Birational Geometry of the Space of Rational Curves in Homogeneous Varieties

by

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Abstract

In this thesis, we investigate the birational geometry of the space of rational curves in various homogeneous spaces, with a focus on the quasi-map compactification induced by the Quot and Hyperquot functors. We first study the birational geometry of the Quot scheme of sheaves on $\mathbb{P}^1$ via techniques from the Mori program, explicitly describing its associated cones of ample and effective divisors as well as the various Mori chambers within the latter. We compute the base loci of all effective divisors, and give a conjectural description of the induced birational models. We then partially extend our results to the Hyperquot scheme of sheaves on $\mathbb{P}^1$, which gives the analogous compactification for rational curves in flag varieties. We fully describe the cone of ample divisors in all cases and the cone of effective divisors in certain ones, but only claim a partial description of the latter in general.

Thesis Supervisor: James McKernan
Title: Professor

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Chapter 1

Rational Curves in $\mathbb{P}^{n-1}$

Describing the behavior of curves in projective space, and particularly counting curves satisfying certain incidence conditions, is one of the most studied topics in algebraic geometry. The simplest case is that of smooth rational curves, i.e. nonsingular subvarieties $C \subset \mathbb{P}^{n-1}$ abstractly isomorphic to $\mathbb{P}^1$. These curves have an associated degree d (its intersection number with a generic hyperplane, or its class in $H_1(\mathbb{P}^{n-1}) \cong \mathbb{Z}$), and the locus of curves with a given degree is finite-dimensional. Indeed, such curves are the images of morphisms $\mathbb{P}^1 \to \mathbb{P}^{n-1}$: parameterizing these maps and quotienting by automorphisms gives

Theorem 1. *The space of smooth rational curves of degree d is quasi-projective of dimension $nd + n - 4$.*

Of course, this space is not closed, an impediment to applying many of the tools of algebraic geometry (which require compactness) to compute curve counts. Thus, we would like to compactify the space of smooth curves, and do it in such a way that the overall space still classifies curves, albeit not just smooth ones.

1.1 Moduli spaces

The formal notion for such a classification problem is a moduli functor and its associated moduli space [14]. Roughly speaking, a moduli space is one whose points correspond to algebro-geometric objects of a predetermined nature, and whose local geometry is given by the behavior of those objects in families:

- A *moduli functor* is a functor $\mathscr{F}$ from schemes to sets which associates to a scheme T the set of families over T whose fibers are the objects being classified, modulo some notion of equivalence.

- $\mathscr{F}$ has an associated *fine moduli space* if it is representable, i.e. if there is a scheme F and an isomorphism of functors $h_F := \{T \mapsto \mathrm{Hom}(T, F)\} \cong \mathscr{F}$.

- $\mathscr{F}$ has an associated *coarse moduli space* if there is a scheme F whose points correspond to isomorphism classes of the desired objects, equipped with a natural transformation $\mathscr{F} \to h_F$ universal among all possible F.

If $\mathscr{F}$ has a fine moduli space F, there also exists a *universal family* $U = h_F(F) = \mathrm{Hom}(F, F)$ over F, i.e. a family with the property that any family of the given type of object over a scheme T is pulled back from U along some morphism $T \to F$. However, if $\mathscr{F}$ has only a coarse moduli space, such a family may not exist.

Remark. One often expands the category of objects from schemes to something more general, in order to allow for the existence of a fine (or at least finer) moduli space. The equivalence relation in the moduli functor is often an obstruction to being representable, as individual objects may have automorphisms: extending to the category of algebraic spaces or stacks often alleviates this particular problem.

1.2 Chow varieties

Returning to the question of compactifying curves of degree d, the simplest approach (and first historically) is to take the associated Chow variety.

Definition-Theorem 1 ([4]). *There exists a variety parametrizing cycles in $\mathbb{P}^{n-1}$ of dimension r and degree d, called the* Chow variety. *Explicitly, it is a closed subvariety $\overline{\mathcal{C}_{n,r,d}} \subset \mathbb{P}^N$ (N depending on $n, r,$ and d) representing the functor which sends a variety T to the set of families of dimension r, degree d cycles in $\mathbb{P}^{n-1}_T$, modulo isomorphism. Moreover, for any subvariety $X \subset \mathbb{P}^{n-1}$, those cycles contained in X form a subvariety $\overline{\mathcal{C}_{r,d}(X)} \subset \overline{\mathcal{C}_{n,r,d}}$, also called a Chow variety.*

Remark. As defined, the Chow variety is typically not normal, so in practice one often passes to the normalization (losing some functorial properties).

If $r = 1$, the Chow variety can be constructed by intersecting a cycle with a pair of hyperplanes H, H' in general position, giving a finite set of d distinct points on each: these points induce a homogenous polynomial called the *Chow form*, whose possible coefficients form a closed subvariety of some projective space.

Example. The first interesting Chow varieties are given by curves of degree d in $\mathbb{P}^3$.

$d = 1$: the Chow variety parameterizes lines in $\mathbb{P}^3$, and is isomorphic to the Plücker embedding of $\mathbb{G}(1, 3) \subset \mathbb{P}^5$ (a quadric hypersurface).

$d = 2$: the Chow variety has two components of dimension 8 corresponding to smooth conics and pairs of lines.

$d = 3$: the Chow variety has four components of dimension 12 corresponding to twisted cubics, planar cubics, pairs of a line and a conic, and triplets of lines [2].

Observe that, even for $d = 2$, the Chow variety does not simply compactify smooth curves, but rather adds a whole component of disconnected subvarieties. For $d = 3$, we also get components of higher genus. Even restricting to a single component of the Chow variety, the local geometry is often too coarse to compute nice intersections. Thus, we look for a more refined moduli problem.

1.3 Hilbert schemes

Recall that the Hilbert polynomial of a degree d curve of genus g is

$$\chi(m) := \chi(\mathcal{O}_C(mH)) = dm + 1 - g. \tag{1.1}$$

Since Hilbert polynomials are constant in flat families, the Hilbert scheme is another potential compactification.

Definition-Theorem 2 ([25]). *There exists a scheme parameterizing closed subschemes in $\mathbb{P}^{n-1}$ with Hilbert polynomial $\Phi(m)$ (presumably, $\chi(X, \mathcal{O}_X(m))$ for some subvariety $X \subset \mathbb{P}^{n-1}$), called the* Hilbert scheme. *Explicitly, it is a closed subscheme $\mathcal{H}_n^\Phi \subset \mathbb{P}^N$ (N depending on Φ and n) representing the functor which sends a scheme T to the set of families of subschemes of $\mathbb{P}^{n-1}$ with Hilbert polynomial Φ, modulo isomorphism. As before, one can extend this to closed subschemes of a given projective variety.*

Remark. For subschemes V of dimension > 1, one can use the formula

$$\chi(\mathcal{O}_{V\cap H}(mH)) = \chi(\mathcal{O}_V(mH)) - \chi(\mathcal{O}_V((m-1)H) \tag{1.2}$$

to see that the Hilbert polynomial precisely specifies the arithmetic genera of V and its intersections with general hyperplanes of varying codimension (up to dim V).

Example. For rational curves (i.e. $g = 0$) in $\mathbb{P}^3$:

$d = 1$: the Hilbert scheme is $\mathbb{G}(1, 3)$, the same as the Chow variety.

$d = 2$: the Hilbert scheme has a single component parameterizing conics, and is isomorphic to a $\mathbb{P}^5$ bundle over $\mathbb{P}^{3\vee}$, the space of planes in $\mathbb{P}^3$.

$d = 3$: the Hilbert scheme has two components of dimension 12 and 15 which respectively parameterize twisted cubics and plane cubics with a point [2].

In the latter case, there is a canonical morphism from the first component to the corresponding one of the Chow variety: it forgets the scheme structure and contracts the locus of curves that have a nonreduced component (dimension 9 in the Hilbert scheme).

For higher degrees, the Hilbert scheme does not give a particularly nice compactification of the locus of smooth curves, as the singularities present become arbitrarily bad.

Theorem 2 (Murphy's Law). *Hilbert schemes and Chow varieties of nonsingular curves exhibit (up to smooth product) every class of singularity of finite type over $\mathbb{Z}$.*

Indeed, limits of many different families of smooth curves may be identified by the Hilbert and Chow compactifications: we would thus like to further rigidify our curves, with the aim of separating these limits.

1.4 Kontsevich spaces

As noted in the opening, we can study embedded rational curves via images of explicit morphisms $\mathbb{P}^1 \to \mathbb{P}^{n-1}$, i.e. as parameterized curves: after quotienting by automorphisms of $\mathbb{P}^1$, we obtain a compactification of the original space. This is the basis for Kontsevich's construction [12] of the space of stable maps, which imports the stability condition from the Deligne-Mumford space of curves to give us a nicer moduli functor.

Definition 1. *A* stable curve *is one which is complete and connected, with nodal singularities and a finite group of automorphisms. A* stable map *is a map $f : C \to X$ is a map whose source is a complete, connected, nodal curve, whose target is projective, and whose group of automorphisms is finite.*

Eliminating infinite automorphisms forces the (relative) canonical divisor (possibly twisted by marked points) to be ample, a key part of the construction of the associated moduli space.

Remark. In each case, one can also consider *marked curves* $(C, p_1, \ldots, p_n)$, curves together with fixed markings $p_i \in C$: this allows one to stabilize curves with infinite automorphisms, e.g. rational curves. Indeed, an unmarked curve of genus zero is never stable, as some component must have automorphisms.

Definition-Theorem 3 ([12]). *There exists a projective coarse moduli space $\overline{M}_g$ (resp. $\overline{M}_{g,n}$), called the* Deligne-Mumford space *of stable curves, for the functor which sends a scheme T to the set of families of stable curves (resp. stable curves with n marked points) over T: it is equipped with a universal family $\mathcal{C}_g$ (resp. $\mathcal{C}_{g,n}$). Similarly, there exists a projective coarse moduli space $\overline{M}_{g,0}(X, \beta)$ (resp. $\overline{M}_{g,n}(X, \beta)$), called the* Kontsevich space *of stable maps, for the functor which sends a scheme T to the set of families of stable maps $f : C \to X$ (resp. stable maps with n marked points) such that $[f(C)] = \beta \in H_2(X, \mathbb{Z})$. $\overline{M}_{g,1}(X, \beta)$ (resp. $\overline{M}_{g,n+1}(X, \beta)$) is itself the universal family for $\overline{M}_{g,0}(X, \beta)$ (resp. $\overline{M}_{g,n}(X, \beta)$).*

Remark. If X is $\mathbb{P}^{n-1}$ (or any Grassmannian), $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$, and we use the degree $d = [\beta]$ in place of β.

For studying rational curves, there are two interesting relevant cases:

- $\overline{M}_{0,n}$, the moduli space of stable rational curves with $n \geq 3$ marked points, and
- $\overline{M}_{0,0}(\mathbb{P}^{n-1}, d)$, the moduli space of stable degree d maps $\mathbb{P}^1 \to \mathbb{P}^{n-1}$.

The former object is a smooth variety birational to $\mathbb{P}^{n-3}$, and can be constructed explicitly as a sequence of blow-ups of projective space: its birational geometry is (conjecturally) described by combinatorics are the subject of much study. The latter object, which compactifies our space of rational curves in projective space, behaves much better than either the Chow variety or the Hilbert scheme, as it is irreducible with finite quotient singularities [16]. Indeed, the analogous space with marked points (which is similarly nice) has been used with great success to compute previously unknown counts of curves satisfying various incidence conditions.

Example. Returning to our study of rational curves of degree d in $\mathbb{P}^3$:

$d = 1$: stable maps are smooth, and we obtain the same compactification as in the Chow and Hilbert cases.

$d = 2$: the boundary (non-smooth locus) of the Kontsevich space is a divisor consisting of maps from a reducible curve $C_1 \cup C_2$ (two $\mathbb{P}^1$s intersecting at a node) with degree one on each: its image is thus a pair of intersecting lines, i.e. a degeneration of a conic. There is a morphism to the Hilbert scheme with contracts this locus down to that of double lines (the image of $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ under the degree two Veronese embedding), which exhibits the Kontsevich space as a blowup of a projective bundle.

$d = 3$: the boundary generically consists of maps from $C_1 \cup C_2$ with degrees 1 and 2 on the two components. There is a further substratum given by maps from reducible curves with three components, each mapping in with degree 1.

For higher d, the boundary will have $\lfloor \frac{d}{2} \rfloor$ components Δ_i, generically consisting of maps from $C_1 \cup C_2$ with degree i on one component and $d - i$ on the other. This boundary is refined by strata of maps from reducible curves with more components.

We can use the techniques of the Mori program to study the geometry of this space in more detail, and understand the relationship between it and the Hilbert and Chow compactifications for small values of d more concretely: to do this, we need to have a more detailed understanding of the Neron-Severi space of divisors N^1 (the torsion-free part of the Picard group).

Theorem 3 ([22]). *$N^1(\overline{M}_{0,0}(\mathbb{P}^{n-1}, d))$ is generated by the Δ_i and H, the divisor of maps whose image intersects a fixed codimension two subspace.*

Within this space, we are interested in two closed cones: the cone of nef divisors (called the *ample cone*), and the larger cone of pseudo-effective divisors (called the *effective cone*). By the fundamental results of Mori theory, if we can understand these two cones, and further decompose the latter cone into Mori chambers, we can determine the birational geometry of $\overline{M}_{0,0}(\mathbb{P}^{n-1}, d)$ completely.

Remark. A *Mori chamber* is, roughly speaking, a cone in N^1 such that divisors in the interior behave similarly: after taking sufficiently high multiples, their linear systems have the same base loci and induce equivalent maps. For instance, the cone of ample divisors A on a projective variety X is a Mori chamber, as $\mathrm{Bs}(|mA|) = \emptyset$ and $\phi_A : X \to \mathrm{Proj}\ (\bigoplus H^0(X, mA))$ is an embedding. In nice cases, one can completely decompose the effective cone into such chambers: the birational geometry of the variety can be described in terms of the facets between adjacent cones, which induce so-called *wall-crossing formulas.*

For the first cone, let T be the divisor of maps whose image is tangent to a fixed hyperplane.

Theorem 4 ([10]). *The nef cone of $\overline{M}_{0,0}(\mathbb{P}^{n-1}, d)$ is generated by H, T, and the pullback of the nef cone of $\overline{M}_{0,d}/S_d$ under the morphism*

$$\overline{M}_{0,0}(\mathbb{P}^r, d) \to \overline{M}_{0,d}/S_d \tag{1.3}$$

which sends a stable map to its stabilized image curve plus d unordered points obtained by intersecting the original image with a fixed hyperplane.

For the second, we assume that $n-1 \geq d$, and let D_{deg} be the divisor of degenerate maps, i.e. those whose image lies in a $(d-1)$−plane $\mathbb{P}^{d-1} \subset \mathbb{P}^{n-1}$.

Theorem 5 ([9]). *If $n-1 \geq d$, the effective cone of $\overline{M}_{0,0}(\mathbb{P}^{n-1}, d)$ is generated by D_{deg} and the boundary divisors Δ_i, $1 \leq i \leq \frac{d}{2}$.*

Example. On $\mathbb{P}^3$:

$d = 1$: there is no boundary divisor, and thus the Picard rank of the Kontsevich space is one (so the ample and effective cones are both the trivial ray).

$d = 2$: the Picard rank is two, the ample cone is spanned by H and T, and the effective cone by D_{deg} and the sole boundary divisor Δ 1-1. Furthermore, for $D \in (H, D_{deg})$, ϕ_D maps the Kontsevich space to the Hilbert scheme by contracting the locus of degenerate (i.e. two-to-one) maps, while for $D \in (T, \Delta)$, it contracts the boundary and produces the space of 1-*stable maps*, in which conics degenerate to lines with a base point, instead of pairs of lines.

$d = 3$: the Picard rank is still two, but the cone between H and D_{deg} is divided into two chambers. The map induced by divisors in (D_{deg}, S) maps the Kontsevich space to the closure of locus of twisted cubics in $\mathbb{G}(2, 9)$ (the space of planes in $\mathbb{P}^9$) by contracting the degenerate maps, while divisors in (S, H) induce a *flip* on the locus of maps which are many-to-one on some component. The base of the flip is the component of twisted cubics in the Chow variety, and its image is the analogous component in the Hilbert scheme. [2]

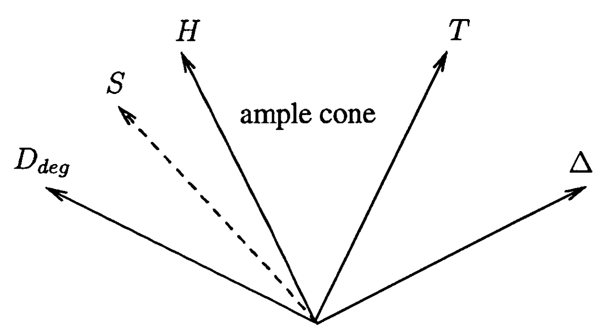

Figure 1-1: Mori chamber decomposition of the Kontsevich spaces $\overline{M}_{0,0}(\mathbb{P}^3, 2)$ (without the dotted ray) and $\overline{M}_{0,0}(\mathbb{P}^3, 3)$ (including it).

1.5 Quot schemes and quasi-map spaces

Prior to quotienting by automorphisms, there is an alternate way to compactify maps from $\mathbb{P}^1$ to $\mathbb{P}^{n-1}$ of degree d, first explored in [15] and [5]. Such maps are induced by line bundles $\mathcal{L}$ on $\mathbb{P}^1$ of degree d, or more precisely by projections $\mathcal{O}^{n\vee} \to \mathcal{L}$ via projectivisation. Thus, a morphism $\mathbb{P}^1 \to \mathbb{P}^{n-1}$ corresponds to an exact sequence $0 \to \mathcal{L}^\vee \to \mathcal{O}^n \to \mathcal{J} \to 0$ modulo automorphisms of $\mathcal{L}^\vee$, where $\mathcal{J}$ is locally free of rank $n-1$. By construction $\mathcal{J}$ has Hilbert polynomial

$$\chi(m) = \chi(\mathbb{P}^1, \mathcal{J} \otimes \mathcal{O}(m)) = (m+1)(n-1) + d \tag{1.4}$$

As with subschemes, there is a natural way to compactify subsheaves of a fixed sheaf (or exact sequences with fixed central term) and specified Hilbert polynomial, called the *Quot scheme.* Although Quot schemes are generally as inscrutable as Hilbert schemes, for rank one subsheaves of the trivial bundle on $\mathbb{P}^1$, they are the projective spaces

$$\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2) \otimes \mathbb{C}^n) \cong \mathbb{P}^{n(d+1)-1}. \tag{1.5}$$

parameterizing those n-tuples of d-forms which determine the inclusion $\mathcal{O}(-d) \cong \mathcal{L}^\vee \hookrightarrow \mathcal{O}^n$. This space has an open locus $U \subset \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2) \otimes \mathbb{C}^n)$ of forms whose associated cokernel $\mathcal{J}$ is locally free: these correspond to honest morphisms $\mathbb{P}^1 \to \mathbb{P}^{n-1}$. We thus obtain the space of smooth rational curves in $\mathbb{P}^{n-1}$ by quotienting U by the action of $SL(2)$ on $\mathrm{Sym}^d(\mathbb{C}^2)$, as well as a compactification by extending this action to the whole Quot scheme. Although the action will no longer be free, we can use geometric invariant theory (GIT) to produce a global quotient.

Definition 2. *The* quasi-map space *of degree d rational curves in $\mathbb{P}^{n-1}$ is*

$$\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2) \otimes \mathbb{C}^n)//SL(2) \tag{1.6}$$

Unlike our previous compactifications, the quasi-map space is not a moduli space for curves. However, being a simple quotient of projective space makes many other attributes (e.g. its cohomology ring, Chow ring, and K-theory) easy to determine via using equivariant torus actions. Furthermore, there is a natural birational map $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2) \otimes \mathbb{C}^n)//SL(2) \dashrightarrow \overline{M}_{0,0}(\mathbb{P}^{n-1}, d)$ which maps an n-tuple of d-forms to the associated morphism $\mathbb{P}^1 \to \mathbb{P}^{n-1}$. On the quasi-map side, the exceptional locus corresponds to points where $\mathcal{J}$ has torsion: these are given by n-tuples of forms with common zeroes. On the Kontsevich side, the exceptional locus is the boundary, i.e. the maps with reducible source.

Example. On $\mathbb{P}^3$:

$d = 1$: The Kontsevich and quasi-map (and Chow and Hilbert) spaces agree.

$d = 2$: The Kontsevich space is the blow-up of the quasi-map space along the locus where the defining forms have two common zeroes, and conversely produces the quasi-map space by blowing down the boundary divisor.

$d = 3$: the Kontsevich space is obtained from the quasi-map space by

(a) blowing up along the $(n-1)$-dimensional locus of three common zeroes,

(b) blowing up along the $(2n-2)$-dimensional locus of two common zeroes,

(c) blowing up along of the $(3n-3)$-dimensional locus of one common zero,

(d) blowing down the exceptional divisor of the second blow-up, and

(e) blowing down the exceptional divisor of the first blow-up,

where, in the last four steps, we take the strict transform of the specified locus: see [KM] for full details. Conversely, one can start with $\overline{M}_{0,0}(\mathbb{P}^{n-1}, d)$ and

(a') Blow up the locus of four irreducible components (a central $\mathbb{P}^1$ with degree 0, attached to three $\mathbb{P}^1$s with degree 1): it is an S_3 quotient of a $(\mathbb{P}^{n-2})^3$ bundle over $\overline{M}_{0,3}(\mathbb{P}^{n-1}, 0) \cong \mathbb{P}^{n-1}$.

(b') Blow up the strict transform of the locus of three irreducible components (a central $\mathbb{P}^1$ with degree 1, attached to two $\mathbb{P}^1$s with degree 1): it is an S_2 quotient of a $(\mathbb{P}^{n-2})^2$ bundle over $\overline{M}_{0,2}(\mathbb{P}^{n-1}, 1)$.

(c') Blow down the strict transform of the locus of two irreducible components (a central $\mathbb{P}^1$ with degree 2, attached to a $\mathbb{P}^1$ with degree 1): it is a $\mathbb{P}^{n-2}$ bundle over $\overline{M}_{0,1}(\mathbb{P}^{n-1}, 2)$ blown up at the locus of three irreducible comonents.

(d') Blow down the strict transform of the locus of three irreducible components.

(e') Blow down the strict transform of the locus of four irreducible components.

This construction, due to Kiem and Moon [15], allows for the recomputation of the cohomology ring of the Kontsevich space and a new calculation of its integral Picard group.

In general, of course, the process of blowing up and blowing down may lead to an arbitrarily complicated space, and thus one cannot infer many results about the birational geometry of Kontsevich space from that of the quasi-map space.

In [5], this study is extended to the space of rational curves in more general homogeneous spaces, with similar results obtained up to $d = 3$. However, the Quot schemes that arise are rather more complex objects: the goal of this work is to understand their birational geometry more thoroughly, towards the aim of getting a better grasp of the structure of the Kontsevich space.

Chapter 2

Rational Curves in Grassmannians

2.1 Background on Grassmannians

We begin by briefly recalling some basic facts about the Grassmannian $G(k,n)$ over $\mathbb{C}$: see e.g. [19] for more details.

- There is a smooth variety $G(k,n)$ of dimension $k(n-k)$ parametrizing k-dimensional subspaces of $V \cong \mathbb{C}^n$ (or projective $(k-1)$-planes in $\mathbb{P}^{n-1}$).

- $G(k,n)$ embeds in $\mathbb{P}^{\binom{n}{k}-1}$ via the top exterior power map, which sends $\Lambda \in G(k,n)$ to $\bigwedge^k \Lambda \subset \bigwedge^k V$ embeds $G(k,n)$, whose image is called the *Plücker embedding* Moreover, this image is the intersection of a set of quadrics defined by the *Plücker relations*, giving an explicit construction.

- $G(k,n)$ can also be realized as a homogeneous space: it is the quotient of $GL(n)$ by the group of matrices stabilizing the span of the first k basis vectors, i.e. matrices with trivial top right $k \times (n-k)$ entries. This simplifies to the quotient $U(n)/U(k) \times U(n-k)$ where $U(m)$ is the space of unitary $m \times m$ matrices.

- $G(k,n)$ is a fine moduli space for the functor which sends a scheme T to the set of rank k subbundles of the trivial bundle of rank n, or equivalently to the set of exact sequences $0 \to S_T \to \mathcal{O}_T^n \to Q_T \to 0$ where S_T is locally free of rank k, $\mathcal{O}_T^n \cong \mathcal{O}_T \otimes V$, and Q_T is locally free of rank $n-k$, modulo isomorphism. It has a universal exact sequence $0 \to \widetilde{S} \to \mathcal{O}^n_{G(k,n)} \to \widetilde{Q} \to 0$, i.e. for any locally free $S' \hookrightarrow \mathcal{O}_T^n$ of rank k (with locally free cokernel Q'), there is a map $f : T \to G(k,n)$ such that $S' = f^*\widetilde{S}$ (and $Q' = f^*\widetilde{Q}$).

- Let $F = \{F^i\}$ be a fixed complete flag of hyperplanes in $\mathbb{C}^n$ (with codim $F^i = i$): given a partition $\lambda = \{\lambda_i\}$ satisfying $n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$, there is an associated Schubert variety

$$\Sigma_\lambda = \{\Lambda \in G(k,n) \,|\, \dim \Lambda \cap F^{k-i+\lambda_i} \geq i \,\forall\, i\} \tag{2.1}$$

of codimension $\sum \lambda_i$, and a corresponding Schubert cycle $\sigma_\lambda = \Sigma_\lambda^\vee$. Note that, for generic Λ, dim $\Lambda \cap F^{k-i} = i$, so only the terms $\lambda_i > 0$ matter.

  - There is one divisorial Schubert cycle σ_1, dual to the space of k-planes that intersect a fixed codimension k-plane in a line.
  - There are two codimension two Schubert cycles σ_2 and $\sigma_{1,1}$, dual to the spaces of k-planes that intersect F^{k+1} in a line and those that intersect F^{k-1} in a 2-plane.

- The cohomology of the Grassmannian is generated by either the Chern classes of $\widetilde{S}$ and $\widetilde{Q}$ (modulo the relation $c(\widetilde{S}) \cdot c(\widetilde{Q}) = 1$), or the Schubert cycles, which are related by the formulas $c_i(\widetilde{S}) = (-1)^i \sigma_{1,\dots,1}, c_i(\widetilde{Q}) = \sigma_i$.

2.2 Rational Scrolls

Given a smooth rational curve $C \cong \mathbb{P}^1 \subset G(k,n)$, the universal property of the Grassmannian gives an associated sequence of vector bundles

$$0 \to K \to \mathcal{O}_{\mathbb{P}^1}^n \to J \to 0 \tag{2.2}$$

on $\mathbb{P}^1$. By a theorem due to Grothendieck, K and J each decompose as a direct sum of line bundles: the given injection and surjection imply that K is a sum of line bundles of degree ≤ 0, J is a sum of line bundles of degree ≥ 0, and the total degree of K is the negative of the total degree of J (say d). We can associate to this sequence a geometric object, called a rational scroll [13].

Definition 3. *Let $a_1 \leq \ldots \leq a_k$ be a sequence of non-negative integers, not all zero, and let $d = \sum a_i$. A k-dimensional* rational scroll *of type $a_1, \dots, a_k$ in $\mathbb{P}^{d+k-1}$ is the join $S_{a_1,\dots,a_k}$ of k rational normal curves of degrees $a_1, \dots, a_k$. If $|a_i - a_j| \leq 1$ for all i, j, the scroll is called* balanced*: if $a_i = a_j$ for all i, j, it is* perfectly balanced *[8].*

More precisely, choosing rational normal curves $f_i : \mathbb{P}^1 \to \mathbb{P}^{a_i}$ and general linear subspaces $\mathbb{P}^{a_i} \subset \mathbb{P}^{d+k-1}$, our scroll will be the union of the $(k-1)$-planes spanned by the points $f_1(z), \dots, f_k(z)$ for $z \in \mathbb{P}^1$. Note that, if one (or more) of the degrees a_i is zero, the corresponding "curve" is a point, and our scroll will be a cone over a lower dimensional scroll. Otherwise, because our spaces are chosen generically, the $(k-1)$-planes will not intersect, and our scroll will fiber over $\mathbb{P}^1$.

The association between rational scrolls and rational curves in $G(k,n)$ arises via the subbundle K given above.

Proposition 1. *If $K \cong \bigoplus_{i=1}^k \mathcal{O}(a_i)$, then $\mathbb{P}K = \mathbb{P}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_k))$ is abstractly isomorphic to the scroll constructed above.*

Now, a rational curve in $G(k,n)$ will actually correspond to a scroll in $\mathbb{P}^{n-1}$, via the map $\mathbb{P}K \hookrightarrow \mathbb{P}\mathcal{O}^n$. If $n = d + k$, it is the embedding of the projective bundle in the above theorem via $\mathcal{O}_{\mathbb{P}K}(1)$. If $n > d + k$, we compose this with an embedding $\mathbb{P}^{d+k-1} \to \mathbb{P}^{n-1}$, while if $n < d + k$, we compose with a projection.

Proposition 2. *The space of isomorphism classes of scrolls in $\mathcal{O}^n$ of type $a_1, \dots, a_k$ has dimension* $\sum_{i=1}^k n(a_i+1) - \mathrm{Aut}(S_{a_1,\dots,a_k}) = n(d+k) - \sum_{i,j=1}^k \max(0, a_j - a_i + 1)$.

Note that for a balanced scroll, the latter term is k^2.

Corollary 1. dim $\mathrm{Mor}_d(\mathbb{P}^1, G(k,n))/\mathrm{Aut}(\mathbb{P}^1) = nd + k(n-k) - 3$.

Example. Consider rational curves in $G(2,4)$, i.e. degree d families of lines in $\mathbb{P}^3$:

1. If $d = 1$, the induced scroll is simply a plane in $\mathbb{P}^3$, and the space of curves is the space of planes with a choice of covering family of lines.

2. If $d = 2$,

   - a general family will induce a scroll isomorphic to a smooth quadric $S \subset \mathbb{P}^3$ determined by two skew lines, and the corresponding exact sequence is

   $$0 \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O}^4_{\mathbb{P}^1} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \to 0 \tag{2.3}$$

   - a family such that every line passes through a fixed point will induce a scroll isomorphic to a quadric cone $F_2 \subset \mathbb{P}^3$, and the corresponding exact sequence is

   $$0 \to \mathcal{O} \oplus \mathcal{O}(-2) \to \mathcal{O}^4_{\mathbb{P}^1} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \to 0 \tag{2.4}$$

   - a family such that every line is contained in a given plane will induce a scroll isomorphic to $\mathbb{P}^2 \subset \mathbb{P}^3$, and the corresponding exact sequence is

   $$0 \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O}^4_{\mathbb{P}^1} \to \mathcal{O}(2) \oplus \mathcal{O} \to 0 \tag{2.5}$$

3. Similarly, for $d = 3$, $K^\vee$ and J both generically decompose as $\mathcal{O}(1) \oplus \mathcal{O}(2)$, with $K^\vee \cong \mathcal{O} \oplus \mathcal{O}(3)$ the conical case (the scroll is the cone over a cubic) and $J \cong \mathcal{O} \oplus \mathcal{O}(-3)$ the degenerate case (the scroll is $\cong \mathbb{P}^2$).

For general d, the induced scroll will be a ruled surface in $\mathbb{P}^3$ associated to an exact sequence of the form

$$0 \to \mathcal{O}(-a) \oplus \mathcal{O}(a-d) \to \mathcal{O}^4_{\mathbb{P}^1} \to \mathcal{O}(b) \oplus \mathcal{O}(d-b) \to 0 \tag{2.6}$$

for some $a, b \leq \lfloor \frac{d}{2} \rfloor$. The subscroll induced by $\mathcal{O}^{4\vee}_{\mathbb{P}^1} \twoheadrightarrow \mathcal{O}(a)$ is a *minimal directrix* of the scroll (unique if $a < d-a$): $\mathcal{O}^{\oplus 4}_{\mathbb{P}^1} \twoheadrightarrow \mathcal{O}(b)$ has the same property for the dual scroll. The space of rational curves in $G(2,4)$ is stratified by these two parameters. In particular, the locus with directrix of degree a is locally closed, irreducible, and has codimension $d - 2a - 1$, and similarly for dual directrix of degree b.

2.3 Chow, Hilbert, and Kontsevich Spaces

Using the Plücker embedding of the Grassmannian, we can define the Chow variety of degree d cycles and the Hilbert scheme of subschemes with Hilbert polynomial $dm + 1$ just as before. These spaces will generally have the same advantages and disadvantages of the analogous varieties for projective space.

Example. The first interesting case is curves with $d = 2$, i.e. conics in $G(2,4)$, which form a space of dimension 9. There is a natural morphism from the space of conics to $G(3,6)$ via the Plücker embedding, which presents $G(2,4)$ as a quadric hypersurface in $\mathbb{P}^5$: this morphism takes a conic to its spanning plane $\mathbb{P}^2 \subset \mathbb{P}^5$, and extends naturally to the Hilbert scheme. Moreover, it is an isomorphism outside the locus $O \subset G(3,6)$ where the spanning plane is contained within the Plücker embedding, and exhibits the Hilbert scheme $\mathcal{H}$ as the blowup of $G(3,6)$ along this locus. The (normalized) component $\mathcal{C}$ of the Chow variety of connected cycles of degree 2 arises as a small contraction on the boundary of the Hilbert scheme, with the locus of double lines (codimension 3 in the Hilbert scheme) forming a $\mathbb{P}^1$-bundle over its image in the Chow variety. See [3] for more details.

We include an additional observation of Chen and Coskun for future reference: as points in the locus O give planes in the Plücker embedding of $G(2,4)$ and have class $\sigma_{1,1}$ or σ_2 as Chow cycles on $G(2,4)$. This partitions $O \subset G(3,6)$ and the exceptional loci in the Chow and Hilbert schemes into two irreducible components, and provide alternate Chow and Hilbert "compactifications" $\mathcal{C}_{1,1}, \mathcal{C}_2$, $\mathcal{H}_{1,1}$ and $\mathcal{H}_2$ by only blowing up the corresponding component of O.

The Kontsevich space $\overline{M}_{0,0}(G(k,n),d)$ gives a somewhat nicer compactification. As before, it is irreducible with finite quotient singularities, and has a divisor theory similar to that for $\mathbb{P}^{n-1}$. Explicitly, letting

- $H_{1,1}$ be the locus of maps whose images intersect a fixed codimension $(k-1)$-plane in a 2-plane, i.e. the divisor associated to the 2-cocycle $\sigma_{1,1}$, and
- H_2 be the locus of maps whose images intersect a fixed codimension $(k+1)$-plane in a line, i.e. the divisor associated to the 2-cocycle σ_2,

one finds that

Theorem 6 ([22]). *The Picard group of $\overline{M}_{0,0}(G(k,n),d)$ is generated by $H_{1,1}, H_2$, and $\Delta_i, i \leq \frac{d}{2}$.*

Next, letting T be the locus of maps whose image is tangent to a fixed hyperplane, and using the morphism to $\overline{M}_{0,d}/S_d$ as before,

Theorem 7 ([11]). *The nef cone of $\overline{M}_{0,0}(G(k,n),d)$ is the product of the cone generated by $H_{1,1}, H_2$, and T with the image of the nef cone of $\overline{M}_{0,d}/S_d$.*

Example. For $\overline{M}_{0,0}(G(2,4),2)$, the nef cone is a triangular cone 2-1 spanned by the three divisors $H_{1,1}, H_2,$, T, and divisors on its boundary gives rise to interesting birational morphisms (as demonstrated in [3]):

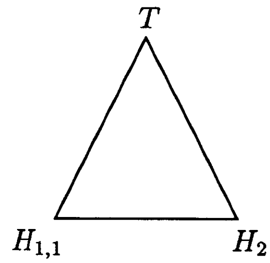

Figure 2-1: Ample cone of the Kontsevich space $\overline{M}_{0,0}(G(2,4),2)$.

- For $D \in (H_{1,1}, H_2)$, ϕ_D contracts the locus of double covers of a line (forgetting the map and remembering the double line only), giving the Chow component $\mathcal{C}$. $\phi_{H_{1,1}}$ itself further contracts the locus of maps whose image is in σ_2, giving the contracted Chow component $\mathcal{C}_2$ (and similarly for ϕ_{H_2}).
- For $D \in (H_{1,1}, T)$, ϕ_D identifies all pairs of lines contained in a given pair of planes in $\Sigma_{1,1}$, and similarly for (H_2, T). ϕ_T contracts the entire boundary by identifying all reducible curves whose node is the same point in $G(2,4)$.

As noted above, the structure of rational scrolls (and thus of rational curves in Grassmannians) is more complicated if $n < k + d$, as it does in the projective case for $n < d + 1$. However, if $n \geq k + d$, we again define D_{deg} to be the set of degenerate maps (whose image spans a space of lower than expected dimension) and D_{unb} to be either

- the locus of maps such that the pullback of the universal subbundle on $G(k,n)$ is not perfectly balanced, if $k|d$, or
- the locus of maps such that the span of the minimal subscroll of the image intersects a fixed codimension qr plane, if $k \nmid d$, where $q = \lceil \frac{d}{k} \rceil$ and $r = qk - d$.

Remark. Explicitly, if $k|d$ and $q = \frac{d}{k}$, D_{unb} will be the image of

$$\overline{M}_{0,0}(F(k-1,k,n),(d-q-1,d)) \to \overline{M}_{0,0}(G(k,n),d) \tag{2.7}$$

where $F(k-1,k,n)$ is the flag variety of $(k-1)$-dimensional subspaces of k-dimensional subspaces of $\mathbb{C}^n$, and the morphism to $\overline{M}_{0,0}(G(k,n),d)$ forgets the first flag. Note that maps from $\mathbb{P}^1$ to $F(k-1,k,n)$ have two degrees, as $H_2(F(k-1,k,n)) \cong \mathbb{Z} \oplus \mathbb{Z}$: these are the degrees of the induced maps to $G(k-1,n)$ and $G(k,n)$. Restricting to the open locus of maps with irreducible source, D_{unb} is the sublocus of maps such that the pullback K of the universal subbundle has a subbundle of rank $k-1$ and degree $-d+q+1$. Equivalently, K has a component of degree $-q-1$, which cannot happen if it splits evenly.

If $k \nmid d$, K never splits evenly, but does generically have a perfectly balanced subbundle of rank r and degree $(q-1)r$, corresponding to the directrix of the associated scroll. Requiring the span of this subscroll to intersect a fixed space of codimension qr gives a divisor D on $\overline{M}_{0,0}(G(r,n), r(q-1))$. We then define $D_{unb} = \pi_{2*}\pi_1^* D$, where π_1 and π_2 are the projections in the diagram

$$\begin{array}{ccc} \overline{M}_{0,0}(F(r,k,n),(r(q-1),d)) & & \\ \downarrow{\scriptstyle \pi_1} & \searrow^{\pi_2} & \\ \overline{M}_{0,0}(G(r,n), r(q-1)) & & \overline{M}_{0,0}(G(k,n),d) \end{array} \tag{2.8}$$

Again, $F(r,k,n)$ denotes the flag variety of r-dimensional subspaces of k-dimensional subspaces of V, and the two morphisms forget the first and second flag respectively. Note that, on the open locus of maps with irreducible source, D_{unb} is precisely as described above.

Example. For $\overline{M}_{0,0}(G(2,4),2)$, D_{deg} is the set of maps whose associated scroll is a plane in $\mathbb{P}^3$, while D_{unb} is the set of maps whose associated scroll is a quadric cone. For $\overline{M}_{0,0}(G(2,5),3)$, D_{deg} is the set of maps whose associated scroll spans a $\mathbb{P}^3$ rather than a $\mathbb{P}^4$, while D_{unb} is generically the set of maps where the directrix of the associated scroll (the join of a line and a conic) intersects a given codimension 2 subspace.

Theorem 8 ([11]). *If $n \geq k+d$, the effective cone is generated by D_{deg}, D_{unb}, and $\Delta_i, i \leq \frac{d}{2}$.*

Example. For $\overline{M}_{0,0}(G(2,4),2)$, the effective cone 2-2 is also triangular, spanned by D_{deg}, D_{unb}, and Δ. In [3], the region between the effective cone and the ample cone is decomposed into seven chambers generated by the given divisors and one non-extremal effective divisor $P = H_{1,1} + H_2 - \frac{1}{2}T$. Geometrically, P is (the closure of) the locus of maps whose associated conic in $\mathbb{P}^5$ (via the Plücker embedding) spans a plane intersecting a fixed $\mathbb{P}^2 \subset \mathbb{P}^5$. One such chamber is the triangular cone spanned by $P, H_{1,1}$, and H_2:

- Divisors in the interior induce rational maps which flip the Kontsevich space to $\mathcal{H}$ over the Chow component $\mathcal{C}$.
- For $D \in (P, H_{1,1})$, ϕ_D flips the space to $\mathcal{H}_2$ over $\mathcal{C}_2$, and analogously for (P, H_2).
- ϕ_P itself contracts the Hilbert scheme to $G(3,6)$.

For the remaining chambers, the stable base loci of divisors in the interior will match those of the adjacent extremal rays of the effective cone, although Chen and Coskun do not construct the associated models explicitly.

Remark. As d increases, the complexity grows rapidly: for $\overline{M}_{0,0}(G(2,5),3)$, Chen and Coskun [3] find fifteen chambers generated by four additional effective divisors, while for $\overline{M}_{0,0}(G(3,6),3)$, they find twenty-two chambers generated by another four divisors. Although the cone decomposition stabilizes for higher k and $n \geq k+d$, the increasing number of extremal rays makes an explicit description intractable.

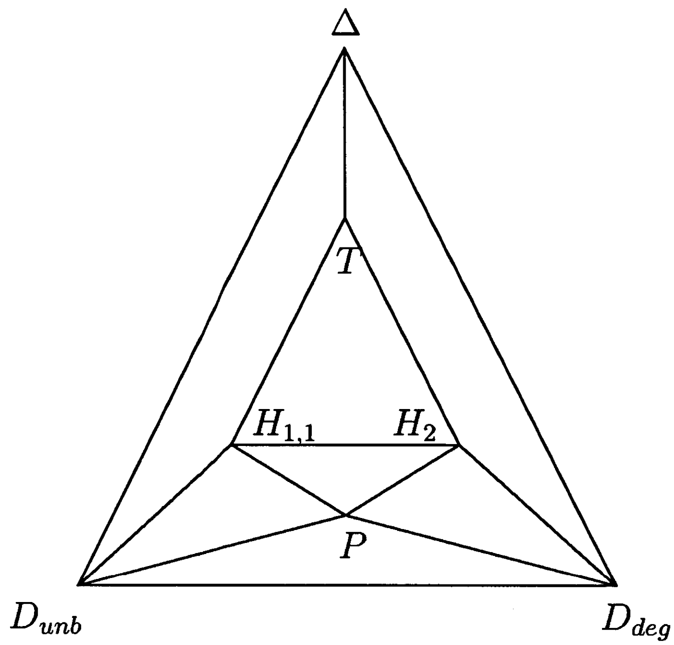

Figure 2-2: Mori chamber decomposition of the Kontsevich space $\overline{M}_{0,0}(G(2,4),2)$.

2.4 Quot Schemes

The Quot scheme compactification of maps to a Grassmannian is not as straightforward as for maps the projective case, so we recall the general definition.

Definition 4. *Given a Noetherian scheme S, a projective scheme X over S, and a coherent sheaf $\mathcal{F}$ on X, the* Quot functor *$\mathcal{Q}uot(\mathcal{F}/X/S) : S - \mathfrak{Sch} \to \mathfrak{Sets}$ assigns to any S-scheme T the set of quotients $\mathcal{F}_T \twoheadrightarrow \mathcal{Z}_T$ ($\mathcal{F}_T$ the pullback of $\mathcal{F}$ to T) which are quasi-coherent and flat over T, with two quotients $\mathcal{F}_T \twoheadrightarrow \mathcal{Z}_T$ and $\mathcal{F}_T \twoheadrightarrow \mathcal{Z}'_T$ equivalent if there is an isomorphism $\mathcal{Z}_T \xrightarrow{\sim} \mathcal{Z}'_T$ commuting with the quotient maps.*

Equivalently, one can define the functor to assign to T the set of short exact sequences $0 \to \mathcal{K}_T \to \mathcal{F}_T \to \mathcal{Z}_T \to 0$, with equivalence based on isomorphisms making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_T & \longrightarrow & \mathcal{F}_T & \longrightarrow & \mathcal{Z}_T & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{=} & & \downarrow{\cong} & & \\ 0 & \longrightarrow & \mathcal{K}'_T & \longrightarrow & \mathcal{F}_T & \longrightarrow & \mathcal{Z}'_T & \longrightarrow & 0 \end{array}$$

Theorem 9. *The functor $\mathcal{Q}uot(\mathcal{F}/X/S)$ is representable by a locally Noetherian scheme* $\mathrm{Quot}(\mathcal{F}/X/S)$*, which is a countable union of projective schemes* $\mathrm{Quot}^{\Phi}(\mathcal{F}/X/S)$ *parametrizing quotients with fixed Hilbert polynomial Φ.*

Let $\mathcal{Q}$ denote the Quot scheme of sheaf quotients $\mathcal{O}^n_{\mathbb{P}^1} \twoheadrightarrow J$ over $\mathbb{P}^1$ with Hilbert polynomial $\chi(J(m)) = (m+1)j + d$, modulo automorphism of kernels: again, $\mathcal{Q}$

equivalently parametrizes short exact sequences $0 \to K \to \mathcal{O}^n \to J \to 0$, with equivalence based on isomorphisms making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \mathcal{O}^n_{\mathbb{P}^1} & \longrightarrow & J & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{=} & & \downarrow{\cong} & & \\ 0 & \longrightarrow & K' & \longrightarrow & \mathcal{O}^n & \longrightarrow & J' & \longrightarrow & 0 \end{array} \tag{2.9}$$

The Quot scheme $\mathcal{Q}$ is a fine moduli space, equipped with a universal sequence

$$0 \to \mathcal{K} \to \mathcal{O}^n_{\mathbb{P}^1 \times \mathcal{Q}} \to \mathcal{J} \to 0 \tag{2.10}$$

over $\mathbb{P}^1 \times \mathcal{Q}$ such that the restriction to a point in $\mathcal{Q}$ is the associated sequence. Moreover, $\mathrm{Hom}_d(\mathbb{P}^1, G(k,n))$ is an open subscheme of $\mathcal{Q}$, and the largest such that this restriction is a sequence of vector bundles. On its boundary, K remains locally free (with rank $k = n - j$ and degree $-d$), but the inclusion into $\mathcal{O}^n$ becomes one of sheaves: that is, the induced map on vector spaces drops rank at certain points of $\mathbb{P}^1$, on which the cokernel J acquires torsion. As K is always locally free, we often consider $\mathcal{Q}$ to parameterize sheaf inclusions $K \hookrightarrow \mathcal{O}^n_{\mathbb{P}^1}$ rather than quotients. Since K decomposes as a sum of line bundles, this allows us to describe points of the Quot scheme explicitly in terms of matrices of polynomials.

Example. For $k = j = 2, n = 4$ (i.e. $G(2,4)$), we find that:

$d = 1$ the map $K = \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{O}^{\oplus 4}_{\mathbb{P}^1}$ is determined by a 4 by 2 matrix with one column of constants (the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}^{\oplus 4}$) and one column of homogeneous linear forms (the inclusion $\mathcal{O}(-1) \hookrightarrow \mathcal{O}^{\oplus 4}$). The rank of such a matrix is at a general point in $\mathbb{P}^1$ is 2: however, if all the 2 by 2 minors (homogeneous linear forms) have a common zero, the rank of the matrix drops at the associated point $p \in \mathbb{P}^1$. In turn, one finds that $J \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}_p$, where $\mathcal{O}_p$ is the skyscraper sheaf at p.

$d = 2$ and $K = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$, the 2 by 2 minors of the inclusion are homogeneous quadratic polynomials which may have a single common zero at p and separate zeroes elsewhere, two common zeroes at p and q, or double common zeroes at p. The associated quotient sheaves J would then be $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}_p$, $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}_p \oplus \mathcal{O}_q$, and $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}_p^2$.

In general, one will obtain quotient sheaves $J \cong \mathcal{O}(b) \oplus \mathcal{O}(b') \oplus E$ where $b \leq a, b' \leq d - a$, and E is a sum of $d - b - b'$ skyscraper sheaves $\mathcal{O}_p$. See [21] for a more detailed exposition.

2.4.1 Construction of the Quot Scheme

Although the global construction of the Quot scheme is complicated in general, in our case (following [24]) we describe $\mathcal{Q}$ as a smooth quotient of an open subset of affine space. Let $\pi_{\mathbb{P}^1}, \pi_{\mathcal{Q}}$ be the two projections of $\mathbb{P}^1 \times \mathcal{Q}$, and set

$$\mathcal{J}_m = \pi_{\mathcal{Q}*}(\mathcal{J} \otimes \pi_{\mathbb{P}^1}^* \mathcal{O}(m-1)) \tag{2.11}$$

Since $\mathcal{J}_m$ is locally free of rank $mj+d$ for $m \geq 0$, there is an associated principal $GL(mj+d)$-bundle X_m equipped with a projection $\rho_m : X_m \twoheadrightarrow \mathcal{Q}$ (via free quotient by $GL(mj+d)$) such that there is a unique isomorphism $\rho_m^* \mathcal{J}_m \cong \mathcal{O}^{mj+d}$ on X_m.

Remark. X_m is analogous to the topological frame bundle associated to a vector bundle. More generally, for E a rank r locally free sheaf over a scheme S, let $\overline{X} = \mathbb{P}(\underline{\mathrm{Hom}}(\mathcal{O}_S^r, E)) \xrightarrow{\pi} S$ be the projective bundle associated to the sheaf of homomorphisms from $\mathcal{O}_S^r$. It is universal with respect to the induced morphism $\zeta : \mathcal{O}_{\overline{X}}^r \to \pi^*(E)$, and there is an open locus $X \subset \overline{X}$ over which ζ is an isomorphism. X is a principal $GL(r)$-bundle over S, and $\pi|_X$ is smooth with connected fibers of dimension r^2. If S is nonsingular, the induced morphism of Chow rings is surjective, with kernel generated by the Chern classes of E (roughly speaking, because the Chern classes measure the failure of E to be already trivial on S).

Set $X = X_0 \times_Q X_1$, and let $\rho : X \to \mathcal{Q}$ be the associated projection:

Theorem 10 ([24]). *X is an open subvariety of affine space.*

Proof. Let Y be the subvariety of the space of pairs of maps

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}^d(-1), \mathcal{O}_{\mathbb{P}^1}^{j+d}) \times \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}^n, \mathcal{O}_{\mathbb{P}^1}^{j+d}) \tag{2.12}$$

(where $\mathcal{O}_{\mathbb{P}^1}^d \cong \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^d$, $\mathcal{O}_{\mathbb{P}^1}^{j+d} \cong \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^{j+d}$) such that first map is injective and the second is surjective onto the cokernel of the first map (both closed conditions). As this cokernel is flat with rank j and degree d, we obtain an induced map $Y \to \mathcal{Q}$ such that the pullback of $\mathcal{J}_0$ and $\mathcal{J}_1$ are trivial. Thus, $Y \to \mathcal{Q}$ factors uniquely through X.

Now, the diagonal embedding $\mathbb{P}^1 \times X \hookrightarrow (\mathbb{P}^1 \times X) \times_X (\mathbb{P}^1 \times X)$ has an associated short exact sequence

$$0 \to \pi_1^* \mathcal{O}_{\mathbb{P}^1 \times X}(-1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1 \times X}(-1) \to \mathcal{O}_{(\mathbb{P}^1 \times X) \times_X (\mathbb{P}^1 \times X)} \xrightarrow{\Delta^*} \mathcal{O}_{\mathbb{P}^1 \times X} \to 0 \tag{2.13}$$

where π_1, π_2 are the two projections. If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^1 \times X$ with $R^1 \pi_{X*}(\mathcal{F}(-1)) = 0$, tensoring by $\pi_1^* \mathcal{F}$ and applying π_{2*} gives

$$0 \to \pi_{2*}(\pi_1^* \mathcal{F}(-1))(-1) \to \pi_{2*} \pi_1^* \mathcal{F} \to \mathcal{F} \to 0 \tag{2.14}$$

By the projection formula, $\pi_{2*} \pi_1^* \mathcal{F} = \pi_X^* \pi_{X*} \mathcal{F}$ (and similarly for $\mathcal{F}(-1)$). Applying this to $\mathcal{F} = (1 \times \rho)^* \mathcal{J}$ and noting that $\pi_{X*} \mathcal{F} \cong \rho^* \mathcal{J}_1 \cong \mathcal{O}_X^{d+j}, \pi_{X*}(\mathcal{F}(-1)) \cong \rho^* \mathcal{J}_0 \cong \mathcal{O}_X^d$, we obtain a diagram

$$0 \to \pi_X^* \mathcal{O}_X^d(-1) \to \pi_X^* \mathcal{O}_X^{d+j} \to (1 \times \rho)^* \mathcal{J} \to 0 \tag{2.15}$$

Of course, there is a natural map $\mathcal{O}^n_{\mathbb{P}^1\times X} \cong \pi_X^*\mathcal{O}_X^n \twoheadrightarrow (1\times\rho)^*\mathcal{J}$, which lifts to a unique map to $\pi_X^*\mathcal{O}_X^{d+j}$: thus, $X \to \mathcal{Q}$ factors uniquely through Y. The maps constructed are clearly inverses, implying the claim. □

Corollary 2 ([24]). *$\mathcal{Q}$ is a unirational, smooth, projective variety of dimension $nd+jk$, with Chow ring generated by the Chern classes of $\mathcal{J}_{-1}$ and $\mathcal{J}_0$.*

Proof. Since X has dimension

$$2d(d+j)+n(d+j) = nd+jk+2d^2+j^2+2dj \tag{2.16}$$

and the projection $X \to \mathcal{Q}$ has relative dimension $d^2+(j+d)^2$, we obtain the given dimension calculation. As noted in the above remark, there is a surjection $A^*(\mathcal{Q}) \to A^*(X) \cong \mathbb{Z}$ with kernel given by the Chern classes of $\mathcal{J}_0$ and $\mathcal{J}_{-1}$, giving the desired statement about the Chow ring. □

Remark. We can extend the action of $GL(d)\times GL(j+d)$ to all of Y, and thus regard $\mathcal{Q}$ as a GIT quotient of Y directly. Indeed, the (semi-)stable locus of the action with equal weighting is precisely X: for $(f,g)\in Y$ and $(\phi,\psi)\in GL(d)\times GL(j+d)$, the action is given by

$$(f,g)\mapsto(\psi^{-1}\circ f\circ\phi, \psi^{-1}\circ g). \tag{2.17}$$

This precisely preserves the induced quotient $\mathcal{O}^n \to$ Coker (f) up to automorphism of the kernel. One-parameter subgroups of this action have fixed points when f is not injective (via an automorphism of Ker $(f)\subset\mathcal{O}^d_{\mathbb{P}^1}$) and when g is not surjective onto the cokernel of f (via an automorphism of Coker $(f)\cap$ Coker $(g)\subset\mathcal{O}^{d+j}_{\mathbb{P}^1}$).

Corollary 3. Pic$(\mathcal{Q})$ *is generated by* $c_1(\mathcal{J}_m)$ *and* $c_1(\mathcal{J}_{m+1})$ *for any integer* $m\geq 0$.

Proof. The statement for $m=0$ follows from the theorem ($c_1(\mathcal{J}_0)$ and $c_1(\mathcal{J}_1)$ are the only Chern classes in degree one), while the general version follows from the pullback of the sequence $0\to\mathcal{O}(-2)\to\mathcal{O}^2(-1)\to\mathcal{O}\to 0$ from $\mathbb{P}^1$ to $\mathcal{Q}\times\mathbb{P}^1$. □

Remark. There are two cases in which these divisors are actually proportional, i.e. $\mathcal{Q}$ has Picard number 1. If $j=n-1$, $\mathcal{Q}$ compactifies maps into $\mathbb{P}^{n-1}$, giving us the projective space $\mathbb{P}^{(d+1)n-1}$ discussed in the last chapter. If $d=0$, we obtain the ordinary Grassmannian $G(k,n)$. Note that, unlike with the Grassmannian, there is a lack of symmetry between j and k. Specifically, if $j=1$ and $k=n-1$, the associated Quot scheme is not a projective space: the generic quotient $\mathcal{O}^n\to J\cong\mathcal{O}(d)$ does give a point in $\mathbb{P}^{(d+1)n-1}$ by duality, but (as we will see below) there will be a locus of codimension one on which J has torsion. Contracting this locus produces the aforementioned projective space, exhibiting the Quot scheme as a blowup.

Going forward, we assume $2\leq j,k\leq n-2$.

2.4.2 Stratification of the Boundary

We now give a stratification of the boundary locus of the Quot scheme, i.e. the points for which the corresponding quotient J has torsion. Let $\mathcal{Q}^{(e)}$ denote the Quot scheme parametrizing quotients of $\mathcal{O}^n_{\mathbb{P}^1}$ with Hilbert polynomial $(m+1)j+d-e$ (thus compactifying $\mathrm{Hom}_{d-e}(\mathbb{P}^1, G(k,n))$). Observe that, on the locus in $\mathcal{Q}$ where the corresponding J has rank e-torsion at some point p, there is a projection to $\mathcal{Q}^{(e)}$ which quotients out this torsion (leaving J to have degree $d-e$). Our goal is to construct the specified locus as a bundle over $\mathcal{Q}^{(e)}$.

Now, as with $\mathcal{Q}$, $\mathcal{Q}^{(e)}$ possess an universal family

$$0 \to \mathcal{K}^{(e)} \to \mathcal{O}^n_{\mathbb{P}^1\times\mathcal{Q}^{(e)}} \to \mathcal{J}^{(e)} \to 0 \tag{2.18}$$

Let $\mathcal{G} = G(\mathcal{K}^{(e)}, e)$ denote the Grassmann bundle of e-dimensional quotients of $\mathcal{K}^{(e)}$. $\mathcal{G}$ is equipped with a structure map ρ to $\mathbb{P}^1 \times \mathcal{Q}^{(e)}$ as well as a universal quotient $\rho^*\mathcal{K}^{(e)} \to \mathcal{F}$. On $\mathbb{P}^1 \times \mathcal{G}$, define $\widetilde{\mathcal{K}}$ to be the kernel of the composition

$$(1\times\rho)^*\mathcal{K}^{(e)} \to \rho^*\mathcal{K}^{(e)}|_{\widetilde{\Delta}} \to \pi^*_{\mathcal{G}}\mathcal{F}|_{\widetilde{\Delta}} \tag{2.19}$$

where $\widetilde{\Delta} = (1\times\rho)^*(\Delta \times \mathcal{Q}^{(e)})$ is the preimage of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{Q}^{(e)}$ and $\pi_{\mathcal{G}} : \mathbb{P}^1 \times \mathcal{G} \to \mathcal{G}$ is the second projection. Since $\widetilde{\Delta}$ intersects each fiber of $\pi_{\mathcal{G}}$ at a single point, $\widetilde{\mathcal{K}}$ is a vector bundle of rank k and degree $-d$, with an inclusion in $\mathcal{O}^n_{\mathbb{P}^1\times\mathcal{G}}$ induced by the one in $(1\times\rho)^*\mathcal{K}^{(e)}$. Thus, we obtain a map $\alpha : \mathcal{G} \to \mathcal{Q}$.

Remark. Pointwise, this map works as follows: $x \in \mathcal{G}$ corresponds to the pair of a $p \in \mathbb{P}^1$ and a rank e quotient $F \cong \mathbb{C}^e$ of E_p for E a subsheaf of $\mathcal{O}^n_{\mathbb{P}^1}$ with rank k and degree $-d+e$. That is, $E_p \twoheadrightarrow F$ is a fiber of the tautological quotient on $\mathcal{G}$. The kernel constructed above is that of the induced map $E \to E_p \twoheadrightarrow F$.

Theorem 11 ([1]). *α surjects onto the locus of points whose associated J has rank at least $j+e$ (i.e. rank e-torsion) at a point $p \in \mathbb{P}^1$, and is an embedding on the open locus $\rho^{-1}(\mathbb{P}^1 \times \mathrm{Hom}_{d-e}(\mathbb{P}^1, G(k,n)))$.*

Proof. For the first assertion, given a point $x \in \mathcal{Q}$ whose associated quotient J has e-torsion at some $p \in \mathbb{P}^1$, let K be the associated subsheaf. Then $\mathcal{O}^{n\vee}_{\mathbb{P}^1} \to K^\vee$ is not surjective, and its cokernel T is torsion of rank at least e at p. Let $E^\vee$ be the cokernel of $K^\vee \twoheadrightarrow T(p) \cong \mathbb{C}^e$: x is the image of the induced point $(p, \mathrm{Coker}\ (K \to E)) \in \mathbb{P}^1 \times \mathcal{G}$.

For the second assertion, let Z be the image of $\rho^{-1}(\mathbb{P}^1 \times \mathrm{Hom}_{d-e}(\mathbb{P}^1, G(k,n)))$ under α: the restriction $\mathcal{O}^{n\vee}_{\mathbb{P}^1\times\mathcal{Q}} \to \mathcal{K}^\vee$ to Z has torsion cokernel $\mathcal{T}$ with rank e on its support. Letting $\mathcal{E}^\vee$ be the kernel of $\mathcal{K}^\vee \to \mathcal{T}$, $\mathcal{E}$ together with the cokernel of $\mathcal{K} \twoheadrightarrow \mathcal{E}$ (a rank e quotient of $\mathcal{E}$) gives our global inverse by the universal property of $\mathcal{G}$. ☐

Corollary 4. *The locus in $\mathcal{Q}$ whose associated J has torsion is codimension $n-k$.*

Proof. $\mathcal{Q}^{(1)}$ has dimension $n(d-1)+k(n-k)$, while $G(\mathcal{K}^{(1)}, 1)$ has $k-1$-dimensional fibers over $\mathbb{P}^1 \times \mathcal{Q}^{(d-1)}$: thus, its total space has dimension $nd + k(n-k) - n + k$. ☐

2.4.3 Generators for the Ample Cone

We denote the complement of this locus by $\mathcal{Q}^0 \subset \mathcal{Q}$: by the corollary, we can perform our divisor constructions on this locus so long as $k, n-k \geq 2$. The universal property of the Grassmannian (as the classifying space for vector bundles) gives an evaluation morphism $\mathrm{ev} : \mathbb{P}^1 \times \mathcal{Q}^0 \to G(k,n)$, which gives an alternate identification of $\mathcal{Q}^0$ as $\mathrm{Hom}_d(\mathbb{P}^1, G(k,n))$. Moveover, the universal short exact sequence $0 \to S \to \mathbb{C}^n \to Q \to 0$ on the Grassmannian pulls back to 2.10 on $\mathbb{P}^1 \times \mathcal{Q}^0$.

Motivated by the construction of Schubert cycles on the Grassmannian, we define the following divisors on the locus $\mathcal{Q}^0$:

- D_1 is set of maps $f \in \mathcal{Q}^0$ such that, for a fixed point $\mathrm{pt} \in \mathbb{P}^1$, $f(p)$ (as a k-plane in V) intersects F^k nontrivially.
- D_2 is set of maps $f \in \mathcal{Q}^0$ such that, for some point $p \in \mathbb{P}^1$, $f(p)$ intersects F^{k+1} nontrivially.
- $D_{1,1}$ is set of maps $f \in \mathcal{Q}^0$ such that, for some point $p \in \mathbb{P}^1$, $f(p)$ intersects F^{k-1} in dimension at least 2.

Using the identification of Schubert cells and Chern classes on the Grassmannian, we obtain:

Theorem 12. *$[D_1] = \pi_{\mathcal{Q}*}(c_1(\mathcal{J})\cdot h)|_{\mathcal{Q}^0}$, where $h = \pi_{\mathbb{P}^1}^* c_1(\mathcal{O}(1))$, while $[D_2] = \pi_{\mathcal{Q}*}(c_2(\mathcal{J}))|_{\mathcal{Q}^0}$ and $[D_{1,1}] = \pi_{\mathcal{Q}*}(c_2(\mathcal{K}))|_{\mathcal{Q}^0}$.*

We thus can complete our divisors to all of $\mathcal{Q}$ as Chern classes (see [20]).

Now, $A^*(\mathbb{P}^1 \times \mathcal{Q}) \cong A^*(\mathcal{Q})[h]/h^2$. Writing $\alpha = \beta + \gamma \cdot h \in A^i(\mathbb{P}^1 \times \mathcal{Q})$ for $\beta \in A^i(\mathcal{Q})$ and $\gamma \in A^{i-1}(\mathcal{Q})$, one sees that

$$\pi_{\mathcal{Q}*}(\alpha) = \gamma, \pi_{\mathcal{Q}*}(\alpha \cdot h) = \beta \tag{2.20}$$

In particular, $c_1(\mathcal{J}) = [D_1] + dh$ ($\mathcal{J}$ generically has degree d on $\mathbb{P}^1$) and $c_2(\mathcal{J}) = \beta + [D_2]h$ for some $\beta \in A^2(\mathcal{Q})$. Applying Grothendieck-Riemann-Roch, we find that

$$\begin{aligned}
\mathrm{ch}(\mathcal{J}_m) &= \pi_{\mathcal{Q}*}(\mathrm{ch}(\mathcal{J} \otimes \mathcal{O}(m-1))\mathrm{Td}(\pi_{\mathcal{Q}})) = \pi_{\mathcal{Q}*}(\mathrm{ch}(\mathcal{J})\mathrm{ch}(\mathcal{O}(m-1))(1+h)) \\
&= \pi_{\mathcal{Q}*}\left(\left(\mathrm{rk}\,(\mathcal{J}) + c_1(\mathcal{J}) + \frac{1}{2}c_1(\mathcal{J})^2 - c_2(\mathcal{J}) + \cdots\right)(1+mh)\right) \\
&= \pi_{\mathcal{Q}*}\left(j + mjh + c_1(\mathcal{J}) + mc_1(\mathcal{J})\cdot h + \frac{1}{2}c_1(\mathcal{J})^2 - c_2(\mathcal{J}) + \cdots\right) \\
&= \underbrace{[mj+d]}_{\text{degree 0}} + \underbrace{[m[D_1] + d[D_1] - [D_2]]}_{\text{degree 1}} + \cdots
\end{aligned} \tag{2.21}$$

Thus, $\mathcal{J}_m$ has rank $mj + d$ and first Chern class

$$(m+d)[D_1] - [D_2] = [D_{1,1}] - (d-m)[D_1], \tag{2.22}$$

since

$$
\begin{aligned}
0 &= \frac{1}{2}c_1(\mathcal{K})^2 - c_2(\mathcal{K}) + \frac{1}{2}c_1(\mathcal{J})^2 - c_2(\mathcal{J}) \\
&= c_1(\mathcal{J})^2 - c_2(\mathcal{K}) - c_2(\mathcal{J}) \\
0 &= 2d[D_1] - [D_{1,1}] - [D_2]
\end{aligned} \tag{2.23}
$$

Next, returning to our sequence 2.10, twisting by $\mathcal{O}(m-1)$, and pushing forward, we obtain the exact sequence

$$
\begin{aligned}
0 &\to \pi_{\mathcal{Q}*}(\mathcal{K} \otimes \pi_{\mathbb{P}^1}^* \mathcal{O}(m-1)) \to \mathcal{O}_{\mathcal{Q}}^n(m-1) \to \mathcal{J}_m \\
&\to R^1\pi_{\mathcal{Q}*}(\mathcal{K} \otimes \pi_{\mathbb{P}^1}^* \mathcal{O}(m-1)) \to 0
\end{aligned} \tag{2.24}
$$

For $m \geq d$, the latter term is zero (on $\mathbb{P}^1$, the degree is ≥ -1 everywhere), and we obtain a morphism γ_m from $\mathcal{Q}$ to the Grassmannian $G(mk-d, mn)$ by taking global sections.

Proposition 3. *For* $m \geq d+1, \gamma_m : \mathcal{Q} \to G(mk-d, mn)$ *is an embedding.*

Proof. γ_m maps a closed point in $\mathcal{Q}$ corresponding to a sequence

$$
0 \to K \to \mathcal{O}_{\mathbb{P}^1}^n \to J \to 0, H^1(\mathcal{Q}, K(m-1)) = 0 \tag{2.25}
$$

to the point in the Grassmannian given by

$$
H^0(\mathcal{Q}, K(m-1)) \subset H^0(\mathcal{Q}, \mathcal{O}^n(m-1)) \tag{2.26}
$$

For $m \geq d$, $H^1(\mathcal{Q}, K(m-1)) = 0$ everywhere, and $K(m-1)$ is generated by global sections and the map is defined globally. Furthermore, for $m \geq d$, $K(m-1)$ itself is the image of the map

$$
H^0(\mathcal{Q}, K(m-1)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}^n(m-1) \tag{2.27}
$$

Thus, we can reconstruct K from its image in the Grassmannian, and γ_m is injective on points. Finally, the differential of γ_m is given by

$$
\mathrm{Hom}_{\mathbb{P}^1}(K, J) \to \mathrm{Hom}(H^0(\mathcal{Q}, K(m-1)), H^0(\mathcal{Q}, J(m-1))) \tag{2.28}
$$

which is injective for the same reason. □

Note that, for $m = d$, $K(m-1)$ still has vanishing H^1, so γ_m will define a morphism: it contracts the locus on which K splits as $\mathcal{O}(-d) \oplus \mathcal{O}^{\oplus k-1}$ (the "most unbalanced locus"), so that

$$
K(m-1) \cong \mathcal{O}(-1) \oplus \mathcal{O}(d-1)^{\oplus k-1} \tag{2.29}
$$

for $m = d$. Moreover, pulling back the ample class on the Grassmannian (the first Chern class of the universal quotient), gives the class

$$c_1(\mathcal{J}_m) = [D_{1,1}] - (d-m)[D_1] \tag{2.30}$$

Thus, $c_1(\mathcal{J}_d) = [D_{1,1}]$ is a semi-ample divisor, while $c_1(\mathcal{J}_{d+1}) = [D_1] + [D_{1,1}]$ is ample (as are $c_1(\mathcal{J}_m)$ for all $m > d$).

Proposition 4. *$[D_1]$ is a semi-ample divisor.*

Proof. We generalize the construction of the Plücker embedding to the Quot scheme (specializing [23]): taking the kth exterior power of $\mathcal{K} \hookrightarrow V_{\mathbb{P}^1 \times \mathcal{Q}}$, twisting by d, and pushing forward under $\pi_{\mathcal{Q}}$ gives an inclusion

$$\mathcal{L} \hookrightarrow \bigwedge^k \mathcal{O}_{\mathcal{Q}}^n \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \cong \mathcal{O}_{\mathcal{Q}}^{\binom{n}{k}(d+1)} \tag{2.31}$$

This induces a morphism $\mathcal{Q} \longrightarrow \mathbb{P}^{\binom{n}{k}(d+1)-1}$ such that the pullback of the hyperplane class is $c_1(\mathcal{L})$: it is an isomorphism on the open locus $\mathcal{Q}^0$, but a contraction on the boundary (thus, $\mathcal{L}$ is not ample). Since

$$\begin{aligned} 1 + c_1(\mathcal{L}) = \mathrm{ch}(\mathcal{L}) &= \pi_{\mathcal{Q}*}(\mathrm{ch}(\wedge^k \mathcal{K}(d))\mathrm{Td}(\pi_{\mathcal{Q}})) = \pi_{\mathcal{Q}*}((1 + c_1(\mathcal{K}))(1+dh)(1+h)) \\ &= \pi_{\mathcal{Q}*}(1 + \underbrace{c_1(\mathcal{K}) + (d+1)h}_{\text{degree one}} + \underbrace{(d+1)c_1(\mathcal{K})h}_{\text{degree two}}) \end{aligned} \tag{2.32}$$

$c_1(\mathcal{L})$ is proportional to $[D_1]$ and thus $[D_1]$ is semi-ample. □

Since the Quot scheme in question has Picard number two, these two semi-ample divisors must generate the ample cone.

There is one more natural operation on the Grassmannian that can be extended to the Quot scheme, namely the duality between $G(k,n)$ and $G(n-k,n)$. Unlike in the degree zero case, given an exact sequence of sheaves

$$0 \to K \to \mathcal{O}^n \to J \to 0 \tag{2.33}$$

over $\mathbb{P}^1$ with J not necessarily locally free, taking duals gives

$$0 \to J^\vee \to \mathcal{O}^{n\vee} \to K^\vee \to \mathcal{J} = \mathrm{Ext}^1(J, \mathcal{O}) \tag{2.34}$$

where $J^\vee$ and $K^\vee$ are now both locally free but the latter object is (non-canonically) isomorphic to the torsion part of J. Thus, we find that dualizing gives a birational map which is an isomorphism on $\mathcal{Q}^0$, and flips the boundary $\mathcal{Q} \setminus \mathcal{Q}^0$ to that of the dual Quot scheme. This gives the fundamental asymmetry of the quasi-map compactification: for instance, the divisor $[D_2]$ will be semi-ample on the dual Quot scheme, but has the boundary as its base locus on the original one.

2.4.4 Generators for the Effective Cone

Following the construction of generators for the effective cone in the Kontsevich space, we separate the cases where $k|d$ and $k \nmid d$:

Proposition 5. *If $k|d$, there is an open locus $\mathcal{Q}^{00} \subset \mathcal{Q}$ where K is perfectly balanced, i.e. $K \cong \mathcal{O}(-q)^k$ for $q = \frac{d}{k}$: its complement, where K is* unbalanced, *is a divisor E.*

Proof. This follows directly from our stated formula for the dimension of the space of associated scrolls, but we give the proof explicitly. First, regardless of the decomposition of K, dim Hom $(K, \mathcal{O}^n) = n(d+k)$: if $K \cong \bigoplus \mathcal{O}(-a_i)$ where $\sum a_i = d$,

$$\text{dim Hom}\,(K, \mathcal{O}^n) = \sum \text{dim Hom}\,(\mathcal{O}(-a_i), \mathcal{O}^n) = \sum n(a_i+1) = n(d+k). \quad (2.35)$$

$\mathbb{Q}^0$ is realized as the open locus of injective homomorphisms modulo automorphisms of K: if K is balanced, $\text{Aut}(K)$ has dimension k^2 ($\text{End}(\mathcal{O}(-q)^k, \mathcal{O}(-q)^k) = \text{Hom}(\mathcal{O}^k, \mathcal{O}^k)$ is the space of nonsingular $k \times k$ matrices). However, the generic unbalanced K decomposes as $\mathcal{O}(-q-1) \oplus \mathcal{O}(-q)^{k-2} \oplus \mathcal{O}(-q+1)$, whose space of automorphisms is

$$\begin{aligned}\text{dim Aut}(K) = \text{dim Aut}(\mathcal{O}(-q-1)) &+ \text{dim Aut}(\mathcal{O}(-q)^{k-2}) \\ &+ \text{dim Aut}(\mathcal{O}(-q+1)) + \text{dim Hom}(\mathcal{O}(-q-1), \mathcal{O}(-q)^{k-2}) \\ &+ \text{dim Hom}(\mathcal{O}(-q-1), \mathcal{O}(-q+1)) + \text{dim Hom}(\mathcal{O}(-q)^{k-2}, \mathcal{O}(-q+1))\end{aligned}$$

since dim $\text{Hom}(\mathcal{O}(a), \mathcal{O}(b)) = 0$ if $b < a$

$$= 1 + (k-2)^2 + 1 + 2(k-2) + 3 + 2(k-2) = k^2 + 1 \quad (2.36)$$

Making K more unbalanced will only increase the dimension of the space of automorphisms, and thus the codimension of the locus in $\mathcal{Q}$. Thus, the locus where K is balanced is open, and its complement has codimension one. □

We can obtain an alternate description of E by generalizing the morphisms γ_m defined above. Although γ_m is not a morphism on $\mathcal{Q}$ for $m < d$, it will be one on $\mathcal{Q}^{00}$ for $m \geq q$: if K is balanced, $K(q-1) = \mathcal{O}(-1)^k$ has vanishing H^1 (and H^0). E is precisely the indeterminacy locus of γ_q, and thus has class $c_1(\mathcal{J}_q) = [D_{1,1}] - (d-q)[D_1]$.

On the other hand, if $k \nmid d$, K generically splits as

$$K \cong \mathcal{O}(-q)^{k-r} \oplus \mathcal{O}(-q+1)^r \quad (2.37)$$

where $q = \lceil \frac{d}{k} \rceil$ and $r = kq - d$. We call this locus the balanced locus, following our notation for scrolls, and denote it $\mathcal{Q}^{00}$ as well. Again, the map γ_q is a morphism on the balanced locus, giving a fiber contraction to the locus of the Grassmannian $G(r, nq)$ parametrizing maps $\mathcal{O}^r \hookrightarrow \mathcal{O}^n(q-1)$. As in the alternate construction of $[D_{1,1}]$, pulling back the ample divisor σ_1 on the Grassmannian gives a divisor $c_1(\mathcal{J}_q) = [D_{1,1}] - (d-q)[D_1]$, which we again denote E.

Remark. This divisor is related to the D_{unb} defined by Coskun and Starr when $k \nmid d$. Indeed, the map $\mathcal{O}^r \hookrightarrow \mathcal{O}^n(q-1)$ is defined by the same polynomials as the minimal subscroll $\mathcal{O}(-q+1)^r \hookrightarrow \mathcal{O}^n$, although the former realizes it as a plane in a larger projective space. Thus, the intersection condition on the subscroll induces a divisor which is a multiple of our divisor E.

Next, repeating this construction on the dual Quot scheme gives an analogous divisor $\check{E} = c_1(\mathcal{K}_p^\vee) = [D_2] - (d-p)[D_1]$, $p = \left\lceil \frac{d}{j} \right\rceil$: this is the divisor of unbalanced $\mathcal{J}^\vee$ if $k|d$, and the pullback of the ample divisor on $G(s, np)$ if $k \nmid d$ $(s = kp - d)$.

Theorem 13. *E and $\check{E}$ are the extremal rays of the effective cone of divisors.*

Proof. Note that if $C, \check{C}$ are moving curves in $\mathcal{Q}$ (i.e. numerically equivalent curves pass through general points) such that

$$C \cdot E = 0, C \cdot \check{E} > 0, \check{C} \cdot E > 0, \check{C} \cdot \check{E} = 0 \tag{2.38}$$

then both E and $\check{E}$ must be extremal rays of the effective cone: since $\mathcal{Q}$ has Picard number two, they are the only ones. By our computation of the classes of E and $\check{E}$, it suffices to find curves C and $\check{C}$ such that

$$C \cdot [D_1] = \check{C} \cdot [D_1] = 1, C \cdot [D_{1,1}] = d - q, \check{C} \cdot [D_2] = d - p \tag{2.39}$$

as such curves will satisfy $C \cdot E = \check{C} \cdot \check{E} = 0$ and

$$\begin{aligned} C \cdot \check{E} &= C \cdot ([D_2] - (d-p)[D_1]) = C \cdot (2d[D_1] - [D_{1,1}] - (d-p)[D_1]) \\ &= p + d - q > 0 \\ \check{C} \cdot E &= \check{C} \cdot ([D_{1,1}] - (d-q)[D_1]) = \check{C} \cdot (2d[D_1] - [D_2] - (d-q)[D_1]) \\ &= q + d - p > 0 \end{aligned} \tag{2.40}$$

We sketch the construction C ($\check{C}$ is obtained similarly): see the following subsection for a more explicit description. A general morphism $f \in \mathcal{Q}^{00}$ corresponds to an inclusion $K \cong \mathcal{O}(-q) \oplus \mathcal{M} \subset \mathcal{O}^n$ with $\mathcal{M}$ balanced. A directrix of the associated scroll (WLOG, induced by the restriction $\mathcal{O}(-q) \hookrightarrow \mathcal{O}^n$) is a degree q rational curve $\mathbb{P}^1 \to \mathbb{P}^{n-1}$. We then extend this curve to a general degree one pencil of degree q rational curves. Adding the subscroll corresponding to $\mathcal{M}$ at each fiber gives a one-parameter family of balanced scrolls of degree d (i.e. a curve in $\mathcal{Q}$) such that one directrix moves and the rest of the scroll (which has degree $d - q$) remains fixed.

Now, intersecting a curve with $[D_1]$ counts the number of fibers of the associated family of scrolls such that, in each such scroll, the preimage of a fixed $p \in \mathbb{P}^1$ intersects a fixed codimension k-plane. In our case, this is the intersection of a general degree one family of $(k-1)$-planes and a codimension k-plane, which is one. Intersection with $[D_{1,1}]$ counts the number of fibers of the family of scrolls such that the preimage of some point $p \in \mathbb{P}^1$ intersects a codimension $(k+1)$-plane: in our case, this depends only on the fixed subscroll, which has degree $d - q$. ☐

2.4.5 Stable Base Locus Decomposition of the Effective Cone

Recall that the stable base locus of a divisor D is the set-theoretic intersection of the base (i.e. indeterminacy) loci of the linear systems $|mD|$ for $m > 0$. We would like to decompose our effective cone into chambers such that every divisor within a chamber has the same stable base locus. We begin by constructing more effective divisors between our ample and effective cones. Repeating the construction above, we obtain a morphism γ_m from $\mathcal{Q}^{00}$ to the Grassmannian $G(mk-d, mn)$ for $m \geq q$, which is an embedding for $m > q$. Thus, we can similarly pull back the ample class from the Grassmannian to obtain an effective divisor $E_m = c_1(\mathcal{J}_m)$ for $q \leq m \leq d$, and compute its class to be $[D_{1,1}] - (d-m)[D_1]$ on $\mathcal{Q}^{00}$. We further define $B_m \subset \mathcal{Q}$ to the closure of the locus where $K \cong \mathcal{O}(-m)\oplus$ terms of degree $\geq -m$.

Example. For $n = 4$ and $k = j = 2$ (compactifying maps to $G(2,4)$), the locally free sheaf K associated to a general closed point splits as $\mathcal{O}\left(-\frac{d}{2}\right) \oplus \mathcal{O}\left(-\frac{d}{2}\right)$ if d is even, and $\mathcal{O}\left(-\frac{d+1}{2}\right) \oplus \mathcal{O}\left(-\frac{d-1}{2}\right)$ otherwise.

- In the former case, twisting $K \hookrightarrow \mathcal{O}^4$ by $\frac{d}{2}$ and taking global sections gives an open embedding $\mathcal{Q}^{00} \hookrightarrow G\left(2, 4\left(\frac{d}{2}+1\right)\right)$ (a space of dimension $2\left(4\left(\frac{d}{2}+1\right)-2\right) = 4d+4 = \dim \mathcal{Q}$): this extends to the locus of points with corresponding $K \cong \mathcal{O}\left(-\frac{d}{2}-1\right) \oplus \mathcal{O}\left(-\frac{d}{2}+1\right)$, which gets mapped to a locus of dimension $4\frac{d}{2}-1 = 2d-1$. Twisting by $\frac{d}{2}-1$, however, leaves one with $K\left(\frac{d}{2}-1\right) \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, which induces the trivial map.

- In the latter case, twisting by $\frac{d+1}{2}$ and taking global sections gives a embedding $\mathcal{Q}^{00} \hookrightarrow G\left(3, 4\left(\frac{d+1}{2}+1\right)\right)$, a space of dimension $6d+3$, which again extends to the locus of points with $K \cong \mathcal{O}\left(-\frac{d+1}{2}-1\right) \oplus \mathcal{O}\left(-\frac{d-1}{2}+1\right)$: this "unbalanced" locus is contracted to a space of dimension $2d-3$. Twisting by one less gives a projection down to $\mathbb{P}^{2d+1}$.

For higher m, one obtains maps from larger domains into other Grassmanians.

Proposition 6. *For $m > q$, B_m has codimension $km - k - d + 1$.*

Proof. This follows directly from our formula for the dimension of the space of associated scrolls, but we again give the proof for bundles explicitly. The codimension of B_m is determined by the minimal dimension of the automorphisms of the associated K. The generic locus on which the lowest degree term of K is $\mathcal{O}(-m)$ is that where the remaining terms (which form a locally free sheaf of rank $k-1$ and degree $m-d$) are as balanced as possible. If $k-1|d-m$, the remaining terms all have the same degree $q_m = \frac{d-m}{k-1}$, and the space of automorphisms is

$$\begin{aligned}
\dim \operatorname{Aut}(K) &= \dim \operatorname{Aut}(\mathcal{O}(-m)) + \dim \operatorname{Aut}\left(\mathcal{O}(-q_m)^{k-1}\right) \\
&\quad + \dim \operatorname{Hom}\left(\mathcal{O}(-m), \mathcal{O}(-q_m)^{k-1}\right) \\
&= 1 + (k-1)^2 + (k-1)(-q_m + m + 1) \\
&= k^2 - (k-1)q_m + (k-1)(m-1) \\
&= k^2 - d + km - k + 1
\end{aligned} \tag{2.41}$$

so

$$\dim B_m = n(d+k) - k^2 + d - km + k - 1 = \dim \mathcal{Q} - (km - k - d + 1) \tag{2.42}$$

as desired. If $k-1 \nmid d-m$, set

$$q_m = \left\lceil \frac{d-m}{k-1} \right\rceil, r_m = (k-1)q_m - d + m. \tag{2.43}$$

Then

$$K \cong \mathcal{O}(-m) \oplus \underbrace{\mathcal{O}(-q_m) \oplus \mathcal{O}(-q_m)}_{k-r_m-1} \oplus \underbrace{\mathcal{O}(-q_m+1) \oplus \mathcal{O}(-q_m+1)}_{r_m} \tag{2.44}$$

and the space of automorphisms is

$$\begin{aligned}
\dim \operatorname{Aut}(K) =& \dim \operatorname{Aut}(\mathcal{O}(-m)) \\
&+ \dim \operatorname{Aut}\left(\mathcal{O}(-q_m)^{k-r_m-1} \oplus \mathcal{O}(-q_m+1)^{r_m}\right) \\
&+ \dim \operatorname{Hom}\left(\mathcal{O}(-m), \mathcal{O}(-q_m)^{k-r_m-1}\right) \\
&+ \dim \operatorname{Hom}\left(\mathcal{O}(-m), \mathcal{O}(-q_m+1)^{r_m}\right) \\
=& 1 + (k-1)^2 + (k - r_m - 1)(-q_m + m + 1) \\
&+ r_m(-q_m + m + 2) \\
=& k^2 - q_m(k-1) + r_m + km - m - k + 1 \\
=& k^2 - d + km - k + 1
\end{aligned} \tag{2.45}$$

giving the same dimension count. □

Theorem 14. *The stable base locus of E_m is B_{m+1} (where $B_{d+1} = \varnothing$), and defines a morphism on its complement in $\mathcal{Q}$. Furthermore, this morphism is a contraction on $B_m \setminus B_{m-1}$, and an embedding on the complement of B_m.*

Proof. Per the construction above, the map associated to E_m takes a point in the Quot scheme associated to a sequence $0 \to K \to \mathcal{O}_{\mathbb{P}^1}^n \to J \to 0$, twists by $m-1$, and takes global sections to get a point in $G(mk-d, mn)$. Given a general point in B_{m+1}, the associated sequence will have $K \cong \mathcal{O}(-m-1) \oplus K'$: thus, twisting by $m-1$ and taking global sections will not give a short exact sequence since

$$H^1(\mathcal{O}(-2) \oplus K'(m-1)) \neq 0 \tag{2.46}$$

Outside this locus, no line bundle in the decomposition of K will have degree $< -m$, so $H^1(K)$ vanishes, implying that the morphism is well-defined. Next, given a general point in B_m, the associated sequence will have $K \cong \mathcal{O}(-m) \oplus K'$: twisting by $m-1$ and taking global sections will ignore the contribution of $\mathcal{O}(-m) \hookrightarrow \mathcal{O}^n$, i.e. identifying all inclusions $K \hookrightarrow \mathcal{O}^n$ with the same restriction $K' \hookrightarrow \mathcal{O}^n$. Finally, outside B_m, K has no components of degree $< -m+1$, so the associated twist is globally generated and we get the desired embedding. □

We can demonstrate the same result numerically by constructing curves C_m whose class covers the loci B_m: these curves will allow us to extend this result to the whole effective cone. Let $\phi : \mathbb{P}^1 \to G(k-1, n)$ be a fixed generic morphism of degree $d-m$, with $\mathcal{M} \hookrightarrow \mathcal{O}^n$ the pullback of the universal subbundle, and let $\psi_1, \psi_2 : \mathbb{P}^1 \to G(1, n)$ be fixed generic morphisms of degree m. Interpolating between the maps ψ_1 and ψ_2 induces a morphism $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to G(1, n) \cong \mathbb{P}^{n-1}$ of bi-degree $(1, m)$ such that

$$\psi((a:b),(c:d)) = a\psi_1(c:d) + b\psi_2(c:d) \tag{2.47}$$

Finally, joining with ϕ via the product map $G(1,n) \times G(k-1,n) \to G(k,n)$, we obtain a morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to G(k,n)$ of bi-degree $(d, 1)$, thus giving a curve $C_m \subset \mathrm{Mor}_d(\mathbb{P}^1, G(k,n)) = \mathcal{Q}^0$.

Remark. Note that the pullback of the universal subbundle restricted to $\{(a:b)\} \times \mathbb{P}^1$ is $\mathcal{O}(-m) \oplus \mathcal{M}$, with the inclusion in $\mathcal{O}^n$ varying over $(a:b)$ only in the map from the first term. Since the morphisms were chosen generically, the class of C_m will cover an open locus of B_m and thus all of it.

Proposition 7. *$C_m \cdot [D_{1,1}] = d - m$ and $C_m \cdot [D_1] = 1$, implying that $C_m \cdot [E_m] = 0$.*

Proof. Recalling our construction of geometric divisor classes (and changing to projective language), $[D_1]|_{\mathcal{Q}^0}$ is the space of maps such that the image of a fixed point in $\mathbb{P}^1$ (as a $(k-1)$-plane in $\mathbb{P}^{n-1}$) intersects a fixed codimenison k-plane F^k. By universal property, the curve C_m induces a family $U_m \subset \mathbb{P}^{n-1}$ over $\mathbb{P}^1 \times \mathbb{P}^1$ such that the fiber over $(a:b) \times \{(c:d)\}$ is the projective $(k-1)$-plane $\mathrm{Span}\{\psi(a:b, c:d), \phi(c:d)\}$. Fixing $(c:d)$ and allowing $(a:b)$ to vary gives a degree 1 family of $(k-1)$-planes, which generically intersects a codimension k plane once. Thus $C_m \cdot [D_1] = 1$.

Similarly, $[D_{1,1}]|_{\mathcal{Q}^0}$ is the space of maps such that the image of some point p intersects a fixed codimension $(k+1)$-plane F^{k+1} nontrivially. We need to show that

$$\{(a:b) \in \mathbb{P}^1 \mid \mathrm{Span}(\psi(a:b, c:d), \phi(c:d)) \cap F^{k+1} \neq \emptyset \text{ for some } (c:d) \in \mathbb{P}^1\} \tag{2.48}$$

(finite by dimension count) is generically of size $d-m$. Because the subscroll induced by ϕ is fixed over $(a:b) \in \mathbb{P}^1$, we can perform our intersection computations on the degeneration where the subscroll becomes a union of $d-m$ general $k-1$-planes (i.e. scrolls of degree 1). Restricting to each such $(k-1)$-plane (WLOG, the interpolation of two $(k-2)$-planes Λ_1, Λ_2), we thus reduce to showing that generically

$$\{(a:b) \in \mathbb{P}^1 \mid \mathrm{Span}(\psi(a:b, c:d), c\Lambda_1 + d\Lambda_2) \cap F^{k+1} \neq \varnothing \text{ for some } (c:d) \in \mathbb{P}^1\} \tag{2.49}$$

is generically of size one. But this family has degree 1 over $(a:b)$, and since we are only counting values of (a,b) over which intersections exist (not the number of intersection points when varying $(c:d)$), we have the result. ☐

Theorem 15. *The stable base locus of any divisor in the chamber between E_m and E_{m+1} is B_{m+1}.*

Proof. Since any divisor in that chamber is a positive linear combination of E_m and the semi-ample divisor $[D_{1,1}]$, its stable base locus must be contained within that of E_m, i.e. B_{m+1}. Now, write such a divisor as $E_m + \varepsilon[D_1]$ for $\varepsilon \in (0,1)$: intersecting with the class of the curve C_{m+1} gives $-1+\varepsilon < 0$, so C_{m+1} must be contained within the base locus of the divisor. But C_{m+1} covers B_{m+1}, so we are done. □

We can repeat our construction of divisors on the dual Quot scheme: ${\mathcal{O}^n_{\mathbb{P}^1\times\mathcal{Q}}}^\vee \to \mathcal{K}^\vee$ is now a map to a locally free rank k, degree d bundle, and we can define twisted projections $\mathcal{K}^\vee_m = \pi_{\mathcal{Q}*}(\mathcal{K}^\vee \otimes \pi^*_{\mathbb{P}^1}\mathcal{O}(m-1))$ of rank $mk+d$. We compute their first Chern classes just as before, using the fact that

$$\begin{aligned} \mathrm{ch}(\mathcal{K}^\vee) = 2k - \mathrm{ch}(\mathcal{K}) \implies c_1(\mathcal{K}^\vee) &= -c_1(\mathcal{K}) = c_1(\mathcal{J}) \\ c_2(\mathcal{K}^\vee) &= c_1^2(\mathcal{K}) - c_2(\mathcal{K}) = c_2(\mathcal{J}) \end{aligned} \tag{2.50}$$

to obtain precisely symmetric divisors $\check{E}_m = c_1(\mathcal{K}^\vee_m) = [D_2]-(d-m)[D_1]$ for $p \le m \le d$, where $p = \lceil \frac{d}{j} \rceil$. However, none of these divisors are semi-ample, as they all have the boundary $\mathcal{Q} \setminus \mathcal{Q}^0$ in their base locus. Beyond that, we can again construct loci $\check{B}_m \subset \mathcal{Q}^0$ corresponding to increasingly unbalanced $J^\vee$ (or equivalently, J), as well as curves $\check{C}_m$ that sweep out these loci. The dimension computations are identical, as are the intersections with the corresponding divisors.

Corollary 5. *The divisors E_m for $q \le m \le d$ and $\check{E}_m$ for $p \le m \le d$ form the boundaries of Mori chambers in the effective cone of divisors of $\mathcal{Q}$.*

2.4.6 Mori Chamber Decomposition of the Effective Cone

Roughly speaking, a smooth variety is an *Mori dream space* if:

1. Its ample and effective cones of divisors are finitely generated.

2. The effective cone can be decomposed into a finite set of subcones (called *Mori chambers*) such that, for any effective divisor D, the birational map

$$\phi_D : X \dashrightarrow X_D := \mathrm{Proj}\ R(X,D) = \mathrm{Proj}\ \bigoplus^{a} H^0(X,aD) \tag{2.51}$$

   depends only on the Mori chamber to which it belongs.

3. Each chamber representing a small modification (rather than a divisorial or fiber contraction) is the pre-image of the ample cone of X_D under ϕ_D.

For a more explicit definition and various associated properties, see [18].

Remark. Since a small modification only changes the positivity properties of the divisors with regards to certain curves, it does not change the combinatorial structure of the effective cone: thus, the models X_D arising from small modifications will have the same Mori chamber decomposition, save that the ample cone will be "moved" to a different chamber.

By the proof of the minimal model program, Fano (and log Fano) varieties are always Mori dream spaces, so we verify when this is the case.

Proposition 8. *The tangent bundle $T_{\mathcal{Q}}$ to $\mathcal{Q}$ is $\pi_{\mathcal{Q}*}\mathrm{Hom}(\mathcal{K},\mathcal{Z})$, and the anticanonical divisor of $\mathcal{Q}$ has class*

$$(n+d(2+n))[D_1]-n[D_2]=n[D_{1,1}]-(d(n-2)-n)[D_1]. \tag{2.52}$$

Proof. We first determine the tangent bundle via the diagonal embedding $\Delta:\mathcal{Q}\hookrightarrow\mathcal{Q}\times\mathcal{Q}$. Let $\mathcal{N}=\pi_{\mathcal{Q}\times\mathcal{Q}*}\mathrm{Hom}(\pi_1^*\mathcal{K},\pi_2^*\mathcal{Z})$ where π_1,π_2 are the two projections from $\mathcal{Q}\times\mathcal{Q}\times\mathbb{P}^1$ to $\mathcal{Q}\times\mathbb{P}^1$, and $\pi_{\mathcal{Q}\times\mathcal{Q}}$ is the projection to $\mathcal{Q}\times\mathcal{Q}$. $\mathcal{N}$ is locally free of rank dim $\mathcal{Q}$, and has a global section induced by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1^*\mathcal{K} & \longrightarrow & \mathcal{O}^n_{\mathcal{Q}\times\mathcal{Q}\times\mathbb{P}^1} & \longrightarrow & \pi_1^*\mathcal{Z} & \longrightarrow & 0 \\ & & & \searrow & \downarrow\cong & \searrow & & & \\ 0 & \longrightarrow & \pi_2^*\mathcal{K} & \longrightarrow & \mathcal{O}^n_{\mathcal{Q}\times\mathcal{Q}\times\mathbb{P}^1} & \longrightarrow & \pi_2^*\mathcal{Z} & \longrightarrow & 0 \end{array} \tag{2.53}$$

Furthermore, the image of the diagonal embedding is the intersection of this section with the zero section, and thus the tangent bundle on $\mathcal{Q}$ is $\Delta^*\mathcal{N}\cong\pi_{\mathcal{Q}*}\mathrm{Hom}(\mathcal{K},\mathcal{Z})$.

We use Grothendieck-Riemann-Roch to compute the canonical class $-c_1(T_{\mathcal{Q}})$:

$$\begin{aligned} \mathrm{ch}(\pi_{\mathcal{Q}*}\mathrm{Hom}(\mathcal{K},\mathcal{Z})) &= \pi_{\mathcal{Q}*}(\mathrm{ch}(\mathcal{Z}\otimes\mathcal{K}^\vee)\mathrm{td}(\pi_{\mathcal{Q}})) \\ &= \pi_{\mathcal{Q}*}\Bigg(\left(\mathrm{rk}\ \mathcal{Z}+c_1(\mathcal{Z})+\frac{1}{2}c_1(\mathcal{Z})^2-c_2(\mathcal{Z})+\cdots\right) \\ &\qquad \left(\mathrm{rk}\ \mathcal{K}+c_1(\mathcal{Z})+\frac{1}{2}c_1(\mathcal{Z})^2-c_2(\mathcal{Z})+\cdots\right)(1+h)\Bigg) \\ &= \pi_{\mathcal{Q}*}\Bigg(jk+\underbrace{jkh+nc_1(\mathcal{Z})}_{\text{degree one}} \\ &\qquad +\underbrace{nc_1(\mathcal{Z})h+\left(1+\frac{n}{2}\right)c_1(\mathcal{Z})^2-nc_2(\mathcal{Z})}_{\text{degree two}}+\cdots\Bigg) \end{aligned} \tag{2.54}$$

Thus, rk $T_{\mathcal{Q}}=jk+nd$ and $c_1(T_{\mathcal{Q}})=(n+d(2+n))[D_1]-n[D_2]$. □

Dividing by n, we obtain the $\mathbb{Q}$-divisor $[D_{1,1}]-(d-\frac{2d}{n}-1)[D_1]$. Thus:

Corollary 6. *$\mathcal{Q}$ is Fano if $(n-2)(d-1)\leq 2$, i.e. if $d=1$ or $d=2$ and $n=4$, and log Fano if $\frac{2d}{n}\geq q-1$.*

Remark. If $j=k|d$, then $\frac{2d}{n}=q$ and the canonical divisor is E_{q+1}: as shown above, it gives the first (and only) divisorial contraction, and thus $\mathcal{Q}$ is log Fano. Similarly, if $j=k\nmid d$, $\frac{2d}{n}=q-\frac{r}{k}\in(q,q-1)$, and the anticanonical divisor lives in (E_{q+1},E_q). In this case, there are no divisorial contractions, and again $\mathcal{Q}$ is log Fano. These extend to whenever $q\leq p$, as $nq-n=kq-k+jq-j\leq kq-k+jp-j<2d$.

We have already determined a few Mori chambers explicitly for general parameters k, n, and d.

- If D is in the interior of the ample cone, ϕ_D is an embedding.
- At the boundary of the ample cone, $\phi_{[D_{1,1}]}$ a morphism which contracts the locus where K is maximally unbalanced (a flipping contraction unless $d = 2$).
- Opposingly, $\phi_{[D_1]}$ contracts the J-torsion locus $\mathcal{Q} \smallsetminus \mathcal{Q}^0$ (always a flipping contraction).
- Crossing the wall $[D_1]$ gives a flip $\phi_D, D \in (D_2, D_1)$ over torsion locus (duality on the open locus $\mathcal{Q}^0$, with J-torsion replaced by $K^\vee$-torsion)
- Finally, $[D_2]$ contracts the locus on the flip where $J^\vee$ is maximally unbalanced.

We explore the other chambers explicitly for $n = 4, k = j = 2$: we only describe the chambers defined by the E_i, as those defined by $\check{E}_i$ will be similar.

$d = 1$: There are no additional chambers in the effective cone, as $[D_{1,1}]$ itself is a fiber contraction to $\mathbb{P}^3$.

$d = 2$: There is one additional chamber between $E_1 = E$ and $E_2 = [D_{1,1}]$. ϕ_{E_2} is a divisorial contraction $\mathcal{Q} \to G(2, 8)$, as is ϕ_D for any $D \in (E_1, E_2)$, while ϕ_{E_1} contracts $\mathcal{Q}$ to a point.

$d = 3$: There are no additional chambers, although subscripts of divisors are incremented by one. ϕ_{E_3} is now a flipping contraction $\mathcal{Q} \to G(3, 12)$ whose image is a $\mathbb{P}^9$ bundle over $\mathbb{P}^7$: the base projectivises inclusions $\Gamma(\mathcal{O}(1)) \hookrightarrow \Gamma(\mathcal{O}^4(2))$, while the fibers are inclusions $\Gamma(\mathcal{O}) \hookrightarrow \Gamma(\mathcal{O}^4(2))$ modulo a $PGL(2)$ action. This map contracts the unbalanced locus, which appears as a $\mathbb{P}^{11}$ bundle over $\mathbb{P}^3$ in the Quot scheme, onto its base. For $D \in (E_2, E_3)$, ϕ_D is the flip of the unbalanced locus over this base $\mathbb{P}^3$, and ϕ_{E_2} contracts the space (outside the unbalanced locus) down to the base $\mathbb{P}^7$.

$d = 4$: There is one additional chamber, and the subscripts are incremented again. $\phi_{E_4} = \phi_{[D_{1,1}]}$ is a flipping contraction into $G(4, 16)$: the balanced locus is an open subvariety of $G(2, 12)$ via the "Veronese" embedding $\{\Gamma(\mathcal{O}^2) \hookrightarrow \Gamma(\mathcal{O}^4(2))\} \mapsto \{\Gamma(\mathcal{O}^2(1)) \hookrightarrow \Gamma(\mathcal{O}^4(3))\}$. Similarly, the locus where $K \cong \mathcal{O}(-1) \oplus \mathcal{O}(-3)$ is an open subvariety of a $\mathbb{P}^{12}$ bundle over $\mathbb{P}^7$, and is embedded by ϕ_{E_4} via $\Gamma(\mathcal{O} \oplus \mathcal{O}(2)) \hookrightarrow \Gamma(\mathcal{O}^4(3))$. The locus where $K \cong \mathcal{O} \oplus \mathcal{O}(-4)$, a $\mathbb{P}^{14}$ bundle over $\mathbb{P}^3$, is contracted to its base. This locus in the indeterminacy locus of ϕ_{E_3}, which maps the Quot scheme into $G(2, 12)$ (isomorphically on the balanced locus, contracting the unbalanced locus E_2 to its base $\mathbb{P}^7$). For divisors D in the chamber (E_2, E_3), ϕ_D is this same contraction, while for $D \in (E_3, E_4)$, ϕ_D is the flip of the most unbalanced locus. Finally, ϕ_{E_2} contracts the space to a point.

One can do a similar procedure for higher degrees d, and thereby obtain the Quot scheme itself as a transformation of either projective space (if $k|d$) or a projective bundle over projective space (if $k \nmid d$) in general. Constructing the flips in question is work in progress.

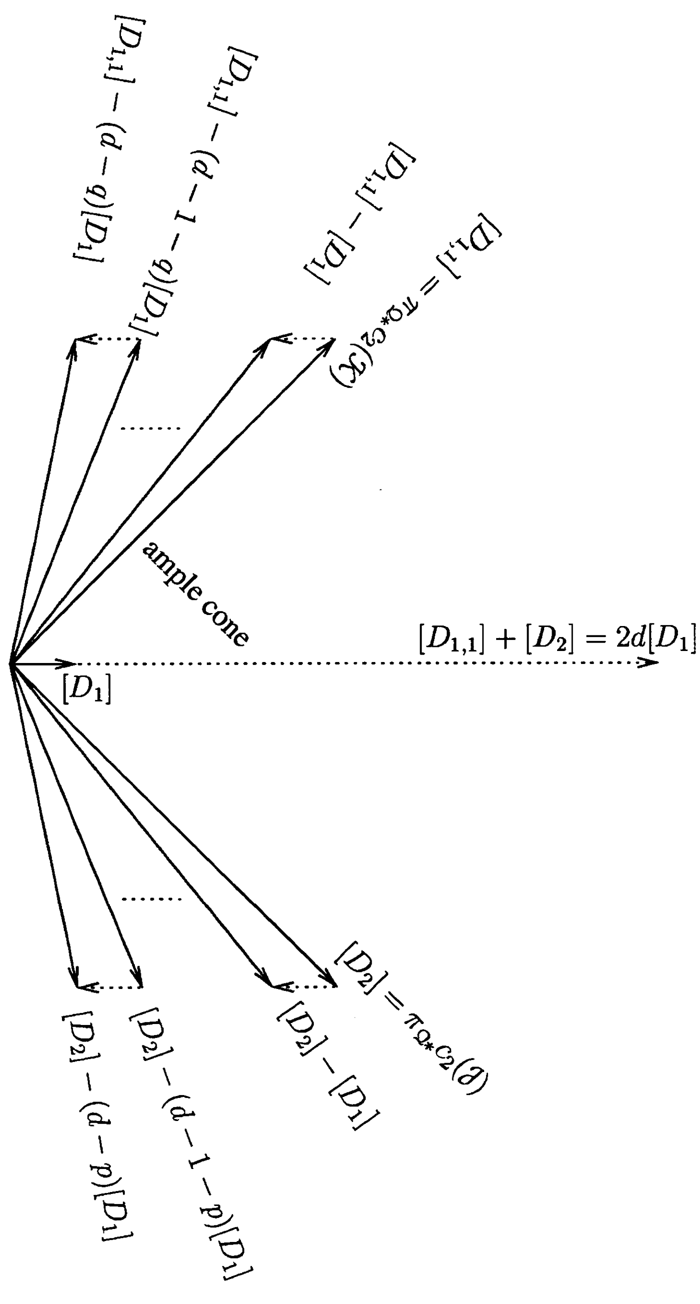

Figure 2-3: The Mori Chamber decomposition of the Quot scheme.

Chapter 3

Rational Curves in Flag Varieties

3.1 Background on Flag Varieties

A (generalized) flag variety is one whose points parametrize *flags* (increasing sequences of subspaces) in a given finite-dimensional vector space $V \cong \mathbb{C}^n$. More precisely, given a natural number n and an increasing sequence of natural numbers

$$\mathbf{k} = \{0 = k_0 < k_1 < \cdots < k_\ell < k_{\ell+1} = n\}, \tag{3.1}$$

- There is a smooth variety $F(\mathbf{k}, n)$ of dimension $\sum_{i=1}^{\ell} k_i(k_{i+1} - k_i)$ parametrizing sequences of inclusions

$$\Lambda_* = \Lambda_1 \subset \cdots \subset \Lambda_\ell \subset V \tag{3.2}$$

  where Λ_i is a subspace of dimension k_i. If $k_i = i$ for each i, this is called the *complete flag variety*, and otherwise a *partial flag variety.*

- There is a natural map from $F(\mathbf{k}, n)$ to the Grassmannian $G(k_i, n)$ given by forgetting all but the i-th piece of the flag. This induces an embedding of the flag variety as the subvariety of the product of Grassmannians $\prod G(k_i, n)$ given by the obvious incidence conditions, and thus an embedding in $\prod \mathbb{P}^{\binom{n}{k_i}-1} \subset \mathbb{P}^{\sum \binom{n}{k_i}-1}$ via the Plücker embedding.

- Like the Grassmannian, $F(\mathbf{k}, n)$ is a homogeneous space: it arises as the quotient of $GL(n)$ by a parabolic subgroup given by nonsingular block upper triangular matrices, whose blocks have size $k_i - k_{i-1}$. This simplifies to the quotient $U(n)/\prod U(k_i - k_{i-1})$ over the complex numbers: in the complete flag case, this is the quotient of $U(n)$ by the maximal torus $T^n = \prod U(1)$.

- $F(\mathbf{k}, n)$ is a fine moduli space for the functor which sends a scheme T to the set of increasing sequences of the form

$$0 \subset S_1 \subset \cdots \subset S_\ell \subset \mathcal{O}_T^n \tag{3.3}$$

where S_i is a rank k_i subbundle of S_{i+1} and $\mathcal{O}_T^n \cong \mathcal{O}_T \otimes V$. Equivalently, it sends T to the set of diagrams of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & S_1 & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & S_\ell & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & Q_\ell & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{3.4}$$

where the S_i and Q_i are subbundles and quotient bundles of $\mathcal{O}_T^n$ of ranks k_i and $n - k_i$ respectively. It has a universal sequence of the form

$$0 \to \widetilde{S}_1 \hookrightarrow \cdots \hookrightarrow \widetilde{S}_\ell \hookrightarrow \mathcal{O}^n_{F(\mathbf{k},n)} \twoheadrightarrow \widetilde{Q}_1 \twoheadrightarrow \cdots \twoheadrightarrow \widetilde{Q}_\ell \to 0 \tag{3.5}$$

which gives the flag variety the obvious universal property. This gives an alternate construction of the morphisms to $G(k_i, n)$ for each i via the universal property of the Grassmannian.

- Let $F = \{F^i\}$ be a fixed complete flag of hyperplanes in $\mathbb{C}^n$ (with codim $F^i = i$). Given a permutation $\omega \in S_n$ such that $\omega(i) < \omega(i+1)$ if i is not one of $k_1, \ldots, k_n$, there is an associated Schubert variety

$$\Sigma_\omega = \{\Lambda_* \in F(\mathbf{k}, n) \,|\, \dim \Lambda_i \cap F^j \geq \#\{i' \leq k_i, \omega(i') > j\} \,\forall i, j\,\}. \tag{3.6}$$

Σ_ω has codimension equal to the *length*

$$\#\{i < j, \omega(i) > \omega(j)\} \tag{3.7}$$

i.e. the number of inversions. As before, there is a corresponding Schubert cycle $\sigma_\omega = \Sigma_\omega^\vee$. In the case of a complete flag variety, we obtain one Schubert variety/cycle for every permutation on n letters.

Remark. To relate this description to our earlier one for the Grassmannian, set

$$\lambda = (\omega(k) - k, \cdots, \omega(1) - 1). \tag{3.8}$$

Since $\omega(i) < \omega(i+1)$ for $i < k$, this is a decreasing sequence of integers between

$n - k$ and 0. Furthermore,

$$\begin{aligned}\#\{i' \leq k, \omega(i') > j\} &= \#\{i' \leq k, i' + \lambda_{k+1-i'} > j\} \\ &= \#\{i' \leq k, k - i' + \lambda_{i'} \geq j\}\end{aligned} \tag{3.9}$$

which equals a given value i when $k - i + \lambda_i = j$, as desired.

Example. Consider the complete flag variety $F((1,2,3),4)$:

- The codimension one Schubert varieties are given by single transpositions $s_i = (i, i+1)$ for $1 \leq i \leq 3$: Σ_{s_i} is the locus of flags where dim $\Lambda_i \cap F^i = 1$.
- The codimension two Schubert varieties are given by pairs of transpositions $\omega = s_i s_j$. If the pair is disjoint, i.e. $|i - j| > 1$, $\Sigma_\omega = \Sigma_{s_i} \cap \Sigma_{s_j}$ (the locus of flags where dim $\Lambda_i \cap F^i =$ dim $\Lambda_j \cap F^j \geq 1$) is irreducible. Otherwise, we have either $j = i - 1$ or $j = i + 1$, and Σ_ω is the locus of flags such that dim $\Lambda_j \cap F^{j+1} \geq 1$ or dim $\Lambda_j \cap F^{j-1} \geq 2$ respectively. These give the two components of the proper intersection $\Sigma_{s_j} \cap \Sigma_{s_j}$.

- The cohomology of the flag variety is freely generated by the Schubert cycles defined above, as well as by the Chern classes of the $\widetilde{S}_i$ and $\widetilde{Q}_i$. More precisely, let L_i be dual of the cokernel of $\widetilde{S}_i \hookrightarrow \widetilde{S}_{i+1}$ (equivalently, the kernel of $\widetilde{Q}_i \twoheadrightarrow \widetilde{Q}_{i+1}$) for $0 \leq i \leq \ell$, where

$$\widetilde{S}_0 = \widetilde{Q}_{\ell+1} = 0, \widetilde{S}_{\ell+1} = \widetilde{Q}_0 = \mathcal{O}^n_{\mathbb{F}(\mathbf{k},n)} \tag{3.10}$$

Next, let x_i, $i = k_i + 1, \ldots, k_{i+1}$ be the set of Chern roots of L_i: that is, the total Chern class of L_i is $c(L_i) = \prod_{j=k_i+1}^{k_{i+1}} (1 + x_i)$. In particular, $c_j(L_i)$ will be the j-th symmetric polynomial in $x_{k_i+1}, \ldots, x_{k_i}$.

Theorem 16. *The cohomology of the flag variety is generated by the Chern classes $c_j(L_i)$ modulo the n symmetric polynomials in $\{x_1, \ldots, x_n\}$, i.e. modulo*

$$\sum_{i=0}^{\ell} c_1(L_i), \sum_{0 \leq i \leq j \leq \ell} c_1(L_i) c_1(L_j) - \sum_{i=0}^{\ell} c_2(L_i), \ldots, \prod_{i=0}^{\ell} c_{k_i}(L_i). \tag{3.11}$$

Schubert cycles correspond to polynomials of Chern roots as follows: given a valid permutation ω, decompose $\omega_0 \omega$ (where $\omega_0 : i \mapsto n - i + 1$ is the order reversing permutation) as a minimal sequence of transpositions $s_{i_1} \cdots s_{i_k}$; then

$$\Sigma_\omega = \partial_{i_k} \circ \cdots \circ \partial_{i_1} \left(\prod_{k=1}^{n-1} x_k^{n-k} \right) \tag{3.12}$$

where

$$\partial_i P = \frac{P(x_1, \ldots, x_n) - P(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}{x_i - x_{i+1}}. \tag{3.13}$$

Example. For the complete flag variety, the L_i are line bundles and $x_i = c_1(L_{i-1})$, so the cohomology ring of F is the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ modulo the symmetric functions in $x_1, \ldots, x_n$. Using the exact sequences

$$0 \to L_i \to Q_i \to Q_{i+1} \to 0, \qquad 0 \to S_i \to S_{i+1} \to L_i^\vee \to 0 \tag{3.14}$$

one finds by induction that the codimension one Schubert variety is

$$\Sigma_{s_i} = \sum_{j \leq i} x_i = -c_1(S_i) = c_1(Q_i) \tag{3.15}$$

while the codimension two Schubert varieties are

$$\Sigma_{s_i s_j} = \begin{cases} \sum_{a \leq i, b \leq j} x_a x_b = c_1(Q_i) \cdot c_1(Q_j) & |i-j| > 1 \\ \sum_{a < b \leq j} x_a x_b = c_2(S_j) & j = i+1 \\ \sum_{a \leq b \leq j} x_a x_b = c_2(Q_j) & j = i-1 \end{cases} \tag{3.16}$$

For a general flag variety,

$$\Sigma_{s_{k_i}} = c_1(Q_i) = -c_1(S_i) = \sum_{j \leq k_i} x_i \tag{3.17}$$

are the codimension one Schubert varieties, while for codimension 2, we have

$$\Sigma_{s_{k_i} s_{k_j}} = \sum_{a \leq k_i, b \leq k_j} x_a x_b = c_1(Q_i) \cdot c_1(Q_j) \tag{3.18}$$

whenever $|k_i - k_j| > 1$, as well as

$$\begin{aligned} \Sigma_{s_{k_i-1} s_{k_i}} &= \sum_{a < b \leq k_i} x_a x_b = c_2(S_i) \\ \Sigma_{s_{k_i+1} s_{k_i}} &= \sum_{a \leq b \leq k_i} x_a x_b = c_2(Q_i) \end{aligned} \tag{3.19}$$

Note that, as in the case of the Quot scheme, $c_2(S_i) = c_1(Q_i)^2 - c_2(Q_i)$.

3.2 Flags of Subscrolls

Just as curves in the Grassmannian are given by rational scrolls in $\mathbb{P}^{n-1}$, curves in flag varieties will be chains of subscrolls.

Definition 5. *A* subscroll *of $S_{a_1,\ldots,a_k}$ is a scroll $S_{b_1,\ldots,b_m} \subset S_{a_1,\ldots,a_k}$ which dominates the base $\mathbb{P}^1$.*

That is, over $\mathbb{P}^1$, the inclusion is given fiberwise as an $(m-1)$-plane in a $(k-1)$-plane. Alternatively, if the containing scroll is given abstractly as $\mathbb{P}K$ for some bundle K, the contained scroll will be $\mathbb{P}K'$ for some subbundle $K' \subset K$.

Remark. Recall that the inclusion $K \subset \mathcal{O}^n$ forced the associated decomposition to be $K \cong \bigoplus \mathcal{O}(-a_i)$ for $a_k \geq \cdots \geq a_1 \geq 0$. If there is a subbundle $K' \subset K$ with decomposition $K' \cong \bigoplus \mathcal{O}(-b_i)$ for $b_m \geq \cdots \geq b_1 \geq 0$, we must further have $a_i \leq b_i$, thereby restricting the set of possible decompositions of K. For instance, if K and K' have ranks 2 and 1 and total degrees 4 and 1 respectively, K can never decompose as $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$.

Proposition 9. *The dimension of the space of subscrolls of the form $S_{b_1,\ldots,b_m} \subset S_{a_1,\ldots,a_k}$ (with $m < k, b_i \geq a_i$) is*

$$\sum_{i=1}^{k}\sum_{j=1}^{m} \max(0, b_j - a_i + 1) - \sum_{j,j'=1}^{m} \max(0, b_j - b_{j'} + 1) \tag{3.20}$$

Now, given a map $\mathbb{P}^1 \to F(\mathbf{k}, n)$, pulling back the universal diagram gives a diagram of sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & \mathcal{O}^n & \longrightarrow & J_i & \longrightarrow & 0 \\ & & \big\downarrow & & \big\| & & \big\downarrow & & \\ 0 & \longrightarrow & K_{i+1} & \longrightarrow & \mathcal{O}^n & \longrightarrow & J_{i+1} & \longrightarrow & 0 \end{array} \tag{3.21}$$

Let d_i be the degree of J_i (so deg $K_i = -d$): then K_i decomposes as $\bigoplus \mathcal{O}(-a_{ij})$ with $\sum_j a_{ij} = d_i$ and $a_{ij} \geq a_{i+1,j}$ for $1 \leq i \leq \ell$, $1 \leq j \leq k_i$. We thus obtain a nested sequence of scrolls

$$S_{a_{1,1},\ldots,a_{1,k_1}} \subset \cdots \subset S_{a_{\ell,1},\ldots,a_{\ell,k_\ell}} \subset \mathbb{P}^{n-1} \tag{3.22}$$

via the inclusions of projectivizations $\mathbb{P}K_1 \subset \cdots \subset \mathbb{P}K_\ell \subset \mathbb{P}\mathcal{O}^n$.

Corollary 7. dim $\text{Mor}_d(\mathbb{P}^1, F(\mathbf{k}, n)) = \dim\ F(\mathbf{k}, n) + \sum_{i=1}^{\ell} d_i(k_{i+1} - k_{i-1})$.

In particular, if $\mathbb{F}$ is the complete flag variety, the space of curves has dimension $\binom{n}{2} + 2\sum d_i - 3$ (after modding by automorphisms of $\mathbb{P}^1$).

Example. Consider rational curves in $\mathbb{F}((1,2),4)$ with multidegree d_1, d_2:

- If $d_1 = d_2 = 1$, a rational curve corresponds to a plane in $\mathbb{P}^3$ (swept out by a family of $\mathbb{P}^1$s) containing a line (with parametrization induced by that of the former family). We thus obtain a bundle over the flag variety $F((2,3),4)$ (just as $\text{Mor}_1(\mathbb{P}^1, G(2,4))$ was a bundle over $G(3,4) \cong \mathbb{P}^{3\vee}$).

- If $d_1 = 1, d_2 = 2$, a general rational curve is a smooth quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ together with a linear section of the $\mathbb{P}^1$ which forms the base of the scroll. That is, this space of rational curves is a $\mathbb{P}^1$ bundle over that of degree 2 rational curves in $G(2,4)$.

- If $d_1 = 2, d_2 = 2$, a general rational curve is a smooth quadric surface together with a degree 2 section of the base. Since the resulting conic is necessarily planar, it generically arises as the intersection of the quadric with a plane.

Remark. Note that, if any $d_i = 0$, the space of maps (and thus any compactification) degenerates to a simpler space or product thereof. In particular, if either d_1 or d_ℓ are zero, we can respectively quotient by or restrict the corresponding fixed flag and obtain a map to a flag variety of lower dimension. If some intermediate d_i is zero, we obtain a map to a product of flag varieties $F((k_1, \ldots, k_{i-1}), k_i) \times F((k_{i+1}-k_i, \ldots, k_\ell - k_i), n-k_i)$. Thus, we assume going forward that all degrees are strictly positive.

3.3 Chow, Hilbert, and Kontsevich Spaces

Combining the embedding of the flag variety into a product of Grassmannians and the Plücker embedding of the Grassmannian, we can define the Chow variety of dimension one cycles and the Hilbert scheme of dimension one subschemes as before. Since

$$H_2(F(\mathbf{k}, n), \mathbb{Z}) \cong \prod H_2(G(k_i, n), \mathbb{Z}) \cong \mathbb{Z}^\ell \tag{3.23}$$

(each piece of the flag has its own degree), these moduli spaces decompose in terms of the multi-degree $\mathbf{d} = (d_1, \ldots, d_\ell)$ of the associated cycle or subscheme. Each of these provide a compactification of the space of maps $\mathbb{P}^1 \to F(\mathbf{k}, n)$ with multidegree $\mathbf{d}$.

Furthermore, we can construct the Kontsevich compactification $\overline{M}_{0,0}(F(\mathbf{k}, n), \mathbf{d})$, which will again be irreducible with finite quotient singularities. Much of the divisor theory carries over from the Grassmannian case: we replace

- $\{H_{1,1}, H_2\}$ with $\{H_{ij} \,|\, i \in \{k_1, \ldots, k_\ell\}, j \in \{k_1, \ldots, \hat{i}, \ldots, k_\ell\} \cup \{i-1, i+1\}\}$, where H_{ij} is the divisor of maps whose image intersects $\Sigma_{s_i s_j}$ nontrivially; and
- $\{\Delta_i\}$ with $\{\Delta_{i_1,\ldots,i_\ell} \,|\, i_j \le \frac{d_j}{2}\}$, where $\Delta_{i_1,\ldots,i_\ell}$ is the divisor of maps whose image has one component of multidegree $i_1, \ldots, i_\ell$ and one of multidegree $d_1 - i_1, \ldots, d_\ell - i_\ell$.

Theorem 17 ([22]). *The Picard group of $\overline{M}_{0,0}(F(\mathbf{k}, n), \mathbf{d})$ is generated by the H_{ij} and the $\Delta_{i_1,\ldots,i_\ell}$.*

Next, let T_i be pullback of the tangency divisor from $\overline{M}_{0,0}(G(k_i, n), d_i)$ (nontrivial if $d_i > 0$), and $\overline{M}_{0,0}(F(\mathbf{k}, n)\mathbf{d}) \to \widetilde{M}_{0,d_1;\ldots;d_\ell} = \overline{M}_{0,d_1+\cdots+d_\ell}//S_{d_1} \times \cdots \times S_{d_\ell}$ the map which intersects each flag with a fixed plane of opposing codimension.

Theorem 18 ([11]). *The nef cone of $\overline{M}_{0,0}(F(\mathbf{k}, n), \mathbf{d})$ is the product of the cone generated by the $\mathcal{H}_{ij}$ and $\mathcal{T}_i$ with the image of the nef cone of $\widetilde{M}_{0,d_1;\ldots;d_\ell}$.*

Note that the dimension of the Picard group and the complexity of the nef cone both grow quite rapidly.

3.4 Hyperquot Schemes

Unlike the Chow, Hilbert, and Kontsevich compactifications, which immediately generalized from projective space to Grassmannians to flag varieties, the Quot scheme compactification used in the previous two chapters needs to be fundamentally modified. Recall that this latter compactification relied on the fact that maps to Grassmannians classify individual vector bundles with certain properties: as mentioned above, maps to flag varieties classify increasing sequences of such bundles. Thus, we need to modify the definition of the Quot scheme (the fundamental tool for forming the quasi-map compactification) to allow for such sequences.

Definition 6. *Given a Noetherian scheme S, a projective scheme X over S, and a coherent sheaf $\mathcal{F}$ on X, the* hyperquot functor

$$\mathcal{HQ}uot(\mathcal{F}/X/S) : S - \mathfrak{Sch} \to \mathfrak{Sets} \tag{3.24}$$

assigns to any S-scheme T the set of flagged quotient sheaves $\mathcal{F}_T \twoheadrightarrow \mathcal{Q}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_\ell$ ($\mathcal{F}_T$ the pullback of $\mathcal{F}$ to T) which are quasi-coherent and flat over T, with two flags of quotients $\mathcal{F}_T \twoheadrightarrow \mathcal{Q}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_\ell$ and $\mathcal{F}_T \twoheadrightarrow \mathcal{Q}'_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}'_\ell$ equivalent if there is are isomorphisms $\mathcal{Q}_i \xrightarrow{\sim} \mathcal{Q}'_i$ commuting with the quotient maps.

Equivalently, one can define the functor to assign to T the set of sequences

$$0 \hookrightarrow \mathcal{K}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{K}_\ell \hookrightarrow \mathcal{F}_T \twoheadrightarrow \mathcal{Q}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_\ell \twoheadrightarrow 0 \tag{3.25}$$

where $\mathcal{K}_i = \text{Ker } (\mathcal{F}_T \twoheadrightarrow \mathcal{Q}_i)$, or diagrams of short exact sequences

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_i & \longrightarrow & \mathcal{F}^n_T & \longrightarrow & \mathcal{Q}_i & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_{i+1} & \longrightarrow & \mathcal{F}^n_T & \longrightarrow & \mathcal{Q}_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{3.26}$$

Equivalence in either case is based on isomorphisms $\mathcal{K}_i \xrightarrow{\sim} \mathcal{K}'_i, \mathcal{Q}_i \xrightarrow{\sim} \mathcal{Q}'_i$ making the obvious diagram commute.

Theorem 19 ([7]). *The functor $\mathcal{HQ}uot(\mathcal{F}/X/S)$ is representable by a locally Noetherian scheme* $\text{HQuot}(\mathcal{F}/X/S)$*, which is a countable union of projective schemes*

$$\text{HQuot}^{\Phi_1,\ldots,\Phi_\ell}(\mathcal{F}/X/S) \tag{3.27}$$

parametrizing flags of quotients with the i-th flag having fixed Hilbert polynomial Φ_i.

We refer to these projective schemes as *hyperquot schemes*: again, although they are very complicated in general, in our case they will behave quite nicely. Specifically, let $\mathcal{HQ}$ denote the hyperquot scheme of flags of sheaf quotients $\mathcal{O}^n_{\mathbb{P}^1} \twoheadrightarrow J_1 \twoheadrightarrow \cdots \twoheadrightarrow J_\ell$ over $\mathbb{P}^1$ with Hilbert polynomial $\chi(J_i(m)) = (m+1)j_i + d_i$, modulo automorphism of kernels: again, $\mathcal{HQ}$ equivalently parametrizes diagrams of short exact sequences of the form $0 \to K_i \to \mathcal{O}^n \to J_i \to 0$ up to isomorphism. As before:

- $\mathcal{HQ}$ is a fine moduli space, with a universal sequence

$$0 \to \mathcal{K}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{K}_\ell \hookrightarrow \mathcal{O}^n_{\mathbb{P}^1 \times \mathcal{HQ}} \twoheadrightarrow \mathcal{J}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{J}_\ell \to 0 \tag{3.28}$$

  over $\mathbb{P}^1 \times \mathcal{HQ}$ such that the restriction to a point in $\mathcal{HQ}$ is the associated sequence over $\mathbb{P}^1$.

- $\mathrm{Hom}_{\mathbf{d}}(\mathbb{P}^1, F(\mathbf{k}, n)) \subset \mathcal{HQ}$ is the largest subscheme over which this sequence is one of vector bundles: on the boundary, the K_i remain locally free of rank $k_i = n - j_i$ and degree $-d_i$, but their inclusions becomes ones of sheaves, and the quotients J_i acquire torsion.

Again, the subsheaves K_i decompose as sums of line bundles $\bigoplus_{j=1}^{k_i} \mathcal{O}(-a_{ij})$ (WLOG, $a_{ij} \le a_{i,j+1}$, so that $a_{ij} \ge a_{i+1,j}$), and we can describe points in the hyperquot scheme as collections of matrices.

Example. We return to our study of rational curves in $F((1,2),4)$. The hyperquot scheme is parameterized by inclusions $K_1 \hookrightarrow K_2 \hookrightarrow \mathcal{O}^4$: since $K_1 \cong \mathcal{O}(-d_1)$ and $K_2 \cong \mathcal{O}(-a) \oplus \mathcal{O}(a - d_2)$ for some $a \le \min(\lfloor \frac{d_2}{2} \rfloor, d_1)$, we find that

- $K_1 \hookrightarrow K_2$ is given by a 2×1 vector A with the first entry a $(d_1 - a)$-form and the second a $\min(0, d_1 - d_2 + a)$-form.

- $K_2 \hookrightarrow \mathcal{O}^4$ is given by a 4×2 matrix B with one column of a-forms and one of $(d_2 - a)$-forms.

As in the Grassmannian case, J_2 will have torsion when B drops rank at certain points, and can acquire torsion up to order d_2. Moreover, J_1 has torsion when the vector of d_1-forms BA drops in rank, i.e. when the forms have a common zero. Note that this does requires either A or B to do so as well, as either the torsion of J_1 surjects onto a torsion component of J_2, or it is in the kernel of $J_1 \twoheadrightarrow J_2$ and thus the cokernel of $K_1 \hookrightarrow K_2$. Moreover, the condition that A drops rank (i.e. the two forms sharing a common factor) occurs in codimension one on the hyperquot scheme: such boundary divisors will occur whenever K_i has rank one less than K_{i+1}.

3.4.1 Construction of the Hyperquot Scheme

Theorem 20 ([17]). *$\mathcal{HQ}$ is an irreducible, unirational, nonsingular, projective variety of dimension* $\dim F(\mathbf{k}, n) + \sum d_i(k_{i+1} - k_{i-1})$.

Proof. As in the case of the regular Quot scheme, we let $\pi_{\mathbb{P}^1}, \pi_{\mathcal{HQ}}$ be the two projections of $\mathbb{P}^1 \times \mathcal{HQ}$, and set

$$\mathcal{J}^i_m = \pi_{\mathcal{Q}*}(\mathcal{J}_i \otimes \pi^*_{\mathbb{P}^1}\mathcal{O}(m-1)) \tag{3.29}$$

For each i, $\mathcal{J}^i_m$ is locally free of rank $mj_i + d_i$ for $m \geq 0$, and there is an associated principal $GL(mj_i + d_i)$-bundle X^i_m equipped with a projection

$$\rho^i_m : X_m \twoheadrightarrow \mathcal{HQ} \tag{3.30}$$

(via free quotient by $GL(mj+d)$) and a unique isomorphism $\rho^{i*}_m\mathcal{J}^i_m \cong \mathcal{O}^{mj_i+d_i}$ on X_m. Construct the product

$$X := \prod_{\mathcal{HQ}}(X^i_0 \times_{\mathcal{HQ}} X^i_1) \tag{3.31}$$

with the induced projection ρ to $\mathcal{HQ}$. We again seek to express this as a subvariety of an abstract space of maps.

Towards this, consider the affine space of 3ℓ-tuples $(\lambda_i, \mu_i, \nu_i)$ of maps in

$$\prod_{i=1}^{\ell} \mathrm{Hom}(\mathcal{O}^{d_i}_{\mathbb{P}^1}(-1), \mathcal{O}^{j_i+d_i}_{\mathbb{P}^1}) \times \mathrm{Hom}(\mathcal{O}^{j_{i-1}+d_{i-1}}_{\mathbb{P}^1}, \mathcal{O}^{j_i+d_i}_{\mathbb{P}^1}) \times \mathrm{Hom}(\mathcal{O}^{d_{i-1}}_{\mathbb{P}^1}, \mathcal{O}^{d_i}_{\mathbb{P}^1}) \tag{3.32}$$

where $j_0 = n$ and $d_0 = 0$, so that the last term is trivial for $i = 1$. Let Y be the open subspace such that each λ_i is injective, each μ_i is surjective onto Coker (λ_i), and Coker $(\pi^*_Y\lambda_i)$ (i.e. the desired $\mathcal{J}_i$) is flat with rank j_i and degree d_i. Finally, we obtain a closed subvariety $Z \subset Y$ of maps such that the following diagram commutes:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow{\scriptstyle \nu_{i-1}} & & \downarrow{\scriptstyle \mu_{i-1}} \\
\mathcal{O}^{d_i}_{\mathbb{P}^1}(-1) & \xrightarrow{\lambda_i} & \mathcal{O}^{j_i+d_i}_{\mathbb{P}^1} \\
\downarrow{\scriptstyle \nu_i} & & \downarrow{\scriptstyle \mu_i} \\
\mathcal{O}^{d_{i+1}}_{\mathbb{P}^1}(-1) & \xrightarrow{\lambda_{i+1}} & \mathcal{O}^{j_{i+1}+d_{i+1}}_{\mathbb{P}^1} \\
\downarrow{\scriptstyle \nu_{i+1}} & & \downarrow{\scriptstyle \mu_{i+1}} \\
\vdots & & \vdots
\end{array} \tag{3.33}$$

Explicitly, Z is a smooth intersection of $\sum_{i=1}^{\ell} 2d_i(j_{i+1} + d_{i+1})$ quadric hypersurfaces defined by the commuting relations $\lambda_i \circ \mu_i = \nu_i \circ \lambda_{i+1}$. Furthermore, the induced sequence of cokernels gives a morphism to $\mathcal{HQ}$ such that the pullback of $\mathcal{J}^i_m$ becomes trivial: by the universal property of the X^i_m, $Z \to \mathcal{HQ}$ factors through X. To invert this map, using the projection formula

$$\pi_X^* \pi_{X*}(1 \times \rho)^* \mathcal{J}_i = \pi_{X*} \rho^* \mathcal{J}_1^i \cong \pi_Y^* \mathcal{O}_X^{d_i + j_i} \tag{3.34}$$

we obtain a diagram

$$\begin{array}{ccccccccc} & & & & & & \vdots & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \pi_X^* \mathcal{O}_X^{d_i}(-1) & \longrightarrow & \pi_X^* \mathcal{O}_X^{d_i + j_i} & \longrightarrow & (1 \times \rho)^* \mathcal{J}_i & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \pi_X^* \mathcal{O}_X^{d_{i+1}}(-1) & \longrightarrow & \pi_X^* \mathcal{O}_X^{d_{i+1} + j_{i+1}} & \longrightarrow & (1 \times \rho)^* \mathcal{J}_{i+1} & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \vdots & & \end{array} \tag{3.35}$$

There exist an induced maps

$$\pi_X^* \mathcal{O}_X^{d_i + j_i} \to \pi_X^* \mathcal{O}_X^{d_{i+1} + j_{i+1}}, \qquad \pi_X^* \mathcal{O}_X^{d_i} \to \pi_X^* \mathcal{O}_X^{d_{i+1}} \tag{3.36}$$

making the diagram commute, and giving the desired inverse by the universal property of Y.

Finally, we obtain the dimension count as

$$\dim X - \dim \prod_{i=1}^{\ell} GL(d_i) \times GL(d_i + r_i) = \dim X - \sum d_i^2 + (j_i + d_i)^2 \tag{3.37}$$

Since

$$\begin{aligned} \dim X &= \sum [2d_i(j_i + d_i) + (j_{i-1} + d_{i-1})(j_i + d_i) + d_{i-1} d_i] - \sum 2d_i(j_{i+1} + d_{i+1}) \\ &= \sum \left[d_i^2 + (d_i + j_i)^2 - j_i^2 + j_{i-1} j_i + d_i(j_{i-1} - j_{i+1}) \right] \end{aligned} \tag{3.38}$$

and $\dim F(\mathbf{k}, n) = \sum k_i(k_{i+1} - k_i) = \sum (j_{i-1} - j_i) j_i$ we are done. □

Corollary 8. Pic($\mathcal{HQ}$) *is generated by the set of Chern classes*

$$\{c_1(\mathcal{J}^i_m) \mid 1 \le i \le \ell, m = a, a+1\} \tag{3.39}$$

for any integer $a \ge 0$.

3.4.2 Stratification of the Boundary

As with the Quot scheme, we would like to stratify the boundary of the hyperquot scheme by loci arising from other hyperquot schemes. Given a sequence of sheaves

$$0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_\ell \hookrightarrow \mathcal{O}^n \twoheadrightarrow J_1 \twoheadrightarrow \cdots \twoheadrightarrow J_\ell \twoheadrightarrow 0 \tag{3.40}$$

on $\mathbb{P}^1$ with the J_i possibly containing torsion, we have rk $J_i = j_i + e_i$ where

1. $e_i + j_i \leq n$ (J_i is a quotient of $\mathcal{O}^n$),
2. $e_i \leq d_i$ (the non-torsion part of J_i must have nonnegative degree), and
3. $j_i + e_i \geq j_{i+1} + e_{i+1}$ (J_i surjects onto J_{i+1}).

Let $\mathbf{e} = (e_1, \ldots, e_\ell) \in \mathbb{Z}^\ell_{\geq 0}$ be an ℓ-tuple of integers satisfying these, and let

- $\mathcal{HQ}^{(\mathbf{e})}$ denote the hyperquot scheme parametrizing flags of quotients with respective Hilbert polynomials $(m+1)j_i + d_i - e_i$,
- $\mathcal{K}_i^{(\mathbf{e})}$ denote its i-th universal subbundle,
- $\mathcal{G}_i$ denote the Grassmann bundle of e_i-dimensional quotients of $\mathcal{K}_i^{(\mathbf{e})}$, with structure map ρ_i, and
- $\mathcal{G} = \prod_{\mathbb{P}^1 \times \mathcal{HQ}^{(\mathbf{e})}} \mathcal{G}_i$ denote their fiber product, with structure map ρ.

On $\mathcal{G}$, there is a universal diagram

$$\begin{array}{ccccccccc} & & & & \vdots & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{J}_i & \longrightarrow & \rho^*\mathcal{K}_i^{(\mathbf{e})} & \longrightarrow & \mathcal{F}_i & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{J}_{i+1} & \longrightarrow & \rho^*\mathcal{K}_{i+1}^{(\mathbf{e})} & \longrightarrow & \mathcal{F}_{i+1} & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \vdots & & & & \end{array} \tag{3.41}$$

where $\mathcal{J}_i$, $\mathcal{F}_i$ are respectively the universal subbundle and quotient of $\mathcal{G}_i$ pulled back to $\mathcal{G}$. Let $\mathcal{V} \subset \mathcal{G}$ be the open subscheme over which the induced maps $\mathcal{J}_i \to \mathcal{O}^n_{\mathcal{V}}$ are injections. Next, let $\mathcal{U} \subset \mathcal{V}$ be the closed subscheme over which the maps $\mathcal{J}_i \to \mathcal{F}_{i+1}$ are trivial. $\mathcal{U}$ can be reconstructed inductively as nested sequences of bundles:

- $\mathcal{G}_1 = Gr(\mathcal{K}_1^{(\mathbf{e})}, e_1)$, with structure map ρ_1 and universal subbundle $\mathcal{J}_1$.
- $\mathcal{G}_2 = Gr(\rho_1^*\mathcal{K}_2^{(\mathbf{e})}/\mathcal{J}_1, e_2)$, now with structure map $\rho_2 : \mathcal{G}_2 \to \mathcal{G}_1$ and universal subbundle $\mathcal{L}_2$: $\mathcal{J}_2$ is the natural extension of $\rho_1^*\mathcal{J}_1$ by $\mathcal{L}_2$.
- $\mathcal{G}_i = Gr(\rho_{i-1}^* \cdots \rho_1^*\mathcal{K}_i^{(\mathbf{e})}/\mathcal{J}_{i-1})$, with structure map $\rho_i : \mathcal{G}_i \to \mathcal{G}_{i-1}$ and universal subbundle $\mathcal{L}_i$: $\mathcal{J}_i$ is the natural extension of $\rho_i^*\mathcal{J}_{1-1}$ by $\mathcal{L}_i$.

Since $\mathbb{P}^1 \times \mathcal{HQ}^{(e)}$ has dimension dim $\mathcal{HQ} - \sum_{i=1}^{\ell} e_i(k_{i+1} - k_{i-1}) + 1$ and $\mathcal{G}_i$ has dimension $e_i(k_i - e_i - k_{i-1} + e_{i-1})$, we find that

Theorem 21. *$\mathcal{U}$ is smooth and irreducible of dimension*

$$\dim \mathcal{HQ} - \sum_{i=1}^{\ell} e_i(k_{i+1} - k_i + e_i - e_{i-1}) + 1 \tag{3.42}$$

On $\mathbb{P}^1 \times \mathcal{U}$, define $\widetilde{\mathcal{K}_i}$ to be the kernel of the composition

$$(1 \times \rho)^* \mathcal{K}_i^{(e)} \to \rho^* \mathcal{K}_i^{(e)}|_{\widetilde{\Delta}} \to \pi_{\mathcal{U}}^* \mathcal{F}_i|_{\widetilde{\Delta}} \tag{3.43}$$

where $\widetilde{\Delta} = (1 \times \rho)^*(\Delta \times \mathcal{HQ}^{(e)})$ is the preimage of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{HQ}^{(e)}$ and $\pi_{\mathcal{G}} : \mathbb{P}^1 \times \mathcal{G} \to \mathcal{G}$ is the second projection. Since $\widetilde{\Delta}$ intersects each fiber of $\pi_{\mathcal{G}}$ at a single point, $\widetilde{\mathcal{K}_i}$ is a vector bundle of rank k and degree $-d$. By the construction of $\mathcal{U}$, we further have inclusions $\widetilde{\mathcal{K}_i} \hookrightarrow \widetilde{\mathcal{K}_{i+1}}$ commuting with the inclusions

$$\widetilde{\mathcal{K}_i} \hookrightarrow (1 \times \rho)^* \mathcal{K}_i^{(e)}, (1 \times \rho)^* \mathcal{K}_i^{(e)} \hookrightarrow (1 \times \rho)^* \mathcal{K}_{i+1}^{(e)} \tag{3.44}$$

By the universal property of $\mathcal{HQ}$, there is a map $\alpha : \mathcal{U} \to \mathcal{Q}$.

Theorem 22 ([6]). *The maps α (1) surjects onto the locus of points $(p, x) \in \mathbb{P}^1 \times \mathcal{HQ}$ whose associated J_is have rank at least $j_i + e_i$ at p and (2) is an embedding when restricted to $\rho^{-1}(\mathbb{P}^1 \times \mathrm{Hom}_{\mathbf{d}-\mathbf{e}}(\mathbb{P}^1, F(\mathbf{k}, n)))$ preimage of the locus where each $\mathcal{J}_i$ has precisely rank $j_i + e_i$ at p and rank j_i elsewhere.*

Proof. For the first assertion, given a point $x \in \mathcal{HQ}$ whose associated quotients J_i have e_i-torsion at some $p \in \mathbb{P}^1$, let K_i be the associated subsheaves, T_i the cokernel of $\mathcal{O}^n \to K_i^\vee$ (torsion of rank at least e_i at p), $E_i^\vee$ the kernel of $K_i^\vee \twoheadrightarrow T_i(p) \cong \mathbb{C}^{e_i}$. Moreover, there are maps inducing a commutative diagram:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_i^\vee & \longrightarrow & K_i^\vee & \longrightarrow & T_i(p) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_{i-1}^\vee & \longrightarrow & K_{i-1}^\vee & \longrightarrow & T_{i-1}(p) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array} \tag{3.45}$$

Then x will be the image of the induced point $(p, \{\mathrm{Coker}\ (K_i \to E_i)\})$.

For the second assertion, let Z be the image of $\rho^{-1}(\mathbb{P}^1 \times \mathrm{Hom}_{d-e}(\mathbb{P}^1, G(k, n)))$ under α: the restriction of $\mathcal{O}_{\mathbb{P}^1 \times \mathcal{Q}}^{n\vee} \to \mathcal{K}_i^\vee$ to Z has torsion cokernel $\mathcal{T}_i$ with rank e on its support. Letting $\mathcal{E}_i^\vee$ be the kernel of $\mathcal{K}_i^\vee \to \mathcal{T}_i$, the $\mathcal{E}_i$ together with the cokernels $\mathcal{F}_i$ of $\mathcal{K}_i \to \mathcal{E}_i$ (a rank e_i quotient by construction) and the kernels $\mathcal{J}_i$ of $\mathcal{E}_i \to \mathcal{F}_i$ give a global commutative diagram

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_i & \longrightarrow & \mathcal{E}_i & \longrightarrow & \mathcal{F}_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{i+1} & \longrightarrow & \mathcal{E}_{i+1} & \longrightarrow & \mathcal{F}_{i+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array} \tag{3.46}$$

and thus global inverse (by the universal property of $\mathcal{U}$). □

Remark. Note that, if $\mathbf{e}$ is the jth-coordinate vector (i.e. $e_i = \delta_{ij}$), $\mathcal{U}$ has dimension $\dim \mathcal{H}\Omega - k_{j+1} + k_j$. As spaces $\mathcal{U}$ of this form cover the boundary of the hyperquot scheme, it will have dimension $\min\{k_{j+1} - k_j\}$. In particular, if there is some j such that $k_{j+1} - k_j = 1$, the boundary will contain a divisor $D_j \subset \mathcal{H}\Omega$, the closure of the image of some $\mathcal{U}$. In the case of the complete flag variety (i.e. $\mathbf{k} = (1, \ldots, n-1)$), we obtain $n-1$ such divisors, which cover the entire boundary locus.

3.4.3 Generators for the Ample Cone

Just as codimension one and two Schubert cycles on the Grassmannian induced divisors on the Quot scheme, so will the analogous cycles on the flag variety give us divisors on the hyperquot scheme. Because our boundary may have divisorial components, however, we give a somewhat more explicit construction. Recall that, in the flag variety $F(\mathbf{k}, n)$, we have ℓ codimension one Schubert cycles $\Sigma_{s_{k_i}} = c_1(Q_i) = -c_1(S_i)$ and

$$\binom{\ell}{2} + 2\ell - \#\{j \in [0, \ell] \mid k_{j+1} - k_j = 1\} \tag{3.47}$$

codimension two Schubert cycles: in the latter case, this counts the classes

$$\Sigma_\omega, \omega = s_{k_i} s_{k_{i'}}, s_{k_i+1} s_{k_i}, s_{k_i-1} s_{k_i} \tag{3.48}$$

discounting the overlaps where $s_{k_{j+1}} s_{k_j} = s_{k_j+1} s_{k_j}$ (whenever $k_{j+1} - k_j = 1$) and the boundary cases $s_{k_i+1} s_{k_i}$ if $k_i = n-1$ and $s_{k_i-1} s_{k_i}$ if $k_i = 1$. As discussed above, each of these is represented as a Chern class (or product thereof), and thus we can define the corresponding divisors

$$\begin{aligned} [D_1^i] &= \pi_{\mathcal{H}\Omega *}(c_1(\mathcal{J}_i) \cdot h) \\ [D_1^{i,i'}] &= \pi_{\mathcal{H}\Omega *}(c_1(\mathcal{J}_i) \cdot c_1(\mathcal{J}_{i'})) \\ [D_2^i] &= \pi_{\mathcal{H}\Omega *}(c_2(\mathcal{J}_i)) \\ [D_{1,1}^i] &= \pi_{\mathcal{H}\Omega *}(c_2(\mathcal{K}_i)) \end{aligned} \tag{3.49}$$

As before, we can write

$$c_1(\mathcal{J}_i) = \alpha_i + d_i h \in H^2(\mathbb{P}^1 \times \mathcal{HQ}, \mathbb{Z}) \cong H^2(\mathcal{HQ}, \mathbb{Z})[h]/h^2 \tag{3.50}$$

and thus find that

$$[D_1^{i,i'}] = \pi_{\mathcal{HQ}*}(d_{i'}\alpha_i h + d_i \alpha_{i'} h) = d_{i'}[D_1^i] + d_i[D_1^{i'}] \tag{3.51}$$

For the other classes, we can more explicitly define them by transporting an alternative form of the degeneracy condition defining the Schubert classes over to the hyperquot scheme. Specifically, take a fixed flag $V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$, and let $\{\mathcal{V}_i = V_1 \otimes \mathcal{O}_{\mathbb{P}^1}\}$ be the induced flag of trivial subbundles. We now define the following loci of points in $\mathbb{P}^1 \times \mathcal{HQ}$:

- $\overline{D_1^i}$ is the locus of points whose corresponding K_i satisfies

$$\text{rk } \{\mathcal{V}_{k_i} \to K_i^\vee\} \leq k_i - 1 \tag{3.52}$$

- $\overline{D_1^{i,i'}} = \overline{D}_i \cap \overline{D}_{i'}$ (for $|k_{i'} - k_i| > 1$),
- $\overline{D_2^i}$ is the locus of points whose corresponding K_i satisfies

$$\text{rk } \{\mathcal{V}_{k_i+1} \to K_i^\vee\} \leq k_i - 1 \tag{3.53}$$

- $\overline{D_{1,1}^i}$ is the locus of points whose corresponding K_i satisfies

$$\text{rk } \{\mathcal{V}_{k_i-1} \to K_i^\vee\} \leq k_i - 2 \tag{3.54}$$

Then $D_1^i = \overline{D_1^i} \cap (\{p\} \times \mathcal{HQ})$ for a fixed $p \in \mathbb{P}^1$ (its Chow class will be independent of choice), while $D_2^i = \bigcup_p \overline{D_2^i} \cap (\{p\} \times \mathcal{HQ})$ (and similarly for the $\overline{D_{1,1}^i}$).

Remark. This latter definition gives us a more concrete understanding of the intersections of these divisors with the boundary of the hyperquot scheme. On a boundary stratum $\mathcal{U}$ given by an ℓ-tuple $\mathbf{e}$, these degeneracy conditions give identical ones on the base variety $\mathbb{P}^1 \times \mathcal{HQ}^{(\mathbf{e})}$. In particular, if $\mathbf{e}$ is the j-th coordinate vector, we obtain a transverse intersection with the boundary if $i, i' \neq j$ and containment otherwise: if our boundary loci are divisorial, the containment becomes an equality. For instance, in the case of the ordinary Quot scheme with $k = n - 1$, $\mathcal{J}$ has rank one but is not locally free, and $\pi_{\mathcal{Q}^*}(c_2(\mathcal{J}))$ gives the locus on which it acquires torsion (i.e. the boundary divisor).

Observe that the conditions defining the loci $\overline{D^i}_1, \overline{D^i}_2$, and $\overline{D^i}_{1,1}$ are equivalent to the conditions defining $[D_1], [D_2]$, and $[D_{1,1}]$ on the Quot scheme corresponding to the Grassmannian $G(k_i, n)$. This implies that each of the divisors is the pullback of its analogous class, and we obtain the induced relation $D_{1,1}^i + D_2^i = 2dD_1^i$. Furthermore, Grothendieck-Riemann-Roch behaves identically on the hyperquot scheme, and we find that

$$
\begin{aligned}
c_1(\mathcal{J}^i_m) &= \pi_{\mathcal{HQ}*}\left(mc_1(\mathcal{J}_i)\cdot h + \frac{1}{2}c_1(\mathcal{J}_i)^2 - c_2(\mathcal{J}_i)\right) \\
&= (m+d_i)[D^i_1] - [D_2] = [D^i_{1,1}] + (m-d_i)[D^i_1]
\end{aligned}
\tag{3.55}
$$

Finally, since the classes $[D^i_1]$ and $[D^i_{1,1}]$ are simply lifted from the divisors $[D_1]$ and $[D_{1,1}]$ on the i-th Quot scheme (via a $\mathbb{P}^1$-fibration), they have the same global sections, and the morphism induced on the hyperquot scheme is simply the composition of the projection to that Quot scheme with the morphisms described in the previous chapter. Thus we obtain:

Theorem 23. *$[D^i_{1,1}]$ and $[D^i_1]$ are semi-ample divisors for each i.*

Assuming each $d_i > 0$, these divisors will be distinct save for $i=1$ when $k_1 = 1$. In this case, $c_2(\mathcal{J}_1) = c_1(\mathcal{J}_1)^2$, so $[D^1_2] = \pi_{\mathcal{HQ}*}(c_1(\mathcal{J}_1)^2) = 2d[D^1_1]$, while $[D^1_{1,1}] = 0$. Thus, we obtain a collection of $2\ell - \delta_{k_1,1}$ semi-ample divisors, a number equal to the Picard number $\rho(\mathcal{HQ})$ (as $c_1(\mathcal{J}^i_m)$ will also be distinct for consecutive m unless $i=1$ and $k_1 = 1$).

Theorem 24. *$\{[D^i_{1,1}]\}$ and $\{[D^i_1]\}$ generate the ample cone of divisors on $\mathcal{HQ}$.*

Proof. Unlike in the case of the Quot scheme, the ample cone will be a cone in a vector space of dimension > 2, and thus can have an arbitrary number of extremal rays. Thus, simply showing that these divisors give extremal rays (which follows from the case of the Quot scheme) is insufficient: we need to construct a facet (codimension one face) corresponding to every subcollection formed by removing one of our divisors. That is, there must be an effective curve which is positive on (i.e. intersects) one of the divisors and is zero on all the remaining ones. To construct this curve, we first give a more explicit realization of the defining curves for the ample cone of the Quot scheme.

Recall first that the locus of the Quot scheme contracted by $[D_{1,1}]$ consisted of points whose corresponding subbundle K decomposed as $\mathcal{O}(-d) \oplus \mathcal{O}^{k-1}$: more precisely, it identifies inclusions $K \hookrightarrow \mathcal{O}^n$, where K has the stated form, which differ only by the choice of degree d rational curve $\mathcal{O}(-d) \hookrightarrow \mathcal{O}^n$. Let $C \in \mathcal{Q}$ be the curve given by the matrix

$$
\begin{pmatrix}
X^d + tY^d & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\tag{3.56}
$$

That is, over each point t, we obtain an inclusion $K \cong \mathcal{O}(-d) \oplus \mathcal{O}^{k-1} \hookrightarrow \mathcal{O}^k$, which we compose with the trivial inclusion $\mathcal{O}^k \hookrightarrow \mathcal{O}^n$. By inspection, $C \cdot [D_{1,1}] = 0$ while $C \cdots [D_1] > 0$ (in the latter case, because the top $k \times k$ minor varies with (s,t)).

Opposingly, we construct a maximally degenerate curve which intersects $[D_{1,1}]$ but not $[D_1]$. Specifically, let K have the same decomposition, and let C' be the curve defined by extending

$$\begin{pmatrix} X^d & t & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3.57}$$

As the only nontrivial minor is independent of t, it is contracted by the Plücker map and thus trivial on $[D_1]$: however, the induced maps $\mathcal{O}^{k-1} \hookrightarrow \mathcal{O}^n$ do vary with t, and $C' \cdot [D_{1,1}] \geq 0$. Thus, these curves define the ample cone of the Quot scheme.

Returning to the hyperquot scheme, we consider the maximally unbalanced locus, i.e. the points where each corresponding K_i decomposes as $\mathcal{O}(-d_i) \oplus \mathcal{O}^{k_i-1}$. Now, let C_i be the curve such that the inclusion of $K_i \hookrightarrow K_{i+1}$ is given by 3.56 composed with the trivial inclusions $\mathcal{O}^{k_i} \subset \mathcal{O}^{k_{i+1}-1} \subset K_{i+1}$, while $K_j \hookrightarrow K_{j+1}$ is similarly constructed using the matrix

$$\begin{pmatrix} X^d & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3.58}$$

By observation, the induced maps $K_j \hookrightarrow \mathcal{O}^n$ are independent of (s,t) for all $j \neq i$: for $j < i$, this inclusion factors through the trivial subsheaf of K_i, while for $j > i$, it does not factor though K_i at all. Thus, C_i is positive on $[D_1^i]$ and zero on the remaining divisors, defining a facet of the ample cone.

If $k_i \neq 1$, we can construct C_i' similarly: it will again live in the maximally unbalanced locus, and the inclusion of $K_j \hookrightarrow K_{j+1}$ will be the same save for $j = i$, where it will be given by the matrix 3.57 defining C' above, and for $j = i-1$ if $k_{i-1} = k_i - 1$, in which case it will be constructed from

$$\begin{pmatrix} t^{-1}X^d & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3.59}$$

This means that $K_{i-1} \hookrightarrow \mathcal{O}^n$ will again be independent of t, and the rest follows as before. This gives the remaining facets of the ample cone. □

Remark. This construction relies on the fact that the maximally unbalanced locus is nontrivial. The existence of intermediate unbalanced loci, which was necessary to determine the chambers of the effective cone, will depend on the parameters.

3.4.4 Generators for the Effective Cone

We begin by determining the choice of ranks k_i and degrees d_i that allow the subsheaves K_i to be balanced. Let $q_i = \lceil \frac{d_i}{k_i} \rceil$, $r_i = q_i k_i - d_i$ as before.

Proposition 10. *The locus where each K_i is balanced is nonempty if, for each i,*

$$q_i \geq \begin{cases} q_{i+1} + 1 & r_i > r_{i+1} \\ q_{i+1} - 1 & r_i = 0, r_{i+1} \geq k_i \\ q_{i+1} & \textit{otherwise.} \end{cases} \tag{3.60}$$

Proof. K_i is balanced if it decomposes as $\mathcal{O}(-q_i+1)^{r_i} \oplus \mathcal{O}(-q_i)^{k_i - r_i}$. If $r_i > r_{i+1}$, we must have some inclusion $\mathcal{O}(-q_i+1) \hookrightarrow \mathcal{O}(-q_{i+1})$. If $r_i = 0$ and $r_{i+1} \geq k_i$, we can all of $K_i = \mathcal{O}(-q_i)^{k_i}$ in the subbundle $\mathcal{O}(-q_{i+1}+1)^{r_{i+1}} \subset K_{i+1}$. In all remaining cases, we must have an inclusion $\mathcal{O}(-q_i) \hookrightarrow \mathcal{O}(-q_{i+1})$. □

Note that this alone does not imply that the projection onto each of the Quot schemes is surjective onto the balanced locus: for instance, if $n = 4, \mathbf{k} = (1,2)$ and $\mathbf{d} = (3,1)$, the hyperquot scheme compactifies planar cubic rational curves (i.e. nodal cubics) in $\mathbb{P}^3$, which clearly does not surject onto the space of twisted cubics. Indeed, in this case J_1 cannot be the balanced bundle $\mathcal{O}(1)^3$, as it must surject onto $\mathcal{O} \oplus \mathcal{O}(1)$. We thus must do a similar procedure on the quotient side: if $p_i = \lceil \frac{d_i}{j_i} \rceil$, $s_i = p_i j_i - d_i$, we have

Proposition 11. *The locus where each J_i is balanced is nonempty if, for each i,*

$$p_i \leq \begin{cases} p_{i+1} - 1 & s_i < s_{i+1} \\ p_{i+1} + 1 & s_{i+1} = 0, s_i \geq k_{i+1} \\ p_{i+1} & \textit{otherwise.} \end{cases} \tag{3.61}$$

This permits the case of the hyperquot scheme with $n = 4, \mathbf{k} = (1,2)$ and $\mathbf{d} = (3,2)$, i.e. rational cubics inside a quadric surface in $\mathbb{P}^3$: since every rational cubic lies in a quadric surface (and every quadric surface contains a rational cubic), this scheme surjects onto both the space of twisted cubics and that of quadric surfaces.

Remark. Combining these two, we obtain bounds on the growth of the degrees d_i given by

$$\frac{j_{i+1}}{j_i}\left(\pm\frac{j_{i+1}}{d_i}\right) \leq \frac{d_{i+1}}{d_i} \leq \frac{k_{i+1}}{k_i}\left(\pm\frac{k_{i+1}}{d_i}\right) \tag{3.62}$$

where the factors in parentheses are contributed by the special cases above. One can see readily that this is satisfied if d_i is some constant d for all i: for any inclusion of sheaves $f : K_i \hookrightarrow \mathcal{O}^n$,

- For $i' < i$, we obtain possible subsheaves by cutting the scroll given by f by general hyperplanes, which produces subscrolls of the same degree.

- For $i' > i$, we set $K_{i'} = K_i \oplus \mathcal{O}^{k_{i'}-k_i}$, and let each inclusion $K_{i'-1} \hookrightarrow K_{i'}$ be the trivial one on the K_i piece and $K_\ell = K_i \oplus \mathcal{O}^{k_\ell - k_i} \hookrightarrow \mathcal{O}^n$ be any extension of f.

Thus, we obtain surjections onto each intermediate Quot scheme. Passing to the extreme values of the bounds, given fixed d_ℓ, we may take

$$d_i = d_{i+1} - q_{i+1}(k_{i+1} - k_i) + \max(0, r_{i+1} - k_i) \tag{3.63}$$

for $i < \ell$, while for fixed d_1, we may take

$$d_i = d_{i-1} - p_{i-1}(j_{i-1} - j_i) + + \max(0, s_{i-1} - j_i) \tag{3.64}$$

for $i > 1$. These sequences give the most increasing and most decreasing possible sequences for $\mathbf{d}$.

Example. For $n = 6$ and $\mathbf{k} = (1, 2, 3, 4, 5)$ (i.e. the complete flag variety), maps from $\mathbb{P}^1$ with uniform degree $d_i = d = 8$ generically give flags of bundles of the form

$$\begin{array}{ccccccccccccccccccccc}
\mathcal{O}(\text{-}8) & \hookrightarrow & \mathcal{O}(\text{-}4) & \hookrightarrow & \mathcal{O}(\text{-}2) & \hookrightarrow & \mathcal{O}(\text{-}2) & \hookrightarrow & \mathcal{O}(\text{-}1) & \hookrightarrow & \mathcal{O} & \twoheadrightarrow & \mathcal{O}(1) & \twoheadrightarrow & \mathcal{O}(2) & \twoheadrightarrow & \mathcal{O}(2) & \twoheadrightarrow & \mathcal{O}(4) & \twoheadrightarrow & \mathcal{O}(8) \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
 & & \mathcal{O}(\text{-}4) & & \mathcal{O}(\text{-}3) & & \mathcal{O}(\text{-}2) & & \mathcal{O}(\text{-}1) & & \mathcal{O} & & \mathcal{O}(1) & & \mathcal{O}(2) & & \mathcal{O}(3) & & \mathcal{O}(4) & & \\
 & & & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & & & \\
 & & & & \mathcal{O}(\text{-}3) & & \mathcal{O}(\text{-}2) & & \mathcal{O}(\text{-}2) & & \mathcal{O} & & \mathcal{O}(2) & & \mathcal{O}(2) & & \mathcal{O}(3) & & & & \\
 & & & & & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & & & & & \\
 & & & & & & \mathcal{O}(\text{-}2) & & \mathcal{O}(\text{-}2) & & \mathcal{O} & & \mathcal{O}(2) & & \mathcal{O}(2) & & & & & & \\
 & & & & & & & & \oplus & & \oplus & & \oplus & & & & & & & & \\
 & & & & & & & & \mathcal{O}(\text{-}2) & & \mathcal{O} & & \mathcal{O}(2) & & & & & & & & \\
 & & & & & & & & & & \oplus & & & & & & & & & & \\
 & & & & & & & & & & \mathcal{O} & & & & & & & & & &
\end{array} \tag{3.65}$$

Note that K_2, K_4, J_2 and J_4 are perfectly balanced (as are K_1 and J_5 trivially), while the remaining sheaves are simply balanced. Fixing $d_5 = 8$ and allowing the degrees to maximally increase, we obtain the vector $\mathbf{d} = (1, 2, 4, 6, 8)$ and a new generic flag of bundles

$$\begin{array}{ccccccccccccccccccccc}
\mathcal{O}(\text{-}1) & \hookrightarrow & \mathcal{O}(\text{-}1) & \hookrightarrow & \mathcal{O}(\text{-}1) & \hookrightarrow & \mathcal{O}(\text{-}1) & \hookrightarrow & \mathcal{O}(\text{-}1) & \hookrightarrow & \mathcal{O} & \twoheadrightarrow & \mathcal{O} & \twoheadrightarrow & \mathcal{O} & \twoheadrightarrow & \mathcal{O}(1) & \twoheadrightarrow & \mathcal{O}(3) & \twoheadrightarrow & \mathcal{O}(8) \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
 & & \mathcal{O}(\text{-}1) & & \mathcal{O}(\text{-}1) & & \mathcal{O}(\text{-}1) & & \mathcal{O}(\text{-}1) & & \mathcal{O} & & \mathcal{O} & & \mathcal{O} & & \mathcal{O}(1) & & \mathcal{O}(3) & & \\
 & & & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & & & \\
 & & & & \mathcal{O}(\text{-}2) & & \mathcal{O}(\text{-}2) & & \mathcal{O}(\text{-}2) & & \mathcal{O} & & \mathcal{O} & & \mathcal{O}(1) & & \mathcal{O}(2) & & & & \\
 & & & & & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & & & & & \\
 & & & & & & \mathcal{O}(\text{-}2) & & \mathcal{O}(\text{-}2) & & \mathcal{O} & & \mathcal{O} & & \mathcal{O}(1) & & & & & & \\
 & & & & & & & & \oplus & & \oplus & & \oplus & & & & & & & & \\
 & & & & & & & & \mathcal{O}(\text{-}2) & & \mathcal{O} & & \mathcal{O}(1) & & & & & & & & \\
 & & & & & & & & & & \oplus & & & & & & & & & & \\
 & & & & & & & & & & \mathcal{O} & & & & & & & & & &
\end{array} \tag{3.66}$$

That is, K_i is the rank i subbundle of K_{i+1} of minimal degree, obtained by removing a piece of maximal degree. Conversely, fixing $d_1 = 8$ and allowing the degrees to maximally decrease, we find that $\mathbf{d} = (8, 6, 4, 2, 1)$ and the flag is

$$
\begin{array}{ccccccccccc}
\mathcal{O}(\text{-}8) \hookrightarrow & \mathcal{O}(\text{-}3) \hookrightarrow & \mathcal{O}(\text{-}1) \hookrightarrow & \mathcal{O} \hookrightarrow & \mathcal{O} \hookrightarrow & \mathcal{O} \twoheadrightarrow & \mathcal{O}(1) \twoheadrightarrow & \mathcal{O}(1) \twoheadrightarrow & \mathcal{O}(1) \twoheadrightarrow & \mathcal{O}(1) \twoheadrightarrow & \mathcal{O}(1) \\
 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \\
 & \mathcal{O}(\text{-}3) & \mathcal{O}(\text{-}1) & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \\
 & & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & & \\
 & & \mathcal{O}(\text{-}2) & \mathcal{O}(\text{-}1) & \mathcal{O} & \mathcal{O} & \mathcal{O}(2) & \mathcal{O}(2) & \mathcal{O}(2) & & \\
 & & & \oplus & \oplus & \oplus & \oplus & \oplus & & & \\
 & & & \mathcal{O}(\text{-}1) & \mathcal{O} & \mathcal{O} & \mathcal{O}(2) & \mathcal{O}(2) & & & \\
 & & & & \oplus & \oplus & \oplus & & & & \\
 & & & & \mathcal{O}(\text{-}1) & \mathcal{O} & \mathcal{O}(2) & & & & \\
 & & & & & \oplus & & & & & \\
 & & & & & \mathcal{O} & & & & &
\end{array}
\tag{3.67}
$$

We concentrate our study of the effective cone on the first two cases for complete flags in $\mathcal{O}^n$ (the third follows from the second by duality). General results for partial flags as well as intermediate sequences $\mathbf{d}$ can be extrapolated from these, but require overly tedious combinatorics.

Uniform $d_i = d$

The simplest case is where $i|d$ for all $i \in [1, n-1]$, i.e. d is a multiple of $\operatorname{lcm}[n-1]$, so that the generic K_i and J_i are perfectly balanced. The associated hyperquot scheme will have dimension $\binom{n}{2} + 2d(n-1)$. For $i > 1$, let E^i be the locus where the sheaf K_i is unbalanced: generically,

$$K_i \cong \mathcal{O}(-q_i - 1) \oplus \mathcal{O}(-q_i + 1) \oplus \mathcal{O}(-q_i)^{i-2}. \tag{3.68}$$

Since $q_{i+1} = \frac{d}{i+1} > \frac{d}{i} = q_i$, the remaining $K_{i'}$ are generically perfectly balanced. Furthermore, a simple computation shows that the dimension of the space of maps $K_i \hookrightarrow K_{i+1}$ and $K_{i-1} \hookrightarrow K_i$ do not change when K_i has this form, but the dimension of $\operatorname{Aut}(K_i)$ increases by one. Thus, E^i will be a divisor. Similarly, there is a divisor $\check{E}^i$ where the sheaf J_i is unbalanced, and the general point in this divisor will have all other $J_{i'}$ perfectly balanced.

Theorem 25. *$[E^i]$ and $[\check{E}^i]$ are extremal rays of the effective cone of divisors.*

Proof. Recall that, for the Quot scheme, we demonstrated extremality by constructing a curve C through a general point which had trivial intersection with E and positive intersection with $\check{E}$. Our goal is to obtain a similar curve C^i for E^i and $\check{E}^i$. First, if $i = n-1$, we can accomplish this directly by lifting via the projection of the hyperquot scheme to the $(n-1)$-st Quot scheme: on the base, the curve C parametrizes a family of inclusions $\mathcal{O}(-q_{n-1})^{n-1} \hookrightarrow \mathcal{O}^n_{\mathbb{P}^1}$, which lifts immediately to a curve C^{n-1}. For $i < n-1$, we consider the locus where K_{i+1} is perfectly balanced, and construct a truncated hyperquot scheme $\mathcal{HQ}^i$ by taking the subsequences $0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_{i+1}$, tensoring by q_{i+1}, and forming all induced cokernels $K_{i+1}(q_{i+1})/K_{i'}(q_{i+1})$ for $i' \leq i$. We can perform the same action globally to obtain a projection morphism from the original hyperquot scheme. We can now do our previous procedure on $\mathcal{HQ}^i$, obtaining a nontrivial family of inclusions

$$\mathcal{O}(-q_i)^i \otimes \mathcal{O}(q_{i+1} \cong \mathcal{O}(q_{i+1}-q_i)^i \hookrightarrow \mathcal{O}^{i+1} \cong \mathcal{O}(-q_{i+1})^{i+1} \otimes \mathcal{O}(q_{i+1}) \tag{3.69}$$

on the Quot scheme where $n = i+1, k = i, d = i(q_i - q_{i+1})$. Lifting this sequence to $\mathcal{HQ}^i$ and then to $\mathcal{HQ}$, we obtain a curve C^i in which K_i (and all other $K_{i'}$) are balanced. □

As with the ample cone, we need to construct a collection of at least $2\ell - \delta 1, k_1 = 2n-3$ extremal rays to describe the cone completely. The above theorem and its dual give $2n-4$ rays generated by $[E^i]$ for $i > 1$ and $[\check{E}^i]$ for $1i < n-1$: for $i = n-1$, we replace $\check{E}^i$ with the boundary divisor (as one does for the Quot scheme with $k = n-1$).

Theorem 26. *The effective cone is generated by the $[E^i]$, the $[\check{E}^i]$, and the $(n-1)$-st boundary divisor.*

Sketch. The construction given above almost gives this immediately, save that the variation of the inclusion of $K_i \hookrightarrow K_{i+1}$ may affect the inclusion of $K_{i'} \hookrightarrow \mathcal{O}^n$ for $i' < i$ and thus whether or not the associated family of cokernels remains balanced. However, since $q_{i-1} > q_i > q_{i+1}$, by dimension count one can always choose a deformation of $K_i \hookrightarrow K_{i+1}$ such that $K_{i-1} \hookrightarrow K_{i+1}$ is fixed: that is, the space of perfectly balanced scrolls of dimension i and degree d which

- contain a fixed, perfectly balanced subscroll of dimension $i-1$ and degree d, and
- are contained in a fixed superscroll of dimension $i+1$ and degree d

is positive dimensional. Choosing each of our families in this manner guarantees that our curves C^i do not intersect $\check{E}^{i'}$ for $i' \neq i$ (as well as the $(n-1)$-st boundary divisor). □

Remark. One can seemingly do virtually the identical construction in the case where d is arbitrary: the K_i will no longer be generally perfectly balanced, but rather just balanced. However, the incidence condition on the directrix (as used for the Quot scheme) will remain independent from other such conditions across the various K_i and J_i. For instance, if $i|d$ but $i+1 \nmid d$, $q_{i+1} - 1 < q_i$, so K_i can become unbalanced while K_{i+1} keeps the same decomposition. Conversely, any incidence condition on the directrix of K_{i+1} cannot force K_i to become unbalanced. Performing an explicit construction of the curves C^i is the subject of ongoing study.

Maximally increasing d_i

We again begin with the case of perfectly balanced splittings: if the degrees d_i increase maximally to some value $d = d_{n-1}$, this implies that $K_i \cong \mathcal{O}(-q)^i$ for $q = \frac{d}{n-1}$, and the hyperquot scheme has dimension $(2q+1)\binom{n}{2}$.

Proposition 12. *If K_i is unbalanced, so is $K_{i'}$ for $i' > i$.*

Proof. Since $i|d_i$ for each i, K_i is unbalanced precisely when it acquires a component of higher (less negative) degree $-q+1$. However, one cannot have an inclusion $\mathcal{O}(-q+1) \hookrightarrow \mathcal{O}(-q)^{i+1}$, so K_{i+1} must also have a component of degree at least $-q+1$. Continuing in this fashion, one finds that $K_{i'}$ has a component of degree at least $-q+1$ for all $i' > i$, and they are all unbalanced. □

Proposition 13. *For each i, the locus where $K_{i'}$ is unbalanced for $i' \geq i$ has codimension one.*

Proof. Note first that, as with the Quot scheme for $k = n-1$, the locus where only K_{n-1} is unbalanced has codimension one: the spaces of maps $\mathrm{Hom}(K_{n-1}, \mathcal{O}^n)$ and $\mathrm{Hom}(K_{n-2}, K_{n-1})$ for general (i.e. perfectly balanced) K_{n-2} have the same dimension, while that of $\mathrm{Aut}(K_{n-1})$ increases by one. Working inductively, we now assume that the locus where $K_{i'}$ is unbalanced for $i' > i$ has codimension one. The generic unbalanced K_{i+1} decomposes as $\mathcal{O}(-q+1) \oplus \mathcal{O}(-q)^{i-1} \oplus \mathcal{O}(-q-1)$: if $K_i \cong \mathcal{O}(-q)^i$, $\dim \mathrm{Aut}(K_i) = i^2$ while $\dim \mathrm{Hom}(K_i, K_{i+1}) = 2i + i(i-1) + 0 = i^2 + i$. However, if K_i is unbalanced as well, and decomposes as $\mathcal{O}(-q+1) \oplus \mathcal{O}(-q)^{i-2} \oplus \mathcal{O}(-q-1)$, we find that

$$\begin{aligned} \mathrm{Hom}(K_i, K_{i+1}) =& \mathrm{End}(\mathcal{O}(-q+1)) \oplus \mathrm{Hom}(\mathcal{O}(-q)^{i-2}, \mathcal{O}(-q)^{i-1} \oplus \mathcal{O}(-q+1)) \\ & \oplus \mathrm{Hom}(\mathcal{O}(-q-1), \mathcal{O}(-q-1) \oplus \mathcal{O}(-q)^{i-1} \oplus \mathcal{O}(-q+1)) \end{aligned} \tag{3.70}$$

which has dimension $1 + [(i-2)(i-1) + 2(i-2)] + [1 + 2(i-1) + 3] = i^2 + i + 1$. Since $\mathrm{Aut}(K_i)$ now has dimension $i^2 + 1$, the loci where K_i are balanced and unbalanced have the same dimension, and the latter remains divisorial. □

Remark. This reflects the fact that, in the Quot scheme, the irreducibility of the the unbalanced locus requires the central term to be balanced. If one considers quotients of a fixed unbalanced vector bundle, one finds that the unbalanced loci of subbundles and quotient bundles similarly decompose.

We now set E^i to be the closure of the locus where $K_{i'}$ is unbalanced for $i' \geq i$ and balanced for $i' < i$. On the dual side, since the degrees will be decreasing, the condition on J_i (balanced if $n - i|d_i$, the incidence condition otherwise) will be independent of that on $J_{i'}$ for $i' \neq i$, following the case of uniform degree d. Thus, we can define $\check{E}^i$ just as above. Following an identical procedure, we find that

Theorem 27. *$[E^i]$ and $[\check{E}^i]$ are extremal rays of the effective cone of divisors.*

Theorem 28. *The effective cone is generated by the $[E^i]$, the $[\check{E}^i]$, and the $(n-1)$-st boundary divisor.*

Remark. For arbitrary $d = d_{n-1}$, if the d_i are maximally increasing, the generic K_i is balanced if it decomposes as

$$K_i \cong \bigoplus_{i'=1}^{i} \mathcal{O}(-a_{i'}), \quad a_i = \begin{cases} q-1 & i \leq s \\ q & i > s \end{cases} \tag{3.71}$$

where $q = \lceil \frac{d}{n-1} \rceil, s = (n-1)q - d$. Thus, it will be perfectly balanced for $i \leq s$ and split unevenly otherwise. In the former case, we use the same unbalanced locus above. If $i > s$, so $i \nmid d_i$, the incidence condition on $\mathcal{O}(-q+1)^s \subset K_i$ gives the same condition on $\mathcal{O}(-q+1)^s \subset K_{i+1}$. That is, the directrix of the scroll associated to K_i maps isomorphically onto that of K_{i+1}, and inductively for $K_{i'}$, $i' > i$. One can thus similarly define the divisors E_i, and the same result follows.

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