On Affine Embeddings of Reductive Groups
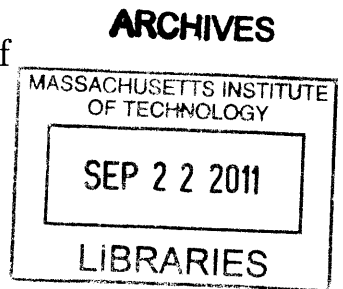
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Abstract

In this thesis, we study the properties and the classification of embeddings of homogeneous spaces, especially the case of affine normal embeddings of reductive groups. We might guess that as in the case of toric varieties, some specific subset of one-parameter subgroups may contribute to the classification of affine embeddings of general reductive group. To check this, we review the theory of affine normal $SL(2)$-embeddings, and prove that the classification cannot be solved entirely based on one-parameter subgroups. We can also show that even though this set does not give a complete answer to the classification problem, but still contains useful information about varieties. We will also give examples of $GL(2)$-embeddings which had not previously been constructed in detail, which might be helpful in understanding the general classification of affine normal G-embeddings.

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Contents

Chapter 1

Embeddings of Homogeneous Spaces

One interesting problem in algebraic geometry is the study of algebraic group acting on algebraic varieties. We are interested in the case when the one of the orbits of this action is dense. It is well known that any orbit is open in its closure, so a dense orbit is automatically open in X. An irreducible algebraic G-variety X is said to be an *embedding* of the homogeneous space G/H, or G/H-*embedding*, if X contains an open G-orbit isomorphic to G/H. (When this variety is normal in addition, we often call it a *quasihomogeneous variety*.) Toric varieties are the most famous examples of G/H-embeddings where G is an algebraic torus and H is the trivial subgroup. Also, flag varieties are another interesting examples of G/H-embedding, and as a generalization of these varieties, spherical varieties have been studied widely. In [22], Luna and Vust give a classification of spherical varieties by suggesting combinatorial structures called colored fan.

Throughout this thesis, we will work over a ground field $\mathbb{C}$, which can be replaced by any algebraic closed field of characteristic 0. G denotes a connected reductive algebraic group unless otherwise specified, and H is an algebraic subgroup of G. All algebraic groups and varieties are assumed to be defined over $\mathbb{C}$.

1.1 First Examples

1.1.1 Toric Varieties

Let $T = \mathbb{G}_m{}^n$ is an algebraic torus defined over $\mathbb{C}$. A *toric variety* X is a normal T-variety which contains T as a dense open subvariety so that the action of T on X extends the regular action of T on itself by multiplication. By a theorem of Sumihiro [30], there is a covering of X by T-stable, affine open subvarieties which are therefore again toric varieties. Hence every toric variety may be obtained by gluing affine toric varieties together. This implies that the classification of toric varieties can be reduced to the problem of classifying affine toric varieties, and observing how those varieties are glued together. The basic theory of toric varieties can be found in many places such as [8], [11], [16].

Suppose X is an affine toric variety, namely $X = \operatorname{Spec} A$. The open embedding $T \hookrightarrow X$ corresponds to an injective homomorphism $A \to \mathbb{C}[T] \cong \mathbb{C}[x_1, x_1^{-1}, \cdots, x_r, x_r^{-1}]$. The action of T on X induces an action of T on A as well via $t \cdot f \mapsto f(t^{-1}x)$ for all $f \in A$.

We denote the set of one-parameter subgroups of T, $\operatorname{Hom}(\mathbb{G}_m, T)$ by $\mathfrak{X}_*(T)$. This is a free abelian group, which is dual to the group of characters of T (which we denote as $\mathfrak{X}^*(T)$), with respect to the perfect paring $\langle \cdot, \cdot \rangle : \mathfrak{X}^*(T) \times \mathfrak{X}_*(T) \to \mathbb{Z}$ determined by

$$\chi(\gamma(a)) = a^{\langle \chi, \gamma \rangle}$$

for all $\chi \in \mathfrak{X}^*(T), \gamma \in \mathfrak{X}_*(T), a \in \mathbb{C}^*$.

Let S be a finitely generated semigroup in $\mathfrak{X}^*(T)$. Set $\mathbb{C}[S]$ equal to the $\mathbb{C}$-subalgebra of $\mathbb{C}[T] = \mathbb{C}[\mathfrak{X}^*(T)]$ generated by characters $\chi \in S$. As S is finitely generated, $\mathbb{C}[S]$ is a $\mathbb{C}$-algebra of finite type. If S generates $\mathfrak{X}^*(T)$ as a group, then the inclusion $\mathbb{C}[S] \subset \mathbb{C}[T]$ induces an equivariant embedding $T \subset \operatorname{Spec} \mathbb{C}[S]$, and

10

every equivariant T-embedding can be obtained in this way:

Proposition 1. *The correspondence $S \mapsto \mathbb{C}[S]$ defines a bijection between the set of finitely generated semigroups $S \subset \mathfrak{X}^*(T)$ which generate $\mathfrak{X}^*(T)$ as a group and the set of isomorphism classes of equivariant affine embeddings of T. Moreover, the morphisms of equivariant affine embeddings correspond (in a contravariant way) to inclusions between semigroups contained in $\mathfrak{X}^*(T)$.*

A semigroup $S \subset \mathfrak{X}^*(T)$ is called *saturated* if $\chi \in \mathfrak{X}^*(T)$ and $\chi^n \in S$ for some positive integer n, implies that $\chi \in S$. The normality condition of toric varieties corresponds to the saturated condition:

Theorem 1.1.1. *The correspondence $S \mapsto \mathbb{C}[S]$ defines a bijection between the set of finitely generated saturated semigroups $S \subset \mathfrak{X}^*(T)$ which generate $\mathfrak{X}^*(T)$ as a group and the set of equivariant normal affine embeddings of T.*

Therefore, we can classify affine toric varieties by classifying all such saturated semigroups of $\mathfrak{X}^*(T)$. For a torus T, we can regard $\mathfrak{X}_*(T)$ as a $\mathbb{Z}$-lattice in the real vector space $\mathfrak{X}_*(T)_\mathbb{R} = \mathfrak{X}_*(T) \otimes_\mathbb{Z} \mathbb{R}$.

Definition 1.1.2. Let $\sigma \subset \mathfrak{X}_*(T)_\mathbb{R}$. We call σ a *convex rational polyhedral cone* if $\sigma = \{\sum_{i=1}^N \lambda_i v_i : \lambda_i \in \mathbb{R}_{\geq 0},$ for all i $\}$ for some finite collection of elements $v_i \in \mathfrak{X}_*(T)$. We say a convex rational polyhedral cone σ is a *strongly convex rational polyhedral cone* if it does not contain any non-zero linear subspace of $\mathfrak{X}_*(T)_\mathbb{R}$.

A strongly convex rational polyhedral cone σ associates to its dual cone $\sigma^\vee \subset \mathfrak{X}^*(T)_\mathbb{R}$, which is defined as

$$\sigma^\vee = \{u \in \mathfrak{X}^*(T)_\mathbb{R} : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

It is easy to show that $\sigma^\vee$ is a convex, rational, polyhedral cone in $\mathfrak{X}^*(T)_\mathbb{R}$ and $(\sigma^\vee)^\vee = \sigma$. The correspondence $\sigma \mapsto \sigma^\vee \cap \mathfrak{X}^*(T)$ defines a bijection between the set

11

of strongly convex rational polyhedral cones and the set of finitely generated semi-groups in $\mathfrak{X}^*(T)$ which are saturated and generate $\mathfrak{X}^*(T)$ as a group by Gordan's Lemma:

Lemma 1.1.3 (Gordan's Lemma). *Given a finite set of homogeneous linear integral inequalities, the semigroup of integral solutions is finitely generated.*

So we can classify affine toric varieties:

Theorem 1.1.4. *The correspondence $\sigma \mapsto \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \mathfrak{X}^*(T)] = X_\sigma$ defines a bijection between the set of strongly convex, rational, polyhedral cones in $\mathfrak{X}_*(T)_\mathbb{R}$ and the set of affine toric varieties with torus T. Moreover, for $\gamma \in \mathfrak{X}_*(T)$, we have $\gamma \in \sigma$ if and only if $\lim_{t \to 0} \gamma(t)$ exists in X_σ.*

Therefore, affine toric varieties X are of the form X_σ, where

$$\sigma = \big\{ \gamma \in \mathfrak{X}_*(T) : \lim_{t \to 0} \gamma(t) \text{ exists in } X \big\}$$

is a strongly convex, rational, polyhedral cone with $\mathfrak{X}_*(T)$.

To construct general toric varieties beside the affine case, we need a new structure called a *fan*. Before we define this, recall $\sigma^\vee \subset \mathfrak{X}^*(T)_\mathbb{R}$ is the set of $u \in \mathfrak{X}^*(T)_\mathbb{R}$ such that $\langle u, v \rangle \geq 0$ for all $v \in \sigma$. For each $\chi \in \sigma^\vee \cap \mathfrak{X}^*(T)$, define $\chi^\perp = \{x \in \mathfrak{X}_*(T)_\mathbb{R} : \langle \chi, x \rangle = 0\}$. If a cone τ can be obtained as $\sigma \cap \chi^\perp$, then we call τ a *face* of σ and write $\tau \prec \sigma$.

Definition 1.1.5. A *fan* Σ in $\mathfrak{X}_*(T)_\mathbb{R}$ is a finite collection of rational strongly convex polyhedral cones σ in $\mathfrak{X}_*(T)_\mathbb{R}$ such that

1. Every face of a cone of Σ is also a cone in Σ

2. The intersection of two cones in Σ is a face of each cone.

From a fan Σ, the toric variety X_Σ is constructed by taking the disjoint union of the affine toric varieties X_σ, one for each σ in Σ, and gluing as follows : for cones σ

and τ, the intersection $\sigma \cap \tau$ is a face of each cone, so $X_{\sigma \cap \tau}$ is identified as a principal open subvariety of X_σ and of X_τ. Hence we can glue X_σ to X_τ by this identification on these open subvarieties. The fact that these identifications are compatible follows immediately from the order preserving nature of the correspondence from cones to affine varieties. (If $\tau \prec \sigma$, then $\mathbb{C}[\tau^\vee \cap \mathfrak{X}^*(T)]$ is the localization of $\mathbb{C}[\sigma^\vee \cap \mathfrak{X}^*(T)]$ at χ, where $\tau = \sigma \cap \chi^\perp$, so X_τ is an open subvariety of X_σ.) Moreover, the torus $T \cong X_{\{0\}}$ is naturally an open dense subset of X_σ since $\{0\}$ is a face of every cone, so we have an open embedding of T onto X_Σ. Therefore, we have following correspondences.

Theorem 1.1.6. *The correspondence $\Sigma \mapsto X_\Sigma$ defines a bijection between fans in $\mathfrak{X}_*(T)_\mathbb{R}$ and isomorphism classes of toric varieties with a torus T.*

A lot of properties of a toric variety can be observed directly from its fan. Some properties of algebraic variety are corresponding to combinatorial properties in its fan in the lattice, so it makes us easier to understand and compute invariants on the variety. Here are some examples of such properties.

Proposition 2.   *1. X is complete if and only if the fan Σ is complete. (The fan Σ is called complete if for all points $x \in \mathfrak{X}_*(T)_\mathbb{R}$, there exists a cone in Σ containing x.)*

   2. X is smooth if and only if all cones of Σ are generated by a part of a basis of the lattice $\mathbb{Z}^r \cong \mathfrak{X}_*(T)$.

   3. X is $\mathbb{Q}$-factorial if and only if all cones of Σ are generated by linearly independent elements of the lattice $\mathbb{Z}^r \cong \mathfrak{X}_*(T)$.

Now we need to check that how the torus T acts on its toric varieties in terms of the cones and fan. Since the fan is composed of cones, it is enough to check the torus action on cones. (The compatibility naturally comes from the conditions to be a fan.) Without using coordinates, we can describe this as following. When A is $\mathbb{C}$-algebra, there is an isomorphism of functors $\operatorname{Spec} A \cong \operatorname{Hom}_\mathbb{C}(A, \mathbb{C})$. Then

$T = \operatorname{Spec} \mathbb{C}[\mathfrak{X}^*(T)] = \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[\mathfrak{X}^*(T)], \mathbb{C})$ can be considered as the collection of semigroup homomorphisms $\operatorname{Hom}(\mathfrak{X}^*(T), \mathbb{G}_m)$, while $X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \mathfrak{X}^*(T)] = \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[\sigma^\vee \cap \mathfrak{X}^*(T)], \mathbb{C})$ consists of $\mathbb{C}$-algebra homomorphisms from $\mathbb{C}[\sigma^\vee \cap \mathfrak{X}^*(T)]$ to $\mathbb{C}$. Therefore, we can express $x \in X_\sigma$ as the element $\underline{x} \in \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[\sigma^\vee \cap \mathfrak{X}^*(T)], \mathbb{C})$ such that $\underline{x}(m) = m(x)$. Then the action of T on X_σ can be described as following :

$$\text{For } t \in T \text{ and } x \in X_\sigma, \ (t \cdot \underline{x})(m) = \underline{t}(m)\underline{x}(m)$$

where the multiplication on the right side is nothing but multiplication in $\mathbb{C}$.

We can describe this action by using coordinates. Let $(a_1, \cdots, a_s)$ be a system of generators of the monoid $\sigma^\vee \cap \mathfrak{X}^*(T)$. With a standard basis of $\mathfrak{X}_*(T)_{\mathbb{R}}$, each a_i may be written in the form $a_i = (a_{i1}, \cdots a_{ir})$ with $a_{ij} \in \mathbb{Z}$ and $t \in T$ is written as $(t_1, \cdots t_r)$ with $t_j \in \mathbb{G}_m(\mathbb{C})$. When a point $x \in X_\sigma$ is written $x = (x_1, \cdots, x_r)$, then the action $T \times X_\sigma \to X_\sigma$ of T on the affine subvariety X_σ is

$$(t, x) \mapsto (t^{a_1} x_1, \cdots t^{a_r} x_r)$$

where $t^{a_i} = t_1{}^{a_{i1}} \cdots t_r{}^{a_{ir}} \in \mathbb{G}_m(\mathbb{C})$.

As the action of T on a toric variety X_Σ can be depicted in terms of fans, the orbits of X_Σ with respect to T can be described using Σ. For each cone $\sigma \in \Sigma$, there is a base point $x_\sigma \in X_\Sigma$ which is the limit point of any one-parameter subgroups in the relative interior of the cone σ. These base points correspond to T-orbits $T \cdot x_\sigma$ in X_Σ, and moreover, the orbits of T in X of dimension d are in bijection with the cones of codimension d of Σ. In particular, fixed points are parameterized by cones of dimension n and T-stable divisors by one-dimensional cones.

14

1.1.2 Flag Varieties

The algebraic group G is called semi-simple when the radical $R(G)$ of G is trivial where $R(G)$ is the maximal closed, connected, normal, solvable subgroup of G. In this section, we will assume that G is a semi-simple, connected algebraic group over $\mathbb{C}$. Such groups are almost classified by their Dynkin diagrams (of type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, and G_2) or equivalently, by their root systems. One fundamental techinque to study algebraic groups is to study its Borel subgroups. A Borel subgroup is maximal, solvable, closed, connected subgroup of G. If G is reductive, all Borel subgroups are conjugate to each other. If the closed subgroup P of G contains a Borel subgroup of G, then we call P a parabolic subgroup of G.

Proposition 3. *For any complete homogeneous space X, we can find G and a parabolic subgroup P such that $X \simeq G/P$, and X is smooth projective space.*

We call a complete homogeneous space G/P a *flag variety*. Let S be the set of simple roots of G. Then the set of parabolic subgroups of G containing B (and then all the set of isomorphism classes of flag varieties with G fixed) is in bijection with the set of subsets of S. Therefore for $I \subset S$, we can associate a parabolic subgroup P_I. In order to understand the geometry of flag varieties, we often use their decomposition into B-orbits.

Proposition 4 (Bruhat Decomposition). *When W is the Weyl group of G and B, we have a decomposition*

$$G = \bigsqcup_{w \in W} B \dot{w} B$$

where $\dot{w}$ is a representative of w in G. In particular,

$$G/B = \bigsqcup_{w \in W} B \dot{w} B/B \text{ and } G/P = \bigsqcup_{w \in W/W_P} B \dot{w} P/P$$

where if $P = P_I$, W_P is the subgroup of W generated by the simple reflections associated to elements of $S \setminus I$.

Since W is finite, we have the following conclusion.

Corollary 1. *Flag varieties have an open orbit under the action of a Borel subgroup.*

The closure of the B-orbits in G/P are called the Schubert varieties (often written as $X(w)$), and they play an important role in the study of G/P. The dimension of $X(w)$ is the length $l(w)$ of w (the minimal number of simple reflections in the expression of w as a product of simple reflections.) In particular, there exists a unique element w_0 of maximal length in W/W_P.

If G is not semi-simple but reductive, then G/P is isomorphic to $G'/(G' \cap P)$ where G' is the semi-simple part (or equivalently, the derived subgroup) of G and then it is still a flag variety.

1.2 Spherical Varieties and the theory of Luna-Vust

From now on, G can be any reductive and connected algebraic group over $\mathbb{C}$. For a closed subgroup $H \subset G$, the homogeneous space G/H is *spherical* if B acts on it with an open orbit. More generally, G/H is spherical whenever H contains a maximal unipotent subgroup of G. Also, we define a *spherical variety* as a normal algebraic variety with an action of G and a dense orbit of B. It can be also described as a G/H-embedding for a spherical homogeneous space G/H. The first examples of spherical varieties are toric varieties (a Borel subgroup of T is T itself), flag varieties, and symmetric spaces. For this section, our main sources are [2], [7], [25].

The rather abstract notion of a spherical variety actually has a very rich geometry, which is only partially understood. It combines features of flag varieties and of symmetric spaces. As for toric varieties, the geometry of fans and convex polytopes play a role, too. Also, spherical varieties are a test case for studying action of reductive groups. More precisely, several phenomena, first discovered for spherical varieties, have been generalized to arbitrary varieties with reductive

group actions. However, many results find a simpler and more precise formulation in the case of spherical varieties.

1.2.1 Basic properties of Spherical Varieties

For an algebraic variety X with an action of B, we define its *complexity* $c(X)$ as the minimal codimension of a B-orbit in X. By a classical result of Rosenlicht [29], $c(X)$ is the transcendence degree of the extension $k(X)^B/k$ where k(X) denotes the function field of X, and $k(X)^B$ its subfield of B-invariants. The set of weights of eigenvectors of B in $k(X)$ is denoted by $\Gamma(X)$. Then $\Gamma(X)$ is a free abelian group of finite rank $r(X)$, and this number is called the *rank* of X. Motivation for these notions is the following result, due to Vinberg in characteristic zero [33], and to Knop in general [19].

Theorem 1.2.1. *For any G-variety X, and for any closed, B-stable subvariety $Y \subset X$, we have $c(Y) \leq c(X)$ and $r(Y) \leq r(X)$.*

Note that spherical varieties are exactly varieties with the complexity is zero. So the theorem implies the following corollary.

Corollary 2. *A G-variety X is spherical if and only if X contains only finitely many B-orbits.*

In particular, any spherical variety contains only finitely many G-orbits, and all of them are spherical. On the other hand, for any nonspherical G-variety X, there exist a G-variety $\widetilde{X}$ that is G-birational to X and that contains infinitely many G-orbits.

The rank of a G-variety is an important invariant; and it generalizes the rank of a symmetric space. The G-varieties of rank zero are just unions of flag varieties. There is a very useful classification of homogeneous spaces of rank one ([1], [26]). Namely, several theorems on spherical varieties use reduction to rank one.

1.2.2 Classification of Spherical Varieties

Recall that an embedding of a homogeneous space G/H is a normal G-variety with an open G-orbit isomorphic to G/H. The embeddings of a given spherical homogeneous space G/H are classified by combinatorial objects called *colored fans*, which generalize the fans associated with toric varieties. This theory, due to Luna and Vust in characteristic zero [22], has been simplified and extended to all characteristics by Knop [18]. Here a basic role is played by the set $\mathcal{V}(G/H)$ of G-invariant valuations of the field $k(G/H)$, with rational values. It turns out that $\mathcal{V}(G/H)$ is identified with a convex polyhedral cone in the $\mathbb{Q}$-vector space $\mathcal{Q}(G/H) := \mathrm{Hom}(\mathfrak{X}^*(G/H), \mathbb{Q})$. In characteristic zero, this cone turns out to be a fundamental domain for some finite reflection group $W(G/H)$ acting on $\mathcal{Q}(G/H)$ [6].

An embedding X of spherical G/H is called *toroidal* if the closure in X of any B-stable divisor in G/H contains no G-orbit. Toroidal embeddings of G/H are classified by fans with support in $\mathcal{V}(G/H)$, i.e. partial subdivisions of $\mathcal{V}(G/H)$ into convex polyhedral cones that contain no line. Smooth, toroidal embeddings are regular, which means that they satisfy the following conditions [5]:

1. Each G-orbit closure is smooth, and is the transversal intersection of the smooth orbit closures that contain it.

2. The isotropy group of any point x acts on the normal space to the orbit $G \cdot x$ with an open orbit.

Conversely, if a homogeneous space G/H admits a *complete* regular embedding X, then G/H is spherical and X is toroidal. The compactifications of symmetric spaces constructed by DeConcini and Procesi are exactly their smooth, toroidal embeddings [9], [34].

The problem of classifying spherical spaces by combinatorial invariants is still open, and only completed in very special cases. (Horospherical varieties, symmetric varieties and wonderful varieties are some of the cases we already understand

well.) Nevertheless, Losev recently proved a uniqueness property for spherical homogeneous spaces.[21]

We have already seen combinatorial invariants of spherical homogeneous spaces: the lattice M of weights of $k(G/H)$, the valuation cone and the set $\mathcal{D}$ of colors of G/H together with a map σ from $\mathcal{D}$ to the dual N of M. We just have to add one natural family of invariants (the stabilizers in G of the colors) in order to have the uniqueness of spherical homogenous spaces. More precisely,

Theorem 1.2.2. *Let G/H_1 and G/H_2 be two spherical homogeneous spaces with the same weight lattice M, the same valuation cone in $N := Hom_{\mathbb{Z}}(M, \mathbb{Z})$ and set of colors $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively (together with maps σ_1 and σ_2 from $\mathcal{D}_1$ and $\mathcal{D}_2$ to N respectively) such that there exists a bijection $\iota : \mathcal{D}_1 \to \mathcal{D}_2$ satisfying, for all $D \in \mathcal{D}_1$, $\sigma_1(D) = \sigma_2(\iota(D))$ and $Stab_G D = Stab_G \iota(D)$. Then G/H_1 and G/H_2 are G-equivariantly isomorphic.*

1.3   Affine Embeddings of Homogeneous Space

Recall that an irreducible algebraic G-variety X is said to be an *embedding* of the homogeneous space G/H if X contains an open G-orbit isomorphic to G/H. We often use the notation $G/H \hookrightarrow X$. Let us say that an embedding $G/H \hookrightarrow X$ is *affine* if the variety X is affine. It is reasonable to study specific properties of affine embeddings in the framework of a well-developed general embedding theory for following two reasons. First, from toric vareities, affine embedding may contribute to understand general embeddings. Secondly, in many problems of invariant theory, representation theory, and other branches of mathematics, mostly affine embeddings of homogeneous space appear.

It is easy to show that a homogeneous space G/H admits an affine embedding if and only if G/H is quasi-affine as an algebraic variety. In this situation, the subgroup H is said to be *observable* in G. A closed subgroup H of G is observable if and only if there exist a rational finite dimensional G-module V and a vector

$v \in V$ such that the stabilizer G_v coincides with H. (This follows from the fact that any affine G-variety may be realized a a closed invariant subvariety in a finite dimensional G-module.) There is a nice group theoretic description of observable subgroups due to A. Sukhanov: a subgroup H is observable in G if and only if there exists a quasi-parabolic subgroup $Q \subset G$ such that $H \subset Q$ and the unipotent radical H^u is contained in the unipotent radical Q^u. (A subgroup Q is said to be *quasi-parabolic* if Q is the stabilizer of a highest weight vector in some G-module V.) It follows from Chevalley's theorem that any subgroup H without non-trivial characters (in particular, any unipotent subgroup) is observable. By Matsushima's criterion, a homogeneous space G/H is affine if and only if H is reductive. In particular, any reductive subgroup is observable. A description of affine homogeneous space G/H for non-reductive G is still an open problem.

One famous class of affine embeddings are affine toric varieties, which we've already seen in the previous section. The classification of affine toric varieties will serve us as a guide to the study of more complicated classes of affine embeddings. Generalizations of a combinatorial description of toric varieties were obtained for spherical varieties, and for embeddings of complexity one. In this more general context, the idea that normal G-varieties may be described by some convex cones becomes rigorous through the method of U-invariants developed by D. Luna and T. Vust. The essence of this method is contained in the following theorem.

Theorem 1.3.1. *Let $\mathfrak{A}$ be a G-algebra and U be a maximal unipotent subgroup of G. Consider the following properties of an algebra:*

1. *It is finitely generated.*

2. *It has no nilpotent elements.*

3. *It has no zero divisors.*

4. *It is integrally closed.*

If (P) is any of these properties, then the algebra $\mathfrak{A}$ has property (P) if and only if the algebra $\mathfrak{A}^U$ has property (P).

Another interesting aspect of affine embedding is the connection with Hilbert's 14th problem. Let H be a closed subgroup $GL(V)$. Hilbert's 14th problem (in its modern version) may be formulated as follows: characterize subgroups H such that the algebra of polynomial invariants $k[V]^H$ is finitely generated. It is a classical result that for H reductive the algebra $k[V]^H$ is finitely generated. For non-reductive linear groups this problem seems to be very far from a complete solution.

Let us assume that H is a subgroup of a bigger reductive group G acting on V. The intersection of a family of observable subgroups in G is an observable subgroup. Define the *observable hull* $\hat{H}$ of H as the minimal observable subgroup of G containing H. The stabilizer of any H-fixed vector in a rational G-module contains $\hat{H}$. Therefore $k[V]^H = k[V]^{\hat{H}}$ for any G-module V, and it is natural to solve Hilbert's 14th problem for observable subgroups.

The following famous theorem proved by F. Grosshans establishes a close connection between Hilbert's 14th problem and the theory of affine embeddings [12], [13].

Theorem 1.3.2. *Let H be an observable subgroup of a reductive group G. The following conditions are equivalent:*

1. *For any G-module V the algebra $k[V]^H$ is finitely generated.*

2. *The algebra $k[G/H]$ is finitely generated.*

3. *There exists an affine embedding $G/H \hookrightarrow X$ such that $\operatorname{codim}_X(X \setminus (G/H)) \geq 2$.*

Definition 1.3.3.   1. An observable subgroup H in G is said to be a *Grosshans subgroup* if $k[G/H]$ is finitely generated.

2. If H is a Grosshans subgroup of G, then $G/H \hookrightarrow X = \operatorname{Spec} k[G/H]$ is called the canonical embedding of G/H, and X is denoted by $CE(G/H)$.

Note that any normal affine embedding $G/H \hookrightarrow X$ with $\operatorname{codim}_X(X \setminus (G/H)) \geq 2$ is G-isomorphic to the canonical embedding. A homogeneous space G/H admits

21

such an embedding if and only if H is a Grosshans subgroup. By Matsushima's criterion, H is reductive if and only if $CE(G/H) = G/H$. For non-reductive subgroups, $CE(G/H)$ is an interesting object canonically associated with the pair (G, H). One can use $CE(G/H)$ to reformulate algebraic problems concerning the algebra $k[G/H]$ in geometric terms. For more about Hilbert's 14th problem and affine embeddings, you may see [3], [10], [31].

Chapter 2

One-Parameter Subgroup of Affine Embeddings

Embeddings of homogeneous spaces are studied well for spherical varieties, but it is not well-known for other cases, especially when the complexity is bigger than 1. Therefore, we want to study embedding of homogeneous space apart from the complexity. It is reasonable to think simple cases first, therefore we are now trying to understand when X is affine normal embeddings of reductive group G, instead of homogeneous space G/H.

Based on knowledge about toric geometry, one might guess that one-parameter subgroup play an important role to determine the $G-$variety. So we are interested in describing and classifying affine G-embeddings X by using one-parameter subgroup of G. From the theory of toric varieties, it is natural to consider the one-parameter subgroups of G whose limit exists in X. One-parameter subgroups and their limits have been studied in several applications, including the Hilbert-Mumford criterion of stability [23], the construction of the spherical building of the group G and the Bialynicki-Birula decomposition of a smooth projective T-variety [4]. For our purpose, we will check that an affine G-embedding X is determined by the set of one-parameter subgroups γ of G such that $\lim_{t \to 0} \gamma(t)x_0$ exists in X. In this chapter, we indicate our base algebraic closed field with characteristic 0 as k

for the simplicity.

2.1 Equivalence Relation on $\mathfrak{X}_*(G)$

A *one-parameter subgroup* of G is a homomorphism of algebraic groups $\gamma : \mathbb{G}_m \to G$, which hence corresponds to a map $\gamma^\circ : k[G] \to k[t, t^{-1}]$. Let $\mathfrak{X}_*(G)$ denote the set of one-parameter subgroups of G. We will denote the trivial one-parameter subgroup $t \mapsto e$ by ε. The group G acts on $\mathfrak{X}_*(G)$ by conjugation, $g \cdot \gamma : t \mapsto g\gamma(t)g^{-1}$. Each one-parameter subgroup $\gamma \in \mathfrak{X}_*(G)$ determines a subgroup

$$P(\gamma) = \left\{ g \in G : \gamma(t)g\gamma(t^{-1}) \in G_{k[[t]]} \right\}$$

of G, which is parabolic if G is reductive. In fact, every parabolic subgroup of G is of the form $P(\gamma)$ for some one-parameter subgroup γ of G. We define an equivalence relation on the set of non-trivial one-parameter subgroups of G by

$$\gamma_1 \sim \gamma_2 \text{ if and only if } \gamma_2(t^{n_2}) = g\gamma_1(t^{n_1})g^{-1}$$

for positive integers n_1, n_2 and an element $g \in P(\gamma_1)$, for all $t \in k^*$. Then the quotient $(\mathfrak{X}_*(G) - \{\varepsilon\})/\sim$ is isomorphic to the spherical building of G [23].

Every parabolic subgroup P of G defines a subset

$$\Delta_P(G) = \{\gamma \in \mathfrak{X}_*(G) : P(\gamma) \supseteq P\}$$

of $\mathfrak{X}_*(G)$. Clearly $\gamma \in \Delta_{P(\gamma)}(G)$ for all $\gamma \in \mathfrak{X}_*(G)$, so $\mathfrak{X}_*(G) = \cup \Delta_P(G)$ where the union is over all parabolic subgroups of G. In the spherical building of G, the images of the sets $\Delta_P(G)$ are simplices and constitute a triangulation of the building [23].

The inclusion of $k[t, t^{-1}]$ in $k((t))$ allows us to view $\mathfrak{X}_*(G)$ as a subset of $G_{k((t))} = Hom_k(k[G], k((t)))$, the set of $k((t))$-points of G. Let $\langle \gamma \rangle \in G_{k((t))}$ denote the

point corresponding to the one-parameter subgroup γ. The group $G_{k((t))}$ contains the subgroup $G_{k[[t]]}$, which consists of all $k((t))$-points of G that have a specialization in G as $t \to 0$. The group $G_{k((t))}$ is the disjoint union of the double cosets of $G_{k[[t]]}$, as described by the Iwahori decomposition:

Theorem 2.1.1 (Cartan-Iwahori Decomposition). *Let G be a reductive algebraic group over k. Every double coset of $G_{k((t))}$ with respect to the subgroup $G_{k[[t]]}$ is represented by a point of type $\langle \gamma \rangle$, for some one-parameter subgroup γ of G. That is,*

$$G_{k((t))} = \bigcup_{\gamma \in \mathcal{X}_*(G)} G_{k[[t]]} \langle \gamma \rangle G_{k[[t]]}.$$

Furthermore, each double coset is represented by a unique one-parameter subgroup.

Using this decomposition, we can replace $k((t))$-points of G with one-parameter subgroups. Now let's say that X is a $G-$variety. For each point x of X, there is a dominant morphism $\psi_x : G \to X$ which is defined by $\psi_x(g) = g \cdot x$. For a point $x_0 \in X$ and a one-parameter subgroup γ of G, we say $\lim_{t \to 0} \gamma(t)x_0$ exists in X if $\psi_{x_0} \circ \gamma : \mathbb{G}_m \to X$ extends to a morphism $\widetilde{\gamma} : \mathbb{A}^1 \to X$. In this case, $\lim_{t \to 0} \gamma(t)x_0$ is defined to be $\widetilde{\gamma(0)}$. That is, the composition of $\psi_{x_0}^\circ : k[X] \to k[G]$ with $\gamma^\circ : k[G] \to k[t, t^{-1}]$ factors through $k[t]$, and the limit, $\lim_{t \to 0} \gamma(t)x_0$, is the $k-$point of X corresponding to the composite map $k[X] \to k[t] \to k$ sending $t \to 0$. This is described by the diagrams:

$$
\begin{array}{ccc}
\mathbb{G}_m & \overset{\subset}{\longrightarrow} & \mathbb{A}^1 \\
{\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle \widetilde{\gamma(0)}} \\
G & \underset{\psi_{x_0}}{\longrightarrow} & X
\end{array}
\qquad
\begin{array}{ccc}
k[X] & \overset{\psi_{x_0}^\circ}{\longrightarrow} & k[G] \\
{\scriptstyle \widetilde{\gamma(0)}^\circ} \downarrow & & \downarrow {\scriptstyle \gamma^\circ} \\
k[t] & \underset{\subset}{\longrightarrow} & k[t, t^{-1}]
\end{array}
$$

Similarly, if λ is a $k((t))$-point of G, then $\lim_{t \to 0} \lambda(t)x_0$ exists in X means $\lambda^\circ |_{k[X]} : k[X] \to k[[t]]$.

The following lemma is straightforward.

Lemma 2.1.2. *Suppose* $\lambda \in G_{k((t))}$ *and* $\alpha \in G_{k[[t]]}$, *so that* α *has specialization* $\alpha_0 \in G_k$. *Let* X *be an affine* $G-$*embedding with base point* x_0. *Then* $\lim_{t \to 0} \gamma(t)x_0$ *exists in* X *if and only if* $\lim_{t \to 0}[\alpha(t)\gamma(t)x_0]$ *exists, in which case*

$$\lim_{t \to 0}[\alpha(t)\gamma(t)x_0] = \alpha_0 \lim_{t \to 0} \gamma(t)x_0.$$

The limit of one-parameter subgroup is closely related with G-embeddings. The following theorem is one of the famous relation.

Theorem 2.1.3. *[17] Let* X *be an affine* $G-$*variety. Suppose that* Y *is a closed* $G-$*stable subvariety of* X *and that* $x_0 \in X$ *is a closed point such that the closure of the orbit* Gx_0 *intersects* Y. *Then there is a one-parameter subgroup* γ *of* G *such that* $\lim_{t \to 0} \gamma(t)x_0 \in Y$.

2.2 One-parameter Subgroup of Affine G-Embeddings

Given a $G-$variety X and a point $x_0 \in X$, define

$$\Gamma(X, x_0) := \left\{ \gamma \in \mathfrak{X}_*(G) : \lim_{t \to 0} \gamma(t)x_0 \text{ exists in } X \right\}.$$

We are interested in the structure of such sets of one-parameter subgroups when X is an affine G-variety and the orbit of x_0 in X is open and isomorphic to G. We call such an $x_0 \in X$ a base point. In the theory of toric varieties, this gives the cone in the lattice $\mathfrak{X}_*(T)$ which gives the one-to-one correspondence to the affine toric varieties. Therefore these sets can be used to solve the classification problem of toric varieties. For any reductive group G, we may ask same question:

Question : Can we classify affine normal G-embedding by using $\Gamma(X, x_0)$ for a reductive group G?

there is a hope that this can contribute to the classification problem of affine G-embeddings for a general reductive algebraic group G. Before we actually check

this, we state some properties which can be observed easily.

Proposition 5. *[24] Let G be a connected reductive group. Suppose X is an affine $G-$embedding and $x_0 \in X$ a base point.*

1. *If $x'_0 = hx_0$, then $\Gamma(X, x'_0) = h\Gamma(X, x_0)h^{-1}$.*

2. *If $\gamma \in \Gamma(X, x_0)$ and $\gamma \neq \varepsilon$, then $\gamma^{-1} \notin \Gamma(X, x_0)$.*

3. *If T is any torus of G, then $\overline{Tx_0} \simeq \overline{T_\sigma}$, where $\sigma = \Gamma(X, x_0) \cap \mathfrak{X}_*(T)$ is a strongly convex rational polyhedral cone in $\mathfrak{X}_*(T)$.*

4. *If $\gamma \in \Gamma(X, x_0)$ and $p \in P(\gamma)$, then $p \bullet \gamma \in \Gamma(X, x_0)$.*

5. *The image of $\Gamma(X, x_0)$ in the spherical building is convex ([23], Definition 2.10).*

From this proposition, we sometimes call $\Gamma(X, x_0)$ a cone structure of X. If this set of one-parameter subgroups can be actually used to solve the classification problem, we should be able to recover the variety X from $\Gamma(X, x'_0)$. Therefore we might need an appropriate valuation on $k(G)^*$ related with each one-parameter subgroup. Now recall that each $\gamma \in \mathfrak{X}_*(G)$ may be viewed as a $k((t))$-point of G. We can find a G-stable valuation v_λ which is associated to every $\lambda \in G_{k((t))}$ in the following way. [22] As λ is a $k((t))-$point of G, we obtain a dominant morphism

$$G \times \operatorname{Spec} k((t)) \xrightarrow{1 \times \lambda} G \times G \xrightarrow{\mu} G.$$

This morphism induces an injection of fields

$$i_\lambda : k(G) \to \operatorname{Frac}(k(G) \otimes_k k((t))) \to k(G)(t).$$

Then $v_t \circ i_\lambda : k(G)^* \to \mathbb{Z}$ is a valuation of $k(G)$, where $v_t : k(G)((t))^* \to \mathbb{Z}$ is the standard valuation associated to the order of t. We define $v_\lambda = \frac{1}{n_\lambda}(v_t \circ i_\lambda)$, where $n_\lambda \in \mathbb{Z}$ is the largest positive number such that $(v_t \circ i_\lambda)(k(G)^*) \subset n_\lambda \mathbb{Z}$. This is

$G-$stable by left translations, i.e., $v_\lambda(s \cdot f) = v_\lambda(f)$ for all $s \in G$, since i_λ is clearly equivariant and $k(G)[[t]]$ is obviously stable for left translations by G in $k(G)((t))$. Here are some properties of these valuations that are proven in [22].

Theorem 2.2.1. *1. Let γ be a one-parameter subgroup of G. For each $f \in k(G)$, there is an open subset $U \subset G$, depending only on f, such that*

$$v_\gamma(f) = \inf_{s \in U} v_t(f(s \cdot \gamma(t))).$$

2. Let γ_1, γ_2 be one-parameter subgroups of G. Then $v_{\gamma_1} = v_{\gamma_2}$ if and only if $\gamma_1 \sim \gamma_2$.

So we have good valuations corresponding to one-parameter subgroup which are stable under the equivalence relation. To answer the question in the beginning of this section, we will now observe specific varieties when $G = SL(2)$.

Chapter 3

Affine Normal $SL(2)$-Embedding

As we have mentioned in chapter 1, when U is the maximal unipotent subgroup of G, the properties of U-invariant ring of $\mathcal{O}(X)$ contains important information of G-variety X. In this chapter we first briefly review the general theory of U-invariant ring of quasihomogeneous varieties, and explain the classification and construction of affine normal $SL(2)$-embedding with more details. By using this, we can show that the set $\Gamma(X, x_0)$ does not contain enough information which enable us to classify all affine normal G-embeddings, unlike the case of toric varieties. In this chapter, we use some results from algebraic group theory, which can be found in many places such as [15].

3.1   U-invariant Ring of Quasihomogeneous Variety

Assume that G is a reductive linear algebraic group. Since $\mathcal{O}(G)^U$ is a finitely generated $\mathbb{C}$-algebra of $\mathcal{O}(G)$, there is an affine G-variety $G(U)$ with point $\bar{e} \in G(U)$ such that the mapping $\phi : G \to G(U)$, defined by $g \mapsto g\bar{e}$, is dominant, and $\phi^*(\mathcal{O}(G(U))) = \mathcal{O}(G)^U$. The stabilizer of $\bar{e} \in G(U)$ is U, and $G(U) \setminus G\bar{e}$ has codimension at least two.

Theorem 3.1.1. *[12], [14] For each G-variety Z, $\mathcal{O}(Z)^U$ is a finitely generated $\mathbb{C}$-algebra.*

We remark several facts here [20].

1. $(\mathcal{O}(G)^U \otimes \mathcal{O}(Z))^G \simeq \mathcal{O}(Z)^U$.

2. The ring $\mathcal{O}(Z)^U$ defines an affine variety $Z(U) := (G(U) \times Z)/G$. There is a canonical dominant morphism

$$\rho : Z \to Z(U)$$

which satisfies $\rho^*(\mathcal{O}(Z(U))) = \mathcal{O}(Z)^U$. ($\rho$ is a composition of $\psi : Z \to G(U) \times Z$ and the natural projection $\phi : G(U) \times Z \to (G(U) \times Z)/G$.

3. If Z is a G-variety and the map $\eta : Z \to Y$ is U-stable (i.e. $\eta(u \cdot z) = \eta(z)$ for each $u \in U$ and $z \in Z$), then η factors through the map ρ and $\bar{\eta}$ as follows:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \rho\ } & Z(U) \\
& \eta \searrow & \downarrow \\
& & Y
\end{array}
$$

If Y is a G-variety and ϕ is a G-equivariant map, then we have a commutative diagram like this:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \phi\ } & Y \\
\rho_Z \downarrow & & \downarrow \rho_Y \\
Z(U) & \xrightarrow[\phi(U)]{} & Y(U)
\end{array}
$$

The map $\phi(U)$ is defined as $(\mathrm{Id} \times \phi(U))/G$.

Now let V be a simple G-module. Suppose that $v \in V$ is a non-zero element and define the map

$$\mu : G \to V$$

by $g \mapsto gv$. Then the corresponding map $\mu^* : V^* \to \mathcal{O}(G)$ is G-equivariant, so it is uniquely determined by the restriction to the U-invariant subalgebra. The vector v define a unique one dimensional vector space in $\mathcal{O}(G)^U$ namely $\mathbb{C}v$. We may take the image of $\mathbb{C}v \subset V$ under G-equivariant embedding $V \to \mathcal{O}(G)^U$. Then we have the following lemma:

Lemma 3.1.2. $\mu^*((V^*)^U) = \mathbb{C}v \subset \mathcal{O}(G)^U$.

Now we consider U-invariant function $f \in \mathcal{O}(G)^U$ and G-module (with the right operation) W generated by f:

$$W := \langle {}^g f | g \in G \rangle \subset \mathcal{O}(G)^U.$$

If O_f is the orbit of f in W, $\overline{O_f}$ is closed and $\mu : G \to W$ is the orbit map, so we can consider a coordinate ring $\mathcal{O}(\overline{O_f})$ as a subring of $\mathcal{O}(G)$ by μ^*.

Theorem 3.1.3. *[20] We have $\mathcal{O}(\overline{O_f}) = \mathbb{C}[g \cdot f | g \in G] \subset \mathcal{O}(G)$. In particular, $\mathcal{O}(\overline{O_f})^U$ contains all components of f in the simple module of $\mathcal{O}(G)^U$.*

Proof. The map $\mu^* : W^* \to \mathcal{O}(G)$ is given by $\lambda \mapsto f_\lambda$ where $f_\lambda(g) = \lambda({}^g f)$. For $h \in G$, call $\lambda_h \in W^*$ *evaluation at the point* h which is defined by $\lambda_h(p) = p(h)$ for $p \in W \subset \mathcal{O}(G)$. Then

$$f_{\lambda_h}(g) = \lambda_h({}^g f) = ({}^g f)(h) = f(hg) = (h^{-1} f)(g),$$

so $\mu^*(\lambda_h) = h^{-1} f$. Since W^* is generated by λ_h, $h \in G$, we have $\mu^*(W^*) = \langle g \cdot f \| g \in G \rangle$, so the theorem is proved. $\square$

The following corollaries follows easily:

Corollary 3. *Suppose that W is a G-module and $W = \bigoplus_{i=1}^t W_i$ is the decomposition into simple modules. For $w \in W$, we can express $w = \sum_{i=1}^t w_i$ with $w_i \in W_i$.*

31

Each component $w_i \in W_i$ defines one-dimensional subspace $\mathbb{C}w_i \subset \mathcal{O}(G)^U$ and set $W' := \sum_{i=1}^{t} \mathbb{C}w_i \subset \mathcal{O}(G)^U$. Then we have

$$\mathcal{O}(\overline{O_w}) = \mathbb{C}[G \cdot W'] \subset \mathcal{O}(G).$$

Corollary 4. *Suppose that $A \subset \mathcal{O}(G)^U$ is a finitely generated subalgebra such that the finitely generated G-module $B := \langle G \cdot A \rangle$ is a subalgebra of $\mathcal{O}(G)$. If $f_1, \cdots f_t \in A$ are linearly independent generators of A, we can set $W := \langle {}^g f_i | g \in G, i = 1, \cdots, t \rangle \subset \mathcal{O}(G)^U$ and $f := \sum_{i=1}^{t} f_i \in W$. Then*

$$\mathcal{O}(\overline{O_f})^U = A \subset \mathcal{O}(\overline{O_f}) = B.$$

Say $f \in \mathcal{O}(G)^U$ is a highest weight vector with weight $w \in \Omega_G$ in terms of the right operation, and let $V := \langle {}^g f | g \in G \rangle$ be a simple module. If O_f is the orbit of f in V, then $\mathcal{O}(\overline{O_f})^U = \mathbb{C}[f] \subset \mathcal{O}(G)^U$. In particular, $\overline{O_f}$ is normal by theorem 1.3.1.

3.2 First Properties and $U-$invariant ring of $\mathcal{O}(X)$

The classification of affine embeddings has been completed only in some cases. For $G = SL(2)$, the simplest algebraic reductive group, the results are well known and there is a specific construction for each case. This construction allows us to answer the question suggested at the end of the previous chapter. Before checking this, we need to understand affine normal $SL(2)-$embeddings more rigorously.

We recall that an irreducible algebraic variety X is called a quasihomogeneous variety of the algebraic group G if it has an open orbit isomorphic to G/H, for some closed subgroup. We are interested in the case when $G = SL(2), H = \{e\}$. Most of this materials can be found in [20] and [28]. When X is an $SL(2)-$embedding, we will call the open dense orbit O_X, and the boundary ∂X is defined as $X \setminus O_X$.

Lemma 3.2.1. *X does not have any $1-$dimensional orbit, and it contains at most*

one fixed point.

Proof. Suppose that $SL(2) \cdot z$ is an orbit in X of dimension equal or less than 1. Then the stabilizer $SL(2)_z$ has dimension greater or equal to 2, and it contains a Borel subgroup, for a connected 2-dimensional subgroup of $SL(2)$ is a Borel subgroup. This implies that z is a fixed point. (For any linear reductive group G, if Z is a G-variety and $z \in Z$ is a point whose stabilizer G_z contains a Borel subgroup, then z is a fixedpoint.) The second assertion directly follows from the following; If Z is a G-variety for a linear reductive group G, and $Gz \subset Z$ is an orbit, then $\overline{Gz}$ contains exactly one closed orbit. $\qquad\qquad\square$

For $SL(2)$, we have an explicit list of all 1-dimensional subgroups.

Lemma 3.2.2. *Every* $1-dimensional$ *subgroup of* $SL(2)$ *is conjugate to one of the following groups :*

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| t \in \mathbb{C}^* \right\}, \quad N = N_{SL(2)}(T) = T \cup \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} T$$

$$U_n = \left\{ \begin{pmatrix} \epsilon & b \\ 0 & \epsilon^{-1} \end{pmatrix} \middle| \epsilon, b \in \mathbb{C}, \epsilon^n = 1 \right\}.$$

Lemma 3.2.3. *A* $2-dimensional$ *orbit in* X *is closed if and only if the stabilizer of the orbit is conjugate to* T *or* N.

Proof. One direction is clear. If the stabilizer contains maximal torus, its orbit is closed. The converse follows from Hilbert's criterion [20]: The closed orbit in X contains the image of a one parameter subgroup, which is a one-dimensional torus. $\qquad\qquad\square$

We call G-embedding X a trivial embedding if X is isomorphic to G.

For any affine G-embedding X, we have the following proposition.

Proposition 6 ([27]). *If* X *is not a trivial embedding, then the boundary* ∂X *is a subvariety of pure codimension 1 in* X.

Therefore when $G = SL(2)$, we have

Lemma 3.2.4. *If X is not a trivial embedding, $dim(X \setminus O_X) = 2$.*

From this result, we have two possibilities for an affine $SL(2)$-embedding X beside the trivial embedding.

Theorem 3.2.5. *If X is not a trivial embedding, X is one of the following types.*

Type 1 $X = O_X \cup O_0$, where $O_0 \simeq SL(2)/T$ or $SL(2)/N$.

Type 2 $X = O_X \cup \cup_{i=1} O_i \cup p$, where p is the fixed point and O_i is a $2-$dimensional orbit isomorphic to U_m with some m.

(We will see later that if X is of type 2, the number of 2-dimensional orbits is always equal to 1.)

From this observation when the variety X is normal, we can decide the singularities of X:

Theorem 3.2.6. *Suppose that X is a normal $SL(2)-embedding$.*

1. *If X is a Type 1 variety, then X is smooth.*

2. *If X is a Type 2 variety, then X is singular only possibly at its fixed point.*

As a matter of fact, the fixed point p is a singular point.

Lemma 3.2.7. *If X is a type 2 variety and p is a fixed point, then $O_{E,p}$ is not a factorial ring, and p is a singular point.*

Proof. We can choose a point e in the dense orbit such that $\lim_{t \to 0} \lambda(t)e = p$ where

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SL(2). \quad \text{Consider the two hyperplanes } D := \overline{Be} \text{ and}$$

$$D_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D, \text{ where } B \text{ is the group of upper triangular matrices in } SL(2).$$

Then obviously $p \in D \cap D_0$. If there is another point $q \in D \cap D_0$, then $\lim_{t \to 0} \lambda(t)q = p$

and $\lim\limits_{t\to 0}\lambda^{-1}(t)q = p$ because $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\lambda = \lambda^{-1}$. Therefore, the stabilizer of q

contains the image of $\lambda(t)$, whence $q = p$. This shows that $D \cap D_0 = \{p\}$.

Now assume that D and D_0 are defined by prime ideals $\mathfrak{p}$ and $\mathfrak{p}_0$ in $O_{E,p}$, respectively. If $O_{E,p}$ is factorial, each prime ideal is a principal ideal so we can find f and f_0 in $O_{E,p}$ such that $\mathfrak{p} = (f)$ and $\mathfrak{p}_0 = (f_0)$. Then $D \cap D_0$ is defined by two equation $f = f_0 = 0$, hence $\mathrm{codim}_X D \cap D_0 \le 2$, which is a contradiction. Therefore, $O_{E,p}$ is not factorial, and p is a singular point. $\qquad\square$

Now let's fix one non-trivial $SL(2)$-embedding X with the general point x_0 in the dense orbit O_X, such that $\lim\limits_{t\to 0}\lambda(t)x_0$ exist when $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SL(2)$.

(It is not hard to see that we can always find such x_0 for non-trivial embedding.) Observe that

$$\mathcal{O}(SL(2)) = \mathbb{C}[x, y, z, w]/(xw - yz - 1),$$

where the maps $x, y, z, w : SL(2) \to \mathbb{C}$ are defined by

$$x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a, \qquad y\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = b,$$

$$z\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = c, \qquad w\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = d.$$

Theorem 3.2.8 ([20]). *If X is a normal variety, then the map $x \in \mathcal{O}(SL(2))$ can be extended to the whole X, and its value at the boundary ∂X is zero.*

Proof. Consider the $SL(2)$-variety $X \times \mathbb{C}^2$ with the natural representation of $SL(2)$ on $\mathbb{C}^2$. Say $x' = (x_0, (1, 0))$ and $X' =: \overline{O_{x'}} \subset X \times \mathbb{C}^2$ which is the closure of the orbit of x'.

35

$$X \xleftarrow{\ pr\ } X \times \mathbb{C}^2 \xrightarrow{\ pr\ } \mathbb{C}^2$$

with maps ϕ and ψ down to X'.

X' is an $SL(2)$-embedding with dense orbit $O' = O_{x'}$, and we have

$$\mathcal{O}(X) \subset \mathcal{O}(X') \subset \mathcal{O}(SL(2)),$$

where the two inclusions are induced by ϕ and the map $SL(2) \to X'$ is given by $g \mapsto gx'$.

To prove the theorem, now it is sufficient to show:

1. $x \in \mathcal{O}(X')$ extends to the zero function on the boundary of X'.

2. ϕ is an isomorphism.

An extension of x, $\widetilde{x} \in \mathcal{O}(X')$, is given by $\widetilde{x}(z,(x,y)) := x$. So to show $\widetilde{x}$ extends to the zero function on the boundary, it is enough to show that $\psi^{-1}(0,0)$ contains all 2-dimensional orbits of X'. The map $\psi : X' \to \mathbb{C}^2$ is equivariant under the $SL(2)$ action, and the fibers over $\mathbb{C}^2 \setminus \{(0,0)\}$ are all isomorphic, so each component of fiber $F := \psi^{-1}((1,0))$ is one-dimensional. The stabilizer of $(1,0)$ is

$$U_1 = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \middle| c \in \mathbb{C} \right\} \subset SL(2), \text{ so } F \text{ is } U_1\text{-stable. Then } F \cap O' = U_1 \cdot x' \text{ is}$$

closed in X' (When U is a unipotent group and Z is $U-$invariant, each orbit in Z is closed), and an component of F. Suppose that there is another component C of F. Other component C of F is also U_1 stable, so it doesn't intersect $U_1 x'$. Since $\psi(SL(2) \cdot C) = \mathbb{C}^2 \setminus \{0\}$, we have $\dim(SL(2) \cdot C) = 3$, which contradicts to $SL(2) \cdot C \subset X' \setminus O'$. It follows that $\psi^{-1}(\mathbb{C}^2 \setminus \{0\}) = O'$, and it proves the first assertion.

For the second claim, we note first that the dense orbits of X and X' are isomorphic, so especially ϕ is a birational morphism. By Richardson Lemma ([20],

II. 3.4), it is enough to show that ϕ is surjective. If X is of type 1, then $f :=$ $\lim_{t \to 0} \lambda(t)x$ is a point of 2-dimensional orbit, and f maps to $f' := \lim_{t \to 0} \lambda(t)x' =$ $(f, 0) \in X'$ under ϕ, so $O_f = \phi(O_{f'})$. If X is of type 2, it is enough to show that the image of ϕ contains $\overline{Bx}$ because ϕ is $SL(2)$-equivariant and $\overline{Bx}$ intersects with each orbit in X. If $y \in \overline{Bx}$, then we can easily find a sequence of matrices such that $g_n = \begin{pmatrix} a_n & b_n \\ 0 & a_n^{-1} \end{pmatrix} \in SL(2)$ with $\lim_{n \to \infty} g_n x = y$ and $\lim_{n \to \infty} a_n = a \in \mathbb{C}$. From this, we have $\lim_{n \to \infty} g_n x' = (y, (a, 0)) \in X'$, hence $y \in \phi(X')$.

$\square$

Define a subgroup U, which is an unipotent subgroup of $SL(2)$:

$$U := U_1^- = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \middle| c \in \mathbb{C} \right\} \subset SL(2).$$

Then we have $\mathcal{O}(SL(2))^U = \mathbb{C}[x, y]$ and $\mathcal{O}(X)^U \subset \mathcal{O}(SL(2))^U = \mathbb{C}[x, y]$, and we already observed $x \in \mathcal{O}(X)^U$ when X is normal.

Lemma 3.2.9. *If $A \subset \mathbb{C}[x, y]$ is a normal homogeneous subalgebra with $x \in A$ and $quot(A) = \mathbb{C}(x, y)$, then A is generated by monomials.*

Proof. We first show that for all s, A contains a monomial $x^r y^s$ for some r. Suppose that $P = x^a y^b + \sum_{i>0} a_i x^{a+i} y^{b-i}$ is a homogeneous element of A with $a_i \in \mathbb{C}, b > 0$. If $k \in \mathbb{N}$ satisfies $kb \geq a$, then we have

$$x^{bk-a} P = (x^k y)^b + \sum_{i>0} a_i x^{(k+1)i} (x^k y)^{b-i} \in A,$$

so $x^k y \in A$ because A is normal. We will now show that $x^a y^b \in A$, and then the claim follows by induction. If $x^a y^b \notin A$ then say that n is a maximal with $x^{n+a} y^b \notin A$. Then for all $0 \leq i \leq b$,

$$(x^{n+a+i} y^{b-i})^b = x^d (x^{n+a+1} y^b)^{b-i},$$

where $d = b(n + a + i) - (b - i)(n + a + 1) = b(i - 1) + i(n + a + 1)$. Therefore, $x^{n+a+i}y^{b-i} \in A$ for $i > 0$. Because $x^n P = x^{n+a}y^b + \sum_{i>0} a_i x^{n+a+i}y^{b-i} \in A$, we have a contradiction. $\square$

Theorem 3.2.10. *If X is a normal $SL(2)-$embedding, there is a positive rational number h such that*

$$\mathcal{O}(X)^U = A_h := \left\{ kx^i y^j \ \middle| \ k \in \mathbb{C}, \frac{j}{i} \le h \right\}.$$

We call this h height of X.

Proof. Suppose that $x^{i_1}y^{j_1}, \cdots, x^{i_s}y^{j_s}$ are monomials which generate $\mathcal{O}(X)^U$. Take $h = \max_s \frac{j_s}{i_s}$. If $x^a y^b \in \mathcal{O}(E)^U$, then $x^i y^j \in \mathcal{O}(E)^U$ for $\frac{j}{i} \le \frac{b}{a}$ because $(x^i y^j)^b = x^{bi-aj}(x^a y^b)^j$, so the assertion is proved. $\square$

Also, we can show that

Theorem 3.2.11. *The height h is invariant under isomorphism.*

3.3 Classification of Affine Normal $SL(2)$-Embedding.

For a G-variety X, we can define an action of G on $\mathcal{O}(X)$ as following :

$$(g \cdot f)(x) := f(g^{-1}x)$$

Now if A is any set in $\mathbb{C}$-algebra with G-action, define $\langle G \cdot A \rangle$ as the $\mathbb{C}$ module generated by $g \cdot a$ for all $g \in G$ and $a \in A$. It is not hard to see that $\mathcal{O}(X) = \langle SL(2) \cdot \mathcal{O}(X)^U \rangle$. Conversely, if a finitely generated subalgebra $A \subset \mathbb{C}[x, y]$ satisfies that the submodule $R := \langle SL(2) \cdot A \rangle$ is a subalgebra (of $\mathcal{O}(SL(2))$), then $R^U = A$ and R is finitely generated, and we can find $SL(2)-$embedding Y such that $\mathcal{O}(Y) = R$. It is therefore important to decide for which h, submodule $\langle SL(2) \cdot A \rangle$ is subalgebra of $\mathcal{O}(SL(2))$.

Lemma 3.3.1. *In $\mathcal{O}(SL(2))$, we have*

$$(\langle SL(2) \cdot x^a y^b \rangle \cdot \langle SL(2) \cdot x^r y^s \rangle)^U \subset \oplus_{i \geq 0} \mathbb{C} x^{a+r-i} y^{b+s-i}.$$

Theorem 3.3.2. *Suppose that X is non trivial $SL(2)-$embedding. Then $h(X) \leq 1$. Also for every $h \leq 1$, there is a normal $SL(2)-$embedding X with $h(X) = h$.*

Proof. 1) Suppose that $f := x^i y^j \in \mathcal{O}(X)^U$. We have to show that $i \geq j$. Consider the right action of f on the simple module $V := \mathbb{C}[x,y]_{i+j}$. The map

$$\mu : SL(2) \longrightarrow V, \quad g \mapsto {}^g f$$

induces a map $\mu^* : \mathcal{O}(V) \to \mathcal{O}(SL(2))$, and together with 3.1.3, we have the following : The image of μ is the orbit O_f of f in V and

$$\mathcal{O}(\overline{O_f}) = \mathbb{C}[g \cdot f | g \in SL(2)] \subset \mathcal{O}(SL(2)).$$

This map μ factors via X as

$$
\begin{array}{ccc}
SL(2) & \longrightarrow & X \\
& \mu \searrow & \downarrow \bar{\mu} \\
& & V
\end{array}
$$

with $\bar{\mu}(e) = f$. Since the limit $\lim\limits_{t \to 0} \lambda(t)e$ exists, $\lim\limits_{t \to 0} \lambda(t)f$ exists. The action of $\lambda(t)$ on f gives us $\lambda(t)f = t^{i-j} x^i y^j$, so we have $i \geq j$.

2) It is enough to show that for given h, $\langle SL(2) \cdot A_h \rangle$ is a subalgebra of $\mathcal{O}(SL(2))$. Suppose $x^a y^b, x^r y^s$ are two monomials in A_h, then $\frac{b}{a}, \frac{s}{r} \leq 1$, so we have $\frac{b+s-i}{a+r-i} \leq h$ for all $i \geq 0$. By the previous lemma, we have $(\langle SL(2) \cdot x^a y^b \rangle \cdot \langle SL(2) \cdot x^r y^s \rangle)^U \subset A_h$, so $(\langle SL(2) \cdot x^a y^b \rangle \cdot \langle SL(2) \cdot x^r y^s \rangle)^U \subset \langle SL(2) \cdot A_h \rangle$. Since A_h is generated by monomials, this proves the first statement. The normal embedding X with

coordinate ring $\mathcal{O}(X) = \langle SL(2) \cdot A_h \rangle$ has A_h as a U-invariant ring, therefore its height is h. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now we can find a structure of $SL(2)$-embedding like the following:

Theorem 3.3.3 ([20]). *Suppose X is a nontrivial affine normal $SL(2)$-embedding with an open orbit $O_X \subset X$ isomorphic to $SL(2)$.*

1. *$X \setminus O_X$ is irreducible and normal. Especially X contains exactly one two dimensional orbit.*

2. *$h = 1$ corresponds to a (unique) smooth $SL(2)$-embedding with two orbits : $X = SL(2) \cup SL(2)/T$.*

3. *If $h = \frac{q}{p}$ and $(p, q) = 1$, then $X = SL(2) \cup SL(2)/U_{p+q} \cup \{pt\}$, and $\{pt\}$ is an isolated singular point in X.*

Proof. Let $h = \frac{q}{p}$ (with $(p, q) = 1$), and $\mathfrak{a} \subset \mathcal{O}(X)$ be the ideal of the closed set $X \setminus O_X$. The exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{O}(X \setminus O_X) \longrightarrow 0$$

is $SL(2)$-equivariant and hence $\mathcal{O}(X \setminus O_X)^U = \mathcal{O}(X)^U/\mathfrak{a}^U = A_h/\mathfrak{a}^U$, where $A_h = \{\mathbb{C}x^iy^j \mid \frac{i}{i} \leq h\}$. By Theorem 3.2.8, we have $x \in \mathfrak{a}$, and $\sqrt{(x \cdot A_h)} \subset \mathfrak{a}^U$. Therefore,

$$\sqrt{(x \cdot A_h)} = \left\{ \mathbb{C}x^iy^j \ \middle| \ \frac{j}{i} < h \right\},$$

and $\overline{A_h} := A_h/(\sqrt{(x \cdot A_h)})$ is a polynomial ring with one variable $t := x^qy^p + \sqrt{(x \cdot A_h)}$. If $\mathfrak{a}^U \neq \sqrt{(x \cdot A_h)}$, then $\mathcal{O}(X \setminus O_X)^U = A_h/\mathfrak{a}^U$ is a quotient ring of $\overline{A_h}$, which is a $\mathbb{C}$-algebra. Then this contradicts the fact that $\dim(X \setminus O_X) = 2$, hence

$$\mathcal{O}(X \setminus O_X)^U = A_h/(\sqrt{(x \cdot A_h)}) = \mathbb{C}[t]$$

40

with $t := x^q y^p + \sqrt{(x \cdot A_h)}$. Especially, $X \setminus O_X$ is irreducible and normal by Theorem 1.3.1.

Now, suppose that $h = 1$. Then $xy \in \mathcal{O}(X)^U = A_1$ and the inclusion $\mathbb{C}[xy] \subset \mathbb{C}[x,y]$ induces an isomorphism $\mathbb{C}[xy] \to \overline{A_1}$. The subalgebra $\mathbb{C}[xy]$ is $T-$invariant under the right operation. So if we define

$$B := \langle SL(2) \cdot \mathbb{C}[x,y] \rangle = \mathcal{O}(SL(2))^T = \mathcal{O}(SL(2)/T),$$

then the exact sequence

$$B \xrightarrow{\ i\ } \mathcal{O}(X) \xrightarrow{\ p\ } \mathcal{O}(X \setminus O_X)$$

is $SL(2)-$equivariant isomorphisms. Therefore, $X \setminus O$ is an orbit isomorphic to $SL(2)/T$ under the morphism $\rho : X \to X \setminus O_X$ defined by $\rho^* = i \circ (p \circ i)^{-1}$ and restriction. This gives us the first type of embedding.

Now assume that $h < 1$. Say $\mathfrak{n} := A_h \cap (x,y)$ is the homogeneous maximal ideal in A_h. Then by lemma 3.3.1, $\mathfrak{m} := \langle SL(2) \cdot \mathfrak{n} \rangle \subset \mathcal{O}(X)$ is a maximal ideal defines a fixed point $e \in X$, so X has a fixed point. Now we have

$$X \setminus O_X = O' \cup \{e\},$$

and O' is isomorphic to $SL(2)/U_n$. Since $\overline{O'}$ is normal and the fixed point has codimension 2, we have

$$\mathcal{O}(\overline{O'}) = \mathcal{O}(X \setminus O_X) = \mathcal{O}(SL(2))^{U_n} \simeq \bigoplus_{i=0}^{\infty} R_{ni}.$$

(R_m is the set of homogeneous polynomials of degree m in $\mathbb{C}[x,y]$ with G-action, which is defined as follows; for $g \in G$ and $f \in R_m$, g acts on $f \in R_m$ as ${}^g f(v) := f(vg)$ when $v = (x,y)$. We can easily show that R_m is isomorphic to the homogeneous polynomials of degree m with the usual G action $g \cdot f(v) = f(g^{-1}v)$

41

via group automorphism $g \mapsto (g^t)^{-1}$.) $U-$invariant ring of $X \setminus O_X$ is given by
$A_h/(\sqrt{(x \cdot A_h)}) = \mathbb{C}[t]$ with $t := x^q y^p + \sqrt{(x \cdot A_h)}$. The weights are occurring on
the multiple of $p + q$, so $n = p + q$.

$\square$

To get a more geometrical aspect of X, we are going to determine the tangent
spaces at the fixed point.

Lemma 3.3.4. *Suppose that $h(X) < 1$. If $\mathfrak{n}$ is the homogeneous maximal ideal of
A_h, define $\mathfrak{m} :=< SL(2) \cdot \mathfrak{n} >$. Then $(\mathfrak{m}^2)^U = \mathfrak{n}^2$.*

Theorem 3.3.5. *Let $M_h := \{(r, s) \in \mathbb{Z} \times \mathbb{Z} | x^r y^s \in A_h\}$ be a monoid of lattice. If
$\{(r_i, s_i)\}_{i=1}^t$ is a minimal generator of M_h, then the tangent space at the fixed point
e_0 has the following $SL(2)-$module decomposition*

$$T_{e_0}(X) \simeq \oplus_{i=1}^t R_{r_i + s_i}.$$

In conclusion, if $h = \frac{p}{q} < 1$, then $T_{e_0}(X)$ contains the representation $R_1 \oplus R_{p+q}$.
In particular, $\dim T_{e_0}(X) \geq 6$, so x_0 is a singular point. Also, if $f \in R_n$ has a
trivial stabilizer and $0 \in \overline{O_f}$, then $\overline{O_f}$ is not normal.

3.4 Construction of Normal Affine $SL(2)-$embedding

From the contents of section 3.1, we can easily have the following theorem:

Theorem 3.4.1. *For $h \leq 1$, let $\{(r_i, s_i)\}_{i=1}^N$ is a minimal generating sets of semi-
group M_h. Define*

$$f := (x^{r_1} y^{s_1}, \cdots, x^{r_t} y^{s_t}) \in R_{r_1 + s_1} \oplus \cdots \oplus R_{r_t + s_t}.$$

Then $\overline{O_f}$ is a normal $SL(2)-$embedding with $h(\overline{O_f}) = h$.

If $f \in R_{n_1} \oplus \cdots \oplus R_{n_t}$ has a trivial stabilizer and $\overline{O_f}$ contains $0 \in R_{n_1} \oplus \cdots \oplus R_{n_t}$,
then $\overline{O_f}$ defines an $SL(2)$-embedding. We want to check that when this is normal,

and as a result, we have that the varieties in the above theorem are all normal varieties.

Definition 3.4.2. For $f = f_{n_1} + \cdots + f_{n_t} \in R_{n_1} \oplus \cdots \oplus R_{n_t}$, we can find a_i, r_i, s_i such that

$$f_i = a_i x^{r_i} y^{s_i} + \sum_{j>0} a_{ij} x^{r_i+j} y^{s_i-j} \text{ where } n_i = r_i + s_i \text{ and } a_i \neq 0.$$

Then define the height of f by $h(f) := \max_{i=1}^{t} \frac{s_i}{r_i}$.

Theorem 3.4.3 ([20]). *1. If $f \in \oplus_{i=1}^{i=t} R_i$ has trivial stabilizer and $0 \in \overline{O_f}$, then $\overline{O_f}$ is an affine $SL(2)-$embedding with $h(\overline{O_f}) = h(f)$.*

2. $\overline{O_f}$ is normal if and only if the monomials composing f generate $A_{h(f)}$ as a $SL(2)$-module.

Proof. 1) Recall that the action of G on R_n is given by $g_f := (g^T)^{-1} f$. It does not change the orbit, and we can identify R_n with $\mathbb{C}[x,y]_n$. Now we can assume that f satisfies $\lim_{t \to 0} \lambda(t) f = 0$ and each component f_i has the form of $f_i = a_i x^{r_i} y^{s_i} + \sum_{j>0} a_{ij} x^{r_i+j} y^{s_i-j}$. Then we have

$$\mathcal{O}(\overline{O_f}) = \mathbb{C}[g \cdots f_i | g \in SL(2), i = 1, \cdots, t] \subset \mathcal{O}(SL(2)).$$

Say $\eta : X \to \overline{O_f}$ is the normalization of $\overline{O_f}$ and $\widetilde{f} \in X$ is a lift of f. Then η is finite and closed, so $\lim_{t \to 0} \lambda(t) \widetilde{f} := x_0$ exist in X, and x_0 is a fixed point. Therefore, $\mathcal{O}(X)^U = A_{h_0} \subset \mathbb{C}[x,y]$ for suitable h_0. According to the definition, $h(f)$ is the smallest h with $f_i \in A_h$ for all i. Since $f_i \in \mathcal{O}(\overline{O_f})^U \subset \mathcal{O}(E)^U$, $h_0 \geq h(f)$. Conversely, $\langle SL(2) \cdot A_{h(f)} \rangle$ is normal and contains $\mathcal{O}(\overline{O_f})$, so $\mathcal{O}(E) \subset \langle SL(2) \cdot A_{h(f)} \rangle$, hence $h_0 \leq h(f)$. This completes the proof of the first statement.

2) Say $\mathfrak{n} \subset A_{h(f)}$ is the homogeneous maximal ideal. We assume that the component f_i has the form $f_i = a_i x^{r_i} y^{s_i} + \sum_{j>0} a_{ij} x^{r_i+j} y^{s_i-j}$. The monomials $\{x^{r_i} y^{s_i} | i = 1, \cdots, t\}$ generate $M_{h(f)}$, so the residue classes $f_i + \mathfrak{n}^2 = x^{r_i} y^{s_i} + \mathfrak{n}^2$ generate the vector space $\mathfrak{n}/\mathfrak{n}^2$, and we have $A_{h(f)} = \mathbb{C}[f_1, \cdots, f_t]$. Hence $\mathcal{O}(\overline{O_f})^U = A_{h(f)}$

43

so $\overline{O_f}$ is normal. Conversely, suppose that $B \subsetneq A_h$ is a proper subalgebra and $A := \gamma[SL(2) \cdot B]$ is $SL(2)$-stable subalgebra of $SL(2)$ generated by B, and $A^U \subsetneq A_h$. (This follows from the proof of Lemma 3.3.1 and Lemma 3.3.4.) So if the monomials composing f doesn't generate the monoid $M_{h(f)}$, then $\mathbb{C}[f_1, \cdots, f_t]$ is a proper subalgebra of $A_{h(f)}$ and so $\mathcal{O}(\overline{O_f})^U = \mathbb{C}[SL(2) \cdot f_i | i = 1, \cdots, t]^U$. Therefore, $\overline{O_f}$ is not normal. $\square$

3.5   Affine Normal $SL(2)$-Embedding and $\Gamma(X, x_0)$

In this section, we will compute $\Gamma(X, x_0)$ for all affine normal $SL(2)$-embeddings by using the classification of $SL(2)$-embeddings. This will give us a negative answer to the question in the section 3.2.

3.5.1   Structures of $\Gamma(X, x_0)$

First we compute $\Gamma(X, x_0)$ for the case when $h = 1$. The group $SL(2)$ acts tautologically on space $\mathbb{C}^2$ and by conjugation on space $\mathrm{Mat}(2 \times 2)$. Consider the point

$$x_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \in \mathrm{Mat}(2 \times 2) \times \mathbb{C}^2$$

and its orbit

$$SL(2)x_0 = \{(A, v) \mid \det A = -1, \mathrm{tr}\, A = 0, Av = v, v \neq 0\}.$$

It is easy to see that the closure

$$X = \overline{SL(2)x_0} = \{(A, v) \mid \det A = -1, \mathrm{tr}\, A = 0, Av = v\}.$$

is a smooth $SL(2)$-embedding with two orbits, and $X \setminus O_X = \{(A, 0) \mid \det A = -1\} \simeq SL(2)/T$.

We can calculate $\Gamma(X, x_0)$ by direct computation.

Note that every one parameter subgroup of $SL(2)$ has the form $g \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} g^{-1}$ for some $g \in SL(2)$. Considering an equivalence relation on one-parameter subgroups, we have:

Lemma 3.5.1. *The set* $(\mathfrak{X}_*(SL(2)) - \{\epsilon\}) / \sim$ *can be described as follows:*

$$\left\{ g \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} g^{-1} \,\middle|\, g = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} (\alpha \in \mathbb{C}) \ or \ g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Proof. Define the right hand side set as W. To show the claim, we need to prove 1) any non-trivial $\gamma(t) \in \mathfrak{X}_*(SL(2))$ is equivalent to one of the elements in W, and 2) any two elements in W are not equivalent. For $\lambda_0(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can compute $\lambda_0(t)g\lambda_0(t^{-1}) = \begin{pmatrix} a & bt^2 \\ ct^{-2} & d \end{pmatrix}$. Therefore this element is in $G_{k[[t]]}$ if and only if $c = 0$, so $P(\lambda_0(t)) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL(2) \,\middle|\, a, b, d \in \mathbb{C} \right\}$.

For any $\gamma(t) = g\lambda_0(t^n)g^{-1} \in \mathfrak{X}_*(SL(2))$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and non-negative integer n, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix}$$

if a is nonzero, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$

if a is zero.

Also it is not hard to see that for any $h \in G$ and $\gamma(t) \in \mathfrak{X}_*(G)$, $P(h \cdot \gamma) =$

45

$hP(\gamma)h^{-1}$, and $\lambda_1(t) \sim \lambda_2(t)$ if and only if $h \cdot \lambda_1(t) \sim h \cdot \lambda_2(t)$. Hence, if a is nonzero,

$$\gamma(t) = g \cdot \lambda_0(t^n) \sim g \cdot \lambda_0(t) \sim \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix} \cdot \lambda_0(t)$$

$$\sim \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \left(\begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix} \cdot \lambda_0(t) \right) \sim \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \cdot \lambda_0(t),$$

and if a is zero,

$$\gamma(t) = g \cdot \lambda_0(t) \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} \cdot \lambda_0(t)$$

$$\sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} c & d \\ 0 & b \end{pmatrix} \cdot \lambda_0(t) \right) \sim \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \cdot \lambda_0(t).$$

Therefore, the first assertion is proved.

Now assume that g_1, g_2 are two different element in W and $g_1 \cdot \lambda_0(t) \sim g_2 \cdot \lambda_0(t)$. Then $l_0(t) \sim g_1^{-1} g_2 l_0(t)$ which implies that $g_1^{-1} g_2$ is an upper triangular matrix. It is not hard to see that this only happens when $g_1 = g_2$, so the second assertion is proved. □

Therefore it is enough to check limits when an one-parameter subgroup is one of the followings:

$$\gamma(t) = \begin{pmatrix} t & 0 \\ \alpha(t - t^{-1}) & t^{-1} \end{pmatrix} \text{ or } \gamma(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}.$$

Now go back to the affine variety X when $h = 1$. For $\gamma(t) = \begin{pmatrix} t & 0 \\ \alpha(t - t^{-1}) & t^{-1} \end{pmatrix}$,

we can compute

$$\gamma(t)x_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 2\alpha(1 - t^{-2}) & -1 \end{pmatrix}, \begin{pmatrix} t \\ \alpha(t - t^{-1}) \end{pmatrix} \right\}.$$
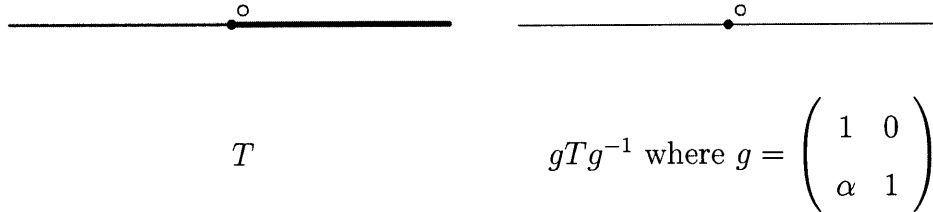
It has a limit in X if and only if $\alpha = 0$.

For $\gamma(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$, we can compute

$$\gamma(t)x_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} t^{-1} \\ 0 \end{pmatrix} \right\}.$$

and the limit doesn't exist regardless of α.

Therefore, the cone structure of X can be described as follows: If $T_0 :=$

$$\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \Big| \; t \in \mathbb{C}^* \right\},$$

$$T \qquad\qquad\qquad gTg^{-1} \text{ where } g = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

Remark 1. When $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $g \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} g^{-1} = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$, so it can

be described together in the first picture. Also, we can show that $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$.

$\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \sim \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$, so negative part of $\Gamma(X, x_0) \cap \mathfrak{X}_*(T)$ for each maximal

torus T should be all same when $G = SL(2)$.

For varieties with $h = \frac{p}{q} < 1$, the descriptions are similar with the above. First, consider $A_h = \{\mathbb{C}x^i y^j \mid \frac{i}{i} \leq h\}$, and define M_h as $\{(i, j) \in \mathbb{Z}^2 \mid x^i y^j \in A_h\}$. Take the generators of the semigroup M_h:

$$\langle (1,0), (r_1, s_1), \cdots, (r_n, s_n), (p,q) \rangle.$$

Consider this as a point x_0 in $V_1 \oplus V_{r_1+s_1} \oplus \cdots \oplus V_{r_n+s_n} \oplus V_{p+q}$, where V_k is the $SL(2)$ module composed of degree k homogeneous polynomials. (Action of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$ on x is $ax + cy$, and on y is $bx + dy$.) Then the affine normal $SL(2)-$embedding corresponding to h is

$$\overline{SL(2)x_0}$$

Now let's compute the cone structure of it.

For $\gamma(t) = \begin{pmatrix} t & 0 \\ \alpha(t - t^{-1}) & t^{-1} \end{pmatrix}$, its action on x is $\gamma(t)x = tx + \alpha(t - t^{-1})y$, so in order to have a limit, α should be 0. If $\alpha = 0$, $\gamma(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and $\gamma(t)(x^i y^j) = t^{i-j} x^i y^j$. Since for any $x^i y^j$ we have $i > j$, the limit is zero point in $V_1 \oplus V_{r_1+s_1} \oplus \cdots \oplus V_{r_n+s_n} \oplus V_{p+q}$ which is obviously inside the closure.

For $\gamma(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$, $\gamma(t)x = t^{-1}x$, so the limit doesn't exist.

Hence, the cone structure of X for any $h < 1$ is the same with the case of $h = 1$, and can be described as following :

$$T \qquad\qquad gTg^{-1} \text{ where } g = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

In conclusion, we can have the following theorem.

Theorem 3.5.2. *For any affine normal $SL(2)$-embedding, the cone structure $\Gamma(X, x_0)$ is unique.*

The solution to classification problems using $\Gamma(X, x_0)$ should include how to reconstruct X from this set. More precisely, as in [24], one can ask that following claim is true:

Let G be a connected reductive group. If X is an affine $G-$ embedding with base point x_0, then $X \simeq \operatorname{Spec} A_{\Gamma(X,x_0)}$, where $A_{\Gamma(X,x_0)} := \{f \in k[G] : v_\gamma(f) \geq 0 \text{ for all } \gamma \in \Gamma(X, x_0)\}$.

Unfortunately, this does not hold in general. As we can see in the above cone structure construction of affine normal $SL(2)-$embedding, the varieties which are not isomorphic give us the same cone structure $\Gamma(X, x_0)$, and the same $A_{\Gamma(X,x_0)}$. So it is obvious that $\Gamma(X, x_0)$ doesn't give us enough information to solve classification of G-embedding completely.

Now to check the claim more precisely, we will verify that for the unique cone structure $\Gamma(X, x_0)$ of affine normal $SL(2)-$ embedding, $\operatorname{Spec} A_{\Gamma(X,x_0)}$ is isomorphic to the unique smooth affine $SL(2)-$embedding as following.

Note that any $\gamma \in \Gamma(X, x_0)$ is equivalent to $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ except the trivial one-parameter subgroup ϵ. For this λ, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$,

$$A\lambda(t) = \begin{pmatrix} at & bt^{-1} \\ ct & dt^{-1} \end{pmatrix}.$$

Therefore, we can easily find all functions in $k[SL(2)] = k[x, y, z, w]/(xz - yw - 1)$ whose $t-$valuation is non-negative are generated (as an algebra) by the monomials x, z, xy, xw, yz, zw. That is,

$$A_{\Gamma(X,x_0)} = k[x, z, xy, xw, yz, zw]/(xy - zw - 1) \subset k(SL(2)).$$

Recall that the unique smooth affine $SL(2)-$embedding can be described as

$$X = \overline{SL(2)x} = \{(A, v) \mid \det A = -1, \operatorname{tr} A = 0, Av = v\}.$$

If we take the coordinate system as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $v = \begin{pmatrix} x \\ z \end{pmatrix}$, we can find the ring of regular function as $\mathcal{O}(X) = k[a, b, c, d, x, z]/(ad - bc + 1, a + d, ax + bz - x, cx + dz - z)$.

Then, we can find the following transformations :

$$A_{\Gamma(X, x_0)} = k[x, z, xy, xw, yz, zw]/(xy - zw - 1).$$

$$\Big\Vert \; xy \mapsto d, \quad xw \mapsto b, \quad yz \mapsto c, \quad zw \mapsto a$$

$$k[x, z, a, b, c, d]/(d - a - 1, ad - bc, dz - cx, bz - ax)$$

$$\Big\Vert \; a \mapsto \frac{a}{2} - \frac{1}{2}, \quad b \mapsto -\frac{b}{2}, \quad c \mapsto \frac{c}{2}, \quad d \mapsto -\frac{d}{2} + \frac{1}{2}$$

$$k[x, z, a, b, c, d]/(a + d, ad - bc + 1, ax + bz - x, cx + dz - z) = \mathcal{O}(X).$$

3.5.2   Some applications using $\Gamma(X, x_o)$

Even though $\Gamma(X, x_o)$ doesn't give us the solution to the classification, still it contains useful information of the G-embedding X. Suppose $f : X \to Y$ is a G-equivariant morphism between affine $G-$embeddings X and Y. If $x_0 \in X$ is a base point for X, then $y_0 = f(x_0)$ is a base point for Y. Moreover, if γ is a one-parameter subgroup of G such that $\lim_{t \to 0} \gamma(t)x_0 = x_\gamma$ exists in X, then $\lim_{t \to 0} \gamma(t)y_0$ exists in Y and is equal to $f(x_\gamma)$ since f is continuous and $f(\gamma(t)x_0) = \gamma(t)f(x_0) = \gamma(t)y_0$ for all $t \neq 0$. Therefore, there is an inclusion $\Gamma(X, x_0) \subset \Gamma(Y, f(x_0))$ whenever there exists an equivariant morphism $f : X \to Y$ of affine $G-$embeddings.

Another interesting properties can be found when X not only has a left G-action, but also right G-action too, which is compatible with the left action. We call X *biequivariant G-variety* in this case.

50

Proposition 7 ([24]). *If an affine $G-$embedding X have both a left and a right $G-$action, then the associated strongly convex lattice cone $\Gamma(X, x_0)$, for any choice of base point $x_0 \in X$, is $G-$stable for the conjugation action of G on $(X)_*(G)$.*

Proof. Suppose that X is a $(G \times G)$-equivariant affine $G-$embedding and let $x \in X$ be a base point. We want to show that $\Gamma(X, h \cdot x) = \Gamma(X, x)$ for any $h \in G$. Let $h \in G$ and assume that $\gamma \in \Gamma(X, x)$, so $\lim_{t \to 0} \gamma(t)x$ exists in X. Then we can check $\lim_{t \to 0}[\gamma(t) \cdot hx] = \lim_{t \to 0}[\gamma(t) \cdot xh'] = \lim_{t \to 0}[\gamma(t) \cdot x] \cdot h'$, for some $h' \in G$, and this limit exists in X. Recall that $h\Gamma(X, x)h^{-1} = \Gamma(X, h \cdot x)$. Thus $\Gamma(X, x) \subset h\Gamma(X, x)h^{-1}$. Now assume that $\gamma' \in \Gamma(X, h \cdot x)$. Then, the same argument implies $\gamma' \in \Gamma(X, x)$. Therefore, for every $h \in G$, $\Gamma(X, x) = h\Gamma(X, x)h^{-1}$. Thus $\Gamma(X, x)$ is $G-$stable for the conjugation action of G on $\mathfrak{X}_*(G)$ for any choice of base point $x \in X$. $\square$

Suppose that X is a normal affine $SL(2)$-embedding. If X is a biequivariant $G-$embedding and x_0 is a base point, $\Gamma(X, x_0)$ is $G-$stable, so $\Gamma(X, x_0) \cap \mathfrak{X}_*(T)$ is equal to $\Gamma(X, x_0) \cap \mathfrak{X}_*(gTg^{-1})$ as a cone. By the structure of $\Gamma(X, x_0)$ we've already observed, we have the following.

Corollary 5. *There is no biequivariant normal affine $SL(2)-$embedding except the trivial embedding.*

Chapter 4

Some examples of Affine Normal $GL(2)$-embedding

The classification of affine normal G-embedding has not been solved except some special cases. We don't even know complete classifications when G is $GL(2)$ or $SL(3)$. To understand more possible relations between the cone structures $\Gamma(X, x_0)$ and G-embedding, we need to observe what happens in specific cases, but the lack of examples make things a bit difficult. So it is worthwhile to construct some family of affine $GL(2)$-embeddings and its cone structures.

When $G = GL(2)$, the situation is more complicated as we might guess. The dimension of orbits now can be 2 or 3, and there are too many kinds of algebraic subgroups of $GL(2)$. Also, the maximal torus has dimension 2, so some of useful facts in $SL(2)$ case do not hold anymore.

We can easily observe that every maximal torus is conjugate to

$$T_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in \mathbb{C}^* \right\},$$

every maximal unipotent subgroup is conjugate to

$$U_0 = \left\{ \left. \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \right| b \in \mathbb{C} \right\},$$

and evry Borel subgroup is conjugate to

$$B_0 = \left\{ \left. \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right| a, d \in \mathbb{C}^*, b \in \mathbb{C} \right\}.$$

Now let X be an affine normal $GL(2)$-embedding. As for the case of $SL(2)$, we can fix one maximal unipotent subgroup U as above. Then from

$$\mathcal{O}(GL(2)) = \mathbb{C}[x, y, z, w]_{(xw-yz)} \simeq \mathbb{C}\left[x, y, z, w, \frac{1}{xw-yz}\right],$$

we have $\mathcal{O}(GL(2))^U = \mathbb{C}\left[x, y, \frac{1}{xw-yz}\right]$, and $\mathcal{O}(X)^U \subset \mathcal{O}(GL(2))^U = \mathbb{C}\left[x, y, \frac{1}{xw-yz}\right]$. Since $A_X := \mathcal{O}(X)^U \subset \mathcal{O}(GL(2))^U = \mathbb{C}\left[x, y, \frac{1}{xw-yz}\right]$, we can construct some examples of $GL(2)$-embeddings when A_X is generated by monomials.

Assume that A_X is generated by monomials. Note that $\mathcal{O}(X)^U$ is finitely generated by Theorem 3.1.1. Also by Theorem 1.3.1, A_X is normal when X is a normal variety. Therefore we can find a rational strongly convex cone σ_X in $\mathbb{R}^3 = \mathbb{Z}^3 \otimes_{\mathbb{Z}} \mathbb{R}$ where

$$M_X := \left\{ (p, q, r) \in \mathbb{Z}^3 \,\middle|\, x^p y^q \left(\frac{1}{xw-yz}\right)^r \in A_X \right\} = \sigma_X \cap \mathbb{Z}^3.$$

Pick a set of lattice points $\{(p_i, q_i, r_i)\}_{i=1\cdots t}$ which generates M_X as a monoid. Now define V_n to be the $\mathbb{C}$-vector space of degree d homogeneous polynomials in $\mathbb{C}\left[x, y, \frac{1}{xw-yz}\right]$, with a action of G as $f \mapsto {}^g f$. (This action can be described as follows : if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2)$, then x maps to $ax + cy$, y maps to $bx + dy$ and $\left(\frac{1}{xw-yz}\right)$ maps to $\left(\frac{1}{(ad-bc)(xw-yz)}\right)$.) Now consider the element

$$f = \left(x^{p_1} y^{q_1} \left(\frac{1}{xw-yz}\right)^{r_1}, \cdots, x^{p_t} y^{q_t} \left(\frac{1}{xw-yz}\right)^{r_t} \right)$$

, in $\oplus_{i=1}^t V_{p_i+q_i+r_i}$.

Lemma 4.0.3. *Assume that σ is a cone in $\mathbb{R}^3$, which does not lie on any plane $ax + by + cz = 0$ with integers a, b, c satisfying $a + b + c = 0$. Then the stabilizer of f is trivial.*

Proof. Suppose that $^g f = f$ holds for some $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2)$. Then for each i,

$$(ax + cy)^{p_i}(bx + dy)^{q_i}((ad - bc)(xw - yz))^{-r_i} = x^{p_i}y^{q_i}(xw - yz)^{-r_i}.$$

Since we can find two linearly independent (p_i, q_i) by assumption, there are nonzero p_i and q_i, which implies that $ab = 0$ and $cd = 0$ (by comparing the coefficients of $x^{p_i+q_i}$ and $y^{p_i+q_i}$.) Because $ad - bc \neq 0$ and there is i such that $p_i \neq q_i$, we have $b = c = 0$. Therefore, $a^{p_i}d^{q_i}(ad)^{-r_i} = a^{(p_i-r_i)}d^{(q_i-r_i)} = 1$ for each i.

By assumption, we can find $i, j \in \{1, \cdots, t\}$ such that $(p_i - r_i, q_i - r_i)$ is linearly independent from $(p_j - r_j, q_j - r_j)$. (If they are dependent for all i, j, there are integers A, B such that $A(p_i - r_i) + B(q_i - r_i) = 0$ for all i, and this leads a contradiction to the assumption.) Hence we can find α, β such that $\alpha(p_i - r_i, q_i - r_i) - \beta(p_j - r_j, q_j - r_j) = (1, 0)$. Then

$$1 = \frac{(a^{(p_i-r_i)}d^{(q_i-r_i)})^\alpha}{(a^{(p_j-r_j)}d^{(q_j-r_j)})^\beta} = a^{\alpha(p_i-r_i)-\beta(p_j-r_j)}d^{\alpha(q_i-r_i)-\beta(q_j-r_j)} = a.$$

Similarly, we can show $d = 1$. Therefore, we can conclude that the stabilizer of f in $GL(2)$ is the trivial group. $\square$

From the lemma, the closure of the orbit $\overline{GL(2) \cdot f}$ in $\oplus_{i=1}^t V_{p_i+q_i+r_i}$ gives $GL(2)$-embedding when σ is the cone in $\mathbb{R}^3$, which does not lie on any plane $ax+by+cz = 0$ with integers a, b, c satisfying $a + b + c = 0$. Let's first consider the case when $r_i = 0$ for all i and the cone σ lies in the ij-plane. Then M_X is generated by $\{(p_i, q_i, 0)\}_{i=1\cdots t}$. Say $f = (x^{p_1}y^{q_1}, \cdots, x^{p_t}y^{q_t})$ is the element in $\oplus_{i=1}^t V_{p_i+q_i}$. As we

discuss above, $X := \overline{GL(2) \cdot f}$ is affine normal $GL(2)$-embedding. By 4, we can conclude that $\mathcal{O}(X) = \langle GL(2) \cdot f \rangle$.

Note that every one parameter subgroup of $GL(2)$ has the form $g \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} g^{-1}$ for some $g \in GL(2)$ and $a, b \in \mathbb{Z}$. Considering an equivalence relation on one-parameter subgroups, we have:

Lemma 4.0.4. *The set* $(\mathfrak{X}_*(GL(2)) - \{\epsilon\}) / \sim$ *can be described as follows:*

$$\left\{ g \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} g^{-1} \;\middle|\; a - b \geq 0, (a, b) = 1, g = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} (\alpha \in \mathbb{C}) \text{ or } g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Proof. Define the right hand side set as W. To show the claim, we need to prove 1) any non-trivial $\gamma(t) \in \mathfrak{X}_*(GL(2))$ is equivalent to one of the elements in W, and 2) any two elements in W are not equivalent. For $\lambda_{(a,b)}(t) = \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix}$ and

$g = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, we can compute $\lambda_0(t) g \lambda_{(a,b)}(t^{-1}) = \begin{pmatrix} p & qt^{a-b} \\ rt^{b-a} & s \end{pmatrix}$. Therefore this element is in $G_{k[[t]]}$ if and only if $r = 0$ when $a > b$, $q = 0$ when $a < b$, and always when $a = b$.

So when $a > b$,

$$P(\lambda_{(a,b)})(t)) = \left\{ \begin{pmatrix} p & q \\ 0 & s \end{pmatrix} \in GL(2) \;\middle|\; p, q, s \in \mathbb{C} \right\},$$

when $a < b$,

$$P(\lambda_{(a,b)})(t)) = \left\{ \begin{pmatrix} p & 0 \\ r & s \end{pmatrix} \in GL(2) \;\middle|\; p, r, s \in \mathbb{C} \right\},$$

56

and when $a = b$

$$P(\lambda_{(a,b)})(t)) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL(2) \;\middle|\; p, q, r, s \in \mathbb{C} \right\} = GL(2).$$

Therefore when $a = b$, we have $g \cdot \lambda_{(a,b)} \sim \lambda_{(a,b)}$, so it is equivalent to either $\sim \lambda_{(1,1)}$ or $\sim \lambda_{(-1,-1)}$.

Now assume that $\gamma(t) = g\lambda_{(a,b)}(t)g^{-1} \in \mathfrak{X}_*(SL(2))$ with $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and non-negative integer n. Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \lambda_{(a,b)} = \lambda_{(b,a)}$, we only need to check when $a > b$. We can observe that

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{r}{p} & 1 \end{pmatrix} \begin{pmatrix} p & q \\ 0 & s - \frac{qr}{p} \end{pmatrix}$$

if p is nonzero, and

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r & s \\ 0 & q \end{pmatrix}$$

if p is zero.

Say $a' = \frac{a}{(a,b)}$ and $b' = \frac{b}{(a,b)}$. If p is nonzero,

$$\gamma(t) = g \cdot \lambda_{(a,b)}(t) \sim g \cdot \lambda_{(a',b')}(t) \sim \begin{pmatrix} 1 & 0 \\ \frac{r}{p} & 1 \end{pmatrix} \begin{pmatrix} p & q \\ 0 & s - \frac{qr}{p} \end{pmatrix} \cdot \lambda_{(a',b')}(t)$$

$$\sim \begin{pmatrix} 1 & 0 \\ \frac{r}{p} & 1 \end{pmatrix} \left(\begin{pmatrix} p & q \\ 0 & s - \frac{qr}{p} \end{pmatrix} \cdot \lambda_{(a',b')}(t) \right) \sim \begin{pmatrix} 1 & 0 \\ \frac{r}{p} & 1 \end{pmatrix} \cdot \lambda_{(a',b')}(t).$$

57

Similarly, if p is zero,

$$\gamma(t) = g \cdot \lambda_{(a,b)}(t) \sim g \cdot \lambda_{(a',b')}(t) \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r & s \\ 0 & q \end{pmatrix} \lambda_{(a',b')}(t)$$

$$\sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} r & s \\ 0 & q \end{pmatrix} \cdot \lambda_{(a',b')}(t) \right) \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \lambda_{(a',b')}(t).$$

Therefore, the first assertion is proved. The second assertion can be proved similarly with the proof of Lemma 3.5.1. □

Hence it is enough to check limits when an one-parameter subgroup is one of the followings:

$$\gamma(t) = \begin{pmatrix} t^a & 0 \\ \alpha(t^a - t^b) & t^b \end{pmatrix}.$$

This one-parameter subgroup acts on the monomial $x^p y^q$ as

$$\gamma(t)(x^p y^q) = (t^a x + \alpha(t^a - t^b)y)^p (t^b y)^q = \sum_{j=0}^{p} \alpha^{p-j} \binom{p}{j} (t^{(aj+bq)})(t^a - t^b)^{(p-j)} x^j y^{p+q-j}.$$

From this we obtain the following:

1. When $\alpha = 0$, the limit $\lim_{t \to 0} \gamma(t)(x^p y^q)$ exists when $\lim_{t \to 0}(t^{(ap+bq)})$ exists, which is equivalent to $ap + bq \geq 0$.

2. When $\alpha \neq 0$, the limit $\lim_{t \to 0} \gamma(t)(x^p y^q)$ exists when for all j, $\lim_{t \to 0}(t^{(aj+bq)})(t^a - t^b)^{(p-j)}$ exists. If $a = b$, the limit exists if and only if $(ap+bq) = 2a(p+q) \geq 0$. If $a \neq b$, the limit exists if and only if $(aj + bq) + a(p - j) \geq 0$ and $(aj + bq) + b(p - j) \geq 0$ for all j. It is easy to observe that this is equivalent to $ap + bq \geq 0$ (which includes the case $a = b$) and $b \geq 0$.
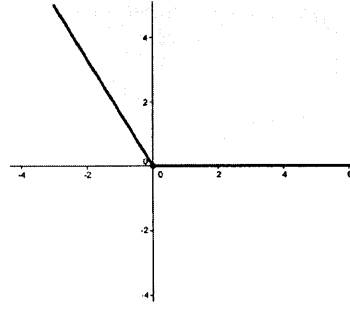
Therefore for $f = (x^{p_1}y^{q_1}, \cdots, x^{p_t}y^{q_t})$, the limit $\lim\limits_{t \to 0} \gamma(t)f$ exists (which automatically implies that $\lim\limits_{t \to 0} \gamma(t)f$ exists in X by definition of X) if and only if, either

1. $\alpha = 0$ and $ap_i + bq_i \geq 0$ for all i, or

2. $\alpha \neq 0$, $b \geq 0$ and $ap_i + bq_i \geq 0$ for all i

Hence the cone structure of X can be described as following:
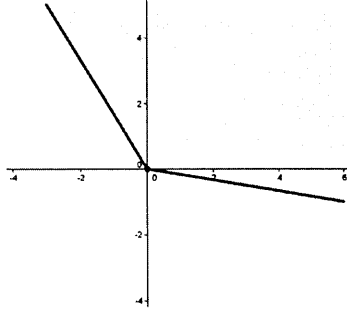
$$T_0 \qquad\qquad gT_0g^{-1} \text{ where } g = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}.$$

Remark 2. When $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $g \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} g^{-1} = \begin{pmatrix} t^b & 0 \\ 0 & t^a \end{pmatrix}$, so it can be described together in the first picture. Also when $a < b$, we have $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in P(\lambda_{(a,b)(t)})$, therefore $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} \sim \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix}$, so $a < b$ part of $\Gamma(X, x_0) \cap \mathfrak{X}_*(T)$ for each maximal torus T should be all same when $G = GL(2)$.

For the general cone σ which satisfies the property in the above lemma, M_X is generated by $\{(p_i, q_i, r_i)\}_{i=1\cdots t}$. Say $f = \left(x^{p_1}y^{q_1} + \left(\frac{1}{xw-yz} \right)^{r_1}, \cdots, x^{p_t}y^{q_t} + \left(\frac{1}{xw-yz} \right)^{r_t} \right)$ is an element in $\oplus_{i=1}^{t} V_{p_i+q_i+r_i}$. Then $X := \overline{GL(2) \cdot f}$ is affine normal $GL(2)$-embedding. Hence we can conclude that $\mathcal{O}(X) = \langle GL(2) \cdot f \rangle$.

The one-parameter subgroup $\gamma(t) = \begin{pmatrix} t^a & 0 \\ \alpha(t^a - t^b) & t^b \end{pmatrix}$ acts on the monomial $x^p y^q \left(\frac{1}{xw-yz} \right)^r$ as

$$\gamma(t) \left(x^p y^q \left(\frac{1}{xw-yz} \right)^r \right) = (t^a x + \alpha(t^a - t^b)y)^p (t^b y)^q (\frac{1}{t^{a+b}(xw-yz)})^r$$

$$= \sum_{j=0}^{p} \alpha^{p-j} \binom{p}{j} (t^{(aj+bq-r(a+b))})(t^a - t^b)^{(p-j)} x^j y^{p+q-j}.$$

From this we obtain the following:

1. When $\alpha = 0$, the limit $\gamma(t) \left(x^p y^q \left(\frac{1}{xw-yz} \right)^r \right)$ exists when $\lim_{t\to 0}(t^{(ap+bq-r(a+b))})$ exists, which is equivalent to $a(p-r) + b(q-r) \geq 0$.

2. When $\alpha \neq 0$, the limit $\lim_{t\to 0}\gamma(t) \left(x^p y^q \left(\frac{1}{xw-yz} \right)^r \right)$ exists when for all j, $\lim_{t\to 0}(t^{(aj+bq)})(t^a - t^b)^{(p-j)}$ exists. If $a = b$, the limit exists if and only if $(ap+bq) - r(a+b) = a(p+q-2r) \geq 0$. If $a \neq b$, the limit exists if and only if $(aj+bq) - r(a+b) + a(p-j) \geq 0$ and $(aj+bq) + b(p-j) - r(a+b) \geq 0$ for all j. It is easy to observe that this is equivalent to $(p-r)a + (q-r)b \geq 0$ (which includes the case when $a = b$) and $-ra + b(p+q-r) \geq 0$.

Therefore for $f = \left(x^{p_1} y^{q_1} + \left(\frac{1}{xw-yz} \right)^{r_1}, \cdots, x^{p_t} y^{q_t} + \left(\frac{1}{xw-yz} \right)^{r_t} \right)$, $\lim_{t\to 0}\gamma(t)f$ exists (which automatically implies that $\lim_{t\to 0}\gamma(t)f$ exists in X by definition of X) if and only if either

1. $\alpha = 0$ and $a(p_i - r_i) + b(q_i - r_i) \geq 0$ for all i, or

2. $\alpha \neq 0$, $-ar + (p_i + q_i - r_i)b \geq 0$ and $a(p_i - r_i) + b(q_i - r_i) \geq 0$ for all i.

Unlike $SL(2)$ case, the cone structures $\Gamma(X, x_0)$ are not unique for affine normal $GL(2)$-embeddings. Therefore, we can expect that this structure implies some characteristics of those embeddings, and it might help us find an explicit way to

60

classify general G-embeddings. Also this $GL(2)$-embedding construction is interesting when we study the properties of embeddings of homogeneous space with complexity 1. Because the codimension of a Borel subgroup is 1, the complexity is 1 for $GL(2)$-embedding. We have a combinatorial description for the complexity 1 case [32], but general theories are not well developed on this kind of variety. For more studies of complexity 1 case, it would be helpful to have a precise description of affine normal $GL(2)$-embeddings. We hope that the cone structure $\Gamma(X, x_0)$ can contribute to many problems related to affine normal G-embeddings, and furthermore, general embedding theory.

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