Topic #3

16.31 Feedback Control

Frequency response methods

- Analysis
- Synthesis
- Performance
- Stability
FR: Introduction

- Root locus methods have:
  - Advantages:
    - Good indicator of transient response;
    - Explicitly shows location of all closed-loop poles;
    - Trade-offs in the design are fairly clear.
  - Disadvantages:
    - Requires a transfer function model (poles and zeros);
    - Difficult to infer all performance metrics;
    - Hard to determine response to steady-state (sinusoids);
    - Hard to infer stability margins

- Frequency response methods are a good complement to the root locus techniques:
  - Can infer performance and stability from the same plot
  - Can use measured data rather than a transfer function model
  - Design process can be independent of the system order
  - Time delays are handled correctly
  - Graphical techniques (analysis and synthesis) are quite simple.
Frequency Response Function

- Given a system with a transfer function $G(s)$, we call the $G(j\omega)$, $\omega \in [0, \infty)$ the **frequency response function** (FRF)

\[
G(j\omega) = |G(j\omega)| \angle G(j\omega)
\]

- The FRF can be used to find the **steady-state** response of a system to a sinusoidal input since, if

\[
e(t) \rightarrow G(s) \rightarrow y(t)
\]

and $e(t) = \sin 2t$, $|G(2j)| = 0.3$, $\angle G(2j) = 80^\circ$, then the steady-state output is

\[
y(t) = 0.3 \sin(2t - 80^\circ)
\]

⇒ The FRF clearly shows the magnitude (and phase) of the response of a system to sinusoidal input

- A variety of ways to display this:

  1. Polar (Nyquist) plot – Re vs. Im of $G(j\omega)$ in complex plane.
     - Hard to visualize, not useful for synthesis, but gives definitive tests for stability and is the basis of the robustness analysis.

  2. **Nichols** Plot – $|G(j\omega)|$ vs. $\angle G(j\omega)$, which is very handy for systems with lightly damped poles.

  3. **Bode** Plot – Log $|G(j\omega)|$ and $\angle G(j\omega)$ vs. Log frequency.
     - Simplest tool for visualization and synthesis
     - Typically plot $20 \log |G|$ which is given the symbol $dB$
• Use logarithmic since if

$$\log |G(s)| = \left| \frac{(s + 1)(s + 2)}{(s + 3)(s + 4)} \right|$$

$$= \log |s + 1| + \log |s + 2| - \log |s + 3| - \log |s + 4|$$

and each of these factors can be calculated separately and then added to get the total FRF.

• Can also split the phase plot since

$$\angle \frac{(s + 1)(s + 2)}{(s + 3)(s + 4)} = \angle (s + 1) + \angle (s + 2) - \angle (s + 3) - \angle (s + 4)$$

• The keypoint in the sketching of the plots is that good straightline approximations exist and can be used to obtain a good prediction of the system response.
Bode Example

- Draw Bode for

\[ G(s) = \frac{s + 1}{s/10 + 1} \]

\[ |G(j\omega)| = \frac{|j\omega + 1|}{|j\omega/10 + 1|} \]

\[ \log |G(j\omega)| = \log[1 + (\omega/1)^2]^{1/2} - \log[1 + (\omega/10)^2]^{1/2} \]

- Approximation

\[ \log[1 + (\omega/\omega_i)^2]^{1/2} \approx \begin{cases} 0 & \omega \ll \omega_i \\ \log[\omega/\omega_i] & \omega \gg \omega_i \end{cases} \]

Two straightline approximations that intersect at \( \omega \equiv \omega_i \)

- Error at \( \omega_i \) obvious, but not huge and the straightline approximations are very easy to work with.

Figure 1: Frequency response basic approximation
To form the composite sketch,

- Arrange representation of transfer function so that DC gain of each element is unity (except for parts that have poles or zeros at the origin) – absorb the gain into the overall plant gain.
- Draw all component sketches
- Start at low frequency (DC) with the component that has the lowest frequency pole or zero (i.e. s=0)
- Use this component to draw the sketch up to the frequency of the next pole/zero.
- Change the slope of the sketch at this point to account for the new dynamics: -1 for pole, +1 for zero, -2 for double poles, . . .
- Scale by overall DC gain

Figure 2: $G(s) = \frac{10(s + 1)}{(s + 10)}$ which is a lead.
• Since \( \angle G(j\omega) = \angle(1 + j\omega) - \angle(1 + j\omega/10) \), we can construct phase plot for complete system in a similar fashion
  
  – Know that \( \angle(1 + j\omega/\omega_i) = \tan^{-1}(\omega/\omega_i) \)

• Can use straightline approximations

\[
\angle(1 + j\omega/\omega_i) \approx \begin{cases} 
0 & \omega/\omega_i \leq 0.1 \\
90^\circ & \omega/\omega_i \geq 10 \\
45^\circ & \omega/\omega_i = 1
\end{cases}
\]

• Draw components using breakpoints that are at \( \omega_i/10 \) and \( 10\omega_i \)

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**Figure 3:** Phase plot for \((s + 1)\)
• Then add them up starting from zero frequency and changing the slope as $\omega \to \infty$

![Phase plot](image)

Figure 4: Phase plot $G(s) = \frac{10(s + 1)}{(s + 10)}$ which is a “lead”.
Bode for $G(s) = \frac{4.54s}{s^3 + 0.1818s^2 - 31.1818s - 4.4545}$.

The poles are at (-0.892, 0.886, -0.0227)

Figure 5: More Complex case
Non-minimum Phase Systems

- Bode plots are particularly complicated when we have non-minimum phase systems
  - A system that has a pole/zero in the RHP is called non-minimum phase.
  - The reason is clearer once you have studied the Bode Gain-Phase relationship
- **Key point:** We can construct two (and many more) systems that have identical magnitude plots, but very different phase diagrams.
- Consider \( G_1(s) = \frac{s+1}{s+2} \) and \( G_2(s) = \frac{s-1}{s+2} \)

![Magnitude plots identical, but phase plots are dramatically different. NMP has a 180 deg phase loss over this frequency range.](image)

Figure 6: Magnitude plots identical, but phase plots are dramatically different. NMP has a 180 deg phase loss over this frequency range.
FR: Summary

- Bode diagrams are easy to draw
- Will see that control design is relatively straightforward as well
- Can be a bit complicated to determine stability, but this is a relatively minor problem and it is easily handled using Nyquist plots
- Usually only necessary to do one of Bode/Root Locus analysis, but they do provide different perspectives, so I tend to look at both in sisotool.