Schramm-Loewner Evolution and Liouville Quantum Gravity

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We show that when two boundary arcs of a Liouville quantum gravity random surface are conformally welded to each other (in a boundary length-preserving way) the resulting interface is a random curve called the Schramm-Loewner evolution. We also develop a theory of quantum fractal measures (consistent with the Knizhnik-Polyakov-Zamolochikov relation) and analyze their evolution under conformal welding maps related to Schramm-Loewner evolution. As an application, we construct quantum length and boundary intersection measures on the Schramm-Loewner evolution curve itself.

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Introduction.—Liouville 2D quantum gravity was initially proposed by Polyakov in 1981 [1] to describe the summation over world sheets of a string (or a gauge-theoretic flux line). The resulting canonical 2D random surfaces, which depend on a real parameter γ, are also expected to arise as continuum limits of the random-planar-graph surfaces developed via random matrix theory, as first became evident (it remains to be proved rigorously) when Knizhnik, Polyakov, and Zamolodchikov (KPZ) [2,3] proposed their famous relation between critical exponents on a random surface and in the Euclidean plane. Via KPZ, Kazakov's exact solution of the Ising model on a random planar graph [4] matched Onsager's results in the plane. The KPZ relation itself was rigorously proven only recently [5].

Schramm-Loewner evolution.—Schramm-Loewner evolution (SLE), introduced by Schramm in 2000 [6], is a family of conformally invariant random curves in the plane, depending on a real parameter κ, which provides a canonical mathematical construction of the universal continuous scaling limit of 2D critical curves (such as percolation or Ising model interfaces). Its invention is on par with Wiener's 1923 mathematical construction of continuous Brownian motion. Critical phenomena in the plane were earlier well known to be related to conformal field theory (CFT) [7], a discovery anticipated in the so-called Coulomb gas approach to critical 2D statistical models (see, e.g., [8]) and now including SLE [9].

When describing a critical model on a random surface, Liouville field theory, itself a CFT, is coupled via KPZ to the corresponding CFT via a specific relation between the Liouville parameter γ and the CFT central charge c [2,3]. The heuristic value of this formalism was checked against manifold instances of exactly solved lattice models [10] and further used to predict properties of SLE [11].

The aim of this Letter is to provide the first direct and rigorous connection between SLE and Liouville quantum gravity: Gluing random surfaces (with the same γ) along parts of their boundaries—and conformally mapping the combined surface to the half-plane—produces an SLE curve with parameter $\kappa = \gamma^2$ as a random seam, also known as conformal welding. This in turn rigorously establishes the relation between γ and c in the Liouville-CFT correspondence mentioned above. (See [12] for mathematical details of this construction, a variant of which was first conjectured by Astala *et al.* [13].)

We also construct quantum gravity fractal measures, using the KPZ formula, and give a quantum gravity interpretation of related SLE processes, thereby providing a rigorous analog of the heuristic "gravitational dressing" of conformal scaling fields in Liouville theory coupled to CFT [2,3,10]. (See [14] for related ideas.)

Liouville quantum gravity.—Any simply connected Riemannian surface can be conformally mapped to a fixed flat domain $\mathcal{D} \subset \mathbb{C}$ and described by the induced area measure on $\mathcal{D}$. (Critical) Liouville quantum gravity consists of changing the (Lebesgue) area measure dz on $\mathcal{D}$ to the quantum area measure $d\mu_\gamma := e^{\gamma h(z)} dz$, where γ is a real parameter and h is an instance of the (zero boundary for now) massless Gaussian free field (GFF), with Dirichlet energy or critical Liouville action $(4\pi)^{-1} \int_{\mathcal{D}} [\nabla h(z)]^2 dz$, and whose two point correlations are given by Green's function on $\mathcal{D}$. The GFF h is a random distribution, not a function, but the measure $d\mu_\gamma$ can be constructed [for $\gamma \in [0, 2)$] [5] as the limit as $\varepsilon \to 0$ of the regularized quantities $d\mu_{\gamma,\varepsilon} := \varepsilon^{\gamma^2/2} \exp[\gamma h_\varepsilon(z)] dz$, where $h_\varepsilon(z)$ is the mean value of h on the circle $\partial B_\varepsilon(z)$, boundary of the ball $B_\varepsilon(z)$ of radius ε centered at z; note, in particular, that $\mathbb{E} e^{\gamma h_\varepsilon(z)} = [C(z; \mathcal{D})/\varepsilon]^{\gamma^2/2}$ [5], where $C(z; \mathcal{D})$ is the conformal radius of $\mathcal{D}$ viewed from z (i.e., up to a constant factor, the distance from z to the boundary $\partial \mathcal{D}$).

Quantum fractal measures and KPZ.—We will now discuss Euclidean and quantum "fractal measures" and provide a new heuristic but genuine derivation of the celebrated KPZ formula [2]. The d-dimensional Euclidean or analogously quantum measure of planar fractal sets is characterized by scaling properties: (i) If we rescale a d-dimensional fractal $X \subset \mathcal{D}$ via the map $z \to \psi(z) = bz$, $b \in \mathbb{C}$ (so that the Euclidean area of $\mathcal{D}$ is multiplied by $|b|^2$), then the d-dimensional Euclidean fractal measure of X is multiplied by $|b|^d = |b|^{2-2x}$, where x (the so-called Euclidean scaling weight) is defined by $d := 2 - 2x(\leq 2)$. (ii) If X is a fractal subset of a random surface $\mathcal{S} := (\mathcal{D}, h)$, and we rescale $\mathcal{S}$ so that its quantum area increases by a factor of $|b|^2$, then the quantum fractal measure of X is multiplied by $|b|^{2-2\Delta}$, where Δ is some analogous quantum scaling weight.

The above assertions suggest that the (γ-dependent) Liouville quantum measure $\mathcal{Q}(X, h)$ of a fractal $X \subset \mathcal{D}$ should satisfy some fundamental scaling properties: (i) If λ_0 is a constant, then

$$\mathcal{Q}(X, h + \lambda_0) = e^{\alpha\lambda_0}\mathcal{Q}(X, h), \tag{1}$$

$$\alpha := \gamma(1 - \Delta). \tag{2}$$

(ii) If $\psi(z) = bz$, then

$$\mathcal{Q}\left(\psi(X), h \circ \psi^{-1}\right) = |b|^{d+\alpha^2/2}\mathcal{Q}(X, h). \tag{3}$$

We explain (3) heuristically: If we can cover X by $\mathcal{N}$ radius-ε balls, then it takes $\mathcal{N}|b|^d$ such balls to cover bX. One next observes that $\underline{h}(\cdot) := h(\cdot) - h_\varepsilon(z)$ on $B_\varepsilon(z)$, given $h_\varepsilon(z)$, is a projected GFF on a disk, which is independent of $h_\varepsilon(z)$ and z (up to negligible effects of $\partial\mathcal{D}$; see [5]), so one can apply (1) to $\underline{h} + \lambda_0$, with the local shift $\lambda_0 = h_\varepsilon(z)$. The expected resulting conformal factor $\mathbb{E}e^{\alpha h_\varepsilon(z)}$ will be $|b|^{\alpha^2/2}$ times larger in the domain $b\mathcal{D}$, because of the scaling $C(bz; b\mathcal{D}) = |b|C(z; \mathcal{D})$. Thus the expected (with respect to h) quantum measure of bX within one of the ε balls covering bX (near bz) should be $|b|^{\alpha^2/2}$ times that of X within one of the ε balls covering X (near z). The law of large numbers then yields (3). (iii) $\mathcal{Q}(\psi(X, h)) = \mathcal{Q}(X, h)$ whenever ψ is conformal and

$$\psi(\mathcal{D}, h) := (\psi(\mathcal{D}), h \circ \psi^{-1} - Q\log|\psi'|),$$
$$Q = \frac{\gamma}{2} + \frac{2}{\gamma}. \tag{4}$$

This is because (see [5,10]), the pair $\mathcal{S} = (\mathcal{D}, h)$ describes the same Liouville quantum surface (up to coordinate change) as the conformally transformed pair $\psi(\mathcal{D}, h)$.

These properties taken together [for $\psi(z) = bz$] imply

$$d = \alpha Q - \alpha^2/2, \tag{5}$$

which by (2) and (4) is equivalent to the KPZ formula [2]: $x = (\gamma^2/4)\Delta^2 + (1 - \gamma^2/4)\Delta$.

SLE definition.—Chordal SLE [6] is a random non-self-crossing path in the complex half-plane $\mathbb{H}$; we mainly use here a (time-reversed) version defined at time $t \geq 0$ by a "zipping-up" conformal map $w := f_t(z)$, from the complex half-plane $\mathbb{H}$ to the slit domain $\mathbb{H} \setminus \eta_t$, with the SLE segment $\eta_t := f_t(\mathbb{R}) \setminus \mathbb{R}$ (or its external envelope) from 0 to the tip $f_t(0)$ (Fig. 1). It satisfies the stochastic differential equation $df_t(z) = -2dt/f_t(z) - \sqrt{\kappa}dB_t$ [with $f_0(z) = z$], where B_t is standard Brownian motion with $B_0 = 0$, and

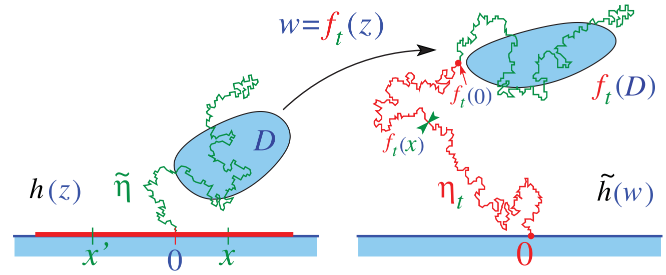

FIG. 1 (color). Chordal zipping-up SLE$_\kappa$ map $w = f_t(z)$ with curve η_t in $\mathbb{H}$. Given f_t, the GFF h can be sampled as the pullback $\tilde{h} \circ f_t$ of a free boundary GFF $\tilde{h}$, plus the process $\mathfrak{h}_t$ (7). Conformal welding: the quantum boundary lengths of any real segments $[0, x]$ and $[x', 0]$ such that $f_t(x) = f_t(x') \in \eta_t$ are equal [12]. The SLE$_\kappa$ $X = \tilde{\eta}$ on the left is h-independent.

$\kappa \geq 0$. If $0 \leq \kappa \leq 4$, then SLE$_\kappa$ is a simple curve, while for $4 < \kappa < 8$ it develops double points and becomes space-filling for $\kappa \geq 8$ [15]. Of particular physical interest are the loop-erased random walk ($\kappa = 2$) [16], the self-avoiding walk ($\kappa = 8/3$), the Ising model interface ($\kappa = 3$ or $16/3$) [17], the GFF contour lines ($\kappa = 4$) [18], and the percolation interface ($\kappa = 6$) [19].

A (reverse) SLE martingale.—Define a real stochastic process for $t \geq 0$ and $z \in \mathbb{H}$, by

$$\mathfrak{h}_0(z) := (2/\sqrt{\kappa})\log|z|, \tag{6}$$

$$\mathfrak{h}_t(z) := \mathfrak{h}_0 \circ f_t(z) + Q\log|f_t'(z)|. \tag{7}$$

By stochastic Itô calculus [i.e., by using the Brownian local covariations $d\langle B_t, B_t\rangle = (dB_t)^2 = dt$, $d\langle B_t, t\rangle = dB_t dt = 0$, and $d\langle t, t\rangle = (dt)^2 = 0$], the particular choice in (7),

$$Q = \sqrt{\kappa}/2 + 2/\sqrt{\kappa}, \tag{8}$$

gives a driftless diffusion process $d\mathfrak{h}_t(z) = -R_t(z)dB_t$, with $R_t(z) := \Re[2/f_t(z)]$. Then $\mathfrak{h}_t(z)$ is a time-changed Brownian motion (called a local martingale) with local covariation $d\langle\mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle = R_t(y)R_t(z)dt$, having the further martingale property $\mathbb{E}\mathfrak{h}_t(z) = \mathfrak{h}_0(z)$.

Consider now the Neumann Green function in $\mathbb{H}$, $G_0(y, z) := -\log(|y - z||y - \bar{z}|)$, and define the time-dependent $G_t(y, z) := G_0(f_t(y), f_t(z))$, i.e., G_0 taken at image points under f_t. A simple calculation of the Green function's variation shows that $-dG_t(y, z) = d\langle\mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle$ (Hadamard's formula). Integrating with respect to t yields the covariation of the $\mathfrak{h}_t$ martingales

$$\langle\mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle = G_0(y, z) - G_t(y, z). \tag{9}$$

Taking the limit $y \to z$ in the latter, one obtains

$$\langle\mathfrak{h}_t(z), \mathfrak{h}_t(z)\rangle = C_0(z) - C_t(z), \tag{10}$$

where $C_t(z) := -\log[\Im f_t(z)|f_t'(z)|]$.

SLE-GFF coupling.—Consider $h := \tilde{h} + \mathfrak{h}_0$, the sum of an instance $\tilde{h}$ of the Gaussian free field on domain $\mathcal{D} = \mathbb{H}$ with free boundary conditions on $\mathbb{R}$ (up to an additive constant) and of the deterministic function $\mathfrak{h}_0$ (6). This h can be coupled [12] with the reverse Loewner evolution f_t described above so that, given f_t, the conditional law of h (denoted by $h|f_t$) is (Fig. 1)

$$h(z)|f_t \overset{\text{(law)}}{=} \tilde{h} \circ f_t(z) + \mathfrak{h}_t(z), \tag{11}$$

where $\tilde{h} \circ f_t$ is the pullback of the free boundary GFF $\tilde{h}$ in the image half-plane and where $\mathfrak{h}_t$ is the martingale (7). This means that, to sample h, one can first sample the B_t process (which determines f_t), then sample independently the free boundary condition GFF $\tilde{h}$, and take (11). Its conditional expectation with respect to $\tilde{h}$ is the martingale $\mathbb{E}[h(z)|f_t] = \mathfrak{h}_t(z)$. Recall that the Green's function $G_0(y, z) = \mathrm{Cov}[\tilde{h}(y), \tilde{h}(z)]$, and thus $G_t = \mathrm{Cov}[\tilde{h} \circ f_t, \tilde{h} \circ f_t]$. The random distribution $\tilde{h} \circ f_t$ and the set of (time-changed) Brownian motions h_t are Gaussian processes, whose respective covariance G_t and covariation $\langle h_t, h_t \rangle$ thus add from (9) to the constant covariance G_0; this in essence yields (11) [12].

Liouville invariance.—Owing to (7), we observe that the right-hand side of (11) is of the form $h \circ f_t + Q \log|f_t'|$. For Q given by (4), this is precisely the transformation law (4) of the GFF h under the conformal map f_t^{-1} [5,10]. Then the pair $(\mathbb{H}, \tilde{h} \circ f_t + \mathfrak{h}_t) = f_t^{-1}(\mathbb{H} \setminus \eta_t, h)$ describes the same random surface as the pair $(\mathbb{H} \setminus \eta_t, h)$: Given f_t, the image under f_t of the measure $e^{\gamma h(z)}dz$ in $\mathbb{H}$ is a random measure whose law is the *a priori* (unconditioned) law of $e^{\gamma h(w)}dw$ in $\mathbb{H} \setminus \eta_t$.

By identifying (4) and (8), we find two dual solutions:

$$\gamma = \sqrt{\kappa \wedge (16/\kappa)}, \qquad \gamma' = 4/\gamma. \tag{12}$$

The first solution $\gamma \leq 2$ corresponds precisely to the famous relation [2,3,10] $\gamma = (\sqrt{25 - c} - \sqrt{1 - c})/\sqrt{6}$, between the parameter γ in Liouville theory and the central charge $c = \frac{1}{4}(6 - \kappa)(6 - 16/\kappa)$ of the SLE's CFT [9] coupled to gravity. The second solution $\gamma' = 4/\gamma \geq 2$ corresponds to a dual model of Liouville quantum gravity, in which the quantum area measure develops atoms with localized area [5,20], and will be discussed elsewhere.

Quantum conformal welding.—In the particular coupling (11) of h and f_t, the two strands of the boundary to be matched along η_t when zipping up by the reverse Schramm-Loewner map f_t have the same quantum length (at least for $\kappa < 4$) (Fig. 1). This property defines a quantum conformal welding and actually determines f_t as a function of h [12].

Let now $\tilde{\eta}$ be an (infinite) SLE$_\kappa$, independent of h (Fig. 1). For each time $t \geq 0$, the forward, "zipping-down" SLE flow map f_{-t}, which obeys the same stochastic differential equation as f_t, but for $dt \to -dt$, maps $\mathbb{H} \setminus \tilde{\eta}_t \to \mathbb{H}$, where $\tilde{\eta}_t$ is the SLE curve segment up to time t. When $\kappa < 4$, $\tilde{\eta}$ divides $\mathbb{H}$ into a pair of welded quantum surfaces that is stationary with respect to zipping up or down via the transformations f_t $(t \in \mathbb{R})$ [12]. The relation (12) between γ and κ is now rigorously clear: Conformally welding two γ-quantum surfaces produces SLE$_\kappa$.

Exponential martingales.—Let us introduce the conditional expectations of exponentials of the field (11), $\mathcal{M}_t^\alpha(z) := \mathbb{E}[e^{\alpha h(z)}|f_t]$, depending on a real parameter α, which are fundamental objects describing quantum gravity coupled to the SLE process. They can be calculated explicitly in terms of (7) and (10):

$$\mathcal{M}_t^\alpha(z) = \exp[\alpha \mathfrak{h}_t(z) + (\alpha^2/2)C_t(z)] \tag{13}$$

$$= |f_t'(z)|^d |w|^{2\alpha/\sqrt{\kappa}} (\Im w)^{-\alpha^2/2}, \tag{14}$$

with $w = f_t(z)$ and d given by the KPZ formula (5). Because of (10), Eq. (13) is an exponential martingale with respect to the Brownian motion driving the reverse SLE process, so that

$$\mathbb{E}\, \mathcal{M}_t^\alpha(z) = \mathcal{M}_0^\alpha(z) = |z|^{2\alpha/\sqrt{\kappa}} (\Im z)^{-\alpha^2/2}. \tag{15}$$

A stronger statement is the identity in law of the conditional exponential measure

$$(e^{\alpha h(z)}|f_t)dz \overset{\text{(law)}}{=} |f_t'(z)|^{d-2} e^{\alpha h(w)} dw, \tag{16}$$

with $dw = |f_t'(z)|^2 dz$, and whose expectations (14) agree.

Expected quantum area.—For $\alpha = \gamma$ (12), $d = 2$ in (5)

$$\begin{aligned} d\mathcal{A} &:= dz \mathbb{E}[e^{\gamma h(z)}|f_t] \\ &= dw|w|^{2-\kappa/2}(\sin\varphi)^{-\kappa/2}, \qquad \kappa \leq 4 \\ &= dw(\sin\varphi)^{-8/\kappa}, \qquad \kappa \geq 4; \qquad \varphi := \arg w. \end{aligned}$$

We now construct explicit invariant SLE quantum measures, by using the martingales (13) for $\alpha \neq \gamma$.

SLE quantum length measure.—An SLE measure recently introduced in the context of the so-called natural parametrization of SLE [21] describes the "fractal length" of the intersection $X \cap D$ of the SLE$_\kappa$ fractal path $X = \tilde{\eta}$ (from 0 to ∞) with an arbitrary domain $D \subset \mathbb{H}$ (Fig. 1). It is shown in Ref. [21] that its expectation with respect to the SLE$_{\kappa \in [0,8]}$ law is finite for any bounded D and given by $\nu(D) := \int_D G(z)dz$, where $G(z) := |z|^a |\Im z|^b$, with $a = 1 - 8/\kappa$ and $b = 8/\kappa + \kappa/8 - 2$, is the SLE Green's function in $\mathbb{H}$. Under the forward direction SLE flow f_{-t} that generates $X = \tilde{\eta}$, the quantity $M_t := (G \circ f_{-t})|f_{-t}'|^{2-d}$, where $d := 1 + \kappa/8$ is the SLE$_\kappa$ (Hausdorff) fractal dimension [22], describes the density of expected Euclidean fractal length of $X \setminus \tilde{\eta}_t$, given the segment $\tilde{\eta}_t$ [21]. This M_t is a local martingale with respect to the forward SLE flow f_{-t} [21]. Geometrically, $\int_D M_t(z)dz$ is the expected length of $X \cap D$ given f_{-t} (a martingale), minus the length of the segment $\tilde{\eta}_t \cap D$ (an increasing process); this so-called Doob-Meyer decomposition is unique and actually determines the latter length as a stochastic process [21].

We extend this construction to the quantum case by defining the expected (with respect to X, given h) Liouville quantum length ν_Q of an infinite SLE path in a domain D:

$$\nu_Q(D, h) := \int_D e^{\alpha h(z)} G(z) dz, \tag{17}$$

where $\alpha = \sqrt{\kappa}/2$ ($= \gamma/2$ for $\kappa \leq 4$ and $\gamma'/2$ for $\kappa > 4$) is chosen to satisfy KPZ (5) for the SLE dimension $d = 1 + \kappa/8$ (and Seiberg's bound $\alpha \leq Q$ [5,23]). Under the forward SLE flow f_{-t}, the corresponding integral $\int_D e^{\alpha h(z)} M_t(z) dz$ yields, by Doob-Meyer, an implicit construction of the quantum length measure. (It exists by Ref. [24] since the second moment $\mathbb{E}[e^{\alpha h(y) + \alpha h(z)} M_t(y) M_t(z)]$ is bounded by $|y - z|^{\mathfrak{d}-2}$, with $\mathfrak{d} = d - \alpha^2 = 1 - \kappa/8$, thus integrable for $\mathfrak{d} > 0$, i.e., $\kappa < 8$. It coincides with the Liouville boundary measure defined on $\mathbb{R}$ by unzipping via f_{-t} [5,12]; this follows rigorously from [21] under a finite expectation assumption.)

Alternatively, using (16), we can condition (17) on the reverse SLE flow f_t and get the transformation law

$$\nu_Q | f_t := \int_D (e^{\alpha h(z)} | f_t) G(z) dz \overset{\text{(law)}}{=} \int_{D_t} e^{\alpha h(w)} N_t(w) dw,$$

where $D_t := f_t(D)$ and where $N_t(w) := G(z) |f_t'(z)|^{d-2}$, with $z = f_t^{-1}(w)$, formally corresponds to replacing in the martingale M_t the zipping-down map f_{-t} by the inverse map f_t^{-1} (which has the same law). The expectation of (17) with respect to h, conditioned on f_t, is from (14):

$$\mathbb{E}[\nu_Q | f_t] = \int_D \mathcal{M}_t^\alpha(z) G(z) dz = \int_{D_t} \mathcal{M}_0^\alpha(w) N_t(w) dw,$$

where $\mathcal{M}_0^\alpha(w) = |w| (\Im w)^{-\kappa/8}$ is the (unconditioned) free boundary GFF expectation $\mathbb{E} e^{\alpha h(w)}$. Finally, taking expectation with respect to f_t via (15) gives the expected quantum length in D, finite for $\kappa \in [0, 8)$ (here $\vartheta := \arg z$):

$$\mathbb{E}\, \nu_Q(D) = \int_D dz \mathcal{M}_0^\alpha(z) G(z) = \int_D (\sin \vartheta)^{8/\kappa - 2} dz;$$

it coincides with the Euclidean area of D for $\kappa = 4$.

Boundary exponential martingales.—Consider now the reverse SLE map $f_t(x)$ restricted to the real axis, with $x \in f_t^{-1}(\mathbb{R}_+)$, such that $f_t'(x) \geq 1$ [15]. The boundary analogs of the exponential martingales (13) are

$$\hat{\mathcal{M}}_t^\beta(x) := \mathbb{E}(e^{\beta h(x)} | f_t) = e^{\beta \mathfrak{h}_t(x)} [f_t'(x)]^{-\beta^2},$$

for any real β, such that $\mathbb{E} \hat{\mathcal{M}}_t^\beta(x) = \hat{\mathcal{M}}_0^\beta(x) = x^{2\beta/\sqrt{\kappa}}$. From (7) one has $\hat{\mathcal{M}}_t^\beta(x) = f_t'(x)^{\hat{d}} u^{2\beta/\sqrt{\kappa}}$ with $u := f_t(x)$ and $\hat{d} = \beta Q - \beta^2$, the boundary analog of KPZ (5) [5].

The expected Liouville quantum boundary length $d\mathcal{L} := dx \mathbb{E}\{\exp[\hat{\beta} h(x)] | f_t\}$ is obtained for $\hat{d} = 1$, with $\hat{\beta} = \gamma/2$ as expected [5] and with the invariant forms $d\mathcal{L} = u du$ for $\kappa \leq 4$ and $d\mathcal{L} = u^{4/\kappa} du$ for $\kappa > 4$.

SLE quantum boundary measure.—A boundary fractal measure $\hat{\nu}$, supported on the intersection of a chordal SLE$_\kappa$ curve $X = \tilde{\eta}$ with the axis $\mathbb{R}$, for $\kappa \in (4, 8)$, has been constructed recently [25], For any interval $I \subset \mathbb{R}_+$, its expectation is the simple integral $\hat{\nu}(I) = \int_I x^{\hat{d}-1} dx$, where $\hat{d} = 2 - 8/\kappa$ is the SLE$_\kappa$ Hausdorff boundary dimension. We define the SLE expected quantum boundary measure $\hat{\nu}_Q$ as

$$\hat{\nu}_Q(I, h) := \int_I e^{\beta h(x)} x^{\hat{d}-1} dx,$$

where $\beta = \sqrt{\kappa}/2 - 2/\sqrt{\kappa}$ satisfies the boundary KPZ relation above for $\hat{d}$ (and the boundary Seiberg bound $\beta \leq Q/2$ [5,23]). As in the bulk case, Doob-Meyer and integrability arguments imply that the measure exists and is nontrivial. Its expectation $\mathbb{E} \hat{\nu}_Q(I) = \int_I x^{2-12/\kappa} dx$ is finite for any $\kappa \in (4, 8]$; it coincides with the Euclidean boundary length for $\kappa = 6$.

We provided a foundational relationship between SLE, KPZ, and Liouville quantum gravity. We hope that it will help to solve the outstanding open problem of rigorously relating them to discrete models and random planar maps.

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