is equal to $L_{0,c}(S^{p-2}\mathfrak{h})$ in graded pieces $0, 1$ and 2, and its character begins as

$$\chi_{M^2}(z) = [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + ([S^p\mathfrak{h} \otimes (\det)^{-2}] - [\mathfrak{h} \otimes (\det)^{-2}])z^2 +$$

$$+([S^{p+1}\mathfrak{h} \otimes (\det)^{-3}] - [\mathfrak{h}^* \otimes \mathfrak{h} \otimes (\det)^{-2}])z^3 + \ldots =$$

$$= [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + [S^{p-2}\mathfrak{h} \otimes (\det)^{-1}]z^2 + ([S^{p-3}\mathfrak{h} \otimes (\det)^{-1}]z^3 + \ldots$$

Finally, Lemma 8.4.4 shows that $M^2_3 \cong S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ is entirely made of singular vectors. From this it follows that the quotient of M^2 by this subspace, which is an $H_{0,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$-module with character

$$[S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + [S^{p-2}\mathfrak{h}(\det)^{-1}]z^2,$$

is irreducible and equal to $L_{0,c}(S^{p-2}\mathfrak{h})$.

This proves the proposition, modulo Lemmas 8.4.2, 8.4.3 and 8.4.4.

□

Lemma 8.4.2. *The space of singular vectors in $M_{0,c}(S^{p-2}\mathfrak{h})_1 \cong \mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ is isomorphic to $S^{p-3}\mathfrak{h}$ and consists of all vectors of the form*

$$x_1 \otimes y_1 f + x_2 \otimes y_2 f, \;\; f \in S^{p-3}\mathfrak{h}.$$

Proof. As a $GL_2(\mathbb{F}_p)$-representation, the first graded piece of the Verma module, $M_{0,c}(S^{p-2}\mathfrak{h})_1$ is isomorphic to $\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$. By Lemma 8.1.2, this is isomorphic to

$\mathfrak{h}\otimes S^{p-2}\mathfrak{h}\otimes(\det)^{-1}$, and by Lemma 8.1.3, it fits into a short exact sequence

$$0\to S^{p-3}\mathfrak{h}\to\mathfrak{h}\otimes S^{p-2}\mathfrak{h}\otimes(\det)^{-1}\to S^{p-1}\mathfrak{h}\otimes(\det)^{-1}\mathfrak{h}\to 0.$$

The irreducible subrepresentation isomorphic to $S^{p-3}\mathfrak{h}$ includes into $\mathfrak{h}^*\otimes S^{p-2}\mathfrak{h}$ by $f\mapsto x_1\otimes y_1f+x_2\otimes y_2f$. Both this subrepresentation and the quotient are irreducible.

If a vector $v\in M_{0,c}(S^{p-2}\mathfrak{h})_1$ is contained in the kernel of B, which is the maximal proper graded submodule $J_{0,c}(S^{p-2}\mathfrak{h})$, then action on it by $y\in\mathfrak{h}$ produces an element of $J_{0,c}(S^{p-2}\mathfrak{h})_0$. However, the form is nondegenerate in degree 0, and $L_{0,c}(S^{p-2}\mathfrak{h})_0=M_{0,c}(S^{p-2}\mathfrak{h})_0$, so $y.v=0$. In other words, such a vector is singular.

To show that $J_{0,c}(S^{p-2}\mathfrak{h})_1\cong S^{p-3}\mathfrak{h}$, we are going to show that:

1. At least one nonzero vector from $S^{p-3}\mathfrak{h}$ is singular;

2. Not all vectors in $M_{0,c}(S^{p-2}\mathfrak{h})_1$ are singular.

The space of singular vectors is invariant under the group action, and both $S^{p-3}\mathfrak{h}$ and the quotient are irreducible, so this proves the claim.

First, let us show that the space $S^{p-3}\mathfrak{h}$ consist of singular vectors. The set of vectors $x_1\otimes y_1f+x_2\otimes y_2f$ is symmetric with respect to changing indices 1 and 2, so it is enough to show that

$$D_{y_1}(x_1\otimes y_1f+x_2\otimes y_2f)$$

is zero. We use parametrization of conjugacy classes from Lemma 7.1.2 and Example 7.1.3 and the definition of Dunkl operator D_{y_1}, and denote $\alpha_b=x_1+bx_2$, and see that the coefficient of $-c_\lambda$ in $D_{y_1}(x_1 \otimes y_1 f + x_2 \otimes y_2 f)$ is

$$\begin{aligned}
&\sum_{\alpha\otimes\alpha^\vee \in C_\lambda} (y_1,\alpha)\frac{1}{\alpha}\left((x_1 - s.x_1)\otimes (s.y_1)(s.f) + (x_2 - s.x_2)\otimes (s.y_2)(s.f)\right)\\
= &\sum_{b,d} 1\cdot \frac{1}{\alpha_b}\left((1-\lambda - bd)\alpha_b \otimes (s.y_1)(s.f) + d\alpha_b \otimes (s.y_2)(s.f)\right)\\
= &\ 1\otimes \sum_{b,d}\left((1-\lambda-bd)s.y_1 + ds.y_2\right)s.f\\
= &\ 1\otimes \sum_{b,d}\frac{1}{\lambda}\left((1-\lambda-bd)y_1 + dy_2\right)s.f.
\end{aligned}$$

The sum is over all $b, d \in \mathbb{F}_p$ if $\lambda \neq 1$, and over all $b, d \in \mathbb{F}_p$ with $d \neq 0 \in \mathbb{F}_p$ if $\lambda = 1$. However, if $\lambda = 1$, then the $d = 0$ term does not contribute to the sum, so let us consider the sum to be over all $b, d \in \mathbb{F}_p$ in both cases. The term $s.f$ is a vector in $S^{p-3}\mathfrak{h}$ with coefficients polynomials in b, d whose degree in b and in d is less or equal to $p-3$. The overall expression is a sum over all $b, d \in \mathbb{F}_p$ of polynomials whose degree in each variable is $\leq p-2$, and it is thus zero by Lemma 6.3.1.

So, the subspace isomorphic to $S^{p-3}\mathfrak{h}$ indeed consists of singular vectors.

To see that the space of singular vectors in $M_{0,c}(S^{p-2}\mathfrak{h})_1$ is not the whole space, it is enough to find one vector which is not singular. For example, the above computation shows that $D_{y_1}(x_1 \otimes (y_1)^{p-2})$ has a coefficient of $-c_1 y_1^{p-2}$ equal to

$$-\sum_{b,d\in\mathbb{F}_p} bd(1-bd)^{p-2} = -\sum_{b,d\in\mathbb{F}_p}\sum_{k=0}^{p-2}\binom{p-2}{k}(-1)^k (bd)^{k+1} = \sum_{b,d\in\mathbb{F}_p}(bd)^{p-1} = 1 \neq 0,$$

so $x_1 \otimes (y_1)^{p-2}$ is not singular. □

In the proof of Proposition 8.4.1 we defined M^1 as the quotient of $M_{0,c}(S^{p-2}\mathfrak{h})$ by the submodule generated by singular vectors $x_1 \otimes y_1 f + x_2 \otimes y_2 f,\ f \in S^{p-3}\mathfrak{h}$ from the previous lemma. It is explained in this proof that M^1 agrees with $L_{0,c}(S^{p-2}\mathfrak{h})$ in graded pieces 0 and 1, and that its character is

$$\chi_{M^1}(z) = [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h}\otimes(\det)^{-1}]z + [S^{p}\mathfrak{h}\otimes(\det)^{-2}]z^2 + [S^{p+1}\mathfrak{h}\otimes(\det)^{-3}]z^3 + \ldots.$$

Next, we find the subspace of M_2^1 which is in Ker$B$, and which is by the same argumentation as in the proof of Lemma 8.4.2 equal to the space of singular vectors in M_2^1.

Lemma 8.4.3. *The space of singular vectors in M_2^1 is isomorphic to $\mathfrak{h}\otimes(\det)^{-2}$. The representatives of these vectors in the Verma module $M_{0,c}(S^{p-2}\mathfrak{h})$ are linear combinations of $x_2^2\otimes y_1^{p-2}$ and $x_1^2\otimes y_2^{p-2}$.*

Proof. The space $M_2^1 \cong S^p\mathfrak{h}\otimes(\det)^{-2}$ fits, by Lemma 8.1.4, into a short exact sequence of $GL_2(\mathbb{F}_p)$ representations

$$0 \to \mathfrak{h}\otimes(\det)^{-2} \to S^p\mathfrak{h}\otimes(\det)^{-2} \to S^{p-2}\mathfrak{h}\otimes\det \to 0.$$

We are claiming that the irreducible subrepresentation consists of singular vectors, but that the quotient is not in the kernel of B. Tracking through all the inclusions, quotient maps and isomorphisms in the previous lemma shows that $x_2^2\otimes y_1^{p-2}$ and $x_1^2\otimes y_2^{p-2}\in M_{0,c}(S^{p-2}\mathfrak{h})_2$ really map to the basis of $\mathfrak{h}\otimes(\det)^{-2}$ under the quotient map $M_{0,c}(S^{p-2}\mathfrak{h})\to M^1$.

We use the following observation. For any rational Cherednik algebra module N, and any $y\in\mathfrak{h}$, $n\in N$, $g\in G$, the relations of the rational Cherednik algebra imply that $g.(y.n)=(g.y).(g.n)$, so the map $\mathfrak{h}\otimes N_i\to N_{i-1}$ given by $y\otimes n\mapsto y.n$ is a map of $GL_2(\mathbb{F}_p)$-representations. So, if N is a quotient of the Verma module and D_y the induced Dunkl operator on the submodule, then the map $y\otimes n\mapsto D_y(n)$ is a map of group representations.

In particular, applying the Dunkl operator is a homomorphism

$$\mathfrak{h}\otimes M_2^1\to M_1^1.$$

Showing that $\mathfrak{h}\otimes(\det)^{-2}\subseteq M_2^1$ consists of singular vectors is equivalent to showing that the restriction of the above map to this space, which is

$$\mathfrak{h}\otimes\mathfrak{h}\otimes(\det)^{-2}\to M_1^1,$$

is zero. To do this, notice that the short exact sequence calculating the composition series of $\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{-2}$ is

$$0 \to (\det)^{-1} \to \mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{-2} \to S^2\mathfrak{h} \otimes (\det)^{-2} \to 0,$$

while the target space of the homomorphism is the irreducible $M_1^1 \cong S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$. By Schur's lemma, this homomorphism has to be zero, and thus $\mathfrak{h} \otimes (\det)^{-2}$ consists of singular vectors.

To see these are all singular vectors in M_2^1, it is enough to find one nonsingular vector, because the quotient of M_2^1 by $\mathfrak{h} \otimes (\det)^{-2}$ is irreducible representation. Direct computation shows that $D_{y_1} x_1 x_2 \otimes y_1^{p-2}$ has c_1 coefficient equal to $x_2 \otimes y_1^{p-2}$, which is nonzero in M^1. ☐

The next module to consider is M^2, defined as the quotient of M^1 by the singular vectors from the previous lemma. The irreducible module $L_{0,c}(S^{p-2}\mathfrak{h})$ is a quotient of M^2, and they agree in degrees $0, 1, 2$. We proceed looking for singular vectors in M_3^2, which turn out to be all of it.

Lemma 8.4.4. *All vectors in* $M_3^2 \cong S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ *are singular.*

Proof. We are going to prove the lemma in two steps: first, we show that the claim follows from showing that the image of $D_{y_1}(x_1^2 x_2 \otimes y_1^{p-2})$ in M_2^2 is zero, and then showing this is true.

First, M_3^1 is the quotient of $M_{0,c}(S^{p-2}\mathfrak{h})_3 \cong S^3\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ by the image of singular vectors from Lemma 8.4.4, isomorphic to $S^2\mathfrak{h}^* \otimes S^{p-3}\mathfrak{h}$. The short exact sequence describing this inclusion is the one from Lemma 8.1.3 combined with Lemma 8.1.2, giving

$$0 \to S^2\mathfrak{h}^* \otimes S^{p-3}\mathfrak{h} \to S^3\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h} \to S^{p+1}\mathfrak{h} \otimes (\det)^{-3} \to 0.$$

Under these morphisms, the image of $x_1^2 x_2 \otimes y_1^{p-2} \in S^3\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ in $S^{p+1}\mathfrak{h} \otimes (\det)^3$ is $y_1^{p-1} y_2^2$.

Second, M_3^2 is the quotient of $M_3^1 \cong S^{p+1}\mathfrak{h} \otimes (\det)^{-3}$ by the image of singular vectors from Lemma 8.4.3, which is the space isomorphic to $\mathfrak{h}^* \otimes \mathfrak{h} \otimes (\det)^{-2}$. The short exact sequence realizing this inclusion and quotient is the one from Lemma 8.1.4, giving

$$0 \to \mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{-3} \to S^{p+1}\mathfrak{h} \otimes (\det)^{-3} \to S^{p-3}\mathfrak{h} \otimes (\det)^{-1} \to 0.$$

The image of $y_1^{p-1}y_2^2 \in S^{p+1}\mathfrak{h} \otimes (\det)^{-3}$ under this quotient morphism is $-y_1^{p-3} \in S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$, which is a nonzero element of it.

Hence, the image of $x_1^2x_2 \otimes y_1^{p-2}$ in M_3^2 is the nonzero vector in an irreducible representation. Hence, if we show this vector is singular, it will follow that the entire space M_3^2 consists of singular vectors.

Third, applying Dunkl operators is a map $\mathfrak{h} \otimes M_3^2 \to M_2^2$, so let us decompose $\mathfrak{h} \otimes M_3^2 \cong \mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$. The short exact sequence doing this is the one from Lemma 8.1.3,

$$0 \to S^{p-4}\mathfrak{h} \to \mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1} \to S^{p-2}\mathfrak{h} \otimes (\det)^{-1} \to 0.$$

As applying Dunkl operator maps this to $M_2^2 \cong S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$, by Schur's lemma the submodule $S^{p-4}\mathfrak{h}$ maps to zero, and the map is zero on $\mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ if and only if it is zero on the quotient $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$. Under this quotient map, the vector $y_1 \otimes (-y_1^{p-2})$ maps to $-y_1^{p-2}$, which is a nonzero element of the irreducible representation $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$. Showing that the entire $\mathfrak{h} \otimes M_3^2$ maps to zero is equivalent to showing that this vector maps to zero,which is equivalent to showing that the image of $D_{y_1}(x_1^2x_2 \otimes y_1^{p-2})$ in M_2^2 is zero.

Finally, we prove $D_{y_1}(x_1^2x_2 \otimes y_1^{p-2})$ is zero in M_2^2 by an explicit calculation using the parametrization of conjugacy classes from Lemma 7.1.2 and Example 7.1.3. The factor of $-c_\lambda$ in $D_{y_1}(x_1^2x_2 \otimes y_1^{p-2})$ is

$$\sum_{b,d} \big(x_1^2 d(\lambda + bd)^2 + x_1x_2(1-\lambda-bd)(1+\lambda-bd+2bd(1-\lambda-bd))+$$

$$+x_2^2(1-\lambda-bd)^2 b(bd-1)\big) \otimes \frac{1}{\lambda^{p-2}} \sum_{i=0}^{p-2} \binom{p-2}{i} (1-bd)^i d^{p-2-i} y_1^i y_2^{p-2-i}.$$

The sum is over $b, d \in \mathbb{F}_p$ if $\lambda \neq 1$ and over $b \in \mathbb{F}_p, d \in \mathbb{F}_p^\times$ if $\lambda = 1$. After quotienting out by vectors from the previous two lemmas, whose images in degree 2 are $x_i(x_1 \otimes y_1 + x_2 \otimes y_2)f, f \in S^{p-3}\mathfrak{h}$, and $x_1^2 \otimes y_2^{p-2}$, $x_2^2 \otimes y_1^{p-2}$, we can write this as

$$\sum_{b,d} x_1x_2 \frac{1}{\lambda^{p-2}} \otimes \Big(-d(\lambda+bd)^2 \sum_{i=1}^{p-2} \binom{p-2}{i} (1-bd)^i d^{p-2-i} y_1^i y_2^{p-2-i} +$$

$$+(1-\lambda-bd)(1+\lambda-bd+2bd(1-\lambda-bd)) \sum_{i=0}^{p-2} \binom{p-2}{i} (1-bd)^i d^{p-2-i} y_1^i y_2^{p-2-i} -$$

$$-(1-\lambda-bd)^2 b(bd-1) \sum_{i=0}^{p-3} \binom{p-2}{i} (1-bd)^i d^{p-2-i} y_1^i y_2^{p-2-i} \Big)$$

Reading off the coefficients of $x_1x_2 \otimes y_1^i y_2^{p-2-i}$ for all $0 \leq i \leq p-2$ and using lemma 6.3.1 multiple times, we see this is indeed 0. □

This completes the proof of Proposition 8.4.1.

8.4.2 Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = p-2$ and $t = 1$

Proposition 8.4.5. *The reduced character of $L_{1,c}(S^{p-2}\mathfrak{h})$ for generic value of c is*

$$H(z) = [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + [S^{p-2}\mathfrak{h} \otimes (\det)^{-1}]z^2,$$

so its character is

$$\chi_{L_{1,c}(S^{p-2}\mathfrak{h})}(z) = ([S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z^p + [S^{p-2}\mathfrak{h}(\det)^{-1}]z^{2p}) \cdot \chi_{S^{(p)}\mathfrak{h}^*}(z),$$

and its Hilbert series is

$$((p-1) + pz + (p-1)z^{2p}) \cdot \frac{(1-z^p)^2}{(1-z)^2}.$$

Proof. It is explained in Proposition 6.2.4, Corollary 6.2.5 and comments between them that the generators of the module $J_{t,c}(\tau)$ for generic c and nonzero t are in degrees divisible by p, and their composition factors are a subset of the composition factors of $(S\mathfrak{h}^*)^p \otimes \tau$. In Lemmas 8.4.6, 8.4.8, and 8.4.10 below we explicitly find these generators for $\tau = S^{p-2}\mathfrak{h}$, and in Lemmas 8.4.7 and 8.4.9 we prove they are the only ones in degrees p, $2p$ and $3p$. The quotient of the Verma module $M_{1,c}(S^{p-2}\mathfrak{h})$ by the submodule generated by these elements is finite-dimensional, and zero in degree $4p$, from which we conclude that they generate the whole $J_{1,c}(S^{p-2}\mathfrak{h})$ for generic c, and that this quotient is irreducible.

The reduced character is calculated in the way explained after Proposition 6.2.4: as we know the generators of $J_{1,c}(S^{p-2}\mathfrak{h})$ explicitly, we evaluate them at $c = 0$ (in fact, they don't depend on c). They are of the form $f_i(x_1^p, x_2^p) \otimes v_i$. The reduced module is then defined to be $S\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h} / \langle f_i(x_1, x_2) \otimes v_i \rangle$. In our case, the generators $f_i(x_1^p, x_2^p) \otimes v_i$ form subrepresentations of type $S^{p-3}\mathfrak{h}$ in degree p, $\mathfrak{h} \otimes (\det)^{-2}$ in degree $2p$, and $S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ in degree $3p$. The quotient of $S\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ by $\langle f_i(x_1, x_2) \otimes v_i \rangle$ is thus equal to the quotient by subrepresentations of type $S^{p-3}\mathfrak{h}$ in degree 1, $\mathfrak{h} \otimes (\det)^{-2}$ in degree 2, and $S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ in degree 3, which is easily seen to have the character

$$H(z) = [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + [S^{p-2}\mathfrak{h}(\det)^{-1}]z^2.$$

□

Lemma 8.4.6. *The vectors*

$$x_1^p \otimes y_1 f + x_2^p \otimes y_2 f$$

in $S^p\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h} \cong M_{1,c}(S^{p-2}\mathfrak{h})_p$ *are singular in* $M_{1,c}(S^{p-2}\mathfrak{h})$ *for all* $f \in S^{p-3}\mathfrak{h}$. *They form a* $GL_2(\mathbb{F}_p)$ *subrepresentation of* $M_{1,c}(S^{p-2}\mathfrak{h})_p$ *isomorphic to* $S^{p-3}\mathfrak{h}$.

Proof. The space of these vectors are symmetric with respect to switching indices 1 and 2, so it is enough to prove D_{y_1} acts on it by zero. A computation very similar to the one in Lemma 8.4.2 gives that the coefficient of $-c_\lambda$ in $D_{y_1}(x_1^p \otimes y_1 f + x_2^p y_2 f)$ is

$$\sum_{b,d} (x_1 + bx_2)^{p-1} \otimes \frac{1}{\lambda}((1 - \lambda - bd)y_1 + dy_2)(s.f).$$

The sum is over all $b, d \in \mathbb{F}_p$ if $\lambda \neq 1$ and over all $b \in \mathbb{F}_p, d \in \mathbb{F}_p^\times$ if $\lambda = 1$. However, if $\lambda = 1$, then every summand is divisible by d so the term with $d = 0$ doesn't contribute and we can consider it as a sum over all $d \in \mathbb{F}_p$. The degree in d of every term of this polynomial is less or equal to $1 + \deg f = p - 2 < p - 1$, so by Lemma 6.3.1, the sum is zero. □

Lemma 8.4.7. *For generic c, the vectors from Lemma 8.4.6 are the only singular vectors in $M_{1,c}(S^{p-2}\mathfrak{h})_p$.*

Proof. We will use Corollary 6.2.5, which in our case states that the space of singular vectors in $M_{1,c}(S^{p-2}\mathfrak{h})_p$ has the same composition series as some subspace of $(\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$. This space is isomorphic to $\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ and fits into a short exact sequence from Lemma 8.1.3

$$0 \to S^{p-3}\mathfrak{h} \to \mathfrak{h}^* \otimes S^{p-2}\mathfrak{h} \to S^{p-1}\mathfrak{h} \otimes (\det)^{-1} \to 0,$$

so the space of singular vectors in degree p can either be isomorphic to $S^{p-3}\mathfrak{h}$ or to its extension by $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$.

We are going to show that the quotient of $(\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$ by $S^{p-3}\mathfrak{h}$ isomorphic to $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$ does not consist of singular vectors, and that there is no other composition factor of $S^p\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ isomorphic to $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$. This will prove the claim.

First, we claim that there is a vector in $(\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$ which is not singular. Namely, we claim that $D_{y_1}(x_1^p \otimes y_1^{p-2})$ is not zero. From the calculation in the proof of Lemma 8.4.6 we can read off that the coefficient of c_1 in it is equal to

$$\sum_{b \in \mathbb{F}_p, d \in \mathbb{F}_p^\times} bd(x_1 + bx_2)^{p-1} \otimes ((1 - bd)y_1 + dy_2)^{p-2},$$

which in turn has a coefficient of $x_1^{p-1} \otimes y_1^{p-2}$ equal to

$$\sum_{b \in \mathbb{F}_p, d \in \mathbb{F}_p^\times} bd(1 - bd)^{p-2} = 1 \neq 0.$$

Second, we claim that $S^p\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ has only one composition factor of type $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$. The quotient of $S^p\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ by the space $(\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$ which we already considered is isomorphic to $S^{p-2}\mathfrak{h} \otimes S^{p-2}\mathfrak{h}$, which by repeated use of Lemma 8.1.3 has subquotients of the form $S^{2p-4-2j}\mathfrak{h} \otimes (\det)^j$. More precisely, the expression in the Grothendieck group of $GL_2(\mathbb{F}_p)$ representations for $[S^{p-2}\mathfrak{h} \otimes S^{p-2}\mathfrak{h}]$ is

$$[S^{2p-4}\mathfrak{h}] + [S^{2p-6}\mathfrak{h} \otimes (\det)] + \ldots + [S^{2p-4-2j}\mathfrak{h} \otimes (\det)^j] + \ldots + [S^0\mathfrak{h} \otimes (\det)^{p-2}].$$

Some of these are irreducible (the ones with $2p-4-2j < p$), and the others decompose further using Lemma 8.1.4 and 8.1.3

$$[S^{2p-4-2j}\mathfrak{h} \otimes (\det)^j] = [S^{p-4-2j}\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^j] + [S^{2j+2}\mathfrak{h} \otimes (\det)^{j+1+p-4-2j}] =$$

$$= [S^{p-5-2j}\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{j+1}] + [S^{p-3-2j}\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^j] + [S^{2j+2}\mathfrak{h} \otimes (\det)^{j+1+p-4-2j}].$$

These are all the composition factors, and none of them is equal to $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$.

□

Next, we consider the auxiliary module $\mathbf{M}^1$, defined as the quotient of the Verma module $M_{1,c}(S^{p-2}\mathfrak{h})$ by the submodule generated by singular vectors $(x_1^p \otimes y_1 + x_2^p \otimes y_2)f$ from Lemma 8.4.6. This module $\mathbf{M}^1$ matches $L_{1,c}(S^{p-2}\mathfrak{h})$ in degrees $0, 1, \ldots 2p-1$, and we search for singular vectors in $\mathbf{M}^1_{2p}$.

Lemma 8.4.8. *The images of the vectors*

$$x_1^{2p} \otimes y_2^{p-2}, \quad x_2^{2p} \otimes y_1^{p-2}$$

in $\mathbf{M}^1$ are singular, and span a subrepresentation isomorphic to $\mathfrak{h} \otimes (\det)^{-2}$.

Proof. The claim that they span a subrepresentation isomorphic to $\mathfrak{h} \otimes (\det)^{-2}$ is easy to check. As this representation is irreducible, it is enough to show that one of them is singular, for example $x_1^{2p} \otimes y_2^{p-2}$. Calculating, as before, the coefficient of $-c_\lambda$ in

$D_{y_1}(x_1^{2p} \otimes y_2^{p-2})$, and denoting $\alpha_b = x_1 + bx_2$, we get it is equal to:

$$\sum_{b,d} \frac{1}{\alpha_b} \left(x_1^{2p} - (x_1 - (1-\lambda-bd)\alpha_b)^{2p}\right) \otimes \frac{(b(1-\lambda-bd)y_1 + (\lambda+bd)y_2)^{p-2}}{\lambda^{p-2}}$$

$$= \sum_{b,d} (1-\lambda-bd)\left((1+\lambda+bd)x_1^p - b(1-\lambda-bd)x_2^p\right)\left(\sum_{i=0}^{p-1}(-1)^i b^i x_1^{p-1-i}x_2^i\right) \otimes$$

$$\otimes \frac{1}{\lambda^{p-2}} \sum_{j=0}^{p-2} \binom{p-2}{j} b^j (1-\lambda-bd)^j (\lambda+bd)^{p-2-j} y_1^j y_2^{p-2-j}.$$

Summing over all d and using lemma 6.3.1, this is equal to

$$\sum_b \left(2\lambda b x_1^p - 2b^2(1-\lambda)x_2^p\right)\left(\sum_{i=0}^{p-1}(-1)^i b^i x_1^{p-1-i}x_2^i\right) \otimes \frac{1}{\lambda^{p-2}} \sum_{j=0}^{p-2}\binom{p-2}{j}(-1)^j b^{j+p-2} y_1^j y_2^{p-2-j} +$$

$$+ \sum_b \left(b^2 x_1^p + b^3 x_2^p\right)\left(\sum_{i=0}^{p-1}(-1)^i b^i x_1^{p-1-i}x_2^i\right) \otimes \frac{1}{\lambda^{p-2}} \Bigg(-2\lambda b^{p-3} y_2^{p-2} - 2(1-\lambda)b^{2p-5}y_1^{p-2} +$$

$$+ \sum_{j=1}^{p-3}\binom{p-2}{j}(-1)^j b^{p-3-j}(-2\lambda - j) y_1^j y_2^{p-2-j}\Bigg).$$

Summing over $b \in \mathbb{F}_p$ using lemma 6.3.1 and reorganizing the terms, we get

$$\frac{-1}{\lambda^{p-2}}\Bigg(2(1-\lambda)\left(x_1^{2p-2}x_2 - x_1^{p-1}x_2^p\right) \otimes y_1^{p-2} + 2\lambda\left(x_1 x_2^{2p-2} - x_1^p x_2^{p-1}\right) \otimes y_2^{p-2} +$$

$$+ 2\lambda x_1^{2p-2}x_2 \otimes y_1^{p-2} + 2\lambda x_1^p x_2^{p-1} \otimes y_2^{p-2} + 2(1-\lambda)x_1^{p-1}x_2^p \otimes y_1^{p-2} + 2(1-\lambda)x_1 x_2^{2p-2} \otimes y_2^{p-2} +$$

$$+ \sum_{i=1}^{p-3}\binom{p-2}{i}\left((2+i)x_1^{2p-2-i}x_2^{i+1} - i x_1^{p-1-i}x_2^{p+i}\right) \otimes y_1^{p-2-i}y_2^i\Bigg).$$

This is nonzero in $M_{1,c}(S^{p-2}\mathfrak{h})$, and the vectors from lemma are not singular there. However, in the quotient $\mathbf{M}^1$, this expression is zero. This can be shown by replacing, in the above long expression, anything of the form $a \cdot x_2^p \cdot y_2 \cdot f$ (meaning any term whose degree of x_2 is at least p and whose degree of y_2 is at least 1) by the equivalent expression $-a \cdot x_1^p \cdot y_1 \cdot f$. The expression then simplifies to 0, showing that in the quotient $\mathbf{M}^1$, $D_{y_1}(x_1^{2p} \otimes y_2^{p-2}) = 0$.

Similarly, the coefficient in $D_{y_2}(x_1^{2p} \otimes y_2^{p-2})$ of $-c_\lambda$ is equal to

$$\sum_{b,d} \frac{b}{\alpha_b} \left(x_1^{2p} - (x_1 - (1-\lambda-bd)\alpha_b)^{2p}\right) \otimes \frac{(b(1-\lambda-bd)y_1 + (\lambda+bd)y_2)^{p-2}}{\lambda^{p-2}} +$$

$$+\sum_a \frac{1}{x_2} \left(x_1^{2p} - (x_1 - ax_2)^{2p}\right) \otimes \frac{(ay_1+y_2)^{p-2}}{\lambda^{p-2}}.$$

The proof that the first part of the sum is 0 in $\mathbf{M}^1$ is very similar to the previous computation, as it differs from the expression calculated there just by one power of b. The second part is equal to

$$\frac{-1}{\lambda^{p-2}} \left(2x_1^p x_2^{p-1} \otimes y_1^{p-2} + 2x_2^{2p-1} \otimes y_1^{p-3} y_2\right),$$

which is also zero in $\mathbf{M}^1$. □

Lemma 8.4.9. *For generic c, the vectors from Lemma 8.4.8 are the only singular vectors in $\mathbf{M}^1_{2p}$.*

Proof. The space of p-th powers in $M_{1,c}(S^{p-2}\mathfrak{h})_{2p}$ is isomorphic to $S^2\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h} \subseteq S^{2p}\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$. Using Lemma 8.1.3, its image in the quotient $\mathbf{M}^1_{2p}$ is isomorphic to the quotient of $S^2\mathfrak{h} \otimes S^{p-2}\mathfrak{h} \otimes (\det)^{-2}$ by $\mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$, which is $S^p\mathfrak{h} \otimes (\det)^{-2}$. So this is the maximal possible space of singular vectors in $\mathbf{M}^1_{2p}$, and the space of singular vectors in $\mathbf{M}^1_{2p}$ is its subspace.

This space decomposes by Lemma 8.1.4 as

$$0 \to \mathfrak{h} \otimes (\det)^{-2} \to S^p\mathfrak{h} \otimes (\det)^{-2} \to S^{p-2}\mathfrak{h} \otimes (\det)^{-1} \to 0.$$

We already showed in that the subspace $\mathfrak{h} \otimes (\det)^{-2}$ consists of singular vectors; to prove that not the entire $S^p\mathfrak{h} \otimes (\det)^{-2}$ does, it is enough to find one nonsingular vector. By another explicit computation, one can show that $x_1^p x_2^p \otimes y_1^{p-1}$ is a p-th power that is not annihilated by D_{y_1}.

Finally, we need to show that there is no other composition factor of $\mathbf{M}^1_{2p}$ made of singular vectors. If such a composition factor existed, it would have to be isomorphic to $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$, and show up as a submodule of the quotient of $\mathbf{M}^1_{2p}$ by the space of p-th powers which we already considered.

Using Lemmas 8.1.3 and 8.1.4, this space is isomorphic to

$$S^{p-2}\mathfrak{h} \otimes S^{p-1}\mathfrak{h} \otimes (\det)^{-1}.$$

Using [31], Theorem (5.3), this is isomorphic to

$$S^{p(p-1)-1}\mathfrak{h} \otimes (\det)^{-1}.$$

The only possible space of singular vectors would be isomorphic to $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$ and contained in the socle (maximal semisimple submodule) of $S^{p(p-1)-1}\mathfrak{h} \otimes (\det)^{-1}$. However, Theorem (5.9) in [31] show that this socle is in fact

$$\bigoplus_{m=0}^{(p-3)/2} S^{2m+1}\mathfrak{h} \otimes (\det)^{p-3-m}.$$

As these are all irreducible modules and none of them is isomorphic to $S^{p-2}\mathfrak{h}\otimes(\det)^{-1}$, we conclude there are no other singular vectors in $\mathbf{M}^1_{2p}$.

□

The second auxiliary module $\mathbf{M}^2$ to consider is the quotient of $\mathbf{M}^1$ by the rational Cherednik algebra submodule generated by the two dimensional space of singular vectors from the previous lemma. Because of Corollary 6.2.5 and the previous lemmas in this section (Lemma 8.4.7 and 8.4.9), $\mathbf{M}^2$ is equal to $L_{1,c}(S^{p-2}\mathfrak{h})$ in degrees up to $3p-1$. In degree $3p$, $\mathbf{M}^2$ contains some new singular vectors, given in the following lemma.

Lemma 8.4.10. *All the vectors of the form $(S^3\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h} \subseteq M_{1,c}(S^{p-2}\mathfrak{h})_{3p}$ are in* $\mathrm{Ker}B$.

Proof. The space of p-th powers in $M_{1,c}(S^{p-2}\mathfrak{h})_{3p}$ is $(S^3\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$. The quotient of this space by the subspace $(S^2\mathfrak{h}^*)^p \otimes S^{p-3}\mathfrak{h}$ generated by singular vectors from Lemma 8.4.6 is by Lemma 8.1.3 isomorphic to $S^{p+1}\mathfrak{h} \otimes (\det)^{-3}$. The quotient of this space by the subspace $(\mathfrak{h}^*)^p \otimes \mathfrak{h} \otimes (\det)^{-2}$ generated by singular vectors from Lemma 8.4.8 is by Lemma 8.1.4 isomorphic to $S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$, and this is the space of p-th powers in the $\mathbf{M}^2_{3p}$.

We are going to show that this space is made of singular vectors by showing that the restriction to $\mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ of the map

$$\mathfrak{h} \otimes \mathbf{M}^2_{3p} \to \mathbf{M}^2_{3p-1}$$

given by $y \otimes m \mapsto D_y(m)$, which is a homomorphism of group representations, has to be zero.

The source space of this map is $\mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$, which fits into the short exact sequence

$$0 \to S^{p-4}\mathfrak{h} \to \mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1} \to S^{p-2}\mathfrak{h} \otimes (\det)^{-1} \to 0.$$

The image of the map is a subrepresentation of the target space $\mathbf{M}^2_{3p-1}$, so if we show that the socle of $\mathbf{M}^2_{3p-1}$ doesn't have $S^{p-4}\mathfrak{h}$ nor $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$ as direct summands, it will follow that the map is zero.

By applying Lemma 8.1.4 twice and Lemma 8.1.3 once, we see that the quotient of $\mathbf{M}^2_{3p-1}$ by the image $S^{2p-1}\mathfrak{h}^* \otimes S^{p-3}\mathfrak{h}$ of singular vectors from Lemma 8.4.6 is isomorphic to $S^{p-1}\mathfrak{h} \otimes S^{p}\mathfrak{h} \otimes (\det)^{-2}$. The quotient of that by the image $S^{p-1}\mathfrak{h}^* \otimes \mathfrak{h} \otimes (\det)^{-2}$ is by Lemma 8.1.4 isomorphic to $S^{p-1}\mathfrak{h} \otimes S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$. Using [31], Theorem (5.3) again, this is isomorphic to $S^{p(p-1)-1}\mathfrak{h} \otimes (\det)^{-1}$, whose socle is by Theorem (5.9) in [31] again equal to

$$\bigoplus_{m=0}^{(p-3)/2} S^{2m+1}\mathfrak{h} \otimes (\det)^{p-3-m}.$$

None of these summands is of the type $S^{p-4}\mathfrak{h}$ nor $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$, so the required map is zero. This proves the lemma.

□

8.5 Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = p-1$

8.5.1 Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = p-1$ and $t = 0$

In this section we will calculate the character of $L_{0,c}(S^{p-1}\mathfrak{h})$ for generic c. We first find a certain space $\mathrm{span}_{\Bbbk}\{v_0, \dots, v_{p-1}\}$ of singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})_{p-1}$. We define an auxiliary module M as a quotient of $M_{0,c}(S^{p-1}\mathfrak{h})$ by the $H_{0,c}(GL_2, \mathfrak{h})$-submodule generated by these singular vectors and by the action of the algebra of invariants $(S\mathfrak{h}^*)^G_+ = \Bbbk[Q_0, Q_1]_+$ (for definitions of Q_0 and Q_1, see section 8.1.2). We calculate the character of M, and finally we show that M is irreducible and isomorphic to $L_{0,c}(S^{p-1}\mathfrak{h})$.

We will extensively use Corollary 8.2.6, which states that all the composition factors of $M_{t,c}(S^{p-1}\mathfrak{h})$ are of the form $L_{t,c}(S^{p-1}\mathfrak{h})$. Because of that, all the singular vectors in $M_{t,c}(S^{p-1}\mathfrak{h})$ form subrepresentations with composition factors isomorphic to $S^{p-1}\mathfrak{h}$.

First, we find singular vectors in degree $p-1$.

Lemma 8.5.1. *The vector*

$$x_1^{p-1} \otimes y_2^{p-1} - x_2^{p-1} \otimes y_1^{p-1}$$

in $S^{p-1}\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h} \cong M_{0,c}(S^{p-1}\mathfrak{h})_{p-1}$ *is singular.*

Proof. The proof is pure computation, using the parametrization of conjugacy classes from Lemma 7.1.2 and Example 7.1.3 and Lemma 6.3.1 extensively.

First, this vector is antisymmetric with respect to indices 1, 2, so it is enough to show that D_{y_1} acts on it by 0. For any λ the coefficient of $-c_\lambda$ in $D_{y_1}(x_1^{p-1} \otimes y_2^{p-1} - x_2^{p-1} \otimes y_1^{p-1})$ is

$$\sum_{s \in C_\lambda} (y_1, \alpha_s) \left(\frac{x_1^{p-1} - s.x_1^{p-1}}{\alpha_s} \otimes (s.y_2)^{p-1} - \frac{x_2^{p-1} - s.x_2^{p-1}}{\alpha_s} \otimes (s.y_1)^{p-1} \right). \qquad (\star)$$

Let us rewrite this using the parametrization of C_λ from Lemma 7.1.2 and Example

7.1.3. We use notation $\alpha_b = \begin{bmatrix} 1 \\ b \end{bmatrix} \in \mathfrak{h}^*$. The sum is over all $b, d \in \mathbb{F}_p$ if $\lambda \neq 1$ and over all $b, d \in \mathbb{F}_p$, $d \neq 0$, if $\lambda = 1$. The above expression is equal to:

$$
\begin{aligned}
(\star) \;=\; & \sum_{b,d} \frac{1}{\alpha_b}\left(x_1^{p-1} - (x_1 - (1-\lambda-bd)\alpha_b)^{p-1}\right) \otimes \frac{1}{\lambda^{p-1}} (b(1-\lambda-bd)y_1 + (\lambda+bd)y_2)^{p-1} + \\
& + \frac{1}{\alpha_b}\left((x_2 - d\alpha_b)^{p-1} - x_2^{p-1}\right) \otimes \frac{1}{\lambda^{p-1}}((1-bd)y_1 + dy_2)^{p-1} = \\
= \; & \frac{1}{\lambda^{p-1}} \sum_{b,d} \sum_{i=1}^{p-1} \binom{p-1}{i} (-1)^{i+1} x_1^{p-1-i} (1-\lambda-bd)^i \alpha_b^{i-1} \otimes (b(1-\lambda-bd)y_1 + (\lambda+bd)y_2)^{p-1} + \\
& + \binom{p-1}{i} (-1)^i x_2^{p-1-i} d^i \alpha_b^{i-1} \otimes ((1-bd)y_1 + dy_2)^{p-1} = \\
= \; & \frac{1}{\lambda^{p-1}} \sum_{b,d} \sum_{i=1}^{p-1} \sum_{j=0}^{i-1} \sum_{k=0}^{p-1} \binom{p-1}{i} \binom{i-1}{j} \binom{p-1}{k} \cdot \\
& \cdot \Big((-1)^{i+1} (1-\lambda-bd)^i b^j x_1^{p-j-2} x_2^j \otimes b^{p-1-k} (1-\lambda-bd)^{p-1-k} (\lambda+bd)^k y_1^{p-1-k} y_2^k + \\
& + (-1)^i d^i b^{i-1-j} x_1^j x_2^{p-j-2} \otimes (1-bd)^{p-1-k} d^k y_1^{p-1-k} y_2^k \Big) = \\
= \; & \frac{1}{\lambda^{p-1}} \sum_{k=0}^{p-1} \sum_{j=0}^{p-2} \binom{p-1}{k} x_1^{p-j-2} x_2^j \otimes y_1^{p-1-k} y_2^k \cdot \\
& \cdot \sum_{b,d} \left(\sum_{i=j+1}^{p-1} \binom{i-1}{j} \binom{p-1}{i} (-1)^{i+1} (1-\lambda-bd)^{p-1-k+i} (\lambda+bd)^k b^{p-1-k+j} + \right. \\
& \left. + \sum_{i=p-1-j}^{p-1} \binom{p-1}{i} \binom{i-1}{p-2-j} (-1)^i d^{i+k} b^{i+1+j-p} (1-bd)^{p-1-k} \right).
\end{aligned}
$$

Reading off the coefficient of $x_1^{p-j-2} x_2^j \otimes y_1^{p-1-k} y_2^k$ and using that $\frac{1}{\lambda^{p-1}}\binom{p-1}{k}$ is never zero, the claim that $(\star) = 0$ is equivalent to showing that for every $0 \leq k \leq p-1$,

$0 \leq j \leq p-2$, the expression $(\star\star)$ is zero, where $(\star\star)$ is

$$\sum_{b,d}\left(\sum_{i=j+1}^{p-1}\binom{i-1}{j}\binom{p-1}{i}(-1)^{i+1}(1-\lambda-bd)^{p-1-k+i}(\lambda+bd)^k b^{p-1-k+j}+\right.$$
$$\left.+\sum_{i=p-1-j}^{p-1}\binom{p-1}{i}\binom{i-1}{p-2-j}(-1)^i d^{i+k}b^{i+1+j-p}(1-bd)^{p-1-k}\right)$$
$$=\sum_{b,d}\left(\sum_{i=j+1}^{p-1}\binom{i-1}{j}\binom{p-1}{i}(-1)^{i+1}(1-\lambda-bd)^{p-1-k+i}(\lambda+bd)^k b^{p-1-k+j}+\right.$$
$$\left.+\sum_{i=0}^{j}\binom{p-1}{p-1-i}\binom{p-2-i}{p-2-j}(-1)^i d^{p-1+k-i}b^{j-i}(1-bd)^{p-1-k}\right)$$
$$=\sum_{b,d}\left(\sum_{i=j+1}^{p-1}\sum_{m=0}^{p-1-k+i}\sum_{n=0}^{k}\binom{i-1}{j}\binom{p-1}{i}\binom{p-1-k+i}{m}\binom{k}{n}(-1)^{m+i+1}\cdot\right.$$
$$\cdot(1-\lambda)^{p-1-k+i-m}\lambda^{k-n}b^{m+n+p-1-k+j}d^{m+n}+$$
$$\left.\sum_{i=0}^{j}\sum_{l=0}^{p-1-k}\binom{p-1}{p-1-i}\binom{p-2-i}{p-2-j}\binom{p-1-k}{l}(-1)^{i+l}b^{j-i+l}d^{p-1+k-i+l}\right)$$

Now we will use Lemma 6.3.1, which states that $\sum_{b\in\mathbb{F}_p} b^N = 0$ and $\sum_{d\in\mathbb{F}_p} b^N = 0$, unless $N \equiv 0 \pmod{p-1}$.

First assume $\lambda \neq 1$. The first part of the sum includes $\sum_b b^{m+n+p-1-k+j}$ and $\sum_d d^{m+n}$, so it is zero unless

$$m+n \equiv 0 \pmod{p-1}$$

$$m+n+p-1-k+j \equiv 0 \pmod{p-1},$$

which implies

$$j \equiv k \pmod{p-1}.$$

The second part of the sum includes $\sum_{b,d} b^{j-i+l}d^{p-1+k-i+l}$, so it is zero unless

$$j-i+l \equiv 0 \pmod{p-1}$$

$$p-1+k-i+l \equiv 0 \pmod{p-1},$$

which again implies

$$j \equiv k \pmod{p-1}.$$

As $0 \leq k \leq p-1$, $0 \leq j \leq p-2$, the possibilities for $j \equiv k \pmod{p-1}$ are $j = 0, k = p-1$ or $j = k$. Let us calculate $(\star\star)$ in those two cases separately.

If $j = 0, k = p-1$, then

$$\begin{aligned}
(\star\star) &= \sum_{b,d}\sum_{i=1}^{p-1}\sum_{m=0}^{i}\sum_{n=0}^{p-1}\binom{p-1}{i}\binom{i}{m}\binom{p-1}{n}(-1)^{m+i+1}(1-\lambda)^{i-m}\lambda^{p-1-n}b^{m+n}d^{m+n} + \\
&\quad +\sum_{b,d} d^{2(p-1)} = \\
&= \sum_{b,d}\sum_{i=1}^{p-1}\sum_{m=0}^{i}\sum_{n=0}^{p-1}\binom{p-1}{i}\binom{i}{m}\binom{p-1}{n}(-1)^{m+i+1}(1-\lambda)^{i-m}\lambda^{p-1-n}b^{m+n}d^{m+n} = \\
&= \sum_{b,d}(\lambda - bd)^{2(p-1)} - (\lambda - bd)^{p-1} = 0.
\end{aligned}$$

If $j = k$, then, using that $a^p = a$,

$$\begin{aligned}
(\star\star) &= \sum_{b,d}\left(\sum_{i=j+1}^{p-1}\binom{i-1}{j}\binom{p-1}{i}(-1)^{i+1}(1-\lambda-bd)^{i-j}b^{p-1}(\lambda+bd)^{j} + \right.\\
&\quad \left. +\sum_{i=0}^{j}\binom{p-1}{i}\binom{p-2-i}{p-2-j}(-1)^{i}d^{p-1+j-i}b^{j-i}(1-bd)^{p-1-j}\right) \\
&= \sum_{b,d}\sum_{i=j+1}^{p-1}\sum_{m=0}^{i-j}\sum_{n=0}^{j}\binom{i-1}{j}\binom{p-1}{i}\binom{i-j}{m}\binom{j}{n}(1-\lambda)^{i-j-m}\lambda^{j-n}(-1)^{m+i+1}b^{p-1+m+n}d^{m+n} + \\
&\quad +\sum_{b,d}\sum_{i=0}^{j}\sum_{l=0}^{p-1-j}\binom{p-1}{i}\binom{p-2-i}{p-2-j}\binom{p-1-j}{l}(-1)^{l+i}b^{j-i+l}d^{p-1+j-i+l}
\end{aligned}$$

Again using that $\sum_{b\in\mathbb{F}_p} b^N = 0$ unless $N \equiv 0 \pmod{p-1}$, $N \neq 0$, this is equal to:

$$\begin{aligned}
(\star\star) &= \sum_{b,d}\binom{p-2}{j}(-1)^{j+1}b^{2(p-1)}d^{p-1} + \sum_{b,d}\binom{p-2}{j}(-1)^{j}b^{p-1}d^{2(p-1)} \\
&= 0.
\end{aligned}$$

Let us now do a very similar computation for $\lambda = 1$. Now $\sum_{b,d}$ is over $b, d \in \mathbb{F}_p$, $d \neq 0$.

$$\begin{aligned}(\star\star) &= \sum_{b,d} \Bigg(\sum_{i=j+1}^{p-1} \sum_{n=0}^{k} \binom{i-1}{j} \binom{p-1}{i} \binom{k}{n} (-1)^{p-k} b^{2(p-1-k)+i+n+j} d^{p-1-k+i+n} + \\ &\quad + \sum_{i=0}^{j} \sum_{l=0}^{p-1-k} \binom{p-1}{p-1-i} \binom{p-2-i}{p-2-j} \binom{p-1-k}{l} (-1)^{i+l} b^{j-i+l} d^{p-1+k-i+l} \Bigg)\end{aligned}$$

Again, this is zero unless $j \equiv k \pmod{p-1}$. If $j = 0, k = p-1$, it is equal to

$$\begin{aligned}(\star\star) &= \sum_{b,d} \sum_{i=1}^{p-1} \sum_{n=0}^{p-1} \binom{p-1}{i} \binom{p-1}{n} (-1) b^{i+n} d^{i+n} + \sum_{b,d} b^0 d^{2(p-1)} \\ &= -\sum_{b,d} \Bigg(\sum_{i=1}^{p-1} \binom{p-1}{i} \binom{p-1}{p-1-i} b^{p-1} d^{p-1} + b^{2(p-1)} d^{2(p-1)} \Bigg) \\ &= -\sum_{i=0}^{p-1} \binom{p-1}{i}^2 \sum_{b,d} b^{p-1} d^{p-1} \\ &= -\sum_{i=0}^{p-1} \binom{p-1}{i}^2 = 0.\end{aligned}$$

If $j = k$,

$$\begin{aligned}(\star\star) &= \sum_{b,d} \Bigg(\sum_{i=j+1}^{p-1} \sum_{n=0}^{j} \binom{i-1}{j} \binom{p-1}{i} \binom{j}{n} (-1)^{p-j} b^{2(p-1)-j+i+n} d^{p-1-j+i+n} + \\ &\quad + \sum_{i=0}^{j} \sum_{l=0}^{p-1-j} \binom{p-1}{p-1-i} \binom{p-2-i}{p-2-j} \binom{p-1-j}{l} (-1)^{i+l} b^{j-i+l} d^{p-1+j-i+l} \Bigg) \\ &= \sum_{b,d} \binom{p-2}{j} (-1)^{p-j} b^{3(p-1)} d^{2(p-1)} + \\ &\quad + \sum_{b,d} \binom{p-2}{p-2-j} (-1)^{p-1-j} b^{p-1} d^{2(p-1)} = \\ &= \binom{p-2}{j} (-1)^{p-j} + \binom{p-2}{p-2-j} (-1)^{p-1-j} = 0.\end{aligned}$$

So, $(\star\star) = 0$ and the vector $x_1^{p-1} \otimes y_2^{p-1} - x_2^{p-1} \otimes y_1^{p-1}$ is singular.

□

Lemma 8.5.2. *There are no singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})_i$ for $i < p-1$, and the space of singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})_{p-1}$ is isomorphic to $S^{p-1}\mathfrak{h}$ as a $GL_2(\mathbb{F}_p)$-representation.*

Proof. From the previous lemma it follows that the space of singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})_{p-1}$ is nonzero, and from Lemma 8.2.6 that all the composition factors of it are isomorphic to $S^{p-1}\mathfrak{h}$. We will now show that for $0 \leq i \leq p-1$, $M_{0,c}(S^{p-1}\mathfrak{h})_i \cong S^i\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$ has no composition factors isomorphic to $S^{p-1}\mathfrak{h}$ unless $i = 0$ or $i = p-1$, in which case it has one. The claim follows from this.

Using Proposition 8.1.3 about the tensor products of symmetric powers, and calculating in the Grothendieck group of the category of finite-dimensional representations of the group $GL_2(\mathbb{F}_p)$ (so, disregarding the question whether the short exact sequence is split or not), we get:

$$\begin{aligned}[S^i\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}] &= [S^i\mathfrak{h} \otimes S^{p-1}\mathfrak{h} \otimes (\det)^{-i}] \\ &= [S^{p+i-1}\mathfrak{h} \otimes (\det)^{-i}] + [S^{p+i-3}\mathfrak{h} \otimes (\det)^{-i+1}] + \ldots + [S^{p-1-i}\mathfrak{h}].\end{aligned}$$

The representation $S^{p-1}\mathfrak{h}$ only appears on this list of representations $S^{p+i-1-2j}\mathfrak{h} \otimes (\det)^{-i+j}$ for $0 \leq j \leq i$, when $i = 0$, which is the trivial case. However, some of the representations on the list are reducible, namely the ones with $i - 1 - 2j \geq 0$. Decomposing them by Proposition 8.1.4,

$$[S^{p+i-1-2j}\mathfrak{h} \otimes (\det)^{-i+j}] =$$

$$= [S^{i-1-2j}\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{-i+j}] + [S^{p-1-i+2j}\mathfrak{h} \otimes (\det)^{-j}]$$

$$= [S^{i-2-2j}\mathfrak{h} \otimes (\det)^{-i+j+1}] + [S^{i-2j}\mathfrak{h} \otimes (\det)^{-i+j}] + [S^{p-1-i+2j}\mathfrak{h} \otimes (\det)^{-j}]$$

Here, we follow the convention that $S^k\mathfrak{h} = 0$ if $k < 0$. In this decomposition all representations are irreducible. Using that $i - 1 - 2j \geq 0$, $p - 1 \geq i \geq 0$ and

$i \geq j \geq 0$, we see that $S^{p-1}\mathfrak{h}$ appears on this list only once, namely when $j = 0$, $i = p-1$. ◻

This space is generated by the singular vector from Lemma 8.5.1. Its explicit basis, which we will need in computations below, is given by $v_0, \dots, v_{p-1} \in S^{p-1}\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$, where

$$v_k = \sum_{i=0}^{k}(-1)^i x_1^{k-i} x_2^{p-1-k+i} \otimes y_1^{p-1-i} y_2^i + \sum_{i=k}^{p-1}(-1)^i x_1^{p-1+k-i} x_2^{i-k} \otimes y_1^{p-1-i} y_2^i.$$

Recall that the algebra of invariants $(S\mathfrak{h}^*)^{GL_2(\mathbb{F}_p)}$ is a polynomial algebra generated by polynomials Q_0 and Q_1 of degrees p^2-p and p^2-1. Also recall that at $t=0$, the space $(S\mathfrak{h}^*)_+^{GL_2(\mathbb{F}_p)} \otimes \tau \subseteq M_{0,c}(\tau)$ is always a subspace of $J_{0,c}(\tau)$, and that the spaces $Q_1 \otimes \tau$ and $Q_0 \otimes \tau$ consist of singular vectors.

Let us consider three spaces of singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})$, all isomorphic to $S^{p-1}\mathfrak{h}$: $\mathrm{span}\{v_0, \dots v_{p-1}\}$ in degree $p-1$, $Q_1 \otimes S^{p-1}\mathfrak{h}$ in degree p^2-p, and $Q_0 \otimes S^{p-1}\mathfrak{h}$ in degree p^2-1. We want to study the $H_{0,c}(GL_2, \mathfrak{h})$-submodule of $M_{0,c}(S^{p-1}\mathfrak{h})$ generated by these vectors, and calculate the character of the quotient M of $M_{0,c}(S^{p-1}\mathfrak{h})$ by this submodule.

Proposition 8.5.3. *Let V be the $H_{0,c}(GL_2(\mathbb{F}_p))$ submodule of $M_{0,c}(S^{p-1}\mathfrak{h})$ generated by singualr vectors $v_0, \dots v_{p-1}$. Then*

$$Q_1 \otimes S^{p-1}\mathfrak{h} \subseteq V,$$

while the intersection of the submodule generated by $Q_0 \otimes S^{p-1}\mathfrak{h}$ and V is generated by $Q_0 v_0, \dots Q_0 v_{p-1}$ in degree $(p^2-1)(p-1)$.

Proof. Let $l = 0$ or 1, and let us study the intersection of the submodule generated by $Q_l \otimes S^{p-1}\mathfrak{h}$ and V. This is a graded submodule of $M_{0,c}(S^{p-1}\mathfrak{h})$, with elements of the form

$$h_0 v_0 + h_1 v_1 + \dots h_{p-1} v_{p-1} = Q_l f,$$

where $h_l(x_1, x_2) \in S^n\mathfrak{h}^*$ for some $n \geq 0$, and $f \in S^{n+p-1-\deg(Q_l)}\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$.

This is a linear equation in $S\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$ with unknowns h_i and f. Reading off the coefficients with $y_1^{p-1-i}y_2^i \in S^{p-1}\mathfrak{h}$ in this equation, we can think of it as a system of p linear equations in $S\mathfrak{h}$, with unknowns h_i and $f_i \in S\mathfrak{h}^*$, $f = \sum_i f_i \otimes y_1^{p-1-i}y_2^i$. The left hand side can then be written as

$$\begin{bmatrix} x_1^{p-1}+x_2^{p-1} & x_1x_2^{p-2} & \dots & x_1^k x_2^{p-1-k} & \dots & x_1^{p-1} \\ -x_1^{p-2}x_2 & -(x_1^{p-1}+x_2^{p-1}) & \dots & -x_1^{k-1}x_2^{p-k} & \dots & -x_1^{p-2}x_2 \\ x_1^{p-3}x_2^2 & x_1^{p-2}x_2 & \dots & x_1^{k-2}x_2^{p-k+1} & \dots & x_1^{p-3}x_2^2 \\ \vdots & \vdots & & & & \vdots \\ (-1)^k x_1^{p-1-k}x_2^k & (-1)^k x_1^{p-k}x_2^{k-1} & \dots & (-1)^k(x_1^{p-1}+x_2^{p-1}) & \dots & x_1^{p-1-k}x_2^k \\ \vdots & \vdots & & & & \vdots \\ x_2^{p-1} & x_1x_2^{p-2} & \dots & x_1^k x_2^{p-1-k} & \dots & x_1^{p-1}+x_2^{p-1} \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_k \\ \vdots \\ h_{p-1} \end{bmatrix}.$$

The i-th row represents the coefficient of $y_1^{p-1-i}y_2^i$, and the k-th column corresponds to v_k. Call the matrix of this system $\mathbf{A}$, denote the vector with entries h_i by $\overrightarrow{h}$ and the vector with entries f_i by $\overrightarrow{f}$. The system of equations in matrix form can then be written as

$$\mathbf{A}\overrightarrow{h} = Q_l\overrightarrow{f}.$$

Next, we need a lemma.

Lemma 8.5.4. $\det \mathbf{A} = (-1)^{p-1/2}Q_1$.

Proof. Factoring out the coefficient -1 from all even rows accounts for the sign $(-1)^{(p-1)/2}$. The remaining matrix $\mathbf{A}'$ has $x_1^{p-1}+x_2^{p-1}$ on the main diagonal, $x_1x_2^{p-2}$ above the main diagonal, $x_1^2x_2^{p-3}$ above that etc, and $x_1^{p-2}x_2$ below the main diagonal, $x_1^{p-3}x_2^2$ below that, etc.

To show $\det \mathbf{A}'$ is invariant under $GL_2(\mathbb{F}_p)$ action, let us show it is preserved by the generators of $GL_2(\mathbb{F}_p)$, for example transformations

$$A\colon x_1 \mapsto x_2, x_2 \mapsto x_1,$$

$$B\colon x_1 \mapsto ax_1, x_2 \mapsto x_2, a \in \mathbb{F}_p,$$

$$C\colon x_1 \mapsto x_1 + x_2, x_2 \mapsto x_2.$$

This proof is direct. The action of A corresponds to transposing $\mathbf{A}'$, which preserves the determinant. The action of B can be calculated as multiplying the entry of $\mathbf{A}'$ in the $(i+1)$-th row and $(k+1)$-st column (the ones corresponding to the coefficient of $y_1^{p-1-i}y_2^i$ in v_k) by a^{i-k}. Factoring out a^i from $i+1$-th row and a^{-k} from the $k+1$-st column, for all rows and columns, we get the determinant we started from multiplied by $a^{(0+1+\dots p-1)-(0+1+\dots p-1)} = 1$. The action of C produces a new matrix, which can be reduced to $\mathbf{A}'$ by a series of row and column operations which don't change the determinant. More precisely, it is possible to use row operations, adding to each row a linear combination of the rows below it, and achieve that the first column is that of the original matrix, and to follow that by a series of column operations, adding to each column a linear combination of the ones before it, and get the matrix we started from.

We concluded that the determinant of $\mathbf{A}'$ is a polynomial in x_1, x_2 of degree $p(p-1)$, invariant under the $GL_2(\mathbb{F}_p)$ action. Hence, it is a multiple of Q_1. The coefficient of $x_1^{p(p-1)}$ in both the determinant and Q_1 is equal to 1, and this finishes the proof of lemma. □

We return to the proof of the proposition and to the system of linear equations

$$\mathbf{A}\overrightarrow{h} = Q_l\overrightarrow{f}.$$

The inverse of $\mathbf{A}$ is $\frac{1}{\det \mathbf{A}}\tilde{\mathbf{A}}$, for $\tilde{\mathbf{A}}$ the adjugate matrix to $\mathbf{A}$. For fixed $\overrightarrow{f}$, the unique rational solution $\overrightarrow{h}$ is given by

$$\overrightarrow{h} = \frac{1}{Q_1}\tilde{\mathbf{A}}Q_l\overrightarrow{f}.$$

This solution will be polynomial if and only if every entry of the vector $Q_l\overrightarrow{f}$ is divisible (in $S\mathfrak{h}^*$) by Q_1.

If $l = 1$, this becomes

$$\overrightarrow{h} = \tilde{\mathbf{A}}\overrightarrow{f},$$

so a polynomial solution exists for every $f \in S\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$. In particular, we can pick $f \in S^0\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$, and it follows that $Q_1 \otimes S^{p-1}\mathfrak{h}$ is contained in V.

If $l = 0$, then using that Q_0 and Q_1 are algebraically independent, it follows that the polynomial solution $\overrightarrow{h}$ will exist if and only if every entry of $\overrightarrow{f}$ is divisible by Q_1. So let $\overrightarrow{f} = Q_1\overrightarrow{f'}$, and notice that

$$\overrightarrow{h} = Q_0\tilde{\mathbf{A}}\overrightarrow{f'}$$

means that every entry of $\overrightarrow{h}$ is divisible by Q_0. From this it follows that the intersection of the submodule generated by $Q_0 \otimes S^{p-1}\mathfrak{h}$ and V is generated by $Q_0v_0, \dots Q_0v_{p-1}$, which are in degree $(p^2-1)(p-1)$.

□

As explained above, the purpose of proving the previous proposition was to conclude:

Corollary 8.5.5. *Let M be the quotient of $M_{0,c}(S^i\mathfrak{h})$ by the $H_{0,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$-submodule generated by singular vectors $v_0, \dots v_{p-1}$ in degree $p-1$, $Q_1 \otimes S^i\mathfrak{h}$ in degree p^2-p and $Q_0 \otimes S^i\mathfrak{h}$ in degree p^2-1. Then its character is*

$$\chi_M(z) = \chi_{M_{0,c}(S^{p-1}\mathfrak{h})}(z)(1-z^{p-1})(1-z^{p^2-1})$$

and its Hilbert series is a polynomial

$$\mathrm{Hilb}_M(z) = p\frac{(1-z^{p-1})(1-z^{p^2-1})}{(1-z)^2}.$$

Proposition 8.5.6. *$L_{0,c}(S^{p-1}\mathfrak{h}) = M$.*

Proof. By Lemma 8.2.6, the irreducible representation $L_{0,c}(S^{p-1}\mathfrak{h})$ forms a block of size one. That means that all the irreducible composition factors that appear in the decomposition of $M_{0,c}(S^{p-1}\mathfrak{h})$ and of M, are isomorphic to $L_{0,c}(S^{p-1}\mathfrak{h})[m]$.

As a consequence, the character of $L_{0,c}(S^{p-1}\mathfrak{h})$ divides the character of M;

$$\chi_{L_{0,c}(S^{p-1}\mathfrak{h})}(z)F(z) = \chi_M(z),$$

for some polynomial $F(z)$ with positive integer coefficients. The character of $L_{0,c}(S^{p-1}\mathfrak{h})$ is of the form

$$\chi_{L_{0,c}(S^{p-1}\mathfrak{h})}(z) = \chi_{M_{0,c}(S^{p-1}\mathfrak{h})}(z)\overline{h}(z)$$

for some polynomial $\overline{h}$ with integer coefficients ($\overline{h}$ is divisible by $(1-z)^2$, as $L_{0,c}(S^{p-1})$ is finite-dimensional and $M_{0,c}(S^{p-1})$ has quadratic growth). Substituting this and the character formula for M in the above equation, we get that

$$\overline{h}(z)F(z) = (1 - z^{p-1})(1 - z^{p^2-1}).$$

Let us use the other version of the character which will enable us to compute $\overline{h}$. Recall that for $V = \sum_k V_k$ a graded Cherednik algebra module, we defined ch_V to be a function of a formal variable z and of a group element g, defined as

$$\mathrm{ch}_V(z, g) = \sum_k z^k \mathrm{tr}|_{V_k}(g).$$

It is then easy to see that

$$\mathrm{ch}_{M_{0,c}(S^{p-1}\mathfrak{h})}(z, g) = \mathrm{tr}|_{S^{p-1}\mathfrak{h}}(g) \cdot \frac{1}{\det_{\mathfrak{h}^*}(1 - zg)},$$

so

$$\mathrm{ch}_M(z, g) = \mathrm{tr}|_{S^{p-1}\mathfrak{h}}(g) \cdot \frac{(1 - z^{p-1})(1 - z^{p^2-1})}{\det_{\mathfrak{h}^*}(1 - zg)}$$

$$\mathrm{ch}_{L_{0,c}(S^{p-1}\mathfrak{h})}(z, g) = \mathrm{tr}|_{S^{p-1}\mathfrak{h}}(g) \cdot \frac{\overline{h}(z)}{\det_{\mathfrak{h}^*}(1 - zg)}.$$

Let $g \in GL_2(\mathbb{F}_p)$. It can be put to Jordan form over a quadratic extension $\mathbb{F}_q$ of

$\mathbb{F}_p$, and assume it is diagonalizable with different eigenvalues, of the form

$$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

with $\lambda \neq \mu \in \mathbb{F}_q$. Then

$$\operatorname{tr}|_{S^{p-1}\mathfrak{h}}(g) = \lambda^{p-1} + \lambda^{p-2}\mu + \ldots \mu^{p-1} = \frac{\lambda^p - \mu^p}{\lambda - \mu} \neq 0$$

and $\operatorname{ch}_{L_{0,c}(S^{p-1}\mathfrak{h})}(z, g)$ is a polynomial in z, so

$$\frac{\overline{h}(z)}{\det_{\mathfrak{h}^*}(1 - zg)} = \frac{\overline{h}(z)}{(1 - z\lambda^{-1})(1 - z\mu^{-1})}$$

must be a polynomial in z as well. By choosing all possible λ and μ in $\mathbb{F}_p \subseteq \mathbb{F}_q$, this implies that $\overline{h}(z)$ is divisible by all linear polynomials of the form $1 - z\lambda^{-1}$, and hence by their product $1 - z^{p-1}$. If λ and μ are in the extension $\mathbb{F}_q$ and not in $\mathbb{F}_p$, then the product $(1 - z\lambda^{-1})(1 - z\mu^{-1})$ is an irreducible quadratic polynomial with coefficients in $\mathbb{F}_p$ with a constant term 1. All such polynomials can be obtained in this way, and $\overline{h}(z)$ is divisible by their product $(1 - z^{p^2-1})/(1 - z^{p-1})$. From this we conclude that $\overline{h}(z)$ is divisible by $1 - z^{p^2-1}$.

Let us write

$$\overline{h}(z) = (1 - z^{p^2-1})\phi(z)$$

for some polynomial ϕ. Then

$$\phi(z)F(z) = 1 - z^{p-1}.$$

However, it follows from Lemma 8.5.2 that $\overline{h}$ is of the form $1 - z^{p-1} + \ldots$, so $\phi(z)$ is of that form as well, and it follows that $\phi(z) = 1 - z^{p-1}$, $F(z) = 1$ and $L_{0,c}(S^{p-1}\mathfrak{h}) = M$.

□

8.5.2 Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = p-1$ and $t = 1$

Computing the character of $L_{1,c}(S^{p-1}\mathfrak{h})$ is very similar to computing the character of $L_{0,c}(S^{p-1}\mathfrak{h})$ in the previous section. We define a set of vectors analogous to v_i:

$$v'_k = \sum_{i=0}^{k}(-1)^i x_1^{p(k-i)}x_2^{p(p-1-k+i)} \otimes y_1^{p-1-i}y_2^i + \sum_{i=k}^{p-1}(-1)^i x_1^{p(p-1+k-i)}x_2^{p(i-k)} \otimes y_1^{p-1-i}y_2^i.$$

Lemma 8.5.7. *The space* $span_{\mathbf{k}}\{v'_0, \dots, v'_{p-1}\} \subseteq S^{p(p-1)}\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h} \cong M_{1,c}(S^{p-1}\mathfrak{h})_{p(p-1)}$ *consists of singular vectors, and isomorphic to* $S^{p-1}\mathfrak{h}$ *as a* $GL_2(\mathbb{F}_p)$ *representation. This is the only space of singular vectors in* $M_{1,c}(S^{p-1}\mathfrak{h})_{p\cdot i}$ *for* $i = 1, \dots p-1$.

Proof. The proof that they are singular is an explicit computation analogous to the one in the proof of Lemma 8.5.1, showing that one vector from this irreducible representation of $GL_2(\mathbb{F}_p)$ is singular. The space spanned by them is only space of p-th powers in degrees $p, 2p, \dots, (p-1)p$ which is isomorphic to $S^{p-1}\mathfrak{h}$ as a $GL_2(\mathbb{F}_p)$ representation; this follows directly from Lemma 8.5.2 and implies that this is the only space of singular vectors for generic c in degrees up to $p(p-1)$.

□

Proposition 8.5.8. *Let* M' *be the quotient of* $M_{1,c}(S^i\mathfrak{h})$ *by the* $H_{1,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$*-submodule generated by singular vectors* $v'_0, \dots v'_{p-1}$ *in degree* $p(p-1)$*,* $Q_1^p \otimes S^i\mathfrak{h}$ *in degree* $p(p^2-p)$ *and* $Q_0^p \otimes S^i\mathfrak{h}$ *in degree* $p(p^2-1)$*. Its character is*

$$\chi_{M'}(z) = \chi_{M_{1,c}(S^{p-1}\mathfrak{h})}(z^p)(1-z^{p(p-1)})(1-z^{p(p^2-1)})\left(\frac{1-z^p}{1-z}\right)^2$$

and the Hilbert series

$$\mathrm{Hilb}_{M'}(z) = p\frac{(1-z^{p(p-1)})(1-z^{p(p^2-1)})}{(1-z)^2}.$$

Proof. The claim is equivalent to the reduced character being equal to

$$\chi_{M_{1,c}(S^{p-1}\mathfrak{h})}(z)(1-z^{p-1})(1-z^{p^2-1}).$$

By definition of M' and the reduced character, it is equal to the character of the $S\mathfrak{h}^*$-module defined as the quotient of $S\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$ by $v_0, \ldots v_{p-1}$ from the previous section, $Q_0 \otimes S^i\mathfrak{h}$ and $Q_1 \otimes S^i\mathfrak{h}$. Corollary 8.5.5 in the previous section shows that the character of this module is as claimed in the proposition.

□

Finally, we have

Proposition 8.5.9. *For generic c,* $L_{1,c}(S^{p-1}\mathfrak{h}) = M'$.

Proof. The character of $L_{1,c}(S^{p-1}\mathfrak{h})$ for generic c is of the form

$$\chi_{L_{1,c}(S^{p-1}\mathfrak{h})}(z) = \chi_{M_{1,c}(S^{p-1}\mathfrak{h})}(z^p)\left(\frac{1-z^p}{1-z}\right)^2 \overline{h'}(z^p)$$

for some polynomial $\overline{h'}$. It divides the character of M', so $\overline{h'}(z)$ divides $(1-z^{p-1})(1-z^{p^2-1})$. Using the same version of the character as in the proof of Proposition 8.5.6, we see that

$$\mathrm{ch}_{L_{1,c}(S^{p-1}\mathfrak{h})}(z,g) = \mathrm{tr}|_{S^{p-1}\mathfrak{h}}(g) \cdot \frac{\overline{h}'(z^p)}{\det_{\mathfrak{h}^*}(1-zg)},$$

and we see that $h(z^p)$ is divisible by $(1-z^{p^2-1})$. From this it follows that $h(z^p)$ is divisible by $(1-z^{p(p^2-1)})$. Finally, it follows from the previous proposition that $\overline{h'}(z)$ is of the form $1-z^{p-1}+\ldots$, and from this, its divisibility by $(1-z^{p^2-1})$ and the fact it divides $(1-z^{p^2-1})(1-z^{p-1})$ it follows $\overline{h'}(z) = (1-z^{p^2-1})(1-z^{p-1})$ and $L_{1,c}(S^{p-1}\mathfrak{h}) = M'$.

□

Appendix A

Computational data

$$\begin{pmatrix}
1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
-y^{-144} & -y^{144} & -2 & 2 & 2 & -3y^{-48} & -3y^{48} & 4 \\
-y^{-144} & -y^{144} & -2 & 2 & 2 & -3y^{-48} & -3y^{48} & 4 \\
-y^{-144} & y^{144} & 0 & 0 & 0 & -y^{-48} & y^{48} & 0 \\
1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\
(\xi^4-1)y^{-96} & (\xi^4-1)y^{96} & -\xi^4+1 & -\xi^4+1 & -\xi^4+1 & 0 & 0 & \xi^4-1 \\
-\xi^4y^{-192} & -\xi^4y^{192} & \xi^4 & \xi^4 & \xi^4 & 0 & 0 & -\xi^4 \\
-\xi^6y^{-216} & -\xi^6y^{216} & -2\xi^6 & 0 & 0 & \xi^6y^{-72} & \xi^6y^{72} & 0 \\
\xi^6y^{-72} & \xi^6y^{72} & 2\xi^6 & 0 & 0 & -\xi^6y^{-24} & -\xi^6y^{24} & 0 \\
\xi^4y^{-48} & \xi^4y^{48} & -\xi^4 & \xi^4 & \xi^4 & 0 & 0 & -\xi^4 \\
(-\xi^4+1)y^{-240} & (-\xi^4+1)y^{240} & \xi^4-1 & -\xi^4+1 & -\xi^4+1 & 0 & 0 & \xi^4-1 \\
(-\xi^5+\xi)y^{-252} & (\xi^5-\xi)y^{252} & 0 & -\xi^6-1 & \xi^6+1 & (\xi^5-\xi)y^{-84} & (-\xi^5+\xi)y^84 & 0 \\
-\xi^3y^{-180} & \xi^3y^{180} & 0 & \xi^6-1 & -\xi^6+1 & \xi^3y^{-60} & -\xi^3y^{60} & 0 \\
(\xi^5-\xi)y^{-108} & (-\xi^5+\xi)y^{108} & 0 & -\xi^6-1 & \xi^6+1 & (-\xi^5+\xi)y^{-36} & (\xi^5-\xi)y^{36} & 0 \\
\xi^3y^{-36} & -\xi^3y^{36} & 0 & \xi^6-1 & -\xi^6+1 & -\xi^3y^{-12} & \xi^3y^{12} & 0
\end{pmatrix}$$

Table A.1: $\underset{\sim}{A}$-matrix for parameter c where $\xi = e^{2\pi i/24}$ and $y = \xi^c$. The columns are labeled by $\widehat{W}$, in the order $\mathbf{1}_+$, $\mathbf{1}_-$, $\mathbf{2}$, $\mathbf{2}_+$, $\mathbf{2}_-$, $\mathbf{3}_+$, $\mathbf{3}_-$, $\mathbf{4}$.

$c = 1/12$	$e_{1/12} = (1,1,0,0,-1,0,0,0)$
$c = 1/4$	$e^1_{1/4} = (1,0,0,0,0,0,1,-1)$, $e^2_{1/4} = (0,1,0,0,0,1,0,-1)$, $e^3_{1/4} = (0,0,0,1,0,-1,-1,1)$
$c = 1/3$	$e_{1/3} = (1,1,-1,0,0,0,0,0)$
$c = 1/2$	$e^1_{1/2} = (1,0,1,0,0,-1,0,0)$, $e^2_{1/2} = (0,1,1,0,0,0,-1,0)$

Table A.2: Nullspace bases for A-matrix at $c = 1/12, 1/4, 1/3, 1/2$

s_{1_+}	$v^{-24}(v-\xi)(v+\xi)(v-\xi^3)^2(v+\xi^3)^2(v-\xi^4)(v+\xi^4)(v-\xi^5)(v+\xi^5)(v-\xi^6)^2(v+\xi^6)^2(v-\xi^7)(v+\xi^7)(v-\xi^8)(v+\xi^8)(v-\xi^9)^2(v+\xi^9)^2(v-\xi^{11})(v+\xi^{11})$
s_{1_-}	$(v-\xi)(v+\xi)(v-\xi^3)^2(v+\xi^3)^2(v-\xi^4)(v+\xi^4)(v-\xi^5)(v+\xi^5)(v-\xi^6)^2(v+\xi^6)^2(v-\xi^7)(v+\xi^7)(v-\xi^8)(v+\xi^8)(v-\xi^9)^2(v+\xi^9)^2(v-\xi^{11})(v+\xi^{11})$
s_2	$2v^{-4}(v-\xi^4)(v+\xi^4)(v-\xi^6)^2(v+\xi^6)^2(v-\xi^8)(v+\xi^8)$
s_{2_+}	$-12v^{-4}(v-\xi)(v+\xi^3)^2(v+\xi^5)(v+\xi^7)(v+\xi^9)^2(v-\xi^{11})$
s_{2_-}	$-12v^{-4}(v+\xi)(v-\xi^3)^3(v-\xi^5)(v-\xi^7)(v-\xi^9)^2(v+\xi^{11})$
s_{3_+}	$v^{-10}(v-\xi^3)^2(v+\xi^3)^2(v-\xi^6)^2(v+\xi^6)^2(v-\xi^9)^2(v+\xi^9)^2$
s_{3_-}	$v^{-2}(v-\xi^3)^2(v+\xi^3)^2(v-\xi^6)^2(v+\xi^6)^2(v-\xi^9)^2(v+\xi^9)^2$
s_4	$3v^{-4}(v-\xi^3)^2(v+\xi^3)^2(v-\xi^9)^2(v+\xi^9)^2$

Table A.3: Schur Elements for $\mathcal{H}_q(G_{12})$, $q=v^2$, $\xi=e^{2\pi i/24}$

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