6.045J/18.400J: Automata, Computability and Complexity Prof. Nancy Lynch

Recitation 1: Math Review

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Definitions and Notation

Problem 1: Define the following words, phrases and symbols.

1. Set $A = \{x, y\}$, subset $B \subseteq A$, proper subset $B \subset A$, multiset $\{x, y, y\}$, power set $P(A)$, cardinality $|A|$, infinite set, natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$[1], integers $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, empty set $\emptyset$, union $A \cup B$, intersection $A \cap B$, Cartesian product $A \times B$, complement $\bar{A}$, sequence (x, y), k-tuple $(x_1, x_2, \ldots, x_k)$.

2. Function $f : D \to R$, domain D, range R, mapping $\to$, one-to-one, onto, bijection (one-to-one, onto).

3. Relation $R = \{(d_1, r_1), (d_2, r_2), \ldots, (d_i, r_i)\}$, reflexive $\forall x, xRx$, symmetric $\forall x, y,\ xRy$ iff yRx, transitive $\forall x, y, z,\ xRy \wedge yRz \Rightarrow xRz$, equivalence (reflexive, symmetric, transitive).

4. Graph $G = (V, E)$, degree, path, simple path, cycle, strongly connected.

5. Alphabet (input/output) $\Sigma = \{a, b, c\}$, symbols a, string $w = baac$, length $|w|$, empty string ϵ, substring (consecutive) baa, concatenation $w||w$ or ww, lexiographic ordering $(\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots)$[2], language $L = \{w_1, w_2, \ldots, w_\ell\}$.

6. Boolean logic $\{0, 1\}$, NOT $\neg p$, AND $p \wedge q$, OR $p \vee q$, XOR $p \oplus q$, implication $p \Rightarrow q$, equality $p \Leftrightarrow q$, distributive law $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$.

7. Theorem, lemma, corollary, proof, intuition, induction (assumes $P(n)$), strong induction (assumes $P(0), P(1), \ldots, P(n)$).

8. $(*)$ Machine, string accepted by a machine, language recognized by a machine.

Proof Techniques

Problem 2: **Set-Theoretic Equivalence**: Recall that in order to prove two sets A, B are equivalent, one must show that $A \subseteq B$ *and* $B \subseteq A$. Prove De Morgan's Law that $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Problem 3: **Proof by Contradiction**: If there are 6 people at a party shaking hands, then there must be at least two people who shook hands with the same number of other people.

Problem 4: **Proof by Induction (Base Case) (Induction Hypothesis) (Inductive Step)**: Problem 0.11 from Sipser's Text.
Find the *error* in the following proof that all horses are the same color.
Claim: In any set of h horses, all horses in the set are the same color.
Proof: By induction on h.
Basis: For $h = 1$. In any set containing just one horse, all horses clearly are the same color.
Inductive Step: For $k \geq 1$, assume that the claim is true for $h = k$ and prove that it is true for $h = k + 1$.

[1] Sipser, pg 4. Zero can also be included in $\mathbb{N}$.
[2] Observe the anomaly that 11 preceeds 000; length takes precedence.

Take any set H of $k+1$ horses. we will show that all horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all the horses in H_2 are the same color. Therefore, all the horses in H must be the same color and the proof is complete.

Problem 5: **Proof by Induction (Base Case) (Induction Hypothesis) (Inductive Step)**: Now *correctly* prove the following statement: $\forall n \in \mathbb{N},\ n^3 - n$ is divisible by 6.

Problem 6: **Proof by Contradiction**: Give a proof by contradiction of the statement of Problem 5. (Start by assuming that for some $n \in \mathbb{N}$, $n^3 - n$ is not divisible by 6).

Problem 7: **Double Induction**:

Let the function $R(s,t)$ (for $s,t \in \mathbb{N}$) be defined by the induction:

$$R(s,t) = R(s,t-1) + R(s-1,t)$$

and the base cases

$$R(s,2) = R(2,s) = s \quad (\text{ for all } s \in \mathbb{N})$$

Prove that $R(s,t) \leq \binom{s+t-2}{s-1}$.